MATHEMATICAL DEFINING AS A PRACTICE: INVESTIGATIONS OF CHARACTERIZATION, INVESTIGATION, AND DEVELOPMENT

By

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To my advisor, Rich Lehrer, for providing infinite support and guidance

and

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CHAPTER I

INTRODUCTION

In recent years, the field of mathematics education has advocated for an expanded view of what it means to know mathematics and participate in mathematics as a practice. The National Research Council (2001) summarized this as a shift away from an entirely procedural mathematics to a more encompassing view that includes developing relations between conceptual and procedural forms of mathematics and learning to participate in epistemic practices of knowledge creation and revision, such as defining, making conjectures and proving. The aim is to support learner agency as creators and doers of mathematics. Here, I present three papers that investigate how students participate in one mathematical practice, defining. In many classrooms, definitions are often treated as given, rather than as negotiated, as they are in the discipline. Historically, mathematicians participated in the co-construction of definitions, and defining often emerged as an adjunct of proving (Lakatos, 1976). Consequently, we need to find ways of engaging students in *defining as a practice* by providing them with opportunities to make sense of and construct definitions, and, in turn, become *authors* of definition (e.g., de Villiers, 1998; Keiser, 2000; Lehrer & Curtis, 2000). Practice refers to recurrent forms of activity that those participating in them identify and recognize as contributing to the accomplishment of a particular goal or experience.

In these papers, I examined the process of instigating and tracing change in students' engagement in the practice of defining via three forms of investigation. First, I conducted a literature review of research in which students participated in defining as a practice. My goal for the paper was to develop a framework identifying key forms of participation in defining that are particular to classrooms, what I term, *Aspects of Definitional Practice*. For example, aspects included *asking definitional questions, constructing and/or evaluating examples,* and *constructing definitional explanations or arguments*. Second, I used the Aspects framework as a lens for investigating how defining was initially established in one middle school classroom within the first few days of instruction. This analysis focused on how defining was realized in interactions among students and between the teacher and students. Third, by looking at the same group of students over a slightly longer period, I expanded and refined my analyses from the second paper to study how students' participation in Aspects of Definitional Practice developed over time and how change in participation influenced the development of mathematical knowledge.

In the first paper, I developed the framework of Aspects of Definitional Practice by reviewing 19 empirical studies in which researchers instigated and/or studied students' engagement in defining as a practice. These studies varied in content, context and in the age of the students. The framework was developed through a method of iterative refinement, using the lens of disciplinary perspectives on definitions and defining to determine what constituted an aspect of practice. These aspects characterize how students from previous studies (of all ages) have participated in defining in ways representative of, yet distinct from, professional mathematicians.

My second paper (Paper 2) consists of two conference papers. The conference papers both present versions of the same analysis aimed at understanding the establishment of definitional practice. Practice is ultimately tied to the production of knowledge, and in the case of defining, tied to the production of definitions, to close examination of the properties of the objects being defined, and to the network of relations by which new definitions build on established definitions. Accordingly, I asked the questions: (1) How are knowledge and practice co-constituted? (2) How do participants in the community contribute to, or support, this co-constitution? To answer these questions, I drew upon data collected as part of a larger study in which sixth-grade students investigated topics in geometry for approximately six months (Lehrer, Kobiela, Weinberg, in press). Our aim in working with these students was to cultivate a culture of inquiry in which students' questions and conjectures guided many of our investigations and in which we leveraged their experiences of moving and walking as resources for reasoning mathematically.

I conducted three forms of analysis, focusing on the first six days of mathematics instruction in this classroom. First, I characterized the development of communal knowledge, representing it as a system of mathematical objects and relations that stood for the mathematical terrain investigated by the class. Second, I characterized interactions around the practice of defining by looking at how they participated in Aspects of Definitional Practice and at how the teacher supported that participation. In doing so, I focused on three 10-minute excerpts that spanned the six days. Finally, I compared my first two analyses side-by-side to develop conjectures of how practice and knowledge were co-constituted. In particular, I described three ways in which this co-constitution occurred.

In the third paper, I extended my methods and analyses from Paper 2 to ask: How might the practice of defining and the knowledge developed by that practice co-develop in a mathematics learning community? To do so, I examined one additional excerpt from the twenty-seventh day of mathematics instruction, about two and a half months later and situated the excerpt within other defining activity that occurred within the larger data corpus. The existing research illustrates that it is possible to engage students in defining, and that doing so supports students' mathematical understandings and provides them opportunities to participate in mathematically productive discourse (e.g., Borasi, 1992; Keiser, 2000; Lehrer, Randle, & Sancilio, 1989; Lehrer, Jacobson, Kemeny, & Strom, 1999; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). However, most of these papers primarily present analyses of very short excerpts of class activity, less than two class periods, and are often illustrations of already established practice. The studies that present longer time scales focus analytically on students' development of conceptions or orientations towards defining rather than on shifts in student participation in practice (Borasi, 1992; Keiser, 2000). Thus, very little is known about how the practice of defining develops. The third paper aims to address this need.

Taken collectively, the three papers provide: (a) an analytic and theoretical framework for examining the mathematical practice of defining as it might be constituted in classrooms; (b) an analysis of the initial establishment of this form of practice as instantiated in interaction among students and their teacher; and (c) an investigation of how knowledge, practice and the interactions that contribute to their co-constitution develop and change over time. These strands of investigation aim to cash in on the promise of the re-conceptualization of school mathematics suggested by the National Research Council and more recently, by the common core standards in mathematics, which interweave mathematical practices and conceptual development. I argue that these three forms of investigation are needed to support ongoing efforts in studying defining in mathematics classrooms. Additionally, I hope the papers will inform efforts to support the design and implementation of similar learning environments by providing teachers and researchers a lens for interpreting student participation and learning.

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CHAPTER II

MATHEMATICAL DEFINING AS A CLASSROOM PRACTICE: A REVIEW

Introduction

In recent years, the field of mathematics education has advocated for an expanded view of what it means to know mathematics and participate in mathematics as a practice. The National Research Council (2001) summarized this as a shift away from an entirely procedural mathematics to one that encompasses several strands of practice. In this expanded view, participants of mathematics develop conceptual understanding of mathematical ideas and entities, strategize about problems solved, develop fluency with procedural rules, engage in mathematical arguments and explanations and develop agency as a “doer” of mathematics. Subsequently, recent research has attended to what Lehrer (2009) calls developing students’ “disciplinary dispositions” (p. 762) as they partake in mathematical practices such as argument, explanation and general disciplinary discourse.

Defining is one mathematical practice that deserves increasing attention for two reasons. First, typical classroom approaches to mathematical definition contrast sharply to disciplinary practices. Mathematical definitions are often treated axiomatically as ideas to be quickly memorized so that students may move on to mathematically “richer” work: activities ranging from applying algorithms and problem solving to exploration, argumentation, and proof (Keiser, 2000, citing Fawcett, 1995). Definitions are typically dictated by the authority of the teacher or textbook, masking the process of how they came to exist. Historically, however, definitions were created and adapted by mathematicians in the process of constructing proofs and creating theory

(Lakatos, 1976), sometimes evolving over long periods of time. Moreover, defining is not a dead practice. The need for new definitions arises through activities such as problem-solving and proof, through opportunities to model the world and through a necessity to characterize new mathematical objects (Ouvrier-Buffet, 2006). Objects being defined evolve in the very act of defining. For example, in the history of investigations of relations among edges, vertices, and faces of polyhedra traced by Lakatos (1976), definitions of polyhedra were continually re-considered as mathematicians proposed new candidates. In these situations, definitions are negotiated among human agents and learning may be thought of as participation in such interactions (Herbst, 2005).

Second, many studies show that typical classroom approaches are inadequate for helping students develop conceptual understanding of definitions. Often students struggle to correctly recall definitions already learned (Vinner, 1983; Vinner & Dreyfus, 1989; Vinner, 1991). When students are able to recall definitions, they do not always use them when reasoning about the concept referred to by the definition (Zazkis & Liljedahl, 2004), but instead use what Vinner (1991) calls their *concept images*. The *concept image* is the image evoked when one hears or thinks of the object and may be represented visually as pictures, symbols, equations, graphs or as a set of properties. Often students develop concept images that emulate prototypical examples they experience repeatedly both in and out of school. From these prototypical examples, students often extract defining features for the object that are not characteristic of the object's definition. This holds true for a diversity of students, including mathematically gifted students in high school and college, and across a range of mathematical topics. Moreover, students' difficulties with definitions have been found to be the root of many students' problems with proofs (Moore,

1994) and generation of appropriate examples when developing proofs (Zaslavsky & Peled, 1996).

However, despite the need to reconsider the role of definition in mathematics education, it is less clear what classroom activities would be entailed by this change in stance toward definition. Some advocate providing students with opportunities to make sense of and construct definitions themselves, and, in turn, become *authors* of definition (de Villiers, 1998; Keiser, 2000). Accordingly, I reviewed the collection of research in which students participated in definition, with a focus on three questions: (1) What is the nature of students' participation in defining? In particular: a) what types of tasks are designed to provide students opportunities to define? and b) what are characterizations of their engagement in the practice? (2) In view of (1), in what ways is defining profitable for students? (3) How is defining supported by learning ecologies, including teachers' practices?

To attend to these questions, I first situate the paper by outlining definitions from a disciplinary perspective and describing three lenses for looking at supporting student engagement in classroom practices. I then detail the methodological considerations for conducting this review. I follow with the Results of the Review, where I attend to each of the three questions. First, I provide an overview of the types of defining tasks scholars have engaged students in. I then highlight particular Aspects of Definitional Practice within the studies, where I consider *Aspects of Definitional Practice* to be forms of participation in defining tasks related to those within the discipline of mathematics. I suggest that it is important to attend to such nuances of practice because they provide a lens for educators and researchers for supporting students' participation in defining. Second, I describe the affordances of these activities, in particular for supporting students' engagement in those disciplinary practices, understanding of the definitions, development of disciplinary dispositions, and for motivating a closer analysis of the objects and relations being defined. Finally, I highlight aspects of instructional design and teacher's roles in orchestrating class discussions that appear to contribute to these affordances. In the Discussion, I conclude by suggesting new directions in research around mathematical defining. I argue that the studies show that defining is a worthwhile endeavor, but nevertheless, more research is needed about teaching practices that support defining and about long-term development when such teaching practices are in place.

Theoretical Perspectives

Throughout the paper, I will refer to students' participation in defining as their engagement in *Aspects of Definitional Practice*. I consider *Aspects of Definitional Practice* to be forms of participation in defining aligned with practices in the discipline of mathematics. I choose this perspective on practice because of the recent movement in mathematics education towards such a lens (Lampert, 1990) and because definitions are typically treated in a manner opposite from disciplinary practice. Due to this disciplinary focus, I first detail how I situate this paper with respect to how definitions and defining are typically framed within the field of mathematics. I then draw upon work that frames my inquiries about how classroom environments are designed to support definition.

Disciplinary Perspectives on Definitions and Defining

Mathematical definitions. I begin by highlighting forms, roles, and properties of definitions noted as relevant by the community of mathematicians. These qualities are significant to note because they guide Aspects of Definitional Practice. A *mathematical definition* is a description of the properties of a mathematical object and the relations among those properties (Lehrer & Curtis, 2000; Polya, 1957). A *mathematical object* is an abstract category produced via reification of activity (Sfard, 1991). Examples of objects include geometric shapes and their components (e.g., circles, ellipses, squares, sides, angles), analytic concepts (e.g., functions, limits) and types of number (e.g., even, odd, composite, prime). Mathematical definitions come in two forms: *structural* and *procedural* (Zaslavsky & Shir, 2005). Structural definitions describe a mathematical object's components. For instance, a structural definition of "angle" might be "two connected sides." Procedural definitions, on the other hand, describe how an object is created. A procedural definition of angle might be "a turn" or, alternatively, a measurement of rotation. Structural and procedural definitions serve equally important roles because they highlight different attributes and relations of mathematical objects.

Mathematical definitions are distinct from other mathematical entities – questions, conjectures, axioms, lemmas, theorems or corollaries – because they are the *negotiated grounds* for mathematical work. Unlike axioms, definitions are contested rather than taken for granted and unlike lemmas, theorems or corollaries, definitions cannot be proven. Definitions, however, resemble theorems in that they may be challenged (Van Dormolen & Zaslavsky, 2003) and are historically the result of mathematical arguments (Lakatos, 1976).

Mathematical definitions serve several purposes. First, definitions are used to introduce new objects to the field of mathematics (Borasi, 1992; Zaslavsky & Shir, 2005). As objects are introduced and used, definitions describe their essential properties and relations. This in turn provides participants in a mathematical community a means of communicating about mathematical ideas (Zaslavsky & Shir, 2005, Citing Borasi, 1992). Other mathematical practices are built directly on systems of definitions. For example, a procedural definition of angle, as turns, supports conversations about walking the perimeter of different polygons. This "path" perspective (Abelson & diSessa, 1980) provides a way of reasoning about the sums of the exterior angles of polygons. Considering only a structural definition of angle may change the types of mathematical work, such as proof, available to a student.

Mathematical definitions also have several distinct features, many of which are influenced by the roles definitions play. As stated above, definitions describe a mathematical object's essential properties and relations. The properties communicated must be *non-contradicting* and *noncircular* (Borasi, 1992; Zaslavsky & Shir, 2005). A non-contradicting definition only includes properties that are able to coexist. A noncircular definition does not use the term being defined. Because definitions are created in a shared community, they inherently must be *unambiguous*. That is, they must always be interpreted in the same way (Zaslavsky & Shir, 2005) and only include *precise terminology* or terms that have already been defined by the community (Borasi, 1992; Van Dormolen & Zaslavsky, 2003). In this way, each definition is part of a larger system of definitions that are related to one another and are grounded in axioms (Van Dormolen & Zaslavsky, 2003). Moreover, *alternate definitions*, those that are different yet equivalent, may exist for the same object (de Villiers, 1998). These definitions vary in form (e.g., textual vs. symbolic or procedural vs. structural) or *minimality* (Van Dormolen & Zaslavsky, 2003; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). Minimal definitions, also referred to as *economic* (de Villiers, 1998), only include descriptions that are necessary for guaranteeing recreation of the object or identification of the object. Minimal definitions are often *hierarchical*, that is, they include definitions already established by the community (Van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). It is important that alternate definitions are *invariant*; their meaning should remain unchanged between representations.

To illustrate the features of definitions, consider two definitions for “triangle:” 1) a polygon with three sides and 2) a closed figure with three connected straight lines, three angles, and three vertices. Both descriptions are valid definitions of triangle. They are noncircular because neither uses the term “triangle.” They are non-contradicting because all the properties described can coexist (as opposed to, for instance, a polygon with three sides and four angles). The terms used, “polygon,” “sides,” “angles,” “vertices,” “lines,” “closed,” are all precise terms that have been defined by the mathematics community at large. At the same time, the definitions differ in their minimality and hierarchy. The first definition is minimal and hierarchical. It uses objects already defined, polygon and sides, and does not include additional, unnecessary information. The second definition is not minimal because it unnecessarily states that a triangle has three angles and three vertices, properties that are already guaranteed when a figure is closed with three connected straight lines. The second definition is hierarchical in some regards, but not others. It uses pre-defined terms such as “angles” and “vertices” but, unlike the first definition, does not take advantage of the definition “polygon.”

Mathematical defining. In his seminal text, *Proofs and Refutations*, Imre Lakatos (1976) provided historical analyses of the development of two mathematical entities: the Euler Characteristic and the proof that the limit of any convergent series of continuous functions is itself continuous. His work provides a lens for thinking about: a) what it means to participate in mathematics generally, b) how mathematical participation develops, c) how *defining* plays a role in that development and d) what it means to participate in defining specifically.

Lakatos (1976) suggested that the discipline of mathematics develops as a practice of what he calls “proofs and refutations.” Mathematical inquiry typically begins with an initial conjecture and a subsequent proof that takes the form of a “thought-experiment.” This thought-experiment decomposes the initial conjecture into sub-conjectures. Other members of the mathematical community often find counterexamples to the initial conjecture. In turn, these lead to re-examination of the proof in order to find the sub-conjecture responsible for the counter-example. This “zig-zag” (p. 42) between proofs and refutations may yield suggestions for an improved conjecture. For example, in the history of the Euler Characteristic, analysis of the original proof led to several counter-examples to the initial conjecture. Because of these counter-examples, some mathematicians suggested modifying the original conjecture to specify that it be true for only a smaller set of polyhedra. The process, Lakatos noted, is counter to how mathematics is typically presented in texts. There, the field is presented deductively, that is, as a logically linear progression, starting with definitions, axioms, lemmas, theorems and finally proofs, often masking the social and organic nature of how mathematics develops.

One significant, but often overlooked, aspect of Lakatos’s (1976) analysis is that it shows how mathematics develops as a system. That is, mathematical objects and entities (such as theorems and proofs) develop in a related way over time, contributing to the development of practices. Although mathematics is often presented systematically, it is usually done so as static representations of knowledge. Lakatos’s historical narrative highlights how the *development* of mathematical entities and objects and their related practices can be represented as a dynamic system. A representation of part of the system Lakatos described in the development of the Euler Characteristic is illustrated in Figure 1. In the figure, arrows are used to show when one mathematical entity, such as a conjecture, proof, counterexample or definition, led to another. For example, when one mathematician provided a proof for the Euler Characteristic, several mathematicians responded to that proof with criticisms of its particular components (referred to as lemmas). The criticism of one lemma was then elaborated with a global counterexample (global because it countered the initial conjecture, as opposed to local counterexamples, which counter a lemma). As Figure 1 illustrates, unlike final deductive presentations of mathematics, relations often follow in non-standard ways. For instance, we see that in the case described by Lakatos, counterexamples followed proofs that led to new definitions that led to new counterexamples.

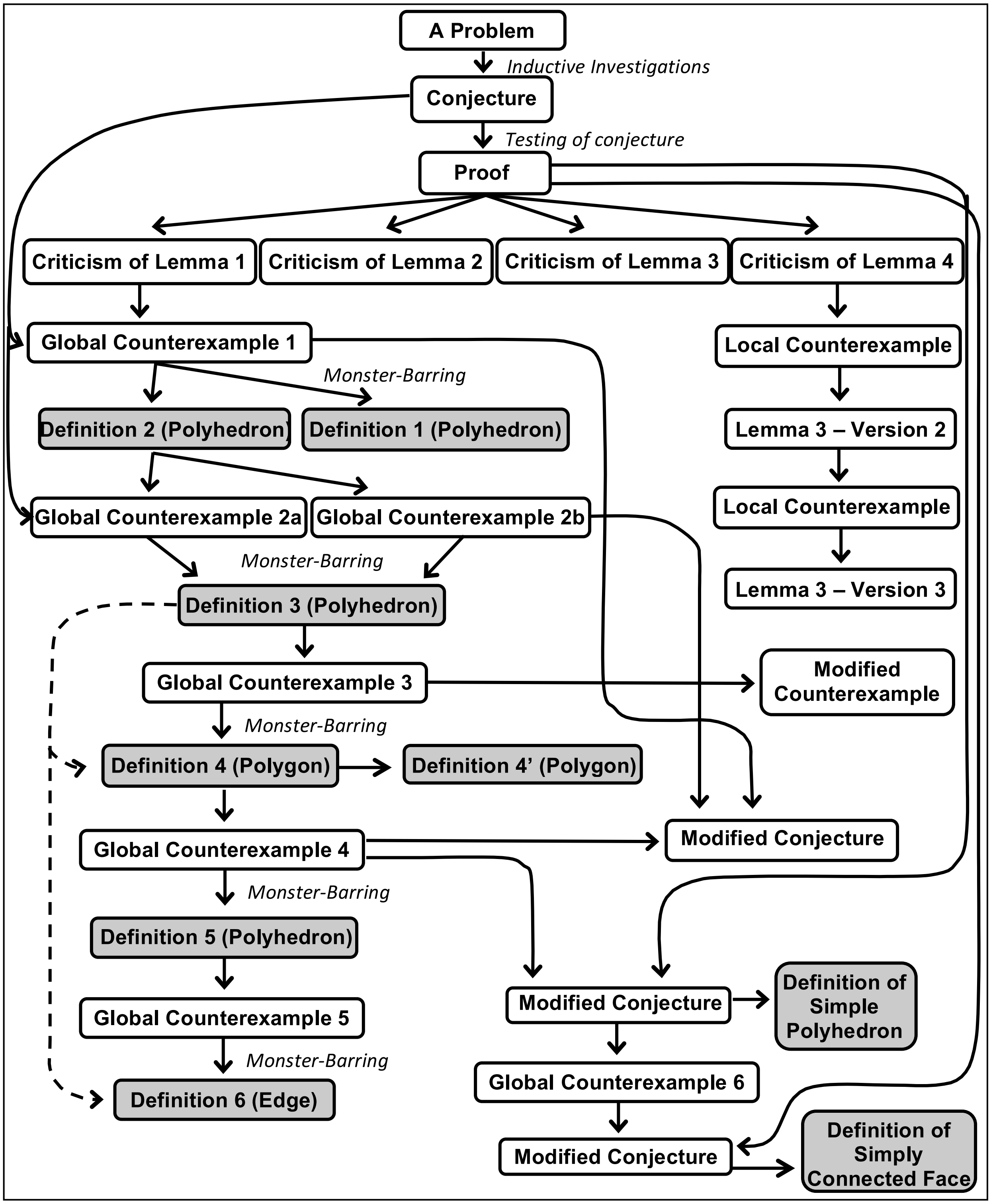

Figure 1. Mathematics develops as a system in Lakatos's *Proofs & Refutations* (1976). The diagram illustrates the systematic relations described by Lakatos in the first part of his book. Arrows indicate instances where one investigation led to another historically, as suggested by Lakatos' analysis. Gray blocks indicate instances involving definitions. Dashed arrows indicate indirect relations between definitions. *Monster-barring* refers to the process where a mathematician proposed a new definition in order to dismiss a counter-example.

Lakatos's (1976) analysis highlights how mathematical defining plays a significant role in the development of a mathematical system. First, defining aids in the refinement of proof. When counterexamples are introduced, contest often arises about the grounding definitions of the proofs. Sometimes new definitions are proposed in order to dismiss the counter-example while still salvaging the proof and the conjecture (a process Lakatos refers to as "monster-barring"); other times the definition remains and the proof is altered. These disagreements highlight a second role of defining: defining itself is a form of mathematical argument. Such arguments are used to dispute inclusion of an aspect within a definition. Definitional arguments are grounded within a community's choice to do one of several things: a) include a case as an example of a particular mathematical object, b) dismiss or keep a proposed counter-example to a proof, c) verify the validity of an object by appealing to a definition or d) justify the equivalence or non-equivalence of two definitions (Van Dormolen & Zaslavsky, 2003). All of these arguments rely on a third role of mathematical defining, that is, that it arises out of the need to communicate within a mathematical community. Thus, the development of mathematics is inherently social and progress hinges upon deliberation among the community's members. Finally, defining also supports the development of other definitions that contribute to new counterexamples. For instance, in the case of the Euler Characteristic, defining "polyhedron" led to a counterexample that, in turn, spurred discussions about the definition of "polygon" and, after further deliberation, the definition of "edge."

Supporting Classroom Disciplinary Practice

In order to understand how defining was supported within the reviewed pieces, I draw upon two lines of work. First, because of my focus on defining as a disciplinary practice, I use

Lehrer's (2009) framework of "design elements" (p. 760). These design elements describe ways to support the development of disciplinary knowledge when designing learning environments. The first design element is the nature of the tasks designed to elicit or support a practice. The design or analysis of tasks should include a focus on how they support the second and third design elements, the inscriptions and material means available as resources for engaging with tasks. As Lehrer describes, inscriptions are essentially "epistemic expressions," (p. 761). In other words, inscriptions embody histories of meaning that enable communication among members in the disciplinary community. For example, in mathematics, notational systems, such as our place value system, carry meaning about the mathematical objects that structure how we look at and talk about the objects. In the case of our place value system, each place indicates a certain number of groups of particular powers of ten (e.g., 23 means "2 groups of ten to the power of 1and 3 groups of ten to the power of 0"); this inscriptional system represents a base ten orientation towards structuring number. Materials are also central to disciplinary activity, but, whereas in disciplines, constructing materials is often central to practice, in schools, students are often provided ready-made products. For instance, Wilkerson-Jerde and Wilensky (2011) found that when unpacking a proof in a new discipline in topology (knot theory), mathematicians often constructed examples to make sense of the mathematical objects at hand, and some examples were visual representations of materials from the world, such as rope. In contrast, students in schools are often provided examples rather than having opportunities to construct their own. Thus, it is important to consider how inscriptions and materials relate to how students participate in disciplinary forms of activity. The fourth element, modes and means of argument, entails disciplinary forms of justification. This form of discourse varies by discipline but also can vary based upon how students experience it and the role they play in participating within it. Finally, if students are provided opportunities to participate in modes and means of argument and engage with inscriptional and material means, they may develop identities as participants within the discipline. In the case of this review, I am interested in how these elements of design are particular to defining. For instance, a) how are tasks structured to encourage defining? b) what forms of inscription are particular to defining and how are those forms used to communicate Aspects of the Practice? c) how does materiality play a role in defining? d) how do forms of argument and participation within argument take shape? and e) in what ways do tasks structure participation to support students' development of disciplinary identities or dispositions?

Lehrer (2009) also acknowledged the significance of the role of the teacher in orchestrating these elements of design. In order to capture the significance of the role of the teacher in supporting defining, I draw upon work related to orchestrating classroom mathematical discussions. I focus on the orchestration of classroom discussions because: a) the discussion is a venue where orchestration of design elements is more visible and b) since defining has historically been a social process between members of the mathematical community (Lakatos, 1976), the discussion is a significant arena for cultivating and observing students' participation in the practice. I draw upon the work of two sets of scholars in particular. The first set of scholars, Engle and Conant (2002) described a framework of four principles for fostering productive disciplinary engagement, where productive disciplinary engagement entails student participation that is significant to a discipline and contributes to a community's collective learning. The first of their principles, *problematizing content*, suggests that teachers should encourage students to probe the conceptual foundations of a discipline, in ways such as justifying conjectures. This relates to the second principle of *giving students authority*, which suggests that teachers should encourage students to also be authors of disciplinary content. The third principle, *holding students accountable to others and to disciplinary norms*, entails that teachers should hold students accountable to the classroom community, both to each other and to established expectations for participating in the discipline at hand. Finally, the fourth principle, *providing relevant resources*, involves providing resources, such as materials, norms, or time, to support the other three principles as well as productive disciplinary engagement in general. For purposes of this review, I use these four principles as a general lens to identify teacher moves when orchestrating discussions that aim at supporting students' participation in defining and development of their identities as definers.

In addition, I also employ O'Connor and Michael's (1996) framework that describes how teachers use one particular talk move, *revoicing*, to support students in participating in disciplinary discourse practices. O'Connor and Michaels use Goffman's (1981) and Goodwin's (1990) notion of participant frames to examine how revoicing shifts, reframes or repositions existing participant roles and structures to place the authority in the hands of the students while also holding them accountable to the social and disciplinary norms of the community. In particular, O'Connor and Michaels note that revoicing serves several functions in a classroom, including: a) repairing (i.e., clarifying reasoning), b) rebroadcasting (i.e., giving students a louder voice), c) reformulating (i.e., advancing the teacher's agenda), d) repositioning student utterances in relation to the content, and e) repositioning students' utterances as opposing stances. These functions are accomplished in three ways, linguistically. First, the teacher reformulates components of the student's talk, either by changing pieces of the content or by changing the language used to describe that content, without correcting the student. Second, the teacher uses indirect speech, namely by using verbs that animate the student as the author of the content (for instance, "so Jane predicts that…" (p. 79)). Finally, the teacher uses markers of warranted inference, such as "so," to create an inference linked to that of the student's previous justification or claim. Whereas Engle and Conant's (2002) framework provides a general lens for identifying key aspects of how the teacher orchestrates discussion, the notion of revoicing as shifting participant frameworks allows a closer description of how those aspects are accomplished linguistically. Thus, I pair the two frameworks in order to capture a broad, yet also detailed, description of the role of the teacher in orchestrating discussion around mathematical definitions.

Method

Inclusion Criteria

Because this review is intended to investigate *mathematical defining* in *learning environments*, I searched for articles in which researchers described attempts at engaging students in mathematical defining. I took *mathematical defining* to include any activity that includes, "formulating, negotiating and revising a [mathematical] definition" (Zandieh & Rasmussen, 2010, p. 59). I took *learning environments* to include any setting in which a task was designed to promote changes in student thinking, with or without the intervention of an instructor. These ranged from one-on-one tutoring sessions to small group working sessions to whole class settings. I included papers that described learning at all ages and in any topic area in order to capture defining in its most general sense. Although papers investigating conceptual understanding of particular mathematical ideas might give insight into students' development of conceptual understanding of definitions, because of my focus on students' participation in defining as a mathematical practice, I did not include such articles unless definitions were part of the activity. Moreover, papers on conceptual development typically characterize nuances of learning for particular content areas, and I was interested in defining more generally. Because I wanted to thoroughly capture all that had been done in defining, I included all forms of writing, including research journal articles, conference proceedings, book chapters and teacher practice journal articles from any year.

Procedure

Search process. I searched for studies via three phases. I first conducted a preliminary search for articles using two methods: a) a keyword search of abstracts in the search engine, *Educational Research Information Center* and b) a keyword search in two of the main mathematics education journals: *Journal for Research in Mathematics Education* and *Mathematical Thinking and Learning*. In both cases, I searched using the keywords "mathematical definition" and "mathematical defining." The *Educational Research Information Center* search yielded 25 articles. The mathematics education journals yielded 6 additional articles, 1 in *Journal for Research in Mathematics Education* and 5 in *Mathematical Thinking and Learning*. In all, this phase led to 31 preliminary articles. In the second phase, I read the abstracts from the original 31 articles to determine which studies described students participating in mathematical defining (as defined above)[1]. When the abstract was not clear, the article was skimmed to determine whether it fit with the criteria. This reading allowed me to eliminate 27 articles, leaving 4 articles remaining. For the final phase, the references of the original set of articles were skimmed for additional studies that were applicable. This final phase also allowed me to identify other forms of writing, namely books or chapters that had not appeared in the search. Additional writings were referred to me.

[1] Except in a few cases in which copies of the documents were not obtainable.

The search led to 19 contributions, described in Table 1. In all, there were 11 peer-reviewed articles, 2 articles from practice journals, 4 conference proceedings, 1 book chapter and 1 book. Participants in the studies ranged from grade 2 to university level. Sample size ranged from 1 teacher with 1 student to multiple classes. Most of the studies were conducted about topics in geometry, especially around two-dimensional shapes and three-dimensional solids. Other topics included fine functions, Fibonacci sequences, equal area, local maximum point of a function, and increasing function. Studies varied in duration, ranging from a single session to several weeks, but most of analyses were conducted of less than 2 sessions.

Table 1. Overview of selected works for the review

Year	**Authors**	**Publication Type**	**Grade Level**	**Sample Size**	**Topic of Investigation**	**Duration of Study**
1989	Lehrer, Randle, & Sancilio	Article, peer-reviewed	4^{th} grade	32 students assigned randomly to 1 of 2 conditions	Pre-proof geometry	17 lessons, each for ½ hour; analysis mainly post interviews
1992	Borasi	Book	2^{nd} grade	1 teacher with 2 students	Circles, Isosceles triangles, Polygons, Variable, Exponentiation	8 instructional sessions, each about 30-40 minutes. Students also engaged in pre-assessment & post instruction take home project. One student had an additional post assessment.
1997	Dahlberg & Housman	Article, peer-reviewed	3^{rd} & 4^{th} year under-graduate math students	11 students in 1-on-1 interview settings	Fine functions	1 session, lasting from 20 minutes to 1 hour
1997	Mariotti & Fischbein	Article, peer-reviewed	6^{th} grade	1 teacher with 3 different classes	Prisms & parallelepiped	About 1 month/ class; analysis of 4 class discussions
1998	de Villiers	Conference Proceedings	10^{th} grade	1 class and a control group	Rhombi & Parallelograms	Unclear; analysis describes piece of instruction & interview results

Table 1, continued

Year	Authors	Publication Type	Grade Level	Sample Size	Topic of Investigation	Duration of Study
1999	Lehrer, Jacobson, Kemeny, & Strom	Book chapter	2nd grade	1 teacher with 1 class	Triangles	2 sessions
2000	Lehrer & Curtis	Practice journal	3rd grade	1 teacher with 1 class	Perfect solids	1-2 sessions (personal correspondence)
2000	Keiser	Practice journal	6th grade	2 teachers, 2 classes	Angles	About 5 weeks of lessons
2001	Leikin & Winicki-Landman	Conference Proceedings	Secondary school math teachers	10 teachers	Fibonacci Sequence	3 year PD course; analysis focused on 1 session
2002	Lin & Yang	Article, peer-reviewed	7th grade	1 teacher with 2 students	Rectangles	2 sessions, conducted several months apart. Each lasting between 1.5 – 2 hours
2002	Furinghetti & Paola	Conference Proceedings	10th grade	1 teacher with 21 students	Quadrilaterals	3 class sessions
2005	Herbst	Article, peer-reviewed	9th - 10th grade	8 classes with 3 teachers (1 teaching 6 of the classes)	Equal Area (in the context of triangles)	2 class sessions for each of the 8 classes. 3 classes were from year 1, 2 from year 2 and 3 from year 3.
2005	Herbst, Gonzalez, & Macke	Article, peer-reviewed	9th grade	2 accelerated geometry classes, 53 students total	Quadrilaterals	2 class sessions
2005	Larsen & Zandieh	Conference Proceedings	University	2 classes – 1 geometry and one group theory	“Small Triangles” on the Sphere, Subgroup	2 class sessions, 1 per class.
2006	Ouvrier-Buffet	Article, peer-reviewed	Freshmen (unclear if University or High School)	1 teacher with 2 groups of 2-3 students	Straight line	Each group participated in a 2-3 hour session.
2005	Zaslavsky & Shir	Article, peer-reviewed	12th grade	4 students	Isosceles triangle, Square, Local Maximum Point of a Function, Increasing Function	4 group sessions, one per concept. Students were also assessed individually before and after each group session. Each session lasted 1.5 – 2.5 hours.

Table 1, continued

Year	Authors	Publication Type	Grade Level	Sample Size	Topic of Investigation	Duration of Study
2009	Ambrose & Kenehan	Article, peer-reviewed	3rd grade	1 researcher teacher & 1 teacher with class of 19	Polyhedra – Pyramid in particular	17 days of instruction; analysis focused on 1 class session & pre & post interviews
2009	Roth & Thom	Article, peer-reviewed	2nd grade	1 teacher & class of 23 students	3-dimensional Solids	3 weeks, 5 lessons per week; analysis focused on 1 class session.
2010	Zandieh & Rassmussen	Article, peer-reviewed	University	1 class of 25 students	Planar & spherical triangles	5 weeks of instruction; analyzed 5 class sessions

Note. The studies varied in the detail they provided about the sample size and duration. Thus, some of the descriptions in the table may be more detailed than others.

Analysis. To initiate analysis, I read the studies with attendance to three broad analytic foci, reflective of my questions: a) the nature of defining, which included *definitional activity* and *Aspects of Definitional Practice*, b) the affordances of defining for mathematical learning and c) instructional supports for defining, including *design elements* and the *role of the teacher in orchestrating discussions*. Initial impressions were documented for each of the three categories, guiding decisions for further analysis. These further details of analysis are described below for each of the three analytic foci.

Nature of defining. For nature of defining, I identified two issues during my initial readings. First, in order to situate the studies, I noted the sequences of activity that students participated in. By capturing what the students did, I hoped to characterize the types of *defining tasks* that might be employed to engage students in defining. I use the term *defining tasks* to describe classroom activities that involve mathematical defining. After my initial readings, I noted 4 types of defining tasks across the studies. These characterizations were checked and confirmed during a second reading of the articles.

Second, I noted *Aspects of Definitional Practice*. Whereas tasks were meant to capture overall activity structure, Aspects of Definitional Practice were intended to characterize particular forms of student participation. During the initial reading, I compared across the studies to generate an initial characterization of Aspects of Practice. To determine what counted as an aspect of practice, I drew upon my readings of disciplinary forms of mathematical definitions and defining, as described earlier. For example, I noted forms of argument around definitions, similar to those described by Lakatos (1976). My initial impressions guided the creation of a set of categories that were refined, elaborated and added to with successive readings, in the tradition of the constant comparative method (Glaser, 1965). Ultimately, this process led to 11 categories that were then used to code the studies. Categories are presented in the results.

For the coding, my unit of analysis was an entire study. Each study was coded across the 11 categories in a binary fashion. In other words, if a study described students participating in an aspect of practice, then it was coded "yes" for that particular aspect, and if not, it was coded "no." In order to be coded as including an aspect of practice, the study had to provide an example of student talk or action that suggested participation in the aspect.[2] I chose to use the entire study as the unit of analysis because studies varied in how extensively they described student activity. Because of this, it would be impossible to make claims about frequency or density of occurrence of the Aspects of Practice. This method gives a base-line portrait to characterize what has been done. Finally, once the studies were coded, I looked across instances of each aspect of practice in order to capture nuances of students' engagement in defining. For instance, I noted that definitional arguments were constructed for different purposes, such as arguing for the inclusion

[2] The requirement that the study provide an example held true in most cases. In the few instances where this did not hold true, I determined that sufficient description was provided to warrant a code.

or exclusion of an example versus arguing for the inclusion or exclusion of a definition. Nuances of Aspects of Practice are also presented in the results section.

Affordances of defining. To capture the educative potential of defining, in my initial reading of the studies, I noted any reference authors made to how engagement in defining might impact students' mathematical thinking or development as a "doer" of mathematics. These claims typically appeared in the results or discussion sections of the studies. During my initial reading, I noted four affordances of defining: a) defining supported students' engagement in the practice of defining, b) defining supported a closer analysis of the objects and relations being defined, c) defining supported students' conceptual understanding of definitions, and d) defining supported students' attendance to aspects of definitions (e.g., roles and features of definitions). As with the analysis of the aspect of the practice of defining, these initial categories were further refined with successive readings (Glaser, 1965). In the final reading, affordances of each study were documented along the 4 categories. In order for an affordance to be documented, the study had to provide evidence to back up their claim. For instance, authors could not simply claim that students improved their understanding of a definition. Rather, they had to either provide an example of one or more cases or provide assessment data indicating so.

Supports for defining. In my initial reading of the articles, I noticed that little was done to characterize support within the studies. Thus, rather than make definite claims about support, I employed two forms of analysis in order to develop a set of conjectures that might guide further research about supporting defining. For the first analysis, in order to characterize how learning environments were designed to support, I compared the studies within each of my codes of affordances (described in the previous section) to identify similarities. For instance, within the code of "defining supports conceptual development," I compared the studies that claimed to support similar things, such as drawing out a range of student thinking. When comparing, I took note of particulars of the five design elements described by Lehrer (2009): tasks, inscriptions, material means, modes and means of argumentation and identity. In the end, I identified 5 particulars of designed environment that appeared to be significant supports, related to tasks, inscriptions, material means and modes and means of argument. Although it was difficult to identify particular of design related to identity, I did find that the teacher appeared to play an important role in supporting this during discussions. I describe this next.

Analysis of the role of the teacher in orchestrating discussion was conducted slightly differently. Most of the articles did not analyze the role of the teacher, and even when they did, did not always do so thoroughly. After initial readings of the articles, I noticed two that when compared side-by-side provided a potentially interesting contrast. Both articles described similar activities conducted with similar ages of students yet resulted in different outcomes in relation to students' development of definitions. In one, by Lehrer and colleagues (1999), students made initial progress in construction of the definition and also participated in definitional practices. In the other article, by Ambrose & Kennehan (2009), the students did not develop definitions for the objects and although the students participated in explanation, only a short account of argument was noted. I thus used these two articles as contrasting cases and compared the role of the teacher, especially in excerpts of transcript of discussion. This comparison highlighted 3 potentially significant roles of the teacher in orchestrating discussion. With this lens in mind, I scanned the rest of the literature for other supportive evidence for these conjectures.

Results of Review

Overview of Results

I present the results in the following sequence. In the first section, I describe mathematical defining in the classroom, first generally describing types of defining activities, including similarities and differences in their execution, and then highlighting aspects of students' definitional practice. The second section then provides an overview of the affordances of these tasks. I conclude the results by considering how those viable defining activities are supported. I start by highlighting aspects of the designed tasks that appear to be significant, drawing upon the affordances described in the second section. Finally, using two studies as contrasting cases, I describe roles of the teacher in orchestrating discussion that I conjecture are significant in promoting defining, supplementing with instances from other studies. The reviewed studies illustrate that the tasks within them provided opportunities for students to participate in a range of aspects of defining that resembled disciplinary Aspects of Practice. Moreover, participation in defining appears to have potential in supporting students' conceptual development and development of disciplinary dispositions.

Nature of Mathematical Defining in Classrooms

Despite differences in contexts and age ranges of the studies, they nonetheless collectively illustrate potential activities for engaging students in defining and how that engagement might play out. In this section, I first provide a general overview detailing the types of tasks students participated in when defining. I then look across the tasks and the studies to highlight aspects of students' participation that appear to be particular to defining.

Types of defining activities. Here I describe 4 types of defining activities described in the reviewed studies: a) sorting and classification, b) evaluating definitions and non-definitions,

c) open-ended construction of definitions and d) defining arising out of problem-solving or proof. All these activities entail formally expressing a mathematical object's qualities in a way that is sharable and upholds the features of definitions detailed previously. Sorting and classification tasks were the most frequently described in the studies (11 of the 19 studies included such activities), followed by defining arising out of problem-solving or proof (7 of the 19), open-ended construction of definitions (4 of the 19), and evaluating definitions and non-definitions (4 of the 19). Note that 3 of the 19 studies described students' engagement in two types of defining activities and 1 study described students' engagement in three types of defining activities.

Sorting and classification. In sorting and classification activities, students were asked to classify objects into one or more groups by describing characteristic properties, either as a whole class or in small groups. These activities varied in how they were structured and proposed to students. In some of the studies, students were provided with examples and non-examples of one particular object (such as "triangle") and were asked to determine which of the set should be included as examples of the object (Ambrose & Kenehan, 2009; Lehrer, Randle, & Sancilio, 1989; Lehrer et al., 1999; Lehrer & Curtis, 2000; Ouvrier-Buffet, 2006). Alternatively, one study provided students with only examples (no non-examples) of an object and asked students to make a list of their common properties (de Villiers, 1998). In many cases, students instead (or additionally) generated what they perceived to be examples of one particular object and then justified its inclusion (Ambrose & Kenehan, 2009; Dalhberg & Houseman, 1997; Furinghetti & Paola, 2002; Lehrer et al., 1999; Lehrer & Curtis, 2000; Zandiah & Rasmussen, 2010). Finally, in other cases (Roth & Thom, 2009; Mariotti & Fischbein, 1997), teachers provided students with a set of different types of objects (e.g., different types of polyhedra) and asked them to place the objects into new or existing groups.

The activities varied in the materiality of the objects that were being classified or sorted, ranging from drawn objects (e.g., de Villiers, 1998; Lehrer et al., 1999), to physical objects (e.g., Ambrose & Kenehan, 2009; Lehrer & Curtis, 2000), to computer animated objects (e.g., Furinghetti & Paola, 2002). In some cases, students worked with objects in one medium and then switched to another. For example, Lehrer and colleagues (1999) engaged students in classification of triangles in order to create a definition for "triangle," first by evaluating a set of drawn triangles and later by evaluating their own constructed triangles made out of sets of three paper strips. As I describe later, differences in materiality were significant in highlighting particular mathematical properties and relations.

Evaluating definitions and non-definitions. In two of the reviewed studies, students were given a list of alternate definitions for an object and asked to comment on them (Borasi, 1992; Leikin & Winicki-Landman, 2001) or determine whether each definition was acceptable (Zaslavsky & Shir, 2005). In both studies, the evaluation activities were conducted with small groups of high school students (either 2 or 4). The activities varied in types of definitions provided to the students. For instance, in Zaslavsky & Shir's (2005) study, the group of four students was asked to collectively evaluate lists of alternate definitions of four different mathematical objects on four different occasions. In two of the sessions, the students evaluated alternate definitions of geometric objects (isosceles triangle and square) and in the other two, they evaluated both definitions and non-definitions of analytic objects (increasing function and local maximum point of a function). The geometric definitions, although all acceptable definitions, ranged in their minimality, form of presentation (procedural vs. structural) and hierarchy.

Similarly, in a different study (Herbst et al., 2005), students indirectly compared and evaluated definitions of *different* objects, through a “Guess my Quadrilateral!” game. In this game, students worked with groups to construct a list of yes or no questions to ask the teacher in order to determine which quadrilateral she had in mind. The goal of the game was to ask as few questions as possible, and, to do so, students had to compare properties and relations of the quadrilaterals. Collectively, the evaluating alternate definitions and non-definitions tasks differed from sorting and classification tasks in the focal topic of evaluation and comparison. In the sorting and classification tasks, the focal topic was the examples whereas in the evaluation of definitions, the focal topic was the definitions themselves (or descriptions of properties and relations of objects). This is not to say that definitions were not compared or discussed in the sorting or classification tasks; rather, the activity was structured around examples. Likewise, for the evaluating definitions tasks, students may have compared or discussed examples, but they were not the focus of students’ analysis.

Open-ended construction of definitions. On a few occasions, students were simply asked to construct a definition of a mathematical object, such as “polygon.” However, the types of definitions students were asked to construct varied. For instance, Lehrer and colleagues (1989) described a learning environment where students were asked to create procedural definitions of geometric objects, either using the computer program LOGO or using traditional construction tools (e.g., pencil, straight edge, protractor). Moreover, the time period of construction varied. Whereas in most studies, students constructed a definition in one or two sessions as an isolated activity (Borasi, 1992; Zandieh & Rasmussen, 2010), Keiser (2000) described two classrooms where students continuously revisited the definition they were constructing (of angle) as they investigated angles in multiple contexts.

Defining arising out of problem-solving or proof. A few studies described defining arising from problem-solving (Herbst, 2005; Lin & Yang, 2002; Mariotti & Fischbein, 1997; Ouvrier-Buffet, 2006) and three described defining arising from proof in a Lakatosian manner (Borasi, 1992; Larsen & Zandieh, 2005; Zandieh & Rasmussen, 2010). In these tasks, the need for modifying or constructing a definition was motivated by the activity at hand. For instance, in Lin & Yang's (2002) study, students worked on the following area word problem: "Conan is going to move to a new home. He has a rectangular swimming pool built in the backyard. When he checked the pool, he said, 'Is it really a rectangular swimming pool?' If you were Conan, what places and what properties would you ask the workers to measure so that you can be sure it is rectangular?" (p. 18). The problem also had a stipulation that each property cost a significant amount of money to check, and students were asked to spend the least amount of money as possible, as a way of encouraging them to consider the minimal properties needed to ensure an object is a rectangle. This problem generated discussions about what a rectangle is and how to construct a minimal definition of one. In this case, as well as many others, discussions about definitions were often encouraged by the teacher. However, unlike the "open-ended construction of definitions," these tasks were situated within the problem or proof at hand.

Nature of student participation in defining. Because defining is a type of mathematical practice, it entails forms of participation in a community of mathematicians. When the community is situated in a classroom composed of learners of mathematics, these forms of participation play out in new ways. In order to support educators in developing such environments, it is important to identity aspects of students' participation in the practice. Here, I attempt to highlight what the reviewed studies suggest such aspects might be and what they might entail. Although I describe these aspects separately, as disjoint entities, in reality defining entails the collective functioning of multiple aspects. Some of these relations are illustrated in the text below. Table 2 provides an overview of the aspects of the practice of defining.

Table 2. Aspects of the Practice of Defining

Aspects of Defining	**Descriptions of Aspects**	**Examples**
Constructing & Evaluating Examples	Constructing examples of the object being defined and/or determining whether an example belongs to a set.	Students were shown geometric solids and asked whether the objects were examples of pyramids. Afterwards, students were then asked to construct their own examples of pyramids and justify why it was an example.
Describing Properties of Objects	Articulating, through talk and/or writing, properties & relations of mathematical objects.	When asked to compare a triangular pyramid and a square pyramid, students offered descriptions such as, they both "look like a triangle" (Ambrose & Kenehan, 2009, p. 165).
Using Definitions to Generate Objects	Generating an object based upon a definition or set of properties.	Students played a game in which they worked in groups to construct a list of "yes" or "no" questions. The goal of the game was to determine the teacher's quadrilateral using as few questions as possible. (Herbst, Gonzalez, & Macke, 2005).
Investigating Fundamental Qualities of Mathematical Objects	Investigating or examining aspects of properties or related properties of an object. Properties need not be part of the definition.	As students evaluated a set of potential "triangles," the issue of orientation arose; that is, if a triangle lays on one side versus another versus a vertex, does it change whether or not it is a triangle? (Lehrer et al., 1999).
Constructing Definitional Explanations & Arguments	Definitional arguments and explanations are justifications in relation to a definition, example of a definition, or qualities of an object being defined.	When defining triangle, one child constructed a "triangle" with one curved side. When her peers rejected her example as a triangle, she disagreed, appealing to their collective definition: "No. It doesn't matter. Look [gesturing to the board], it has three corners and three sides" (Lehrer et al., 1999, p. 78).
Revising Definitions	Adding properties to, eliminating properties from, or modifying elements of a definition.	When defining "perfect solid," students added the property that faces needed congruent sides to their definition (Lehrer and Curtis, 2000).
Asking Definitional Questions	Asking questions about definitions or about qualities, properties or relations of the objects being defined.	"Okay, but do you have to have endpoints [to form a triangle]? [sketches three rays that intersect to form a triangle] Is that not a triangle? Can you form a triangle with rays?" (Zandieh & Rasmussen, 2010, p. 62)
Negotiating Criteria for Judging Adequacy or Acceptability	Negotiating which features or roles of definitions should be used to determine whether a definition is adequate or acceptable.	A group of students, in evaluating a definition of square, discuss whether a definition needs to be minimal: "Yoav: Too many details, but it is still a definition. Omer: What do "too many details" have to do with that? Mike: In which definition here don't you have too many details?" (Zaslavsky & Shir, 2005, p. 329)

Table 2, continued

Aspects of Defining	**Descriptions of Aspects**	**Examples**
Considering Definitions in New Forms or Contexts	Re-defining or re-considering a definition of an object in a new form (such as procedural) or a new context. (such as a new space)	After defining quadrilaterals structurally, students constructed procedural definitions using the software program LOGO. Definitions took the form of sets of instructions for "walking" along a shape's edges (Lehrer, Randle, & Sancilio, 1989).
Engaging in Definitional Conjectures, Experiments & Tests	Begins with making conjectures about properties to include in a definition and/or about potential examples of the object being defined. Conjectures are followed by experiments that are then tested in some manner.	In trying to define "perfect solid," students constructed a conjectured definition, experimented by creating possible examples for perfect solids and then tested their candidates by comparing them to existing examples and non-examples and to their definition (Lehrer & Curtis, 2000).
Establishing and/or Investigating Systematic Relations	Considering the meaning of new mathematical objects or new mathematical questions, conjectures, theorems or proofs that are related to an object being defined.	When a class investigated two-dimensional representations of three-dimensional solids, because words like "sides" and "corners" meant different things to children, they negotiated agreed upon meanings (Lehrer & Curtis, 2000).

Defining involves constructing and evaluating examples. In many of the studies (14 of the 19), students constructed and/or evaluated examples and/or non-examples of the object being defined. Evaluation involved determining whether or not a case should be included as part of the set in question. In some cases, construction and/or evaluation was the focal activity, organized as a sorting or classification task, in effort to generate classes of objects or descriptions of one particular class (Ambrose and Kenehan , 2009; Furinghetti & Paola, 2002; Lehrer & Curtis, 2000; Lehrer et. al, 1989; Lehrer et. al, 1999; Mariotti & Fischbein, 1997; Roth & Thom, 2009; Zandieh & Rasmussen, 2010). For example, in both Ambrose and Kenehan's (2009) and Lehrer and colleague's (1999) studies, elementary-aged students were asked to evaluate cases in a collection of objects, justifying their choices to include or exclude the case as a member of the main object (e.g., pyramids and triangles, respectively). Afterwards, children constructed their own example of the object and justified why they considered it to be a member of the class.

In other cases, construction of examples arose in service of constructing definitional arguments or evaluating a current definition (Borasi, 1992; Herbst, 2005; Keiser, 2000; Ouvrier-Buffet, 2006; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). For instance, in Zandieh & Rasmussen's (2010) study, when the students constructed definitions for planar triangles, one of them questioned whether "endpoints" was a necessary property to include in their definition. To illustrate his point, he drew three intersecting rays as an example of a triangle without endpoints. Sometimes, examples also served as sense-making devices for students. For example, when Dahlberg & Housman (1997) asked students to make sense of a definition provided to them, several students spontaneously constructed examples. In fact, the authors found that constructing examples, either spontaneously or when prompted, supported learning of the concept. Constructing and evaluating examples was a significant aspect of the practice of defining because it helped students consider what the class of objects being defined should include and provided a set of objects to describe. Moreover, the latter cases show that constructing and evaluating examples play a significant role in students' participation in other Aspects of Definitional Practices, such as describing, and thus may be an important aspect to cultivate.

Defining involves describing properties. Often when students constructed and evaluated examples, they also described and articulated properties and relations of the examples. This descriptive quality is what pushes example construction and evaluation towards definitional activity and beyond simply building and making decisions of "in" versus "out." Despite this co-occurrence, I include description as a separate aspect of definitional practice from example construction and evaluation because they both play important roles in defining. In a few cases, teachers started a lesson with pure description of examples and/or non-examples (Ambrose & Kenehan, 2009; de Villiers, 1998; Lehrer & Curtis, 2000; Lehrer et. al, 1989). For instance, Ambrose & Kenehan (2009) describe an introductory activity in a lesson in which students were shown a large triangular pyramid and a large square pyramid and asked what the solids had in common. Children offered descriptions such as, they both “look like a triangle” or their “bottoms are different” (p. 166).

However, most of the time, description serviced other goals, such as constructing a definitional argument, explaining a particular classification or writing a definition for an object (Ambrose & Kenehan, 2009; Borasi, 1992; de Villiers, 1998; Herbst, 2005; Lehrer & Curtis, 2000; Lehrer et. al, 1989; Lehrer et. al, 1999; Keiser, 2000; Mariotti & Fischbein, 1997; Roth & Thom, 2009; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). For example, in trying to convince his classmates that a constructed example was a case of a spherical triangle, one college student argued, “First of all, use the first one as the equator, and you come around and you stop on the opposite side. So it **goes completely around, like 300 degrees** or something like that. Another line segment **on the great circle**, and you have a third line segment **on the great circle**, and **they all intersect** with each other only once, only once, only once [pointing to each of the line segments in turn]” (underlining added to highlight his use of descriptions) (p. 65). In all of these forms of activity, it is important that descriptions go beyond “lists of properties” but instead contribute to the construction of a definition. For instance, in one second grade class, as the students classified a collection of examples and non-examples of triangles, the teacher kept a running list of their agreed upon “rules” for triangles on the front board and rules that were tentative and belonged to individuals on the side board (Lehrer et. al, 1999). However, in other cases (e.g., Ambrose & Kenehan, 2009), although descriptions may have served immediate goals (such as classification), they were not simultaneously repurposed for the construction of definitions. Rather, it was only later that the students constructed definitions. The importance of this will be discussed further later.

Younger children's descriptions often varied in their attendance to mathematically significant properties and relations. For instance, Ambrose & Kenehan (2009) note that often students' initial descriptions were holistic, describing the overall looks of a geometric solid (e.g., both "look like a mummy house," (p. 165)), whereas later children more frequently attended to the parts of a solid and the relations between them. Because articulation of properties and relations is critical for constructing definitions, it is important to consider how one might cultivate mathematically relevant descriptions. This will be discussed later.

Defining involves using definitions to generate objects. As described, many of the activities involved students in generating descriptions for known objects. However, a few studies illustrated that defining may also involve generating *objects* (e.g., "square" or "triangle") for a given definition or set of properties. In two cases, generation occurred as students collectively constructed definitions for an unfamiliar object. Zandieh and Rasmussen (2010) describe college students constructing a definition for a subset of triangles on the sphere, which they termed "small triangles," whereas Lehrer and Curtis (2000) describe third graders constructing a definition for "perfect solid." In the case of the college students, their investigations were guided by their desire for the object to uphold a particular theorem. For the elementary students, their investigations were motivated by the desire to find all five of the perfect solids and were guided by the teacher who informed students whether their constructed polyhedra were in fact perfect solids.

In a study by Herbst and colleagues (2005), generating objects took the form of a game called "Guess my Quadrilateral!" In the game, students worked in groups to construct a list of "yes" or "no" questions that they would ask to determine which quadrilateral the teacher was thinking of. Their goal in the game was to guess the quadrilateral using as few questions as possible. Because of this, when students generated questions, they often considered which quadrilaterals the question would exclude. Several of the groups created tree diagrams to illustrate how responses to their questions related to possible quadrilaterals. In both the game setting and the cases where students were generating objects when constructing definitions, students had to consider how properties or rules related to one object rather than others. In other words, this aspect of defining requires students to consider an object not as an isolated case, but in relation to other objects, whether a set of quadrilaterals or the set of objects which are not perfect solids.

Defining involves investigating fundamental qualities of mathematical objects. One aspect that contributes to description and example construction and evaluation is the investigation of fundamental qualities of mathematical objects. In several of the studies, as students constructed definitions, they also examined more carefully particular qualities of the objects they were defining (Borasi, 1992; Furinghetti & Paola, 2002; Herbst, 2005; Keiser, 2000; Lehrer & Curtis, 2000; Lehrer et al., 1999; Zandieh & Rasmussen, 2010). Typically students' investigations were not directly expressed in definitions but were critical in their evaluation of potential examples. For instance, in Lehrer and colleagues (1999) study, as students evaluated a set of potential "triangles," the issue of orientation arose; that is, if a triangle lays on one side versus another versus a vertex, does it change whether or not it is a triangle? They investigated this quality of the object by constructing triangles with paper strips and considering whether the triangles changed when rotated, an investigation that led to agreement that orientation does not matter. Sometimes, investigating qualities unveils equivalent relations that allow for new ways of defining objects. For instance, one group of third graders noticed that satisfying the constraint of congruent sides for examples of perfect solids were met by equilateral polygons where the number of rotational symmetries was the same as the number of sides (Lehrer, personal communication). Although the examination of qualities of objects may occur in classrooms outside of defining, in the context of defining, it arises out of the need to make sense of the object at hand. Thus, such investigations may be motivated by students' inquiries (rather than suggested by the teacher) and also promote unpacking relations between an object's properties.

Defining involves constructing definitional explanations and arguments. The most thoroughly documented aspect of defining practice was that of constructing definitional arguments and/or explanations. Definitional arguments and explanations are justifications in relation to a definition, to an example of a definition or to qualities of an object being defined. Arguments and explanations took similar forms, but, unlike explanations, arguments arose from contest and were used to resolve that contest. This distinction is significant because historically the need to resolve disagreements led to advancement in the field (Lakatos, 1976). Despite this, because the definitional arguments and explanations took similar forms and because it was not always possible to discern whether a justification was an argument or explanation, I chose to include them in the same category of practice while still distinguishing them when possible.

I noted four types of definitional arguments and explanations that students engaged in. First, some definitional arguments and explanations were used to justify the inclusion or exclusion of a definition (de Villiers, 1998; Borasi, 1992; Larsen & Zandieh, 2005; Leikin & Winicki-Landman, 2001; Mariotti & Fischbein, 1997; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). These types of justifications often occurred when students were evaluating alternate definitions. Zaslavsky & Shir (2005) further delineated such justifications into five types. The first type, *mathematical arguments*, involved invoking logical concerns, that is, evaluating the definitions based on their correctness. In Larsen & Zandieh's (2005) study with undergraduates, students employed this type of justification by proving that one definition was equivalent to another. The second type, *communicative arguments*, were those in which students evaluated definitions based mainly on clarity, comprehensiveness and their accessibility to the audience. *Figurative arguments* occurred only when evaluating geometric definitions. These arguments mainly focused on the issue of whether or not it is acceptable to define a geometric figure based on its latent parts. *Example-based reasoning* used examples to convince others about an aspect of including or excluding a definition. These justifications mainly took the form of counter-examples in order to reject a definition. Finally, *definition-based reasoning* argues for or against a definition by invoking features or roles of mathematical definitions.

The second general type of definitional argument or explanation was used to negotiate aspects of properties or relations of an object (Mariotti & Fischbein, 1997; Keiser, 2000; Lehrer et al., 1999). This type of justification appeared related to the first, except rather than arguing for the inclusion or exclusion of an entire definition, students attended to the discussion of emergent aspects of an object that might be included in the definition. Thus, this form of justification was less directly related to a completed definition but was still significant in contributing to the construction. For example, Keiser (2000) described students' arguments related to one class member's question about whether increasing the physical size of the angle increased its measure. In this case, the argument was resolved by one child's use of two examples (same angles formed by the watch hands and the clock hands) to illustrate that size did not matter. This argument example is similar to the example-based arguments that Zaslavsky & Shir (2005) described. As another example, in Lehrer and colleagues study (1999), students' initial investigations with triangles led to arguments about the qualities of sides, such as their necessity to be straight. This type of definitional argument or explanation may occur when students are investigating qualities of mathematical objects, illustrating a potential link between two Aspects of Definitional Practice.

The third and most frequent type of definitional argument or explanation was used to argue for the inclusion or exclusion of an example (Ambrose & Kenehan, 2009; Dalhberg & Houseman, 1997; Herbst, 2005; Keiser, 2000; Lehrer & Curtis, 2000; Lehrer et al., 1999; Lin & Yang, 2002; Mariotti & Fischbein, 1997; Roth & Thom, 2009; Zandieh & Rasmussen, 2010). This type of justification, not surprisingly, occurred frequently in the sorting and classification tasks since the nature of the task centered on evaluation of examples and non-examples. These justifications appeared to range in sophistication. Although there was some variation of arguments and explanations within studies, variation was most clearly observed across the studies. The least sophisticated justifications did not attend to any aspect of the object: [it's not a pyramid] "because it just doesn't look like one…I can't quite put my finger on it" (Ambrose & Kenehan, 2009, p. 172). Other less sophisticated arguments and explanations attended to non-mathematical attributes of the object to suggest inclusion or exclusion. For instance, these included justifications that attended to shape or size (e.g., "cause this one is sort of bigger than the other ones," Roth & Thom, p. 66) or justifications that describe the overall appearance (e.g., "because it looks like a triangle," Ambrose & Kenehan, 2009, p. 167). More sophisticated justifications attended to particular components of the objects, such as a geometric object's parts, but argue based on prototypical beliefs. For instance, one student explained to her class during their investigation of triangles that, "this one isn't a triangle. Because these things [pointing at the long sides] are going way up high, and they have to be kind of smaller" (Lehrer et al., 1999,

p. 75). In contrast, other justifications attended to *mathematically relevant* properties of an object's components (e.g., those are all pyramids because "they all have a pointy part at the top," Ambrose & Kenehan, 2009, p. 169, or it belongs with the cubes "cause these are more squares…they are all squares I think, Roth & Thom, 2009, p. 72).). Other justifications considered relations between features (e.g. "some of the triangles don't touch the base," Ambrose & Kenehan, 2009, p. 169). However, the most sophisticated form of justification was that which privileged mathematical properties that were part of the current definition. For instance, Lehrer and colleagues (1999) describe one child's argument during a class's construction of a definition of triangle. The child had constructed a triangle with 3 paper strips, with one curved strip. When the class rejected her example as a triangle, she disagreed, appealing to their collectively constructed definition of "3 corners, 3 sides," "No. It doesn't matter. Look [gesturing to the board], it has three corners [gesturing to each vertex] and three sides [gesturing to each strip of paper]" (p. 78). This type of justification is sophisticated because it resembles those described by Lakatos (1976) as emblematic of arguments within the mathematical community. Such arguments are emblematic because the forms of evidence used are agreed upon components of the community's mathematical system (i.e., the definitions), rather than, for example, opinions. Moreover, they also contribute to the overall goal of constructing a definition and may lead the community to consider what needs to be revised about a current definition.

The final type of justification was used to justify that conditions were minimal. Although this only clearly occurred in one study, it is important to note because it requires students to consider relations between properties. For instance, Lin & Yang (2002) describe two students investigation of minimal properties for a rectangle. When the teacher asked them whether four right angles implies that the opposite sides will have the same length, one student responded, "Because they are right angles, two sides of the right angles will be parallel, it won't be stretching out, because these two lines will always be cut off like this, then because it a right angle too, so it must be straight" (p. 22). Lin and Yang noted that their students' arguments typically employed natural language or gestures, for instance, moving a pencil along a horizontal line to show that AB equals CD and AD equals BC in a rectangle). With all the forms of argument, often argument led to revision of definition, another aspect of practice, described next.

Defining involves revising definitions. Defining also involves the revision of definitions to serve the needs of the mathematical classroom community. Revision often resulted from definitional arguments or from evaluating examples or non-examples. Often, definitions were expanded to include additional properties or relations (Borasi, 1992; de Villiers, 1998; Mariotti & Fischbein, 1997; Kieser, 2000; Larsen & Zandieh, 2005; Lehrer & Curtis, 2000; Lin & Yang, 2002; Zandieh & Rasmussen, 2010). Sometimes, students expanded definitions unnecessarily, but incorporated newly learned attributes. Other times definitions were expanded to include necessary properties that were needed for the definition to be correct. For instance, in Lehrer & Curtis's (2000) study with third graders constructing a definition for "perfect solids," in trying to find the final perfect solid, the teacher suggested that they compare a non-example they had constructed with an example. This comparison helped them realize that, although both solids were built with triangular pieces, unlike the non-example, the triangle pieces in the example had congruent sides. They thus added this property to their list of rules. In other cases, students revised definitions to make them more minimal (Borasi, 1992; de Villiers, 1998; Herbst et al., 2005; Lehrer & Curtis, 2000; Lin & Yang, 2002). For instance, de Villiers (1998) describes an activity where high school students were given a set of examples of rhombi and were asked to list their common properties and then create a definition for them. Because this usually resulted in long definitions that were not minimal, the students were then asked to shorten their definitions by considering eliminating some of the properties. Sometimes, definitions were neither expanded nor reduced but instead modified, mainly to improve their correctness (Lehrer & Curtis, 2000; Zandieh & Rasmussen, 2010). For instance, towards the beginning of their investigation of "perfect solids," when students in Lehrer & Curtis's (2000) class found new examples and non-examples, because one of the examples had 3 faces coming together at each vertex, one student noted that their conjecture of "three faces at each vertex" (p. 326) could not be true. Rather than eliminating that property all together, students suggested modifying it, either to "three or four faces come together at each vertex" or "the number just has to be the same at each vertex, but could be any number" (p. 326).

Defining involves asking definitional questions. On occasion, students also asked questions about definitions or about the qualities, properties or relations of the objects being defined. Students' questions were not frequently described within the studies, but, nevertheless, those questions that were varied in their purpose. Some questions were about the nature of definitions or defining. For instance, when Leikin & Winicki-Landman (2001) asked teachers to construct procedural definitions, when reflecting on the activity, one of the teachers asked, "What is considered to be a definition?" (p. 71), spurring a discussion about whether procedural definitions should be considered definitions or not. A couple questions were about the qualities of an object, asked in the process of trying to make sense of examples or the construction of examples. For example, in one study, the child was constructing procedural definitions of shapes using LOGO when he asked, "Will this still be a rectangle if I make these sides longer and longer and these shorter and shorter?" (Lehrer et. al, 1989, p. 166-167). Other questions asked about which properties of an object are necessary and/or sufficient for inclusion in the definition.

For instance, in Borasi's (1992) study, when the students were considering the definition of isosceles triangle, one student asked whether the property of "two equal angles" was sufficient for determining a triangle to be isosceles: "Does it really guarantee that if a triangle has two equal angles then it is isosceles?" (p. 34). As another example, in Zandieh & Rasmussen's (2010) study, when the undergraduates worked in groups to write definitions of planar triangles, one student questioned whether the property of "endpoints" was necessary for an object to be a triangle: "Okay, but do you have to have endpoints?" (p. 62). The student then proceeded to draw three rays, intersecting to form a triangle and asked another type of definitional question, one about including or excluding the drawn case within the class of triangles: "Is that not a triangle?" (p. 62). All of the question types occurred within multiple studies, occurring at least twice and at most four times in all. Thus, although definitional questions were not frequent, they illustrate the potential for the mathematical inquiry that defining may encourage.

Defining involves negotiating criteria for judging adequacy or acceptability. Students also negotiated which features of definitions should be used to determine the acceptability of a definition (Borasi, 1992; Leikin & Winicki-Landman; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). In all of these cases, such discussions arose when students were either evaluating multiple definitions or constructing their own definition and were motivated by the need to determine whether to accept a particular definition or part of a definition. Topics of negotiation included: a) what constitutes a definition (Leikin & Winicki-Landman, 2001), b) whether a definition needs to be minimal (Borasi, 1992; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005), c) whether procedural definitions are acceptable (Leikin & Winicki-Landman, 2001; Zaslavsky & Shir, 2005), d) whether any concept may serve as a basis for a definition or if those concepts must first be defined (e.g., can you define "square" using the notion of “rectangle,” Zaslavsky & Shir, 2005), e) whether *any* property (e.g., the latent parts, such as the diagonal of a square, or properties of objects that are the result of proof, such as the sum of the angles in a triangle) may serve as the definition or part of the definition (Borasi, 1992; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005), f) whether correctness of a statement guarantees its acceptance as a definition (Zaslavsky & Shir, 2005), and g) whether multiple definitions for an object may exist (Zaslavsky & Shir, 2005). For instance, when the students in Zaslavsky & Shir’s (2005) study were evaluating alternate definitions for square, they came across a longer definition. This definition spurred discussion about whether a definition needs to be minimal to be acceptable.

Erez: It’s correct, but it’s not a definition.
Yoav: It’s correct, and it is a definition.
Erez: It has too many details.
Yoav: Too many details, but it is still a definition.
Omer: What do “too many details” have to do with that?
Mike: In which definition here don’t you have too many details? (p. 329)

In this example, Erez argued that a definition is not “correct” because it “has too many details.” Omar and Mike then questioned more generally why having extra information is unacceptable. The students later came to agree that although extra details are not preferable, they are nonetheless acceptable. Criteria are often negotiated in service of definitional arguments. However, negotiations of criteria is still worth separating as a distinct aspect of definitional practice because they allow norms to be established for future definitional arguments and thus serve to construct shared understandings of practice within a mathematical community.

Defining involves considering definitions in new forms or contexts. Defining may also involve the consideration of definitions of existing objects in new forms or in new contexts. Students considered definitions in new forms when they evaluated and/or constructed procedural definitions. Evaluation of procedural definitions led to discussions about whether such definitions are acceptable forms of definition (Leikin & Winicki-Landman, 2001; Zaslavsky & Shir, 2005) whereas construction of procedural definitions provided students opportunities to reason about the properties and relations of the mathematical object being defined (Lehrer et al., 1989; Leikin & Winicki-Landman, 2001). For example, in Zaslavsky & Shir's (2005) study, the group of four students evaluated a list of definitions for square, one of which was procedural: "An object that can be constructed (in the Euclidean Plane) as follows: Draw a segment; from each edge erect a perpendicular to the segment, in the same length as the segment (both in the same direction). Connect the other 2 edges of the perpendiculars by a segment. The 4 segments form a quadrangle that is a square" (p. 345). When students evaluated this definition, they discussed whether a set of instructions should be allowed to be a definition. One student noted, "It's an instruction, it's not-," and his peer followed with, "It's a description of how to construct a square…You should write that we don't accept it [as a definition of a square]" (p. 329). On the other hand, Lehrer and colleagues (1989) engaged two groups of students in the construction of different types of procedural definitions. One group used protractors and rulers to create instructions for constructing two-dimensional geometric figures while the other group used LOGO (a software program) to create sets of instructions for "walking" along a geometric figure's sides. These two situations differed in more than material – the LOGO context allowed students to take a "path perspective" (Abelson & diSessa, 1980) that required them to consider relations between adjacent sides and angles. Using rulers and protractors required articulating positional aspects (e.g., this side is connected to this one) but did not necessitate articulating relations between angles and sides.

Students considered definitions in new contexts by expanding an existing definition to new domains or by using a definition of an object in a new space. For instance, Borasi (1992) asked students to expand their existing definition of exponents in the whole numbers to include new domains, namely fractional exponents and negative exponents. This expansion required students to rethink their notion of multiplication as repeated addition and to consider patterns within the system of numbers. That is, in order to agree upon new, additional rules, they had to take into consideration how patterns within the existing system could be carried into the new domain. On the other hand, when applying an existing definition of an object to a new space, students had to re-conceptualize the object because it's appearance, qualities and properties often changed (in other words, change their concept images). This definitional habit of mind is like that required for some definitional arguments. For instance, Zandieh & Rasmussen (2010) asked a group of college students to use their definitions of triangles in the plane to construct triangles on a new surface – the sphere. In deciding upon what constituted a triangle on the sphere, students continuously revisited their definition. As one student noted when evaluating a non-prototypical triangle, "It's not a traditional triangle, but it's correct by the definition" (p. 63). As a contrast, Borasi (1992) asked her students to consider two-dimensional geometric figures in the context of taxicab geometry, where distances and points are constrained to a grid (resembling a grid of streets in a city). The students were asked to draw the collection of points a distance five from one point. Although one of the students correctly drew this set of points, forming a diamond shape, she refused to acknowledge that it was a circle, even though it fit their agreed upon definition of circle. Thus, this habit of mind is at the center of discussions involved in considering objects in new contexts, especially new spaces.

Defining involves engaging in definitional conjectures, experiments and tests.
In a few cases, defining took the form of cycles of conjectures, experiments and tests (Larsen & Zandieh, 2005; Lehrer & Curtis, 2000; Lehrer et. al, 1999; Zandieh & Rasmussen, 2010). Cycles

began with making conjectures about properties to include in a definition and/or about potential examples of the object being defined. Conjectures were then followed by experiments that were then tested in some manner. For instance, Lehrer and colleagues (Lehrer & Curtis, 2000; Lehrer et. al, 1999) describe two elementary school classrooms, where in each, students worked collectively to construct a definition ("perfect solid" and "triangle," respectively). After students had done some initial work in formulating a conjectured definition, in both cases, they were asked to construct an example of the object being defined. Thus, in these cases, conjectures took two forms: the definition and the examples. Students then shared their "experiments" with the class. Experiments were tested against the class's definition and against existing examples and non-examples. Experiments and tests took different forms in Larsen & Zandieh's (2005) work with college students. In this setting, students wanted to make a simpler definition of "subgroup" that would be easier to use in proofs. Students, working in a group, came up with a conjectured definition. They immediately tested it by trying to prove that their new definition was equivalent to the original definition. When this failed, the teacher suggested a counter-example to their definition. Students then "experimented" by analyzing the counter-example with the goal of improving their definition. This contrast between the elementary and college settings illustrates how participation in defining may shift when students also have greater experience participating in other mathematical practices, such as proof.

Defining involves establishing & investigating systematic relations. As illustrated in Lakatos's (1976) historical analysis, defining involves establishing systematic relations, both among definitions and between definitions and other mathematical entities, such as proof. A few of the studies hint at how students' participation in defining may involve unpacking relations between definitions (Lehrer & Curtis, 2000; Lehrer et al., 1989; Herbst, 2005; Ouvrier-Buffet, 2005). For example, in two of the studies, when students created definitions of objects, they also defined any objects that were needed in their initial definitions. Lehrer and Curtis (2000) describe how when the third graders investigated two-dimensional representations of three-dimensional solids, they found that words like "sides" and "corners" meant different things to children. Because of this, the students negotiated agreed upon meanings for these objects. In Ouvrier-Buffet's (2005) study of older children defining "straight line," the students suggested making a glossary to establish definitions of commonly used words, such as "pattern." Investigating such systematic relations may also entail unpacking fundamental qualities of the objects being defined, illustrating how the two Aspects of Practice may be related. Systematic relations can also be established between objects in a hierarchical manner (e.g., a square is a type of rectangle). Lehrer and colleagues (1989) described students doing so when they constructed both structural and procedural definitions for quadrilaterals. When constructing structural definitions, the students considered whether squares were kinds of rectangles, facilitated by comparing their properties. When constructing procedural definitions, students instead compared procedures for constructing squares to those for constructing rectangles. These relationships were made clear by the LOGO environment; whereas procedures for rectangles could produce squares, procedures for squares did not produce rectangles other than squares.

In their work with college undergraduates, Zandieh and colleagues (Larsen & Zandieh, 2005; Zandieh & Rasmussen, 2010) illustrated how defining led to questions and conjectures and how proofs alternatively contributed to the need to create new definitions. In this case, when students applied their definition of planer triangle to the sphere and constructed examples of spherical triangles, they began noticing new properties of the triangles. These discussions led to questions about the sums of the angles in spherical triangles ("Right, but is there a relationship between the sum of angles in the triangles") and to conjectures about what those sums might be. In this case, defining contributed not to definitional questions and conjectures, but rather to provable conjectures that could ultimately become theorems. Later, the students were asked to prove or disprove whether the property of "side-angle-side" held for triangles and, if not, to define a subset of triangles for which it did. In this case, rather than changing the theorem, the students defined an object that would fit the theorem, much in the manner that Lakatos (1976) described in his analysis of the history of the proof of the Euler characteristic.

Potential of Mathematical Defining in Classrooms

Defining contributed to engagement in Aspects of Definitional Practice. The studies collectively illustrate that allowing students to participate in defining provides opportunities for learning in several ways. First, as illustrated in the previous section, the activities provided students opportunities to participate in aspects of the practice of defining. As shown in Table 3, in all of the studies, students participated in two or more aspects of the practice of defining. Because these studies are representations of the students' participation in defining, it is possible that they do not include all aspects in which students participated in and, thus, activity might have been even more mathematically richer than portrayed. Moreover, Table 3 also shows that in almost all of the studies (16 of 19), students participated in definitional argument and/or explanation. Explanation and argument are forms of mathematical discourse that are prevalent in reform mathematics classrooms where the emphasis has been on promoting discourse-rich environments (Chapin, O'Connor, & Anderson, 2003) and encouraging reasoning (National Research Council, 2001). Furthermore, because definitions are often the base for further mathematical work in which students participate in explanation and argument, such as problem-solving and proof, participation in definitional argument and explanation might provide an entrée into these forms of mathematical discourse. Other discourse-rich Aspects of Definitional Practice, such as revising definitions and describing properties, were also reported in at least half of the studies.

Table 3. Occurrence of Aspects of Definitional Practice among the reviewed works

	Gen. Obj.	Conj., Exp., Tests	Neg. Crit.	New Form, Cont.	Syst. Rel.	Ask Ques.	Fund. Qual.	Rev. Def.	Desc. Prop.	Con. Eval. Ex.	Expl. & Arg.	**SUM**
D&H (1997)										**X**	**X**	**2**
H,G&M (2005)	**X**										**X**	**2**
L&Y (2002)								**X**			**X**	**2**
O-B (2006)					**X**					**X**		**2**
D (1998)								**X**	**X**		**X**	**3**
F&P (2002)						**X**	**X**			**X**		**3**
A&K (2009)									**X**	**X**	**X**	**3**
R&T (2009)									**X**	**X**	**X**	**3**
M&F (1997)								**X**	**X**	**X**	**X**	**4**
L&W-L (2001)			**X**	**X**		**X**					**X**	**4**
L&Z (2005)		**X**			**X**			**X**			**X**	**4**
L,R&S (1989)				**X**	**X**	**X**			**X**	**X**		**5**
H (2005)					**X**		**X**		**X**	**X**	**X**	**5**
Z&S (2005)			**X**	**X**					**X**	**X**	**X**	**5**
L,J,K,S (1999)		**X**					**X**		**X**	**X**	**X**	**5**
K (2000)						**X**	**X**	**X**	**X**	**X**	**X**	**6**
L,C (2000)	**X**	**X**			**X**		**X**	**X**	**X**	**X**	**X**	**8**
B (1992)			**X**	**X**		**X**	**X**	**X**	**X**	**X**	**X**	**8**
Z&R (2010)	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**11**

SUM	3	4	4	5	6	6	7	8	12	14	16

Table 3, continued

Note. Articles are denoted by the first initial of the last name of each author, in the order in the reference (e.g., L,R&S stands for Lehrer, Randle & Sancilio). Year of publication follows. Aspects of Practice are abbreviated according to the following: Investigating Fundamental Qualities of Mathematical Objects (**Fund. Qual**), Describing Properties of Objects (**Desc. Prop.**), Using Definitions to Generate Objects or Examples (**Gen. Obj.**), Asking Definitional Questions (**Asking Ques.**), Engaging in Definitional Conjectures, Experiments & Tests (**Conj., Exp., Tests**), Constructing Definitional Explanations & Arguments (**Expl. & Arg.**), Revising Definitions (**Rev. Def.**), Constructing & Evaluating Examples (**Con. Eval. Ex.**), Negotiating Criteria for Judging Adequacy or Acceptability (**Neg. Crit.**), Considering Definitions in New Forms or Contexts (New Forms, Cont.), Establishing & Investigating Systematic Relations (**Syst. Rel.**).

Defining motivated closer analysis of the objects and relations defined. One aspect of definitional practice, investigating fundamental qualities of mathematical objects, illustrated that defining has the potential to motivate investigations about the properties and relations of objects. As mentioned previously, this motivation helps make such investigations authentic and warranted rather than separate and dictated by the teacher. Moreover, such investigations may encourage systematic investigations that encourage development of relations between definitions (such as between "polyhedron" and "side"). In addition, closer analyses of objects often reveal attributes of objects that may not be articulated in definitions but that are significant in identifying examples and non-examples and contribute to a multi-faceted understanding of the object. As described next, closer inspection may support students in developing deeper understanding of the objects they are defining.

Defining contributed to conceptual development. In many cases, participation in defining also supported students' conceptual understanding of definitions. As mentioned earlier, one concern with traditional approaches to definition is that often students do not develop understanding of the concepts being defined, leading to difficulties when engaging in problem solving and proof. Moreover, a major goal for mathematics educators in recent reforms is to promote mathematics as a sense-making enterprise, one in which proficiency in mathematics involves a deeper understanding of concepts (National Research Council, 2001). At the very least, when students participated in defining, it exposed their thinking about seemingly simple mathematical objects, especially in relation to what they considered to be examples of an object (Ambrose & Kenehan, 2009; Keiser, 2000; Lehrer et al., 1999; Mariotti & Fischbein, 1997; Roth, 2009). For instance, Keiser (2000) describes the varied ideas sixth grade students initially articulated about angles: a) considering the vertex to be the angle, b) considering the rays to be the angle and c) considering interior space to be the angle. Moreover, as students explored angles, they continued to express other ideas, such as suggesting that the size of the angle might impact its measure. Exposing students' thinking early on and throughout allows teachers to center their instruction on students' ideas.

At the same time, defining appeared to help broaden students' initial images of the objects they were investigating, often entailing that they were able to more correctly generate and/or evaluate examples or non-examples of objects (Borasi, 1992; Dalhberg & Houseman, 1997; Keiser, 2000; Lehrer et al., 1989; Lehrer et al., 1999; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). For instance, in Keiser's (2000) study, many of the students initially thought that angles only existed *within* shapes (not on the outside), but after their five-week investigation, more of them began to accept exterior angles as angles. Defining also appeared to support students' description of objects, moving them away from holistic descriptions towards more mathematical descriptions that focused on relevant parts and properties (Ambrose & Kenehan, 2009; Roth & Thom, 2009). In other cases, students already attended to mathematically relevant features, but instead, became more aware of which of those features were *necessary* in order to construct a definition (Borasi, 1992; de Villiers, 1998; Herbst, et al., 2005; Keiser, 2000; Lehrer & Curtis, 2000; Lehrer et al., 1999; Lin & Yang, 2002; Ouvrier-Buffet, 2006; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). For example, when defining "perfect solid," the students in Lehrer & Curtis's (2000) study continuously revised their conjectured "rules" as they evaluated new examples and non-examples. One conjecture, that three faces needed to meet at each vertex, was eliminated once a student noticed that the octahedron had four faces coming together at each vertex and was thus not a necessary property. They later realized, after comparing an example with a non-example, that the faces all needed to have the same lengths of sides, a property that was later further revised to include the same angle size as well. In a couple of these cases, students also improved in writing minimal definitions (Borasi, 1992; de Villiers, 1998; Lin & Yang, 2002). De Villiers (1998) compared two instructed groups, one who engaged in learning definitions in the traditional way and one who engaged in constructing definitions and revising them to make them more minimal. When tested at the end of instruction, a higher percentage of the instructional group gave correct, minimal definitions. Furthermore, in a couple of cases, the defining activity supported students in thinking about relations between properties and/or between objects, such as hierarchical relations between geometric shapes (Furinghetti & Paola, 2002; Lehrer et al., 1989).

Defining encouraged developing disciplinary identities. Third, in some cases, students appeared to develop dispositions towards what it means to participate in defining. That is, they appeared to develop authority as a participant in the practice. For instance, as mentioned under Aspects of Practice, in a few cases, defining provided a venue for students to negotiate what counts as a definition and important properties of definitions (Borasi, 1992; Leikin & Winicki-Landman; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). Unlike in traditional classes, where the authority rests in the teacher or textbook, here students were taking it upon themselves to determine what properties were significant to conduct their investigations with definitions. And, in some of these cases, students showed evidence of learning about the properties of definitions, such as the fact that multiple definitions may exist for the same object. Moreover, although not frequently documented, there were instances in which students' talk appeared to suggest that students were taking on authorities as "definers." For example, in Lehrer & Curtis's (2000) study of the third graders defining "perfect solid," students appeared to uptake the goal of defining. When they were struggling to find the fifth perfect solid and had a solid that was surprisingly rejected, several students proclaimed, "we must not have found all the rules [properties] yet!" (p. 328). This example illustrates that students had tied the activity of constructing and evaluating examples to the overall goal of constructing a definition, without prompting from the teacher. In Borasi's study (1992), when the students were proving a theorem about polygons, after the teacher asked "How do you think we can prove something like this?" one student remarked, "I don't know. Take a polygon as an example. [[the teacher] immediately draws one, a convex pentagon] **We never really got to the definition of a polygon. We think this is a polygon**" (bold added for emphasis) (p. 50). This instance illustrates the student's appreciation of the role that definition plays in other mathematical practices and their authority in determining that definition. This is notable because this student had reported having horrible past experiences with mathematics and had recently failed the standardized test in geometry. In Zandieh & Rasmussen's (2010) study with undergraduates, the students had previously engaged with definitions and, in the cases described, they already appeared to have developed dispositions as authors of definitions. When they were working in their small groups, they frequently reminded one another of their goal of defining and, even for the familiar object of "triangle" on the plane, they negotiated which properties should be part of the definition by appealing to examples and features of definitions rather than appealing to a pre-determined definition. Although this study may not show how such dispositions develop, it does illustrate what participation might look like after students have taken on more authoritative roles with defining.

Supporting Defining

The previous section illustrates that defining has potential to support students in several ways, and thus the question remains of how that might be done. Although the studies in general focused very little on supports for defining, they still suggest some important directions for further investigation. In the next two sections, I illustrate what such conjectures might look like, first in regards to designed supports and then with regards to how a teacher might orchestrate classroom discussion.

Designing tasks to support defining. Here, I employ Lehrer's (2009) framework of designing for disciplinary practices to describe some supports for defining along the lines of four of the five elements of design: nature of tasks, inscriptions, material means, modes and means of argument. These are suggestions for defining broadly, and it is expected that particulars of design would vary depending on the particular mathematical topic of investigation. Supports the fifth element, identity, along with additional supports about modes and means of argument, are discussed in the following section about the role of the teacher in orchestrating discussions around definition.

Tasks should include opportunities to construct new forms of definitions. As described earlier, one aspect of the practice of defining that occurs less frequently in classrooms is that in which students consider definitions in new forms, namely procedural, and within new contexts, such as a new space. Lehrer and colleague's (1989) study illustrates that engaging students in the construction of procedural definitions, in particular those where students create instructions for walking polygons, has the potential to help students notice relations between figures. Additionally, constructing procedural definitions might provoke students to ask definitional questions that may be less likely with structural definitions, such as questions about modifying properties of objects described above. Besides supporting conceptual understanding of the objects being defined, procedural definitions are also significant to the discipline and constructing them is an important part of disciplinary practice. However, as Zaslavsky & Shir (2005) pointed out in their study, procedural definitions are not naturally accepted forms of definitions for students. Although they described a small sample, the students were mathematically advanced and yet still rejected procedural definitions as acceptable definitions. Moreover, Leikin & Winicki-Landman (2001) found that teachers too questioned the validity of procedural definitions. Thus, students should be provided opportunities to construct procedural definitions as well as discuss their role as definitions, perhaps by experiencing their use in solving problems or proving theorems.

Tasks should include opportunities to evaluate examples and definitions. When students were asked to evaluate examples and non-examples or definitions and non-definitions, the conversation appeared to support development of students' understanding of the definitions. As Dahlberg & Housman (1997) found in their one-on-one sessions, generating examples appeared to support students' learning of a new definition. This result is consistent with work done with mathematicians that shows that examples play a significant role in their sense-making of new ideas (Wilkerson-Jerde & Wilensky, 2011). In the studies, many times, evaluation was promoted through classification or sorting tasks of examples or by giving students a list of definitions and non-definitions. In both cases, having non-examples and non-definitions appeared to be critical for supporting changes in students' thinking. For example, in Zaslavsky and Shir's (2005) study, students changed their thinking mainly when they evaluated definitions of analytic concepts rather than geometric concepts. Although this difference may be attributed to the difference in the concepts, at the same time, the analytic concepts included definitions and non-definitions whereas the geometric definitions included only examples and non-examples.

Evaluating examples and definitions also appeared to generate definitional argument among students. Argument is an important part of mathematical practice, and as the studies illustrate, can constitute a significant part of defining as well. Argument, however, is rooted in contest within a mathematical community, and thus examples and definitions must be designed to promote contest. Classification or sorting tasks should include a range of examples and non-examples and evaluation activities should likewise include a range of definitions and non-definitions. Especially important is to include non-prototypical examples and/or definitions. Examples and/or definitions that differed from students' images often generated contest that resulted in revision of the definition at hand (e.g., Lehrer et al., 1999). Moreover, particular definitional activities appear to promote particular types of definitional argument, so tasks should be designed with these in mind. For instance, sorting and classification activities appear to be fertile grounds for promoting arguments over the inclusion or exclusion of examples of the mathematical object being defined. Alternatively, as Zaslavsky and Shir (2005) illustrated, having students evaluate alternate definitions and non-definitions and justify choices led to arguments over inclusion or exclusion of the definitions. Moreover, when definitions varied in form and length, students also argued about the nature of definitions, such as whether definitions must be minimal or can be procedural. This suggests that when students are asked to evaluate alternate definitions that vary in their features, it may encourage such discussion.

Tasks should take advantage of leveraging students' everyday experiences. In the studies with young children, many of the sorting and classification tasks took advantage of children's existing experiences with shape as a starting point for description. For instance, in Lehrer and colleague's (1999) work with second graders, the students had pre-existing notions of what a triangle should look like. Because of this, they had resources with which to evaluate and describe the set of triangles provided by the teachers. At the same time, their different opinions provided contest to motivate conversations around their choices.

Inscriptions should position defining at the forefront. The need for communication about an object's properties and relations is what distinguishes defining activities from classroom activities centered solely on conceptual development. Moreover, simply identifying an object's properties may be a part of defining but is not a defining activity all on its own. When they sort, students may discuss similarities and differences of the objects and then refine their classifications. However, this task is only considered a defining activity if the students attempt to describe what constitutes membership to a group of objects (see Lehrer et. al., 1998 for an example). In studies by Lehrer and colleagues (Lehrer & Curtis, 2000; Lehrer et al., 1999), the teacher used inscriptions to make the definition salient and at the center of attention. For instance, in one classroom, on one board, she wrote the students' conjectured rules on the side board and their agreed upon rules on the front board. Thus, in this case, changes in inscription represent changes in definitions, thus making revision visible to students.

Materials should be selected to highlight particular properties and relationships. Finally, materials that teachers use should be selected in order to highlight properties and relations of the objects being definition. For example, in Lehrer and colleague's (1989) study in which students constructed procedural definitions, they found that students who constructed procedural definitions with LOGO versus with compasses, rulers and pencil recognized relations between objects more readily on a post assessment than those who had worked with the other materials. The authors conjectured that because LOGO requires students to construct directions for "walking" a geometric object, students must think about relations between angles and lengths of sides. Certain directions can be modified to create new shapes and modifications highlight important similarities and differences between the shapes that promote, for example, seeing hierarchical relations between quadrilaterals. In contrast, when constructing with rulers and compasses, procedural definitions do not need to take into account relations between properties. For instance, such a procedural definition for a triangle might be: "Draw a straight line of a particular length. Draw another straight line, connected to one vertex of the first line. Draw a third line that connects the remaining two vertices of two lines." Note that this definition does not specify relations between the angles (internal or external), nor relations between sides and angles. As another contrast, Furinghetti & Paola (2002) found that when they had students work with Cabri, a dynamic geometry software, students talked about relations between geometric objects, but in reverse hierarchical order. That is, they thought squares were the largest set because with the program, it was easiest to start with a square and "stretch" it into other shapes. Perhaps because students were manipulating the overall shape rather than particular properties that they programmed (as in LOGO), the conventional hierarchical relations were not as salient.

The role of the teacher in orchestrating discussion in defining. All of the class activities that researchers produced in the reviewed studies involved some form of discussion around definitions. However, facilitating such discussions is not trivial. Teachers play a critical role in orchestrating classroom discussions in mathematics classrooms (Stein, Engle, Smith, & Hughs, 2008), but what about this role is particular to supporting defining? In order to highlight potentially significant ways in which the teacher facilitates discussion, I contrast excerpts of classroom transcripts from two studies about classroom defining. Both of these studies involve elementary-aged students participating in classification activities in geometry, yet with varying results. In the first study, Ambrose and Kenehan (2009) engaged third graders in the classification of polyhedra in order to construct a definition of pyramid. Although students began to notice mathematical features of pyramids, on the whole, the researchers found that the children did not develop mathematical definitions. In the second study, Lehrer and colleagues (1999) describe a class of second graders classifying a set of examples and non-examples in order to construct a definition of triangle. In this case, students progressed in the development of a definition and also participated in definitional practices, especially that of argument.

Here, I compare excerpts of transcript from whole class discussion from the two studies. Each illustrates a piece of classroom discussion where students negotiated inclusion or exclusion of examples. The goal of this comparison is not to criticize the teacher; nor am I claiming that these are typical excerpts of classroom practice. Rather, this contrast may provide some initial conjectures of significant teacher moves for promoting defining, in particular, how teachers create opportunities for defining. Using the frameworks of revoicing as shifting participant frameworks (O'Connor & Michaels, 1996) and the four principles for fostering productive disciplinary engagement (Engle & Conant, 2002), I claim that the teacher from Lehrer and colleague's (1999) study played a significant role in making the activity a *defining* activity in three ways: a) by positioning defining at the forefront, b) by positioning defining as a form of argument, and c) by encouraging precise language. Each of these includes specific talk moves, described below. For purposes of this comparison, I use “Teacher A” to refer to the teacher from Lehrer et al. (1999) and “Teacher B” to refer to the teacher from Ambrose and Kenehan (2009).

Positioning defining at the forefront. The first noticeable difference between the two excerpts is that unlike Teacher B, Teacher A positioned defining at the forefront of the class discussion (see Figure 2). Teacher A did this in two ways. In her classroom, the purpose of the activity was to construct a set of sharable, agreed upon “rules” for triangles, and she made this goal explicit by reminding the students of it during discussion. As the students were discussing the set of examples and non-examples of triangles, students implicitly expressed that they thought the relative lengths of the sides and the orientation were both significant for determining triangles from non-triangles. In response, Mrs. Curtis asked the students, “Could that possibly be a **rule for triangles**? (bold added for emphasis).” In doing so, she related the current activity of evaluating the examples and non-examples to their ultimate goal of making rules for triangles, thus holding them accountable to the discipline. Moreover, Teacher A also gave defining a voice by giving the “rules” agency and “revoicing” the class’s established rules: “**Our rules for triangles say** that a triangle needs three sides” (bold added for emphasis). In the sense of participant frames for revoicing (O’Connor & Michaels, 1996), Teacher A rebroadcasted the class’s rules by using indirect speech, and thus positioned the definition as a significant participant in the discussion. I will further describe this significance in the next section.

(In evaluating the examples and non-examples on the board, the students initially selected prototypical triangles oriented on a base (such as equilateral triangles). Teacher A: Could that possibly **be a rule for triangles**? All the sides of a triangle have to be the same length. Children: No, yes, no… Beth: Only the diagonal sides. … Teacher A: OK, touch the sides you are calling the diagonal sides. [Beth touches the two slanted sides] Teacher A: OK, then what are you calling the other side? Beth: This one? That is the bottom. Teacher A: **Our rules for triangles say** that a triangle needs three sides. So, I would say, side, side, side [marking each of the sides].	(The class is looking at a trapezoidal prism. One student, Mikey, thinks the solid is a pyramid "because it has triangles.") Janet: It doesn't have the triangle at the top. That's one special thing that pyramids have. Teacher B: **What do you mean by a triangle at the top?** Janet: That at the top. Teacher B: **So how about this one**, where's the triangle at the top of that one (hands Janet a hexagonal pyramid). Janet: They all make a… Rick: Point. Teacher B: Rick, say what you mean. Rick: They all have a pointy part at the top. Teacher B: They all have this pointy part at the top. **Does everyone agree?** Where is the pointy part at the top of that one Mikey?

Figure 2. Positioning defining at the forefront. The two pieces of transcript provide a contrast for how teacher may or may not make defining a focal activity. Teacher A (left) positioned the definition as a key participant in the discussion and thus made defining a focal activity. Bold words are used to highlight differences in talk between the teachers. The transcript on the left comes from Lehrer et al. (1999, p. 74-75) and the transcript on the right comes from Ambrose and Kenehan (2009, p. 167).

As a contrast, when the students in Teacher B's classroom proposed important properties, she asked questions such as, "What do you mean triangle at the top?" or "So what about this one?" or "Does everyone agree?" (p. 167). These questions are important for eliciting students' descriptions and holding students accountable, and are considered productive talk moves in promoting mathematical discussion more generally (Chapin et al., 2003). However, they are not as directly tied to *constructing* a definition as Teacher A's question and comment were. This difference may seem subtle, but may be important for moving the activity from solely *description* to *defining*. This is not to say that teachers should not ask questions similar to those of Teacher B, and, in fact, description is an important aspect of the practice of defining and should be encouraged. Rather, teachers should pair such questions with questions that position defining at the forefront. In activities where defining arises out of problem-solving, it is especially essential for the teacher to initiate defining by asking students to consider definitions.

Positioning definition as a form of argument. Teacher A also played an important role in positioning defining as a form of argument (see Figure 3). She did so in two ways. First, she positioned students' utterances as competing with the class's definition. For instance, when Beth stated that one of the sides of the triangle was the "bottom," Teacher A replied, "Our **rules** for triangles **say** that a triangle needs three sides" (bold added for emphasis). As described in the previous section, by using indirect speech, Teacher A imparted agency to the class's definition and, in a sense, positioned it as a member of the community. Because she placed this utterance in response to Beth's claim, it then positioned the definition in contest to her claim. Furthermore, the teacher then proceeded to position defining as an argument in a second way, by placing Beth's utterance in contrast to her own: "So I would say, side, side, side…But why is it the bottom? Why does it get a special name?" She then furthered the contest by presenting a counter-argument: "Why can't it be side, side, side?" In making her argument, the teacher invoked the class's definition, and, thus, modeled a more sophisticated form of definitional argument. Recall that that later Sadie made the argument referred to previously about her triangle with curved sides, and, in doing so, also invoked the class's definition. This suggests that, perhaps, the teacher's modeling had some uptake. Moreover, by invoking the definition to counter Beth's point, Teacher A held her accountable to disciplinary norms. In other studies, other teachers similarly positioned defining as a form of argument by providing counter-examples to contest students' definitions. For instance, during Borasi's (1992) study, when the students were defining circle, Borasi pointed out to them that a ball would satisfy their definition. This counter-example helped the students realize they needed to modify their definition to include that a circle must lie in the plane.

Teacher A:	OK, then what are you calling the other side?
Beth:	This one? That is the bottom.
Teacher A:	**Our rules for triangles say** that a triangle needs three sides. **So, I would say, side, side, side** [marking each of the sides].
Beth:	But this is the bottom of the triangle [pointing at the bottom of the triangle]
Teacher A:	But **why is it the bottom**? **Why does it get a special name? Why can't it be side, side, side?**
Beth:	These two are the sides [pointing to the two slanted sides] because this one is laying flat [pointing at the bottom] but these ones are going up [gestures, showing how the sides slant.]

Janet:	They all make a…
Rick:	Point.
Teacher B:	Rick, say what you mean.
Rick:	They all have a pointy part at the top.
Teacher B:	They all have this pointy part at the top. **Does everyone agree?** Where is the pointy part at the top of that one Mikey? (Hands him the large tetrahedron.)
(The class is looking at a polyhedron with a hexagonal base. It is not a pyramid.)	
Ernesto:	I think it goes in the not one (referring to the not-pyramid pile).
Teacher B:	**And why does it belong in the not one?**
Ernesto:	The pointy top doesn't show that much.
Teacher B:	Okay, the pointy top's not quite up the way these are. And why else not, Rick?

Figure 3. Positioning definition as a form of argument. The two pieces of transcript provide a contrast for how teacher may or may not position defining as argument. Bold words are used to highlight differences in talk between the teachers showing, in particular, that whereas Teacher A positioned defining as an argument, a form of contest, Teacher B positioned it as explanation. The transcript on the left comes from Lehrer et al. (1999, p. 75) and the transcript on the right comes from Ambrose and Kenehan (2009, p. 167, 169).

Teacher B also pushed students to justify their thinking by asking probing questions. However, her questions encouraged *explanation* rather than argument. For instance, when a student suggested a claim, she asked, "And why does it belong to the not one?" and later followed with, "…And why else not, Rick?" Although these questions are important for promoting articulation of students' claims, they do not go further to position claims as competing, thus creating contest. Although Teacher B had earlier asked students whether they agreed with a claim, she did not ask them to defend their stances or ask if for disagreements. Not only is argument is an important disciplinary practice, but contest forces members of a community to consider opposing viewpoints and come to a resolution, one that helps to push knowledge forward. And, in several of the reviewed studies, argument promoted revision (e.g., Borasi, 1992).

Encouraging preciseness in descriptive language. An essential part of defining is the articulation and description of an object's properties and relations. Although students may come with everyday forms of language for describing mathematical objects, one of the key roles of the teacher is to help students move towards more mathematical descriptions of objects. One way Teacher A did this is by encouraging preciseness in the students' descriptive language (see Figure 4). For example, in her conversation with Beth, rather than accepting Beth's description of "diagonal sides," Teacher A requested that she elaborate on what she meant by diagonal sides: "show us **what you mean** when you say the diagonal sides?" She further supported Beth's communication by suggesting that she point to the diagonal sides. Moreover, later when Beth referred to the "bottom" of the triangle, Teacher A again encouraged her to expand on her description: "But **why is it the bottom? Why does it get a special name**? Why can't it be side, side, side?" As these examples illustrate, Teacher A pushed on precise language by asking Beth questions that probed into what she meant. At the same time, questions like this that inquire about a related aspect of the object push students towards developing a mathematical system. In a mathematical system, as was described with Lakatos (1976), relations between mathematical relations are investigated and fleshed out.

Teacher A: Beth, can you come up to one of these triangles and **show us what you mean when you say the diagonal sides**? [Beth goes to the board and points at shape 5, an equilateral triangle.] Teacher A: OK, **touch the sides** you are calling the diagonal sides. [Beth touches the two slanted sides] Teacher A: OK, then what are you calling the other side? Beth: This one? That is the bottom. Teacher A: Our rules for triangles say that a triangle needs three sides. So, I would say, side, side, side [marking each of the sides]. Beth: But this is the bottom of the triangle [pointing at the bottom of the triangle] Teacher A: But **why is it the bottom**? **Why does it get a special name? Why can't it be side, side, side?**	Janet: It doesn't have the triangle at the top. That's one special thing that pyramids have. Teacher B: **What do you mean by a triangle at the top?** Janet: That at the top (traces her finger around the apex at the top of one of the pyramids). Teacher B: So how about this one, **where's the triangle at the top of that one** (hands Janet a hexagonal pyramid). Janet: They all make a… Rick: Point. Teacher B: Rick, say what you mean. Rick: They all have a pointy part at the top. Teacher B: They all have this pointy part at the top. Does everyone agree? **Where is the pointy part at the top of that one** Mikey? (Later, discussing a new solid) Elizabeth: It has a pointy part. Teacher B: This part, it has **sort of** a pointy part, doesn't it? Dwayne: They all have triangles like, like that, hmm, look like a pyramid. Teacher B: It's got **a bunch of triangles**, yup. Ernesto: I think it goes in the not one. Teacher B: And why does it belong in the not one? Ernesto: The pointy top doesn't show that much. Teacher B: Okay, **the pointy top's not quite up the way these are**. And why else not, Rick?

Figure 4. Encouraging preciseness in descriptive language. The two pieces of transcript provide a contrast for how teachers may encourage descriptive language. Bold words of the left are used to highlight that Teacher A pressed the student for further description. Bold words on the right show that Teacher B either did not press students (second excerpt), or, when she did, continued by asking pointed questions (first excerpt). The transcript on the left comes from Lehrer et al. (1999, p. 74-75) and the transcript on the right comes from Ambrose and Kenehan (2009, p. 167 & 169).

Teacher B also encouraged children to describe the objects in their own words. However, when children offered descriptive language such as "pointy," "bunch of triangles," or "not quite up the way these are," she did not always push them to elaborate on their descriptions. Instead, she sometimes restated what they said with positive affirmation: "This part, it has sort of a pointy part, doesn't it?" When the teacher did attempt to push children to elaborate on their descriptions, she did not always press students to describe what they meant. For instance, when a child noted that the solid was not a pyramid because “it doesn’t have the triangle at the top,” the teacher first asked, “What do you mean by a triangle at the top?” When the student explained that “that, at the top,” tracing her finger around the top of a pyramid, using gesture to describe what she meant, Teacher B asked the student to show “the triangle” on another solid. Although this move is good for supporting students’ articulation, she does not also go further and ask the student to elaborate in her own words on what made both examples the top (and not something else). When a child offered the description of “pointy,” rather than pushing on what makes something “pointy,” she again asked the student to identify the “pointy part” on another solid. This observation reflects the authors’ description of how the teacher supported descriptiveness in language. They give the example of a student noting that a solid has hexagons and in response, the teacher asked a series of pointed questions: “How many does it have?...Where are the hexagons? Are they attached to each other?” These questions are mathematically directed, but at the same time, they are so directed, they may potentially reframe the activity from articulating and constructing a definition to answering questions about particular aspects of a polyhedron. Thus, it is possible that such probes in isolation are not enough. To make the activity definitional, the probes must encourage students to further describe and articulate their descriptions of properties.

Another way teachers can support precise language is by selecting fruitful comparisons when students are evaluating examples and non-examples. For instance, in Lehrer & Curtis’s (2000) study of third graders investigating perfect solids, when the students reached an impasse, the teacher presented two contrasting solids – one example and one non-example they had constructed – and asked the students to compare and contrast the two. The teacher’s selection was significant because she chose two solids that had many of the same features, helping students isolate the mathematical property that they needed for refining their definition.

Discussion

Although mathematics educators have started to rethink what it means for students to make sense of and construct definitions, little has been known about the potential of these new avenues. This review suggests that defining is, in fact, a worthwhile endeavor to pursue. By providing students opportunities to make sense of definitions, and perhaps construct their own definitions, they generally develop richer understandings of the concepts at hand. Sometimes, this engagement generates conversations about notions of definitions more generally, including their key features and roles. Moreover, the studies illustrate that defining is a complex, yet accessible practice, resembling in many ways the practice that professional mathematicians engage in (Lakatos, 1976; Wilkerson-Jerde & Wilensky, 2011). I identified 11 Aspects of Practice described by the reviewed studies. These included: constructing and evaluating examples, describing properties of objects, using definitions to generate objects, investigating fundamental qualities of mathematical objects, constructing definitional explanations and arguments, revising definitions, asking definitional questions, negotiating criteria for judging adequacy or acceptability, considering definitions in new forms or contexts, engaging in definitional conjectures, experiments and tests, and establishing and/or investigating systematic relations.

The Aspects of Definitional Practice may provide a lens for future research in defining. For instance, some of the aspects, such as considering definitions in new forms, occurred less frequently, yet they were still shown to be significant. Therefore, aspects that have been studied less would be important avenues to pursue. In particular, it would be important to consider how such aspects interact with other definitional aspects. Although some of the Aspects of Definitional Practice appeared to play integral roles collectively (e.g., argument often led to revision of definitions), other relations were less clear. Ideally, these aspects, and their relations, will be further investigated and refined so as to ultimately provide a lens for instructional designers and teachers. Jacobs and colleagues (Jacobs, Lamb, & Philipp, 2010) described the importance of developing teachers' "professional noticing" of student thinking, a practice which first involves characterizing how students think and then using those characterizations to inform teaching. Likewise, I suggest that if we are to take students' engagement in disciplinary practices seriously, then teachers should also develop "professional noticing" of those practices.

At the same time, however, in order to support teachers' development of professional noticing of practices, the field needs to be able to characterize how students' *develop* those practices. Having a lens to look at development would allow teachers to see what progress would look like and base instructional choices on understanding that progress. In general, very little is known about how defining develops. The reviewed studies collectively hint at development of definitional argument, but they tell us more about how *arguments* might develop rather than how *arguing* develops as a socially situated practice. Moreover, most of the studies illustrate very short time scales, even when instruction lasted longer. In fact, 13 of the 19 studies described only 1 or 2 sessions of a particular class[3] and 3 other studies described 3 to 5 sessions, but only provided snapshots of class activity (rather than pictures of development).[4] Some of these studies, such as Lehrer and colleague's (1999), illustrate initial entrée into the practice of

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[3] Some of these studies may have analyzed more sessions, but not for the same group of students (for instance, Herbst, 2005).

[4] Duration was unspecified in one study.

defining but do not show later engagement. On the other hand, Zandieh and Rasmussen (2010) show what the practice might look like for students with histories of engagement in defining and other mathematical practices, but do not show preliminary work. Two of the studies described longer development, but still leave more to be learned. One of these studies (Keiser, 2000) describes students' learning over 5 weeks, but focuses on changes in their conceptions rather than in their engagement in the practice. The other study (Borasi, 1992) describes the learning of 2 students with one teacher over 8 sessions in an after-school setting, rather than a whole class setting. Although the study illustrates how the students change in their orientation towards defining, there is less focus analytically on how Aspects of Practice shift as they participate.

Furthermore, Lakatos's (1976) analysis highlights how mathematics develops as a system and the role definition plays in that development. A few of the studies (e.g., Herbst, 2005) illustrated instances of systematic relations. Despite this, studies have yet to illustrate the *development* of defining through the lens of system. Mathematical systems, as an analytic lens, might allow one to see how students create connections among definitions as well as definitions and other entities in mathematics, such as proof. In this sense, future studies should investigate students' participation in defining over longer periods of time in order to understand: a) how multiple aspects of defining develop (particularly through a system view) and b) how defining participates in other mathematical practices.

At the same time, studying students' development of mathematical defining would also allow for studies of support. This review provided an analysis that suggests ways in which the teacher plays a significant role in supporting authentic participation in defining. Unlike previous work about orchestration of classroom discussions in mathematics, the focus of the comparison here was to identify teacher moves to support the practice of defining in particular. Thus, these moves, as well as the aspects of design described, provide a potential starting place that may initially guide future analyses and classroom work. As noted previously, the teacher moves are preliminary conjectures and would need to be more thoroughly investigated in subsequent studies.

To close, this review provides a first step towards developing a language for describing students' participation in the mathematical practice of defining and how one would support that practice. Although the review suggests some general directions, it does not address nuances such as particular content areas. Nonetheless, it provides a direction for the field in pursuit of supporting students' participation in doing mathematics.

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CHAPTER III

ESTABLISHING A MATHEMATICAL PRACTICE IN A MIDDLE SCHOOL CLASSROOM

Introduction

Recently, reform efforts in mathematics education have attempted to provide students with opportunities to participate in mathematics in ways that more closely reflect practices in disciplinary mathematics (Lampert, 1990). Central to these efforts is how such practices are established within classrooms. In this paper, we attend to the establishment of one particular practice, mathematical defining. Our focus on mathematical defining is motivated by the fact that in many classrooms, definitions are often treated in ways that are counter to how they are treated in the discipline of mathematics. Historically, mathematicians have participated in the co-construction of definitions, and defining often emerged from proving (Lakatos, 1976). Some scholars have thus suggested that we instead engage students in *defining as a practice*, by providing them with opportunities to make sense of and construct definitions themselves, and, in turn, become *authors* of definition (e.g., de Villiers, 1998; Zandieh & Rasmussen, 2010). Although such studies provide examples of students' engagement in the practice of defining, very little has been done to show how the practice is established.

In this paper, we investigate how the practice of defining was established in one middle school mathematics classroom. We take the view that a practice is a recurrent activity structure governed by normative expectations about appropriate forms of participation. Practices are tied to the production of knowledge. The practice of defining, in particular, is tied to (a) the production of definitions, (b) the close examination of the properties of the objects being defined, and (c) the network of relations by which new definitions build on established definitions. Thus, our investigation of establishment involved a close look at the co-constitution of the practice of defining with communal knowledge. Accordingly, we were interested in the following two questions: 1) How are knowledge and the practice of defining co-constituted? and 2) How do participants in the community contribute to, or support, this co-constitution? We were particularly interested in the teacher's role in initially supporting emergent forms of definitional practice and how students, in turn, became participants in the practice. To attend to these questions, we first present a framework for characterizing the practice of defining in classroom communities. We then use this framework to illustrate how the co-constitution of defining and knowledge was established in three excerpts of classroom interaction.

Characterizing Defining as a Practice

To describe the lens we used to examine defining as a practice in classrooms, we begin by describing from a disciplinary perspective what we mean by mathematical definitions and defining. We then outline a framework for characterizing forms of participation in defining in classrooms, what we refer to as *Aspects of Definitional Practice*. The first author created this framework by reviewing 19 studies in which researchers instigated and/or studied students' engagement in defining as a practice. These studies varied in content, context and in the age of the students. The Aspects of Practice were developed through a method of iterative refinement, using the lens of disciplinary perspectives on definitions and defining to determine what constituted an aspect of definitional practice.

Disciplinary Perspectives on Definitions and Defining

A *mathematical definition* is a description of the properties of a mathematical object (such as a geometric shape) and the relations among those properties (Polya, 1957). Mathematical definitions are distinct from other mathematical entities – questions, conjectures, axioms, lemmas, theorems or corollaries – because they are the *negotiated grounds* for mathematical work. Unlike axioms, definitions are contested rather than taken for granted and unlike lemmas, theorems or corollaries, definitions cannot be proven. In order to characterize defining as a mathematical practice, we draw upon the work of Imre Lakatos (1976), who analyzed how mathematics developed historically in the profession. Essentially, mathematicians create systems of mathematical objects and relations between objects. Defining serves several functions in creating these systems. It contributes to the refinement of proof and to the development and refinement of other definitions. For instance, in Lakatos's example of the Euler Characteristic, defining "polyhedron" led to a counterexample that, in turn, spurred discussions about the definition of "polygon" and, later, the definition of "edge." Defining is also a form of argument, in that it arises out of contest about the meaning of particular objects motivated by the need for members of the mathematical community to communicate and develop a shared understanding.

A Framework for Analyzing Defining in Classrooms: Aspects of Definitional Practice

In reviewing the literature, we identified multiple Aspects of Definitional Practice. These aspects characterize how students from previous studies (of all ages) have participated in defining in ways representative of, yet distinct from, professional mathematicians. We created this framework to provide a lens for investigating students' participation in defining in classrooms. Although there are more, we briefly describe five of the Aspects of Definitional

Practice below that are most relevant to the results discussed here. While we describe these aspects separately, in reality, defining entails their collective functioning.

Definitional arguments and explanations are used to justify (a) inclusion or exclusion of a definition, (b) inclusion or exclusion of an example of a definition, (c) aspects of qualities of the object being defined, or (d) whether conditions in a definition are minimal. For example, Lehrer and colleagues (Lehrer, Jacobson, Kemeny, & Strom, 1999) describe one child's argument for the inclusion of an example during her class's construction of a definition for triangle. The child had constructed a triangle with 3 paper strips, one of which was curved. When the class rejected her example as a triangle, she disagreed, appealing to their collectively constructed definition of "3 corners, 3 sides:" "No. It doesn't matter. Look [gesturing to the board], it has three corners [gesturing to each vertex] and three sides [gesturing to each strip of paper]" (p. 78). This type of argument is emblematic of those within the discipline of mathematics (Lakatos, 1976) because it takes as evidence agreed-upon definitions. Arguments and explanations may take similar forms, but, as illustrated in the example above, unlike explanations, arguments arise from contest and are used to resolve that contest. This distinction is significant because historically the need to resolve disagreements led to advancement in the field (Lakatos, 1976).

Defining also involves the *construction and/or evaluation of examples and/or non-examples* of the object being defined, where evaluation involves determining whether or not a case should be included as part of the set in question. Constructing and evaluating examples is significant to the practice of defining because it helps students consider what the class of objects being defined should include and provides a set of objects to describe (see, for example, Zandieh & Rasmussen, 2010).

Defining may also involve *revising definitions* to serve the needs of the mathematical classroom community. Revision often results from definitional arguments or from evaluating examples or non-examples. When revising, definitions are sometimes expanded to include additional properties or relations while, at other times, reduced to become more minimal. In other instances, definitions are instead modified, mainly to improve their correctness.

Proposing definitions about the properties to include in a definition is another aspect of defining. Proposed definitions may then be tested, for instance, against examples of the object being defined, and possibly revised.

Finally, defining involves *asking questions* about definitions or about the qualities, properties or relations of the objects being defined. Some definitional questions are general (e.g., what is a polygon?). Others are more about particular qualities of an object, often asked in the process of trying to make sense of examples (e.g., "Will this still be a rectangle if I make these sides longer and longer and these shorter and shorter?" from Lehrer, Randle, & Sancilio, 1989, p. 166-167). Questions may also be asked about which properties of an object are necessary and/or sufficient for inclusion in the definition, such as, "Does it really guarantee that if a triangle has two equal angles then it is isosceles?" (Borasi, 1992, p. 34).

Method

We present data from video records of whole class activity where sixth-grade students created and refined mathematical definitions of geometric objects. Our instructional design capitalized on students' everyday experiences and conceptions of space, especially bodily motion, and on everyday forms of argument, especially propensities to categorize and classify. For example, we anchored students' learning about polygons to paths that they walked (Abelson

& diSessa, 1980; Lehrer et al., 1989) and related familiar properties of polygons, such as "straight" sides, to experiences of unchanging direction while walking. Working from these embodied forms of activity, we cultivated students' dispositions toward posing questions and making conjectures. We privileged forms of explanation that were oriented toward the general and that appealed to mathematical system. Although our focus on spatial mathematics was informed by the school's grade-level standards for mathematics, the conduct of any particular class was informed by our interpretations of students' questions and by our judgments of their current levels of understanding.

Participants, Setting and Data Collection

Participants (n=18, 10 male, grade 6, ethnically diverse) attended an urban school serving primarily underrepresented youth in the southeastern region of the United States. Half of the students came from traditional classrooms that emphasized procedural mathematics. The other students had been with the classroom teacher the year before, and had engaged in some conversations about definitions related to mathematical symmetries. Despite this, we still considered the context to be good for studying establishment of practice because (a) norms surrounding participation in practice still needed to be established for new members, (b) old members varied in their participation in practice, and (c) the content was not trivial to students.

Our participants came from a contained classroom, that is, they remained in the classroom with one teacher for all their core academic subjects. The second author served as a visiting classroom instructor for mathematics during the school year, and the regular classroom teacher occasionally interacted as well. The first day of instruction occurred during the second week of school, after a week working within a Connected Mathematics Project curriculum unit on polygons. When the visiting mathematics instructor (who we shall from here on refer to as the teacher) first visited, he intended simply to have a conversation with the students about what they had learned. It was only after this first class, when it was clear that students' ideas about polygons were still developing, that he decided to continue to teach mathematics. Mathematics class was conducted twice each week, for 1.5 hours per class. We videotaped each lesson and then digitally rendered the video for further analysis. We also took field notes of whole group interactions in order to contextualize the video recordings, serve as a platform for reflection, and guide the next day's instruction. Students also wrote summaries of their thinking at the end of every lesson and took periodic assessments, and both were additional sources of data.

Analysis

For our analysis, we traced initial explorations that emerged as students pursued the question, "What is a polygon?" We focused on the first six days of instruction because the activity largely involved defining and because it allowed us to see how initial forms of definitional practice arose and were supported. To do so, we divided the data into *definitional episodes* – segments of (possibly overlapping) time in which the class participated in making sense of one particular object (e.g., polygon or side). We limited definitional episodes to whole class discussion in order to capture collective activity. When creating definitional episodes, we identified three 10-minute excerpts of class discussion for careful analysis of the establishment of the practice of defining. We chose the excerpts (from days one, four and six) because they were similar in activity structure (open-ended construction of definitions) and topic (all began with the question, "what is a polygon?") and served as good representations of shifts in classroom interaction. We wanted the excerpts to be long enough to span multiple definitional episodes, in order to see the development of the mathematical system, but short enough to look carefully at interaction. The excerpts were then transcribed, taking into account both talk and gesture.

We then conducted four phases of analysis. First, we created a representation of the development of collective knowledge as a mathematical system. To create this representation, we looked across neighboring definitional episodes to identify moments of talk, gesture and inscription about interrelationships between mathematical objects and/or qualities of objects. For instance, defining “polygon” created the need to establish what a “side” was, suggesting a link between “polygon” and “side.” Our intention in making this representation was not to make claims about what individuals were thinking, but rather to represent the terrain investigated by the class. Second, using our theoretical framework, we coded when a member of the classroom community (teacher or student) participated in an aspect of definitional practice, using one or more speaker turns as the codable unit. Third, we mapped uses of Aspects of Definitional Practice onto the representation of the mathematical system. Finally, we characterized patterns of interaction within each excerpt in relation to the map between the coded Aspects of Definitional Practice and the mathematical system, and then looked for shifts in these patterns across the excerpts. In particular, we considered the roles taken on by students and the teacher in these interactions. Our choices for determining their roles were guided by the lens of participant frameworks (Goffman, 1981), and in particular O’Connor & Michaels’ (1996) framing of revoicing as positioning. We chose to use this framework because we were interested in how the class’s activity might be positioned as defining and how participants might be positioned as definers. To do so, we looked at how participants (both teachers and students) used talk and gesture to position their collective activity and roles within that activity.

Establishing the Co-Constitution of Practice and Knowledge

Here, drawing upon the three excerpts, we highlight a few ways in which defining and the construction of a mathematical system co-emerged and how that emergence was supported. First, definitional questions served to encourage the investigation of new and related mathematical objects, and thus supported development of a mathematical system. That is, when a new object was introduced, the teacher asked the students for the definition of the new object. For instance, when the class was making sense of a definition containing “angle,” the teacher asked, “What makes an angle again?” The teacher often further highlighted the importance of new objects by writing the names of the objects on the board. Later, on the fourth day, students began to appropriate these types of questions. For example, after students revised their definition of polygon to include not only “sides” and “angles,” but also “closed,” the teacher asked, “if we take this definition, can there be a polygon with two sides?” One student, Kate, suggested that as long as the two sides were *connected*, it was possible, and then suggested an oval as an example. When Kate’s example caused many in the class to protest, a group of students asked their peers, “What’s a side, people?” By asking definitional questions, students were beginning to take on the role of supporting one another in their collective activity.

The teacher also played a large role in modeling Aspects of Definitional Practice. As time progressed, the teacher modeled different aspects in order to serve the emergent needs of the community. Initially, as noted above, the teacher modeled the *asking of definitional questions* that supported development of the mathematical system. Later, he also modeled *constructing definitional arguments*, and, in doing so, encouraged preciseness in students’ definitions. For instance, when students defined a polygon as having “sides” and “angles,” the teacher drew three connected, but not closed lines, and said, "I want to know what makes something a polygon. I know it has sides and it has angles SO…this then is a polygon right?" In making his argument, he positioned the counter-example in relation to their definition, and, in turn, caused students to *revise* their definition to include the property of connectedness. In the last class, the teacher's modeling of definitional practice shifted to address new mathematical relations. For instance, the teacher asked a new type of *definitional question*, one that encouraged students to think about the economy of their definition: "Can you make any closed figure with sides that does NOT have angles?" At the same time, students continued to appropriate forms of participation that the teacher had been modeling. For instance, in response to the teacher's question, one student, Ned, *constructed the example* of a football-shaped figure. When asked to explain his thinking, he pointed to the lines and noted, "two sides," then pointed to the vertices and said, "no angles." He continued, "They can't be angles cause an angle has to be a straight line, two straight lines make an angle." What is noteworthy about Ned's *definitional argument* is that it appealed to his conceived definition of angle in a similar manner as had been earlier modeled by the teacher.

Finally, the teacher also played a large role in positioning both students and content. Initially, the teacher positioned students as participants in Aspects of Definitional Practice. For instance, when one student suggested that a polygon "has the same angles and the same length of uh, same lengths of sides," the teacher revoiced the student's utterance as a "claim," thereby positioning his activity as *proposing definitions.* Another student, in response, suggested, "all regular polygons." The teacher referred to this suggestion as an "amendment," in turn positioning her contribution as participating in *revising definitions*. Later, as the class developed a need to remember their agreed upon definitions, the teacher positioned definitions at the forefront. For instance, when students proposed definitions, he wrote them on the board, and when those definitions were revised, he indicated those changes as well. He also often also requested that students write agreed definitions in their notebooks.

Discussion

In this paper, we provided an illustration of the initial establishment of a mathematical practice, defining. We do not mean to claim that by the end of the six days, the practice was fully established. Rather, we illustrate how in establishing this practice, the roles of the teacher and the students were constantly shifting as the students gained more authority and began to appropriate forms of participation. Our analysis suggests the importance of the teacher in modeling Aspects of Definitional Practice, in initially positioning students as participants in those aspects, and in positioning definition at the forefront of discussion. As students began to appropriate particular forms of participation, the teacher in turn modified what he modeled and positioned to fit the new goals of the community and to support investigation of new mathematical properties and relations. Controversies about definition led to elaboration of mathematically important ideas such as side, angle, polygon, and straight that contributed to the development of a mathematical system. These ideas were then taken up and used during the remainder of the year. Figure 1 illustrates the relation between students' engagement in Aspects of Practice, teacher supports and the development of a mathematical system.

Our paper has two contributions. First, the use of our framework of Aspects of Definitional Practice illustrates a potentially significant analytic tool for characterizing student engagement in the practice of defining. This framework has the potential to be refined and expanded as it is used in relation to new classroom environments. Although others have parsed mathematical practices tied to particular content (Cobb, Stephan, McClain, & Gravemeijer, 2001), this paper illustrates how this may be done in regards to an epistemic practice that *spans* mathematical content. Likewise, the framework, along with the supports we identified, have the potential for supporting teachers interested in developing similar learning environments and supporting students in engaging in the practice of defining. The Aspects of Definitional Practice may allow a teacher to identify what types of activity to model and encourage with her students. We focused on collective activity, but this framework may also be useful for capturing changes in how individual students participate in the practice of defining and develop identities as definers. In our ongoing analysis, we are investigating how roles of individual students shift, taking into account their particular histories within the classroom community.

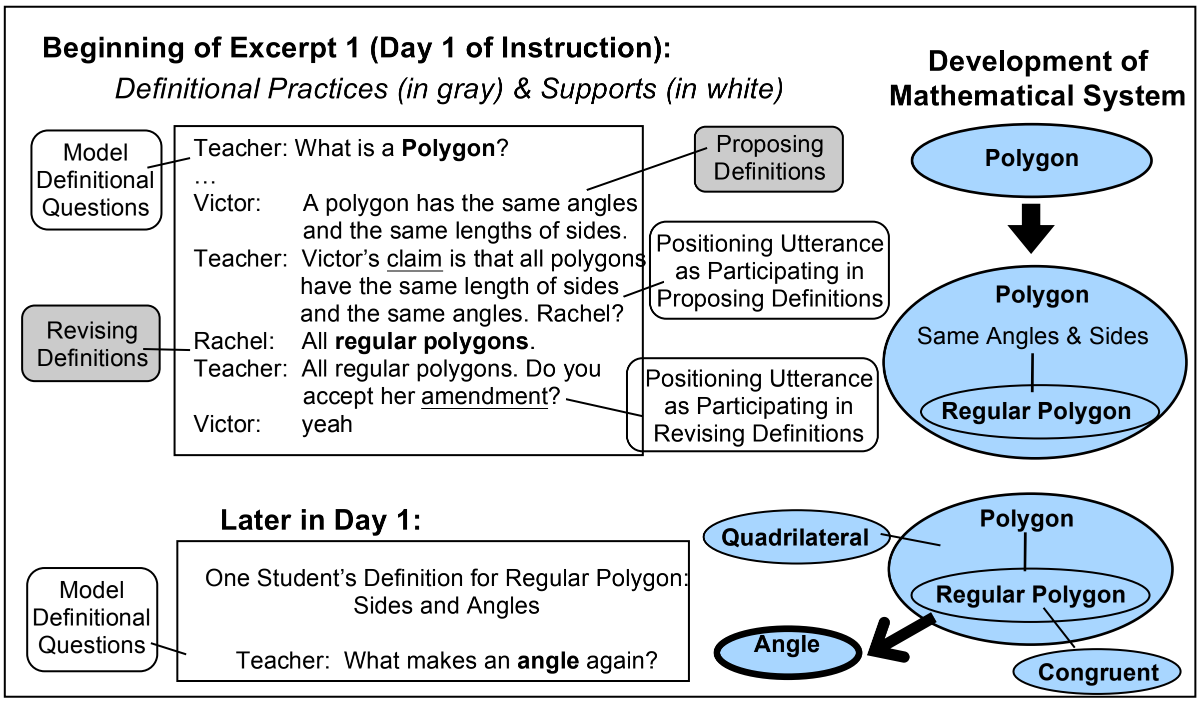

Figure 1. Establishing definitional practice on the first day of instruction. The left side presents transcript from two time points in Excerpt 1. The right shows the mathematical system concurrently developed. Aspects of Definitional Practice and supports are highlighted in the transcript. Nodes indicate objects that were defined or whose qualities were explored. Solid lines in the system indicate relations discussed between objects.

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CHAPTER IV

CHARACTERIZING AND SUPPORTING PRACTICES OF DEFINING IN A MATHEMATICS CLASSROOM

Introduction

Recently, reform efforts in mathematics education have attempted to provide students with opportunities to participate in mathematics in ways that more closely reflect practices in disciplinary mathematics (e.g., Lampert, 1990). Central to these efforts is how such practices are established within classrooms. In this paper, we attend to the establishment of one particular practice, mathematical defining. Our focus on mathematical defining is motivated by the fact that in many classrooms, definitions are often treated in ways that are counter to how they are treated in the discipline of mathematics. Historically, mathematicians have participated in the co-construction of definitions, and defining often emerged from proving (Lakatos, 1976). Some scholars have thus suggested that we instead engage students in *defining as a practice*, by providing them with opportunities to make sense of and construct definitions themselves, and, in turn, become *authors* of definition (e.g., de Villiers, 1998; Zandieh & Rasmussen, 2010). Although such studies provide examples of students' engagement in the practice of defining, very little has been done to show how the practice is established.

In this paper, we investigate how the practice of defining was established in one middle school mathematics classroom. We take the view that practice is a recurrent activity structure governed by normative expectations about appropriate forms of participation. Practice is ultimately tied to the production of knowledge, and in the case of defining, tied to the production of definitions, to close examination of the properties of the objects being defined, and to the network of relations by which new definitions build on established definitions. Thus, our investigation of establishment involved a close look at the co-constitution of practice with communal knowledge. Accordingly, we were interested in the following two questions: 1) How are knowledge and practice co-constituted? And 2) How do participants in the community contribute to, or support, this co-constitution? We were particularly interested in the teacher's role in initially supporting emergent forms of practice and how students, in turn, became participants in the practice. To attend to these questions, we first present a framework for characterizing the practice of defining in classroom communities. We then use this framework to illustrate how the co-constitution of defining and knowledge was established in three excerpts of classroom interaction that span the first six days of instruction.

Characterizing Defining as a Practice

To describe the lens we used to examine defining as a practice in classrooms, we begin by describing from a disciplinary perspective what we mean by mathematical definitions and defining. We then outline a framework for characterizing forms of participation in defining in classrooms, what we refer to as *Aspects of Definitional Practice*. The first author created this framework by reviewing 19 studies in which researchers instigated and/or studied students' engagement in defining as a practice. These studies varied in content, context and in the age of the students. The Aspects of Practice were developed through a method of iterative refinement, using the lens of disciplinary perspectives on definitions and defining to determine what constituted an aspect of practice.

Disciplinary Perspectives on Definitions and Defining

A *mathematical definition* is a description of the properties of a mathematical object (such as a geometric shape) and the relations among those properties (Polya, 1957). Mathematical definitions are distinct from other mathematical entities – questions, conjectures, axioms, lemmas, theorems or corollaries – because they are the *negotiated grounds* for mathematical work. Unlike axioms, definitions are contested rather than taken for granted and unlike lemmas, theorems or corollaries, definitions cannot be proven. In order to characterize defining as a mathematical practice, we draw upon the work of Imre Lakatos (1976), who analyzed how mathematics developed historically in the profession. Essentially, mathematicians create systems of mathematical objects and relations between objects. Defining serves several functions in creating these systems. It contributes to the refinement of proof and to the development and refinement of other definitions. For instance, in Lakatos's example of the Euler Characteristic, defining "polyhedron" led to a counterexample that, in turn, spurred discussions about the definition of "polygon" and, later, the definition of "edge." Defining is also a form of argument, in that it arises out of contest about the meaning of particular objects motivated by the need for members of the mathematical community to communicate and develop a shared understanding.

A Framework for Analyzing Defining in Classrooms: Aspects of Definitional Practice

In reviewing the literature, we identified multiple Aspects of Definitional Practice. These aspects characterize how students from previous studies (of all ages) have participated in defining in ways representative of, yet distinct from, professional mathematicians. We created this framework to provide a lens for investigating students' participation in defining in classrooms. Although there are more, we briefly describe six of the Aspects of Definitional

Practice below that are most relevant to the results discussed here. While we describe these aspects separately, in reality, defining entails their collective functioning.

Defining involves constructing and evaluating examples. Defining involves the *construction and/or evaluation of examples and/or non-examples* of the object being defined, where evaluation involves determining whether or not a case should be included as part of the set in question. Constructing and evaluating examples is significant to the practice of defining because it helps students consider what the class of objects being defined should include and provides a set of objects to describe (see, for example, Zandieh & Rasmussen, 2010).

Defining involves describing properties. Often when constructing and evaluating examples, members of a mathematical community may also *describe* and articulate properties and relations of the examples. This descriptive quality is what pushes example construction and evaluation towards defining and beyond simply building and making decisions of "in" versus "out." Description often supports other goals, such as constructing a definitional argument, explaining a particular classification, or writing a definition for an object (e.g., Borasi, 1992; Lehrer & Curtis, 2000). It is important that descriptions go beyond a "list of properties" and instead contribute to the construction of a definition.

Defining involves constructing definitional explanations and arguments. *Definitional arguments and explanations* are used to justify a) inclusion or exclusion of a definition, b) inclusion or exclusion of an example of a definition, c) aspects of qualities of the object being defined, or d) whether conditions in a definition are minimal. For example, Lehrer, Jacobson, Kemeny, and Strom (1999) describe one child's argument for the inclusion of an example during her class's construction of a definition for triangle. The child had constructed a triangle with 3 paper strips, one of which was curved. When the class rejected her example as a triangle, she disagreed, appealing to their collectively constructed definition of "3 corners, 3 sides:" "No. It doesn't matter. Look [gesturing to the board], it has three corners [gesturing to each vertex] and three sides [gesturing to each strip of paper]" (p. 78). This type of argument is emblematic of those within the discipline of mathematics (Lakatos, 1976) because it takes as evidence agreed-upon definitions. Arguments and explanations may take similar forms, but, as illustrated in the example above, unlike explanations, arguments arise from contest and are used to resolve that contest. This distinction is significant because historically the need to resolve disagreements led to advancement in the field (Lakatos, 1976).

Defining involves revising definitions. Defining also involves *revising definitions* to serve the needs of the mathematical classroom community. Revision often results from definitional arguments or from evaluating examples or non-examples. When revising, definitions are sometimes expanded to include additional properties or relations while, at other times, reduced to become more minimal. In other instances, definitions are instead modified, mainly to improve their correctness. For example, during their investigation of "perfect solids," students in Lehrer & Curtis's (2000) class found a new perfect solid with 3 faces coming together at each vertex. In response, one student noted that their conjecture of "three faces at each vertex" (p. 326) could not be true. Rather than eliminate that property all together, students suggested modifying it, either to "three or four faces come together at each vertex" or "the number just has to be the same at each vertex, but could be any number" (p. 326).

Defining involves proposing definitions. *Proposing definitions* about the properties to include in a definition is another aspect of defining. Proposed definitions may then be tested, for instance, against examples of the object being defined, and possibly revised, as illustrated in the previous example from Lehrer & Curtis (2000).

Defining involves posing definitional questions. Defining may also involve *posing questions* about definitions or about the qualities, properties or relations of the objects being defined. For example, some definitional questions are general (e.g., what is a polygon?). Others are more about particular qualities of an object, often asked in the process of trying to make sense of examples (e.g., "Will this still be a rectangle if I make these sides longer and longer and these shorter and shorter?" from Lehrer, Randle, & Sancilio, 1989, p. 166-167). Questions may also be asked about which properties of an object are necessary and/or sufficient for inclusion in the definition, such as, "Does it really guarantee that if a triangle has two equal angles then it is isosceles?" (Borasi, 1992, p. 34).

Method

We present data from video records of whole class activity where sixth-grade students created and refined mathematical definitions of geometric objects. Our design for instruction capitalized on students' everyday experiences and conceptions of space, especially bodily motion, and on everyday forms of argument, especially propensities to categorize and classify. For example, we anchored students' learning about polygons to paths that they walked (Abelson & diSessa, 1980; Lehrer et al., 1989) and related familiar properties of polygons, such as "straight" sides, to experiences of unchanging direction while walking. Working from these embodied forms of activity, we cultivated students' dispositions toward posing questions and making conjectures. We privileged forms of explanation that were oriented toward the general and that appealed to mathematical system.

Participants, Setting and Data Collection

Participants (n=18, 10 male, grade 6, ethnically diverse) attended an urban school serving primarily underrepresented youth in the southeastern region of the United States. The percent of children attending the school who qualify for free or reduced lunch ranges from 60 to 80 from year-to-year. Half of the students came from traditional classrooms that emphasized procedural mathematics. The other students had looped up with the classroom teacher from the year before, and had engaged in some conversations about definitions related to mathematical symmetries. Despite this, we still considered the context to be good for studying establishment of practice for a number of reasons. First, because there were a considerable number of new members to the community, norms surrounding participation in practice still needed to be established. Old members were also varied in their participation in practice, suggesting that they, despite past experiences, were still making sense of what it meant to participate in defining. Moreover, the content, as will be illustrated, was not trivial to students, and thus served as a good context for investigating the co-constitution of knowledge with practice.

Our participants came from a contained classroom, that is, they remained in the classroom with one teacher for all their core academic subjects. One of us served as a visiting mathematics instructor, and was the primary classroom instructor for mathematics during the school year. The regular classroom teacher remained in the classroom during math class, and occasionally interacted as well. The first day of instruction occurred during the second week of school, after a week working within a Connected Mathematics Project curriculum unit on polygons. When the visiting mathematics instructor (who we shall from here on refer to as the teacher) first visited, he intended simply to have a conversation with the students about what they had learned. It was only after this first class, when it was clear that students' ideas about polygons were still developing, that he decided to continue to teach mathematics.

Mathematics class was conducted twice each week, for 1.5 hours per class. Each lesson was videotaped and digitally rendered for further analysis. Field notes were taken of whole group interactions. The aim of the notes was to contextualize the video recordings and to serve as a platform for reflection. At the end of each lesson, field notes were compiled, and these served to guide the next day's instruction. Although our choice of mathematical topics was informed by the school's grade-level standards for mathematics, the conduct of any particular class was informed by our interpretations of students' questions and by our judgments of their current levels of understanding. The latter were informed both by classroom interaction and by the results of periodic assessments. Students also wrote summaries of their thinking at the end of every lesson, and these student journals were an additional source of data.

Analysis

For our analysis, we traced initial explorations that emerged as students pursued the question, "What is a polygon?" We focused on the first six days of instruction because the activity largely involved defining and because it allowed us to see how initial forms of practice arose and were supported. To do so, we divided the data into *definitional episodes* – segments of (possibly overlapping) time in which the class participated in making sense of one particular object (e.g., polygon or side). We limited definitional episodes to whole class discussion in order to capture collective activity. When creating definitional episodes, we identified three excerpts of class discussion for careful analysis of the establishment of practice. We chose the three excerpts (from days one, four and six) because they were similar in activity structure (open-ended construction of definitions) and topic (all began with the question, "what is a polygon?") and served as good representations of shifts in classroom interaction. Each excerpt was approximately 10 minutes long. We wanted the excerpts to be long enough to span multiple definitional episodes, in order to see the development of the mathematical system, but short enough to look carefully at interaction. The excerpts were then transcribed, taking into account both talk and gesture. We were particularly interested in gestures with inscriptions and materials that were central to the activity of defining.

We then conducted four phases of analysis. First, we used the frame of mathematical system in order to create a representation of the development of collective knowledge. When considering knowledge, we were mainly interested in the properties and relations of mathematical objects that the class discussed or explored. To create this representation, we looked within and across neighboring definitional episodes to identify moments of talk, gesture and inscription about interrelationships between mathematical objects and/or qualities of objects. For instance, defining polygon created the need to establish what a side was, suggesting a link between "polygon" and "side." Our intention in making this representation was not to make claims about what individuals were thinking, but rather represent the terrain investigated by the class.

Second, using our theoretical framework, we coded when a member of the classroom community (teacher or student) participated in an aspect of definitional practice, using one or more speaker turns as the codable unit.

Third, via a process of iterative refinement, we developed categories to describe ways in which the teacher supported the development of practice and then used our categories to code the teacher turns. Our creation of categories was influenced by two bodies of work. The first was the lens of participant frameworks (Goffman, 1981), and in particular O'Connor & Michaels' (1996) framing of revoicing as positioning. We chose to use this framework because we were interested in how the teacher might position the class's activity as defining and position participants as definers. Similarly, we were guided by Goodwin's (1994) notion of professional vision because we were interested in how the teacher, as a disciplinary representative, might highlight or code participation in practice. To use these two frames, we looked at how the teacher used talk and gesture to position or highlight collective activity and roles within that activity. For example, teacher moves included modeling Aspects of Definitional Practice or requesting participation in Aspects of Practice. We will exemplify the teacher moves further in the results.

Finally, we mapped uses of Aspects of Definitional Practice along with teacher supports onto the representation of the mathematical system developed by the class. We then characterized patterns of interaction within each excerpt in relation to the map between the coded Aspects of Definitional Practice and supports and the mathematical system, and then looked for shifts in these patterns across the excerpts.

Establishing the Co-Constitution of Practice and Knowledge

Here, drawing upon the three excerpts, we highlight three ways in which defining and the construction of a mathematical system co-emerged and how that emergence was supported.

Posing Questions that Elaborated System Components

The first way in which the practice of defining and the mathematical system investigated by the class was co-constituted was through the *posing of definitional questions*. Definitional questions encouraged the investigation of new and existing mathematical objects and relations between objects, and thus supported development of a mathematical system. Three types of questions appeared to be especially important in contributing to this co-constitution. The first type of question asked about relations between two different classes. For instance, towards the beginning of the first day, students had started to discuss the definition of polygon. One student suggested that, "a circle wouldn't be a polygon cause a circle doesn't have any sides." The teacher then re-positioned the student's utterance as a *definitional question*, saying, "okay so QUESTION. Circle is? A polygon?" Here, the question opened up the conversation to further discussion about relations between circle and polygon, inviting other members of the class to participate.

A second *definitional question* that served to expand and deepen the mathematical system was that which asked about a property of the object being defined. That is, when a new object was introduced, the teacher asked the students for the definition of the new object. For instance, a little later on the first day, when the class was making sense of a definition containing "angle," the teacher asked, "What makes an angle again?" This *definitional question* invited discussion about the definition of angle, an important part of the existing definition.

The final type of *definitional question* that pushed on developing the mathematical system was that which asked about extreme cases or about economic definitions (those definitions that use the least amount of properties to describe a mathematical object). These questions served to push on relations between the properties that constituted a definition. For example, on the fourth day of instruction, after students revised their definition of polygon to include not only "sides" and "angles," but also "closed," the teacher asked a question about extreme cases, "if we take this definition, can there be a polygon with two sides?" One student, Kate, suggested that as long as the two sides were *connected*, it was possible, and then suggested an oval as an example. When Kate's example caused many in the class to protest, a couple of students asked their peers, "What's a side, people?" Note that the teacher's question motivated students to think about the effect of minimizing the number of sides on the other properties of a polygon (angles and closed), and, in turn, sparked conversation about the meaning of one of those properties.

These examples also illustrate the role of the teacher in modeling the posing of definitional questions. Whereas earlier, the teacher had asked many questions about properties of the objects being defined, here we see students appropriate this particular aspect of definitional practice. Similarly, later, when discussing the definition of "side," the class realized the need to specify that a side must be "straight." When the teacher noted, "But I don't know what I mean by side yet. I heard the word STRAIGHT," a student asked, "What does straight mean?" By asking definitional questions, students were beginning to take on the role of supporting one another in their collective activity. Moreover, as the class began to develop a mathematical foundation of definitions, the teacher asked new questions that served emergent needs of the community, namely to push on relations among existing objects. Figure 1 presents the three examples illustrated above and highlights how they contributed to the development of a mathematical system.

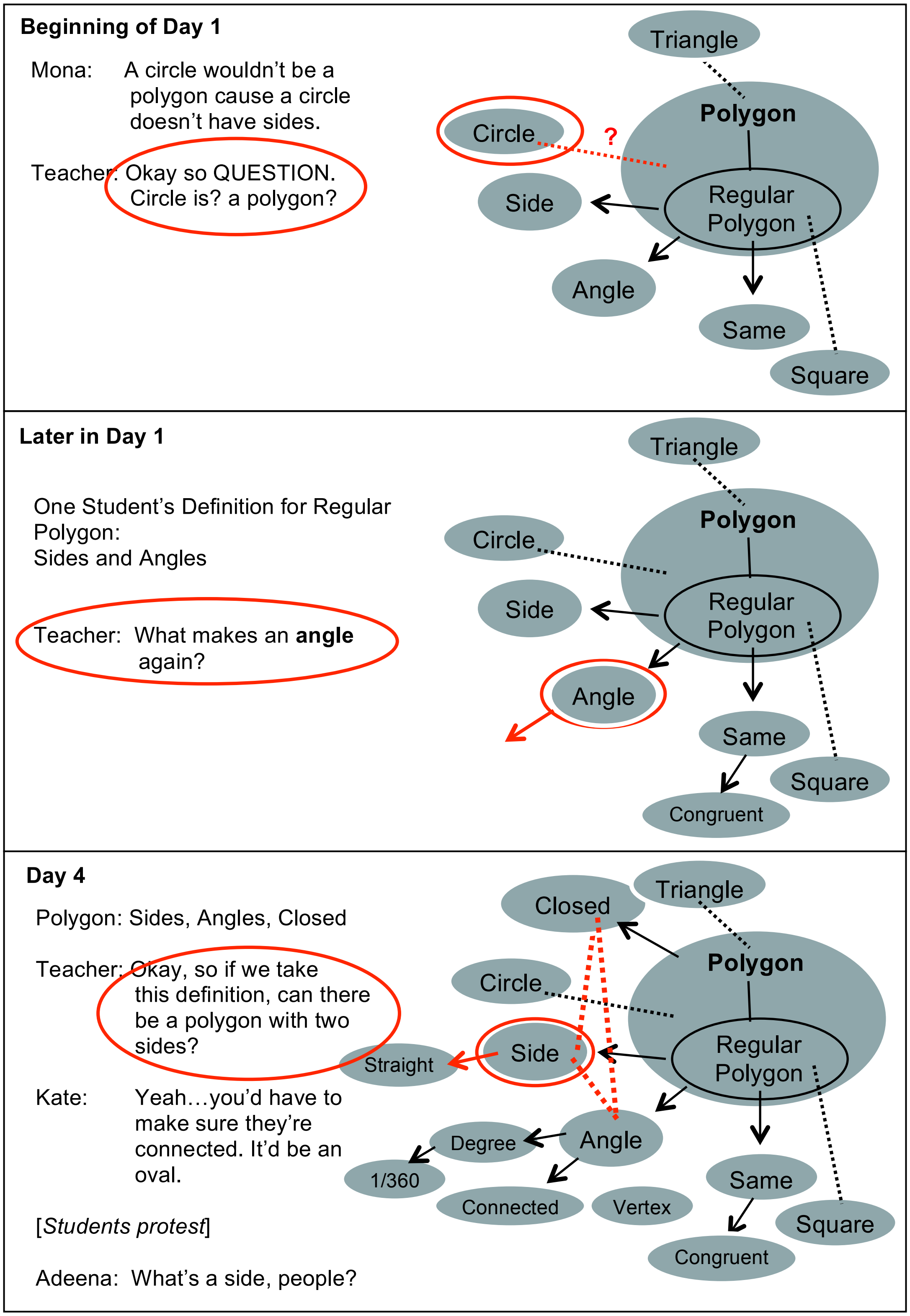

Figure 1. Establishing practice on the first day of instruction. Each portion presents transcript from the alongside a representation of the mathematical system being developed. Nodes indicate objects that were defined or whose qualities were explored. Solid lines in the system indicate relations discussed between subsets of classes. Dotted lines indicate relations between two classes. Arrows indicate when a property

of an existing definition is expanded upon. Red is used to highlight which relations or properties are elaborated on in the example.

Provoking Contest through the Generation of Examples and Non-Examples

A second way in which practice and knowledge were co-constituted was through contest provoked by the *generation of examples and non-examples* (what Lakatos, 1976, would refer to as "monsters"). This happened in two ways. First, the teacher sometimes *constructed counter-examples* to contest students' definitions. For instance, on the fourth day of instruction, when students defined a polygon as having "sides" and "angles," the teacher drew three connected, but not closed lines (see Figure 2, left side), and said, "I want to know what makes something a polygon. I know it has sides and it has angles SO…this then is a polygon right?" In making his *definitional argument*, he positioned the counter-example in relation to their definition, and, in turn, caused students to *revise* their definition to include the property of connectedness, which the teacher then relabeled as "closed."

In other cases, the teacher *asked definitional questions* about relations that motivated students to construct examples that provoked contest. For example, on the sixth day of instruction, the students again revisited their definition of polygon. This time, Michelle, reading from her notebook, explained that their definition now contained the properties of "sides," "angles" and "closed." The teacher then *asked a definitional question* about the economy of this definition: "Can you make any closed figure with sides that does NOT have angles?" In response to the teacher's question, one student, Ned, *constructed the example* of a football-shaped figure (see right of Figure 2). When asked to explain his thinking, he pointed to the lines and noted, "two sides," then pointed to the vertices and said, "no angles." He continued, "They can't be angles cause an angle has to be a straight line, two straight lines make an angle." Ned's *definitional argument* provoked contest among members of the class. One student, Kate, protested, saying, “I don’t [agree], cause that’s not a polygon…and Michelle forgot to say that it has to have straight lines.” The teacher noted that the difference in opinion between Kate and Ned was the definition of “side.” Ned, in response, asked, “What did we say a side is?” Here, Ned’s example, much like Kate’s “oval” described previously, pushed upon relations of what constituted a definition of polygon. The contest provoked by Ned’s example motivated the need to further discuss the definition of side and it’s relation to straight.

What is noteworthy in the last example is that the teacher’s question provoked Ned to participate in the aspect of practice, *constructing an example*, in turn placing authority of defining in the hands of the students. This highlights another teacher support: requesting participation in Aspects of Practice. Moreover, by looking across the two examples provided, we again see the role of the teacher in modeling Aspects of Practice, this time the aspect of *constructing a definitional argument*. In the first example, the teacher modeled using the definition in his argument: “I know it has sides and angles. SO, this then is a polygon, right?” He then proceeded to label his figure as he spoke, further highlighting the components of the definition in his counter-example: “side one, side two, side three. Angle one, angle two-.” Likewise, Ned’s *definitional argument* also appealed to his conceived definitions of polygon and angle in a similar manner as had been earlier modeled by the teacher. Like the teacher, Ned pointed to the components of the figure as he argued his case and then referred to the definition of angle to justify this *description of the properties*. Thus, each of these examples illustrate the role of the teacher in supporting student participation.

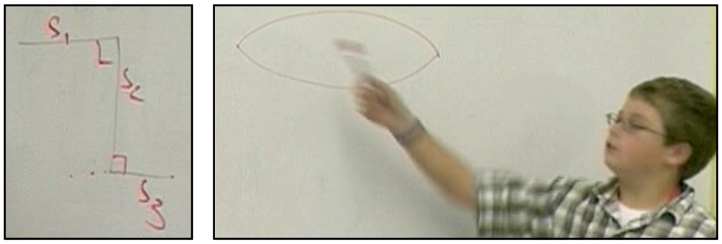
Figure 2. Teacher counter-example (left) and student example (right).

Keeping Definition at the Forefront

Finally, the practice of defining and the development of mathematical system co-emerged by keeping definitions at the forefront. This occurred in two ways. On the one hand, participants verbally positioned definitions in relation to arguments or examples. For example, in the last section, both the teacher and Ned did so as they referenced the definition in defending their examples. A second way this occurred was by recording and highlighting definitions. When the teacher drew his counter-example of a polygon with sides and angles (left of Figure 2), he also labeled the figure in relation to the components of the definition, and, in doing so, highlighted those components. Moreover, the teacher also often wrote definitions on the board or requested that students write definitions in their notebooks. The teacher especially pushed on the writing of definitions starting on the fourth day of instruction, when it became apparent that students continued to revisit the same questions and definitions. By the sixth day, more students were using their notebooks as resources for referencing agreed upon definitions, as Michelle had done in the previous example. Having a record allowed for the stabilization of definitions, and stabilization, in turn, allowed for the investigation of new objects and relations.

Discussion

In this paper, we provided an illustration of the initial establishment of a mathematical practice, defining. We do not mean to claim that by the end of the six days, the practice was fully established. Rather, we highlight three ways in which the class's participation in the practice of defining allowed for investigations of qualities of mathematical objects that that contributed to the development of a mathematical system. Posing definitional questions motivated the introduction of new objects and properties and the elaboration of relations between those objects and properties. Controversies about examples of definitions led to elaboration of mathematically important ideas such as side, angle, polygon, and straight. These ideas were then taken up and used during the remainder of the year, partly supported by verbal and written reference to the definitions. Moreover, we illustrate how in establishing this practice, the roles of the teacher and the students were constantly shifting as the students gained more authority and began to appropriate forms of participation. Our analysis suggests the importance of the teacher in modeling Aspects of Definitional Practice, in requesting participation in Aspects of Practice, and in positioning definition at the forefront of discussion. As students began to appropriate particular forms of participation, the teacher in turn modified what he modeled and positioned to fit the new goals of the community and to support investigation of new mathematical properties and relations.

Our paper has two contributions. First, the use of our framework of Aspects of Definitional Practice illustrates a potentially significant analytic tool for characterizing student engagement in the practice of defining. This framework has the potential to be refined and expanded as it is used in relation to new classroom environments. Although others have parsed mathematical practices tied to particular content (Cobb, Stephan, McClain, & Gravemeijer, 2001), this paper illustrates how this may be done in regards to an epistemic practice that *spans* mathematical content. Likewise, the framework, along with the supports we identified, have the potential for supporting teachers interested in developing similar learning environments and supporting students in engaging in the practice of defining. The Aspects of Definitional Practice may allow a teacher to identify what types of activity to model and encourage with her students. We focused on collective activity, but this framework may also be useful for capturing changes in how individual students participate in the practice of defining and develop identities as definers. In our ongoing analysis, we are investigating how roles of individual students shift, taking into account their particular histories within the classroom community.

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CHAPTER V

INVESTIGATING THE CO-DEVELOPMENT OF MATHEMATICAL KNOWLEDGE AND THE PRACTICE OF DEFINING IN A MIDDLE SCHOOL CLASSROOM

Introduction

In recent years, the field of mathematics education has advocated for an expanded view of what it means to know mathematics and to participate in mathematics as a practice (National Research Council, 2001). This paper investigates one practice that has received increasing attention: mathematical defining. Whereas definitions are often treated in school mathematics as rote, unchanging entities, recently mathematics educators have suggested that classroom mathematics should instead treat definitions as they are treated in the discipline. In the discipline, definitions are co-constructed by mathematicians with the goal of creating a system of mathematical objects, properties and relations. Definitions are subject to revision depending on the emergent needs of the mathematical community, such as the desire to reject a particular case of an object (a “monster”) (Lakatos, 1976). Unlike other forms of mathematical argument, definitions are negotiated, and not taken as shared, as are axioms, or as contested, as are conjectures, or as settled, as are proofs.

In light of this recent trend~~s~~, scholars have conducted several studies that essentially provide existence cases suggesting that it *is* possible to engage students in this form of practice, and that doing so provides students with opportunities to participate in productive mathematical discourse, which in turn nurtures the growth of students’ mathematical understandings (e.g., Borasi, 1992; Keiser, 2000; Lehrer, Randle, & Sancilio, 1989; Lehrer, Jacobson, Kemeny, & Strom, 1999; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). However, most of these studies primarily present analyses of very short excerpts of class activity, less than two class periods, and are often illustrations of already established practice. The studies that present longer time scales focus analytically on students' development of conceptions or orientations towards defining rather than shifts in student participation in practice (e.g., Borasi, 1992; Keiser, 2000). Thus, very little is known about how the practice of defining develops. This paper aims to address this need.

Here, I present analyses of how students in one sixth grade classroom participated in the mathematical practice of defining and how that practice developed over time. I take the view that a practice is a recurrent activity structure governed by normative expectations about appropriate forms of participation. Epistemic practices result in the production of knowledge, and the practice of defining, in particular, is tied to (a) the production of definitions, (b) the close examination of the properties of the objects being defined, and (c) the network of relations by which new definitions build on established definitions. Thus, this investigation of development involves a close look at the *co*-development of the practice of defining and of communal knowledge. I investigated three questions: (a) How does the practice of defining develop? (b) How does communal knowledge develop? (c) How do practice and knowledge co-develop, and how is such development reflected in the forms of participation generated by teachers and students?

In the following section, I outline three theoretical perspectives that helped shape my work: one about what it means to participate in defining from a disciplinary perspective, what I term, *Aspects of Definitional Practice*; one about learning, and one about forms of interaction that are key to supporting the orchestration of classroom discussions in mathematics. I then follow by describing the context of the study, my sampling method, and three phases of analysis

– analysis of knowledge, analysis of practice, and analysis of how knowledge and practice were co-constituted. I follow with the research findings, first presenting an overview of how practice and knowledge co-developed. I then illustrate the evolution of the co-development of practice and knowledge by describing four episodes of classroom interaction at varying points of time. I conclude with implications for the field and suggest how similar classroom environments can be designed.

Theoretical Perspectives

In this section, I outline the perspectives that guided my analyses. The first perspective describes a lens for determining what constitutes participation in the practice of defining, what I call *Aspects of Definitional Practice*. The second perspective describes a general lens I employ to look at learning of disciplinary practice – as situated. The third perspective describes a lens for looking at other forms of interaction that are important for establishing and maintaining communal practice, drawing upon four bodies of work. This lens focuses on the role of individual participants, especially the teacher, in shaping what it means to interact with others around practice.

Characterizing Defining as a Practice

Disciplinary perspectives on definitions and defining. To situate defining as a practice in classrooms, I first describe from a disciplinary perspective what I mean by mathematical definitions and defining. A *mathematical definition* is a description of a mathematical class or property (e.g., "polygon," "function," "straight"). Functionally, definitions allow members of a mathematical community to distinguish between classes of objects and determine whether cases are members of a class (Lakatos, 1976). Definitions come in two forms (Eylon & Reif, 1984; Zaslavsky & Shir, 2005). Structural definitions communicate the properties that constitute a mathematical object and the relations among those properties. For instance, "equilateral triangle" may be defined structurally as "a polygon that has three congruent sides." Here, the class is "equilateral triangle." Its properties are "sides" and any properties that comprise a polygon. Moreover, it is characterized by the relation between "polygon" and the class of polygons with "three congruent sides," where sides are understood to be straight. Note that the properties that constitute a mathematical object are themselves mathematical objects that may be defined. In contrast, procedural definitions describe how to construct a class of objects. For example, a procedural definition for "equilateral triangle" could be "walk *n* number of straight steps, turn 120-degrees right, walk *n* straight steps, turn 120-degrees right, walk *n* straight steps, turn 120-degrees right."

Mathematical definitions are distinct from other mathematical entities – questions, conjectures, axioms, lemmas, theorems or corollaries – because they are the *negotiated grounds* for mathematical work. Unlike axioms, definitions are contested rather than taken for granted and unlike lemmas, theorems or corollaries, definitions cannot be proven. In order to characterize defining as a mathematical practice, I drew upon the work of Lakatos (1976), who suggests that mathematicians create systems of mathematical objects. Defining particular classes of mathematical objects often leads to refinement of definitions as potential cases and counter-examples are investigated. Often, the grounds of proof are challenged as mathematicians work to modify the scope of a definition, occasionally by contracting it ("monster barring") and more commonly, by expanding it. For instance, in Lakatos's example of the Euler Characteristic, defining "polyhedron" led to a counterexample that, in turn, spurred discussions about the definition of “polygon” and, later, the definition of “edge.” Defining is also a form of argument, in that it arises out of contest about the meaning of particular objects motivated by the need for members of the mathematical community to communicate and develop a shared understanding.

A framework for analyzing defining in classrooms. In reviewing the literature for Paper 1, I identified 11 Aspects of Definitional Practice. The Aspects of Definitional Practice were the foundation for a coding scheme I created for Paper 2 to analyze how members of the class participated in defining. This coding scheme was again used for part of my analysis in this paper. Rather than describe the aspects here, I will describe how I developed operational definitions for a subset of them in the Methods Section. To read more about how the aspects were developed and grounded in the literature, please refer to Paper 1.

A Situative Approach to Learning

A useful frame for studying the co-development of defining with knowledge development is what Greeno (1996) describes as “situative.” One of the key tenants of the situative approach is that knowledge development and practice are tightly related, and that it involves learners’ development of disciplinary dispositions (Boaler, 2002; Lehrer, 2009). Thus, the situative approach motivates a focus not only on knowledge and practice, but also on how the two co-develop in interaction. Such an analysis must be evidenced by talk, gesture, inscription and other forms of communication about meaning. This focus on interactions requires understanding how practices are negotiated and how joint activity becomes taken-as-shared (Yackel & Cobb, 1996). In this process, members of a group take on different roles as they are positioned in terms of “competence, authority, and accountability” (Greeno, 1996, p. 88), both by themselves and others. Next, I describe the work I drew upon for investigating how such joint activity is negotiated among members of a community.

Supporting Disciplinary Practice in Classroom Discussions

For my final lens, I drew upon four bodies of work that describe how participants contribute to the development of communal understanding for what it means to engage in interactions around practice. This lens informed my analysis of interactions pivotal in supporting the creation of community norms, expectations, and understandings about what it means to participate in defining. Much of the work I draw on comes from scholars interested in how math talk communities develop and how discussions are orchestrated in relation to mathematical practices such as collaboration, explanation or argumentation. Despite differences in topic, this work still provided a way to look generally at interactions in order to consider what is significant for establishing a classroom culture for the practice of defining. I focused on the orchestration of classroom discussions because defining has historically been a social process between members of the mathematical community (Lakatos, 1976), and discussion is a productive means for cultivating and observing students' participation in the practice. I describe each strand of this lens in what follows.

Articulating expectations for participation in practice. The first strand describes how participants in a classroom culture play a pivotal role in establishing norms by articulating expectations for participation in mathematics practice. Expectations are often negotiated between members in interaction, but in a classroom, these negotiations are heavily scaffolded by the teacher. The teacher, standing in as a disciplinary representative, often articulates rules or expectations for participating in communal mathematics practices (Horn, 2008; Lampert, 2001;

Wood, 1999; Yackel & Cobb, 1996). For one, teachers may communicate expectations for the roles participants should take on in certain moments and how they should act in those roles (Horn, 2008; Wood, 1999). Wood (1999) describes a teacher who created expectations for roles in class presentations, and articulated not only what the speaker should do, but also how the audience should participate. On the first day of school, the teacher she was studying began the year by explaining the importance of disagreements in math class. When they participated in their first math discussion, the teacher requested that students vocalize agreement or disagreement. As students learned to do so, the teacher articulated new expectations for listeners, such as "decide if you have a question, so that you can ask it." This example illustrates how the teacher's expectations shifted in relation to the emergent needs of the community.

A teacher may also implicitly communicate expectations through her reactions to students' contributions (Enyedy et al., 2008, Strom, Kemeny, Lehrer, & Forman, 2001). For example, a response of "I like that!" or "listen to her" suggests that a student's contribution is important and legitimizes it (Yackel & Cobb, 1996). Moreover, expectations do not need to be communicated by talk. Horn (2008) describes how Deborah Ball, as a classroom teacher, asked a student to stay to the center of the classroom after he presented in order to receive questions from students. By physically placing him at the center of the class, where the teacher usually stood, she implicitly articulated two expectations: (a) that students need to take a justified position in a discussion and actively defend that position and (b) that the other students also need to engage in the mathematics work.

Modeling practice. A second strand in supporting disciplinary practice describes how participants play a role in modeling for other members of the community how one should participate in practice. Sometimes modeling comes in the form of providing examples, and might be coupled with the articulation of expectations (Lampert, 2001; Wood, 1999). For instance, in the example provided above from Wood (1999), when the teacher requested that students begin to ask questions to the presenter, she illustrated several examples: “You might think, ‘I’m not sure of what you’re saying?’ or ‘I’m not sure how you did it?’ or ‘You don’t count the way I thought you should’” (p. 186). Similarly, Lampert (2001) describes how she as the teacher initially modeled how one should respectively disagree with others, building on an interaction between two of the students, “If you disagree, like Anthony just disagreed with Eddie, that’s very very important to do in math class. But, when you disagree or think somebody misspoke, you need to raise your hand and say, I think he must have meant plus, not times” (p. 70).

Teachers may also model participation by participating alongside students in ways that they hope students will appropriate. For example, Hufferd-Ackles, Fuson, and Sherin (2004) describe how one teacher modeled how to ask questions to presenters during whole group discussions, and eventually the students began to use the same questions. It is important to note, though, that learning by watching someone model entails more than copying or imitating. Rather, because of the interactive nature of discussions, it requires learning *when* to participate in those forms of interaction as well.

Positioning practice & participants. A third component of supporting participation in classroom practices is the notion of positioning other members as participants in disciplinary practice or positioning particular forms of activity as significant to disciplinary practice. Positioning is important to supporting disciplinary engagement because it gives students authority by highlighting them as authors of disciplinary content. One way in which this is done is what O’Connor and Michaels (1996) call *revoicing*. They examine how revoicing shifts, reframes or repositions existing participant roles and structures to place the authority in the hands of the students, while also holding them accountable to the social and disciplinary norms of the community. In particular, O'Connor and Michaels, and several scholars since, have noted that revoicing serves several functions in classrooms. First, by restating a student's utterance and attributing authorship to the student (e.g., "Jim said…"), revoicing serves to rebroadcast a student's statement, thereby giving them a more prominent voice and positioning their contribution as important (Enyedy et al., 2008; Forman & Ansell, 2002; Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Strom et al., 2001). Similarly, some of the ways expectations are implicitly articulated, as described earlier (e.g., "Tim, do that again. *Watch*. Do that again," p. 759, Strom et al., 2001), also serve to position a comment as important and the student as an important contributor. Second, by changing components of the student's talk when restating the student, the teacher may repair (clarify) or reformulate the utterance, often in order to advance the teacher's agenda (Enyedy et al., 2008; Forman & Ansell, 2002; Forman et al., 1998; Jacobson & Lehrer, 2000; Strom et al., 2001). For instance, a teacher may replace a student's word with a more mathematically precise word. Sometimes, such revoicing may be non-verbal. For instance, one person may repeat the gesture of another person, but modify it slightly to highlight a new feature (Strom et al., 2001). Third, by using indirect speech, namely verbs that animate the student as the author of the content (e.g., "so Jane predicts that…" (O'Connor & Michaels, 1996, p. 79)), the speaker positions a student's utterance in relation to content, such as an argumentative stance (Forman et al., 1998; Strom et al., 2001). Note in the example provided, the teacher used a marker of warranted inference, "so," to link to the student's previous justification or claim. And, finally, in a similar way, revoicing can be used to pit two stances as competing (Forman et al., 1998; Horn, 2008; Strom et al., 2001). This has been found to be especially important in supporting interactions around mathematical argumentation in classrooms.

Although in many of these studies, the focus was on how the teacher used revoicing to establish or maintain interactional norms, students may also participate in revoicing. In fact, even after initial norms and expectations for participation have been established, revoicing and other positioning moves may still be used as a way to negotiate interaction (Forman & Ansell, 2002; Horn, 2008; Strom et al., 2001). For instance, Horn (2008) describes how members of Ball's class positioned themselves and others into argumentative roles, such as "principal of a controversy," the person who makes the initial claim, and "dissenter," the person who voices initial disagreement. Positioning into roles is accomplished in several ways: (a) by assuming a role (e.g. "I disagree with Joe"), (b) by designing others into roles (e.g., "I disagree with Joe" designs him as the principal of controversy), (c) by ratifying a role, and (d) by animating others into roles, for instance, by juxtaposing positions of two participants.

Disciplining perception. A final strand draws upon Goodwin's (1994) construct of "professional vision," what he defines as the "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606). Building professional vision consists of the use of three practices, namely: (1) coding (labeling events or artifacts in discipline-specific terminology), (2) highlighting (marking important aspects of events or artifacts), and (3) producing and articulating material representations. Goodwin, drawing upon his studies of archeologists and lawyers, argued that these three practices are important forms of communication and serve as resources for apprenticing and essentially "teaching" non-members important aspects to focus on in discipline-specific work.

Stevens and Hall (1998) extended Goodwin's notion of professional vision to illustrate how visual practices, and the teaching of those visual practices, are often tied to particular forms of representation. In looking at the case of a tutor and student and another of two professional engineers, they found that when a disruption occurred, for instance in the form of the representation, it triggered what they term a "breakdown" in shared understanding. At this point, one person typically stepped in to "discipline the other person's perception." Disciplining perception usually involved a directive such as, "look at it this way" (p. 141), followed by embodied interactions with the representation, or representations, under discussion. Goodwin's notion of professional vision has also been used to look at teachers learning to notice and interpret student thinking (e.g., Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2008). However, other than Stevens and Hall (1998), little has been done to show how the lens might inform how teachers discipline the perception of students into ways of "seeing" characteristic to the discipline of mathematics. A couple of studies provide some additional insight into what that might entail. For instance, part of Lampert's (2001) work in establishing a classroom culture involved helping students understand what a conjecture was by "labeling" their assertions as such, a form of what Goodwin might call "coding." I used Goodwin's framework to investigate how the teacher and students helped discipline others' perceptions in ways similar to this.

Method

Instructional Context

Participants & setting. Participants (n=18, 10 male) were an ethnically diverse class of sixth grade students who attended an urban school serving primarily underrepresented youth in the southeastern region of the United States. The percent of children attending the school who qualify for free or reduced lunch ranges from 60 to 80 from year-to-year. The participants came from a contained classroom, that is, they remained in the classroom with one teacher for all their core academic subjects. Half of the students came from traditional classrooms that emphasized procedural mathematics. The other students had looped up with the classroom teacher from the year before, and had engaged in some conversations about definitions related to mathematical symmetries. Despite this, because there were a considerable number of new members to the community, norms surrounding participation in practice still needed to be established and old members were also varied in their participation in practice, suggesting that they, despite past experiences, were still making sense of what it meant to participate in defining. I mention this because, as I describe later, part of what I looked for in development was how members' roles in the class shifted over time, and how their histories played a part in that.

Instructional design. The students' work with definitions was situated within a larger project, aimed at engaging students in authentic mathematical inquiry about geometry and spatial mathematics (Lehrer, Kobiela, & Weinberg, in press). That is, we encouraged students to pose mathematical questions and conjectures, and, in turn, those questions and conjectures guided many of their investigations. At the same time, students also constructed definitions, formulated arguments, and wrote about and inscribed aspects of their explorations. Topics included definitions of polygons and related objects and properties, interior and turn angle sums for polygons, relations between the number of sides of a polygon and the number of diagonals, triangle congruency theorems, symmetries, and area measure of polygons. The research goal was to find out if it would be possible to engage students in this more open-ended form of inquiry, and, if so, what that development would entail.

Lehrer began working with the students as a visiting mathematics instructor on the second week of school. During the first week of school, students had worked within a Connected Mathematics Project unit on polygons. When Lehrer visited the following week, he intended only to find out how the students were thinking about what they had learned, and, in a whole group setting, asked, “What is a polygon?” When this question spurred much interesting debate and discussion, Lehrer and the classroom teacher decided he should continue to work with the students exploring topics in geometry.

Because investigations were generally guided by students’ questions and conjectures, there was no written curriculum per se. Nonetheless, there were several key features of the instruction. First, the work began with asking students to define “polygon” and related properties, such as “side,” “angle,” “straight.” These definitional investigations comprised the first few weeks of the school year and took advantage of students’ previous experiences with the geometric objects, in that they had enough familiarity with them to propose initial definitions that could then be revised and expanded. This aspect of instruction was consistent with their work in Connected Mathematics, although Connected Mathematics did not position students as generators of definitions.

Second, mathematical questions were highlighted as important. Starting early on, the teacher asked students to pose questions about particular mathematical objects (such as a square drawn on the board). These questions, and others that arose in class, were then documented in a large visible class list of questions, on which student authorship was denoted. Moreover, questions that could be investigated with available resources and knowledge and related to the overarching topic of polygons were privileged. For example, those that could not be investigated, such as, “Did Mr. Einstein make one (an octagon)?” were immediately answered and given less floor time. The teacher also verbally noted when a question was a “good” question.

Third, students routinely investigated questions from the questions list. Most of the time, the teacher asked students to investigate the same question, but on a couple of occasions, students selected a question to investigate with their small group. When investigating, students had access to their math notebooks and a variety of tools (e.g., protractors, straight edges, etc.). At times, investigations were open-ended and students chose their approach, whereas other times, the teacher guided investigations, for example, by suggesting a set of cases to test.

Fourth, because questions and conjectures often led to investigations that led to new questions or conjectures, students were engaged in the creation of a mathematical system of definitions, conjectures, questions, investigations, theorems and proofs (Lakatos, 1976).

And, finally, throughout the instruction, the teacher capitalized on students’ everyday experiences of space to help them reason about objects, properties and relations among them. For example, students experienced angles as portions of full rotations and “straight” as a constant heading while walking.

Data Collection Procedure

As previously mentioned, Rich served as a visiting mathematics instructor, and was the primary classroom instructor for mathematics during the school year. The regular classroom teacher remained in the classroom during math class, and occasionally interacted as well. Mathematics class was conducted twice each week, for 1.5 hours per class. There were a total of 46 geometry lessons during the year. Each lesson (except for one) was videotaped and digitally rendered for further analysis. One camera captured whole group discussion, generally maintaining a wide shot and zooming in momentarily for inscriptions, materials, gestures or other bodily movement. During small group work, this camera roamed between the different table groups with the intent of providing snapshots of work from a range of students. Starting in October of the school year, a second camera was mounted onto the wall to capture the interactions of one small group of three students for the remainder of the school year. At the same time, field notes were taken of whole group interactions in order to supplement the video and serve as a platform for reflection among the research team members. At the end of each lesson, field notes were compiled, and these served to guide the next lesson. In addition, students kept math journals that provided an additional insight into their thinking and into their developing dispositions towards the mathematics. The intention of the mathematics journals was to support writing mathematics as a form of self expression. We gave the students periodic written assessments about what they had been investigating in class and, at the end of the year, conducted one-on-one semi-structured interviews to get a sense of what they had learned and their dispositions towards mathematics class. For purposes of this paper, because my focus is on collective classroom activity, I mainly used the field notes and the video records of whole group activity.

Analysis

My analysis consisted of three parts: (a) characterizing the mathematical knowledge developed by the class, (b) characterizing changes in interactions around practice, including how the students participated in mathematical defining and ways in which discussions were orchestrated specific to defining, and (c) using the first two parts of analysis, characterizing how practice and knowledge co-developed. I begin by describing my sampling procedure for selecting episodes of classroom activity to analyze. Then, for the remainder of the section, I describe each of the three parts of analysis.

Sampling Procedure. My sampling procedure consisted of four phases of data reduction. I describe each in turn below.

Phase 1. First, I selected seven days from the original 46 days of geometry instruction to focus on. To do so, I watched video from about half of the days, distributed throughout, and read field notes for the other half. When doing so, I noted instances when students were engaged in the negotiation of a definition (either structural or procedural) for more than a few turns of talk. That is, students had to have competing ideas about the definition. This resulted in approximately 21 potential days (note, there may be other instances not captured in the field notes). On these days, students constructed structural and/or procedural definitions for “scalene,” “triangle,” “circle,” “pentagon,” “polygon,” “rhombus,” “square,” and “diagonal.” From there, I chose to select the first six days of instruction because students were engaged in defining “polygon” and its related properties (e.g.. “sides,” “angles,” “straight”) for an extended period of time, and those days allowed me to see initial development unfold. I then selected the 26^{th} day of instruction, during which students constructed definitions of “triangle.” I chose this point because it occurred two and a half months after the sixth day (later than most of the episodes) and thus allowed me to see if practice was sustained and/or changed over an extended period of time. Moreover, “triangle” was different from “polygon,” but similar in that its definition relies on many of the same properties and relations, and this similarity allowed me to see how students came to use those properties and relations (unlike a definition of “symmetry”). Some definitions, like “pentagon,” were constructed as procedural definitions and others, like “rhombus,” were influenced by the use of dynamic tools and could not be as easily compared. The “triangle” episode also showcased multiple students' definitions, providing a more representative account of students' constructed definitions.

Phase 2. To select excerpts from the seven days for analysis, I divided the data into *Definitional Episodes.* I defined Definitional Episodes to be segments of whole group discussion that involved one or more of the following: (a) the negotiation of a mathematical definition, (b) discussion of relations between two or more classes or properties, or (c) discussion of relations between a case and a class. I limited Definitional Episodes to moments from whole class discussions because, although small group activity may have influenced whole group activity, I was mainly interested in how knowledge and practice became taken-as-shared (Yackel & Cobb, 1996). *Negotiation of mathematical definitions* are segments of time when competing or alternate definitions were proposed and discussed. Proposals involved at least two members of the class, and could occur between the teacher and a student or between students. This included negotiating a new definition or the revision of an existing definition. These segments usually began with the question, "What is a ____?" *Discussions of relations between two or more classes or properties* were segments of time when the class discussed relations between two or more classes or properties. For instance, such segments began with questions like, "What is the difference between an angle and a vertex?" In addition, on a few occasions, students discussed what terms described a situation (for example, "congruent" was used to describe when two objects were the same). To count as an episode, discussion had to include students' justifications of the relations. Justifications moved the conversation from simply identifying whether or not a relation existed (e.g., "a square is a polygon") to discussing the properties underlying those relations. *Discussions of relations between case and class* were segments of time when the class discussed relations between a specific case and a class. For instance, such segments began with questions like "is this (drawn rectangle) a regular polygon?" To be counted as an episode, discussion had to include students' justifications of the relations.

Definitional Episodes were framed around one or two main mathematical objects or properties that were under discussion. In addition, other objects or properties were often mentioned in reference. I chose to parse Definitional Episodes in this way because it helped highlight the main object(s) being defined, which will, as discussed later, aided in my analysis of the development of mathematical knowledge.

I used the following procedure for determining and documenting Definitional Episodes. First, for each of the seven days, I watched the video of whole class discussions and made general descriptive notes of the activity. As I made the notes, I parsed the activity into Definitional Episodes. I started a new episode when a new mathematical object or property was introduced and, as described above, (a) its definition was negotiated, (b) it was discussed in relation to another class or property, or (c) it was discussed in relation to a case. Segments of whole group discussion that were not counted as Definitional Episodes included times when a definition was put forth, but not negotiated (that is, multiple student contributions were not elicited), moments when the class discussed something not related to mathematics for longer than 30 seconds, times when a definition of a non-mathematical object was discussed (e.g., "convention"), or times when the class learned how to use a tool (for example, a protractor). Although Definitional Episodes often began with a question (e.g., "What is a polygon?"), not all questions automatically guaranteed the start of a new episode. Sometimes, questions were posed but not immediately investigated. Other times, questions were related to the same object under discussion and did not provoke the exploration of a new mathematical object. For example, one student posed the question, "Can a regular polygon be an irregular polygon too?" after they had been discussing the difference between regular and irregular. In this instance, the question was not about a new mathematical object, and, thus, did not warrant a new episode. In some instances, questions were immediately resolved and did not result in the presentation of multiple ideas. This also happened to be the case for the question “Can a regular polygon be an irregular polygon too?” and was a second reason it did not become the start of a new episode. In that instance, one student responded “no” and one other provided an explanation. The conversation then shifted to discussing the definition of polygon.

For each Definitional Episode, I documented (a) the start and end times of the episode , (b) the object being defined in the episode (e.g. “polygon”), (c) how the episode began, that is, what triggered discussion of the particular definition or relation (e.g., a teacher question, etc.), (d) a summary of the Definitional Episode, (e) the definitions discussed during the episode, and (f) the consensus definition, if one was reached. In cases when two Definitional Episodes about the same topic (e.g., “polygon”) were separated only by small group work, I documented them as one episode instead of two. This process resulted in a total of 48 Definitional Episodes. These Definitional Episodes were used for the analysis of knowledge development.

Phase 3. Third, in order to conduct a detailed analysis of interactions around practice, I further narrowed the sample of Definitional Episodes to four 10-minute excerpts, three from the first six days and one from the 26^{th} day, with a total of 16 Definitional Episodes (one of these episodes was only partly included). The excerpts from the first, fourth, and sixth days all involved discussions of the structural definition of “polygon” and all began with the question of “what is a polygon?” Similarly, in the excerpt from the 26^{th} day, the discussion was motivated by the question of “what is a triangle?” and also involved structural definitions.

I selected the excerpts based upon several criteria. First, in order to characterize development, the selected excerpts represent interaction at progressive points in time. Second, to allow for parallel comparisons, I selected excerpts that were (a) centered around the definition of the same or similar object, (b) involved investigations of the same form of definition (*either* structural definitions or procedural), and (c) were similar in activity structure. Third, I selected the excerpts to each span multiple Definitional Episodes to allow for analyses of how practice led to investigations of new relations. “Polygon” had the advantage over other objects that were frequently defined (e.g., “straight,” “side,” “angle”), because its definition had many similar features to “triangle” and allowed for easier comparisons of practice. Within the first six days of instruction, there were four days in which students pursued the question “what is a polygon?” Because these occurred on the first, fourth, fifth and sixth days, I chose to focus on the first, fourth and sixth to represent that span. I used other Definitional Episodes, including the one from the fifth day, to look for confirming or disconfirming evidence about my findings from the selected episodes. Table 1 shows the selected Definitional Episodes (highlighted in gray) in relation to the entire sample of Definitional Episodes.

Table 1. Definitional Episodes from Days 1, 4, 6, and 26

DE	Day	Main Object(s) Defined	Starter of Episode	Form of Definition
1	Day 1	Polygon	T asks, “Who can help me understand what a polygon is?”	Structural
2	Day 1	Quadrilateral	T says, "Now someone will tell me what the heck a quadrilateral is, cause I haven't heard that word yet."	Structural
3	Day 1	Circle, Polygon	T asks: "Okay so question. Circle is a polygon?"	Structural
4	Day 1	Regular Polygon	T says, “So so far, I can't, the only thing I know is that there are some polygons that are regular. And they have equal sides and equal angles. So now I know what a regular polygon is and I'm very happy. Cause if I see a square, what will I say?"	Structural
5	Day 1	Same, Congruent	Right as T is about to write the word "same," he says (pointing to the board), "We used a word last year, now it-."	Structural

Table 1, continued

DE	Day	Main Object(s) Defined	Starter of Episode	Form of Definition
6	Day 1	Regular Polygon	T: "...So a lot of people said this show that this figure is not regular. But Shaunee objects. So we need to listen to Shaunee's objection."	Structural
7	Day 1	Angle	T asks, "What makes an angle again?"	Structural
8	Day 1	Regular Polygon	T: " IF you don't like Shaunee's definition, what would you do to it to make sure that this does not get in?"	Structural
9	Day 1	Vertex, Angle	Adeena: "You was asking what was that called. That was the angle, that you were trying to get us to say. Not a vertex, but the point was the angle."	Structural
10	Day 1	Regular Polygon	T: "IF I say that the sides have to be congruent and the angles have to be congruent, Shaunee, is this a regular polygon?"	Structural
11	Day 1	Angle	T asks Ned to present first: "You tell me you've got 4 different angles up there. Tell me what you're thinking. Tell us."	Structural
12	Day 1	Degree	T: " What's a degree?"	Structural
13	Day 1	Vertex	T: "Where's the vertex, Ned?"	Structural
14	Day 1	Angles (what makes angles equivalent)	T draws the same angle, except one side is longer than on the original angle, and then asks, "Do you agree or disagree that I have now drawn two different angles?"	Structural
15	Day 1	Straight	T: "I never did ask you this question. Everyone keeps talking to me about straight sides. I never did hear what made something straight."	Structural
16	Day 1	Angle	Kira asks if her group could present their angles.	Structural
17	Day 2	Degree	T: "Now when you say degree, what's one degree?"	Structural
18	Day 2	Angle	T asks Ned to interpret Kira's drawing: "what is she trying to show us about what I did?"	Structural
19	Day 3	Fifty Degrees	T: "What is fifty? What part of a circle?"	Structural
20	Day 3	Angle	T: "Will the angle measure when I extend the lines be less, the same or greater and why?"	Structural
21	Day 3	Angle	T: "Someone said 180 is the largest angle. And I asked you what were they thinking."	Structural
22	Day 4	Octagon	T: "What makes it an octagon?"	Structural
23	Day 4	Obtuse	T: "How do you know these angles are obtuse?"	Structural
24	Day 4	Polygon	T: "You know, I have to say, I've been here for two weeks now and I've never heard you once tell me what you meant by polygon."	Structural
25	Day 4	Circle	T: "Is a circle a polygon?"	Structural
26	Day 4	Polygon	T: "well how bout tell me what a polygon is before you tell me what it not is."	Structural
27	Day 4	Side	Mona, Kate, Adeena: "what's a side?"	Structural
28	Day 4	Straight	Vern: "What does straight mean?"	Structural/ Procedural

Table 1, continued

DE	Day	Main Object(s) Defined	Starter of Episode	Form of Definition
29	Day 5	Straight	T: "what did we conclude about straight? What was one meaning of straight?"	Structural/ Procedural
30	Day 5	Angle	T: "What do we call it when we introduce the zig-zag?....When we have a line meeting another line?"	Structural
31	Day 5	Irregular; Regular Polygon	T: "so what's the difference between a regular polygon and one that is not regular?"	Structural
32	Day 5	Polygon	T asks them what a polygon is.	Structural
33	Day 5	Closed	T: "What does closed mean again?"	Structural
34	Day 5	Angle	He asks Nicholas to come up and show the angles: "Where are the angles?"	Structural
35	Day 5	Polygon vs. Circle	T: "Do we have to say anything about angles? Is there any way we could generate something that was closed with sides without making the same number of angles as there are sides?"	Structural
36	Day 5	Polygon	Kira suggests an answer to T's question. T repeats his question: "Can I just say that to make a polygon, I need to have it 3 or more sides and the figure has to be closed? Do I have to say anything about angles or not?"	Structural
37	Day 5	Regular 4-Sided Polygon	T: "On Tuesday, I asked you to try to figure out how you would walk to make a polygon…I will give you 5 more minutes to write directions."	Procedural
38	Day 6	Polygon	T: "Okay, what is a polygon?"	Structural
39	Day 6	Straight	T: "How did we define straight?"	Structural/ Procedural
40	Day 6	Polygon	T returns the conversation to the original question: "How were you thinking about this Kira?"	Structural
41	Day 6	Closed	T asks Shaunee: "So closed means what?"	Structural
42	Day 6	Polygon	T asks Shaunee: "If something is closed and has sides, must it have angles or not?"	Structural
43	Day 6	Regular vs. Polygon	T asks: "Are there more polygons or are there more regular polygons?"	Structural
44	Day 6	Rectangle	T asks them to write directions either for a rectangle or for a regular triangle.	Procedural
45	Day 6	Regular Triangle	T asks: "But what about (directions for walking) the triangle?"	Procedural
46	Day 26	Triangle	T: "What's a triangle?"	Structural
47	Day 26	Regular	T: "What's the definition of a regular polygon again?"	Structural
48	Day 26	Triangle	T turns to another definition: "A triangle has 3 straight sides, 3 angles, interior angles of 180. You mean each interior angle is a hundred and eighty degrees? What do you mean?"	Structural

Phase 4. Finally, I selected a few additional excerpts of definitional activity from the days between Day 6 and Day 26 to provide additional and/or confirming evidence about co-development of practice and knowledge. From the 21 days described in Phase 1, I chose three additional days – the eighth day, the 17th day and the 19th day. I selected the eighth day because students spent a little time constructing definitions of "triangle," and this would provide a contrast to their work on day 26. On the 17th day, students revisited the definition of "polygon" and related properties, providing a comparison point to their earlier work. On the 19th day, students began constructing definitions of rhombi. Because "rhombus," like triangle, is a sub-class of polygon, it provided a way to compare practice and knowledge development.

Transcription of sampled data. I conducted two levels of transcriptions. I first did a rough transcription of all the Definitional Episodes, capturing talk and descriptions of inscriptions, bodily motion, and gesture. To capture gesture or bodily motion, I used parentheses to denote descriptions of each as they occurred during talk. By doing so, I was able to see how gesture or bodily motion highlighted meaning in talk, such as messages about practice, and vice versa. Moreover, at times, embodied communication existed without talk, and reflected how a participant thought about a mathematical idea (e.g., turning one's body to communicate an amount of turn, see Figure 1). For inscriptions, I noted in parentheses anything written on the board and described any diagrams. I then added additional detail to the transcripts for the four 10-minute excerpts, borrowing conventions from Dressler & Kreuz (2000), to highlight stressed and overlapping talk. I focused mainly on stressed and overlapping talk in order to see what participants positioned as important about practice, and also to gauge the level of engagement from students. For forms of stress, I noted elongated syllables (::), emphasized words (CAPS), and rising (/) and falling (\) intonations. I used brackets ([ ]) to indicate talk spoken by at the same time.

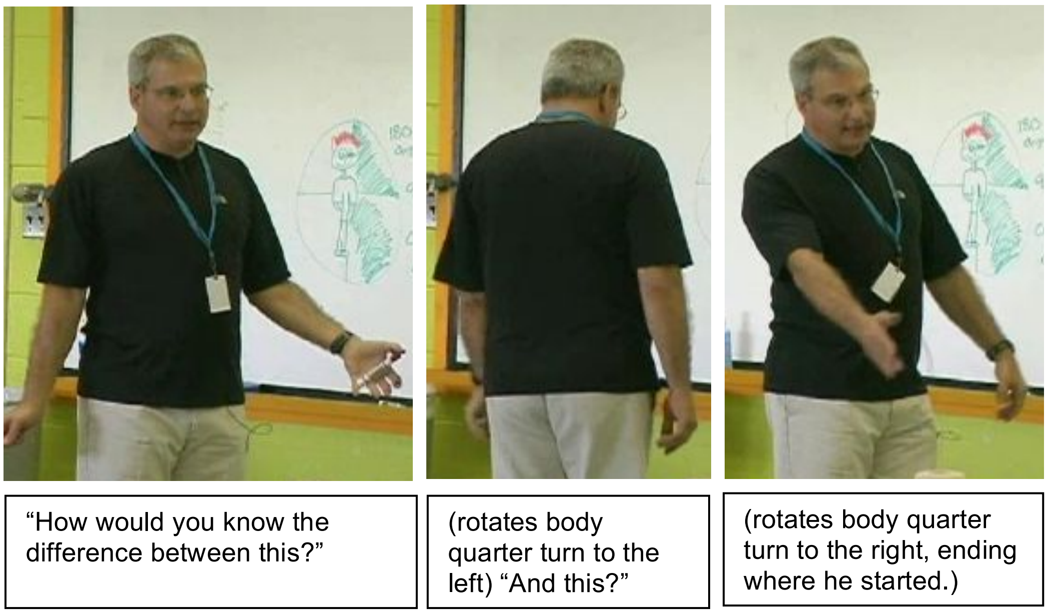

Figure 1. Example of transcribing talk and bodily rotation.

I parsed all transcripts into turns of talk, which served as my unit of analysis for both sets of analyses. The turn of talk allowed me to look at how definitional practice developed in interaction, including changes in the roles of the teacher and students. Moreover, it allowed me to see how in-the-moment choices and forms of interaction helped to influence the development of communal knowledge. All transcriptions are located in the Appendix.

Characterizing mathematical knowledge. To characterize mathematical knowledge, I documented several features of the mathematical ideas explored by the class. My intention was not to make claims about what individuals were thinking, but rather to capture the nature of the mathematical system explored by the class. Using members’ talk, gesture and/or inscription, I documented three features of communal knowledge development: (a) the objects investigated by the class, (b) the types of relations discussed, and (c) the frequency with which an object or property was discussed. In what follows, I describe each of the three representational components, what they illustrate about communal knowledge development, and the evidence I drew upon to document their existence at the level of the turn of talk.

Mathematical objects. In representing knowledge development, I denoted the mathematical objects (or properties) defined or discussed in order to capture *what* the class investigated. To document classes of objects, for each turn of talk, I noted all mathematical objects mentioned (e.g., "polygon," "side," "angle") and any proposed cases of the objects (e.g., a drawn square). For instance, in the turn of talk, "A **polygon** has the same **angles** and the same length of uh, same length of **sides**," the mathematical objects, bolded, are "polygon," "angles," and "sides." Alternatively, cases were drawn on the board or illustrated with bodily motion or gesture. For example, a student provided an example of a polygon with two sides by gesturing an oval in the air. For procedural definitions, objects or properties often took the form of an action (e.g., "a step").

Nature of relations. I also noted how members related mathematical objects. This included relations between: (a) a class and a sub-class (e.g., "polygon" and "regular polygon") or a case of the class (e.g., "polygon" and a drawn rectangle), (b) a class and the properties that describe that class, what I refer to as inclusive relations (e.g., "polygon" in relation to "sides" and "angles"), and (c) a class and another class (e.g., "side" and "angle"). Additionally, I noted *how* classes, sub-classes, or properties are related. That is, in the definition, "a regular polygon has the **same** sides," "same" relates "regular polygon" to "sides."

To document the nature of relations, I drew upon several cues. First, relations between a class and sub-class or a class and a potential case were often indicated with a "be" verb (e.g., "a square **is** a polygon" or "a square **is not** a polygon"), an adjective (e.g., "**regular** polygon" implies that it is a type of polygon), or an inscription (e.g., an arrow) or a gesture. Second, I noted inclusive relations when a "has" verb linked a class and its properties, as in "a polygon **has** angles" or "a polygon **does not have** angles." I also documented any adjectives describing the nature of the inclusive relationship. For example, in the statement "a regular polygon has the **same** angles," the adjective "same" indicates the kind of angles that constitute a regular polygon. Inclusive relations were also communicated via inscription or with the body, for instance, by turning a quarter turn to describe a ninety-degree angle. Finally, I noted relations between classes when descriptions linked two mathematical terms (e.g., when the teacher asked for "another word" for "same," the students responded with "congruent") or when a member reasoned about relations between properties that constituted a definition, such as when reasoning about economic definitions. These cues were meant as general guidelines, and did not guarantee the existence of the relations.

Frequency an object, property, or relations are discussed. I also documented the relative frequency with which objects, properties or relations were discussed in order to see which mathematical ideas reoccurred and how. For example, some ideas were dismissed and never brought up again while others returned. To document frequency, within each Definitional Episode, I noted whether an object (or case of an object), property, or relation was mentioned. Frequency was then defined as the number of Definitional Episodes within which an object, property or relation arose. Because Definitional Episodes are organized around the discussion of a mathematical object, this gave me an estimate of how frequently a topic was re-introduced.

Characterizing interactions around practice. To characterize interactions around the practice of defining, I analyzed two forms of participation. One form of participation, *Engaging* *in Aspects of Definitional Practice*, was intended to characterize how participants engaged in the practice of defining, and, as the name would suggest, drew upon my framework of Aspects of Definitional Practice. The other form of participation, *Orchestrating Definitional Discussions*, was intended to capture interactional moves that may potentially support orchestration of discussions around the practice of defining. My analysis of orchestration drew upon the literature I outlined earlier, describing how individuals come to understand what it means to participate in mathematics practices. My goal for this paper is to distinguish how these forms of interaction are constituted in the context of defining.

To analyze these two forms of participation, I developed coding schemes that were used to code the teacher's and students' turns of talk during whole group discussion. Coding at the level of turn of talk allowed me to trace the roles the teacher and the students take on as they participated in definitional practice and how those roles changed over time. Moreover, I was able to look carefully at how moment-to-moment choices in participation contributed to supporting knowledge development.

In what follows, I first describe my method for developing coding schemes. Second, I describe my procedure for conducting the coding and synthesizing the codes. In the last two sections, I describe the coding schemes, first for *Engaging in Aspects of Definitional Practice*, and then for *Orchestrating Definitional Discussions*.

Coding scheme development. For both forms of participation, I developed coding schemes that allowed me to characterize participation at the level of turn of talk. The coding schemes were initially developed using the sample of three 10-minute excerpts of whole group activity, taken from days one, four and six of mathematics instruction. Using this sample, I developed and revised my coding schemes via an iterative process. That is, for each form of participation, I parsed the turns of talk into initial categories, and then used the categories to code the turns of talk. My initial categories for *Engaging in Aspects of Definitional Practice* were the Aspects of Definitional Practice that I attempted to operationalize. My initial categories for *Orchestrating Definitional Discussions* were created by categorizing forms of talk that did not fall under *Engaging in Aspects of Definitional Practice*, with an eye toward the theoretical work described earlier. When coding, I noted turns of talk that did not fit my initial coding scheme, and this led to the addition of new categories or subcategories, the splitting of existing categories, and clarification or elaboration of existing categories. Likewise, as I refined the coding schemes, initial categories that were not used were eliminated. Finally, I checked the codes using the fourth 10-minute excerpt from the 26th day of instruction and made slight revisions to the coding scheme.

Coding and synthesis procedures. Once the coding scheme was solidified, I coded the entire sample once more, first coding for *Engaging in Aspects of Definitional Practice* and then going back through and coding for *Orchestrating Definitional Discussions*. This process allowed me to maintain consistency within each coding scheme. When coding, a turn of talk could receive multiple codes within each coding scheme. In instances when an utterance spanned multiple turns of talk (for instance, if the speaker was interrupted), then both turns of talk received the code.

Once the sample was coded, I synthesized the data in two ways. First, I noted the frequency of codes within each coding in order to find general trends of participants' use of particular Aspects of Definitional Practice or ways in which discussions are orchestrated. Second, I looked at codes assigned to the teacher versus those assigned to the students in order to see if there were shifts in the roles that the participants take on. For instance, did the teacher model particular Aspects of Practice early on? Likewise, did students begin to appropriate participation in these Aspects of Practice? Who was doing most of the work orchestrating discussion and did that change over time? Further analyses of the coded data were done to look at co-development of knowledge and practice. Those are described later.

Preliminary coding scheme: Engaging in Aspects of Definitional Practice. I created the coding scheme for *Engaging in Aspects of Definitional Practice* by expanding, cutting, and operationalizing my theoretical framework of Aspects of Definitional Practice. In the process, I found seven of the initial 11 Aspects of Definitional Practice to be prevalent and describable at the level of turn-of-talk. These aspects included *asking definitional questions*, *describing properties and/or relations*, *constructing and/or evaluating examples*, *constructing definitional explanations and arguments*, *revising definitions*, *establishing and reasoning about systematic relations*, and *negotiating criteria for judging adequacy or acceptability of definitions*. Additionally, I created an eighth category, *proposing definitions*, that was related to part of one of the original Aspects of Definitional Practice, *engaging in cycles of definitional conjecture, experiments, and tests*.. The other aspects were not included because they either were not relevant to the sample of data (e.g., *considering definitions in new forms or contexts*) or were too inclusive and difficult to operationalize (e.g., *investigating fundamental qualities of mathematical objects*).

In the following sections, I describe each of the eight categories that comprise my coding scheme for Engaging in Aspects of Definitional Practice. An abbreviated version of the coding scheme is presented in Table 2.

Table 2. Coding scheme for *Engaging in Aspects of Definitional Practice*

Aspect of Practice	**Description**	**Examples**
Asking Definitional Questions	Speaker asks a question about a definition or about qualities, properties, relations, or examples of the object being defined.	"[When we define polygon] do we need to say sides and angles or is it enough to say sides?"
Proposing Definitions	Speaker proposes properties or relations to include in a definition.	"A polygon has the same angles and the same length of uh, same lengths of sides."
Describing Properties and/or Relations	Speaker articulates, through talk and/or writing, properties & relations of a class of mathematical objects or a particular case of a class. Properties and relations may be described in service of other goals, such as constructing an explanation or proposing a definition.	A student was asked what the definition of polygon is. In responding, he also described properties and relations of the object: "A polygon has the **same angles** and the **same length** of uh, **same lengths of sides**."
Constructing &/or Evaluating Examples	Speaker constructs an example of the object being defined and/or determines whether a **particular** example belongs to a set. May be in service of constructing definitional arguments or explanations or in service of evaluating a definition.	A student suggests that a polygon is defined as "sides and angles." The teacher draws an example using their definition of three connected but not closed lines (a "Z" like figure). A student then evaluates the example: "that's not a polygon."
Establishing & Reasoning about Systematic Relations	Speaker establishes, considers, or reasons about relations between two or more general classes of objects or properties OR unpacks a definition of an object that is part of the definition of another object being defined (e.g., unpacking sides because it is part of the definition of polygon).	*[Example of reasoning about relations between two pairs of classes of objects: (a) circles and polygons and (b) circles and objects with sides. The relation being examined here is one of class inclusion.]* "A circle wouldn't be a polygon cause a circle doesn't have sides."
Constructing Definitional Explanations & Arguments	Speaker justifies a claim about a definition, example of a definition, qualities of an object being defined, or relations between two classes of objects.	"A circle wouldn't be a polygon **cause a circle doesn't have sides**."
Revising Definitions	Speaker adds properties to, eliminates properties from, or modifies elements of a definition. May also include re-assigning a definition to a new set (see example).	One student claims that a polygon "has the same angles and the same length of uh, same lengths of sides." Another student notes instead, "all regular polygons," suggesting that the definition is not relevant for polygons but instead for regular polygons.
Negotiating Criteria for Judging Adequacy or Acceptability of Definitions	Speaker negotiates with another speaker which features or roles of definitions should be used to determine whether a definition is adequate or acceptable.	One group defined a triangle as 3 sides, 3 angles and closed. A student said their definition needed to include, "straight sides." Two of the group members protested, and in doing so negotiated about the features of a definition. One argued, "But we already said sides" and the other followed, "That's the definition of sides."

Asking definitional questions. Defining involved *asking questions* about definitions or about the qualities, properties or relations of the objects being defined. The types of questions that the students and teacher asked varied. For instance, some questions simply requested the definition for an object (e.g., "What is a polygon?"). Other questions asked about the inclusion or exclusion of a particular case in relation to a class, such as asking whether a rectangle drawn on the board is a regular polygon: "Is it a regular polygon or isn't it?" In contrast, some questions asked about the existence or nature of *general* relations between classes of objects: "Circle IS a polygon?...Why can't a circle be a polygon?" Questions also probed into relations among the properties that constituted a single definition. Sometimes this was done by asking about the economy of the definition. That is, such questions asked about which properties of an object are sufficient (versus necessary) for inclusion in the definition: "[When we define polygon] do we need to say sides and angles or is it enough to say sides?" Other times, relations among the properties of a definition were questioned about extreme cases. For example, one student asked, "Can there be a polygon under two lines? Under three lines?"

Some questions promoted a focus on the clarity or preciseness of the language used in a definition. This is important because definitions historically serve a communicative purpose in mathematics communities (Lakatos, 1976) in that they contribute to a common, agreed upon language. For instance, at one point the teacher asked a question to encourage students to use the word "congruent" in place of "same:" "What's that word we use when we mean lay down on top of one another?" Note that in asking the question, he used the language of "we" to indicate that the word was an agreed upon, established term. Finally, although less frequent, questions also probed into epistemic issues, such as about what it means to participate in defining. For example, when one student noted that she did not need to include the property of "straight" in her definition of “triangle” because it was implicit in their definition of “sides,” the teacher asked, “And, can we assume that? Because we have done this?”

I coded a turn of talk as *asking a definitional question* when the speaker (a) used upward inflection to indicate a questioning tone (this would be denoted in the transcript by a “?”) and (b) asked one of the types of questions denoted above. I did not code questions that probed about a speaker’s thinking (e.g., “How were you thinking about that?”), asked for clarification (e.g., “What did you mean when you said….?”,) or asked for confirmation (e.g., the student says, “all the same sides” and the teacher asks, “All the same sides?”). Although these types of questions are important for collaborative work (Staples, 2007), they are not particular to the practice of defining.

Proposing definitions. Defining also involved *proposing definitions,* that is, proposing what to include in the definition of a mathematical object. Proposed definitions may be refuted and then possibly revised. The category of proposing definitions was not one of the original Aspects of Definitional Practice, but resembled the “conjecture” part of the aspect of *engaging in cycles of definitional conjecture, experiment and tests*. Such cycles are not represented by a single turn of talk, but, because students were often asked about definitions of objects, there were many utterances that resembled a “conjecture” or “claim.” I initially started to code for “definitional conjectures” but soon found that it was difficult to determine what constituted a conjecture (e.g., does any statement or opinion constitute a conjecture?). Thus, I created the code of *proposing definitions.*

I coded a turn of talk as *proposing definitions* when the utterance included a stated definition or part of a definition. In some cases, this was indicated because the turn of talk was in response to a question asking for a definition. For example, when the teacher initially asked what a polygon is, one student replied, “A polygon has the same angles and the same length of uh, same lengths of sides.” A turn of talk was also coded as *proposing definitions* even when a question is not posed. In these cases, the utterance usually included a declaration that a mathematical object “is” or “has” certain properties or relations. The properties or relations did not have to be conventionally correct in order to be coded as *proposing definitions*. For instance, one student proposed that “additionally, all polygons have five sides,” a statement that was quickly refuted by other students. Similarly, the properties or relations proposed sometimes attended to features that were not mathematically significant but instead described an object’s appearance or were cyclical in nature. For example, students first proposed definitions for straight that included “no zig-zags” and “straight is straight.” I wanted to include less mathematical proposals of definitions in order to capture changes in their proposed definitions. What students choose to include in a definition may reflect what they consider important or acceptable features to include.

Describing properties and/or relations. Often, members of the classroom community also *described properties and/or relations* of the examples. Description is central to definition construction in many ways, including when constructing a definitional argument, explaining a particular classification, evaluating an example of a definition, or writing a definition for an object. Although *proposing definitions* and *describing properties and/or relations* might seem similar, and often occur simultaneously, they occur separately as well. For instance, students sometimes proposed a definition that did not actually describe properties, like the definition “straight is straight” described earlier. This distinction allowed me to distinguish between proposed definitions that included properties and relations and those that did not. Moreover, sometimes properties were described in moments that were not immediately tied to proposing a definition, such as when describing properties of a particular example, such as a drawn rectangle.

I coded a turn of talk as *describing properties and/or relations* when the speaker stated a property or a relation with reference to a mathematical object. This included describing which properties or relations an object does *not* have. I considered a property to be linguistically a noun that describes a component of the object (e.g., “a polygon has *sides*”). In contrast, in the statement “[a quadrilateral] is a square,” I did not consider “square” to be a property because it is not a noun describing a part of a quadrilateral. I considered a relation to be a verb linking the object with its properties or properties with other properties (e.g., “a regular polygon *has* the same sides.”) or an adjective (e.g., “the sides are *straight*”). When referencing the mathematical object, the speaker did not have to explicitly mention the object as long as there existed evidence for the referent. For instance, a speaker might point to an object while describing its properties or may respond to a teacher’s question about a particular object (e.g., if the teacher asks, “how many sides does this have?” and the student responds, “four.”). The mathematical object can be concrete, such as a drawing of a rectangle, or an abstract class, such as the class of rectangles.

As with the *proposing definitions* category, descriptions of properties and/or relations did not need to be mathematically correct to be coded as *describing properties and/or relations*. For instance, the turn of talk, “A polygon has the same angles and the same length of uh, same lengths of sides,” was coded as *describing properties and/or relations* because although not all polygons have the same angles and same lengths of sides, he specified the properties “angles” and “sides” and the relation of “same.” Moreover, descriptions of properties or relations did not have to be mathematically precise, relevant or conventional to be coded. For example, utterances that used the word "same" instead of "congruent" to describe the sides and angles or those that used the word "point" instead of "vertex" were still coded.

Constructing and/or evaluating examples. Members of the class also *constructed and/or evaluated examples and/or non-examples* of the objects they were defining, where evaluation involved determining whether or not a case should be included as part of the set in question. Constructing and evaluating examples is significant to the practice of defining because it helps students consider what the class of objects being defined should include and provides a set of objects to describe.

I coded a turn of talk as *constructing and/or evaluating examples* when the speaker (a) constructed a case of a mathematical object, either by drawing it, gesturing it, or using physical materials to build it or (b) voiced a claim regarding the exclusion or inclusion of a particular case into a general class. For instance, when the students considered the definition of regular polygon, the teacher drew a rectangle on the board and asked them if that figure was regular. Students' responses were evaluations of the case, ranging from "yes," "no," to "it is not a regular polygon." In this situation, the teacher's turn of talk in which he drew the rectangle as well as each of the students' separate turns of talk were coded as *constructing and/or evaluating examples*. This code was only assigned when the utterance was about a *particular* case. Instances when a speaker relates two general classes, such as "a square is a regular polygon," were instead assigned the code *establishing and reasoning about systematic relations*, described next. As with earlier categories, evaluations or constructions of examples did not have to be accurate or conventional to be coded as such.

Establishing and reasoning about systematic relations. The students and the teacher also *established and reasoned about systematic relations.* This occurred in two ways. First, members noted or described a relation between two or more general classes of mathematical objects or properties. For example, one student noted that "a circle wouldn't be a polygon cause a circle doesn't have sides." In this statement, the student related the general class of "polygons" to the general class of "circles," and in her explanation related the class of "circles" to the class of objects without sides. Second, members sometimes unpacked a definition that was part of the definition of another object being defined. For instance, when the students were defining "polygon," a question arose of what a "side" is. This, in turn, led to the defining of "side." Because "side" was part of the definition being discussed, defining it led to unpacking implicit relations between the definition of side and the definition of polygon.

To be coded as *establishing and reasoning about systematic relations*, the turn of talk either (a) needed to include a statement relating two general classes of object, linguistically connected with a "be" verb (e.g., "a square's a polygon"), (b) needed to be in response to an inquiry about a class relation (e.g., the teacher asks if a circle is a polygon and the students respond "no"), or (c) contain a proposal of a definition of an object or property that is part of a definition currently being discussed. Statements about relations between particular cases and classes or between versions of descriptive language (e.g., "points" versus "corners" versus "vertices") were not coded in this category. Again, as with earlier codes, relations did not have to be correct or conventional to be coded in this category (e.g., "all shapes are polygons except for the squares and quadrilateral").

Constructing definitional explanations and arguments. This category refers to turns of talk in which members *constructed definitional explanations and arguments* related to a definition or an example of the object being defined. For instance, one student explained why circles should not be included within the class of polygons: "A circle wouldn't be a polygon cause a circle doesn't have sides." Definitional arguments and explanations varied in the extent to which they attended to the definition. That is, some used the definition as part of their argument, whereas others less so. For instance, in the previous example, the student justified her claim that circles are not polygons by appealing to a property within the definition of polygons, "sides." In contrast, she could have justified her claim by describing the appearance of circles (e.g., they are too curvy) or by appealing to properties or relations not yet agreed upon as part of the definition. This distinction will be highlighted further in the results.

A turn of talk was coded as *constructing definitional explanations and arguments* when a member of the class justified a claim about a) inclusion or exclusion of a definition or part of a definition in relation to a class (e.g., when one student claims that "all polygons have five sides," another student disagrees and argues that "because um if all polygons have five sides, but we also had the square was a polygon and the triangle was a polygon…and they've only got three and four [sides]."), b) inclusion or exclusion of an example or class in relation to a class (e.g., "a circle wouldn't be a polygon cause a circle doesn't have sides"), or c) whether or not conditions in a definition are economical (e.g., the definition of triangle, "three sides and closed" does not need to include "angles" because "won't it come with angles?"). Justifications were often indicated by the use of causal language, such as "so," "then," or "because." A claim or stance without any justification (e.g., "I agree") was not coded as constructing definitional arguments and explanations. Often such claims were coded as *constructing and/or evaluating examples* or e*stablishing or reasoning about systematic relations*.

Revising definitions. Members of the class also *revised definitions*. Revising definitions involved changing proposed definitions, for instance, by adding properties or relations to, eliminating properties or relations from, or modifying elements of a definition. Additionally, at times a definition was not changed but was reassigned to a new object. For example, when one student proposed that a polygon "has the same angles and the same length of uh, same lengths of sides," another student suggested that instead the definition is true for "regular polygons." Definitions were revised for many reasons, including (a) to improve the definition's clarity, (b) to eliminate a case from the class (usually presented as a potential example), (c) to include a case in the class, (d) to make the definition more economical, (e) to make the definition more accurate or (f) to include mathematically agreed upon terminology.

I coded a turn of talk as *revising definitions* when it included a proposal for changing a definition in one of the ways described above. In order to determine whether an utterance was a change from a previous definition, there had to be a previous, different version of the definition stated by a member of the class in the same Definitional Episode. For example, at one point, the teacher revoiced the class's definition for polygon, "I want to know what makes something a polygon. I know it has sides and it has angles. SO, this then is a polygon, right?" He then proceeded to draw three connected sides, roughly forming a "Z." One student, in protest, offered a revision that involved adding new properties to their definition: "It has to be CONNECTED." Because all revisions were essentially proposals of definitions, if an utterance was already coded as *revising definitions*, then I did not code it as *proposing definitions*. An exception to this would be if the turn of talk were especially long and contained a separate proposed definition.

Negotiating criteria for judging adequacy or acceptability of definitions. On a few occasions, members of the class *negotiated criteria for judging adequacy or acceptability of definitions*. That is, they negotiated which features or roles of definitions should be used to determine whether a definition is adequate or acceptable. In order to be coded as *negotiating criteria for judging adequacy or acceptability of definitions*, two or more members of the class had to engage in explicit dialogue about the criteria. In other words, I did not code turns of talk in which one member made a statement about what he or she thought about criteria unless another member responded. In these cases, each turn of talk was given a code. For example, when a group of students presented their definition for triangle as “three sides, three angles and closed,” another student responded that they needed to include “straight sides.” In response, the students countered that they did not need to do so because “straight” is implicit in their definition of “sides.” One argued, “but we already said sides,” and another followed with “that’s the definition of sides.” The teacher then picked this up and revoiced the students’ argument. Although implicit, in this excerpt, the students negotiated the rule that once a definition is agreed upon, it does not have to be articulated in another definition, a rule that had been discussed earlier in the class.

Orchestrating definitional discussions. In addition to coding forms of definitional practice, I characterized other ways that members of the class participated in talk around mathematical defining. My goal in denoting these forms of participation was to see if and how particular moves potentially supported interactions around defining. Although most of these moves were conducted by the teacher, I did on occasion notice students participating in them as well. I noted six forms of orchestration particular to defining: (a) using meta-talk to communicate about practice, (b) requesting participation in Aspects of Practice, (c) positioning a student utterance as participating in an Aspect of Practice, (d) encouraging precise language or agreed upon terms, (e) positioning definition or Aspects of Defining at the forefront, and (f) modeling participation in Aspects of Practice. Members also participated in more general discourse moves, for instance, negotiating the social norms for participating in the class (Yackel & Cobb, 1996). However, because these forms of interaction were less frequent and because my focus was on understanding interactions particular to the practice of defining, I did not include them in my analysis. The codes are summarized in Table 3, and I elaborate on each of them in the following sections.

Table 3. Coding scheme for *Orchestrating Definitional Discussions*

Orchestration Move	**Description**	**Examples**
Using Meta-Talk about Practice	States explicitly or implicitly expectations for one of the following: (a) the purpose of defining, that is, why one would engage in defining, (b) what features a definition should have or the functions it should serve, or (c) the rules for participation in defining.	"Remember the goal is that I need to be able to tell the difference between a polygon and a carrot… Carrots. Circles. Anything else. Anything that you don't want to call a polygon, I have to be able to look at your definition and say oh thank you. Now I know."
Requesting Participation in Aspect of Practice	Requests participation in an aspect of practice, either through a direct statement or via a question.	*[In this example, the teacher requests that the student participate in proposing definitions.]* "What's the definition of a regular polygon again? Rhonda?"
Positioning a Student as Participating in an Aspect of Practice	Revoices an utterance while at the same time describing the utterance in terms of an aspect of practice.	When asked for a definition of polygon, one student said, "A polygon has the same angles and the same length of uh, same lengths of sides." Teacher then positioned the student utterance participating in *proposing definitions*: "Vern's **claim** is that all polygons have the same length of sides and the same angles."
Encouraging Precise Language or Agreed Upon Terms	Encourages use of precise language or agreed upon terms in one of the following ways: (a) Revoices an utterance, inserting more mathematically precise language, (b) Adds verbal, gestural or written stress to highlight a particular mathematical term, (c) Suggests an alternate word to use that is more mathematically precise and/or aligned with agreed upon terms, OR (d) Solicits mathematical language that has been previously discussed.	When asked about regular, students said they had to have same sides, same angles. Another student then added "all the sides are congruent." The teacher highlighted this agreed upon term: "All the sides are congruent. THANK YOU Ted… Okay, **that math word says it all**. All the sides are congruent. All the angles are congruent. Yeah, good."
Positioning Definition or Aspects of Defining at the Forefront	Highlights the definition or an aspect of definitional practice via talk, gesture or inscription.	*[In this example, the teacher requests that students participate in keeping track of the definition.]* "Does everyone have this definition (taps on the board) in their math notebook?.. Well I think you better put it in there…"

Table 3, continued

Orchestration Move	**Description**	**Examples**
Modeling Participation in Aspects of Practice	Participates in an aspect of practice, and in doing so, models participation.	[*See table of Engaging in Aspects of Practice for example*]

Using meta-talk to communicate about practice. The teacher in particular often used *meta-talk to communicate about practice*. That is, he explicitly or implicitly stated messages about or expectations for engaging in the practice of defining. Sometimes, these messages were about the purpose or the goal of defining, that is, why one would engage in defining. For instance, when he first asked students to define "polygon," he communicated to students that the purpose of constructing the definition was to help him distinguish between two objects, "Okay give me the most general definition you can. So that I can recognize a polygon and **I could tell the difference** between a polygon and a turnip." The teacher's messages sometimes also communicated what features should be included in a mathematical definition. For instance, in the example just provided, the teacher had noted that the definition should be "general," suggesting that definitions should communicate properties for all cases of the class. At other times, the teacher communicated rules for participating in defining. However, rather than making a list of declarative statements, he situated these rules within the class's defining activity. Rules included: (a) if you want to rule an object out from a set, you must provide justification for doing so, (b) when constructing a definitional justification, you should appeal to the definition, (c) if a decision to include or exclude an object from a class is contrary to the definition, then the definition should be revised, (d) when defining, it is important to write in order to keep track of agreed upon definitions, (e) when we define, we first need to know the definition of an object in order to discuss which objects are not members of the class, (f) when we define, we use our minds, not dictionaries, (g) when we define, we do not guess, and (h) once we agree on a definition, then we need to stick with it. To illustrate the last rule, the students had just re-visited the definition of side. Once they determined that sides must be straight, the teacher noted, "alright so a side, if we're going to agree. Now Ned, **once we say this, then this is what we mean**."

I coded a turn of talk as *using meta-talk to communicate about practice* when it communicated a message about the practice of defining or about definitions in one of the ways articulated above. Sometimes, the teacher did so by communicating his thoughts out loud in the manner of a soliloquy, indicated linguistically with the pronoun "I." For instance, when articulating to the students that their definition of polygon needed more in order to distinguish objects, he said, "I have to be able to look at your definition and say oh thank you. **Now I know…so far I can't**. The only thing **I know** is that there are some polygons that are regular and they have equal sides and equal angles. So now **I know** what a regular polygon is and **I'm** very happy." Other times expectations were phrased as directives that the teacher either requested that the students do (marked by pronoun "you"), requested that they as a group do (marked by the pronoun "we"), or requested that that the students direct to him (marked with a combination of "I" and "you"). For example, when one student countered another student's proposed definition with a counter-example, the teacher followed with, "Okay, so as soon as **we** find something that **we'd** like to call a polygon that has other than 5 sides, **we KILL that conjecture**." Note in this example, the teacher also positioned the directive as a hypothetical using the phrase "as soon as we," suggesting it as a rule for the future. Other times he linked a message to a piece of talk about communicating in practice with a linking word, such as "so" or "because." For instance, when one student introduced a new object, "quadrilateral" to the discussion, the teacher requested, “Now someone will tell me what the heck a quadrilateral is **cause** I hadn’t heard that word yet.” Here, the teacher requested that students *propose a definition* and used the word “cause” to suggest that a new word implies the need to for doing so. Other times, the teacher implicitly articulated important features of a definition by highlighting or coding them (Goodwin, 1994). For example, when looking at different groups’ definitions for “triangle,” he “coded” them describing their varying degrees of “sparseness,” a way of indicating their minimality: “So, this is like, this (points to a definition). Very slim. I would call this one (points to another definition) somewhat slim…This (points to another) is an expanded one.”

Requesting participation in Aspects of Practice. At times, members requested that students participate in Aspects of Practice. For instance, the teacher requested that students *propose definitions* by asking them about the definition of an object: “What is a polygon?” Other times, the teacher requested that they *construct or evaluate an example*. For instance, when the class was trying to construct a definition of “regular polygon,” the teacher drew a rectangle on the board and asked, “Is that a regular polygon?” The teacher also requested students to participate in *constructing definitional explanations or arguments*, sometimes in response to other students’ utterances. For instance, when one student had presented an example to illustrate his stance that a polygon could have sides but no angles, the teacher asked, “Does anyone have a **counter-argument** for Ned?...Can you **argue** with Ned? Do you, do you agree with Ned or not?” I coded turns of talk as requesting participation in Aspects of Practice if it (a) was phrased as a command or a question and (b) the response assumed by the request was something that I could identify as an Aspect of Definitional Practice, even if the actual response was not one.

Positioning a student as participating in an aspect of practice. The teacher also often *positioned a student as participating in an aspect of practice*. For instance, when the teacher asked for the definition of polygon, Vern offered the definition, "same sides and same angles." The teacher then revoiced Vern's utterance, calling it a "claim" and thereby positioned him as participating in the aspect of *proposing definitions*: "Vern's **claim** is that all polygons have the same length of sides and the same angles." In response, another student, Rachel, then offered the following suggestion: "All regular polygons." The teacher revoiced Rachel's utterance, and by calling the utterance "her amendment," positioned her as participating in the aspect of practice of *revising definitions*: "All regular polygons (points at Rachel and looks at Vern). Do you accept her **amendment**?"

I coded a turn of talk as *positioning a student as participating in an aspect of practice* when it took one of two forms. First, the teacher revoiced student utterances (O'Connor & Michaels, 1996) in ways that re-described student participation in terms of Aspects of Definitional Practice, as illustrated in the examples above. Linguistically, this was often indicated with a restatement of the student's utterance, an attribution of authorship to the student, and a statement or descriptor related to an Aspect of Practice (e.g., "claim" or "amendment" as bolded in the examples). In the examples above, the teacher did not change the students' utterance, but rather labeled them, or as Goodwin (1994) might say, coded them in terms of Aspects of Definitional Practice. Second, in a few instances, when students mentioned a new object, the teacher sometimes changed the syntax of the utterance from a statement into a *definitional question*. For instance, at one point, a student called out "irregular polygon." The teacher then changed the student's utterance into the question, "Can polygons be irregular?" In doing so, the teacher re-positioned the student's seemingly unrelated contribution as participating in an aspect of practice. Note that at the same time, he also sent a subtle message that we must define relations before using them. This illustrates how utterances were often related to multiple codes.

Encouraging precise language or agreed upon terms. The teacher, and sometimes students, also *encouraged precise language or agreed upon terms*. This move is important because it shows what language the member highlights and values and thus contributes to a shared mathematical language. Members encouraged precise language and agreed upon terms in a number of ways. Sometimes, they revoiced student utterances, inserting more mathematically precise language. For instance, when one student defined a "regular polygon" as having "all angles are the same," the teacher followed with, "and all angles are the same, are **congruent**." Although others have described such moves to be important generally in math class, I include them here because it appears especially relevant to the construction of mathematical definitions, both in relation to what terms are used within the definition, but also what terms are privileged for those objects being defined. Other times, members emphasized mathematical language by adding verbal, gestural or written stress to highlight a particular mathematical term. For instance, they emphasized terms (e.g., "REGULAR"), repeated words multiple times (e.g., "oh. This is for a kind of polygon called **regular**. **Regular**. It's a **regular** polygon."), wrote them on the board, or underlined terms already written. The teacher in particular also suggested alternative language that was either more mathematically precise or aligned with agreed upon terms. For instance, when the students had used the word "size" to define "regular polygon," the teacher asked, "Okay, can we use the word length?" Other times, he solicited previously agreed upon language from the students. For instance, when the students used the word "same" to define regular, the teacher asked, "what's that word we use when we mean lay down on top of one another? (shows with markers. One student says “congruent.”) All sides are congruent. Cause that's what we mean by equal here. They're the same length.”

Positioning definition or aspects of defining at the forefront. The teacher, and sometimes students, also highlighted definitions or Aspects of Definitional Practice via talk, gesture or inscription. Whereas in the previous orchestration move, *encouraging precise language or agreed upon terms*, members highlighted particular *words*, here they highlighted the entire definition or parts of the definition. Sometimes the teacher positioned definitions by writing them on the board. Often, in doing so, he simultaneously positioned students’ participation in Aspects of Practice as important. For instance, when a student, Lavona, *proposed the definition* for polygon, “I think all shapes are polygons except for squares and quadrilaterals,” the teacher wrote her definition on the board and annotated it with an “L” to attribute authorship to Lavona. By writing this definition, he not only made it accessible to all the students, but he also highlighted it as something worth talking about. Similarly, when students *revised definitions*, the teacher often marked those changes on the board. Other times, the teacher would ask the students to write definitions in their notebooks, usually suggesting that it was important for keeping track. For example, when the students had constructed a definition for “regular polygon” in attempts to define “polygon,” the teacher stopped and asked, “Does everyone have this definition (taps the definition written for “regular” on the board) in their math notebook?... Well I think you better put it in there, cause we have to get a definition for polygon, and so far, WE don’t have one.”

Other times, members positioned a definition at the forefront by verbally using it when engaging in Aspects of Definitional Practice. In these cases, the teacher in particular sometimes revoiced a definition and then related it to engagement in an Aspect of Practice, often using a subordinate conjunction such as “so” or “if.” For instance, when the students had “sides” and “angles” as their definition of polygon, the teacher said, “I want to know what makes something a polygon. I know **it has sides and it has angles** SO, this then is a polygon right?” He then *constructed an example* using the students’ definition – three connected but not closed lines. Here, the teacher positioned the students’ definition at the forefront by stating it and then relating it with the conjunction “SO” to his *construction of the example*. In a later example, the teacher *asked a definitional question* in relation to the class’s definition: “okay so now I’m beginning to get an ideas that **a polygon that is something that has sides, angles and is connected**. That is it’s closed. Okay, **if we take this definition**, can there be a polygon with two sides?” Here, the teacher again revoiced the class’s definition and then related it to his question with the conjunction “if” and the statement “take this definition.”

Modeling participation in Aspects of Practice. The teacher and students also frequently participated in Aspects of Definitional Practice and, in doing so, *modeled participation in Aspects of Practice*. Anytime an utterance was coded as participating in an Aspect of Practice, I also coded it as *modeling participation in Aspects of Practice*. Although this might seem redundant, by assigning it a code, I marked its importance analytically.

Characterizing the co-development of knowledge and practice. Once complete, I compared the analyses of mathematical knowledge and the interactions around practice side-by-side in order to develop conjectures of how interactions around definitional practice contributed to the development of communal knowledge and how the knowledge developed, in turn, informed participation in practice. To do so, I first looked at points in my analysis of knowledge development when (a) an object, relation or property was added to knowledge system representation or (b) an object, relation or property was revisited. I compared these instances to what was happening at the same moment with respect to members’ participation in practice.

Likewise, I identified points of shift in practice and compare these instances to my analyses of knowledge development. I termed these moments in which knowledge and practice informed one another points of *contact between knowledge and practice*. Finally, I looked to see *who* was contributing to the creation of points of contact in order to identify the roles of the students and teacher and whether those roles shifted over time. My main goal was mostly to identify differences in the teacher versus the students, but I noted some differences that existed between students, especially between those who had been in the class the previous year versus those who had not. I checked my conjectures generated from my four excerpts with other Definitional Episodes in order to generate confirming or disconfirming evidence.

Results

Overview of Results

Multiple interactions contributed to the co-development of communal knowledge and defining, illustrated in Figure 2. In what follows, I provide a broad overview of these interactions and the ways in which knowledge and practice changed over time and how different members contributed to those changes. I then illustrate nuances of the interactions and changes by describing the sampled excerpts from the first, fourth, sixth and 26th days of math instruction. I conclude by illustrating additional interactions that occurred between Day 6 and Day 26 in order to provide further confirming evidence for the changes and suggest possible continued forms of support for students' development.

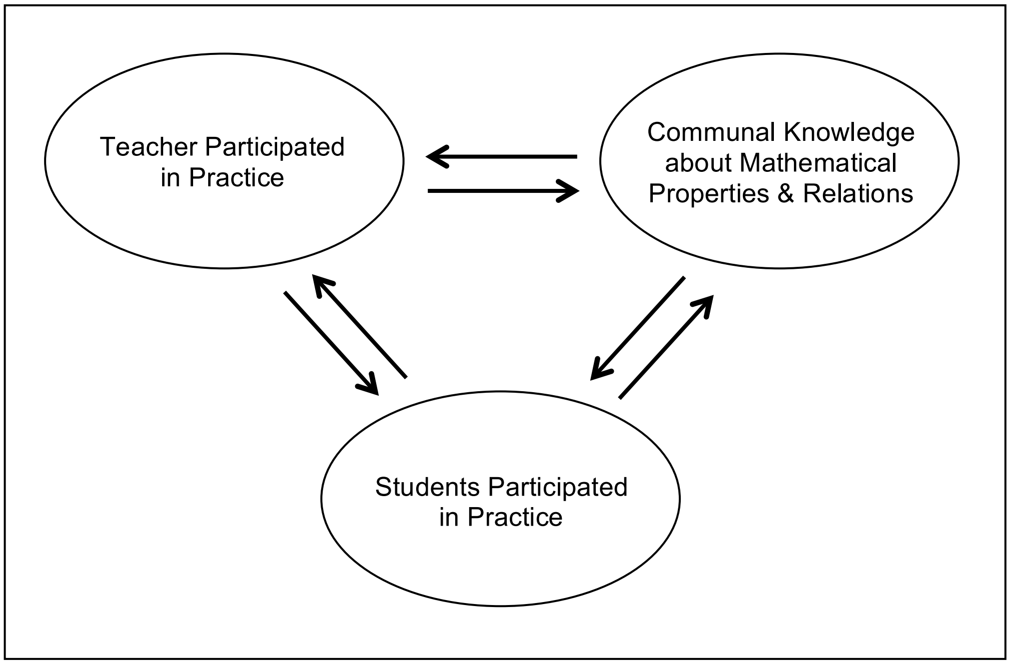

Figure 2. Interactions contributing to the co-development of the practice of defining and communal knowledge.

Throughout the course of instruction, the teacher participated in Aspects of Definitional Practice and in forms of Orchestrating Definitional Discussions that made contact with mathematical practices and in ways that supported the development of communal knowledge. Initially, he achieved these forms of contact by asking definitional questions about properties of the object being defined ("What is a polygon?") or about class relations ("Is a circle a polygon"). At the same time, by participating in this and related Aspects of Practice, the teacher supported *students'* participation in aspects of the practice of defining. Asking general questions positioned students to *propose definitions* and *reason about systematic relations.* He further positioned students as definers by labeling their contributions, thus attributing agency, *engaging in meta-talk about practice*, and *encouraging precise language and agreed upon terms*. These forms of support allowed the students immediate access to the practice, a form of scaffolded participation.

At the same time, through these interactions, he continuously modeled participation in defining, and, in particular, ways of participating that made contact with knowledge development.

The initial forms of interaction provided a space for students to present their ideas about polygons and related properties. These ideas helped inform the teacher's next moves in practice, illustrating how the development of knowledge, in turn, informed practice. For example, in order to problematize a feature of a student's definition, he often proposed an example that provoked contest. By doing so, he participated in the Aspects of Practice of *constructing and evaluating examples* and *constructing definitional arguments* in ways that again made contact with the mathematics and prompted students to further expand the system of mathematics objects and relations they were exploring.

In later classes, students began to appropriate participation in the Aspects of Practice the teacher had been modeling, and, in particular, the ways in which he made contact with knowledge. Through their initial explorations of the mathematical properties and relations, definitions and examples of objects stabilized and served as resources for students' participation in practice. For instance, they now were able to use definitions as sources of justification when *constructing definitional arguments*. As students participated in these ways, they in turn modeled participation for their peers. Moreover, although discussions were mainly orchestrated by the teacher, there were a few instances in later classes, especially the 26th class, in which students appropriated these forms of orchestration.

In the excerpts that follow, I bold pieces of transcript to highlight how members participated in Aspects of Definitional Practice, the ways in which they Orchestrated Definitional Discussion, and the ways in which practice and knowledge made contact. Italics are used to indicate the forms of interaction described.

Excerpt 1: Initial Forms of Practice and Knowledge

The first excerpt, from the beginning of the first day of instruction, illustrates how the teacher initially scaffolded students' participation in practice while also supporting the development of communal knowledge. This excerpt began with the teacher, Dr. Rich, asking students for their definition of polygon. As he *asked this definitional question*, he also engaged in *meta-talk* about features of definitions, that they should be "general," and the functions they should play, that is, they should allow one to distinguish between objects: "**What is a polygon?**...Okay give me the most **general** definition you can. So that I can **recognize** a polygon and I could **tell the difference** between a polygon and a turnip." At the same time, by posing a question and then following with a command ("give me") he also *requested* that students participate in the Aspect of Practice of *proposing definitions*. This orchestration move opened up the floor to student participation. Vern responded by proposing that "a polygon has the same angles and the same length of uh, same lengths of sides." The teacher then revoiced Vern's proposal, labeling it as a "claim" and allowed another student to respond.

T: Vern's claim is that all polygons have the same length of sides and the same angles. Rachel.
R: All regular polygons.
T: All regular polygons (pointing at Rachel and looking at Vern) Do you accept her amendment?
V: yeah.
T: All REGULAR polygons

In the above exchange, by revoicing Vern's statement and labeling it as a "claim," the teacher *positioned* him as participating in the Aspect of Practice of *proposing definitions*. Similarly, the teacher also revoiced Rachel's contribution and labeled it as an "amendment." This move again served to *position* her as participating in practice, albeit this time in the aspect of *revising definitions*. Moreover, the label of "claim" also suggested a non-permanent status, and perhaps aided in inviting others to refute it. He then used verbal stress to emphasize the newly agreed upon term ("REGULAR").

In this example, the teacher *posed a definitional question* about the properties that constituted the object and thus *prompted elaboration of system components*. By doing so, potential properties of "polygon" were introduced, including "same sides" and "same angles." The rebuttal by another student introduced yet another object to the class, "regular polygon," and the qualifier of "regular" implicitly implied that this new class was related to polygons in some way. Although in this case, the student volunteered the rebuttal, at times, the teacher encouraged students to do so (e.g. "does anybody have a counter-argument?").

Students continued to present their ideas about polygons, both by *proposing potential definitions* as well as suggesting *systematic relations* between polygons and other classes of objects. One student, Kira suggested that "all polygons have 5 sides." This idea was immediately revoked by several students and another student, Kate, offered the *definitional counter-argument* that "if all polygons have five sides but we also had the square was a polygon and the triangle was a polygon and they've only got 3 or 4." The teacher then used Kate's argument as an opportunity to *articulate rules* for participating in defining: "okay so as soon as we find something that we'd like to call a polygon that has other than five sides, we KILL that conjecture." In this statement, he also *positioned* Kira's contribution as participating in the Aspect of Practice of *proposing definitions* by labeling it as a "conjecture." Other students suggested types of polygons: decagon, septagon, octagon, hectagon, hexagon and pentagon.

After hearing her classmates suggest relations, one student, Lavona, *proposed a new definition*: "I think all shapes are polygons except for…uh a quadrilateral." The teacher once again *positioned* the contribution as participating in *proposing definitions* by revoicing it and labeling it as a "conjecture." He further highlighted its significance by writing it on the board, thus *positioning the definition at the fore*. As he did so, he remarked that he was writing "so I can keep track." The teacher then once again took the opportunity to *request that students propose definitions* in a way that, similar to his earlier question, served to elaborate on system components: "Now someone will **tell me what the heck a quadrilateral is**, cause I hadn't heard that word yet." As before, the teacher's request turned the floor to the students and provided them an opportunity to voice their ideas. Students suggested various relations. More than one student proposed that a quadrilateral was the same thing as a square. At the same time, other students appeared to reject Lavona's proposed definition. Some argued that "a square is a polygon" whereas one group of girls, Mona, Kate, and Adeena, called out that "a circle wouldn't be a polygon cause a circle doesn't have sides." In *constructing this definitional argument*, the girls introduced a new object, "circle," to the discussion and *reasoned about systematic relations* between it and polygon, suggesting that it did not have sides, and as Adeena added "or angles."

Rather than accepting the girls' proposition, the teacher revoiced their comment as a question, thus *positioning* their contribution as participating in the Aspect of Practice of *asking definitional questions*: "Okay so QUESTION. Circle is? A polygon?" In doing so, he emphasized the word "question" and wrote it on the board, further highlighting its importance. This verbal and written positioning opened the conversation up to other students, who also unanimously rejected the relation ("NO::"). The teacher then *asked another definitional question* in order to request that students participate in the Aspect of Practice of *constructing a definitional argument*. At the same time, he also articulated the need to provide justification when ruling out one class from another: "No::? No. Alright. Well I like circles. So **if I'm going**

to rule circles OUT from polygons, why can't a circle be a polygon?" Students all at once yelled out similar arguments, claiming that circles did not have sides or angles, and, in doing so, *described properties* of the class.

The teacher calmed the students down and returned the discussion to their original goal of understanding what made something a polygon. He reminded them that until now, they had only established what made something regular. In the remainder of the excerpt, the students further discussed the notion of regularity and the related idea of congruency. To consider regularity, the students considered whether a square and rectangle drawn by the teacher were regular and why. Once they had established that regular was defined as "same sides" and "same angles," the teacher prompted the students to remember another word for "same" that they had learned last year. Through this discussion, "congruent" was introduced as a way of describing the nature of "sameness," that they could lay one on top of the other and completely overlap. The teacher continued to *encourage this agreed upon term* throughout the class period whenever they returned to the definition of regular. He did this by revoicing students' descriptions of "same" sides or angles as "congruent" (e.g., "can I use these words?...all the sides are congruent? All the angles are congruent?") and writing the definition on the board to refer to throughout the class. In fact, in a couple of instances, students began using the word themselves.

Recap of excerpt 1. Although this was students' first entrée into defining as a group, they participated in almost all Aspects of Definitional Practice. They *proposed potential definitions*, *reasoned about systematic relations* between classes, *revised definitions*, *evaluated potential examples*, *asked definitional questions*, *constructed definitional arguments or explanations*, *described properties* of examples or classes of objects, and *negotiated criteria for judging adequacy and acceptability of definitions*

The teacher played an important role in scaffolding this participation. Perhaps most prominent, he frequently *requested that students participate in various Aspects of Practice*, by posing a question, by revoicing a student statement as a question, or by directly requesting. Some students, especially those who had been in the class the year before, more readily participated on their own. Kate, Mona, Adeena, Lavona, Rachel and Kira had all been members of the class previously, and played an important role in volunteering proposed definitions and providing counter-arguments to others' proposed definitions. Nonetheless, it was the teacher's questions that invited students' initial proposals for definitions. Moreover, by revoicing student comments as questions or conjectures, and highlighting these contributions by writing them on the board, he positioned the definitions to the forefront and made them visible and accessible to other students. Questions also served to invite other students to contribute, and often these invitations were met with great energy and enthusiasm, indicated by the large amount of overlapping talk. Additionally, by *positioning* and labeling students' contributions, he further emphasized their importance and their role in their collective endeavor of constructing definitions. This served as an important form of supporting students in becoming "definers."

Many of the *definitional questions* posed or revoiced by the teacher probed about the properties constituting an object or particular class relations. Thus, he participated in this Aspect of Practice in a way that encouraged *elaborating system components*. Within the first class period, the teacher asked or revoiced such questions about "polygon," "quadrilateral," "circle," "angle," "degree," and "straight." In doing so, he modeled participation not only in this Aspect of Practice, but, more importantly in doing so in a way that made contact with the mathematics students were exploring. The teacher often further highlighted the importance of new objects by writing the names of the objects on the board, *emphasizing agreed upon terms*. As a result of the teacher's questions, new objects, properties and relations were introduced and discussed. And, although the teacher initiated these conversations, he gave the students opportunities to contribute, allowing them agency in the process. Students proposed potential properties of polygon, including "same sides," "same angles," "angles," as well as related classes of objects, "regular," "octagon," "circle," "square," etc. "Regular" was also defined as "same angles" and "same sides" and later as "congruent sides" and "congruent angles." Students also discussed whether a rectangle the teacher had drawn on the board should count as regular, and in doing so, again revisited its properties. The objects, properties and relations described within approximately the first 10 minutes of class are shown in Figure 3. In the figure, ovals represent objects or properties, sold lines represent sub-class relations, dashed lines represent class relations, and arrows represent inclusive relations between an object and the properties that possibly constitute it. Words or numbers on the edges describe the nature of the relation. The shading illustrates the frequency with which objects were discussed, with darker shading indicating they were mentioned in more Definitional Episodes.

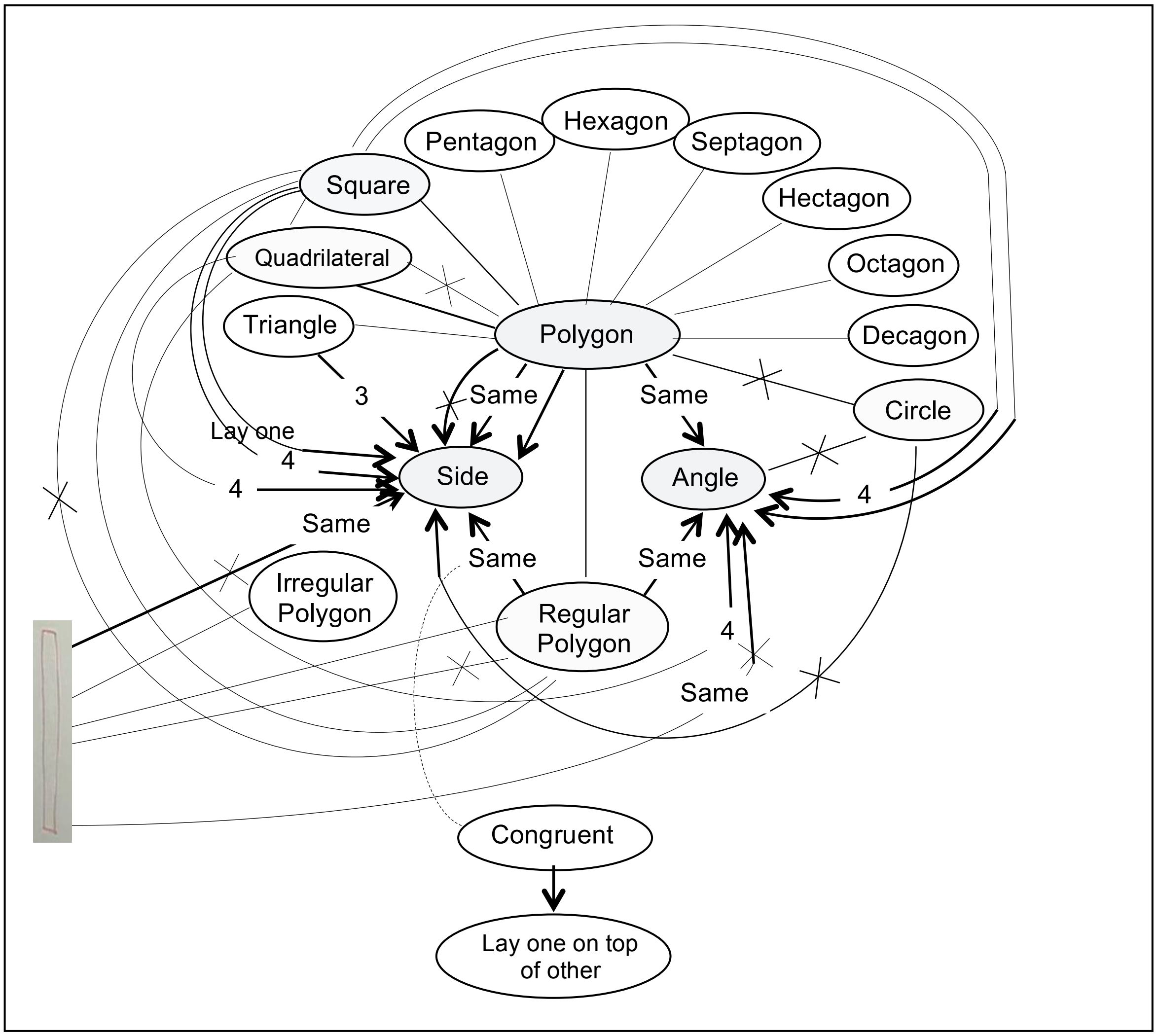

Figure 3. Knowledge development during the beginning of Day 1. Ovals represent objects or properties mentioned by members of the class during the first 10 minutes of class discussion. Solid lines represent sub-class relations, dashed lines represent class relations, and arrows represent inclusive relations between an object and the properties that possibly constitute it. Words or numbers on the edges describe the nature of the relation. The shading illustrates the frequency with which objects were discussed, with darker shading indicting they were mentioned in more Definitional Episodes.

Excerpt 2: Students Take on Authority for Expanding the Mathematical System

Students spent much of the remainder of Day 1 and Days 2 and 3 investigating angles and degrees in greater depth and learning how to use protractors as a tool for reasoning about angles. Their inquiries about polygons resurfaced on the fourth day of instruction. Although many of the ideas were the same as on the first day, as new needs of the community arose, students and the teacher began to take on new roles. Students began to appropriate some of the forms of interaction the teacher had modeled earlier and he began to model additional forms of interaction that encouraged introduction of new mathematical properties and relations.

Towards the beginning of the fourth day, the teacher elicited students' questions. When a student, Kira, asked, "Can there be a polygon under three lines?" the teacher returned to their previous conversation about defining polygon: "…I still don't know what you mean by polygon, I STILL if I went to Mars and read your ideas about polygon, I might think it's a bottle." As on Day 1, this question again served to elaborate on system components by opening up the floor to students to discuss their ideas about "polygon." Moreover, the teacher made the choice to redirect discussion to defining "polygon" instead of having students respond to Kira's question, sending the implicit message that they needed to first establish what a polygon was before asking questions about it. He asked Rachel to respond and as she stated her definition, he wrote it on the board: "A polygon is a…something that has all the same sides. Has the same sides and the same angles." Although the definition was not correct, he still used this as an opportunity to emphasize agreed upon terms by eliciting the previously discussed property of "congruent."

T: (writing Rachel's definition) All sides
R: Are the same.
T: Are the-what's the word we use when we mean lay down on top of one another? (lays one marker on top of the other to demonstrate).
J: Congruent. Congruent.
T: All sides are congruent. Cause that's what we mean by equal here. They're the same length.
R: And all angles are the same.
T: And all angles are the same. (continues writing) Are congruent. Okay, we could pick one up and stick it on the other.

In this exchange, the teacher supported the students' participation in defining in several ways. By writing Rachel's definition on the board, he acknowledged her contribution as important and *positioned the definition at the forefront*, making it accessible to students. Moreover, by asking students about “the word we use when we mean lay down,” he allowed students to reason about the relation and further contribute to the construction of the definition. When Rachel again used the word “same” to describe the angles, the teacher revoiced her definition, inserting the word “congruent,” further *emphasizing the use of agreed upon and precise language*.

The teacher then opened up the floor for other contributions: “Okay so:: Is that it. Is that all we need?” Here, he *requested further contributions* towards a proposed definition while also *asking a question that encouraged the elaboration of systematic relations*. Kate raised her hand and stated that she disagreed and “that’s for a regular polygon.” In doing so, Kate participated in the Aspect of Practice of *revising definitions* while also reminding her classmates of an object and relationship they had discussed earlier. The teacher then revoiced Kate’s suggestion, repeating “regular” multiple times, as if to emphasize its importance. He also modified the definition on the board, *positioning the revised definition to the forefront*. Another student, Jomerd, then called out two different contributions that the teacher once again revoiced as questions. Note “Ss” refers to multiple students below.

T: Oh. This is for a kind of polygon called REGULAR. (edits the definition on the board) Regular. It’s a regular polygon. Alright well.
J: Irregular polygon.
T: If you – Can polygons be irregular?
Ss: Yes.
J: A circle. A circle.
T: Is a circle a polygon?
Ss: No::
T: Well. Question. (writes on the board, “Is a circle a polygon?”)

In the above interaction, the teacher again revoiced contributions into *definitional questions*, opening them up for conversation by others in the class and directing discussion to consider a new object, “irregular polygon,” and re-consider the relation between the classes, “circle” and "polygon." Students again disagreed with the notion that a circle could be a polygon. Jomerd argued that it had zero sides whereas Vern suggested that it might have one.

The teacher then redirected the discussion back to their original goal of defining polygon, noting that "we have to get a definition for a polygon, and so far, WE don't have one…What are we gonna do?" Lavona suggested that they "list why we think a circle's not a polygon," essentially attempting to *negotiate* their rules for defining as well as their *criteria for determining acceptability of a definition*. The teacher replied by reiterating a message similar to one he had said earlier, "How can you do that when you don't know what a polygon is yet? How do you know what it's not?" He again *requested that they propose a definition* for "polygon," and when Jomerd replied that "there are only two kinds, a regular and an irregular," the teacher requested that they *propose a definition* for "irregular." Mona offered the definition "it has different s-sizes of sides. The sides aren't congruent…and it has to have angles." What is noteworthy about her proposed definition is that she used the class's agreed upon language of "congruent" to describe the relations among the sides. Jomerd suggested an alternate definition that "nothing is the same, like the angles aren't the same, the sides aren't the same." Before the class could expand upon the notion of irregular more, Lavona *posed a definitional question* that resembled the teacher's questions. She asked, "what makes it regular?" Her question suggests that she had begun to pick up on the type of questions that the teacher had been asking. However, in this case, the class had just defined regular and their definition was on the board. The teacher acknowledged Lavona's question, and asked another student to respond. Vern noted, "it's on the board. All sides are congruent and all angles are congruent," and the teacher followed with "so we have an answer to that question." Thus, although her contribution was recognized, the teacher also sent the implicit message that definitional questions should address definitions not already agreed upon.

Up until this point, many of the ideas and interactions and responses were similar to those on Day 1. The teacher then shifted to a different approach to problematize their definition of "polygon" and expand the mathematical landscape they were exploring. He drew three connected, but not closed lines, and said, "I want to know what makes something a polygon. I know it has sides and it has angles SO…this then is a polygon right?" (see Figure 4). In making his argument, he *positioned their definition at the fore* by relating it to his example ("side one, side two, side three, angle one, angle two"). At the same time, his argument also modeled how one might use the Aspects of Definitional Practice, *constructing and evaluating examples* and *describing properties*, in service of a *definitional argument*. Moreover, by asking his question of "this then is a polygon right?" he *requested that students participate in evaluating his example*. Students, with much emotion, protested his example all at once, arguing that it was not a polygon. Finally, one student, Owen, stated that "it has to be CO::nnected." The teacher added this *revision* to their definition on the board and then suggested alternate language they could use to express the same idea: "sometimes we say that it's closed. Meaning that is have an inside, and an outside."

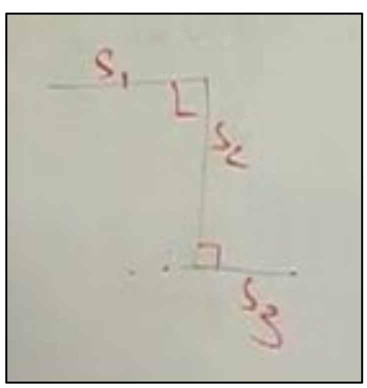

Figure 4. Teacher constructed example using students' definition of "3 sides, 3 angles."

Now that they had revised their definition of "polygon," the teacher then returned to Kira's initial question, asking, "if we take this definition, can there be a polygon with two sides?" Kate, suggested that as long as the two sides were **connected**, it was possible, and then suggested an example of an oval. Note that in reasoning about her example, Kate appealed to the class's newly revised definition. In doing so, Kate's definitional argument resembled that of the teacher's when he justified the validity of his zig-zag example. Another student gestured an oval to illustrate what she thought Kate meant and the teacher drew her interpretation on the board, making it accessible to others in the class (see Figure 5). Much like the zig-zag, the drawn oval caused many in the class to protest. Amidst the disagreement, Mona, whispered to her table mates, "What's a side?" and the other girls chimed in, with Adeena asking loudly to their peers, "What's a side, people?" This *definitional question* resembled that the teacher had been modeling in that it asked about the properties of an object and served to *elaborate on the mathematical system* the class was exploring. In this case, the students now were responsible for navigating the conversation to investigating new relations.

Figure 5. Student example of a polygon with two sides, drawn by the teacher.

Students then *proposed definitions* of "side." One student, Diego, said, "I think a side is a line that's connected to another line." The teacher, like before, drew an example in order to provoke contest. This time, he drew a closed figure with one curved line (see Figure 6) and noted, “I had a line, and there’s I connected it and then I connected it again. Do we want to call this thing (points to the curved side) a line?” Again, his question invited students to evaluate his example, and Lavona noted, “it has to be STRAIGHT.” Note in her response, she added *emphasis* to the term. Students continued to discuss sides and whether they needed to be straight. When they reached an impasse, the teacher noted, “But I don’t know what I mean by side yet. I heard the word STRAIGHT.” Vern then followed with the *definitional question*: “What does straight mean?”

Figure 6. Teacher constructed example using student definition of side.

Recap of excerpt 2. In this second excerpt, many of the initial interactions resembled those from the first day. The teacher continued to aid the class in making contact between practice and knowledge by *asking definitional questions that elaborated on system components* and *modeling* participation in that form of practice. Students, in turn, began to appropriate the teacher’s moves by asking similar questions and, in doing so, were also making contact with knowledge development. Like the teacher, those students who asked questions were now *modeling* an Aspect of Definitional Practice for their peers. However, at the same time, students were possibly still constructing normative understandings for when such questions are appropriate and the purpose they serve. This was evident when Lavona asked, “What makes [a polygon] regular?” even though the class had a few minutes before defined “regular” and written it on the board. In addition to asking questions, more students contributed to the discussion, including students, such as Vern, Diego and Owen, who had not been in the class during the previous year. In this brief excerpt, students participated in all of the Aspects of Definitional Practice: *asking definitional questions*, *proposing definitions*, *describing properties and /or relations*, *constructing definitional arguments or explanations*, *constructing and/or evaluating examples*, e*stablishing and reasoning about systematic relations*, *negotiating criteria for judging adequacy and acceptability of definitions*, and *revising definitions*.

During the first day and the beginning of the fourth day, the teacher's questions had motivated the introduction of new ideas and properties. However, when they reached a stalemate, the teacher implemented a new tactic for promoting the expansion of their mathematical system, illustrating how the mathematical ideas posed by the class coupled with their engagement in practice, informed his next moves. In this case, the teacher introduced or highlighted examples that problematized students' definitions. In three instances such *examples provoked contest* from the students and prompted them to introduce new properties and relations. These examples all shared two features that appeared to support this interaction: (a) they all were counter to what students viewed as polygons and (b) the feature that caused them to be undesirable was exactly what the students needed to add or describe in their definitions. In other words, the examples contrasted to polygons in one or two ways. For example, the zig-zag consisted of straight sides but was not closed and the 3-sided figure was closed but had one side curved. These examples resembled what Lakatos (1976) referred to as "monsters" – extreme examples mathematicians historically presented in order to counter particular proofs or theorems. These monsters had in turn caused mathematicians to reconsider their definitions. In the case of the sixth graders, the "monsters" prompted students to expand their ideas about polygons. They introduced the idea of "closed," defined "side" as needing to be "straight," and spent the next part of class constructing definitions of "straight" in their table groups.

The teacher also continued to Orchestrate Definitional Discussions by *engaging in meta-talk*, by *emphasizing agreed upon terms* and by *positioning definitions at the fore*. While he continued to stress and encourage the use of mathematical language such as "congruent," he also emphasized new language, such as "closed," to help students describe the properties they were trying to articulate. He continued to write students' definitions and contributions on the board, a way of positioning them to the fore, and stressed the need for students to also keep track of the definitions in their notebooks. For instance, at one point during this excerpt, the teacher noted, "Does everyone have this definition (taps on the board) in their math notebooks?...Well I think you better put it in there cause we have to get a definition for polygon." This message was reiterated at other points in time, both by Dr. Rich and by the regular classroom instructor.

Excerpt 3: Student Positioning of Definition at the Fore

After the question of "straight" had arisen on the fourth day, the class defined and investigated qualities of straightness, in particular by leveraging their experiences of walking in straight paths. They then used their path definition of straight (as "no turns") to write directions for walking particular polygons. They began the fifth day by revisiting their definitions of straight (ranging from "no bumps or lumps or zig-zags" to "180-degrees" and "it goes on and on in like one direction") and discussed whether a zig-zag should be considered straight or not. They then discussed differences between regular and irregular polygons and returned again to defining polygon. Vern proposed the definition of "sides and angles," and with prompt and reminder from the teacher, Kate added the property of "closed." This led the teacher to ask about the definitions of "closed" and "angle." Mona additionally asked their reoccurring question of "is a circle a polygon?" which led to further discussion of the relation, this time leveraging their newly constructed definition of straight as "180-degrees." The teacher then posed a new question, about the economy of their definition of polygon: "Can I just say that to make a polygon, I need to have it three or more sides and the figure has to be closed? Do I have to say anything about angles or not?" Kira suggested that a circle might be an example, but others countered that it was not a polygon. Ned then asked another definitional question, "Is the circle the only non-polygon? What about an oval?" After a brief diversion to address Ned's question, they concluded the original question about economy with Kate's suggestion that the polygon would have to have angles. The rest of the fifth day was spent looking at their directions for walking a square.

At the beginning of the sixth day of instruction, the teacher returned to the question of defining polygon once more. In this instance, students continued to appropriate forms of participation that the teacher had been modeling and articulating. The teacher once again opened with asking the definitional question of "what is a polygon?" This time, however, they established the definition more quickly. One student, Mataya, appeared to read from her notebook: "It is a closed figure that has angles and sides." The teacher wrote the definition on the board, again *positioning it at the forefront*, and then returned to the *definitional question* about economy he had posed the day before: "Can you make any closed figure with sides that does NOT have angles?" This time, two students, Ned and Kira, suggested that they could and the teacher asked them to draw an example, saying "if it's possible, draw it on the board." This move not only held them accountable for their claims, but placed them at the front of the classroom and in the center of the discussion. At the same time, by insisting they draw and then describe their drawings, he *requested that they participate in the Aspect of Definitional Practice* of *constructing and evaluating examples* of objects being defined.

Ned drew a football-shaped figure (see Figure 7) and when prompted by the teacher to "help us understand how you're thinking," defended his example. His example, however, was met with disagreement from Kate.

N: (points to the two "sides") Two sides. (points to the vertices) No angles. They can't be angles cause an angle has to be a straight line, two straight lines make a angle (uses his hand to show two potential straight lines - see Figure 6)
T: And angle has the intersection of two, lines? Two straight lines? Okay. (several students raise their hands – Kate, Vern, Diyari) Does anyone have a counter-argument for Ned? Kate. (she looks confused) Well, can you argue with Ned? Do you, do you argree with Ned or not?
K: Um I don't cause that's not a polygon.
T: Okay.
K: And Mataya forgot to say [that it has to have straight lines.]
T: [I think you need to say that to Ned] though.
K: (turns to Ned) That's not a polygon.
N: Did he say it had to be a polygon?
Ss: Yeah.
K: Cause based on, based on Mataya's um thing.

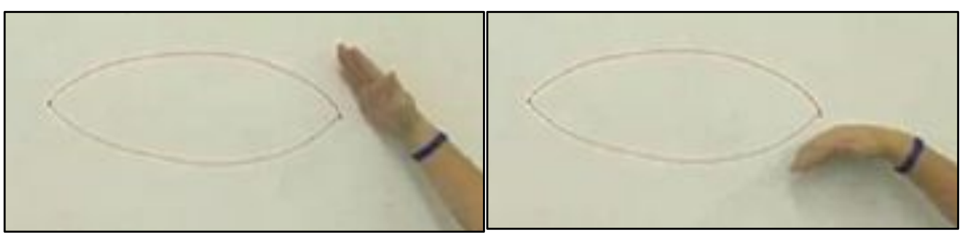

Figure 7. Ned's example of a polygon with sides but no angles. Here his gestures are meant to show that the angles are not made up of straight lines.

There are several noteworthy points in the above interaction. First, in constructing his argument to defend his example, Ned *described the properties* of the figure and then appealed to the definition of angle to make his case ("they can't be angles, cause an angle has to be two straight lines, two straight lines make an angle"). In doing so, he *positioned the definition at the* *forefront*, suggesting that he considered it to be an important form of evidence. His appeal to the definition resembled how the teacher had earlier *modeled* this Aspect of Practice. Ned's football construction also resembled the teacher's earlier constructions in that it was counter to what students considered a polygon to look like. Thus, although not intentional, it too provoked contest and prompted Kate to engage in c*onstructing a definitional argument*. In her counter-argument, she too *positioned definition at the forefront* by noting that Mataya should have included the notion of straight in their definition. In this interchange, the teacher supported the students' *construction of definitional argument* by asking Kate to address Ned and not him ("I think you need to say that to Ned though"), thus *positioning* the students in more authentic contest.

The teacher then pointed out that the point of difference in Kate and Ned's thinking was what they considered a "side" to be. Ned furthered this point by *posing the definitional question* of "What did we say a side is?" This question, like others before, once again directed the conversation to consider the definition of side. The teacher followed with: "What did we decide if you don't want to have that as a side, what must you define as a side, what must you define a side to be so you can rule it out? Cause right now, until, there's nothing wrong with what Ned has done. He has a start and an end and it makes a beautiful curve and it closes just like polygons, it's closed. So I see no reason yet to reject that figure." In this message, the teacher *articulated an expectation for participating in defining*, that to rule out an example, you must appeal to the definition and potentially revise it. At the same time, by asking the definitional question of "what must you define a side to be?" he requested that students *revise the definition*. Cordell responded that what Ned had drawn was not a polygon "because the sides have to be congruent." Although faulty, in his contribution, Cordell had appropriated the language of “congruent” to describe the side lengths. The teacher pointed out that what Ned had drawn could be considered congruent if one folded the sides onto each other. Kate then reiterated her earlier argument that the “lines have to be straight.” A couple other students also agreed. The teacher articulated another rule for defining, namely that once they agreed upon the definition, they would need to stick with it: “**once we say this, then this is what we mean**. A side is a line. And we said it usually has a beginning and an end point. A line segment that is STRAIGHT.”

The teacher asked the students how they had defined straight. Diego explained that “it had to have no curves, creases, bends,” reiterating a common way students had defined it two classes earlier. The teacher wrote this definition on the board and then added that “if we walked in a path, we would have no turns.” Jomerd added another student-invented definition, that straight could mean 180-degrees. They then return to the initial conversation of whether there could exist a closed figure with sides but no angles. Kira presented the example she had drawn, a depiction of a “marker cap” (Figure 8). She argued that “the inside of this marker cap is circular at the top (moves finger along the top rim of the cap) and it has no angles on the side (points to her drawing of the top of the marker cap) cause that line is curved and if you look down on the inside of here, it has sides.” Vern disagreed with Kira and argued that “the marker doesn’t have sides because um a marker top goes circular all the way down (makes spiral gesture). It doesn’t have (gestures up and down with finger) just a normal side.” Although Kira and Vern’s *definitional arguments* did not appeal to the definition like Ned and Kate had, they still *described the properties* of the example as they *evaluated* it. The teacher revoiced Vern’s argument, *positioning it in relation to the definition* while still attributing authorship to Vern: “so **he’s** saying that when you have this cylinder…it’s like one of these (draws a circle). And we decided that a circle, is a circle a polygon? (students reply “no”) Okay. Okay so it doesn’t HAVE sides **in**

the way that we define it cause if you went here (places marker on the circle), you would have to turn (turns marker as if it is walking along the path)…you'd have turning in order to make that." Kira argued in response that the marker cap did have sides going down the sides. The teacher pointed out that what they were talking about were two-dimensional objects and suggested that they add that property to their definition.

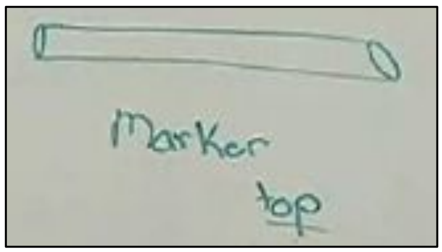

Figure 8. Kira's example of a polygon with sides but no angles.

In this remainder of the excerpt, the teacher turned their attention back to the original question of whether "closed" and "sides" guaranteed "angles." He asked a student, Shaunee, to reason about the relation, first prompting her to talk about their definition of "polygon" and "closed." When Shaunee needed help defining "closed," Adeena offered the definition of "when two lines are touching each other." The teacher drew another example to problematize her definition showing two lines connected (Figure 9) and asked, "So is this closed?" Adeena laughed, as if to suggest that she knew this game by now, and said, "no. um. When things say like things can't get out…like a back door." The teacher added, "sometimes we call this the interior, inside and the exterior, outside. That's what closedness does. Separates things. Inside and outside."

Figure 9. Teacher constructed example using student definition of closed.

Recap of excerpt 3. In this excerpt, the students were beginning to take on yet again more responsibility and agency in supporting the class's development of mathematical knowledge. Whereas earlier, the teacher's examples had largely been the source of contest and revision of definitions, here, *students'* constructed examples motivated reconsideration of the ideas they had been exploring. Moreover, Kate and Ned's contributions illustrated an awareness of the significance of *keeping the definition at the forefront*, something the teacher had been consistently modeling and emphasizing through meta-talk and writing. Although Kate had used definitions to justify inclusion and exclusion of examples and other definitions even as early as Day 1, here she explicitly referred to their communal definition by noting that "Mataya" forgot to say" and "cause based on, based on Mataya's um thing." Although subtle, this reference to their definition resembled the teacher's previous talk (e.g., "according to our definition").

Students' engagement in practice appeared to be supported by the teacher's earlier modeling. In addition, during the end of the fourth day of instruction and for much of the fifth day of instruction, students had been asked to construct directions for walking particular regular polygons, a form of procedural definition. They had exchanged their directions for walking squares and then shared their experiences of trying to use others' directions. When one group claimed that their directions were easy to follow, the teacher followed their directions in a way that showed what features the directions lacked. In response, the students collectively revised this set of directions. This activity, although a different form, reflected the cycle of definition posing, example generation and evaluation, argument, and revision that they had engaged in with the structural definitions. Thus, it is possible that this activity further contributed to the adaptation of these forms of practice. In this excerpt, students once again participated in most of the Aspects of Practice, including *asking definitional questions*, *proposing definitions*, *describing properties and /or relations*, *constructing definitional arguments or explanations*, *constructing and/or evaluating examples*, e*stablishing and reasoning about systematic relations*, and *revising definitions*.

The students' participation in practice supported the expansion and elaboration of the mathematical ideas they had been exploring. Here, they began with a more refined definition of polygon, as "a closed figure with sides and angles." However, Ned's example and his question about sides encouraged the class to revisit their definition of "side" and enforce the notion that it implied straightness. In turn, this provided an opportunity for the class to revisit their ideas about straightness that they had extensively constructed on the fourth day. Kira's example prompted the class to add the property of "2D" to their definition. Although the property was suggested by the teacher, Kira's example and argument motivated its addition. Moreover, because of students' discussion of the relation of polygon and circle during the previous class, they were more readily to reject a circle as a polygon. This consensual idea served as a resource for evaluating Kira's example.

Despite these student contributions, the teacher still played an important role in helping students make contact between practice and knowledge. In particular, he *asked a new type of definitional question*, one that encouraged students to think about the economy of the definition:

“Can you make any closed figure with sides that does NOT have angles?” This type of question would be difficult to investigate without first establishing what constituted a polygon and having some initial discussion about the properties of closure, sides, and angles. This question, coupled with the examples the students created, prompted them to make further make contact with the system of mathematical ideas they had been exploring. At the same time, because this question probed more deeply into the relations they had been investigating, it leveraged their initial explorations. In this way, the knowledge developed by the class informed the teacher’s next moves in instruction.

Excerpt 4: Student Agents in Orchestrating Defining

During the rest of the sixth day of instruction and for the next class period, students continued to construct procedural definitions of polygons. They then shifted to investigating interior and turn angle sums, first for triangles and then for polygons more generally. After several other investigations, including symmetries, rhombi and diagonals, on the twenty-six day of math instruction, the students transitioned to studying triangles and their properties in more detail. Before starting their investigations, the teacher asked that they first construct definitions for “triangle.” He began by *asking the definitional question* of “What’s a triangle?” Students immediately began calling out responses. One student said “a shape” and another, Terrance, started saying “three-sided-.” The teacher stopped the students and *requested* that they work with their table groups to come to consensus about one definition: “I want you to work in table groups and write me a definition of a triangle so that, so that we can know for sure, given a triangle an anything else that we might generate in 2D, or in 3D, that, what we’re looking at is a triangle.” As in the first day of math instruction, in this turn of talk, he again reiterated the purpose of definitions, that they help distinguish objects. After students spent a few minutes *proposing definitions* in their table groups, the teacher stopped them and added one more request – that they create an *economical* definition, one that used as few words as possible. His goal in doing so was to encourage the students to think about the relations among the properties of triangles.

After students worked in their table groups, the teacher asked each group to write their definition on the board. As a class, they then went through each definition and evaluated it. The teacher began by reading off Kate, Mona and Adeena's definition: "three sides, three angles only and it is closed." He then *asked a definitional question* that encouraged the students to consider whether their definition was inclusive enough: "Can anyone think of something that their definition, it would wouldn't work for it? Or something that is not triangular but their definition would seem to fit it?" Several students raised their hands and the teacher called on Vern who suggested "straight sides." The three girls immediately protested at once, arguing that their definition of "sides" implied the notion of straightness.

A: But we already said sides.
T: [So this assumes that the]
M: [That's the [definition of sides.]]
K: [definition of sides.]
T: def[inition of side means] straight. (draws from "side" and writes "straight")

What is noteworthy about this interaction is that it resembles the interaction from the sixth day of instruction when students had discussed Ned's football example. In that instance, Kate had been the student to suggest that their definition of "polygon" needed to include "straight," and the teacher had then noted that once they establish that a "side" means "straight," then they do not need to specify so. Here, Kate and her table mates took on the role of the teacher and *negotiated* with Vern about whether or not to include "straight," sending the message that there was no need to based upon their definition. At the same time, however, Vern's contribution was still important because he not only *described the properties* of "triangle," but also *reasoned about the systematic relation*s between its properties and sub-properties, Aspects of Practice that the students had spent extensive time developing within the first few classes of the semester.

The teacher then *requested* the definition of straight, "just so we're all on the same page." Mataya responded that "it means a line going 180, NO turns." Her quick response contrasted with many students' earlier inclinations to define straight as "no zig-zags" and suggested that these were now the consensual definitions. These two definitions had been encouraged by the teacher, in part because they had been used when the students constructed definitions for walking polygons and when they investigated sums of angles.

They returned to Kate, Mona and Adeena's definition of "triangle" and the teacher asked if there was "anything that this doesn't cover?" Kira noted that it did not include the fact that the turn angles are 360. The teacher, in trying to encourage the students to think about the economic relations among the properties asked, "we don't have other properties, but are these properties good(/) enough?" When a couple students responded "yes," he noted, "so that is a definition that works."

They then moved on to the next definition: "three sides, three angles, can be regular or irregular polygon and it's closed." The teacher once again *asked a definitional question* to push the students to think about the economic relations: "do they need to say closed if they say polygon?" Several students replied "no" but Ned replied "yes" and explained that "cause regular polygon is always closed." The teacher used this as an opportunity to revisit the definition of "regular," asking students, "what's the definition of regular again?" Rachel responded, "I think it was straight lines, with straight lines, angles and it's closed?" The teacher probed by asking, "but what makes it regular?" Lavona replied, "all the sides, same sides" and Jomerd and others added, "same sides, same angles." Terrance then instead suggested, "all the sides are congru::ent," adding emphasis on "congruent." This contribution was noteworthy because, without prompt, Terrance suggested a modification that the teacher had often encouraged early on. The teacher acknowledged this contribution by responding, "All the sides are congruent. THANK YOU Terrance…Okay, that math word says it a::ll."

They continued to go through the definitions in a similar manner. When they arrived at Diego's definition, "three straight lines and has to be connected," the teacher noted, "NOW, that's a really sparse definition. That's the sparsest one so far. Does it work? Or do we HAVE to say angles? What do you think?" One student agreed, "yes" and Diego followed by arguing, "but won't it come with angles?" The teacher revoiced "as soon as Diego says, three sides and closed?" and Rachel followed with "it al-it already has angles." Thus, although many of the definitions included "angles," when prompted about their necessity, at least some students seemed to readily accept that they were implied. The teacher then went back over several definitions and described them with varying degrees of "slimness:" "So, this is like, this. Very slim. I would call this one somewhat slim. I'd call this one pretty slim, right? This is an expanded one, but it works." Two groups had included that the sides had to be "congruent," further examples of appropriation of the word. In both cases, the teacher asked students whether all triangles had congruent sides and they quickly suggested that those were only for "regular" triangles. One group had included that triangles had "three points," and when the teacher asked them for another word "that we've been using," the class chorused, "vertex."

Recap of excerpt 4. Although several weeks had passed since their initial work with defining "polygon," the students readily appealed to the objects, properties and relations that they had spent several classes investigating. Their definitions varied in economy, but all attended to necessary properties and relations of triangles and leveraged ideas they had explored during the first few days of the semester (see Table 4). All the definitions included the properties of “three sides” and “closed” and most included straightness. Some definitions included properties from their recent investigations, including angles sums and diagonals. Unlike their definitions of “polygon,” their triangle definitions were created with little scaffold and within a much shorter time frame. Figures 10 and 11 illustrate the differences in the class’s initial definitions of “polygon” and their definitions of “triangle.” Figure 10 shows the objects, properties and relations explored by the class during most of the first day of instruction. During that day, although the students generated some properties, such as “side” and “angle,” properties such as “closed” were absent. On Day 26, the conversation was more focused on properties and relations, most of which were student-initiated, such as “sides,” “angles,” “closed,” “congruent,” and “straight.” Moreover, there were fewer deviations on Day 26, that is, the conversation was more focused on definition construction. In contrast, on Day 1, the teacher had to remind students of their goal of creating a definition of “polygon” and later “regular.” When asked to define polygon, students listed many examples of polygons (e.g., “octagon,” “quadrilateral”), but without contributing directly to the creation of a definition. On Day 26, students mentioned only one such class relation (to “quadrilateral”), and when they did so, they specified the relative properties. This contrasts too to Lavona’s definition of polygon on Day 1: “I think all shapes are polygons except for a quadrilateral.” The students’ attendance to properties and relations on Day 26 suggests that they had developed an inclination to seeing definitions as a means of distinguishing a class of objects from others. Moreover, this propensity was not limited to a select few students. All the groups of students constructed definitions, and students who had been new to the class, such as Terrance, Diego, and Mataya, were important contributors during the discussion. In face, during both days of instruction, about 78% of students contributed to the discussion in ways that supported the development of communal knowledge, as illustrated in the figures.

Table 4. Student definitions of triangles

1	Triangle: 3 sides, 3 angles only, and closed
2	Triangle = 3 sides, 3 angles, can be a regular or irregular polygon, and it is a closed figure.
3	A triangle has 3 straight sides, 3 angles, Interior angles of 180-degrees, Exterior angles of 360-degrees and it's enclosed!
4	A triangle has 3 straight lines and has to be connected.
5	A triangles is a polygon. It has 3 congruent sides. A triangle has no diagonals. It is closed with three interior angles and 3 exterior angles. It has 3 straight lines with three points. If you add another side it becomes a quadrilateral.
6	A triangle has 3 closed sides and a polygon. All sides have congruent sides. And 3 turn angles and 3 interior angles. And 3 turn angles. The sum of the turn angles is 360. A system triangle has all turn angle is 120-degrees.

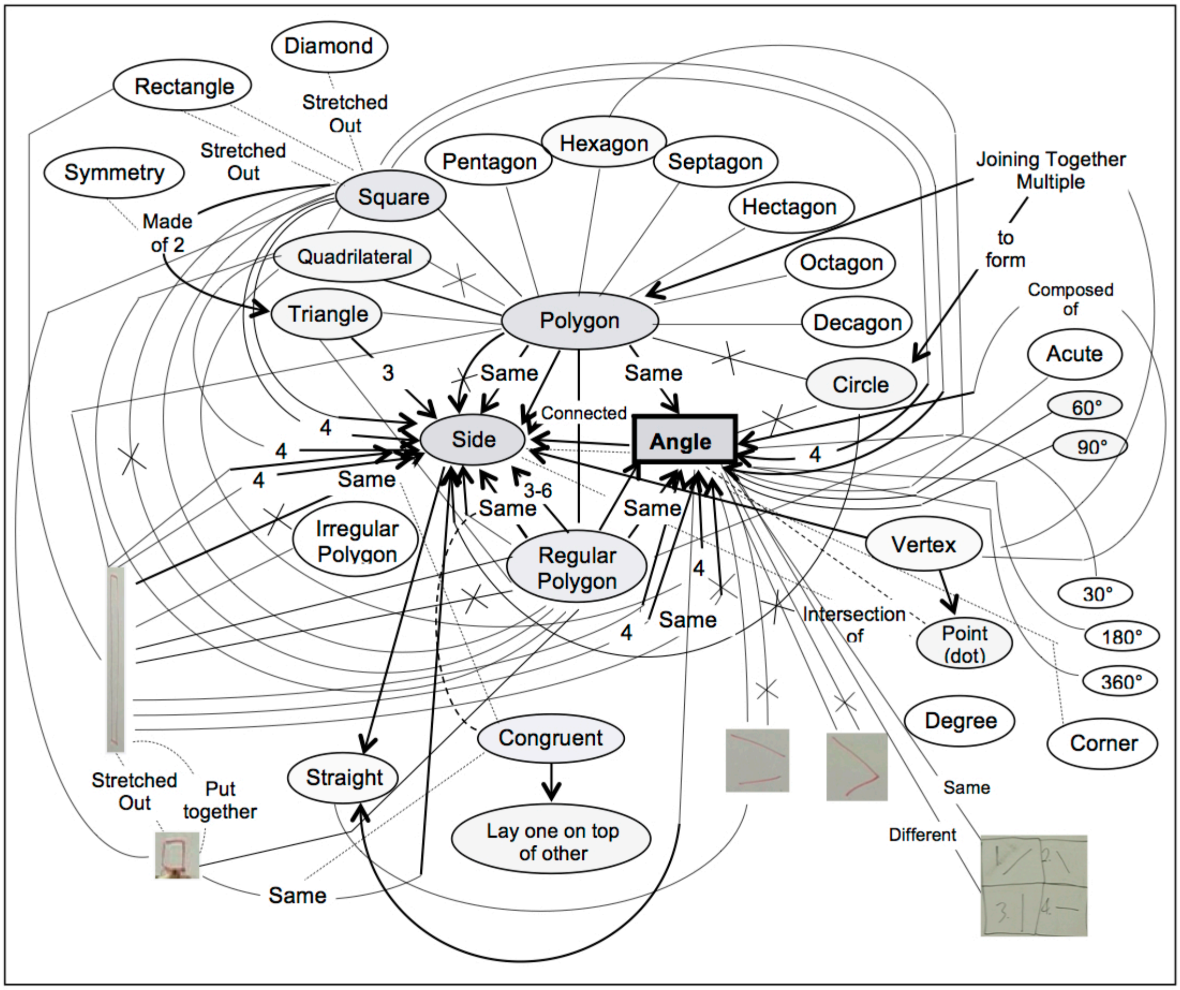

Figure 10. Knowledge development later in Day 1. Ovals represent objects or properties mentioned by members of the class. Solid lines represent sub-class relations, dashed lines represent class relations, and arrows represent inclusive relations between an object and the properties that possibly constitute it. Words or numbers on the edges describe the nature of the relation. The shading illustrates the frequency with which objects were discussed, with darker shading indicating they were mentioned in more Definitional Episodes.

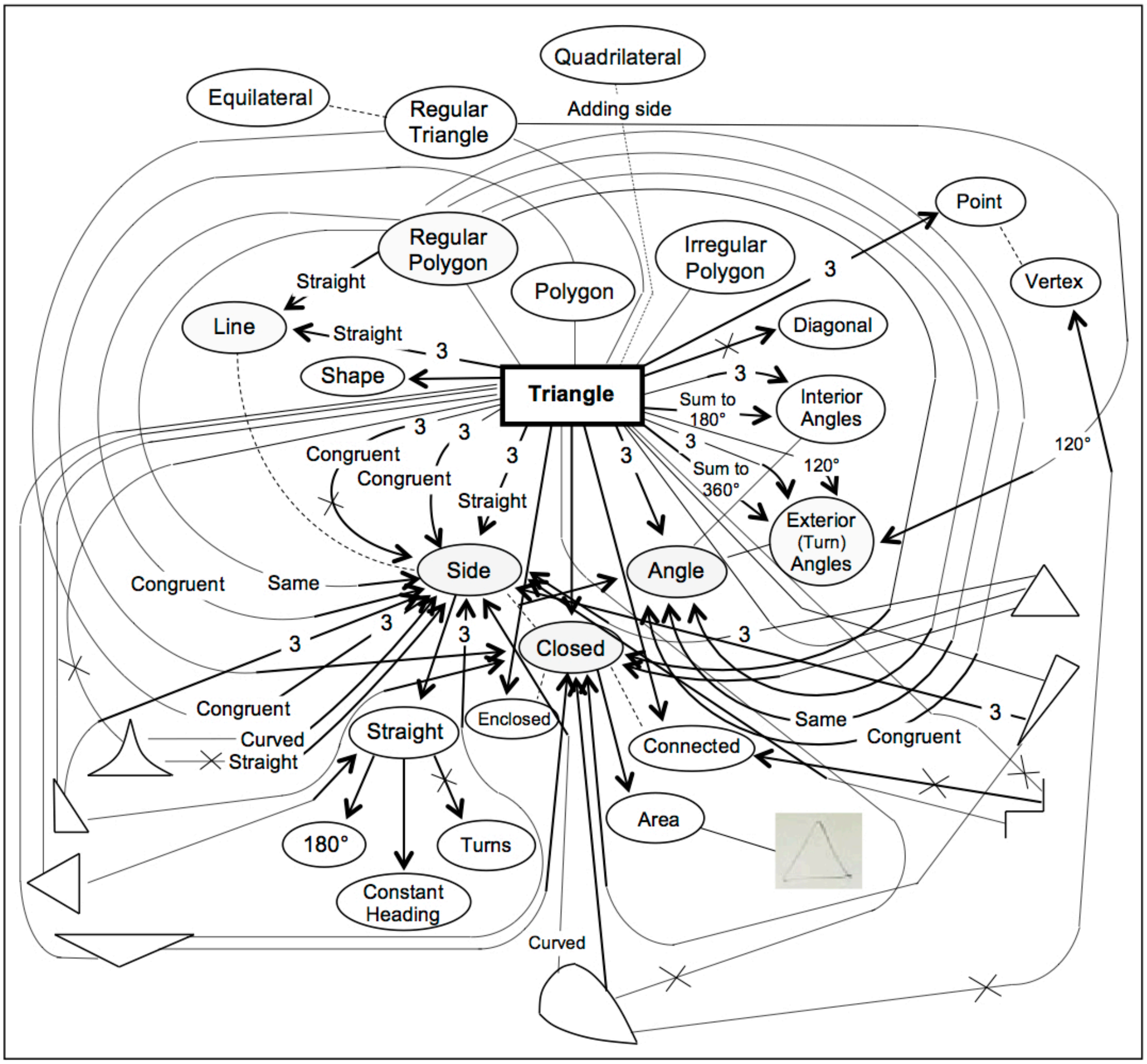

Figure 11. Knowledge development during Day 26. Ovals represent objects or properties mentioned by members of the class. Solid lines represent sub-class relations, dashed lines represent class relations, and arrows represent inclusive relations between an object and the properties that possibly constitute it. Words or numbers on the edges describe the nature of the relation. The shading illustrates the frequency with which objects were discussed, with darker shading indicating they were mentioned in more Definitional Episodes.

Students also continued to participate in Aspects of Definitional Practice, including *proposing definitions*, *describing properties and/or relations*, *constructing definitional arguments*, *establishing and reasoning about systematic relations*, *negotiating criteria for judging adequacy of definitions*, and *revising definitions* when needed. Moreover, in the next part of the lesson, students used their definitions to evaluate a set of potential triangles, providing them an opportunity to engage in the aspect of *constructing and/or evaluating examples.* Although students did not construct definitions during whole group discussion, video of two of the small groups reveals that their interactions resembled their whole group interactions from the fourth and sixth days of instruction. Here, a few students appeared to take on the role of the teacher in orchestrating discussion. In Kate's group, they almost immediately constructed a definition of "only three sides, only three angles and closed." When Mona suggested that sides needed to be straight, Kate reminded her that "we already knew sides were straight…that's the definition of side." Later when Adeena proposed the same idea, Mona reiterated the same argument. When Vern later in whole group also proposed that "straight" needed to be included in their definition, all three girls insisted that "sides" implied straightness, suggesting that Mona and Adeena readily accepted Kate's argument. Their reminder resembled the message the teacher had communicated on the sixth day of math class. Also during small group time, Adeena proposed that they needed to specify that the sides and angles be "equal." Mona quickly countered this proposal, stating, "No that's for a reg(/)ular. Does everything ha::ve to be regular? No::. I don't think." A similar interaction occurred between Diyari and Cordell. In their group, Diyari started by suggesting that a triangle was "a 3-sided figure." Jomerd followed with "a 3-sided, closed." As they argued over whether to use "polygon" or "figure," Cordell continued to write a definition. When he shared his version, "a triangle is a three-sided figure that has a turn angle of 120." Diyari then immediately presented a counter-example, in a manner similar to how the teacher had in the initial days.

D: nu-uh, not all of them do. This is a triangle (draws something). That's a triangle.
C: That's not a regular triangle.
D: But you just wrote a triangle (points to Cordell's notebook). You didn't write a regular triangle.
C: (writes something in his notebook) A regular triangle.

In the interaction between the boys, Diyari, taking on the role the teacher had earlier modeled, prompting Cordell to *revise* his definition. He did so by presenting a counter-example and by *positioning Cordell's definition at the forefront* by pointing to his notebook and noting, "But you just wrote a triangle." Although video is not available for all groups, these two groups further suggest evidence that students were inclined to attending to the properties in their definitions, and, in these cases, with no prompt from the teacher. In a different way, Terrance also appropriated the role of the teacher. In whole group discussion, his contribution and verbal emphasis of "congru::ent" to the definition of "regular" served to encourage his classmates to use an agreed upon term.

The teacher, in turn, played a similar role to earlier excerpts, but again, the students' participation in practice and the mathematical ideas they proposed informed his instructional moves. He again initiated defining by requesting that students *propose definitions*. However, this time, the focus was more on economical definitions. Although he had asked students definitional questions about economy during the fifth and sixth days, this time he started with a more open-ended request and then followed up during the discussion with particular probes (e.g., "do they need to say closed if they say polygon?"). He again engaged in *meta-talk about participation in practice*. Although some messages were similar to earlier ones (e.g., "write me a definition of triangle so that, so that we can know for sure…that, what we're looking at is a triangle."), others differed given the greater focus on economy. The teacher additionally labeled or coded (Goodwin, 1994) students' definitions using descriptors such as "slim," "sparse," "works," and "good enough" to highlight the degree to which they were necessary or sufficient.

Other Contributions to Creating a Culture of Defining

The four excerpts illustrate how over time, students participated in practice in ways that resembled the teacher's participation and served to make contact with mathematical ideas. At the same time, students had developed descriptions of a rich set of properties and relations that they were able to leverage as resources for constructing definitions of triangles. Some of these interactions had begun to occur during those initial six days. But how were these forms of participation sustained and furthered developed in the time between the sixth day and twenty-sixth day? Here, I describe students' activity immediately following the third excerpt and then illustrate interactions during three different points of time in the days leading up to Day 26. These examples show that students continued to engage in practice in ways similar to what was described on the sixth day. At the same time, they readily described some properties (such as sides and angles) whereas others (e.g., "closed") required some prompt from the teacher. However, unlike earlier, the teacher's prompts more quickly reminded students of these properties. Many of the teacher's moves resembled those from earlier episodes, and he continued to reinforce similar messages about practice.

Students spent the remainder of Day 6 and the following class period constructing procedural definitions of polygons, including squares, rectangles, and regular triangles. The students repeated this exercise with regular pentagons during the 13th and 14th days. These experiences contributed to students' defining in two ways. As previously mentioned, constructing directions followed a pattern of proposal, example construction and evaluation, argument and revision that resembled that which students had experienced when constructing structural definitions. Thus, students' inclinations towards proposing, countering, and revising definitions, as witnessed in the small groups, was only further reinforced through these activities. Moreover, when constructing the polygons, they had to reason about the angle measures and the relations among them. This close look at properties extended into their investigations of interior and turn angles sums. As Table 4 illustrates, multiple groups included these properties in their definitions, suggesting that the experiences provided them resources for definition construction. Throughout the two and a half months, there were also moments when teacher (or other students) prompted students to recall definitions (e.g., "What's equilateral mean?"). These conversations, although brief, possibly served as important reminders of properties and their relations. Likewise, there were points when students used definitions in service of arguments, serving as additional reminders. For instance, on the tenth day of math instruction, when explaining by the turn angle needed to be 90 if the interior angle was 90, Vern explained, "because a straight line is 180-degrees."

In addition to these experiences, students on several occasions engaged in short discussions defining new objects and properties or revisiting existing definitions. These episodes were usually motivated by a definitional question, asked both by students and the teacher. Sometimes, questions were asked when a student introduced a new object. For example, on the 8^{th} day of instruction, students had started to explore Diyari's conjecture that the interior angles of a triangle sum to 180-degrees and the turn angles sum to 360-degrees. The teacher asked students to create a triangle and test out Diyari's conjecture with the triangle. When presenting his group's triangle, Terrance called it a "scalene" and the class chorused "what's a scalene?"

Terrance and Shaunee both responded that "it's a triangle," suggesting a relation between the class and sub-class. Although students did not delve more into this definition, later in the same class, when presenting his triangle, Cordell started to doubt whether what he had drawn was a triangle: "this, it has uh these two are the same size but these (points to the third side) ain't and I think a triangle supposed to have congruent sides." The teacher then asked him if his figure is a triangle. Cordell responded, "I don't know what it's called. What is this called?" Instances such as these two suggest that students had started to develop inclinations towards posing definitional questions when it is not clear what the meaning of an object is, a habit of mind that the teacher had actively been promoting early on through modeling and meta-talk.

The conversation that followed about the definition of triangle also illustrated yet another instance in which contest over an example motivated discussion about the definition and its properties. This was furthered by the teacher's request that students needed to back up their claim with a definition: "Does everyone agree with that? If it doesn't have congruent sides, it's not a triangle? (students disagree) Okay, so if you don't agree, <u>you'll have to give Cordell a definition</u> that would allow that to be a triangle." Students' proposed definitions included some properties that they had been exploring. Others were quickly prompted with questions from the teacher, as illustrated below.

A: "Well, um, I think if it doesn't, just because it doesn't have the same size, um that it's not a REGULAR triangle, um like, regular triangle sided polygon, but it can also be like an irregular triangle.

T: so how would you define a triangle, Lavona, that would allow Cordell's triangle to be called a triangle?

L: I would say a triangle is a triangle that has <u>straight sides</u> down and doesn't slant or like anything and it's got the <u>two sides</u> that are going down are <u>equal</u>.

T: So what do you mean straight sides down? I'm not sure I understand.

L: ...it's got like the <u>two sides</u> on both sides are <u>straight</u>.

T: So will any three lines make a triangle?

L: Not unless if they're like –

J: unless it's <u>closed</u>.

The teacher then pointed out that what Cordell had drawn was a polygon, based upon their definition of “three or more sides and closed.” Some students agreed, but others, like Jomerd, suggested it was “not a regular polygon, but irregular.” They eventually agreed that the triangle was an irregular triangle. In contrast to the students’ later work with triangles, their immediate definitions, although containing some properties, lacked others and also contained some visual descriptions such as Lavona’s “doesn’t slant.” However, in contrast with their earlier work, when prompted by the teacher with the question “will any three lines make a triangle?” Jomerd immediately responded with the property of closure. Recall that in the later episode of triangles, students all included the property of closure in their definitions.

On the 17th day, at the end of their work looking at angle sums, the students revisited their definition of “polygon.” Towards the end of class, the classroom teacher asked, “Can some somebody give me a definition of a polygon?" Cordell responded, “A polygon is a figure that has more than 3 sides and um, has congruent sides, has angles and all sides are congruent." Note in his definition, he described the properties of “more than 3 sides” and “angles” and appropriated the language of “congruent.” The classroom teacher then questioned him, asking whether he meant “regular.” Interestingly, Cordell stated, “I know what it is,” and proceeded to look in his notebook, suggesting an inclination towards using the notebook as a resource. Meanwhile, other students suggested definitions. Lavona proposed, "a polygon is a shape that has 360 degrees and more than 3 sides and the interior angles are all going to be 180." She too attended to properties of the object, and, as was evident on the 26th day, included their newly investigated properties of angle sums (although not all completely conventional). Dr. Rich, as in the earlier days, positioned the definition at the fore by writing it on the board, thus making it accessible to others in the class. The classroom teacher asked the students “why does it have to have at least 3 sides?” and one student responded, “cause it has to be closed.” Recall that to prompt closure, the teacher drew examples of the zig-zag figure and later two connected lines, suggesting that perhaps these earlier examples helped students reason about these relations.

Later, the teacher posed the question: "If the polygon has all sides congruent, must it have all angles congruent?" Louisa immediately responded yes and suggested, “What about irregular polygons?” The teacher noted, "irregular polygons are polygons that DON'T have all sides congruent and DON'T have all angles congruent." Jomerd added, "it must be closed, right?" This illustrates increasing awareness of the necessity to included closure. The teacher followed with the same message the three girls later articulated on the 26th day: "yes, i'm saying cause **I'm calling it polygon, right**? So for us, polygon means 3 or more sides and closed. **That's our definition of polygon**.” This conversation also provided the opportunity for the teacher to ask students about related definitions, including regular polygon and straight. Students’ definitions of regular appropriated the language of “congruent” as was witnessed later in the triangle episode. Students’ definitions of “straight” included “180-degrees.” When the teacher probed “if I were walking,” Diego added that “it has to have a starting point and an ending point.” The teacher probed further by noting, “if I walk do I ever change direction if I’m straight?” and the students responded “no.”

During the 19th day, the object of “rhombus” was brought up by students, leading to a conversation of what a rhombus was. Here, their definitions were negatively influenced by a tool they had been using to investigate the teacher’s previous question of whether congruent sides implied congruent angles. The tool, four paper strips connected at the vertices with brad fasteners, allowed students to see that a square could be tilted and no longer have the same angles (while maintaining the side lengths). However, the notion of a rhombus as a “tilted square” became their definition. To encourage students to focus on properties once again, the teacher redirected their attention to them, "what do you know about the **properties** of a rhombus then? What can you tell me?" This prompted students to describe the sides and angles and their relations. This move sent the message that properties were important in defining and redirected students’ attention to those properties.

These excerpts illustrate that students were continuing to attend increasingly to properties when constructing definitions and continued to appropriate agreed upon language. The teacher’s questions, like those posed earlier, prompted students to revisit existing definitions. At the same time, with probing from the teacher, the property of “closed” was quickly accessible. Alongside the definitional work, students were also engaged in experiences investigating questions and conjectures about polygons and properties. In these experiences, they were played increasingly prominent roles in the class. These experiences no doubt only reinforced students’ development as authors of mathematics.

Discussion

The goal of this paper was to investigate how the practice of defining and communal knowledge each changed over time and co-developed within interaction. I presented several relations that I suggest contributed to the co-development of practice and knowledge. Initially, the teacher asked definitional questions that prompted students to propose ideas about polygons, sides, angles, and other related entities. Additional definitional questions encouraged discussions about definitions of properties and relations between “polygon” and related objects, such as “circle” and “regular polygon.” Revisions to initial definitions were motivated by contest from students' competing ideas. In order to problematize students' definitions further and encourage the consideration of new properties, such as "closed" and "straight," the teacher presented examples (monsters) created based upon their definitions. As definitions stabilized, this opened up opportunities to ask new questions, such as those of economy, that probed into relations among properties. By Day 26 of math instruction, students developed definitions that attended to mathematical properties with much less scaffolding from the teacher.

To illustrate this co-development, I drew upon two frameworks for describing how members of the class participated in the practice of defining. The framework of Aspects of Practice allowed me to describe how the teacher and the student participated in defining in ways reflective of the discipline of mathematics. From the beginning of Day 1, most of these forms of participation were accessible to students with varying scaffolding and support from the teacher. Students had opportunities to p*ropose definitions*, to *describe properties and relations* of objects being defined, to construct *definitional arguments* for or against definitions or examples of definitions and to *revise definitions* in lieu of arguments. Students were asked to *evaluate examples* constructed by the teacher and were later prompted to *construct their own examples* in order to reason about the definitions they were constructing. Students also increasingly *asked definitional questions*, often reflecting those the teacher had earlier modeled. The Aspect of Practice of *negotiating criteria for judging adequacy or acceptability of definitions* occurred less frequently. However, among the few instances in which it did occur, earlier ones were between the teacher and a student whereas the later one occurred between students, suggesting that students were developing greater authority for their practice.

The second framework for investigating practice, Orchestrating Definitional Discussions, described other forms of interaction particular to defining. Although the teacher was the main participant in this form of practice, over time, students also began to appropriate certain forms of interaction, such as positioning definitions at the forefront and emphasizing agreed upon terms. Moreover, their interactions on the 26th day suggested increased inclinations towards these forms of interaction, especially in small groups where the teacher was absent.

Through their interactions in practice, students had opportunities to explore mathematical properties and relations of the objects they were defining. Unlike traditional approaches to definitions, students had opportunities to first express their ideas and articulate ways of describing objects and then revise their ideas in lieu of arguments or counter-examples. Over time, students developed inclinations towards definitions that distinguished objects from others, as evidenced by their property-rich triangle definitions on the 26th day. Through this approach, students derived definitions for “polygon,” “side,” “angle,” “straight,” “triangle” and other objects. By investigating economic definitions, students probed more deeply into relations among properties. Moreover, by allowing students to present multiple ideas and negotiate those ideas, students generated multi-faceted notions of the objects and properties. Angles were defined not only as two connected lines, but also as “turns.” Likewise definitions of straight included “no bends,” “no turns,” “constant heading,” and “180-degrees.” These results reflect those of others who have given students opportunities to negotiate definitions (e.g., Keiser; Lehrer et al., 1999) and suggests that the experiences of these students is not an isolated case.

By centering defining around students’ ideas and participation in practice, the teacher created multiple opportunities for students to become authors of definition and develop greater mathematical authority (Boaler, 2002). He did this in part by highlighting students’ contributions in talk and writing, requesting that they participate in Aspects of Practice, and continuously acknowledging their authority. Students’ increased authority was evidenced by their readiness to contribute definitions, examples and counter-arguments, including arguments counter to the teacher. “Monsters” often provoked explosions of contributions, suggesting that students were invested in the activity and eager to participate. In the excerpts, students who had been in the class the year before were initially the more prominent participants, but as time when on, participation expanded to other students. The teacher supported this shift in part by recruiting the participation of students. In other excerpts, he created many opportunities to present their work on the board, allowing a greater range of students to participate and also positioning their contributions as important. Students such as Kate, Mona and Adeena were important in initially aiding the teacher in modeling practice. However, these extra aids do not suggest that this class is not replicable. Rather, in other classrooms, teachers may have to spend more time initially modeling, requesting participation and positioning students in practice.

Although this paper focused exclusively on the students’ work in defining, students were also developing other practices. Their early work in defining provided a foundation that supported their participation in other practices. Their communal understandings of the mathematical objects allowed them a common ground for asking questions about them and posing conjectures related to those properties. Likewise, several aspects of defining were similar in nature to other forms of practice and potentially provided an accessible arena for these forms of interactions. Moreover, students’ investigations of questions and conjectures opened up doors for further discussions of definitions as new objects and relations were introduced.

So, then, what can be learned from this case for other teachers and other classrooms? Here, I suggested several teacher supports that were important in cultivating and facilitating students’ participation in practice in ways that also promoted the development of mathematical ideas. These forms of interaction included: (a) asking definitional questions that elaborated on system components, (b) constructing monstrous examples that provoked contest, (c) emphasizing agreed upon terms, and (d) positioning definition at the forefront by highlighting definitions through writing and juxtaposing them with arguments and descriptions of examples. In addition, orchestration moves helped to provoke student participation in defining. Practices are forms of knowledge and they are equally important to cultivate within classrooms. Although these interactions were situated within the context of geometry, I argue that they are applicable to other subject areas, grade levels and classrooms. In fact, other work in defining has illustrated similar interactions. For instance, Lehrer and colleagues (1999) describe the initial work of a teacher cultivating defining in her second grade class. This teacher too positioned defining at the forefront by using two boards to post the definitions of "triangle." One board showed the class's agreed upon definition whereas the other served as a platform for emerging proposals. The teacher frequently redirected her students' attention to their agreed upon definition, especially when they were evaluating a set of potential triangles on the board. The set of examples created contest and motivated discussion over what constituted a triangle. Like the examples Dr. Rich had posed, these examples varied from students' visions of what a triangle should look like, and they were presented in a way that highlighted contrasting features. In another study, Zaslavsky and Shir (2005) showed a small group of high school students engaged in the evaluation of sets of definitions, including non-geometric definitions such as "function." These definitions had been designed to too cultivate contest among the students so that contrasting features were prominent (that is, they varied by important features that the researchers wanted to be the center of discussion), illustrating the potential for such interactions to generalize to other settings. Nonetheless, geometry provides particular affordances in that the mathematical objects are easily drawn and described. For this reason, geometry might be an ideal early entrée into defining, as illustrated by the case of the second graders.

I began this paper by pointing to the need for better understanding how the practice of defining develops in classroom environments. This works builds upon others who have studied students' participation in defining in three ways. First, the framework of Aspects of Definitional Practice suggests a way to describe and analyze practice. Because the framework was created by reviewing definitional work in various classrooms, it suggests that these forms of participation are not isolated to this one classroom, teacher, age group or content area, and might be a useful way to communicate analytically about students' engagement. Second, whereas others have tended to present illustrations of developed practice, here I suggest how those forms of activity might come about. Moreover, this analysis illustrates how this practice develops alongside knowledge and illustrates interactions that are significant to this development. The forms of interaction I present that helped encourage this contact (questioning, examples that provoked contest, and positioning definition to the fore) are not new to math classes. However, here I illustrate the role they play in merging practice and knowledge. Although one case, these interactions provide initial conjectures to test out in other classrooms and with other teachers and provide an important first step in theory development.

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CHAPTER VI

CONCLUSION

As mentioned previously, these three papers collectively provide: (a) an analytic and theoretical framework for examining the mathematical practice of defining as it might be constituted in classrooms; (b) an analysis of the initial establishment of this form of practice as instantiated in interaction among students and their teacher; and (c) an investigation of how knowledge, practice and the interactions that contribute to their co-constitution develop and change over time. In them, I aimed to provide detailed analyses that might contribute to a larger theory describing classrooms that promote student engagement in mathematical practices. My hope is that this initial theory will provide grounds for further research and work within classrooms. Although the data describes one classroom and one teacher, I argue that the interactions described here have implications for other classrooms and other lines of research.

First, the papers present a *theoretical language* for describing how members of classroom communities interact around definitional practice and illustrate the utility of such a language. I presented two frameworks, *Engaging in Aspects of Definitional Practice*, and *Orchestrating Definitional Discussions*, that together provide a means for describing how members of the classroom, both teacher and students, participated in defining. Both frameworks were grounded within other empirical work, suggesting their potential relevance to other classrooms. The Aspects of Practice framework was initially created by reviewing empirical studies where students participated in the construction and negotiation of definitions. These studies were conducted with different age levels, in different topics, and in different countries. Yet in each, multiple, common Aspects of Practice were at play. The framework for Orchestrating Definitional Discussions was influenced by previous descriptions of math talk communities and how teachers in these communities orchestrate productive discussions. Whereas the Aspects of Definitional Practice describe forms of participation in defining from a disciplinary perspective, the goal in creating the Orchestrating Definitional Discussions framework was to characterize forms of participation that support interactions in defining. In Papers 2 and 3, these frameworks not only allowed me to characterize and describe how participants interacted around practice, but they also provided a lens for looking at interactions between practice and knowledge. That is, I was able to identify specific forms of participation and characterize how those forms aided in the development of communal knowledge. In this way, the frameworks have the potential to serve as ways to communicate analytically among researchers.

Similarly, the language for describing defining may also provide a resource for working with teachers to establish similar classroom environments. Others have created frameworks to describe student ways of thinking in various content areas (e.g., Franke, Carpenter, Levi, & Fennema, 2001; Lehrer, Jacobson, Kemeny, & Strom, 1999; Lehrer, Kim, Ayers, & Wilson, in press). Such frameworks have been used as forms of support for teachers in professional development settings (Franke et al., 2001; Kim, 2012) in order to help teachers develop what Jacobs, Lamb and Philipp (2010) term "professional noticing" of student thinking. That is, teachers learn how to pay attention to and interpret student thinking and use those interpretations to inform their next steps in teaching. Likewise, they provide a basis for communication among teachers in professional communities. Here, I propose frameworks that instead characterize student engagement in practice. In this sense, the frameworks for definitional practice provide initial starting places to work with teachers to create environments in which to further study defining.

Likewise, the three papers also contribute to a growing body of work looking at describing *teacher* mathematical practices. Recently, mathematics educators have described "high leverage practices" (e.g., representing concepts with examples) that teachers can rehearse and implement within their classrooms (e.g., Kazemi, Franke, & Lampert, 2009). The forms of participation described in the Orchestrating Definitional Discussions framework. as well as the forms of contact between knowledge and practice identified in the second and third papers can be considered "high leverage practices" for defining. Although Dr. Rich's teaching moves were situated in geometry, they still have the potential to travel across settings. For instance, teachers may present examples that create contest in other areas of mathematics, although the forms may vary. Ball (1993) describes a class where presenting the case of "0" as even or odd provoked contest and discussion about their definitions. Moreover, positioning definitions at the forefront allows students to have the same frame of reference and also allows the teacher to continuously relate activity to the overarching goal of creating a definition. The types of questions Dr. Rich asked are also easily transferable to other contexts (e.g., "what's a function?" "what is odd?"). In other settings, educators need to consider the resources students might bring to the table for such discussions and how to leverage those resources. For instance, if students have not previously had experiences with a mathematical object, they may need to first explore the object. Curtis (Lehrer & Curtis, 2000) did this in her second grade classroom when she wanted to introduce students to "perfect solids." She presented different solids, two of which were Platonic or "perfect." Students used these examples to generate initial definitions that were then used to construct their own examples and further revise their definition.

Finally, this work illustrates a case of how knowledge and practice interact. Too often, studies focus on one or the other. Here, I suggest not only how knowledge and practice interact in this context, but also suggest a method for analysis. By looking at the level of turn of talk, I was able to look for moments when participation in practice informed practice and vice versa. This approach also afforded an analysis of the roles of different members in the class and how those roles shifted as students began to take on more authority and as their mathematical explorations grew.

Like other studies, this set of studies has limitations. Although the cases provide conjectures for generalization, these conjectures need to be tested and refined in other settings. In particular, it would be useful to explore these ideas in a setting where students have had fewer opportunities to talk about and reason about mathematics. In Dr. Rich's class, half the students had been in the class previously and often served as an additional support to reinforce norms and practices. Moreover, in the third paper, I only sampled a few points in time, and more sampling would allow for a richer and more nuanced picture of development. For instance, students' construction of procedural definitions was not a focal part of the analysis, and might add additional insight into how Aspects of Definitional Practice vary. No doubt, students' experiences in practice were influenced by their engagement in procedural definitions and, thus, a richer analysis would be worthwhile.

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APPENDIX A

TRANSCRIPTS

DAY 1

DEFINITIONAL EPISODE #1
[00:05:34.09]

RL: polygons and vertexes. uh:: who can help me understand, what a polygon is? Just so I can kind of
Mic: a regular or a regular or polygon?
RL: A regular or [irregular?]
Ama: irregular]
RL: Okay give me the most general definition you can. So that I can recognize a polygon and I could tell the difference between a polygon and a turnip.
[Students: what's a turnip?
RL: Turnip? Uh it's uh how bout a carrot. I want to know the difference between a polygon and alright Vincent (points to Vincent who has his hand raised).
Vin: A polygon has the same angles and the same length of uh (pause), the same lengths of sides.
RL: Vincent's claim is that all polygons have the SAME length of sides and the SAME angles. Rhonda.
Rho: All regular polygons.
RL: All REGULAR polygons (pointing at Rhonda and looks at Vincent). Do you accept her amendment?
Vin: yeah
RL: All REGULAR polygons. Kenjra (points at Kenjra)
Ken: (reading from notebook) Additionally all polygons have 5 sides.
RL: All polygons have 5 sides.
Mic: No (raises hand)
RL: Who can make-uh- Someone says no.
Mic: I say no. (raises hand)
Kay: I say no. (rasies hand)
RL: Okay. The troublesome trio (referring to Kayla, Amani, and Micah) say no.
Jee: Oh we too. We say no.
RL: Why not?
Kay: um because um if all polygons have 5 sides but we also had the square was a polygon and the triangle was a polygon.
RL: Okay so, [as soon as we find]
Kay: and they've only got 3 and 4]

RL: something that we'd like to call a polygon that has other than 5 sides, we KILL that conjecture. Okay we-so we have to not say 5 sides. Alright but what do we say? Cause so far the only kind of polygons I know from what you've said are what kind Nicholas?
Nic: Uh, sir? The only kind of polygons are?
RL: are what kind?
Nic: not 5 sided but-
RL: no but what kind do we no so (shrugs) far? Cause someone (points to Vincent) Vincent said
Nic: Squares, octagon
Vin: (?)
RL: huh?
Vin: I talked about regular polygons.
RL: REGULAR polygons. I thought you said that. I thought I heard that INSTANTLY. Yes, regular polygons. Okay, Jeewar.
Jee: There are lots of other polygons. There's a decagon a septagon a octagon uh::
Ken: A hectagon.
Jee: A he-HEX agon.
Cou: a pentagon
Lou: I think all shapes are polygons except for the squares and quadrilateral.
Jee: lots a gons
RL: Okay Louisa's conjecture is that all shapes are polygons except for what?
Lou: uh a quadrilateral.
RL: Except for quadrilaterals.
RL: alright I'm going to write that up here so I can keep track
[interuption]
RL: alright. All shapes (pauses and writes an "L" above what he is writing--signifies "Louisa") all shapes are polygons except for quadrilaterals.

DEFINITIONAL EPISODE #2
[00:08:40.19]

RL: Now someone will tell me what the heck a quadrilateral is cause I hadn't heard that word yet.
Lou: (raises her hand) It's a square.
Vin: (looks at Louisa) That's a polygon
Ama: That's a square, isn't it?
Vin: a square is a polygon.
RL: You mean (draws arrow going down from "quadrilateral" and draws a square) quadrilateral and square are synonyms? (gestures between the two representations)
Ama: Yeah cause they have 4 angles and 4 sides.
Vin: But a square's a polygon (speaking to Lousia)
Lou: [So what it's
Kay: but a square's a polygon.]
Mic: a circle
Mic, Kay, Ama: A circle wouldn't be a polygon cause a circle doesn't have sides.
Ama: A circle has no sides or no angles.

DEFINITIONAL EPISODE #3
[00:09:09.27]

RL: Okay so QUESTION. Circle is? a polgyon? (writes this on the board)
SS: NO::
RL: No::? No. Alright. Well I like circles. So if I'm going to rule circles OUT from polygons, why can't a circle be a polygon?
SS: [(?)]
Cou: [Because it doesn't have any sides]
Lou: [because it doesn't have any angles
Sha: No sides.
RL: No sides?
Sha: No sides?
(students still talking)
RL: alright. Okay. You have to have sides?
Lou: No
SS: Yes
Mic: because pol-

DEFINITIONAL EPISODE #4
[00:09:41.17]

RL: Alright (they quiet down) Alright let me see those of you who want-okay we're trying to create a definition of a polygon. Remember the goal is that I need to be able to tell the difference between a polygon and a carrot.
S: carrot?
RL: Carrots. Circles. Anything else. Anything that you don't want to call a polygon, I have to be able to look at your definition and say oh thank you. Now I know. Okay so that's what we're doing here. So, so far I can't. The only thing I know is that there are some polygons that are regular and they have equal sides and equal angles. So now I know what a regular polygon is. And I'm very happy. Cause if I see a square, what will I say?
Ken: That it's not a regular polygon.
Ama: it is.
RL: I WHAT?
Ama: It IS a regular polygon.
Ken: what I said a regular polygon.
RL: Is it a regular polygon or isn't it?
SS: YES.
Cou: Yes cause it got sides and angles.
RL: Okay well. We can kinda have a situation like this, right Louisa? I have a dog. And her name is MINI. Okay can I call her both dog and Mini?
SS: No
RL: Isn't-no? No I can't? You mean Mini's not a dog?

SS: Yes (they talk at once-hard to discern)
RL: Okay so if I say Mini can I think of her as well as a dog?
SS: Yes.
RL: Okay so just because something is a square doesn't mean it couldn't ALSO be something else right? Is that right Louisa?
(silence) Okay so. I want to know, when I see a square (points behind him to the drawn square) and I'm thinking about your definition of REGULAR polygon, I want to know is a square a regular polygon? Michaela.
(Michaela says something quietly)
RL: What?
Mic: Square is a regular polygon.
RL: Is it or isn't it? It is? How do you know? (Jeewar raises his hand) Jeeward, let Michaela answer this.
Mich: Because it has sides and it has angles.
RL: Okay. So::. Um. Let me draw something else that has sides and angles.
(draws long rectangle) And just pretend that I can draw. And those are straight. Is that a regular polygon?
SS: Yes
Vin: No it's not.
Lou: No no no.
Vin: No cause it's an irregular polygon.
Ken: Isn't it too long?
Cou: It don't have-
RL: Alright I want those of you who think that this is a regular polygon to stand up (gestures up)
(Nobody stands)
RL: I want those of you who think that this is NOT a regular polygon to stand up.
(Everyone except for Daniel and Shatteryia)
RL: I want those of you who neither stood up on either occassion, what are you?
Sha: We don't know.
RL: You don't know?
Sha: We don't-
RL: You don't know? Okay. So
Sha: It's like part of the (?)
RL: Alright so there are two people who don't know and I suspect there might be more than two. So, those of you that are standing, how could you convince Shatteryia and Daniel? How could you convince them that this in fact is a regular or is NOT regular polygon. Is not. Rhonda.
Rhonda: It is not a regular polygon.
RL: Not? (writes on the board) Why not? (Push on argument)
Rho: Cause it doesn't have the same size sides.
RL: The same::?
Rho: SIZE sides.
RL: Okay can we use the word length? So all sides (writes on the board as he speaks) are

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DEFINITIONAL EPISODE #5
[00:13:27:09]

RL: (turns to the class) we used a word last year.
S: Equivalent.
RL: Equivalent? Equivalent with respect to? Length? Or we used another word. (points to Jeewar)
Jee: Length.
Lou: He just said that.
S: Height or width
S: Area. Width area.
Ama: Circumference.
Jee: circumference
RL: This thing has a circumference?
Jee: No
Sha: No it's not. It's uh. What's it called.
S: Ver:::
S: Area?
Jee: Vertical?
Nic: Perimeter?
Tim: Height.
RL: Alright I just don't want you to call out every word that we talked about last year. okay. So what. um. We say here that all the sides are NOT the same length. Or is there a way of doing it if I didn't even if I couldn't even measure the lengths? Is there a way of establishing whether or not two things have the same length? How would I do that? If I had something I didn't have a ruler and I say you know I think these things are both the same length. How would you establish that? You can sit down. Courtland? How would you do that?
Cou: Dr. Rich. You can tell um how it looks?
RL: Huh?
Cou: You could tell how it looks because one side can be uh longer than the other one.
RL: Right. But suppose I claimed that the sides were the same length. How could I establish that even without a ruler? Rhonda?
Rho: With another object.
RL: With another object? Okay. So if this has the same length as this (holds up pieces of paper) what should I be able to do? Nicholas?
Nic: Put one beside each other.
RL: Okay we should lay them right on top of one another? And have nothing sticking out? Nothing leftover? Do you remember the word that we used when we had this
Nic: Overlap?
RL: situation? Where it just (puts arms together) stuck right and we couldn't tell the difference?
S: Symmetry
RL: Well a symmetry is a certain KIND of indifference, right? When we turn or slide or flip something. But what about when we just (holds hand up as if holding something)
Lou: Flip?
RL: No we just lay it right on top of.
S: Mirror symmetry?

RL: Okay I'm trying to establish-maybe we're. I'm trying to-these are all ways of establishing when things are equivalent so that's good right? But I have something very simple in mind. Nothing too fancy. Devalon?
Dil: Slide Symmetry?
RL: Okay. A Slide symmetry means that (puts paper rectangle on the board) right if I and I did this (slides it to the right), I couldn't tell the difference. It has exactly the same look. Same shape. Same look. But I want to know if two things have the same length and I don't have a ruler and we slide them to uh. We put one right on top of the other or:: think about this I pick this length up (indicates length of the rectangle on the board) Right over here. Right on top of it. Okay what's the word for that (gestures up and down) I just want to establish the word so that we can kinda keep it in mind. Shatteryia?
Sha: Can I ...I believe it's a regular polygon.
RL: Okay. Let me finish this and then we'll come back to your issue. Okay how if I started this (writes on the board) C-O (writes cong)
Rho: Congruent.
RL: congruent (writes it on board) this was our little special word when we said, lengths were congruent, or areas were congruent we said that, not only is it the SAME but that we can literally put one on top of another and establish, that they are identical. okay? so we're going to try to remember this word. put it in your notebooks… think about it… I'm going to give everybody a minute to put it in their notebooks. to think about it…and to be sure that you kind of understand, what we're, getting at.
(students write in notebooks. One student tries to say something)
RL: just a second. I want to make sure we're all on the same page on this. we're going to use this. if I wanted to establish that A:: [holds up rectangular piece of paper labeled "A" in right hand] was congruent to B:: (holds up different rectangular piece of paper labeled "B" in left hand) in some, in some way? then say the amount of space covered or the area? what would I do?...(looks around) Tim? what would I do?
Tim: °put em' together°
RL: put em' together? (overlaps the two pieces at the ends) and what should I what should happen?
Cou: it should be-
RL: have I established it yet?
Cou, S: no
Jee: >°nonono°< (raises hand)
RL: (moves papers to overlap a little more) have I established it yet?
Lou, S, S: no (Kayla shakes head no)
RL: (moves papers to overlap a little more) how bout now?
Lou, Mic, Tim, S, S: no
RL: (moves papers to overlap a little more) how bout now?
SS (about 5-6): no
RL: (moves papers to overlap completely) how bout now?
SS (same 5-6): yes
RL: okay. alright so that's what we mean by congruent.

DEFINITIONAL EPISODE #6

[00:19:01:06]

RL: alright NOW, let’s get back <to Shateryia’s has uh a suggestion for us.> (turns and walks to board) we say (points to rectangle on board) that this is NOT a regular polygon, <because (points to text written on board as he reads) the sides are not all of the same length?> (looks at class)...okay or <the sides are not CON gruent?> they would fail this test (turns back to board). if I took THIS piece (indicates width of rectangle) and (rotates fingers clockwise and places them on the length of rectangle) laid it on this piece would they be congruent? if I put one (writes 1 above the width) on two (writes 2 to the left of the length) would one be congruent with two?
S: °no°
SS: °no°
RL: °okay?°…Why not? [Epistemic Message--need for justification]
Tim: cause they different, sides.
RL: right cause they’re different lengths good. okay SO. so a lot of people said that this shows that this figure (points to rectangle) is <NOT REGULAR. but SHATTERYIA objects>, so we need to listen to Shatteryia’s objection. [Epistemic Message--Importance of ideas of community]
Sha: Because a square, that's a square it's just stre::tched. like okay all of these (holds up bag of plastic shapes) are polygons right? so if we had (she empties bag on her desk and then picks up a piece) this one and (picks up another piece and puts it next to the first shape) this one (?)
RL: uh huh. You want to hold that up so everyone can see it?
(Shatteryia holds up the two pieces)
Sha: If I had B and G. B like just took this part and slid it down to make G.
RL: Okay Omari, can you restate what Shatteryia's trying to tell us? [Norm of accountability]
Oma: I think what Shatteryia's saying is that (pause)
RL: Daniel can you restate what Shatteryia is trying to say?
Dan: I think what she's trying to say is just that that like that square on the board
RL: uh huh
Dan: It just got stretched to make that other, congruent (?)
RL: okay. Amani?
Ama: I think what she's also trying to say is that like all they did is take like probably 3 squares and like put em all in to make like one.
RL: okay so. Is Shatteryia right? Could we make a rectangle in this way?
SS: yes
RL: okay. so Shatteryia says we can take a square (draws square on the board) and one thing we can do with it is we can stretch it (draws arrow going down from the square). We can pull on a side (gestures the motion). This one right here (points to side in the drawing). I'm going to pull on it. And I'm gonna transform (pulls arm down) it. Okay you're-do you see that? (enacts movement again) pshhhh. And she says when you do that that's one way of thinking about (draws rectangle next to the square) a rectangle (points to drawn rectangle) okay Amani suggested that another way to think about what Shatteryia is saying is that we could take the square and join other squares to it. (draws 3 squares adjacent and then errases connecting lines) and that would create a rectangle. We could glue two squares together. And that would create a rectangle as well. (..) Okay. Louisa?
Lou: so Shatteryia's saying that a square is more like a rectangle? Or like kinda like?=
RL: you'll have to ask Shatteryia.
Lou: A family? a cousin to the rectangle or something?

Shatteryia: (giggle) like okay. Yeah.
RL: mmm hmmm.
Shatteryia: okay like.
Lou: so a square IS kinda like a rectangle.
Shat: it IS a rectangle. It's just first like this and then like stretch it and it's kind of like this (holds up rectangle).
RL: so Shatteryia says that-and I think Louisa's helping us see this. (points to drawings on teh board) That Shatteryia says look. a square IS related to a rectangle (points back to original rectangle of discussion). Because you can make rectangles by stringing squares together. Or by pulling on one of the sides keeping everything else the same. just pulling (gestures down) everything down. Or. okay. Kenjra?
Ken: Dr. Rich what if you decided to pull on the triangle inside of the square because two triangles does make a square. It'll go either that way or that way.
RL: Can you draw for us what you mean?
(RL erases the board)
Ken: you have a square (draws a square). Not exactly a square but you can get the.
RL: okay
Ken: inside of that square you find two triangles. what if you decided to pull one square that way (draws an arrow from one corner) and one square that way.
RL: Uh huh. What would happen then?
Ken: It would enlarge (?)
RL: you mean when you say square do you mean?::
Ken: I mean one triangle.
RL: one triangle. you want to pull a triangle this way (RL gestures over the drawn arrow) and pull the other triangle the same distance teh other way? okay you want to pull the vertex? Is what we call-do we call that a vertex?
SS: mmm no.
RL: Oh no? no. okay. what do we call that?
Vin: that's um. (..)
Jee: stretching. enlarging.
RL: I mean. I want to know this point right here (points to vertex) where this side meets this side. does that have any? have we talked about that?
Lou: oh it is like a vertex. it's just that.
RL: oka
Vin: a symmetry?
RL: uh a symmetry?
Vin: yes.
RL: okay. how so?
Vin: um if you have a square and you put it in half like that you have two triangles but they're the same size.
RL: same exact si-so if you um (closes marker top) hmm. where's our square? Do we have a square? a nice big one Mrs. Lucas?
DL: uh no.
RL: no. okay. let's hold on to that idea. i'm going to say square, triangle and symmetry (writes all three on board) okay SO. i'm not sure Kenjra if we do that what it is that we're trying to say.

that's what i'm. now i understand what you're trying to do but I wasn't sure that we understood yet.
Ken: what i'm trying to say is sometimes squares don't always make rectangles. i think maybe if you pull it out that way some, it'll kind of make a diamond I think.
RL: okay so you could take a square and depending upon WHERE you did this stretching idea, it doesn't HAVE to make a rectangle. It could make something else. Is she right?
SS: yes.
RL: So that's another interesting observation. okay so thank you Kenjra. So. now I wonder. um if that rules OUT the idea, if that necessarily makes this (points to the board) a regular polygon. that's what we have to go back to because Shatteryia is right. And Kenjra is right. we can take a square. and we can stretch it or transform it in different ways and we can make other shapes and we can see that with our heads and we can even do this on our computers. and we'll be sure that we will do this on our computers so this is very-right so we can see these things. BUT, I want to go back to this. just because we can do that (turns and walks to board) does that MEAN, that (points to rectangle on board) this rectangle is now a regular polygon?
(Micah shakes her head no)
Lou: °no°
Mic: °no°
RL: okay Shatteryia what has to be true for a polygon to be regular?
(Courtland raises his hand)
Shatt: it have to be like it’s a three or more sides.
RL: okay what else?
Shatt: (   ) it has to be a polygon.
RL: oka::y so a regular polygon has to be a polygon. okay. what else?
(Shatteryia looks down, smiles slightly)
Shatt: Can you restate your question?
RL: yes…um I’m just asking you what’s your definition of a regular polygon?
Shatt: it’s like a regular polygon is like, like up to:: (look down at something in front of her) six sides. three up to six sides and, like, regular polygon is like a hexa-a hexagon a qaud-whatever’s it’s called and a triangle.
RL: so you know KINDS of regular polygons. <what has to be true of all of those kinds?> what makes them regular?
(Shatt looks down at what’s in front of her)
Shatt: cause they all have °sides, and angles?°
RL: they all have sides and angles? so. I’m going to write down what you said over here. (walk to other board on the side of the classroom) um (writes as he speaks) all, regular…polygons…have, sides…and angles. (turns to face class) okay…um::….so Shatteryia. from that point of view (walks to other board) does this (points to rectangle) have sides and angles?
Shatt: yes.
RL: okay does this have sides and angles? (can’t tell what he’s pointing to)
(Shatteryia nods)
RL: so, according to your definition (points to definition written on other board) are::-is this (points to rectangle) a regular polygon and is this (points to square drawn on board) a regular polygon?
Shatt: yes

RL: yes. (nods once) how many people agree? with Shatteryia? that IF we define a regular polygon as having sides and angles (points to definition on board), THE::SE two, are regular polygons.
Jee: °no:: ° (shakes head no)
RL: …(Jeewar raises his hand) well:: (looks toward Jeewar) no? yes? how many agree? stand up if you agree with Shatteryia. (Michaela, Kayla, Daniel stand) NOT that you agree that this is your definition of a regular polygon but rather (Rhonda, Tim stand) IF we define a regular polygon to have a polygon that has sides and angles (Courtland & Amani stand) would we have to agree? (Justin stands) that this (points to rectangle on board) and this (points to board) are regular polygons? (walks to other board) okay (Shatteryia, Kenjra stand up) I'm going to say it again. (writes "polygons" on board with the rest of the definition)
RL: Shatteryia's definition has three pieces. she says that a regular polygon IS a polygon. okay it has, (waves hand) <sides, and, angles.> so:: I want to know if you use that as your definition of regular polygon, is this a regular polygon? (Courtland sits and writes, Dilovan stands) those of you that agREE, sit down. those of you who disagree stand up (Louisa, Jeewar, Courtland, Nick, Vincent, Omari, Brandon, Micah stand) okay. um:: Brandon. <why do you disagree?>
Bra: well uh. I would say cause regular polygons have um equal sides.
RL: well Shatteryia's definition, says-it doesn't say anything about equal sides…<I-I'm not saying that, everyone accepts, Shatteryia's definition> but I'm saying IF we did I want you to play, like a PRETEND game. IF we accept it. okay IF we decided to call regular polygons those things that had sides and angles, then I want to know whether or not we would have to call this square (points to square on board) and this rectangle (points to rectangle on board) regular. (Louisa shakes her head. Students are talking to one another quietly.)
RL: okay according to Shatteryia's definition…the only requirements are that they have sides, a::nd angles.
Vin: but she's saying a REGULAR polygon.
RL: well she did say that but I-I just want you to go with it. IF we accept Shatteryia's definition of a regular polygon. IF. IF. then I want to know whether or not, we have to accept THIS (points to square on board) as regular and THIS (points to rectangle on board) as regular.
Vin: but you said the SQUARE is regular. But um but um.
RL: well, let's look let's reason with the definition. okay how many sides do you see here Vincent?
Vin: four.
RL: how many angles do you see?
Vin: two.
RL: two angles? can you show them to me?
Vin: I mean four.
RL: you mean four?
Vin: yeah
RL: where are they?
Vin: (points) over:: across
(RL points to the top left corner of the rectangle on the board)
RL: here? is this one?
Vin: yeah

DEFINITIONAL EPISODE #7
[00:32:15:23]

RL: what makes an angle again? I-I don't know if I ever got that. wha-what makes?
Lou: oh the sides.
RL: sides? okay (throws arms up) I'm going to do it. (turns and walks to the board) here's a side (draws a line) and here's another side (draws a line separate from the first one).
(Micah jumps up and down raising her hand)
Lou: no, that's not a side it has to be straight.
S: connected
RL: This is very straight and this is very straight.
(Many students speaking at once)
(Jeewar raises his hand)
Jee: uh:::
Lou: but it has to be the same
Sha: but it has to be connected
Ama: so that's what he was-
RL: oh CONNECTED. oh. connected sides. (draws two lines that are connected at one point)
Ama: that DOT was an angle.
RL: so::
(many students begin talking at once)
Jee: (waving hand frantically) no Dr. Rich,
RL: not an angle?
Mic: yeah it is.
Jee: Dr. Rich (starts walking to board)
Dil: It has to be 90-degrees.
RL: oh it has to be NINETY DEGREES. okay.
Mic: no.
Ama: no
(many students say no. many people are talking at once)
Ama: there's ACUTE angles
RL: (draws right angle on board and points to it) only these are angles?
Kay: there's different kinds of angles. there's like sixty so it doesn't have to be.
(noise quiets)

DEFINITIONAL EPISODE #8
[00:33:29:18]

RL: alright SO. we have to get untracked here a little bit SO. let's just say this. if we follow the definition, that Shatteryia proposed. we would have to accept THIS (points to rectangle drawn on board) as regular.=
Ama: =yes=
RL: =because <it has sides.> and it has angles. a::nd it's a polygon…and that's that. it's regular. according to Shatteryia's definition. if you don't like Shatteryia's definition what would you do to it? to make sure that this (knocks to rectangle on board) does NOT get in. (Micah, Kayla,

Amani, Jeewar, Courtland, Nick, Rhonda, Omari, Vncent raise hands—can't see Kenjra, Michala or Dilovan)
RL: um::. um::. I haven't called on Micah yet.
Mic: um. you would have to say all regular polygons have, the-e sa::me…all the sides have sa::me, (RL is writing as she speaks)
Dan: equal
Ama: equal
Mic: uh length
Dan: equal sides
Mic: <and um> the angles all:: meet one degree. oh not one degree but uh…uh like all of them have ninety degrees or all of them have sixty degrees or all of them-
RL: okay. can I use these words? (points to board)
Dan: °sides congruent°
Mic: yeah
RL: all the=
Mic: =yeah
RL: sides are congruent? all the angles are congruent? if I lay those angles on top of one another (shows with arms) I couldn't tell?

DEFINITIONAL EPISODE #9
[00:34:43:24]

RL: A-Amani?
Ama: um::…yeah but um I was waiting for you to like, pass some of that part so I could tell you when you was talking about that point part. when Vincent was talking about symmetries, you asking what was that called? that, was the angle, that you were trying to get us to say. not a vertex, that's- the point was an angle.
RL: okay. so. (turns to board, points to the square that Kendra had drawn) what's the difference (turns to look at Amani with confused look on his face) between a vertex and an angle?
Ama: well::, the vertex, well is like, when two things come together and makes like um, like four-more than um two things fittin' together it makes like a little,
Dan: °circle°
Vin: circle
Ama: <middle> circle
RL: it makes a circle?
Ama: mmmhmm
RL: so. if I have three things (draws an "x" on the board) four things. how many things? do I have here. THINGS? what are THINGS? I'm-I'm confused (shakes head)
Ama: like this (takes out piece of paper) like what we did, um…is put different shapes together=
RL: =yup
Ama: like the square and the triangle.=
RL: =yup.
Ama: and I think this was a hexagon (points to paper)
RL: you were trying to see how they would fit? yeah.
Ama: like these circle (points to paper) like if it makes like

RL: oh those CIRCLES.
Ama: (   ) that's the vertex.
RL: okay. so what is a cir-is a circle a vertex?
Lou: >yes<
Ama: not just like a
Mic: no um
Ama: plain circle it's like if it MAKES a circle.
Mic: see when all of them are together (gestures with hands in circular motion)
Ama: all of them (   ) together.
Lou: the middle where it like joins together is a vertex.
Mic: where all the shapes connect, yeah.
RL: okay what if I just have a square (points to drawing of square on board) like here.
Lou: that's not a vertex.
(lots of students talking at once)
Mic: you have to have more than one.
Ama: it's like if you have=
Kay: =you have to have more than one shape.
Dan: more than one polygon (   )
Ama: like <if you draw four squares together> when angles,
Kay: when all angle touch
Ama: together.
Kay: when all the angles touch (laughter)
RL: so, let's play. let's play. u::m

DEFINITIONAL EPISODE #10
[00:36:39:27]

RL: if I change my definition let's apply that and let's get back to this angle vertex thing? Alright. So. let's play the definition. IF I say that the sides have to be congruent and the angles have to be congruent? SHATTERYIA is this a regular polygon?
Sha: yes.
RL: look at the definition. what does the definition say Shatteryia?
Lou: sides and angles congruent which means they have to be the SAME.
RL: what does it say?
Sha: all polygons. all regular polygons have a polygon
RL: okay they're polygons WITH?
Sha: with sides (   )
RL: yeah and maybe I should say ALL sides? congruent. ALL angles congruent?
Sha: °yes°
RL: alright so Shatteryia. are all the sides here (points to rectangle on the board) congruent? huh?
Sha: yes if you draw it correctly.
RL: wh-if I draw it correctly?
Sha: mmhmm

RL: you mean? how could I draw it correctly and make one (points to side labeled "1") the same as two (points to side labeled "2")?
Vin: it's impossible. it's impossible.=
Sha: =no
RL: huh?
Sha: never.
RL: never, right? okay. SO. Shatteryia. looking at the definition that the class is using? the classes' definition, what most people are using?....°okay.° draw me something…that is::…regular. oh you put your glasses on good idea…come on up here and draw me something that is regular.
Sha: like a regular polygon?
RL: I want a regular polygon.
(short talk about sitting down)
RL: okay Shatteryia's going to draw a regular polygon. let's see if we agree. (Shatteryia draws a square) Courtland, is that a regular polygon?
Cou: yes sir.
RL: you think it is? how do you know?
Cou: because it has same sides and uh, well the sa::me sides <uh congruent and the angles.>
RL: and the angles are all the same? they're congruent too?
Lou: mmmhmmm.

DEFINITIONAL EPISODE #11
[00:39:27:00]

RL: alright. I want to get back to this angle thing. I would like you to take out a piece of paper and work with your partner and your table to draw outside of a shape, I just want you to draw, not a shape, but 5 different angles and how do you know that they're different.

[00:39:57.03]

[00:52:35:16]
RL: Nicholas, I'm going to start with you. You tell me you have 4 different angles up there, tell me what you're thinking. Tell us.
Nic: Well I was thinking if you have 4 of the same angles and they're turned different ways, it's not really the same angle, I mean they're the same angle, they're just turned different ways.
RL: Ok, can you -
Nic: I thought maybe they'd be different angles.
RL: What make them an angle? Tell me, tell us about how you're thinking that when you go like this, you make an angle, how you thinking about that?
Nic: (pause) I was just thinking if you had 4 for example, 4's just a straight line, that's an angle.
RL: A straight line is an angle?
Nic: Yeah.
RL: Why do you think so?
Nic: because say you're going to draw a square (draws a rectangle).

RL: Yup.
Nic: This part right here (circles one side) this (points to one of his angles) is part of the angle of the square.
RL: Okay, it's part of the angle of the square? How many angles does that thing have that you just drew?
Nic: 4
RL: 4, can you point to them?
Nic: (points to each of the sides)
RL: Okay, can you point to the 4 sides?
Nic: (points to the same things)
RL: Okay, what's the difference, in your thinking, between a side and an angle?
Nic: (pause) I don't really know.
RL: Okay, so for you right now, the way you're thinking about it, a side and an angle could have the same meaning.
Nic: correct
RL: Okay, so you were thinking that because these were oriented in different directions, they would be 4 different kinds of angles. Okay good, thank you Nicholas. Thank you for sharing your thinking. I'm going to ask, uh yes, Tim? You want to talk to us about yours? no? did you want to ask nick a question? No?
Tim: (?)
RL: Well why don't we let Courtland and Devalon (discussion about how to pronounce his name). Okay go ahead.
Jee: We are saying that this is an angle (points to angle labeled 60), mostly when you have a square, the si-the corner, for example, like here's one side (gestures over the arc he has drawn for the angle). This (points to the 90-degree angle) is what it would be like if there was like a square. One of the corners of the square.
RL: Okay would you point to the angles on Nicholas's drawing, just so I can, on the thing that he just drew - the shape that he just drew. In your view, where are the angles?
Jee: (points to a vertex) right there.
RL: How many are there?
Jee: there are 4 angles.
RL: they're what?
Jee: four angles.
RL: can you show me please?
Jee: (draws in arcs at each angle)
RL: does everyone agree with jeewar? four angles? thank you jeewar. So, continue with you
Jee: you can have a 60-degree angle, a 30-deg angle, a 90-deg angle, a 180-deg angle, and a 360-degree angle.
RL: So you agree with nicholas then that those are 4 180-degree angles? that he wrote?
Jee: yeah
RL: ok. any questions for this group from anyone else? louisa has a question.
Lou: why did you put, why did you put just half of a circle instead of like 90-degree. like a square that's saying it's a right angle.
Jee: you can do both ways
Lou: yeah, but if you did that in like a regular classroom, they would think that you're saying that's a different angle.

RL: Vincent would you like to comment upon Louisa's comment or do you have a different question.
Vin: I have a different question. Now if, now those 4 that nick did have the same are the same angles, just pointing in different directions, but you're agreeing with him. I don't know how you could agree with him, but they're the same exact angles, but just pointing in different directions.
Jee: well nick said that this (gestures over the side lenght?) not this (points to angle). We're saying that this is an angle and he's saying that that's an angle.
RL: Well nick was saying that was a side and he wasn't quite sure about how to distinguish a side from an angle, but he drew 4 different lines and it looks to me that what he drew is a lot like what you drew for number 4.
Jee: that's a 180-degree angle.
RL: that's a 180-degree angle. So. Is that what you're saying Vincent or are you saying something else?
Vin: I'm like asking, like how could, like those, like those four lines are the same exact angles, they're just pointed in different directions.
RL: okay, so in your view, they're not 4 different angles, they're just pointed in 4 different directions.
Vin: yes.
RL: so is that the point you're trying to make?
Vin: yes.
RL: okay, good.
Mic: I kinda agree and disagree cause if it they could be different angles if you added another line to it, where the corners met, and it could be different angles.
RL: Okay, so you could consider a way that you could adjust it so it might be different angles, but the way they stand right now-
Mic: they're all the same.
RL: they're all the same.
Jee: here's an example of one way (?) a 60-degree angle. A triangle (points to a triangle he has built around the 60 angle)
RL: uh huh.

DEFINITIONAL EPISODE #12
[00:59:30:14]

RL: um may I ask what makes something an uh what does it mean when we say that it's ninety degrees. what's a degree? I'm not sure I have understood THAT. what's a DEGREE?
(Courtland and Jeewar raise their hands)
RL: uh:: Courtland?
Cou: I think it's the size.
RL: it's what? the size? and what about how do you measure the size of an angle?
Cou: by uh::
RL: by what?
Cou: uh:: (shrugs) I don't know.
RL: okay uh (Jeewar raises his hand) yes Jeewar?

(shows that one starts at zero and by rotating quarter turns, the angles increase by 90-degrees each time)
RL: okay, so what's ONE degree?
Jee: a tiny turn.
RL: a very tiny turn? how MUCH of a very tiny turn?
Ken: not even
RL: not even half of a turn or a quarter of a turn? I agree. how much of a turn is a degree?
…
Dil: one eighth?
RL: one EIGHTH? one eighth of a turn? does everyone agree?
Jee: or:: or:: or a ten, out of three hundred sixth degrees.
RL: ten out of three sixty where'd you get ten?
Jee: like I was just thinking if I'm going here (rotates a quarter turn) that's like
RL: how many is that?=
Jee: =if that's 90-degrees-
RL: okay so we say a quarter turn is the same or equivalent to ninety degrees? (writes this on the board) okay so if I have a circle (draws a circle) and I start here (draws arrow pointing up) and I turn a quarter of a turn (draws arrow pointing to the right and motions a quarter turn with his marker) right? that's the same thing as 90-degrees? if I turn another quarter of a turn how much? (draws an arrow pointing downward)
Jee: one eighty. (writes "180°")
RL: if I turn another quarter of a turn? (draws an arrow pointing to left)
Jee, Vin: two seventy. (writes "270°")
RL: if I turn all the way?
SS: three sixty (writes "360°")
RL: three sixty? so if I:: so they're ninety what?
SS: degrees.
RL: DEGREES. ninety DEGREES. (writes "90 degrees" on board) degrees so:: HOW much is one degree? How much of a turn?
Jee: one out of three hundred and sixty.
RL: one out of what?
Jee, SS: three hundred sixty.
RL: okay. another way to think about a degree it's one out of three hundred and SIXTIETH of a turn. (writes "1/360 turn" on the board) everyone get up. >up up up<
(students stand up, talking amongst themselves) alright. I want you to hold your right hand up (holds up arm in a right angle. Students follow his lead) alright. I want you all to TURN one fourth of a turn in the right direction (everyone turns one quarter turn as indicated) okay lets go back (turns so that he faces the board. The other students do the same) okay. okay I'm gonna turn to my right (waves right arm). okay I'll start turning. you tell me when to stop when I reach a quarter of a turn. ready (rotates tiny steps at a time) ch-ch-ch-ch-ch-ch (he reaches about a quarter turn)
SS: stop.
RL: how many degrees have I come?
SS: ninety.
RL: okay watch this turn. (he rotates back to facing the board and then rotates with little steps again) ch-ch-ch-ch-ch-ch-boom (stops at somewhere in between 0 and 90)

SS: eh:: (sounds like a buzzer)
RL: how much of a turn is that?
Jee: sixty. sixty.
Ama: sixty.
Tim: about three six.
RL: about sixty you think?
S: yeah.
RL: alright.
Jee: maybe you
RL: (rotates back to face the board) tell me when to stop. I want to turn a half turn. ready? (he turns a quarter turn)
Ken, S: stop.
Jee, Lou, SS: no::.
SS: no.
Ama: come on, keep going.
RL: (rotates more until he is facing them)
SS: stop.
RL: alright. how many degrees was that?
SS: one eighty.
RL: how much? how many degrees are in an entire circle?
Vin: thirty-three sixty.
SS: three sixty.
RL: okay. I want everyone to turn…right…three fourths of a turn. let me see you do it. (students turn, several counting the quarter turns) how many degrees did you turn all together?
Lou: three sixty.
SS: two seventy.
RL: HOW many?
Mic: two seventy.
SS: two seventy.
RL: two seventy.
Jee: who said three sixty?
RL: alright now. I want to go back to where they started. (the students turn to face the front of the class. RL remains facing them) I want you to turn, one whole turn around.
(the students and RL rotate until they reach their starting place) (turns as he talks) one:: whole turn.
(students turn along with RL) okay. HOW many degrees did you turn?
Jee: three sixty.
SS: three sixty.
RL: alright now I want everyone to turn ONE three sixtieth of a turn.
Lou: what?
Ama: we just did it
RL: ONE three sixtieth
Lou: oh::
Ken: oh no we gotta do it just a tiny bit.
Ama: just a ti::ny little bit.
RL: uh uh. I should hardly be able to see the motion.

Vin: look. I did it.
Tim: oh. like this like this.
RL: okay a quarter of a turn would be like this (rotates a quarter turn) but a three SIXTIETH? <very very> small part of a turn. alright? so that's another way to think about what an an-what a MEASURE. what a degree is. we can think of how much (walks towards board) if this is a right angle…(draws a right angle on the board) we can think of it as:: how much we turn to go from here (points to one side with marker) to here (draws arc down to other side). how much do we turn? okay. and that is one fourth or NINETY three sixtieths. (walks to board again) so ninety three sixtieths (writes 90/360) … okay. how many nineties in 360?
SS: four.
RL: okay. (writes as he talks) ninety plus ninety plus ninety plus ninety
Vin: equals
SS: three sixty
RL: alright so:: if I divide the numerator by ninety? and the denominator by ninety?
Lou: you can't just
RL: what'd I get?
Dan: one
RL: one what?
SS: one eighty
RL: (shakes head) huh? how many nineties in three sixty?
SS: four.
Lou: you could have just multiplied ninety times four.
Vin: one twenty. Nevermind.
RL: nono. so. these are some things we need to think about when we think about what this measure means

[01:06:23.00]

DEFINITIONAL EPISODE #13
[01:08:39.24]

RL: And a vertex. where's the vertex nicholas? Come and point to it.
Nic: (gestures along the arc drawn inside the angle)
Lou: that's a vertex?
Nic: oh I forgot. I forgot.
Lou: you can't see a vertex. you can only see it going all (?).
RL: Vincent, where would you see a vertex here?
Lou: I think it's right there (points to vertex)
Vin: yeah, it's right there (points to vertex)
RL: you think it's right there? you agree with Louisa.
Tim: yes it is.
RL: Alright so. From now on, we'll say the point where these two lines meet, we'll call that a vertex. I'm just gonna say from now on, write this down in your math notebooks, we're going to call that a vertex.

[01:09:23.25]

DEFINITIONAL EPISODE #14
[01:13:23.04]

RL: Now here's a couple of questions for you. Suppose I had one line and it met another like this. ready? (draws an angle) and then someone said to me I'll draw you a different angle, watch this. (draws same angle with lines extended). Do you agree or disagree that I've now drawn 2 different angles.
Jee: disagree
Ken: disagree
RL: okay, if you disagree, who disagrees?
Jee: me
(a few students raise their hands)
RL: alright Daniel, why do you disagree. Oh. Rhonda disagrees too. First Daniel, then Rhonda.
Lou: Me, me too.
RL: yeah how come? Why do you disagree?
Dan: Why I disagree is that it's uh, can you say that again?
RL: Yes, I say that i've drawn two different angles here. And you disagreed with me. You said no you didn't. They're the same angle. So now I say to you. How - why do you think they're the same angle?
Dan: I think they're the - uh
RL: Cause look how much longer this is.
Ken: but it's still the same angle.
Vin:; but it's still the same thing
RL: what do you mean it's the same angle. Watch this, watch this. see I'm going to measure the angle from here to here (draws a line through the width of the angle) look how much longer that is than from here to here.
Ken: but look at the actual angles themselves.
Vin: why don't you scoot that one up.
Ama: yeah, scoot it up.
RL: oh you want me to scoot it - oh, if I scooted it up it'd be-
Vin: so (?) if scooted it up, it'd still be the same angle.
Ama: not the way
S: it'd be half of.
Ama: like half of it.
(students talk all at once)
RL: so I would measure from here to here and here to here (can't see gesture) and have to put it in the same place, if I'd wanted to use that as a measure.
Lou: If you want to use that to measure, you have to make it the same.
RL: alright so. alright. i'm gong to get rid of that and i'm going to get rid of that (appears to erase the lines). So, tell me if i thought about it without using these lines, how could these be the same angles.
Jee: right here (points to something - can't see)
Vin: yeah, right there. right where he's (?)

Jee: (draws in arcs in the angles)
RL: what do you mean -
Vin: it's the same exact angle.
RL: look all I see are squiggly lines there. What do you mean? but how do you know that? what would have to be true for them to be the same?
Cou: if uh, they was the same uh, same size.
S: same degrees.
RL: Same degrees? alright so if I wanted to know if
Vin: they have the same vertex.
RL: Alright, so here's a vertex. here's a vertex (darkens in the vertices). okay you guys have to
Vin: but they have the same exact vertex.
RL: you have to back off so that. So. if I measured how much of a turn I did (draws in a dotted arc for one angle), from here to here (rotates marker to show angle), right? ready? ch-ch-ch (rotates marker again) let's call that 90?
Lou: but Dr. Rich if you wanted to measure that-
Ken: uh uh, that's not 90-degrees
Lou: it's not, if you wanted to measure that, you'd have to make the line longer so then it would be like fair.
RL: why do i have to make the line longer? Why can't i just use this point and orient it on the line and just turn it like that?
Lou: it wouldn't make sense. Like
RL: what do you mean it doesn't make sense?
Lou: it wouldn't
RL: well. we'll have to come back to whether or not we actually believe this, but if I turn and I have the same amount of turn, let's say that in each case, I turned 85 three sixtieths (writes 85/360 under each angle), would they have the same - would they be the same angles?
S: yes
S: yes, no
SS: no::
Jee: yes.
S: yes.
Jee: what did you say?
RL: i said if I wnated to move, if I wanted to rotate this onto the other line or side, and I moved 85 three sixtieths of a circle this time, and 85 three sixtieths of circle that time, I want to know if those angles are the same or different.
Jee: same
RL: okay, they're the same. they have the same measure?
S: yeah.
RL: alright.
Jee: what's hte difference?
RL: well, i'm just asking.
Vin: it's measured
Ama: (?) magnifying glass
Vin: it's dependant on what's you, it's dependant on what -
RL: Suppose I took this angle (points to a ninety degree angle on the board), and I turned it (rotates hand) but I kept everything else exactly the same. So in other words, I did this (holds two

markers together at 90). Ready? I do this and then i do this (rotates markers, keeping angle). Have I changed the angle?
SS: no::
RL: What have I changed and what remains the same?
SS: direction.
RL: I changed direction
Lou: yeah, but the thing is -
RL: But the measure remained the same?
Vin: yes.
RL: so something did change and something did stay the same.
Ken: but it's not the angle that changed.
RL: okay it's not hte angle measure that changed, but the orientation did.

DEFINITIONAL EPISODE #15
[01:19:04.07]

RL: So. I never did ask you this question. Everyone keeps talking to me about straight sides. I never did hear what made something straight.
Lou: You have to make it straight, or else it wouldn't be the correct measurement.
RL: Now, look. I'm from Mars. I don't know what straight means. Someone tell me what straight is.
Lou: Straight means that
Ken: wait a minute. what does straight mean?
RL: yeah, what does straight mean?
Lou: straight means that
Vin: it's straight!
SS: (laughter) yeah
Ken: it's a line that goes down without (?)
Lou: it's a line that goes down without
Jee: parallel.
(students talking at once)
Vin: straight is straight.
RL: straight is straight? what?
Lou: it's a line that goes down without curving or
RL: (to Vin) that doesn't help me. Daniel. Alright chill. chill. chill. that's 60s talk for calm down. It's a little late, but here's what I want you to think about for the next time you have class. How could you tell somebody who didn't know what straight meant and couldn't actually see it, they would just have to do something to draw it, what would they do to make something straight? How would they know?

[01:20:21.16]

DEFINITIONAL EPISODE #16
[01:20:59.28]

Lou: we were discussing about the angle and how it could be different and how like the angles would be used in shapes and so like what I did (points to drawing of straight angle labeled "180") was that I used 180 um, agreed as an angle used for squares and rectangles. And, I guess like 205 degrees, mostly used for polygons, hexagons, octagons-
RL: where's the 205, can you show us? How do you know that's 205?
Lou: well how do you know that's 90?
RL: well, because, 90 is about a quarter of a turn, so if i were looking for 90 in this, i'd say that's about there, from there to there is ninety (draws a line in her angles and gestures along the arc). So that's why i'm asking you that.
Lou: are you saying that there's 90 lines in that area?
RL: what am I saying when i say 90? 90 what?
Ken: 90 degrees.
RL: it's a part of a rotation, right? not a line.
Vin: I got a question. How can, how do you know something is 360-degrees?
Lou: I know.
Vin: how do you know that?
Lou: do you have to count?
RL: alright that's a good point. we would have to make a definition, right? Because unless we define things, you're right. we never know what it is we're talking about. So we just said that we would like the total num-amount of turn to partition that circle into 360, but we could have partitioned it in some other way. Right? so. Just because we've all agreed in the past that a total, one whole turn will be the same as 360ths, 360 360ths. Right? and the reason for that actually kind a goes way back a couple of thousand years ago from the people who originally were thinking about this. they were operating in a different grouping. we grouped in ten, they grouped in 60s.
Vin: okay like, so but, so i could say a whole entire circle could be a 180.
RL: you could say it and then you'd have to show us what you mean. You'd have to define it. And once you did that, just like remember Shatteryia said to us, I would like a regular polygon to be, to have sides and angles. And we said, IF you agree to define a regular polygon that way, then we would have to allow this rectangle and this square (draws rect and square) both to be regular. But we said that we would like to define a regular polygon as having all sides congruent, all angles congruent. That meant that this was regular (points to square) and this was not (points to rectangle). But it is a matter of convention. And we start somewhere, like with these conventions, and then from there, we build, but unless we get out definitions right, unless we know what we're talking about, all our buildings are shakey. So, that's why i'm asking you these question. What is straight? Cause so far, everyone seems to be using it, but i'm not sure that we've actually decided what it means. As long as we've decided what it means, and we agree, then we can all use it the same way.
Vin: I wonder how these people come up with this.
RL: hmm?
Vin: i said i wonder how these people come up with these things.
Ken: yeah like the alphabet.
Tim: or like words
RL: well, we're going to try to give you an opportunity to come up with stuff. and then we'll see how you come up with stuff. I bet the way you come up with stuff is a lot like the way other

people come up with stuff. So, yeah, that's a good point right, how do they do that? Well i wonder if it's so mysterious or if we can do it too.
Mic: I bet we could do (?)
RL: huh?
Mic: I bet we could do it.
RL: I can't hear you Micah.
Mic: I bet we could do it.
RL: I bet you could do it too. We did some of it last year, didn't we?
Mla: okay. we got acute angles. (reading from what she had written on the board) acute angles are less than ninty degrees and are mostly used with triangles.
Lou: nuh-uh. also with some other ones.
RL: okay so you have a classification system for angles the same way we have one for polygons?
Mla: yes
RL: alright so:: what makes an angle acute? I wasn't sure sure I understood that.
Mla: it has to be less than ninety degrees.
RL: less than ninety? what makes it obtuse?
Lou: it has to be more than nin-ninety.
RL: more than ninety?
Lou: ninety or three hundred sixty degrees (shrugs) or something like that.
RL: or something like that?
Mic: I have a question.
RL: alright. Micah has a question for ya.
Mic: um well this just came to me.
RL: could you speak in a loud math voice because of this blower here?
Mic: I just thought of this and how is there an angle above three sixty? if there is. I don't know.
SS: (speaking all at once)
Mic: Is there an angle ABOVE 360?
Ken: well
Mic: like over like 360, 370 degrees.
Ken: it won't be a straight line
Vin: well since those people whoever they were made that up.
Lou: well since they created-
Ama: well why can't we make it up.
Vin: yeah that's i'm asking.
RL: (has written question on the board) we'll take as given that one full turn around a circle is 360-degrees.
Lou: it's like ABC, you go all the way to Z and you have to start all over again. That's the same thing as that.
Ken: I got it, I got it.
RL: so. that's something worth investigating.
Lou: It's like ABC, you start from A to Z and then you start all over again.
DL: Is there a situation in which you might see that sort of thing? Come up with (?)
(Kenjra has drawn a circle and said something - hard to follow)
RL: well we're saying you measure in terms of a turn.
Lou: unless if you made it a whole different style. You know words go on for- I mean numbers go on forever. But.

Vin: (?)
Ken: Who made up numbers? Why did they make up numbers?
RL: Jeewar, you have the closing comment of the class. Go ahead.
Jee: that's 360 and then you could go to 4 hundred and 50.
RL: how?
Jee: you could 360 turn (gestures in an arc) you make another type of bigger circle, same thing again.
RL: so jeewar - wait a minute, you're saying a bigger circle has more degrees in it?
Lou: but it'll still have 4 if you divide it. So it make no difference, it's still going to be the same.
Jee: no difference
Lou: unless you made more um parts into it, that'll make it different but
Ken: but wait a minute, don't it depend on the circle size? of 360 angle?
RL: you mean, so are you saying that if the angle
Lou: yeah, it makes more
RL: if the circles are bigger (gestures in circle) they have more degrees?
Lou: yes you make more degrees that (?)
Vin: the smartest person in the world who made this up.
RL: no no I don't think it's the smartest person in the world.
Vin: Okay, it's not the smartest person in the world, but you had to be pretty good to come up with these
Jee: dr. Rich
RL: well um, you think that we could come up with some of this stuff?
Lou: yeah, and then give it to the government and put it into the little education.
RL: Alright we are now going to investigate this question. we're going to see if we can come up with anything. The question is. And this is what math will be next time. We're going to investigate this (points to question written on board). Is there an angle above 360? More than 360?
Lou: unless you make more degrees.
RL: how do we make angles again? what do we do?
Ken: by squares.
RL: well, there are other ways of making angles. Not all the things here were squares.
Lou: But Dr. Rich, circles are mostly divided by even numbers.
RL: wait a minute. Let me ask this question. How do we make, how did you make an angle. What did you do? I didn't ask you how you measured it. How did you make it? What?
Vin: I drew straight lines.
RL: okay
Ama: putting them (?)
RL: okay you had two
Mic: putting two lines together.
RL: lines that met. And where they met, we called that a what?
SS: vertex
RL: a vertex. So you had two angles that met at a vertex. And we measured them by trying to think of how much of a turn it would be to move one, rotate one, onto the other. Okay? So that's, those are the conceptual tools I want you to think about when you think about this question. Is there an angle greater than 360?
Lou: can we write our explanation about what we think?

Vin: but i don't get how if 360 is a full turn, why would they create numbers over that?
Lou: numbers go on forever, so there's no (?)
Vin: so there has to be an angle over 360 if numbers go on and on. It just doesn't stop at 360.
RL: I don't know. It depends on how we think about it. Right?
Ama: And children are going to (?)
RL: yeah, they are.
(students talking)
RL: okay so next class we have to do 2 things. We have to figure out how we're going to investigate this question, okay? And then how we would reach a conclusion about it. That's what we'll be doing. We need to investigate this question. I'm coming back on Tuesday. I think. Mrs. Lucas. Thank you. I'll be back on Tuesday. In the meantime, I don't want you daydreaming, I want you thinking. So. Think about how you might answer this question.

[01:31:50.05]

DAY 2

DEFINITIONAL EPISODE #17
[00:02:54.20]

RL: Now when you say degree, what's one degree? Would you remind me again? Someone remind me. Oh alright go ahead Jeewar.
Jee: Uh a tiny bit like not even your body moves at all.
RL: So if I'm standing here. You stand and show me a degree. A turn of a degree.
(Jeewar stands and slightly turns his body to the left)
RL: Just barely moved? Barely rotated. Alright so if I wanted to be a little bit clearer about what a degree is cause we can't all get up and just move a little bit. hmmm. Shatteryia.
Lou: (raises hand) oh I know, I know.
RL: Tim? Help me here. What's a degree? How much of a turn is a degree? How can I think of it. I know it's just like a little bit, but how much?
Tim: One degree.
RL: One degree. Yeah. That's what I want to know. What's one degree?
Lou: I know
Tim: ...
RL: Just a slight bit I agree. (several students have hands raised) Yeah. Good. Who can add on to that? Dilovan.
Dil: One three sixty.
RL: One three sixty?
Lou: (raises hand again) I know.
RL: One three sixty of:: the notebook? (holds up notebook)
Lou: No it's one three sixteith of a circle.
RL: Oh of a circle. Okay. So.

[00:04:17.04]

DEFINITIONAL EPISODE #18
[00:18:45.28]

RL: Nick, come and interrpret this one for us. What is Kenjra trying to show us? Interrpret. What is she trying to show us? Other than I have a full head of hair. What is she trying to show us? About what I did.
Nic: That on the first one, you turned 180-degrees and on the second, you turned 90 degrees.
S: That's backwards
Sha: Cause she got a
Vin: That's backwards.
RL: Okay so we're not sure what this 180 refers to?
Nic: (shakes head) uh uh.
RL: But we see that there are two 90s. So it looks like we might (gestures over the drawing)
Nic: and there's a 2 (points to something on the drawing).
RL: Right, and maybe she means for us to grab the two 90s and make 180? Okay. Now, that's good. Alright, so this has some things that I did, right? I did, you can see htat it's me. Right? very clearly it's me. Who else could it be? and I turned this time, how much did I turn?
SS: 90
RL: 90. And then this time?
Vin: 160.
Jee: 170.
Vin: 160
S: 180
Lou: no
RL: somewhere between 90 and 180?
SS: yes
RL: alright.
Nic: since you put the 180 up there, since you put hte 90 right there by the 90, why would you put the 180 up here and not down here?
RL: Okay, so thank you. Have a seat. And, Tim, come on up and help us udnerstand what this person's trying to show and how's it alike and different than what Kenjra did?
Tim: This person's trying to show that the first time you turned 90-degrees (writes 90).
RL: okay?
Tim: And this one (points to Kenjra's) What did you want abou this one again?
RL: Well I just wanted to see what was alike and different. So, like for example, Kenjra has the 90 represented right? The same way this person does, but it's not quite as clear that the 90 refers from here to here as it is in this one (gestures over arc). But what about this (points to other drawing). What are they trying to show here?
Tim: They're trying to show like a second turn, you turn like a slightly turn after you turn 90 degrees, you like turn 179.
RL: I turn something less than 180?
Tim: yeah
RL: yeah. OK. Thank you.
Ken: Dr. Rich.
RL: Yes.
Ken: That 90 in the middle that means you turn 90-degrees twice.

RL: But did i? If i turn 90-degrees twice, once, twice, wouldn't I be facing in this direction?
SS: yes
RL: Okay, let's see what i did. I went 90, and then I went (rotates)
S: less than 180
RL: some more, right? So somewhere. Okay? Uh what othere diagrams are like this one. Who did this one? (points to one drawing.)
Lou: (raises hand)
RL: okay so Louisa. What did you have in mind when you made this?
Lou: well um you didn't really turn like all the way, like 180-degrees. So I was like, you turned less than (?)
RL: okay. now. Let's see (points to drawing.)
Vin: yeah that was mine. That was mine (laughter)
RL: this is different.
Vin: I didn't totally make it that way
RL: So Vincent, would it have been a little better maybe to show this? (points to another drawing) since we're talking about a part of a circle?
Vin: oh okay.
RL: yeah okay. Michaela? Did you do this one? In this one, I see that a circle has how many 90-degree pieces? We can see that with Michaela's. And she said the second was over 90-degrees, but she didn't tell us how much over. Right, but she's telling us that it's over. Alright. So. Now, what I want to know is, before we get to Micah's question. I want to know how you know. You all told me that I turned in this direction. And I'm looking at this and you know what I think I did? I think I started right here (puts marker on one line) and went here (rotates marker counterclockwise).
SS: What?
RL: What? What.
Lou: He's rewinding himself.
RL: No. Suppose. How would you know the difference between this (rotates body counterclockwise) and this (rotates CC). Are they the same thing?
SS: yes.
Lou: it could be.
RL: Well what's the same about them? Nick
Nic: they both make 90 degrees
RL: thank you but what's different? Vin?
Vin: cause one goes the, one goes to 360 or 0 degrees and one goes to 90 degrees.
RL: what do you mean?
Vin: okay, if
RL: if i start? how would i tell?
Vin: okay, okay when you do this (stands up) now when you do this, you go back to 360. (stands facing the board)
RL: or?
SS: 0.
RL: okay so I agree that they're different motions. Lou?
Lou: um, 90-degrees can potentially start anywhere because they all have the same size, like (?) a perfect circle, so.
RL: Okay, so you're saying look, the amount of the turn is 90 degrees (writes "amt of turn = 90")

Lou: and so it's the same
RL: right? you're saying that.
Lou: cause like you just added 90 plus 90 plus 90.
RL: okay but i want to know this. And I want to know how i do this. Omari. How would I tell somebody the difference between this (rotates 90). Here, everyone get up. Alright, so you're facing the wall. Face the wall. Alright now do what I did the first time. (students rotate 90 CC). Now, now that you're there, go back to where you started from. The way I did. Would you agree that those are two different motions?
SS: no
SS: yes.
RL: they're exactly the same?
(students talk all at once)
RL: alright. you can have a seat. have a seat. Shatteryia, how are they the same?
Shatt: cause like rotate like this is like 90-degrees and then you turn to 0.
RL: okay.
Shatt: It's like you going back and then you go from 0-degrees to 90-degrees.
RL: okay so in both cases, I turned 90. A total of 90? 90 360ths of that circle.
Shat: yeah
RL: okay. But. What did i do that was different?
Lou: All the way to 180.
RL: Did I turn in the same direction? Rhonda?
Rho: No you went from 90 to 0 and from 0 to 90.
RL: okay. So I went, if I label this 0, this turn is 0 to 90 and this turn is 90 to zero (writes "0 --> 90" and "90--> 0") Okay so if I wanted to make a quick way of representing this so that people could follow which direction I was turning? What would you do? Do it on your paper. I want a way of knowing which direction I turned. And I want to be able to look at it and see right away oh what direction people turned.

[00:28:01.15]

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[00:33:45.21]

RL: Alright, let's start out with Justin.
Jus: um i had
RL: big loud math voice Justin. Got to be able to hear you back here.
Jus: um we went from the start to the, to 90-degrees, then you went back and went from start to 170-degrees.
DL: I didn't hear that.
Jus: you went from teh start to 90-degrees and then back to the start and then you went to that (points to the end).
RL: okay. okay so, how would you tell the difference from when I started and wound up at 90-degrees and I was at 90-degrees and I went and rotated and I wound up back where I started? how would you show the difference?
Jus: because you went to 90-degrees, then you went back to start then.

RL: okay so you would show me with your hand what I did? yeah and then what? (Just gestures along the arc from start to end and then back)
Jus: like that
RL: okay fine. thank you justin. um. okay so justin, i want you to take a look at what other people did and then i'm going to ask you to compare what they were thinking to how you were thinking. Rhonda, could you describe what you did. where's Rhonda?
Rho: you went from 0 to 90 and then you went back from 90 to 0 and you said is it the same? I said no and you said draw (?)
RL: and then so you used arrows?
Rho: yes
RL: to show a difference in direction?
Rho: yup.
RL: okay, thank you. Justin, what do you think of that?
Jus: she used arrows. she went in a circle but i didn't.
RL: well she's just trying to show you two of them, the difference between going from 0 to 90 and 90 to 0.
Jus: (?)
RL: are they the same number of degrees?
Vin: yes.
RL: huh?
Jus: yes.
RL: yes, no. No? how do you know? can you show me?
Jus: this (points to rhonda's drawing) all the same degrees but if you turn the whole thing it is 360 degrees.
RL: but how much did I turn? use your marker there and show me what I did. \
Jus: (draws a line along the path) you went like that and then you went back. (gestures along the arc CCW).
RL: okay, did I walk along that, out of that circle
Jus: yes (?)
RL: did i? (pause) did I walk that circle? I mean it's okay to, but look (places marker on line and rotates it). Yeah so i turned that much of the circle? and then i turned back (rotates marker CCW). Okay so Rhonda has shown this direction and then this direction (writes over her arrows). So how much did I turn each time? how many degrees?
Jus: 90
RL: okay but what was different?
Jus: it was different turns?
RL: well different, can i use the word direction? Alright. where's yours?
Cou: the small one.
RL: here? so you're showing me the first time i did it. the difference between the first turn and the second. okay, but what i would like you to think about is what's the difference between, i want you to get up courtland. i want you to turn 90.
Cou: (rotates 90 CC)
RL: now turn back to where you started 90.
Cou: (rotates 90 CCW) I don't know
RL: okay turn with me. Okay we're facing this way. Now i'm going to turn in this direction. ready? turn 90.

Cou: (turns 90 CC)
RL: okay. now turn back to where you started
Cou: (turns 90 CCW)
RL: how much did you turn back?
Cou: 90.
RL: 90. okay what was different about the two turns?
Cou: (?)
RL: huh?
Cou: I went in different directions.
RL: different directions. yeah. yeah?
Cou: uh huh
RL: okay so one way we could show different directions are with arrows. Right? okay. Great. Now you have a question.
Vin: I have a question for Justin. With his picture. Why did you go to uh, not, why did you uh do the thing where he did 175 degrees or 160 degrees or whatever it was. Why didn't you just do the 90 degrees.
Jus: Cause I put it together.
RL: yeah he was showing us all 3 things that I did. That's why. Yeah, that's fine. Okay. have a seat everybody. thank you. Now we have to get back to the question. is it possible to turn. okay you had questions here (speaking to Ken) are those questions, do you have questions still about this idea of the direction? Alright. Okay. Micah. Amani. Yeah you two. Uh, what do you think of your question now? What have you concluded?
Mic: I've concluded that no. That there may be, but i don't know really.
RL: that's a heck of a conclusion. I maybe I don't know.
Mic: first I thought no. And I've been thinking about it and i'm maybe now.
RL: alright so you're on the maybe side.
Mic: yeah.
RL: you've gone from no to maybe. I guess, is this a group opinion?
Kay: yeah
RL: or this is group of no maybe?
Mic: I have another question.
RL: another question.
Mic: yeah kind of like that one.
RL: okay
Mic: Is there a um degree in negative, and if there is, is there a degree a hundred and sixty negative? and add on to that one
RL: (starts writing) Is there a
Kay: If three hundred and sixty is negative, would it still mean zero?
Ama: yeah.
RL: Is - uh if we turn, uh can I rephrase it this way? If we turn a negative 360 degrees, will that still be equivalent to
Ama: zero
RL: (writes the questions on the board as "Is there a negative degree?" and "If we turn -360^o = 0^o?") Is that right?
Mic: yes.

RL: okay Kenjra. What do you think? Do you think it's possible that we could have negative degrees?
Ken: if you get a positive degree, then you should get a negative degree.
RL: okay you think it's possible. Louisa, what do you think?
Lou: i think it's possible cause i tried it on (?) and um
RL: Louisa i can't hear you and that means that no one else can.
Lou: I think it's possible like Kenjra's saying, if there's a positive degree, then there's a negative degree. And I think it's, I kind of think it's impossible where Micah asked the question about can there be more than 360 in a circle. I think it's impossible.
RL: you think it's impossible or possible?
Lou: I think it's impossible.
RL: IMpossible. She said not, not 360 in a circle, cause we just said we just mad ethat us and we're going to divide that up into 360 pieces, but what she's asking is could any ANGLE be more than 360.
Mic: yeah
RL: we could divide the circle as many times as you like, it's just that people have to agree about something so we all said that we're going to divide it into 360 parts and we're going to call each part a degree. Rhonda.
Rho: If they're negatives and they go that way (gestures in a circle) wouldn't hte (?) go that way?
RL: uh Rhonda has a conjecture. She has a way of interpretting this for us. Would you stand up so that people can see the gesture you're going to make? Cause i think it might be hard otherwise.
Rho: if the circle's that way (gestures clockwise), wouldn't the negatives have to be that way (gestures counter-clockwise)?
RL: Here's what I want you to do. First. I want you to discuss with your table group and I'm going to give you five minutes and I'm going to ask one of you to act as the person who represents the table, I want to know what you think of Micah's first question. Is there any angle greater than 360-degrees, if we accept that their are 360-degrees in a circle. Second, is there such thing as a negative degree and if so, what does it mean?

[00:44:19.06]

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[00:53:53.10]

Nic: what we decided is like when Dr. Rich turned around twice and he went 720-degrees, we thought like if you took a 360-degree spinner and you added it with another 360-degree spinner, you get a 720-degree spinner (he writes this as 360 + 360 = 720 on the board) So it'd be like (draws circle) this is 360 right here, and that's 720 right there (writes 720 on top), and you turn that much (draws arc around), then it's just like going on a 360 spinner (draws a circle and labels it 360). You turen that much, it's like going on a whole circle.
RL: so. are your, is your conclusion that there are angles greater than 360 or not? what's your conclusion?
Nic: there are.
RL: there are. Alright. How many people follow Nick's argument? (a few raise their hands). Alright. Rhonda could you restate the argument please?

Rho: they're saying like if you go around the circle twice, then you'll have a degree higher than 360.
RL: okay so their argument is, that if I, let me know if I have this right guys. I'm going to take this circle here and I'm going to mark it red where we start at zero, so we can all see. alright. (holds up protractor) And their argument is this I believe. (rotates the protractor) How much have we turned?
SS: 360.
RL: (rotates again) how much have we turned?
Jee: negative 360.
SS: no
Jee: it went backwards.
Lou: no it didn't.
S: 720
RL: so just in terms of, forget about sign, we haven't talked about that yet, but just in terms of the number of degrees, I'll do it again. Ready? (rotates the protractor).
Lou: 360.
S: negative
RL: forget the negative. forget the positive. How much?
Jee: 720
RL: 720. Is there anyone who disagrees with that? (a few raise hands) okay, go ahead Kayla.
Kay: when you turn that way, it's negative, so it'd be negative.
RL: okay if we just forget positive or negative for just a moment, what about the amount? is the amount. Do you agree with the amount as 720? Okay Micah do you disagree?
Mic: I disagree because if you turn one time, 360 adds to 0 because you move nothing, so like 360, zero.
RL: but did I move nothing? watch this? (rotates the protractor) ch-ch-ch.
Mic: yeah you move, but it doesn't LOOK like it at the end.
RL: so sometimes, we might want to know about what did it all amount to, what's the result. And the result is as if I didn't move, but could there be other times when it would matter? Do you ever see anything called RPM on a car?
Ken: Rose Park Magnet
RL: Rose Park Magnet. no. sometimes we could think about times when we might want to know right?
Lou: the speed when driving?
RL: yeah sometimes we might want to know just how many times in a period of times or just how many times we've gone around right? but lots of other times we might not want to know. Okay so. Is it possible that we might have a rotation more than 360?
S: maybe an angle
RL: sure right? but we still have this question of whether that's useful. we can certainly see it's possible. alright, thank you. Your next, what'd you decide about the negative business?
Dan: We decided there is a negative cause like when we were like, when we heard everybody talking, cause when you were going to, like a normal circle will be going to the right side. So, it would be, so would be gong to the right to make a whole circle. So, everybody's saying that if (?) a positive degree, then going to the left side might be a negative degree and sound slike it would make sense. So in conclusion we decided that there is a negative degree.

RL: there is a negative degree. And for you, a negative degree is a matter of which direction that you're turning? Okay yeah, how many people agree with that interpretation? Courtland do you agree?
Cou: hold on say that again.
RL: ok i think you better restate that cause courtland's not sure if he agrees or not.
Cou: can you say it over?
RL: say it again please?
Dan: if like, if the beginning of the circle you can go to the right side until they make a whole circle (gestures in a circle) and I guess people were saying that that would make a positive degree or something? And then if you go to the left side (gestures the other way) it would make a negative degree.
S: I agree.
RL: okay how many people agree, put your hands up. (most students raise their hands.) How many people think this idea is like absurd? (no one raises hand) Last year for those of us who were here last year, what does this remind you of?
SS: number walks.
RL: when we did our number walks, when we went in this direction we called it positive. when we went in this direction, we called it negative. right? when we did our number wallks? right.

[01:00:51.08]

DAY 3

DEFINITIONAL EPISODE #19
[00:12:57.00]

RL: Alright. So what is uh:: fifty? What part of the circle? Anybody?
Lou: Three hundred and sixty.
RL: No uh, we know that three hundred and sixty degrees is all the way around right? But I'm not asking you that. I'm asking you a very simple question. What does fifty degrees mean? How much of the whole circle?
(Courtland raises his hand)
RL: Kayla.
Lou: It's like
Kay: Fifty percent.
RL: Well fifty percent, wouldn't that be the same thing as 50 over 100 (writes on the board)?
Jee: (waves hand) I know I know.
RL: Wouldn't that be the same thing as one half? Wouldn't that be the same thing as one eighty over three sixty? Divided by three sixty? Wouldn't it? Yeah. So we wouldn't say fifty percent would we?
Kay: No (shakes head)
RL: No. Okay. Jeewar?
Jee: Fifty out of three sixtieth.
RL: Thank you. Fifty three sixtieths. (writes on the board "50/360")

RL: Fifty three sixtieths. What's another fifty three sixtieths? How much is that? (writes "+ 50/360" to the right of the "50/360")
Jee: One hundred three sixtieths.
RL: One hundred three sixtieths or? How many degrees?
Jee: One hundred degrees. (RL writes "=100/360")
RL: One hundred degrees. (draws an arrow and writes "100 <degree symbol>")

DEFINITIONAL EPISODE #20
[00:14:15.00]

RL: Alright. So an acute angle, since someone brought it up, is an angle that's less than ninety. Now I'm going to ask you this. Okay. I am going, to, find a ruler (walks around)
Sha: Here.
RL: Thank you Shatteryia. And I'm going to extend this line like this. (extends one of the angle's sides) And I want you to do the same thing. I want you to take the angle that you had and extend one of the lines. Come on.
(Students draw)
RL: Okay now extend the other line. Make it even longer. (makes a gesture--holding out his arms) Extend those lines. Longer, longer.
Mic: I can't go much longer.
RL: You can't go much longer? Okay. Question. Question for Nicholas. Will the angle measure when I extend the lines be less, the same or greater? And why?
Nic: The same.
RL: Nicholas says the same. Stand up if you agree with Nicholas.
(Micah, Rhonda, Shatteryia, Justin, Brandon, Jeewar and Courtland stand up. Can't see Dilovan or Kenjra or Michaela.)
RL: If you are not sure, stand up.
(Dilovan, Courtland, Michaela, Louisa, Brandon, Omari, Micah, Kayla, Tim, Amani, Shatteryia, Vincent, Daniel, Justin)
Nic: I'll tell you why::.
RL: Alright Nicholas has an explanation for you doubters. See if this is persuasive. Get up there Nicholas.
Nic: Alright. (Walks to the board and points his protractor to the vertex) If you have zero right here (taps the protractor near the vertex) And you've got your thing up there and zero's right here (taps protractor near the vertex again), and you extend the line (gestures toward the top line)
RL: When you say zero's right there and you go (taps his hand on the board near the vertex)
Nic: When you have it there (places protractor so that it is centered at the vertex) When you have it like that.
RL: When you have it like what? I see zero right there (points at the protractor, towards the top-zero is not on the angle side). Is that how you measure it?
Nic: Uh I forgot. Uh dang. What I'm saying is that if you have it like this (places protractor momentarily on the vertex), that line (gestures upward in the path of the top side of the angle) it can-

RL: Now wait a minute. (makes the same gesture, except with his arm). I don't know what that means.
Nic: This line (points to the top side of the angle) can be as long as it wants to be but it's not going to change.
RL: You say it's not going to change it but how come?
Nic: Because it's not going to change the degree. If it, if you turn (moves his hand in a curve on the board)
RL: Why wouldn't it change the degree? It looks longer to me. It's got to change the degree.
Nic: No.
RL: No.
Nic: This would change the degree. If you move this (points to top side of the angle) up here (gestures in an arc to the right) That would change the degree.
RL: Okay so if I actually took this (lays marker over the side) and moved it like this (rotates the marker) then you could claim the degree would change but otherwise it won't?
Nic: No.
RL: Well show me with the protractor that it's still the same.
Nic: (places protractor centered at the vertex and pauses)
RL: What's he doing up here? What should he be doing?
Nic: I forgot.
(Amani has her hand raised)
RL: Someone come up and help him. He's, he's having trouble. I need a protractor user. Amani get up there.
Cou: Oh me.
RL: You're elected.
Nic: Anybody who knows how to use this.
RL: A-mani. Amani, that's your name right? Help.
Ama: No. Just Mani (laughs as she walks up there)
Nic: Do you know how to use a protractor?
RL: Okay, those of you that are standing can sit down because you're going to find this persuasive in just a minute I think. Maybe.
Ama: (places protractor on angle, but not centered at vertex)
RL: Okay wai-she's putting the protractor at the very end.
S: No.
Ama: (moves the protractor so that it's centered)
RL: Okay, you put it. You line the center up with the vertex. You put it up with zero (points to the protractor, where it aligns with the bottom side of the angle) and:: you find out how many. Okay. So. If you do it this way then you kind of have to go backwards. Zero, three twenty.
Ama: (flips protractor over)
RL: If you do it the other way it gets a little easier. And I like to keep it easy, right? Okay, is that about the same?
Ama: Uh
RL: Well we have to get it on the, right? I don't know where the vertex went off to. Could you (waves his hand)
Ama: (removes the protractor from the board) I was making that the (inaudible)

RL: Kind of like (tries to redraw the angle) We kind of lost part of our angle. Alright you try it. Try it with your protractor and your angle. What happens to the angle measure? Nick says it's going to stay the same.
Nic: It will.
RL: Because?
Vin: It stayed the same.
RL: Because why Vincent?
Vin: It stays the same because um because the angle is still like the, it's still, the angle hasn't changed, it's only the lines that have changed.
RL: So if we measure the angle with rotation, has the amount of rotation changed?
S: No.
RL: Look. (places marker on a side of the angle) Everyone look. (rotates the marker, keeping one end fixed to the vertex) Okay? So the amount of rotation. Do it with your pencil on your angle. Look at the amount of rotation. Does the amount of rotation change when you change the lengths of those lines?
S: No.
RL: If the amount of rotation doesn't change, then the measure of the angle can't change. Okay so. Nick. Justin, sorry. Justin.
Jus: (shows RL something in his notebook)
RL: (speaking to Vincent) What we'll do. Okay. This is your job. When Justin has a question, you're supposed to help answer. Okay. That's why you're table partners. We'll put it right at the center here, right buddy? And then what we'll do is we'll follow how much it goes around. See? How much it rotates. Can I borrow your pencil? So we'll go from here, which is zero, all:: the way to about there. So that's fifty-five to fifty-eight. Does that make sense?
Jus: mmm hmm.
RL: Okay, do another one. Vincent, help Justin out to do the next one. Alright. Now. We've established I think, but Micah has a question so I'm not sure we all agree.
Mic: Nah I'm the same way.
RL: You what?
Mic: Nevermind.
RL: Okay. Now I have a question. Alright. Does anything change? Maybe the angle measure stays the same, but does anything change when I extend those lines? Is there some way that anything changes?
Jee: No. No. (shakes head)
RL: You can't think of anything that changes?
SS: The length.
RL: Okay the length is changing. Right? (silence)
Lou: (raises her hand)
RL: (walks over to get a meter stick and then walks back to the board and lays the meter stick over the angle, vertically) What about the distance from here to here (draws a line from one side of the angle to the other).
S: It's getting bigger.
RL: mmm hmm. (draws another line further out)
Lou: Dr. Rich does the length really matter in (inaudible).
RL: Louisa asked a question. She addressed it to me. Someone else has to answer it. Okay. Omari, you're elected. Thank you for volunteering. Louisa, address your question to Omari.

Lou: Does the length really matter? When like when you do the little angle thing (waves her hand in the air). Because mostly, so people that do like the buildings and stuff, they think that (inaudible).
RL: Well for the measure of an angle Louisa, and that's what we're talking about, (draws angle on the board) Deople that do like the buildings and stuff, they think that (inaudible). Well for the measure of an angle Louisa, and that's what we're talking about, (draws angle on the board) Does length matter?
Lou: (inaudible)
RL: Pardon?
Lou: No.
RL: Why not?
Lou: Because (inaudible)
RL: Because what?
Lou: It still would be the same.
RL: Come up and show me. (points to the angle he just drew)
Lou: (walks up to the board)
RL: Nick
Nic: In certain cases it does.
RL: Show me. You find a case in which you think you changed the length and the angle's gonna change?
Nic: (inaudible)
RL: Okay right there. You've got a protractor, Louisa? (hands her a protractor) Measure that. How much is this angle (points to the angle drawn on the board).
Lou: (places protractor with center far from vertex)

[00:23:16.02]

DEFINITIONAL EPISODE #21
[00:48:56.05]

RL: Someone came along and said, 180 is the largest angle and I asked you what were they thinking and Rhonda I never did give you a chance to talk about that. I want to do that now.
Rho: we said, well, our table said that uh in a circle, when they have it in a circle, how they have it, they usually have it 360, 90, 180 then 270. We thought that they're thinking it went 180, 90, then 180, then back again.
RL: so you were noticing a pattern? 90, 180, you add another 90, you add another 180?
Rho: (nods)
Tim: yeah. Like if you go straight down to 180, and then you turn again, you have another 180.
RL: okay good. very nice. So you're thinking about putting the 2 pieces together. Somethign like this, right? (points to an angle labeled with interior and conjugate measures)
Tim: yeah
RL: anybody else have other ideas about this? kayla, Micah, Amani? What'd you come up with about this 180 idea.
Kay: We came up with that
RL: loud voice please kayla

Kay: we came up with that 180 is like the 0 in the number line cause it's like in the middle and um, when you draw the circle and you split it into 4 different part, 180 is a straight line down the middle.
Mic: and then like 90 is kinda like 270 because it's just, it's just flipped around.
RL: okay so you're saying you go a 90 here, you go another 90 here (draws arc arrows around the circle) and you get back to this 360 or 0. Okay? very nice thinking. Now let me tell you exactly what people are thinking about. Um. it's something like what you're thinking but they say this. Because we measure things in terms of the circle, they say that if you know one angle, you automatically know the other one. What do they mean by that? if you have 124, could you find the other angle? What was the other angle? Nicholas.
Jee: 236.

[00:51:49.07]

DAY 4

DEFINITIONAL EPISODE #22
[00:05:27.10]

S: octagon.
RL: Okay it's an octagon. What makes it an octagon?
SS: 8 sides.
RL: okay so it's a polygon, it has 8 sides. (lists on the board "polygon," "8 sides") Is that it? Any 8-sided polygon will do?
SS: no
RL: no? Well you made it you can't talk. Alright, someone else. Omari. What do we have to worry about? what else do we hav eto consider?
Oma: sides.
RL: What about the sides? Yeah? Who said that? I heard someone say something. Tim? Did you.
Tim: Nah, she said (referring to the sub)
RL: oh you said that. okay. thank you. you would be right. alright so, we're going to call this thing where the sides are all equal, they're equivalent, they're congruent, right was the word we used?
SS: yes
RL: meaning that if you put one right on top of the other, can't tell the difference? They're exactly the same length?

DEFINITIONAL EPISODE #23
[00:06:41.08]

RL: How do you know tha::t…these angles are obtuse? Micah?
Mic: They're not in 90-degrees. They're above 90-degrees? °I think.°

RL: They’re not in 90-degrees they’re above 90-degrees? Is that what you’re saying? (positions marker on the octagon Vincent drew) so ninety would be like this and you’re thinking they’re greater than that? okay.
Mic: °yes::°
RL: okay:: so your test is LOOKING. Omari? (walks over to where Omari had drawn his polygon and points to it) you had a similar idea?
(Jeewar raises his hand)
RL: Jeewar?
Jee: about the eight sides. like how is it congruent? because (holds up hand) like an angle like that (motions to form an angle that looks ninety)
RL: yeah=
Jee: =(pointing to Vincent’s drawing) on the six, in the six and the five, on it.
RL: (looking at the drawing on the board) I’m not sure everyone over here understood what you meant. Maybe you could SHOW us.
Jee: (walks to the board and draws an arc in the angle.)
RL: oh oh oh, who did this? Dilovan. Are you saying what Dilovan is telling us over here?
Jee: right here, look.
RL: so Dilovan pulled it out so we could see it. And you're doing the same thing? Okay
Vin: that's an error, I just accidently did a little dent in there.
RL: oh you didn't mean to.
Vin: no.
RL: that's okay, so that's a drawing that we all understand what you mean.

[00:08:21.27]

DEFINITIONAL EPISODE #24
[00:14:58.15]

RL: well you know I’ve gotta tell ya. I have to say. I’ve been here for two weeks and I never heard you ONCE, tell me what you meant by polygon. You know, first it was vertex and we kind of got that squared away I still don’t know what you mean by polygon, I STILL if I went to Mars and read your ideas about polygon, I might think it's a bottle. Rhonda?
Rho: What?
(some students talking)
Rho: (quietly) A polygon is a=
RL: =Okay wait a minute. Rhonda's on, uh has the floor. I'm gonna - Omari may I erase your piece of art just for now? Nice thinking. Okay Rhonda. A polygon is? I'm going to write a, polygon is. (writes this on the board) Yup.
Rho: Something that has all the same sides. Has the same sides and the same angles.
RL: All sides
Rho: Are the same.
RL: Are the-what's that word we use when we mean lay down on top of one another? (shows with markers)
J: Congruent. Congruent.

RL: All sides are congruent. Cause that's what we mean by equal here. They're the same length.
Rho: and all angles are the same.
RL: And all angles are the same. (adds this to the written definition) Are congruent. Okay, we could pick one up and stick it on the other. Okay so:: Is that it. Is that all we need?
K: I disagree (raises her hand).
RL: Kayla.
K: That's for a regular polygon.
RL: Oh. This is for a kind of polygon called REGULAR. (edits the definition on the board) Regular. It's a regular polygon. Alright well.
Jee: Irregular polygon.

[00:16:54.04]

DEFINITIONAL EPISODE #25
[00:16:59.02]

Jee: A circle. A circle.
RL: Is a circle a polygon?
SS: No::
RL: Well. Question. (writes on the board, "Is a circle a polygon?")
Vin: It doesn't even have sides.
RL: Al::right.
Vin: It has one.
Jee: It has zero.
RL: Still that, if you tell me that they're irregular and they can be polygons, what do I, how do I have to change this definition? Does everyone have this definition (taps on the board) in their math notebook?
SS: no, yes
RL: Well I think you better put it in there, cause we have to get a definition for a polygon, and so far, WE don't have one. So we're not sure exactly what we're talking about when we say polygon. We have not yet agreed. Okay some people said they want all the sides to be congruent and all the angles to be congruent and people said yeah, we like that, but it's a special KIND of polygon, it's called regular. Okay. But you know what. I still -then people there could be irregular ones. But, according to that definition so far, you can, we don't have irregulars, we only have regulars. (pause) So what are we gonna do? What are we gonna do?
Jee: I don't know. Do you know?
RL: Not if you don't. Since it's up to you. We right now do not know what we're talking about.
S: Nope.
RL: Right.
Lou: Dr. Rich.
RL: Yes.
Lou: We can list why we think a circle's not a polygon.
RL: Well how can you do that when you don't know what a polygon is yet? How do you know what's not?

Lou: What we THINK. What we THINK.
RL: What you think?
Lou: Yes.

DEFINITIONAL EPISODE #26
[00:18:33.05]

RL: Well how bout tell me what a polygon is before you tell me what it not is.
Jee: A polygon is all sides - it says on the board
Lou: A polygon is-
RL: No. We said here that REGULAR polygon has these properties Jeewar. But-
Jee: There are only two kinds, a regular and an irregular.
RL: Good. Tell me what makes a polygon irregular. (about 3-4 students raise hands) Okay Micah.
Mic: Um, it has different s-sizes of sides. The sides aren't congruent.
RL: Okay, so it's not necessary, it has to have sides. Yeah, okay. (writes)
Mic: And it has to have angles.
RL: And it has to have angles. (writes on the board) Okay. Jeewar, did you want to add on to that? Yes?
Jee: That all uh - nothing is the same, like the angles aren't the same, the, sides aren't the same.
RL: So are you tell me that in an irregular - a polygon that is NOT regular, no two angles can be the same?
Jee: (?) No angles can be the same. (?)
RL: Are ya?
Jee: If it's 2 out of 5, yeah.
Lou: What makes it regular? What makes it regular?
RL: Okay, Louisa has a question. Who can answer Louisa's question? Okay, Vincent?
Vin: It's on the board. All sides are congruent and all angles are congruent.
RL: Alright, so we have an answer to that question. Okay, come one people, you're not thinking.
Ama: We are thinking.
RL: You're not thinking about math, is what I meant to say.
Kay: oh
(laughter)
Lou: All the shapes are not the ones you (?) around.
RL: I want to know <what makes something a polygon>. I know it has sides and it has angles SO…this then is a polygon right? (draws a Z-shape)
RL: nice polygon, huh?
S: that’s not a polygon.
RL: what’d you mean it’s not a polygon?
S: it has to be CONNECTED.
RL: (labels the figure as he talks, students are talking) side one side two side three angle one, angle two
Vin: it has to be
RL: look at that. it’s beautiful.

Lou: it has to be a SHAPE.
RL: what?
Lou: it has to be a shape
RL: this is a shape. This is a shape. I love it
(students are yelling back at RL)
Dil: it has to be connected.
RL: oh CONNECTED. (class calms down) it has to be connected. (writes "connected" underneath "sides" and "angles") So sometimes we use a word when we mean that, like if I draw this thing (draws a triangle), sometimes we say that it's closed (writes "closed" next to triangle). Meaning that it has an inside, and an outside. Okay? So sometimes when we want to talk about that idea of CLOSED, we talking about that all the sides are CO-nnected. One to the other, all the way back to where we started.
RL: okay so now I'm beginning to get an idea, that a polygon that is something that has sides, angles and is connected. That is it's closed. Okay, if we take this definition, can there be polygon with two sides?
Kay: yeah
S: no
Kay: yeah
RL: okay. What would that look like? Kayla?
Kay: you'd have to make sure they connected.
RL: the two sides connected? okay uh::
Kay: it'd be an oval.
RL: an oval?
Ken: like this.
RL: oh::. (draws oval on board)
Mic: (talking to Ama and Kay) °what's a side though? what's a side?°
RL: you want to do that?
S: no.
Jee: nuh-uh

DEFINITIONAL EPISODE #27
[00:22:36.25]

Ama: what's a side?
RL: you don't like this? (points to the oval)
(students talking all at once)
Vin: no I do not like it.
Kay: what's a side?
Ama: what's a side people?
Mic: What's a side?
RL: oh:: thank you.
Jee: and the angle. And the angle.
RL: What do we mean by side? (writes on the board as a question) What do we mean by side? Yeah, what do we mean by side? Everyone's yelling side. Nicholas, what do we mean by it?

Nic: Like if you have a four-sided polygon, one side - or if you have, a polygon that has 4 parts.
RL: Yeah.
Nic: Uh, I don't know.
RL: I [want to know] what a side is.
Jee: [Dr. Rich,] can I go get a dictionary.
RL: No, you can use your head. Instead of someone elses. Daniel.
Dan: I think a side a line that's connected to another line.
RL: Okay. So Daniel says it's a line (writes "line" on the board) connecting (writes "connecting") to another line (writes "to another line" on the board). Okay, so. Daniel.
Dan: mm hmm
RL: Ready? (draws triangle with one curved line and then looks at Daniel)
Dan: (has confused look on his face)
Jee: huh?
RL: Well I had a line, and there's I connected it and then I connected it again. Do we want to call this thing (points to curved line) a line?
S: No
RL: No?
Lou: No Dr. Rich it has to be STRAIGHT.
RL: It has to be STRAIGHT. Okay. So we have straight lines. Or parts of lines, okay. Kayla.
Kay: What is the biggest, line. What is the smallest um side? And, um it doesn't have to be a STRAIGHT side.
SS: Yes it does (students start talking all at once)
Mic: It can be a squiggle.
Ama: Cause my SIDE is not straight.
Kay: Amani (with annoyed tone)
(girls laugh)
RL: Alright, longest line, shortest, these are other questions (writes on board "longest line? shortest line?") BUT, Kayla then you said it DOESN'T have to be straight?
Kay: No
RL: To be a polygon?
Kay: Oh, a polygon?
RL: Yeah, we're talking about polygons, right?
(girls talking to each other)
RL: We're talking about something other people - so what do they mean when they say polygon?
S: Has (?)
RL: Well only for regular ones (underlines something on the board).
S: Oh.
RL: Otherwise, we have sides, that are, closed. That form a closed figure. But I don't know what I mean by side yet. I heard the word STRAIGHT.

DEFINITIONAL EPISODE #28
[00:25:31.14]

Vin: What does straight mean?
RL: Yes, what does straight mean? How do you know?
Mic: 180-degrees.
RL: oh:: A hundred and 80 degrees.
Mic: yeah. what?
RL: What do you mean by 180-degrees?
Lou: well a 180-degrees is straight across.
RL: So if I have the two parts. Okay this would be a 180-degree angle. (draws something) Are there straight sides that can meet at angles other than a 180-degrees?
Vin: yes
RL: okay so here's an example of one (draws two lines connected.)
Sha: no
Vin: no.
RL: that's not straight?
Vin: no that is not straight.
Jee: yes it is
Mic: yes
SS: yes
Jee: no it's not no it's not.
Vin: that is not straight.
RL: Hold on. In your table groups, I want you to come up with a definition of straight in the next 2 minutes. I'll give you til 130.

[00:26:35.20]

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[00:35:46.11]

RL: alright now. Let's start out with Rhonda and Shatteryia. Could you help us understand. Well actually let me not ask the question that way. Rhonda and Shatteryia, yours is up here?
Tim: That's mine too.
RL: okay, and you too Tim? The 3 of you. Okay, so Tim, Shatteryia and Rhonda. And I'm going to ask um Kenjra to interpret for us what do you think they're trying to say.
Ken: Now?
RL: yeah. Now would be a really good time.
Ken: a line is a line that is straight. not a line that has zig-zags or curves.
RL: Okay, so, what's their definition of straight?
Ken: up and down.
RL: you think they think up and down is the only possible straight line?
Ken: yes.
SS: no.
Ken: by their thing (points to the board). by their conjecture, what is that called? Diagram.
RL: how many of you said something like this, no zig-zags or curves. (reads the board) How many of you had something like that? Raise your hand if you had something like this idea. Okay. So Kenjra's group, you have a similar idea. Straight. (they have written "without any humps,

lumps, bumps, zig-zags, or loop-te-loops. so straight is something without ziging and zagging").
No bumps, lumps, zig-zags, loop holes, oh so it's not broken at all? okay, so straight is something without zigging and zagging. Alright now. What does that mean? Okay what does it mean if you, If you were walking. Okay let me ask this question. If you were walking, how, what would it mean to walk in a straight path? What would it mean? If you were a person who could either move or turn, what does it mean?
Lou: it means to walk ongoingly and your foot is in front of each other like really close.
RL: Okay. Does everyone agree with that? So I don't change my direction. I don't turn at all. I just keep going in a constant direction?
S: No.
Ama: yes.
RL: no?
Jee: if you want to take a turn you just turn.
Vin: could have an ending point.
RL: alright is could have an ending point. Okay so if I have an ending point (walks) uh stop.
Dan: and then you restart again.
RL: okay and then I could restart again? but as long as I didn't change direction, would I still be straight?
S: you could stop and change direction.
RL: I could stop and change direction. How can I change direction? What do I have to do?
SS: (talking all at once)
S: like this turning.
RL: turn?
S: yeah
RL: alright, so maybe one way to think about straight that comes from what they're done (points to Ken's def) is to think that there are no turns, like it's a path without any turns. Okay. And can I walk in any direction and just as long as I keep the same direction, is that okay?
SS: yes.
RL: So they don't have to be just vertical or horizontal? they can be like this (positions pencil in different orientations) or like that? okay.
Vin: it can be like this (holds up pencil)
RL: alright um. Who had a different kind of idea. Uh so. um. Nick. what are these people thinking about? This is um
S: Dilovan.
RL: and also, you guys too? no
Jee: that's Daniel, Vincent and Justin.
Dan: we're the orange ones.
RL: they want us to have a starting point
Vin: and it could have an ending point.
RL: and an ending point.
Vin: it COULD.
RL: alright.
Vin: i could have an ending point. it doesn't have to be.
RL: If it has
Vin: but it has to have a starting point.

RL: if we start walking (illustrates with a diagram showing "starting point" connected to "endpoint" with a line. he has drawn a point and a line connected) and we stop somewhere (draws a point at the end of the line) versus if we keep walking forever and ever? (draws a line with an arrow at one end)
Vin: yeah
RL: alright. that's what you're trying to get at?
Vin: yeah but it has to have a starting point.
RL: yeah okay, we have to start walking somewhere? a point?
Dan: yup
RL: alright. so, sometimes people in math, the idea that you have (writes "line segment" next to first line), they call it, they make a distinction. they say, oh you mean a segment versus a line. Alright but let's get back to Dilovan. How is he thinking Nicholas?
Nic: he's thinking that a line should be a 180-degree angle.
RL: Okay so. he's defining straight by angularity. He says that it's a hundred, it's a straight, It's even called a straight angle. okay. Are there any other ways?
Cou: I kinda disagree.
RL: Courtland you have disagree with that?
Cou: on my problem too. Because on one (?) it stops at one point, but a line, a line it keeps going forever.
RL: okay so. um could we keep going forever this way if we wanted to?
Cou: yeah on both ways
RL: if we left a trail, what angle would the trail be at? if we left a path?
Cou: a line.
RL: huh? right here (points to something on the board - can't see)
Cou: ahh, (?)
Vin: (?)
Cou: but it's still goes
Vin: yeah but all it is is it's going longer. the lines.
RL: okay so we have (interruption by intercom) a couple of ways of thinking about straight. Those people who haven't yet talked, what are those ways? Omari. tell us how we're thinkign about straight.
Oma: hmm. straight.
RL: straight.
Oma: my opinion of straight?
RL: yeah, well what have we been talking about? your opinion of what we've been talking about. so what are the, how would we know whether or not something was straight?
Oma: I would say that we'd know that something is straight is by like it having, like it could go on forever, but like it having no curves, no bends, no creases. Just going in one path.
RL: How do we produce a path like that that has no bends, no curves?
Lou: at a starting point?
Oma: well first you got to find a starting point. Then where the starting point is, then that's where, then if you're going to do a line, then that's where your line with start off. Then your line can either end where you want it to go or you can just have it keep going.
RL: okay, but how do I know? - suppose I do this. okay i'm going to do this. i'm going to start at a point. ready? you tell me if i'm going straight. and I ended. (He walks, starting at a point and curves and notes he ended)

Oma: that wouldn't be straight.
RL: why not?
Oma: because I'd say you took a, well, you took a turn and a line can't have a turn.
RL: okay, so we follow a constant direction? And we don't have any turns? Okay. Daniel.
Dan: you could have a turn.
Oma: well you could have a turn speaking of straight.
RL: At the point that we turn and that we go, what does it make, when we make a path and then we turn and we continue to make a path. What is that? What do we call that?
Jee: obtuse or
RL: obtuse or acute what? okay we go this way and then we turn (draws an angle) i'll say, look if i kept going, i would go like that, but instead I turn from here to here (draws an arced arrow showing the turn angle). okay that would be?
Jee: obtuse
RL: okay, but these two things meet to form?
Jee: an angle.
RL: an angle right?

[00:44:36.07]

DAY 5

DEFINITIONAL EPISODE #29
[00:03:07.01]

RL: what did we conclude about straight? What was one meaning of straight? Daniel.
Dan: it could have, oh no, it has to have a starting point and it could have an ending point.
RL: okay, so we made a distinction between lines and line segments, didn't we? We said if we started some place and kept going forever, we call those a line. If we stop somewhere we call that a line segment. But what made a line segment or a line straight?
Vin: we never got to that.
RL: yeah I think we did.
Vin: what?
RL: yeah, Louisa?
Lou: um do you want the definition of straight line?
RL: yeah what's a straight line?
Lou: it's a line that goes on and on and um, in like one direction and it's a 180-degrees.
RL: okay, so there are two ways we thought about, right? you can go on and on without changing direction. That was straight. You could walk without turning. And also we thought of it as a hundred eighty degree angle. And you know what we call those angles?
Jee: straight.
RL: straight, right? they're called straight angles to remind us of that. Kenjra.
Ken: Adding on to Louisa's, they also had no bumps or lumps or zig-zags or loop-te-loops.
RL: no, right, cause we wanted to rule out any kind of turning thing, any kind of action. Kayla?
Kay: Me, Amani and Micah found out that a zig-zag can be a straight line.
RL: a zig-zag CAN be a straight line?
Ama: cause it's made up of

Kay: cause it's made up of little straight lines.
RL: so you can think of zig-zags as being composed of straight lines but not A straight line. It has more than one straight lines in it. or line segment?

DEFINITIONAL EPISODE #30
[00:04:58.12]

RL: What do we call it when we introduce the zig-zag? What do we call that? Micaela? What do we call this (holds hands together at a point) When we have a line meeting another line? Alright Vincent?
Vin: a vertex.
RL: they meet at a vertex and what do they make? Nicholas?
Nic: a bend.
RL: A what?
Nic: a bend
RL: a bin?
Nic: a vertex?
RL: where they meet make a vertex, that is what-
Nic: a point?
RL: A vertex is a point. I agree. let's see someone who has not yet contributed. Rhonda. What do we call this where the two meet? Tim, I know that you're listening cause I'm going to call on you next.
Rho: it's a vertex when two points meet.
RL: okay so when these two line segments meet, we call this a vertex. what do they form together?
S: (?)
RL: a what? they do connect. what is the whole thing together called?
Rho: an angle.
RL: an angle, yes! Thank you. good that's what we call it, we call it an angle.

[00:06:06.23]

DEFINITIONAL EPISODE #31
[00:07:37.01]

RL: so what's the difference between a regular polygon and one that is not regular? alright, who's going to - Micah.
Mic: a regular polygon, all their sides are um, uh, I can't remember the word
SS: congruent
Mic: congruent.
RL: what does that word congruent mean?
Mic: It means they'll lay on top of each other
RL: okay
Mic: and match

RL: okay so all the sides are congruent. is that it?
Mic: and all the angles
RL: oh and all the angles are congruent.
Jee: like this (holds us two pencils at a 90-degree angle)
RL: okay, thank you jeewar for that drama. alright. what makes a polygon not regular then? alright. you haven't gone yet daniel.
Dan: I have.
RL: you what?
Dan: i have
RL: a half?
Dan: no i have gone.
RL: you have?
Dan: yeah.
RL: that's okay, i'll call on you again. you know why that is?
Dan: why?
RL: cause i'm a generous kind of guy.
Dan: (laughter) I think that irregular polygons that all the sides aren't congruent.
RL: yeah, all the sides are not congruent.
Vin: and all the angles aren't congruent.
RL: and all the angles are not congruent. one angle, at least one angle is different from others, and maybe they're all different. we don't know.
Vin: they could all be the same
RL: what?
Vin: they could all be the same, but they could have different sides, lengths.
RL: ahh so that's a conjecture you have. okay i'm going to put this up here. it's question number seven. And i'm gonna say. I'll call this the VF. The VF conjecture. What is the conjecture again? This is actually a conjecture.
Jee: statement?
RL: give me a statement. What's your statement?
Vin: I saidwell maybe an irregular polygon can all have the same angles, but it could have different lengths of sides. Or it could be the other way around.
RL: Irregular polygon can have all angles congruent, but not all sides congruent. (writes it on the board as the "VF conjecture: Irregular polygon can have all angles congruent but not all sides congruent.") Or irregular polygon, or not regular, can have alll sides
Vin: no
RL: you don't like that one? I'll ask, this will be the RL conjecture.
Vin: what's RL.
Lou: rich.
RL: alright, the DR conjecture. Not regular polygon can have all sides congruent but not all angles - i'm going to use that for angles- congruent (writes on board). I hope this is in everybody's notebook. There are now two conjectures, along with our questions. (to DL) these are polygon questions and conjectures. we have 6 of them so far Mrs. Lucas. They're on the board, but Jeewar has them in his notes. Does everyone have them in their notes?
SS: yeah.
RL: okay Kayla?
Kay: Can a regular polygon be an irregular polygon too?

RL: okay so (writes on board) Can an irregular polygon be a regular polygon also.
(students write - DL asks Louisa what her question is) Alright let's try tackling number nine right away. Can an irregular polygon be a regular one as well?
S: no.
Vin: no it can't because a regular polygon has all same sides and angles and an irregular polygon has to have different sides or angles.
RL: so we defined a regular polygon -

DEFINITIONAL EPISODE #32
[00:13:36.00]

RL: What does a polygon have? What does polygon have?
Vin: sides and angles.
RL: (writing on the board) okay a polygon we said had sides, at least three sides we said? And angles. And we said that a side means a straight line or line segment. And what other property did it have? Cause remember that didn't quite work. (pause) Can anyone use this definition and make something that's not a polygon?

[00:14:28.01]

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[00:16:53.03]

RL: At least 3 sides and angles we said was a polygon and we said a side means straight line segment. Kayla?
Kay: it has to be closed.
RL: Closed. Okay. So it's got to have all of these. At least 3 sides and angles.

DEFINITIONAL EPISODE #33
[00:17:11.28]

RL: What does closed mean again? Michaela?
Mla: it means like, it means, closed figure means something that's not open at all.
RL: not open. okay. And what else can you know about it Omari?
Oma: i would say that closed would mean like all connected,
RL: all connected?
Oma: lno gaps that lead to the outside shape.
RL: no gaps? so here's a way I could do something with sides and angles right? (draws a zig-zag) but it wouldn't be a polyogn. But if i did this. (draws a triangle) then it's closed and it has 3 angles.

DEFINITIONAL EPISODE #34
[00:18:01.18]

RL: And it has 3 angles. Where are the angles? Nicholas
Nic: sir?
RL: up.
Nic: on the board?
RL: yeah.
Lou: Dr. Rich, what do you mean by (?)
RL: that's what nicholas is about to show us on that figure i just drew.
Nic: (draws in arcs at each angle.)
RL: how is? what was Nicholas thinking right there that he knew that? What's an angle again?
Ken: a vertex.
Vin: a vertex.
Lou: like half a vertex.
RL: okay, Shatteryia? (putting his fingers together at a point) What is this point called Shatteryia honey?
Sha: vertex.
RL: vertex. What does two lines meeting, what does it form?
Sha: a vertxex.
RL: it forms a vertex and what else? What else? what else? Someone help Shatteryia out.
SS: angles.

[00:19:05.10]

DEFINITIONAL EPISODE #35
[00:21:29.08]

Mic: But is a circle is a polygon?
RL: okay we haven't decided that, if a circle isn't a polygon. Is a circle a polygon?
SS: no::
RL: okay why not?
SS: cause it doesn't have sides or angles.
RL: so circle is made by taking something from the center and just tracing a path. No sides. According to our definition.
Ken: you can make a circles with angles.
Nic: Does it have a side except one is an angle?
RL: Well if we, Does a circle have sides? (writes this on the board)
SS: no
SS: yes.
Ken: yes one circle does have sides.
(students talk)

Lou: a circle does not have sides. only when you divide it. it really (?)
RL: well, we said that a polygon though was 3 or more.
Ken: (?)
RL: well can a circle, can, If our definition of a side is that it's a straight line. We said straight meant the angle was a hundred and eighty. If you go around in a circle, is the angle a hundred and eighty?
SS: no
Lou: no, it's three hundred and sixty.
RL: So does it have a side or doesn't it?
Lou: no it doesn't have a side.
Vin: yes it
RL: well we said a side forms a straight line segment. Does a circle form a straight line segment?
S: it has a curve.
Vin: i'm saying that a side (?) the angle, cause if you think of a circle it's like this here's a half of a circle (gestures a semi circle) and here's a half of a circle (gestures a semi circle)
RL: yeah
Vin: that's congruent.
RL: yeah. but
Vin: i'm saying the circle (?)
RL: Well, let's think about walking. We said straight meant no angles. Or a hundred and eighty degree angle. So if I walk in a circle, what do I have to do?
(student says turn)
RL: How many times do I have to turn?
SS: 360.
RL: well at least 360 times. so can it be straight according to our definition of straight?
SS: no.
RL: okay, so if we're going to agree that the polygon has three or more sides and they're closed, then a circle can't be a polygon.

DEFINITIONAL EPISODE #36
[00:24:04.01]

DL: Can I put a question down?
RL: okay so Kenjra has a question to the question i posed. what was my question? Omari? what was my question? that we're considering I hope. Wait a minute, first we have to get the question out. I don't think everyone has the question. What is the question? What question did i just ask you?
S: what's the question?
Jee: does a circle have sides?
RL: we kind of got into that from Vincent.
Vin: yeah
RL: and we decided that according to our definition of side, no.
Jee: no there's not anything on the board.

RL: but there are different ways of thinking about it, but if use this definition, we have to rule it out. but what I asked was this. Just to remind us. I ask, Can I just say that to make a polygon, I need to have it 3 or more sides and the figure has to be closed? Do I have to say anything about angles or not? What do you think?
SS: no
Lou: you have to say something about angles.
RL: okay how many think you think I DON'T have to say anything about angles? (No one raises their hands) Kayla
Kay: I'm confused.
RL: I'm asking would it be enough to know if something was a polygon, would it be enough to know that it had three or more sides and that it was closed? Kayla?
Kay: if it's closed doesn't it automatically have the angles there?
RL: well i'm asking you that.
Lou: it still does but (?)
Ken: no a circle is closed but it doesn't have angles.
Lou: but it's not even a polygon (?)
Kay: but that's not a polygon and it doesn't have at least three sides.
Lou: it's not part of a polygon. All polygons have at least well,
RL: wait a minute, all polygons have at least 3 sides and they're closed
Lou: no
RL: nicholas?
Nic: Is the circle the only non-polygon? What about an oval? Is that not a polygon?
RL: Well I don't know. Is it a polygon?
SS: no
RL: What's the definition of polygon? That we have right up here.
SS: sides and angles
Mic: straight but it's curved.
RL: Sides. And we said a side meant that we didn't have to turn at all. Just kept going in the same direction. If you keep going in the same direction, will you make an oval?
S: no
RL: So I guess an oval according to our definition can't be a polygon. It doesn't mean that it's not a nice form. But it's not a polygon. Alright so. I'll just leave that. But Kayla thinks not. If you tell them there are at least 3 sides and they're closed, you automatically know it has to make angles. You don't have to say it explicitly. But you know, just something for you to consider.
DL: Is that something written in your notebooks?
SS: yes, no
RL: it should be written in your notebooks.
DL: kayla could you repeat your statement?
Kay: I think you don't have to say anything about the angles cause if it's a polygon, it has 3 or more sides and it's closed, then it's automatically going to have those angles.
DL: so you're saying it's a polygon.
Kay: mm hmm
DL: you're saying it's a polygon and it has 3 or more sides, that you don't have to say anything about angles?
RL: well if it's 3 or more sides and it's closed.
Kay: and it's a polygon

RL: well that's the definition of polygon. 3 or more sides and closed.
Kay: and if you're going by our rules
RL: yeah, well our rules are very close to the rules that people outside of the classroom have too.
DL: so that's a question, would it be true that a closed figure with 3 or more sides, must, you don't have to worry about angles.
RL: must have angles.
DL: must have angles.
RL: yeah i like the way you said that. Must have angles.
DL: so everyone needs to have that in your notebooks cause I'm going to ask how you would go about proving one way or the other.
Jee: can we write it as a "KF statement."
RL: as a statement?
Jee: a KF statement.
RL: a KF statement.
DL: a Kayla Frank statement.
RL: yes, Mrs. Lucas will write it as a KF statement.
DL: (writes it on the board) If a closed figure has 3 or more sides, it must have angles. Is that it?
RL: yes.

[00:29:57.07]

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[00:31:56.20]

Ken: Dr. Rich I have an answer to your question. Your question was can we make an 8-sided polygon that (?) I say yes you can. (they clarify that it is Lou's question).
RL: Okay, so Kenjra, you say the answer to this question is yes. What's your justification?
Ken: i don't know what that means.
RL: well you should. justification means what's your explanation.
Ken: oh. you made (counts something). Can I draw it up on the board?
RL: mm hmm.
Ken: (goes to the board and draws a figure with 8 sides that looks like a hexagon on top of a square) And at the same time it's combineded.
RL: So 1, 2, (labels the sides with numbers) you actually mean this?
Ken: yes.
RL: 3, 4, 5, 6, 7, 8. So, Kenjra has drawn, an example. If there's one example, then the answer to that question is yes. She's just drawn one. Would you agree that the angles are not all the same? Would you agree that the sides are not all congruent?
SS: yes.
Vin: no
RL: Point is they're NOT congruent. Because a regular 8-sided polygon would have all sides congruent, all angles congruent. Kenjra has shown us an example of something that has 8 sides, it's closed. Therefore is a polygon. And doesn't have all angles congruent, doesn't have all sides congruent. Therefore, is not regular, right?
Ken: but at the same time it's a combination. it tiles.

RL: what do you mean it's a combination?
Ken: a hexagon has 6 sides cause a (?)
RL: oh the way you made it is you thought about a hexagon and then you erased one of the sides of a square and stuck that up there? So right here is what you mean. (draws a line to separate the shapes) and what you did is this (erases line) right? That's another way we can think about a shape, as being composed of two or more other shapes. Very nice thinking.
Ken: thank you.
RL: you're welcome. Okay Omari.
Oma: I have a question.
RL: go ahead.
Oma: My question is Is there any polygon that is, that has more sides than 10 sides.
RL: than has more sides than what?
Oma: More sides than 10. And if there is, than what are they?
RL: Okay so another question we could ask ourselves. Question 13. Would you get that there from Omari please? Omari's question is can we make a polygon with more than 10 sides? Is that your question Omari?
Oma: A polygon with more than 10 sides and if there is, what are they?
Dan: His question is can you make a polygon with more than 10 sides?
RL: yes
Dan: would it be regular or irregular?
RL: he didn't say.
Dan: (to Omari) would it be regular or irregular?
Vin: that's his question.
RL: no it isn't his question. His question is does any polygon exist that has more than 10 sides? okay. nicholas.
Nic: What's the biggest polygon?
RL: What do you mean by biggest?
Nic: One with the most sides.
RL: so that's like Omari's question (and restates it as "is there any limit to the number of sides that the polygon can have?") If that's what you mean by biggest. Cause you know before Amani said what;s the area of that octagon? and we never did get to that but we will.
Vin: I know the answer to that question.
RL: to which question?
Vin: both of them.
RL: okay go on.
Vin: Yes because numbers go on and on.
RL: okay so Vincent has an explanation and an answer. Who can restate Vincent's thinking? Courtland must be able to do this for us. Courtland, Vincent had an answer and an explanation. You need to say it again and Courtland will try to restate what you're saying. Try to be clear in your explanation so that Courtland can understand you.
Vin: I said yes to their question because
RL: to which question?
Vin: both of their questions.
RL: but would you restate the question please?
Vin: okay. that polygons, that there's a number over. how do i say this. Yes, there is a polygon over a 10-sided polygon because numbers go on and on.

Cou: but do you know any?
Jee: yes, you could make one right now.
RL: so the question is yes there should be cause numbers go on and on so why couldn't the number of sides go on and on. and your challenge is well could you make one for us? but that's a different question.

DEFINITIONAL EPISODE #37
[00:38:48.28]

RL: On Tuesday, I asked you to try to figure out how you would walk to make a polygon. I would like you, I will give you 5 more minutes to write directions. Directions. Because other people, I don't want you to put it on the floor and then show other people, I want you to write directions.

[00:39:25.18]

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[00:55:37.16]

RL: How many of you found it very easy to follow somebody else's directions? Just Nicholas. Okay Nicholas, would you read the directions that you followed and it was easy to follow?
DL: Cause the rest of your group is not agreeing.
Oma: i agree.
DL: you agree? Michaela are you saying
Mla: (?)
DL: Okay, so would one of you, would one of you read the directions and Nicholas I want you to do exactly what the directions say so we can see that it's easy to follow.
Oma: One, take 3 straight steps
DL: Take what?
Oma: 3 straight steps and then turn 90-degrees.
Nic: (takes 3 steps and turns left 90)
Oma: Take another 3 straight steps and then turn 90-degrees.
Nic: (takes 3 steps and turns left 90)
Mla: Take another 3 straight steps and then turn 90-degrees.
Nic: (takes 3 steps and turns left 90)
Mla: Take your last 3 straight steps and then turn 90-degrees.
RL: can I do it? Okay thank you.
Vin: your feet are bigger.
RL: go on. read them.
Oma: 3 straight steps and then turn 90-degrees.
RL: Okay 3 straight steps. 1, 2, 3. (takes 3 steps) So what I like about it so far is, I am a robot. and now i know how many steps to take. there are 3. Now what do you want me to do?
Oma: turn 90-degrees.

RL: turn 90-degrees. (turns right 90)
Oma: then take another 3 straight steps and then turn 90-degrees.
RL: 1, 2, 3. (takes 3 steps)
Oma: then turn another 90-degrees. (turns left 90)
Mla: Take another 3 straight steps and then turn 90-degrees.
RL: 1,2,3. 90 degrees (takes 3 steps, turns right and hits desk). Error. Error.
DL: Did he follow the directions?
SS: yes
SS: no.
Vin: no he didn't.
RL: what do you mean I didn't follow the directions. You tell. I'm gonna do the directions again. You tell me that I didn't follow em. When I don't follow em, you yell out okay? Go ahead, give me the directions.
DL: I'll do them too.
RL: Mr. and Mrs. Robot?
DL: yeah.
Mla: Take 3 straight steps and then turn 90-degrees.
RL: (takes 3 steps, turns left)
DL: (takes 3 large steps, turns left)
Oma: Take another 3 straight steps and then turn 90-degrees.
RL: (takes 3 steps, turns right)
DL: (takes 3 large steps, turns left)
Vin: Stop.
SS: (talk at once)
DL: (holds hands up) following directions.
Vin: Stop.
Bra: Error.
Vin: Error.
SS: Error.
Mic: no they're weren't specific enough.
Kay: they weren't specific enough.
Mic: right or left or how big the steps were.
DL: Did Dr. Rich and I both follow the directions?
SS: yes
SS: no
DL: we took 3 straight steps and we turned 90 degrees each time, is that correct?
SS: yes.
RL: okay straight means we didn't change our direction, but I saw Mrs. Lucas, cause she has a big memory bank. It said steps and some of her steps were like this and then she computed a smaller step like this, so her steps, my steps in my bank are a little smaller, they're just like my foot. And then sometimes I went to the left and sometimes I went to the right. Why?
SS: cause you didn't know
RL: I didn't know. you didn't tell me direction. you didn't tell me direction of the rotation. Alright how could we fix these directions so that any robot on the planet, even Dr. Rich could do it well? Okay, what's the first direction?
Oma: Take 3 straight steps and then turn 90-degrees.

RL: (writes on the board) Take 3 straight steps
Ken: forward.
(intercom interuption)
Ken: forward. Take 3 straight steps forward.
RL: okay, take 3 straight steps forward.
Ama: heel-to-toe steps.
RL: heel toe steps? So step means heel to toe.
Ken: Then turn 90-degrees right.
RL: Turn 90 right. Okay now what?
Ken: take another 3 straight steps heel to toe forward.
RL: okay so i'll say, i'm going to abbreviate this, i'm going to say forward 3. then what?
S: South
Ken: you got to turn this way (rotates body left)
RL: do I need to say south?
SS: no
Ama: it's right.
Ken: left.
SS: right
RL: everybody get up. Back up, give yourselves some room. Alright now. We're going to take just a couple of steps. Go forward 2 steps
Kay: 2 steps, 2 steps
Ama: heel to toe
RL: heel to toe. we defined steps over here. alright now. turn right 90. Follow your right hand. right 90. i meant your other right for some of you. it says go forward. Now the question is some people said you should turn left, some people said you should turn right. What do you think it is?
SS: Right
SS: left.
RL: do whatever you think you should do. left or right 90. then take another 2 steps.
Nic: i don't agree with that.
(students talking all at once)
Tim: you gotta turn right. look Dr. Rich. (shows him)
RL: everyone have a seat. Okay so Omari has an observation. Woiuld you please make that for all of us Omari?
Oma: I think you have to, whatever way, like to go straight first, but whatever way you turn, that's the way that you're going to have to keep going. So if you go right, then you have to keep going right till you start back. If you go left, then you have to keep going left till you start back to where you started.
DL: why?
RL: let's see if that's true.
Oma: because if you went right and then went left (?)
RL: (writes the rest of the directions on the board: "RT 90. Forward 3. RT 90. Forward 3. Right 90.") Okay so i'll do it and let's see if it works. Okay, i'll start here. One, two, three. Turn right a quarter turn? or 90. 1, 2, 3. Turn a quarter turn right. 1,2,3. quarter turn right. 1,2,3. that's where i started. i just turned once more to get back to where i started. OR, Omari says I could use lefts. Okay, let's see if he's right so to speak. 1,2,3. Left 90, 1,2,3. Left 90. 1,2,3. Left 90. 1,2,3. Back to where I started.

Cou: yes.
RL: Okay?
DL: so were you correct Omari? what would happen if you mixed?
Oma: if you mixed, you'd probably be going in all sorts of directions. Like you go down, then you turn right, and then you turn left, and then you turn right again and then you turn left. It's all messed up.

[01:05:19.27]

DAY 6

DEFINITIONAL EPISODE #38
[00:03:37.05]

RL: Okay what is a - What is a polygon? Let's get that first maybe. Let's (Jeewar raises his hand) [back up].
Jee: [What type?]
RL: What is a polygon. Michaela.
Mich: (appears to be reading from notebook) It is a closed figure that has angles and sides.
RL: Okay. It's a closed figure. And it has sides.
Mich: And angles.
RL: And angles. (writes on the board. finished product says, "closed figure, sides & angles") Can you make any closed figure with sides that does NOT
have angles?
S, Ken: Yes.
RL: You can. Okay. Who said yes? Kenjra. Draw one on the board for me. Who else said yes? Nick?
Draw one on the board for me.
Nick: Did you say a closed figure that?
RL: It has to have closed figure and has to have sides, but no angles.
S: that's impossible (?)
RL: Is that possible?
SS: Yes, no
RL: Okay, If it's possible, draw it on the board.
Ken: Well.
RL: Draw your's over there (points Nick to other whiteboard)
Ken: I can't draw this.
RL: Draw it on the board.
Sha: Yes you can.
RL: You can. You can do it.
Sha: I know way you can draw it.
RL: Okay, the rest of you should be thinking of this, and whether or not YOU think it's possible.
(Nick draws a football shape. Ken draws hers. Says, "it's the inside of a marker cap.")
RL: Alright Nicholas. Help us understand how you're thinking.
Nick: (points to the two "sides") Two sides. (points to the vertices) No angles.
RL: [Two sides, no angles.]

Nick: [They can't be angles] cause an angle has to be a straight line, [two straight lines] make an angle (gestures--clip).
RL: [Has to have]
RL: An angle has the intersection of two, lines? Two straight lines? Okay. Does anyone have a counter-argument for Nicholas? Kayla. (Kayla looks confused) Well, can you argue with Nicholas. Do you, do you agree with Nicholas or not?
Kay: Um I don't cause that's not a polygon.
RL: Okay.
Kay: And Michaela forgot to say [that it has to have straight lines.
RL: [I think you need to say that to Nicholas] though.
Kay: (turns to Nick) That's not a polygon.
Nick: Did he say it had to be a polygon?
SS: Yeah.
Kay: Cause based on, based on Michaela's [um thing.]
RL: I said that, okay what I said Nicholas. And I think here's the point of difference. (draws a circe around "sides") I said that it's a closed figure with sides. Now, Nicholas is saying that this is a side. Okay I'm going to outline in blue
Nick: What did we say a side is?
RL: (outlines one of the sides of the football as he talks) What Nicholas is calling a side.
Vin: (quietly) You don't know what a side is?
RL: Okay What did we decide if you don't want to have that as a side, what must you define as a side, what must you define a side to be so you can rule it out? Cause right now, until, there's nothing wrong with what Nicholas has done. He has a start and an end and it makes a beautiful curve and it closes just like polygons, it's closed. So I see no reason yet to reject that, figure. Okay, um, uh uh (points to Courtland who has his hand raised).
Jee: Courtland.
RL: Courtland.
Cou: Uh well, um, them a (inaudible).
RL: What's that?
Cou: Well=
RL: We're addressing Nicholas's figure here, right?
Cou: I was on hers (points to Kenjra).
RL: You what?
Cou: Was doing hers (points to Kenjra).
RL: No let's do Nicholas right now cause that's what - that's what we're focusing on right now. Then we'll get to Kenjra's.
Ken: I disagree with my[self.]
RL: [Well she just] erased it.
Ken: I disagree with myself.
RL: Well put it up cause it helps us think. I don't care if we later disagree. We want stuff to think with. Go ahead.
Cou: I think it's not (RL: Huh?) because the sides have to be congruent and
RL: Okay so you say that the sides have to be congruent. When I look at that. I bet I could flip that and it would, be the same length and everything and sit right on top of it. I'll bet that could be congruent.
Jee: Yup.

RL: Pretty close. Alright.
Cou: Well it's not a poly[gon.
RL: Ka][yla.]
Kay: [Lines] have to be straight.
RL: Lines have to be straight. Daniel, do you agree with that?
Dan: Yes.
Vin: I do.
RL: Okay. How many people agree that we said that, alright so a side, if we're going to agree. Now Nicholas, once we say this, then this is what we mean. (writes on the board as he speaks) A side is a line. And we said it usually has a beginning and end point. A line segment, that is STRAIGHT.
Nick: (inaudible)

DEFINITIONAL EPISODE #39
[00:08:30.08]

RL: Okay and a st-how did we define straight? Daniel?
Dan: It had to have no curves, creases, bends.
RL: Okay so we said straight was no bends (writes this on the board). And if we walked in a path (gestures out with hand), we would have no turns. (writes this on the board)
Dan: (quietly) Yeah, no turns.
RL: Okay, is there any other way that we defined straight?
Jee: (quickly) No zig-zags.
RL: Don't just call out. You know what the rules are here Jeewar. Any other way? Anybody who hasn't gone yet? Jeewar?
Jee: one eighty.
RL: Okay. So IF we think about the line like this look (holds up two markers, end to end). Everyone looking? Some people are not looking. One of their names is Louisa. Look. (pause) (rotates one marker) Okay if we rotate it till it's a hundred eighty degrees (shows markers end to end in line), this is another way we can think about straight. (draws two lines on the board, connected end to end at a point). The two line segments are at this angle (writes "180" with degree symbol). They'll form a straight (gestures out with arms) line segment.

DEFINITIONAL EPISODE #40
[00:09:43.16]

RL: Okay, so. When I get back to my question. Closed figure with sides that doesn't have angles. Alright. How were you thinking about this Kenjra?
Ken: Um. Where is that marker? Oh here it is. The inside of this marker cap (points to marker) is circular at the top (moves finger along the top rim of the cap) and it has no angles on the side (points to her drawing of the top of the marker cap) cause that line is curved and if you look down on the inside of here, it has sides. (points marker cap out, presumably towards RL)
RL: Where are the sides?

Ken: They're going down (puts finger in marker cap. looks inside marker cap for a few seconds, frowning).
RL: Vincent? You have a comment on that?
Vin: Yes, but um=
RL: =Please use a loud math voice.
Vin: (louder) Okay, it can't um- the marker doesn't have sides because um a marker top goes circular all the way down (makes spiral gesture). It doesn't have (gestures up and down with finger) just a normal side. It has (makes spiral gesture), it goes circular all the way down.
RL: Okay so if I can rephrase what I understand Vincent to be saying? Yes, I know who could do that for us? Rhonda. Rhonda. What is Vincent's argument? Would you restate it for us?
Rho: (pause) That he's saying that it doesn't have sides cause it's circular.
RL: That what?
Rho: It doesn't, that it- that it doesn't have sides.
RL: Okay. So he's saying that when you have this cylinder, and of course it's also a 3D object. It's like one of these (draws a circle on the board).
Vin: (quietly) uh yeah and [it goes straight.]
RL: [And we] decided that a circle, is a circle a polygon?
SS: No
RL: Okay. Okay so it doesn't HAVE sides in the way that we define it cause if you went here (places marker on the cirlce), you would have to turn (turns marker as if it is walking the path of the circle) and turn a little bit (turns marker more, moving along path), and turn a little bit (turns marker more, moving along path). So you'd have turning in order to make that. Okay Kenjra thank you. (Kenjra has her hand raised) Oh. Yes, Kenjra?
Ken: Um. On the other side, on the inside of here, it has little prongs like that on the inside of them. But they do make SIDES [(inaudible)]
RL: Right but what WE'RE talking about is a two-dimensional object, right? Okay, we maybe should SAY that about polygons. That they're two dimensional, closed figures. (writes "2D" on the board next to definition) Now. To answer the question, Nicholas. What do you think about this? If it's two-dimensional and closed and it has sides, it must have angles? (pause) Kenjra you can sit down. Thank you. Shatteryia, what do you think? (long pause) Just hang on Daniel, I want to give everyone some think time. Shatteryia, what do you think? If it's closed and it has sides, does it have angles or not? You can say yes, you can say no. You can say i don't know.
Shatt: (shakes head) I don't.
RL: Okay. How are you think-what? Shatteryia. What's a polygon?
Shatt: A [polygon?]
RL: [What's our] definition of a polygon?
(Shatteryia looks through her notebook)
RL: Shatteryia, if you look on the board, we've been defining that. What does that say?
Shatt: A closed figure with sides and angles.

DEFINITIONAL EPISODE #41
[00:13:32.16]

RL: Okay. So closed means what?

Shatt: Um. Polygon.
RL: Okay what does closed mean though all by itself with or without- we can have polygons that are closed but we can have other things that are closed. Amani, would you help out Shatteryia? What is closed mean please?
Ama: Closed is like when um, is like when two lines are touching each other.
RL: Okay. So is this closed? (draws two lines connected).
Ama: Okay. (laughs) No. Um. (pause) When things say like things can't get out.
RL: They can't get out? [K.]
Ama: [Yeah like] a back door.
RL: So. We have lines or curves somehow that make something that has, sometimes we call this the interior, inside, and the exterior, outside. That's what closedness does. Separates things. Inside and outside.

DEFINITIONAL EPISODE #42
[00:14:36.27]

RL: Shatteryia? Are you okay with that one? Alright. Now. Shatteryia. Coming back to you. What about the angles. If something is closed and has sides, must it have angles or not? What do you think? [Shatteryia.]
Shatt: [Yeah.] (nods)
RL: Yes. Why do you think so?
Shatt: Because like the uh, triangles. (points to the board)
RL: Yup triangle I'll draw one here. Yup.
Shatt: Like at the top it has angle.
RL: mmhmm.
Shatt: The top has angles. The side has angles -the two sides have angles.
RL: Okay how many angles does a triangle have?
Shatt: Three (holds up 3 fingers)
RL: Where are they?
Shatt: The top (points to the board)
RL: Yeah. Here? (draws in curved line to denote angle)
Shatt: The side.
RL: Yup. (draws in curved line to denote a second angle)
Shatt: and (inaudible)
RL: (draws in curved line to denote remaining angle) Okay. So it has three angles, that are INside the figure. You know what we sometimes call the angles that are INside the figure?
Ken(?): Interior?
RL: Interior. (writes "interior" on board) So these are INTERIOR, angles. (pause as he finishes writing) Okay so. We're pretty good now on how we want to understand polygon, right? We know that it has sides and it's closed. Is there any limit to the number of sides? (initial silence)
Jee: Nope. No not any.
RL: Okay so we can have as many sides as we like (writes "sides - as many as we like") I'll write here as many as we like. Let's go back toward one that is pretty familiar.

DEFINITIONAL EPISODE #43
[00:16:20.14]

RL: Oh. We never answered this question. Now are there more polygons or are there more regular polygons? Kayla?
Kay: Polygons.
RL: Why do you say so?
Kay: Um because a polygon, it only has to have sides and angles. It doesn't have to um, it doesn't have to be closed=
RL: =oh
Kay: =well it has to be closed but it doesn't have to be uh like, the same angles and the same sides. [Congruent.]
RL: [Okay.] So these can be ANY (writes "--> any"), many sided figure (writes "sided"). Any of them (writes "figure"). And the regulars are the special ones. What's special about em Louisa?
Lou: They have number of sides.
RL: What makes them regular?
Lou: Their shapes are (inaudible)
RL: That what?
Lou: Their shapes are (inaudible)
RL: Yeah they are pretty common. That's a good observation. What else? What else do we know about regular polygons? Tim? Regular polygons. What do we know about them?
Tim: We know that they have sides, angles, and
RL: Okay all polygons have sides and angles. What's special about REGULAR polygons?
Tim: It can be=
RL: =Micah?
Mic: All the sides and the angles are congruent.
RL: Alright all (writes as he speaks) sides and angles, are congruent. Tim. What does congruent mean? What does that mean to you?
Tim: The same.
RL: Yeah, very good. Same right? Put one right on top of the other. Okay. Now. Does anyone disagree with this? Does everyone see how this works? Okay it's like if you have a bunch of dogs. And then you have some special kinds of dogs like German Shephards. There are many more dogs in the world than there are dogs that are German Shephards. So regular polygons just work like that. Okay so that's just a matter of convention. So so far we're not too excited so let's get excited.

[00:18:33.00]

DEFINITIONAL EPISODE #44
[00:29:44.08]

RL: Now that you're primed and ready to go. In your table groups, here are the problems that I want you to solve. You pick the one that you want to solve. Alright you ready?
Mic: Almost.
RL: Okay. Either write directions to make a rectangle. OR. Write directions to make a regular, triangle. A triangle with all sides congruent, all angles congruent.
Vin: We already did that.
RL: I don't think anyone has done that just yet.

[00:30:24.13]

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[00:57:56.26]

RL: Alright uh, I would like you to share your solutions with the rest of the class. How many people figured out how to make the rectangle? (Kenjra, Brandon, Courtland, Kayla, Amani, Michaela raise their hands) Okay let me start out with Louisa and Kenjra. Two different directions alright. Louisa, what are your directions please?
Lou: They're different.
RL: What?
Lou: They're different.
RL: Are they using the compass? North, south, east, west?
Lou: Well it's also usign the compass and right and [left.]
RL: [And] right and left? Okay. Uh, go ahead.
Lou: Five steps um, north.
RL: Okay, five steps
Lou: North.
RL: North. Yup.
Lou: Two steps west. Or you can call it left.
RL: If I go five steps north, okay. (enacts) Okay now what?
Lou: Two steps left.
RL: How can I? Oh, so I just turn (rotates body clockwise) and go two steps west? Okay good.
Lou: Then five steps uh down. Left again.
RL: So five steps down, wouldn't that be in the opposite direction of north? What's the opposite direction of north?
Lou: I mean right.
RL: South.
Lou: Two [steps]
RL: [And then], two steps east?
Lou: Yeah.
RL: Alright. Good. So that would work if we knew north, south, east, west. Okay, can anyone else give me a different way of doing it?
Jee: The same way.
RL: Pardon?
Jee: The same way (inaudible). Like five steps right. Two steps right. [Five steps right]
RL: [I don't know] what you mean by right.

Jee: Right ninety-degrees.
RL: Okay so I can go forward five steps, (writes as he talks) right ninety. Then what? Forward [how many steps?]
Jee: [Two steps]
RL: Two steps?
Jee: Right ninety.
RL: Right ninety.
Jee: Five steps, right ninety.
RL: Forward five. Right ninety. Forward two. Right ninety.
Jee: Then turn back=
RL: Does anyone notice a pattern to this? Amani?
Ama: First you're walking, then you turn and then you're um walking again and turning.
RL: Alright. Anybody else add on to that? Here I'll just write this part again. (writes something on the board) (silence) Jeewar?
Jee: Five two, five two.
RL: Okay so. If I repeat this, two times (writes something on the board), that'll make a rectangle, right? pshh, kkk, pshh, kkk. Okay. (silence) Micah and Amani and Kayla?
Ama: mmm hmmm.
RL: Are your directions like this?
Mic: (nods) mmm hmmm.
RL: Do you use a different metric? Instead of steps you were using blocks?
Mic, Kay: mm hmm.
RL: [Okay.]
Mic: (quietly) And i[nches.]
RL: And you were still using degrees though for the amount of turn?
Kay: yeah
Ama: and we used inches.
RL: okay, did uh what about you rhonda and shatteryia?
Shat: we used tiles
RL: tiles?
Shat: on the sheet we used inches and on the floor we used tiles.
RL: okay so you used tiles on the floor? what were your directions with teh tiles? can you give them to us please?
Shat: yeah you go one tile forward.
RL: (writes) forward one tile?
Shat: four
RL: forward four tiles. yup. then what?
Shat: then you turn a whol 90 degrees.
RL: then what?
Shat: you go 2 more tiles.
RL: forward 2 tiles.
Shat: then you turn another 90 degrees.
RL: okay then what?
Shat: go forward 4 tiles.
RL: forward 4 tiles. See that pattern again?
Shat: then go 2 more tiles

RL: 2 more tiles? (finishes writing) Okay. So, how are a square and a rectangle the same and how are they different? Amani?
Ama: the rectangle and the square are the same because both have 4 sides, both you have to turn 90-degrees.
RL: okay so that's how they're alike. Four sided 90-degree turners. Kayla?
Kay: they're different because on a rectangle, you have two different sides
RL: the lengths aren't? okay so in a rectangle
Kay: and in a square they're all the same.
RL: the lengths are not all congruent. So a square has everything a rectangle has plus it has all the sides congruent. Alright now. That was challenging to think about.

DEFINITIONAL EPISODE #45
[01:03:52.06]

RL: But what about the triangle? Alright vincent.
Vin: you want me to tell you the direcitons?
RL: yes, please tell me the directions for triangle. And i'll write them over here. Okay go ahead, ah and i need a volunteer to follow what vincent is telling us to do. Courtland.
Vin: go FD 3 feet heel-to-toe.
RL: fd 3 steps (writes)
Vin: heel to toe
RL: whatever. right. good enough.
Cou: (walks 3 steps)
RL: now what?
Vin: FORWARD.
Cou: (walks forward)
RL: okay.
Vin: turn 60-degrees left.
RL: turn left 60-degrees. Alright go ahead turn left 60-degrees (helps Cou)
Cou: (turns 60)
RL: that's good. about there. just bring your other foot there so your don't trip over yourself. put your feet together. okay so now we turn leftt 60. okay now what?
Vin: go FD 3 feet heel-to-toe.
RL: FD 3 steps again?
Vin: yes.
RL: alright.
Cou: (walks 3 steps)
RL: now what?
Vin: turn left 60-degrees again.
RL: turn left 60-degrees again.
Cou: (turns 60)
RL: alirght now what?
Vin: now go FD 3 feet heel-to-toe.
RL: FD 3 steps?

Cou: (walks 3 steps)
RL: alright now what? do i turn left 60 degrees again just to get back to where I started?
Vin: uh huh.
RL: did it make a triangle?
SS: no
S: yes.
RL: no? no i don't think it did. right. I mean let's draw, let's draw teh path it made. I think i saw Courtland go 3 steps and if we kept on going it'd be like this (draws a line with dotted line extending from it), but we turn right or left 60-degrees (draws the turn angle). So that's about like that. And then we got to here and we had to turn left again. If we kept going it'd be like this. We turn left. 90'd be there, so 60 would be about there. And we went another 3 steps. Did not seem to me to make a triangle.
SS: no
RL: it made sommething but it doesn't look like it's a triangle
Vin: you have to turn right.
RL: oh you think it's if you turn right it'll work.
Vin: you have to turn right.
RL: okay i'll do this one okay. What do you think Micah. If you turn right will that solve the problem?
Micah: No
SS: no
RL: alright (walks) 1,2,3. turn left 60. now what?
Vin: go 3 steps.
RL: 1,2,3. Okay
Vin: no turn right.
RL: alright how much?
Vin: 60 degrees.
RL: 60. alright that's about right there. now what.
Vin: no i mean 90-degrees.
RL: alright now it's 90. now what?
Vin: FD 3 steps.
RL: okay. (walks) I still don't get a triangular feeling out of this. Remember we're trying to make a regular triangle. All the angles have to be teh same. All the sides have to be the same. Everything, angles an sides are congruent. Ah.
Jee: I made one.
RL: alright you pick somebody to direct and i'll write your directions down. Alright Daniel you're up. Jee has the floor.
Jee: FD (?)
RL: how bout we just go fd 3 steps so we can all?
Jee: heel to toe.
Dan: (walks 3 steps)
RL: FD 3 steps. Okay now what? alright put your feet together that's fine. alright now what?
Jee: turn left 60-degrees.
Dan: it'll be about there?

RL: i don't know what do you think? you know 90 is about a quarter of a turn so 60 you pointed out to me was 2/3 of 90 before. alright, now what? Jee? so far your directions are looking an aweful lot lilke this table's directions.
SS: I know!
Ken: you go to turn 50 degrees.
RL: have a seat, have a seat. alright now that you all have an idea, I want you to all look at your directions again for a triangle and I want you to rethink it. If you didn't do a triangle. That's okay. Alright. so we've figured out how not to make a triangle.
Jee: i know.
RL: hey i put your directions there. i want an equilateral, equiangular triangle. i want to be able to walk it. you had a chance. You have to prove that to me
Mic: 30- degrees
(students talking all at once)
RL: alright i'm going to give somebody else a chance. Omari, you're on. Omari has a diff way of thinking about this. If I were you I'd listen to Omari. Omari has the floor. Get up there please. pick somebody who you're going to direct Omari.
Oma: brandon.
RL: brandon you're up. alright this is for a triangle. Alright. I'll write down what Omari tells me to do. I mean what Om tells Bra to do. alright go ahead.
Oma: go forward 3 steps.
RL: FD 3 steps. (writes)
Bra: (walks 3 steps)
Oma: then turn right
RL: turn right.
Oma: a 120-degrees
RL: a 120-degrees. Okay there's 90, where's 120. there, there we go
Bra: (turns 120)
Oma: take another 3 steps.
RL: FD 3 steps.
Bra: (Walks 3 steps)
Oma: turn another 120-degrees.
RL: turn Right 120-degrees.
Bra: (turns 120)
RL: now what?
Oma: take another 3 steps.
RL: FD 3 steps
Bra: (walks 3 steps)
Ken: and you're back to where you started.
Oma: turn right 120-degrees
RL: turn right another 120-degreees. back to where you started
Bra: (turns 120)
Jus: that's not where you started
Dan: that's not where you started
RL: alright. Let me try the walk.
(argument over who gets to walk - they decide Shat and Lou)
RL: now let's see if they both do the same thing? ready? okay, go ahead omari.

Oma: FD 3 steps.
Shat: (walks 3 steps)
Lou: (walks 3 steps)
RL: put your feet together so no one trips. now what?
Oma: turn 120-degrees
RL: turn right 120-degrees
Tim: shat don't know what she's doing.
Rho: yes she does.
RL: i just think you turned 180 degrees louisa, shatt. Where's 120?
Shat: (readjusts)
Lou: (readjusts)
RL: alright
Oma: take another 3 steps.
Shat: (walks 3 steps)
Lou: (walks 3 steps)
Oma: turn right 120-degrees.
Shat: (turns right 120)
Lou: (turns right 120)
Oma: take another 3 steps forward.
Shat: (walks 3 steps)
Lou: (walks 3 steps)
Oma: turn 120.
Shat: (turns right 120)
Lou: (turns right 120)
Oma: and then go 3 steps one more time.
Shat: (walks 3 steps)
Lou: (walks 3 steps)
RL: so we're not exactly following your directions to the T here, are we? we have, we're winding up in different places.
Cou: Dr. Rich, I got a triangle.
S: i got a triangle.
RL: Omari's directions, we have to see if we can follow them first.
Cou: I can just let me.
RL: alright tim?
(they discuss who will follow directions - decide on Vin and Tim. they start walking, but students are talking all at once)
RL: hang on stop. everybody get up. now. I want everyone to raise their right hand. and I would like you to turn right 90-degrees. Alright go back to where you started. Turn left 90. Now I want everyone to turn 120-degrees about where you think that is.
Kay: right or left?
RL: doesn't matter. Alright so what you have to figure out for yourselves is the difference between 90, 180, 120
Lou: oh i think i know
RL: and i want to see it in your math notebooks. we will pick this back up on thursday. You might want to consider Why does Omari's directions, why do they work? They will make a triangle that has all angles and all sides the same length. Your homework is to figure that out."

[01:17:33.08]

DAY 26

DEFINITIONAL EPISODE #46
[00:20:40.28]

RL: While we're thinking about why we might want to explore 3D, we're going to stay in 2D for a little bit longer. And here's my question. What's a triangle?
S: A shape.
Tim: Three-sided-
RL: - uh uh uh. I want you to work in table groups and write me a definition of a triangle so that, so that we can know for sure, given a triangle and anything else that we might generate in 2d, or in 3d, that, what we're looking at is a triangle. okay you can talk in your table groups. I want you to come up with ONE defintion per table group. That means that it will be the Rhonda, Omari, Louisa definition. Okay? It will be, okay so each person has to agree about the definition. I want you to work to(/)ge(\)ther. To(/)ge(\)ther. Alright.

[00:21:52.15]

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[00:32:01.08]

RL: so. The idea is that we can read this definition and it will allow us to recognize anything that's a triangle and exclude anything else. And we want to see who can do it with the fewest possible ideas, but it could still work. So I want you to look at every definition. I want you to track, just read it and see whether or not the defintion works. Cause good definition do what? What kind of work do they do for us?" (No one answers. He lets the students still at the board finish.) Alright. I'm going to start over here. This one says that a triangle has 3 sides, 3 angles only and it is closed. So their definition says a triangle has 3 sides, 3 angles, and it is closed. Okay. Can anyone think of something that their definition, it would be triangular but their definition wouldn't work for it. Or something that is not triangular but their definition would seem to fit it? Okay, some people think they can do this. Vincent.
Vin: Straight sides.
Ama: But we already said sides.
RL: [So this assumes that the]
Mic: [That's the [definition of sides.]]
Kay: [definition of sides.]
RL: def[inition of side means] straight. (draws an arrow from side and writes "straight")
Vin: (looking at the girls) [Three sides, three angles.]
Mic: (quietly, possibly responding to Vin) We just said that.
RL: And, can we assume that? Because we have done this? Our definition of side (circles "sides --> straight") has included the notion of straight. Remind me of what straight means though? Just so we're all on the same page. Michaela.
Mich: it means a line going 180, NO turns.=
RL: = 180, no turns. And what if I'm walking? Give me the walking definition of straight.
Jee: You never ever turn (inaudible).

RL: Never, ever turn, right? No turns, you keep a constant heading. (walks to illustrate). Alright, so this - can you think of anything that this doesn't cover then? Kenjra?
Ken: That it shows the exterior angles have to be 360.
RL: Okay. I'm uh -
Ama: Yeah, we forgot that, [that's what we were about to]
RL: [we don't have] a other properties, but, are these properties good(/) enough? Is this set of properties good enough?
S: Yes.
S: Yes.
RL: Okay. So that is a definition that works. Alright. Let's look at this one. 3 sides, 3 angles, can be regular or irregular polygon and it's closed. Do they need to say closed if they say polygon?
SS: No.
RL: Why do you say yes Nicholas?
Nick: cause a regular polygon is always closed.
RL: Is an irregular polygon open?
Dan: (quietly) No. (shakes head)

DEFINITIONAL EPISODE #47
[00:36:18.06]

RL: What's the definition of a regular polygon again? Rhonda?
Rho: Uh, I think it was straight lines, with straight lines, angles and it's closed?
RL: Okay, straight lines, so we call those sides. And it's closed. But, what makes it regular?
Lou: All the sides, same sides.
RL: All the sides are the same? And what [else?]
Jee: [Same sides,] same [angles.]
S: [Angles]
Vin: [Angles.]
RL: [And all] the angles are the same. [Okay]
Tim: [all the] sides are congru::ent.
RL: All the sides are congruent. THANK YOU Tim.
Jee: All the [ANGLES are congu(/)ent(\).]
RL: [Okay, that math word] says it a::ll. All the sides are congruent. All the angles are congruent.

DEFINITIONAL EPISODE #48
[00:36:59.01]

RL: Yeah, good. So, we could say it could be regular or irregular, and, as long as we say POLYGON, we know it has to be closed. Good. Alright. A triangle has 3 straight sides, 3 angles, interior angles of 180. You mean each interior angle is a hundred and eighty degrees?
S: No::

RL: What do you mean?
Vin: I mean all of them add u::p to.
RL: O::kay, the interior angle, the SUM (writes something) of the interior angles is 180. Sum of the exterior angles 360, and it is
S: Enclosed.
RL: Or, (erases something) closed. We just use the word closed. Okay, it encloses a space, and you know if you think about it, if I draw this triangle, (draws an equilateral triangle) this has an area, and we could figure it out, cause that's one of our questions, we have to learn to figure this out, and outside of it (gestures outside of the triangle) is just basically infinite. So closed, closed matters. It closes something, it makes the area have a definite value. and outside of it, pshhhew (throws his arms out to the side). It's the whole rest of the plane. Infinite. Alright. So, this works. Right, this works. (points to definition)
S: Yes.
RL: Good. Alright let's look at this one. [00:38:24.21] 3 straight lines and has to be connected or closed. (note that definition just said "connected" so "closed" was RL's addition. So Daniel's definition (writes "closed") is three straight lines, or we could say three sides, and closed. (rewrites as "3 sides, closed.") NOW, that's a really sparse definition. That's the sparsest one so far. Does it work? Or do we HAVE to say angles? What do you think?
S: Yes.
RL: Well.
Dan: But won't it come with angles?
RL: As soon as Daniel says, three sides and closed?
Rho: It al-it already has angles.
RL: It already has the angles. So, this is like, this (points to a different definition - can't see). Very slim. I would call this one (points to another) somewhat slim. I'd call this one (points to another) pretty slim, right? This (points to another) is an expanded one, but it works.
Vin: (inaudible)
RL: No, no. Expansion is - but we're just trying to see what we can get away with. [00:39:30.13] Alright let's look at this one. A triangle is a polygon. Okay it has 3 congruent sides. It has no diagonals. No one's mentioned that before. It is closed. So as soon as they said polygon (points to the word "polygon" in their definitions), could we assume closed?
S: Yes.
RL: Okay, is it true that every triangle has 3 congruent sides?
Dan: No.
SS: No.
Dan: Not every [triangle.]
RL: [Okay] so, here might be a triangle [right] (draws something - can't see)
Nic: [A] regular triangles.
RL: Okay, a regular triangle DOES have 3 congruent sides. What is another name for a regular triangle? Did we ever - did we ever write that down? So -
Ken: An equilateral.
RL: An equilateral triangle. E:: qua:: lateral (writes word on board). So write that down okay? Put that in your notebooks, so that from now on when we say equilateral, we all know what we're talking about. So it's an equilateral triangle. It's CLOSED. 3 interior angles. 3 exterior angles. 3 straight lines with 3 points. What's another word for points that we've been using?
SS: Vertex.

RL: Vertices, right? (writes something on the board.) And, interesting tha::t this definition says hey if you added another side it would no longer be a triangle but something called a quadrilateral. Okay. So that's nice. [00:40:50.26] Alright, let's look at this one. (Courtland is still at the board.) Courtland has been busy editing here. Okay, I'm gonna let him finish. Courtland, then you present yours and see if it works. (They wait for Courtland to finish writing.) Alright, let us know what you got there Courtland.
Cou: A triangle has 3 closed
RL: You got to face us and speak in a loud math voice because of that fan.
Cou: A triangle has 3 closed sides and a polygon. All sides have congruent sides. Uh:: and 3 turn angles and 3 interior angles. Turn angle is, the turn angles is 360.
RL: The sum of the turn angles is 360?
Cou: Uh huh, the sum of the turn angles is 360. A system, a system of triangle has all turn angles of 120.
RL: Okay, who are you representing there? Is this Dilovan and Jeewar?
Cou: Yes.
RL: Okay, so, um, are you claiming that the only triangles are the ones that are equilateral?
Dil: (shakes had no)
RL: Okay cause that's what that definition says. It says all sides are congruent. Do you mean that?
Dil: (shakes head no) I messed up.
Jee: [R::egular]
RL: [Then you said] all the turn angles are a hundred and 20. Which of the [triangles?-]
Dil: [For a regular.]
RL: For a REGULAR. So for an equilateral triangle that's true. Right, okay? we have a bunch of definitions that work and we find that we can be very economical and state just a few things about a triangle and we can make a good definition. Now what I'd like you to do is use your definition and in your group decide which of these things that I just gave you, which are triangles?

[00:45:04.29]

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[00:49:17.07]

RL: (RL has put an overhead of the shapes up.)
RL: Alright um. This One (points to the triangle with curved in sides.) Omari.
Oma: no
RL: Why not?
Oma: because it has curved sides.
RL: okay curved sides. so if I traveled along this, would I have to change my direction? Okay, so that's out. doesn't have sides, they're not straight. so this is no. Alright vincent.
Vin: yes.
RL: yes why?
Vin: closed figure with 3 sides.
RL: closed figure, 3sides. Louisa how bout this one?

Lou: yes.
RL: why?
Lou: because it's straight and (?).
RL: okay. Rhonda. this one.
Rho: yes.
RL: why?
Rho: it's closed shape and (3 sides??).
RL: closed and 3 sides. alright. um. justin. what about this one?
Jus: yes
RL: why?
Jus: because closed and (3 sides??).
RL: closed and what?
Jus: (?)
RL: closed and 3 sides. alright. How bout this one right here. Jeewar.
Jee: closed and have 3 sides.
RL: so? is it or not?/
Jee: yes.
RL: alright what about this one? it's closed.
SS: no
RL: why not?
SS: (?)
RL: it look,s like a what?
SS: (?)
RL: so is it a triangle or what?
Ama: no because it has curved sides.
RL: so this looks like a piece of candy?
Tim: no halloween corn.
RL: alright, but it's stil no according to the definition?
Nic: no vertex (?)
RL: alright what about this one. it's got 3 sides
SS: but they're not connected.
RL: good.

[00:51:11.19]