CHAPTER I

INTRODUCTION

1.1 Motivation

Mathematical optimization methods are useful in developing cost-effective designs of large systems with many design parameters, objectives and constraints. However, when the optimization is deterministic, it cannot assure satisfactory performance over the entire range of operating conditions. Many parameters of the design problem are random and cannot be controlled. In such cases, the optimal solution obtained from a deterministic optimization may violate the imposed constraints with an unacceptable probability. If the worst case scenario is used, then the solution obtained may not be as cost effective. Empirical safety factors are used with deterministic optimization to account for the uncertainties. However, such an approach may miss the true optimum if some constraints are too conservative or be unreliable if some constraints are not conservative enough.

This makes reliability-based design optimization (RBDO) a valuable tool that can help obtain cost effective designs under uncertain conditions. It can assist in decision-making and provide quantification of the uncertainty associated with the design. Robustness is also a very important issue, i.e. it is desirable to have a design which is insensitive to variations in the system variables. This can be achieved in RBDO by optimizing the mean value of the objective as well as minimizing the variance of the objective. Formulations to handle such multi-objective robust design problems need to be developed.

The RBDO methodology has been successfully used to solve mathematical problems but due to its high computational cost its use in the industry is still minimal. This creates a need for developing algorithms which solve the RBDO problem efficiently.

1.2 Objectives

Several procedures have been developed in the literature for reliability-based design optimization (RBDO), including the Reliability Index Approach (RIA), the Performance Measure Approach (PMA), and more recent techniques wherein the reliability and optimization calculations are decoupled. This study seeks to address the motivations and issues presented in the previous section through the following objectives:

(i) This study extends the decoupled RBDO approach to include standard deviations as design parameters and wherein simulation or other methods can replace the traditional first order analytical method for reliability assessment. The accuracy and computational efficiency of the various RBDO methods are compared.

(ii) The methods are extended to robust design and their applicability is investigated. Formulations to handle the multi-objective robust design problem are given. The study also investigates a single loop method and extends it for the robust design problem.

(iii) The developed methods are finally implemented on an automotive crash safety problem.

Objective 1: The decoupled approach which is discussed in Chapter 2 does not have the ability to use different reliability assessment techniques. For highly non-linear constraints the first order reliability method will not give accurate results hence simulation based methods might be required. Hence there is a need to develop algorithms wherein simulation-based reliability assessment can replace the first order reliability assessment when necessary. Also current techniques do not have the ability to consider standard deviations of variables as design parameters. This problem is also addressed in the study.

Objective 2: Robust design requires that both the optimization of the mean value of the objective and minimization of the variance of the objective happen simultaneously. Efficient multi-objective reliability-based optimization techniques are required for this purpose. An accurate and efficient method is required to estimate the variance of the objective. In this study, formulations which help in solving such multi-objective RBDO problems are given and applied to mathematical and practical problems.

Objective 3: The developed methods are applied to a complex automotive crash safety problem. The objective of the problem is to maximize the star rating and constraints are the FMVSS criteria for neck chest and head injuries coefficients for two different load cases. This problem shows the suitability and applicability of the developed methods to complex practical problems.

1.3 Organization of the Thesis

The remainder of the thesis is organized as follows. Chapter 2 provides a brief description of various design optimization techniques. The state of the art reliability-based design optimization methods are also discussed and their formulations explained. In Chapter 3 the improved RBDO technique is introduced and applied to two simple examples. Chapter 4 discusses the new formulations for robust design. Finally Chapter 5 applies the techniques developed in Chapter 3 and 4 to the crash safety problem. The results are graphed and tabulated for comparison. The conclusions of the study are given in Chapter 6.

CHAPTER II

DESIGN OPTIMIZATION

2.1 Introduction

The process of finding the most optimal design is complicated by the uncertainty, which is caused by random variations in material properties, loading, and environmental conditions. Compromises have to be made between performance, cost, risk and quality attributes to obtain a reliable and cost effective system, including the variations in design parameters.

Design optimization can be done by any of the following techniques.

1. Deterministic optimization.
2. Safety factor based design optimization.
3. Reliability-based design optimization.
4. Robust design

A description of each is provided here with respect to a hypothetical two-dimensional problem. The problem consists of two design variables represented by the design vector x=[X_1, X_2]. The aim is to minimize the objective function, $f_o(x)$ subject to two constraints, $G_1(x)$ and $G_2(x)$.

2.2 Deterministic Optimization

Deterministic optimization obtains the solution by performing a simple mathematical optimization procedure. This can be represented as

$$\begin{aligned} &\min \ f_o(x) \\ &s.t. \\ &g_1(x) = \left(G_1(x) - G_1^{Target}\right) > 0 \\ &g_2(x) = \left(G_2(x) - G_2^{Target}\right) > 0 \end{aligned} \qquad (2.1)$$

If the randomness in the involved variables is considered then the optimum will not occur exactly at the point A but will have high probability of occurring in an area around A. Most of this area lies in the infeasible domain. This indicates a high probability of failure of the system. This leads to the use of safety factors in design optimization.

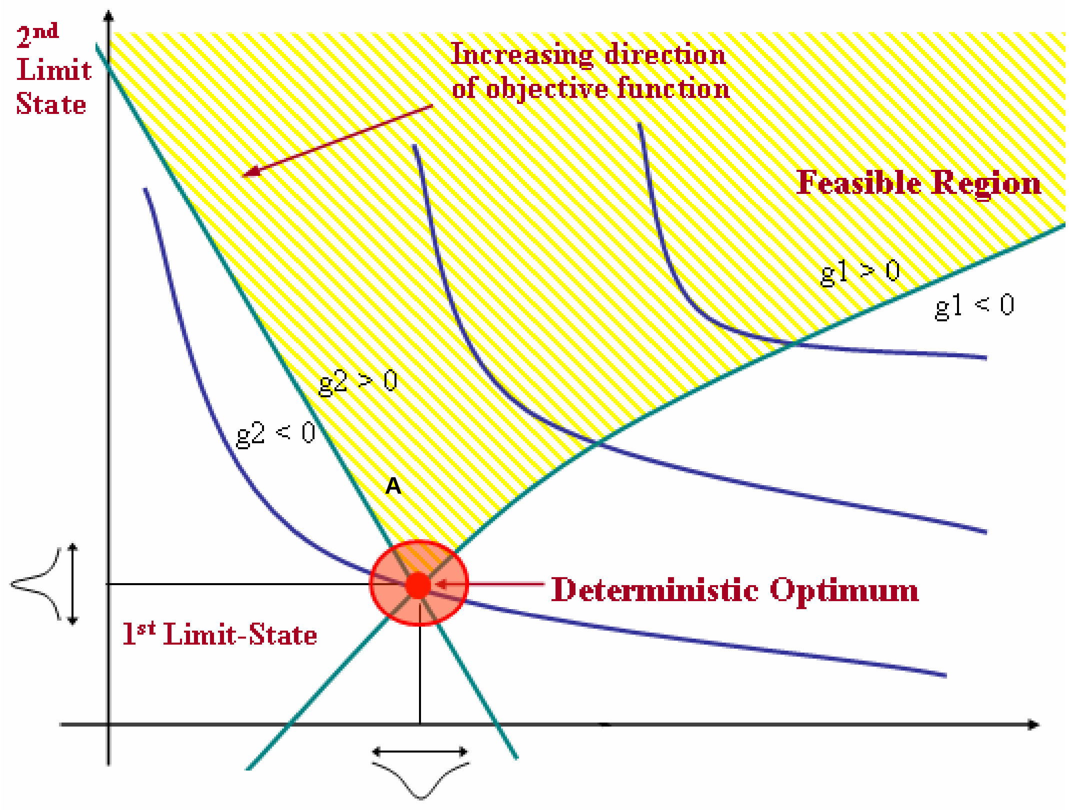

Figure 2.1: Deterministic optimization

2.3 Safety-factor based design optimization

In safety-factor-based design each of the limit states is offset by a factor of safety. This makes the feasible domain smaller. Deterministic optimization now gives a solution that is more feasible than the one obtained earlier. But if the safety factor is too high then we might get an over safe design which has an unnecessarily high cost and if the safety factor is too low then we still have a high probability of failure.

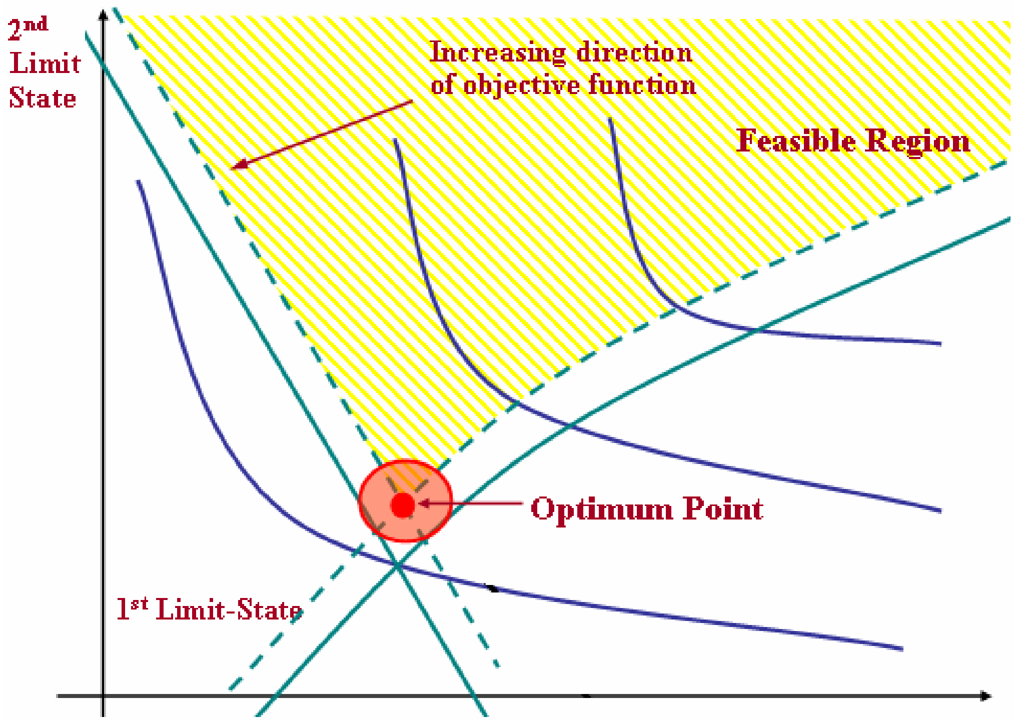

Figure 2.2: Safety factor based deterministic optimization

2.4 Reliability-based design optimization

Due to the uncertainty in determining the appropriate safety factors, direct reliability based design, where an optimum solution is achieved with respect to a target system reliability, is more desirable. This ensures both cost effectiveness and a low probability of failure. In such an optimization procedure the constraints are not deterministic but in fact are replaced by reliability assessment for those limit states. Since the reliability assessment, which can be done by several methods such as FORM, inverse FORM or simulation, is an iterative procedure, the number of function evaluations increases in reliability based optimization when compared to deterministic or safety factor based optimization.

Fig 2.3 shows a comparison between the reliability based optimum (B) and the deterministic optimum (A).

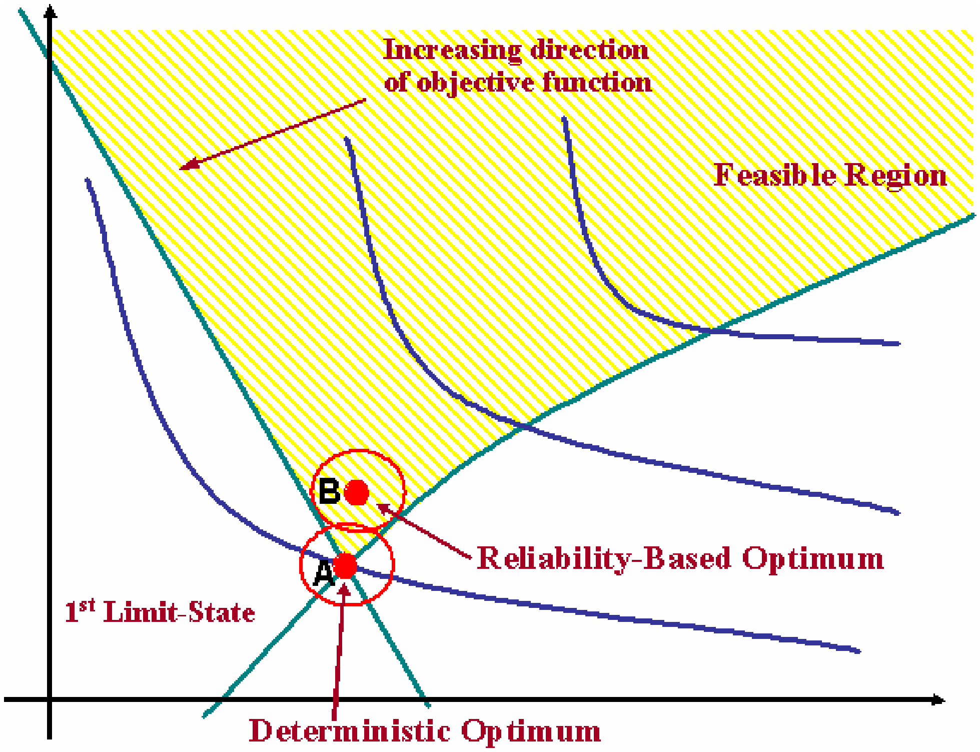

Figure 2.3: Comparison between RBDO and deterministic optimization

The basic formulation of an RBDO problem is shown in Eq. (2.2).

$$\begin{aligned} &\min \ \{f_0(x,d)\} \\ &s.t. \\ &f_j(x,d) < 0 \\ &P(g_i(x,d) > 0) > R_t, i = 1 \ldots n \end{aligned} \tag{2.2}$$

In Eq. (1) f_0 is the objective function and R_t is the target reliability associated with the i^{th} constraint function (g_i). f_j are the deterministic constraints, x is the vector of random design variables and d is the vector of deterministic design variables. The reliability of a limit state is related to its distance from the origin in the standard normal coordinate system .This will be discussed in detail later.

Earlier research in RBDO has been done using minimization of weight, minimization of cost and minimization of probability of failure as objective functions. More recently Royset et al (2001) used life cycle cost as the objective function to be minimized. The reliability assessment in the constraint, which can be done by several methods such as FORM, inverse FORM or Monte Carlo simulation, is an iterative procedure. Therefore the number of function evaluations increases considerably in reliability-based optimization when compared to deterministic or safety factor based optimization. In order to improve the computational efficiency of RBDO, several approaches have been developed in the literature where the reliability calculations are decoupled from the optimization loop.

The reliability constraints in RBDO techniques have mostly been computed by using the FORM method. The probability in the reliability constraint is replaced by the reliability index (β), which is defined as the minimum distance from the origin to the limit state surface (failure surface) in the standard normal coordinate system. The reliability index is related to the probability of failure (p_f) and the reliability (R) of the linearized constraint function as

$$p_f = 1 - R = \Phi(-\beta) = 1 - \Phi(\beta) \qquad (2.3)$$

where $\Phi(\beta)$ is the cumulative standard normal distribution. Tu et al (1999) developed a performance measure approach where inverse reliability analysis was used to define the reliability constraint. All nested double loop methods follow a procedure similar to the one shown in Fig. 2.4.

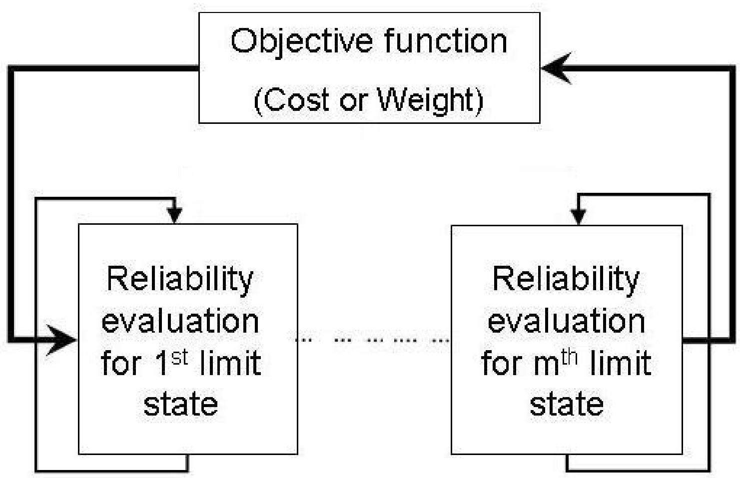

Figure 2.4: Concept of nested double loop RBDO methods

The bold arrows show the iterations caused by the optimization process and the thin arrows signify FORM or inverse reliability-based iterative procedure. For each iteration of the outer loop, the reliability constraints need to be evaluated, thus increasing the number of function evaluations.

2.4.1 Reliability Index Approach (RIA)

This is the traditional approach of reliability-based optimization where the FORM analysis of the limit state is used to evaluate the reliability constraints of the optimization problem. This can be mathematically represented as

$$\begin{aligned}
&\min \{f_o(\mu_{x,} d)\} \\
&s.t. \\
&f_j(\mu_x, d) \leq 0 \quad \text{Structural Constraints} \\
&\beta_i > \beta_{ti} \quad \text{Reliability Constraints}
\end{aligned} \tag{2.4}$$

The reliability constraints are met by ensuring that the reliability index (β) of each limit state (or failure mode) of the system is greater than a target value of reliability index (β_t). The first order reliability index β is obtained by performing the FORM analysis , which in itself is an optimization problem in the standard normal space and can be written as

$$\begin{aligned}
&\min \|u\| \\
&s.t.\ g_i'(u) = 0 \\
&where \\
&g_i'(u) = g_i(x, d)
\end{aligned} \tag{2.5}$$

The minimum distance point obtained from the above optimization is called the most probable point (MPP). Note that a separate iterative analysis needs to be performed to find the MPP and reliability index for each limit state. Many MPP search algorithms are available; the Rackwitz-Fiessler (1978) method is most commonly used.

2.4.2 Performance Measure Approach (PMA)

This method was proposed by Tu et al (1999) and is computationally more efficient than RIA for inactive constraints. For active constraints, Tu et al (2001) found RIA to be more efficient. PMA is based on the inverse FORM, where the performance function value is computed corresponding to a given value of the reliability index. The RBDO problem under PMA can be mathematically represented as

$$\begin{aligned}
&\min \{f_o(\mu_x, d)\} \\
&s.t. \\
&f_j(\mu_x, d) \leq 0 \quad \text{Structural Constraints} \\
&\mathrm{g_i(x, d)} \geq 0 \quad \text{Reliability Constraints}
\end{aligned} \tag{2.6}$$

In Eq. (2.6) the first order probabilistic measure $g_i(x,d)$ is the solution of a non-linear optimization problem in the standard normal space, as shown in Eq. (2.7). In Eq. (2.6) only the active reliability constraints will converge to zero. The rest will converge to values denoted by g_i* which is the performance measure of the i^{th} limit state corresponding to the target reliability index (β_{ti}).

$$\begin{aligned} &\min\ g_i'(u) \\ &s.t.\ \|u\| = \beta_{ti} \end{aligned} \tag{2.7}$$

$g_i'(u)$ is the function $g_i(x,d)$ in the standard normal space. The optimum point obtained on the β_t hyper-sphere is the MPP with a prescribed target reliability index β_t.

The reliability constraint in Eq. (2.6) can be written as $g_i(x) \geq g_i*$ because

$$\text{Prob}\{\text{g}_\text{i}(\text{x}) \geq \text{g}_\text{i}^{\ *}\} = \Phi(\beta_{\text{it}}) = \text{R}_\text{t} \tag{2.8}$$

where R_t is the target reliability that needs to be achieved. The active limit state will have $g*$ value as zero. Since the MPP search, done by inverse reliability analysis, is also an iterative procedure therefore PMA is also a nested-double loop method.

2.4.3 Decoupled Methods

The objective of decoupled RBDO methods is to reduce the number of function evaluations. Methods to separate the reliability calculations from the optimization loop are desirable to increase the efficiency of the optimization procedure. Also, if decoupling is achieved it will be easier to implement different reliability analysis procedures for different constraints. A valuable approach is to use deterministic optimization to get a good initial design and then use the reliability analysis techniques to shift the solution to a point where it can satisfy the reliability constraints as well.

The general concept of decoupled RBDO is shown in Fig. 2.5.

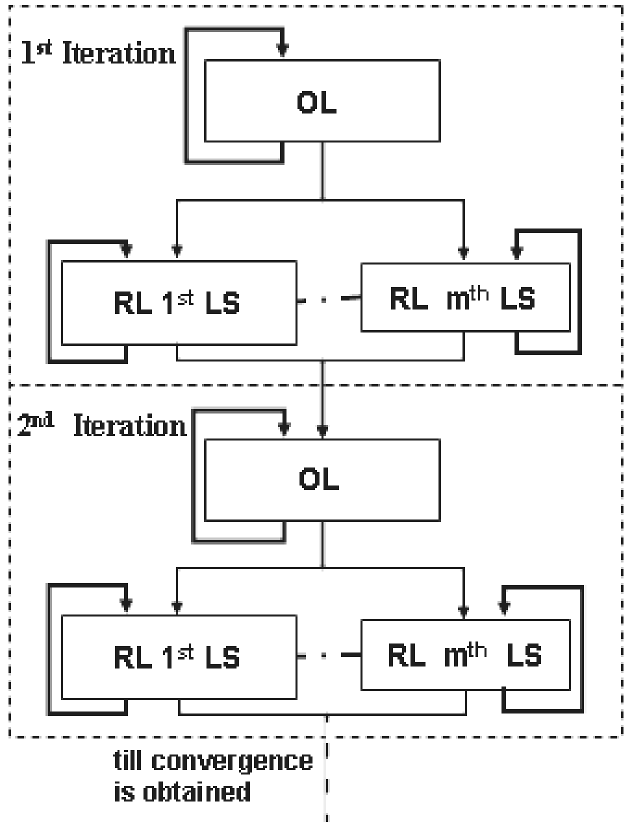

Figure 2.5: Decoupled RBDO methods

In Fig. 2.5, OL denotes optimization loop and RL the reliability loop. The optimization loop does not contain any reliability loops within and hence the decoupling is achieved. The MPP obtained from the previous reliability loop is used to define the performance of the limit state for the particular values of means of random design variables (μ and the deterministic design variables (d). Decoupled RBDO can be done using either direct FORM (Zou and Mahadevan, 2004) or inverse FORM (Du and Chen, 2001, Wu et al, 2001, Royset et al, 2001). Decoupled methods based on the RIA approach have also been developed. This thesis follows the inverse FORM-based decoupling, i.e., methods based on PMA.

The procedure followed in existing decoupled PMA methods can be understood by the flow diagram in Fig. 2.6.

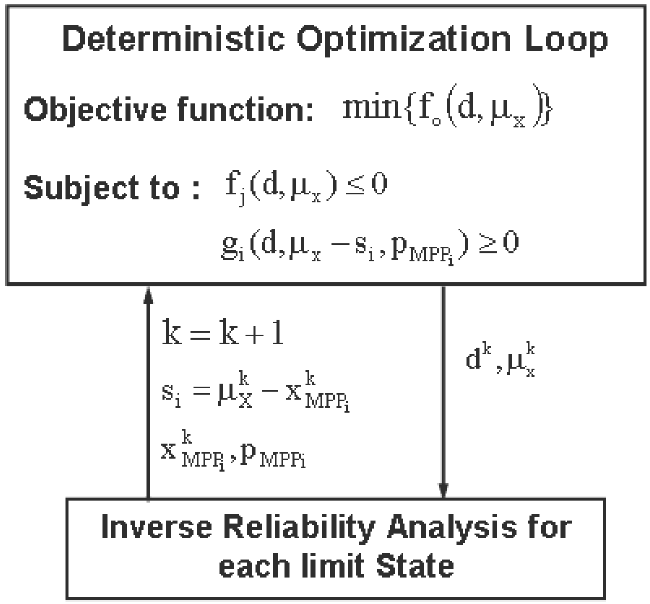

Figure 2.6: Flow diagram of the decoupled PMA

In Fig. 2.6, s_i is the shifting vector and measures the performance of the limit state i. As x tends to the design point x^*, the shifting vector s_i attains a constant value, and so we obtain g_i^* which is the performance value of the limit state. If there are multiple limit states then the other limit state boundaries should also be shifted towards the feasible region by the distance equal to the difference between the optimal values and the most probable points. Thus the shifting vector value will be different for each limit state. Other random parameters (p) associated with limit states can also be included in the calculations. Note that the reliability loop also gives MPP (p_{MPP}) values for these parameters after every reliability iteration and this will be the starting value for the next reliability iteration. The decoupled PMA approach is used in this thesis.

The decoupled PMA method is extended in Chapter 3 to handle the following situations:

1. Standard deviations can be included as design parameters which helps in robust design.
2. Reliability assessment is done using methods other than FORM when necessary, e.g. Monte Carlo simulation, allowing a modular approach for different limit states.

2.4.4 Single Loop Methods

These methods convert the RBDO problem into a deterministic single loop optimization. The constraints in the single loop methods are approximated by the first order method (FORM or inverse FORM). The nested optimization is eliminated by satisfying the optimality conditions at the MPP. Such methods based on the FORM approximation (RIA type) have been developed by Friis-Hansen and Madsen (1992) and Kushel and Rackwitz (1997, 2000). Single loop methods using inverse FORM (PMA type) have been formulated by Chen et al (1997) and more recently by Liang et al (2004). In this thesis the PMA-based method developed by Liang et al (2004) is used, as shown in Eq. 2.9.

$$
\begin{aligned}
&\min\{f_o(\mu_X)\} \\
&s.t. \\
&f_j(\mu_x) \leq 0 \quad \text{Deterministic Constraints} \\
&\mathrm{g_i}(x) \geq 0 \\
&where \\
&x - \mu_x = -\beta_t * \alpha * \sigma \\
&\alpha = \nabla g_u(d,x) / \| \nabla g_u(d,x) \| \\
&\quad = \sigma * \nabla g_x(d,x) / \| \sigma * \nabla g_x(d,x) \| \\
&\mu_{xl} \leq \mu_x \leq \mu_{xu}
\end{aligned}
\tag{2.9}
$$

This method does not perform the reliability assessment iteratively, instead makes use of derivatives which are calculated only when the optimizer changes any of the design variable values. Thus the method gets rid of unnecessary gradient evaluations and gives accurate results efficiently. The choice of initial design affects the stability of this method. A good initial design such as the deterministic optimum might be helpful in reducing this instability. The single loop method is extended in Chapter 4 for robust design problems and the results obtained are compared with the decoupled approach.

CHAPTER III

IMPROVED DECOUPLED APPROACH

3.1 Introduction

The proposed decoupled RBDO approach has two features:

1. Inclusion of standard deviations of variables as deterministic design parameters.
2. Inclusion of simulation based reliability assessment to replace the existing FORM-based approach when necessary.

3.2 Decoupled approach with standard deviations as design parameters

The inclusion of standard deviations as design parameters will help deal with robust design problems. Here we find the shifting vector in the standard normal space, as opposed to the origional decoupled RBDO method where we calculate the MPP in the design space. The standard deviations are treated as determinisitic design parameters.

The algorithm is as follows:

1. Initialize K=1 , $s_i = 0$, $p_{MPPi} = \mu_p^o$, $U_{MPPi} = 0$, $\sigma_x = \sigma^0$, $\mu_x^K = \mu_x^o$.
2. The shifting vector can be defined as $s_i = U_{MPP_i} \times \sigma_x$
3. Perform deterministic optimization.

$$\begin{aligned} &\min f(d,\mu_x,\sigma_x) \\ &s.t.\ g_i(d,\mu_x - s_i, p_{MPPi}) \geq 0 \end{aligned} \quad = \quad \begin{aligned} &\min f(d,\mu_x,\sigma_x) \\ &s.t.\ g_i(d,\mu_x + U_{MPP_i} \times \sigma_x, p_{MPPi}) \geq 0 \end{aligned}$$

4. K=K+1, d^K = d , $\mu_x^K = \mu_x$ and $\sigma^K = \sigma_x$
5. Reliability Assessment of each limit state is done using inverse reliability analysis and the new value of most probable point (U_{MPPi} and p_{MPPi}) corresponding to the target reliability is found.
6. Find the objective function value and check its convergence. Check feasibility of limit states.
7. Stop if the convergence and feasibility criteria are fulfilled; otherwise repeat steps 2 to 6.

The reliability assessment in step 3 is done using the inverse reliability method where the MPP (inverse reliability based) is found corresponding to the target reliability (β_t value) as shown in Eq. 2.7.

The MPP obtained will lie on the β_t circle. The MPPs for both violated as well as inactive constraints are shown in Fig. 3.1. If at this MPP point the value of the limit state is less than zero then it implies that the design solution has not yet achieved the target reliability (If for that limit state a positive value denotes safe region).

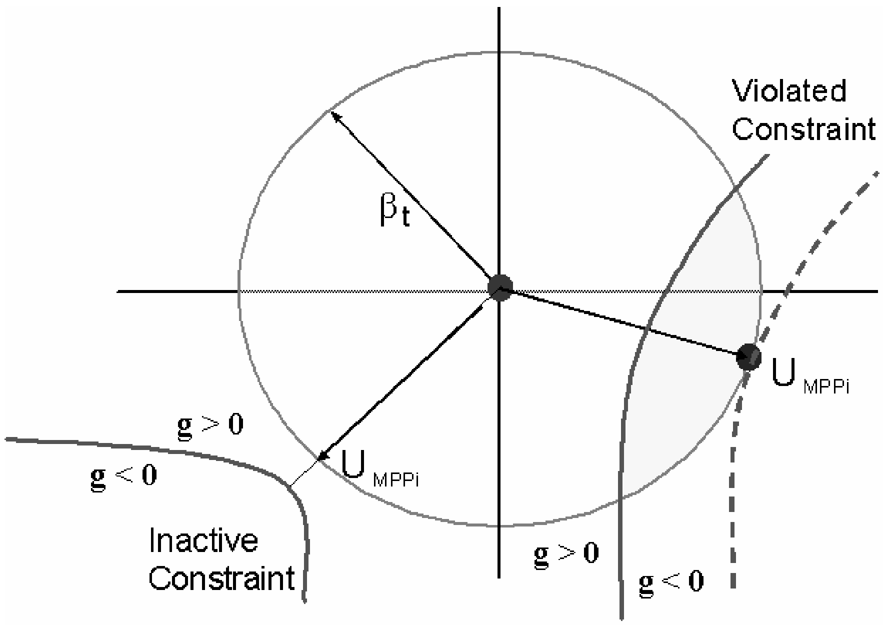

Figure 3.1: Reliability assessment in standard normal space

The standard normal values of the inverse reliability-based MPP are passed on to the next optimization calculation:

$$\begin{aligned} &\min \ f(d,\mu_x,\sigma_x) \\ &s.t.\ g_i\left(d,\mu_x + U_{MPP_i} \times \sigma_x, p_{MPPi}\right) \geq 0 \end{aligned} \tag{3.1}$$

The MPPs will again have to be updated for the new values of design variables and the sequential calculations will continue till the converged objective function value is obtained. At the converged solution the target reliability is achieved with a first-order approximation.

3.3 Simulation-Based Reliability Assessment

The above method can be extended to include the option of reliability assessment by Monte Carlo Simulation (MCS) using the following procedure. In step 5 in the above algorithm, MCS is used instead of inverse FORM for calculating the failure probability of the limit state. Then, the simulation based sensitivities (S_{Xi}) are given by

$$S_{X_i} = E[(U_i)]_{\Omega} \tag{3.2}$$

Thus the sensitivity of the i^{th} variable is approximated by the average of the standard normal variables in the failure samples (Wu, 1999). For low probability of failure or large U_i* value the directional cosines proportional to the simulation based sensitivities. Therefore the direction cosines can be computed as

$$\alpha_i = \frac{S_{X_i}}{\sqrt{\sum_{i=1}^{n} \left(S_{X_i}\right)^2}} \tag{3.3}$$

These values can then be used in the calculation of the most probable points using Eq. 3.4.

$$U_{MPP_i} = \alpha_i \times \beta_t \tag{3.4}$$

These standard normal MPPs can then be used in calculating the shifting vector and thus used in the SORA method.

3.4 Using Correction Factors in Simulation-based SORA

The simulation method shown above makes a first order approximation in Eq. 3.4. This drawback can be removed by using a corrected β_t. The correction in β_t depends on the amount of nonlinearity in the associated limit state. The amount of nonlinearity in a limit state can be estimated by comparing the probability of failure values obtained from both FORM and Monte Carlo simulation. For this we follow a semi-heuristic procedure to ensure convergence.

1. Obtain the reliability index (β_{FORM}) from FORM analysis of the limit state.
2. Obtain the reliability index (β_{SIM}) from the simulation analysis by using

$$\beta_{SIM} = \Phi^{-1}\left(\frac{n}{N}\right) \tag{3.5}$$

where:

n = Number of failed samples in the simulation.

N= Total number of samples.

Φ = Cumulative standard normal distribution.

3. Find the error in simulation by

$$e_{SIM} = \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{(1-P_f^T)}{NP_f^T}} \times 100\% \tag{3.6}$$

4. If β_{SIM} and β_{FORM} are approximately equal i.e.

$$\left|\beta_{\mathrm{SIM}} - \beta_{\mathrm{FORM}}\right| \le \frac{e_{SIM} \times \beta_t}{100} \tag{3.7}$$

Then Correction Factor CF = 1.

5. If β_{SIM} or β_{FORM} are much greater than $2*\beta_{Target}$ then take them to be equal to $2*\beta_{Target}$ This is done to reduce the computational effort to calculate the reliability index of inactive constraints having high reliability.

6. If $\beta_{SIM} < 0.2*\beta_{Target}$ OR $\beta_{FORM} < 0.2*\beta_{Target}$ Then

$$CF = \frac{1+\beta_{FORM}}{1+\beta_{SIM}} \tag{3.8}$$

7. For all other cases the correction factor is given by

$$CF = \frac{\beta_{FORM}}{\beta_{SIM}} \tag{3.9}$$

For convergence we make sure that all the constraints satisfy the target reliability restriction. We assume failure of the limit state if

$$\beta_{SIM} < \beta_{Target} - \frac{e_{Simulation} \times \beta_{Target}}{100} \tag{3.10}$$

Once the correction factors for each of the limit states are obtained we get the standard normal MPP for each limit states using Eq. 9 instead of Eq. 4.

$$U_{MPP_i} = \alpha_i \times \beta_{Target} \times CF \tag{3.11}$$

3.5 Examples

The above described improved methods are tested on two examples.

3.5.1 Example 1 (Tu et al, 1999)

Consider a system defined by two independent, normally distributed system parameters $X = [X1, X2]^T$ with constant standard deviations σ_1=0.3 and σ_2=0.3. The design vector is d = $[d1, d2]^T = [\mu_{X1}, \mu_{X2}]^T$. The RBDO problem is defined for a target reliability of P_{ft} in Eq. 3.12.

$$
\begin{aligned}
&Cost = d1 + d2\\
&s.t.\\
&P(G_i(X) \le 0) \le P_{ft} = \Phi(-\beta_t), \qquad i = 1, 2, 3\\
&0 \le d1 \le 10, \qquad 0 \le d2 \le 10
\end{aligned}
\tag{3.12}
$$

where β_t is the target reliability index. The three limit states for the optimization problem in Eq. 3.12 is given in Eq. 3.13.

$$
\begin{aligned}
G_1(X) &= \frac{X_1^2 X_2}{20} - 1\\
G_2(X) &= \frac{(X_1 + X_2 - 5)^2}{30} + \frac{(X_1 - X_2 - 12)^2}{120} - 1\\
G_3(X) &= \frac{80}{(X_1^2 + 8X_2 + 5)} - 1
\end{aligned}
\tag{3.13}
$$

<u>Solution:</u>

The RBDO is performed using three different initial design points i.e. (1, 4) (5, 5) and (7, 9). The standard deviations of *X1* and *X2* in the above problem were treated as constants. The target reliability index, β_t is set at 3.0 which corresponds to a reliability of 99.87%. The improved decoupled method is employed for optimization using two reliability assessment approaches: (1) Inverse FORM; and (2) Monte Carlo Simulation with correction factors (*CF*); both of which have been discussed in section 4.

Table 3.1: Example 1: RBDO solution

	SORA with inverse FORM				Simulation based SORA with CF			
	d1	d2	Cost	Function Evaluations	d1	d2	Cost	Function Evaluations
1	3.439	3.287	6.726	569	3.439	3.286	6.725	120575
2	3.439	3.287	6.726	549	3.436	3.285	6.721	180797
3	3.439	3.287	6.726	549	3.437	3.286	6.723	150696

Discussion: The results above show that the improved method works efficiently. Simulation can effectively replace the Inverse FORM analysis. In this case the FORM approximation of the limit states gave a good estimate of the reliability so the simulation result is similar to that of inverse FORM. Note that simulation is used only when FORM is either inaccurate or fails to converge.

3.5.2 Example 2

Ford side impact beam problem: The optimization problem is defined in Eq. 3.14 The design variables and parameters involved are defined in Table 3.2.

Minimize : Weight (W)

s.t.

$$
\begin{aligned}
&V_{B\text{-}Pillar} \le 9.9 \text{ (m/s)} \\
&V_{front\ door} \le 15.69 \text{ (m/s)} \\
&V^*C_1 \le 0.32 \text{ (m/s)} \\
&V^*C_2 \le 0.32 \text{ (m/s)} \\
&V^*C_3 \le 0.32 \text{ (m/s)} \\
&D_{upper\ rib} \le 32 \text{ (mm)} \\
&D_{middle\ rib} \le 32 \text{ (mm)} \\
&D_{lower\ rib} \le 32 \text{ (mm)} \\
&\text{Abdomen Load} \le 1.0 \text{ (KN)} \\
&\text{Pubic Symphysis Force} \le 4.0 \text{ (KN)}
\end{aligned}
\tag{3.14}
$$

where

V = Velocity

D = Deflection

Table 3.2: Example 2: Variables

Variables		Lower Bound	Mean	Upper Bound	Standard Deviation
X1	Thickness of B-Pillar inner (mm)	0.500	TBD	1.500	0.030
X2	Thickness of B-Pillar reinforcement (mm)	0.450	TBD	1.350	0.030
X3	Thickness of floor side inner (mm)	0.500	TBD	1.500	0.030
X4	Thickness of cross member #1 & #2 (mm)	0.500	TBD	1.500	0.030
X5	Thickness of door beam (mm)	0.875	TBD	2.625	0.050
X6	Thickness of door belt line reinforcement (mm)	0.400	TBD	1.200	0.030
X7	Thickness of roof rail (mm)	0.400	TBD	1.200	0.030
X8	Material property of B-Pillar inner	-	0.345	-	0.006
X9	Material property of floor side inner	-	0.192	-	0.006
X10	Barrier height (mm)	-	0.000	-	10.00
X11	Barrier hitting position (mm)	-	0.000	-	10.00

Variables X1-X7 are design variables. X8-X11 are random variables. All the constraints are treated as probabilistic with the target reliability as 90% which corresponds to a β value of 1.28. The response surfaces for all the constraints are available.

Solution: The RBDO is performed with three different initial designs (nominal values, lower bound, and upper bound). The improved decoupled approach is employed. The results, obtained using the Inverse FORM method for reliability assessment, are given in Table 3.3.

Table 3.3: Example 2: RBDO results using inverse FORM

S.No.	X1	X2	X3	X4	X5	X6	X7	Weight	Function Evaluations
1	0.500	1.306	0.500	1.323	0.875	1.200	0.400	24.586	1414
2	0.500	1.306	0.500	1.323	0.875	1.200	0.400	24.586	1338
3	0.500	1.306	0.500	1.323	0.875	1.200	0.400	24.586	1390

The RBDO was also performed using the simulation method with correction factors. The results obtained for four different initial designs are shown in Table 3.4.

Table 3.4: Example 2: RBDO results using simulation with correction factors.

S.No	X1	X2	X3	X4	X5	X6	X7	Weight
1	0.500	1.309	0.500	1.397	0.875	1.200	0.400	24.901
2	0.500	1.308	0.500	1.395	0.875	1.200	0.400	24.894
3	0.500	1.308	0.500	1.408	0.875	1.200	0.400	24.945
4	0.500	1.309	0.500	1.386	0.875	1.200	0.400	24.856

The computational effort is very high for this method (takes between 100,000 to 200,000 function evaluations to reach a converged solution) but the results meet the reliability requirement more accurately. The comparison of the achieved reliability indices for each limit state as obtained from inverse FORM and the four simulation runs are given below. 100,000 samples were used for each limit state to calculate the achieved reliability.

Table 3.5: Example 2: Comparison of Reliability Index achieved for each limit state

Method	LS1	LS2	LS3	LS4	LS5	LS6	LS7	LS8	LS9	LS10
Simulation 1	2.560	**1.269**	2.560	2.560	2.560	2.560	1.283	2.560	2.560	2.264
Simulation 2	2.560	1.286	2.560	2.560	2.560	2.560	**1.263**	2.560	2.560	2.209
Simulation 3	2.560	**1.264**	2.560	2.560	2.560	2.560	1.291	2.560	2.560	2.245
Simulation 4	2.560	**1.261**	2.560	2.560	2.560	2.560	**1.273**	2.560	2.560	2.135
Inverse FORM	2.560	**1.245**	2.560	2.560	2.560	2.560	**0.828**	2.560	2.560	2.346

Discussion: As expected the simulation based RBDO required higher computational effort but satisfied the target reliability conditions to a greater extent. In Table 3.5 the limit states not satisfying the target reliability are shown in bold font. It can be observed that the simulation results give a reliability index closer to the target. The difference is more noticeable in limit state 7 (LS7).

3.6 Summary

The decoupled PMA-based RBDO method is extended in this chapter to replace the first order reliability method with Monte-Carlo simulation to satisfy the feasibility of the nonlinear limit states more accurately. Due to the use of Monte-Carlo simulation the number of function evaluations is more than in the traditional methods. The proposed method also helps to include standard deviations as design parameters

CHAPTER IV

ROBUST DESIGN

4.0 Overview

Uncertainties in the design and system variables create variability in the objective function and constraints in design optimization. RBDO does not consider this variability and in effect optimizes the mean value of the objective function. Robust design under uncertainty however is a bi-objective optimization:

1. Optimize the mean value of cost/objective function (μ_f), and
2. Minimize the variance of the objective function (σ_f^2).

The constraints for the above optimization will stay probabilistic to ensure the reliability requirements. In this section we suggest formulations for implementing the decoupled and single loop methods for the robust design problem. The applicability of these formulations is also discussed. The two issues in reliability-based robust design problems are

1. Compute the variance of the objective function.
2. Select a suitable algorithm for bi-objective optimization.

4.1 Variance of the Objective Function

Several formulations used to compute the variance of the objective function (*f*) are discussed below.

1) The first approach is to use the first order variance using the Taylor series expansion. This gives the variance of the objective as

$$Var(f) \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial f}{\partial X_i}\frac{\partial f}{\partial Xj}COV\left(X_i, X_j\right) \tag{4.1}$$

If there is no correlation between the design variables then Eq. (4.2) can be rewritten as

$$Var(f) \approx \sum_{i=1}^{n}\left(\frac{\partial f}{\partial X_i}\right)^2 Var(X_i) \tag{4.2}$$

The drawback of using this method is that the value of the derivatives, computed in the above expressions, are zero around the point of minima or maxima. This results in the value of variance to be close to zero.

2) The second way to represent the variation is through a percentile formulation (Du et al 2003). The variation in the objective in this case can be represented by

$$\Delta f_{\alpha_1}^{\alpha_2} = f^{\alpha_2} - f^{\alpha_1} \qquad (4.3)$$

where f^{α_2} and f^{α_1} are the values of objective function evaluated at the α_2 and α_1 percentile values. α_1 and α_2 lie on either side of the mean as shown in the Fig. 4.1.

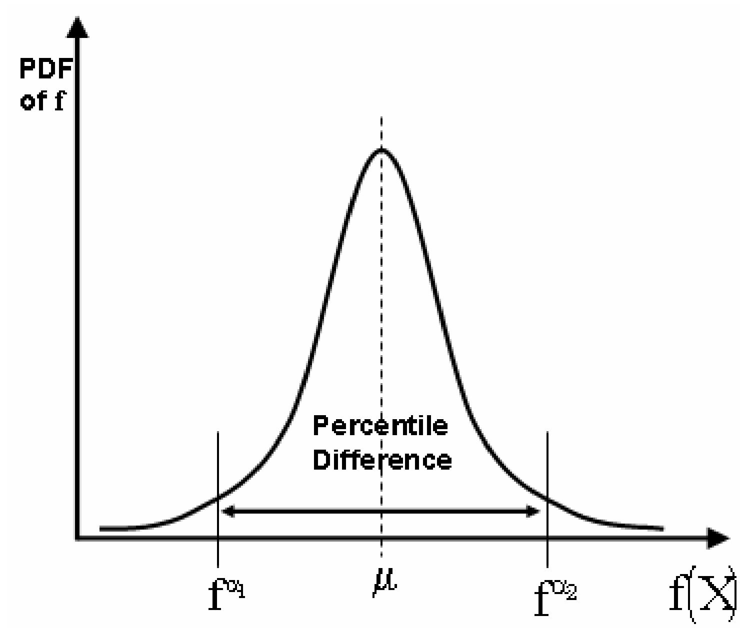

Figure 4.1: Percentile difference formulation.

When their difference is minimized, it automatically leads to the minimization of variance. It should be noted that Eq. 4.3 is not an estimate of the variance but is a function which when reduced leads to the reduction of variance. If the mean of the objective also needs to be minimized then we just need to minimize $f^{\alpha 2}$. On the other hand if the mean needs to be maximized then we need to maximize just the $f^{\alpha 1}$ value. This can be better explained in Fig. 4.2.

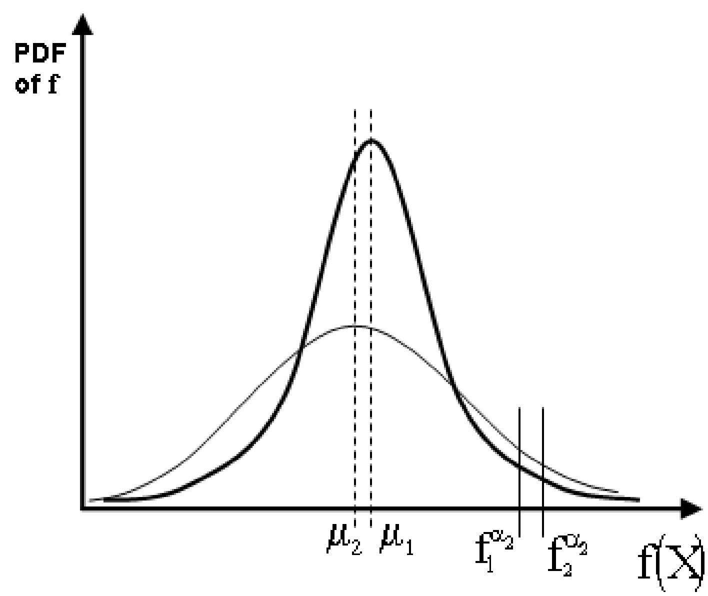

Figure 4.2: Comparison of PDFs of objective function for two designs.

With reference to Fig. 4.2 (Du et al, 2003), if we have two designs then, on performing the minimization, the percentile formulation will choose design 1 over design 2 because design 1 has a lower α_2 percentile value $\left(f_1^{\alpha_2} < f_2^{\alpha_2}\right)$ even though design 2 has a lower mean (μ_2<μ_1). Similarly if the mean of the objective function needs to be maximized then we just need to maximize the lower percentile value (i.e. $f^{\alpha 1}$) of the objective function. Thus the percentile formulation gives the more robust solution as the optimal one and reduces the bi-objective optimization to a single objective optimization.

3) Another method used to represent the variance uses a variation of the Simpson's rule (Youn and Choi, 2004; Taguchi. 1989; D'Errico and Zaino, 1988). A weighted three level numerical integration is performed on the first two moments of the objective function, thereby giving an estimate of its mean and variance. In this method, the function is first evaluated using the mean values of the design variables. Then the function's 95.84th and 4.16th percentile values, which correspond to a reliability index of ±√3, are calculated using the inverse reliability analysis. The weighted sum of these three function values gives us an estimate of the mean of the objective function. The value of the weights is given in Eq. 4.4.

$$\mu_{f(x)} = \frac{1}{6} f_{\beta_t=\sqrt{3}} + \frac{4}{6} f_{\mu_x} + \frac{1}{6} f_{\beta_t=-\sqrt{3}} \tag{4.4}$$

The variance about this mean value is given by Eq. 4.5.

$$\sigma^2_{f(x)} = \frac{1}{6}\left(f_{\beta_i=\sqrt{3}} - f_{\mu_x}\right)^2 + \frac{1}{6}\left(f_{\beta_i=-\sqrt{3}} - f_{\mu_x}\right)^2 \qquad (4.5)$$

The objective function for robust design will be the weighted sum of the mean and variance values obtained in the above equations. This will be discussed later in the chapter.

4) The most accurate method to estimate the variance is by Monte Carlo Simulation. Random samples are generated for the input design variables and the objective function is evaluated for all these samples. The mean and variance of the samples can be calculated using simple statistical analysis.

The variances obtained from the first two methods are based on sensitivities obtained using derivatives about just the mean value. For obtaining close to true variances of the objective, sensitivities will need to be derived using a more global technique. These techniques are mostly simulation-based hence require more computational effort. In the third method the variance is calculated using function and sensitivity evaluations at three points to incorporate a more accurate behavior of the function about the mean value.

The above techniques use the local sensitivities of the variables to estimate the variance. The local sensitivities are derivative-based and involve changing one variable at a time keeping the other variables constant. On the other hand, during global sensitivity analysis several variables are changed at the same time and their effect on the output is observed. This helps in capturing the interaction effects as well. Therefore, global sensitivity analysis can be defined as the study of a response or output for the entire range of an input variable. Some methods based on conditional variance estimation have been developed where the sensitivity with respect to a variable x_i is defined as

$$S_{x_i} = \frac{Var\left[E\left(y/x_i\right)\right]}{Var\left[y\right]} \qquad (4.6)$$

But, most global sensitivity analysis techniques stem from the Sobol indices (Sobol, 1993) where the function is decomposed into increasing order terms through functional ANOVA. The first order terms represent the main effects while the higher order terms correspond to the interaction effects. Consider an integrable function $y = f(x)$ defined in an *n*-dimensional unit hypercube I^n and, $x \in I^n$. It is also assumed that the function is square integrable. This function can be written as

$$y = f_o + \sum_i f_i(x_i) + \sum_{i<j} f_{ij}(x_i, x_j) + ... + f_{12...n}(x_1, x_2, ..., x_n) \quad (4.7)$$

Eq. (4.7) can be simplified to give

$$y = f_o + \sum_{s=1}^{n} \sum_{i_1<...<i_s}^{n} f_{i_1...i_s}(x_{i_1}, ..., x_{i_s}) \quad (4.8)$$

where $1 \le i_1 < ... < i_s \le n$ and all member functions are orthogonal, therefore they can be written as integrals of $f(x)$. This property is illustrated in Eq. (4.9).

$$\int f(x)dx = f_o \quad (4.9a)$$

$$\int \left(f(x) \left(\prod_{k \ne i} dx_k \right) \right) = f_o + f_i(x_i) \quad (4.9b)$$

$$\int \left(f(x) \left(\prod_{k \ne i,j} dx_k \right) \right) = f_o + f_i(x_i) + f_j(x_j) + f_{ij}(x_i, x_j) \quad (4.9c)$$

Eq. (4.8) leads to the necessary condition required for Eq. (4.7) to be called an ANOVA decomposition, i.e.

$$\int_0^1 f_{i_1...i_s}(x_{i_1}, ..., x_{i_s}) dx_k = 0 \text{ for } k = i_1, ..., i_s \quad (4.10)$$

Since the function $f(x)$ is square integrable, therefore all its member functions are also square integrable. Therefore on squaring (4.8) and integrating over I^n we get

$$\int f^2 dx - f_o^2 = \sum_{s=1}^{n} \sum_{i_1<...<i_s}^{n} f_{i_1...i_s}^2 dx_{i_1} \cdots dx_{i_s} \quad (4.11)$$

The above equations can be rewritten as

$$D = \sum_{s=1}^{n} \sum_{i_1<...<i_s}^{n} D_{i_1...i_s} \quad (4.12)$$

The sensitivity with respect to a particular term is defined by

$$S_{i_1...i_s} = \frac{D_{i_1...i_s}}{D} \quad (4.13)$$

The $D_{i_1...i_s}$ can be calculated using Monte-Carlo simulation based techniques (Saltelli et al. 1997). It has also been shown that

$$S = \sum_{s=1}^{n} \sum_{i_1<...<i_s}^{n} S_{i_1...i_s} = 1 \quad (4.14)$$

The sensitivity with respect to a particular variable can then be defined by the total sensitivity index which is the sum of the sensitivities of all the terms containing that particular variable. Further, global sensitivity analysis methods have some common features with robust design. In robust design, we need to minimize the variance of the objective function involved. This objective can be a complex response surface for which the variance calculation can be cumbersome and inaccurate. Chen et al. (2004) represent this complex multivariate response surface as a product of some standard univariate basis functions. These functions satisfy the conditions for the ANOVA decomposition explained above and thus the global sensitivity analysis based methods can be employed for variance estimation. Basis functions for some standard response surfaces are given by Jin (2004).

This study implements the local techniques discussed above, and compared them using one linear and two nonlinear functions. The first two functions are constituted of three variables $X = [X_1, X_2, X_3]$. All three are normally distributed with means as [9, 5, 3] and standard deviations of [1, 0.5, 0.25]. The third function is consists of only one variable i.e., X_2. The functions are given in Eq. 4.15.

$$\begin{aligned} F_1(X) &= 9X_3 - 4X_1 + 7X_2 \\ F_2(X) &= \frac{10}{(X_1 X_3^2)} - X_3 X_2^2 + 9X_1 + 12 \\ F_3(X) &= -0.03 * X_2^3 + 6.548 * X_2^2 - 62.37 * X_2 + 152.31 \end{aligned} \quad (4.15)$$

The values of variances obtained are given in Table 4.1. Ten thousand samples were used in the Monte Carlo estimation of the variance.

Table 4.1: Comparison of variance estimation techniques.

	F_1			*F_2*			*F_3*		
Variance Estimation Method	$\mu_{F_1(X)}$	$\sigma^2_{F_1(X)}$	*N_1*	$\mu_{F_2(X)}$	$\sigma^2_{F_2(X)}$	*N_2*	$\mu_{F_3(X)}$	$\sigma^2_{F_3(X)}$	*N_3*
First order	26.00	33.31	4	18.12	345.07	4	0.38	0.178	2
3-point numerical integration	26.00	33.31	21	17.29	351.46	33	1.66	4.97	17
Monte Carlo Simulation	26.04	33.97	10000	17.30	343.59	10000	1.86	5.01	10000

In Table N_1, N_2 and N_3 correspond to the number of function evaluation for F_1, F_2 and F_3 respectively. It is seen that the first order method gives a good estimate of the variance for a linear function and is most computationally efficient. The estimate is reasonable for any arbitrary nonlinear function as well. For a non-linear function where the means of the design variable are close to its minima or maxima (F_3), the first order method fails as is shown by the results of the third function. In such a case, the numerical integration method should be used.

4.2 Multi-objective Optimization

Many techniques are available for performing multi objective optimization; these convert multiple objectives into a single objective. Two of these methods are discussed here.

1) The weighted sum approach is a simple, but popular method to combine the two objectives

$$Min \quad w \times \mu_{f_o(x)} + (1-w) \times \left(\sigma_{f_o(x)}\right) \tag{4.16}$$

The weight (w) need to be decided subjectively. The weight can be varied to generate a set of non-dominant solutions also called the pareto set.

2) In the ε -constraint method, one of the objectives is converted into a constraint, by setting it to a target value. For the robust design problem the resulting optimization problem becomes

$$\begin{aligned} & Min \quad \left(\sigma_{f_o(x)}\right) \\ & s.t. \\ & \mu_{f_o(x)} \le \mu_{f_{TARGET}} \end{aligned} \tag{4.17}$$

when the target is set for the mean of the objective function. Alternatively, if the target is set for the standard deviation of the objective then the problem can be formulated as

$$\begin{aligned} & Min \quad \left(\mu_{f_o(x)}\right) \\ & s.t. \\ & \sigma_{f_o(x)} \le \sigma_{f_{TARGET}} \end{aligned} \tag{4.18}$$

The other probabilistic constraints will remain unchanged. Finding a target value for the mean or standard deviation of objective can be difficult hence the weighted sum formulation is applied in this study. The percentile formulation and the weighted sum formulation are compared with the help of two examples, using both the decoupled and single loop formulations.

4.3.1 Example 1

This problem is same as the Example 1 discussed in Chapter 3. The robust design procedure for the problem was performed using three different initial design points i.e. (1, 4) (5, 5) and (7, 9). The standard deviations of X_1 and X_2 in the problem were treated as constants. A range of results is obtained for different combinations of initial design and weights (0.5, 0.1) as is shown in Table 4.2. The first order method was used to evaluate the variance as the objective function in this problem is linear. The target reliability index, β_t is set at 1.28 corresponding to 90th percentile formulation. The decoupled PMA approach is used to address the robust design problems. For the percentile formulation, the α_2 percentile value to be minimized is taken as 90th percentile, for the sake of illustration.

Table 4.2: Example 1: Robust design without σ's as design parameters

Method	Cost	Function Evaluations
Weighted Sum + Decoupled method	6.726	570-579
Weighted Sum + Single loop method	6.726	132-177
Percentile formulation + Decoupled method	6.726	827-877
Percentile formulation + Single loop method	6.726	166-196

The number of function evaluations for the different initial points were different hence a range is provided in the above table. On comparing the results in Table 4.2, it can be seen that both the formulations give the same solution. Also, the results are the same as the RBDO. This is because for a linear objective function with constant standard deviations the robust design problem collapses to a RBDO problem. Note that the number of function evaluations for percentile formulation is more than the weighted sum formulation when both objectives have equal weights.

The above procedure was performed again, considering standard deviations as deterministic design parameters. The standard deviations were given bounds as σ_1,σ_2= [0.3,0.5]. The results obtained are shown in Table 4.3.

Table 4.3: Example 1: Robust design with σ's as design parameters

Method	Cost	Function Evaluations
Weighted Sum + Decoupled method	6.192	729-2247
Weighted Sum + Single loop method	6.192	114-261
Percentile formulation + Decoupled method	6.192	1036-1086
Percentile formulation + Single loop method	6.147-6.192	443-621

When the standard deviations are considered as design parameters, for this problem, they reach their lower bounds. Once again, the computational efficiency of the single loop method is higher than that of the decoupled approach. Some convergence issues were noted for certain initial designs in the case of the single loop method.

4.3.2 Example 2

This problem is same as the Example 2 discussed in Chapter 3. The formulation from Eq. (4.16) is used as the objective for weighted sum formulation. The variance is evaluated, like in Example 3, using the first order method since in this case also the objective function is linear. The 3-point numerical integration method could also be used but wasn't because of its computationally more intensive than the first order method. First, the standard deviations are not treated as design variables and all variables are assumed to be uncorrelated normals. The results using the different approaches are given in Table 4.4.

Table 4.4: Example 2: Robust design without σ's as design parameters

Method	Weight	Function Evaluations
Weighted Sum + Decoupled method	24.586	2172-2496
Weighted Sum + Single loop method	24.562	570-1002
Percentile formulation + Decoupled method	24.586	3931-5016
Percentile formulation + Single loop method	24.562-25.585	867-1235

A range in values of the number of function evaluations is provided as different initial designs lead to different number of function evaluations. It can be observed that both the methods (i.e. percentile and weighted sum) result in the same solution. In fact the results are the same as the RBDO solution obtained in Table 3.3. The reason for this is the same as in Example 1, i.e. linear objective function. Also the computational effort for the percentile formulation was found to be higher than the weighted sum method. This was also observed previously for Example 1. In the weighted sum approach, when the weight is kept unreasonably low there is difficulty in the convergence of some variables when using the single loop method.

Next, the standard deviations are treated as deterministic design parameters and their bounds are given in Table 4.5.

Table 4.5: Bounds on σ's of the design variables

Variable	Description of Variable (in mm)	Lower Bound	Upper Bound
X1	Thickness of B-Pillar inner	0.020	0.040
X2	Thickness of B-Pillar reinforcement	0.020	0.050
X3	Thickness of floor side inner	0.020	0.040
X4	Thickness of cross member #1 & #2	0.020	0.040
X5	Thickness of door beam	0.040	0.060
X6	Thickness of door belt line reinforcement	0.020	0.040
X7	Thickness of roof rail	0.020	0.040

The results for the robust design optimization including standard deviations as design variables are shown in Table 4.6.

Table 4.6: Example 2: Robust design with σ's as design parameters

Method	Weight	Function Evaluations
Weighted Sum + Decoupled method	24.556	3738-4512
Weighted Sum + Single loop method	24.555-25.345	1668-2202
Percentile formulation + Decoupled method	24.554-24.555	2744-3269
Percentile formulation + Single loop method	24.557	2706-2806

When standard deviations are treated as design parameters they move to their lower bound. For all cases the single loop was found to be superior in computational efficiency to the decoupled method. For some initial designs and very low weight values the decoupled loop was able to converge to a lower optimal solution than single loop but used more function evaluations.

4.4 Summary

The above observations and comparisons between the RBDO and robust design results lead to the following conclusions. The robust design problems can be classified into three groups based on the order of the objective function and the choice of design parameters.

1. Linear objective function and constant standard deviations for the design variables: For such problems the robust design solution will be equal to the RBDO solution.

2. Linear objective function and constant COV : If the objective function is linear then it has the form

$$\text{Min} \quad f = \sum \alpha_i X_i \tag{4.19}$$

This case can be divided into two sub-cases:

   a. All α_i's are positive: In this case, the RBRD (reliability-based robust design) solution will collapse to the RBDO solution.
   b. If some α_i's are negative: In this case, the RBRD solution should be different as the first order variance will become a competing objective to the first order mean.

3. Nonlinear objective function: If the objective function is nonlinear then RBRD will give a solution different from RBDO, because the derivatives are not constant.

CHAPTER V

CRASH SAFETY PROBLEM

5.1 Introduction

The crash safety problem provided by General Motors involves maximizing the NCAP (new car assessment program) Star Rating for the design. The design objective is to satisfy the FMVSS (Federal motor vehicle safety standard) 208 occupant safety criteria as well.

5.2 Importance of the problem

In 1978 the New Car Assessment Program (NCAP) in the United States was initiated with the primary purpose of providing consumers with a measure of the relative safety potential of vehicles in frontal crashes. NCAP now supplies consumers with important comprehensive information, including frontal and side crash test results, to aid them in their vehicle purchase decisions. Vehicles that are rated higher on these safety ratings are considered safer. The increasing reliance of consumers on these ratings to make decisions regarding their purchase necessitates that GM maximize the star rating on their products.

5.3 Background

Researchers have used crash test data to determine the likelihood of injuries that may be sustained in a crash. In addition, that data was used to create the NHTSA's star system. This system makes automobile safety ratings easier for consumers to understand when buying a car.
In frontal crashes, the star rating is determined by the worst score on two criteria:

- Head Injury Criteria (HIC)
- Chest deceleration (Chest G)

In order to receive a five-star rating, both of these criteria must be below the level that indicates a 10-percent chance of severe injury. There is a star rating for each of the front seat passengers, for each type of test that was run (frontal or side impact).

Table 5.1: Rating for frontal-impact tests

# of Stars	Result
5	10% or lower chance of serious injury
4	11% to 20% chance of serious injury
3	21% to 35% chance of serious injury
2	36% to 45% chance of serious injury
1	46% or greater chance of serious injury

The star rating shown in Table 5.1 is what we wish to maximize in this study.

5.4 Problem setup

The objective function chosen is to minimize the total probability of serious injury or the total injury probability. The Star rating can be calculated from Table 1 knowing the total injury probability. The total injury probability is the union of the probability of serious head injury (P_{HIC}) and the probability of serious chest injury (P_{CG}). P_{HIC} and P_{CG} can be obtained from empirical equations which require the head injury coefficient (HIC) and chest deceleration (CG) value. The objective function used is computed as shown in Eq. 5.1.

$$
\begin{aligned}
&P_{HIC} = 1.0/(1.0 + \exp(5.02 - 0.00351 * (HIC))) \\
&P_{CG} = 1.0/(1.0 + \exp(5.55 - 0.069 * (CG))) \\
&\text{valuex} = P_{HIC} + P_{CG} - P_{HIC} * P_{CG} \\
&\text{valuex} = \text{valuex} * 100 \\
&\text{Total\_Injury\_Probability} = \text{valuex}
\end{aligned} \qquad (5.1)
$$

Once a converged solution for the total injury probability is obtained, the star rating is determined from Eq. 5.2.

$$
\begin{aligned}
&\text{valuex} = \text{Total\_Injury\_Probability} \\
&\text{if(valuex} < 11.0) \text{ tempx} = 5.0 + ((11.0 - \text{valuex})/11.00) \\
&\text{if(valuex} >= 11.0 \ \& \ \text{valuex} < 21.0) \text{ tempx} = 4.0 + ((21.0 - \text{valuex})/10.01) \\
&\text{if(valuex} >= 21.0 \ \& \ \text{valuex} < 36.0) \text{ tempx} = 3.0 + ((36.0 - \text{valuex})/15.01) \\
&\text{if(valuex} >= 36.0 \ \& \ \text{valuex} < 46.0) \text{ tempx} = 2.0 + ((46.0 - \text{valuex})/10.01) \\
&\text{if(valuex} >= 46.0) \text{ tempx} = 1.0 \\
&\text{Star Rating} = \text{tempx}
\end{aligned} \qquad (5.2)
$$

The constraints are based on the FMVSS 208 safety criteria, which give the critical values of chest deceleration, head injury coefficient value, various neck injury coefficients and chest compression obtained from crash tests for dummies of various sizes. We also have additional constraints based on the chest deceleration and head injury coefficient we get for the NCAP load case where the crash tests are done with speeds different from those used for the FMVSS 208 criteria. A total of nine constraints were identified for the design at hand. Response surfaces were made available for all the nine constraint functions. These response surfaces will be used to estimate the response of the limit states. All constraints are probabilistic and need to satisfy a 97% reliability criterion. This corresponds to a reliability index of 1.89.

Now that the objective and the constraints are listed, the optimization problem is written as

Minimize (***Total Injury Probability***)
s.t.
Prob (NCAP Chest g 3ms < 60) > 0.97
Prob (NCAP HIC 36 ms < 1000) > 0.97
Prob (50^{th} %ile Unbelted NTF < 1) > 0.97
Prob (50^{th} %ile Unbelted NTE < 1) > 0.97
Prob (50^{th} %ile Unbelted HIC 15 ms < 700) > 0.97
Prob (50^{th} %ile Unbelted Chest Compression < 63.0) > 0.97
Prob (5^{th} %ile Unbelted NTF < 1) > 0.97
Prob (5^{th} %ile Unbelted NCF < 1) > 0.97
Prob (5^{th} %ile Unbelted Chest Compression < 53.0) > 0.975

Here NTF, NTE and NCF are the neck injury coefficients for the different load cases. The corresponding design and other random variables and their distributions are given in Table 5.2 and Table 5.3.

Table 5.2: Design variables

Variable	*Range for Means*		*Range for σs*		*Nominal Values*	
	LB	*UB*	*LB*	*UB*	*Mean*	*Std*
Tether Length (mm)	175	300	8.75	15	237.5	11.875
Vent Area Scale	1.366	3.534	0.0683	0.1767	2.45	0.1225
Twist Shaft Level (N)	2500	6000	125	300	4250	212.5
Knee Bolster Stiffness Scaling	1	1.5	0.05	0.075	1.25	0.0625
Inflation Output	1	1.4	0.05	0.07	1.2	0.06
Pre-Tension Spool (mm)	25	55	1.25	2.75	40	2
Pre-Tension Firing Time (s)	0.005	0.01	0.00025	0.0005	0.0075	0.000375
Column Strokes (mm)	25	75	1.25	3.75	50	2.5
Air Bag Size Scaling	1	1.25	0.05	0.0625	1.125	0.05625

Table 5.3: Noise variables

Variable	*Mean*	*Std. Dev.*
FBD	0.45	0.1166667
FSD	0.45	0.1166667
FDB	0.3	0.06667
FDT	0.3	0.06667
FAC	0.45	0.1166667
FAH	0.45	0.1166667
DTx1	3.029	0.0127
DTx3	2.884	0.0127
Dty	-0.395	0.0127
DTz1	0.885	0.00635
DTz3	0.903	0.00635
AFT1	0.15	0.001
AFT2	0.02	0.001
CBL	4459	148.273333
PvA1	-0.38397	0.02181945
PvA2	-0.35779	0.021819443
PvA3	0.357793	0.021819443

5.5 Solution

5.5.1 Preliminary Analysis

First a preliminary analysis was done to check the feasibility of the constraints. In order to do this, the response surface of the constraint function was minimized within the bounds provided for the design variables, using the mean values of the random variables. All the limit states were found to be feasible as shown in Table 5.4.

Table 5.4: Preliminary analysis of limit states

Limit state	*Target*	*Minimum*	*Maximum*
NCAP HIC (36 ms)	1000	63.37	6168.55
NCAP Chest G (3ms)	60	35.6278	53.6995
50th %ile Unbelted Chest Compression	63	35.5471	63.1908
50th %ile Unbelted NTE	1	-0.03017	1.23058
50th %ile Unbelted NTF	1	0.06844	1.55763
50th %ile Unbelted HIC (15 ms)	700	25.7583	5776.41
5th %ile Unbelted Chest Compression	52	46.2757	61.958
5th %ile Unbelted NCF	1	-0.22032	1.91607
5th %ile Unbelted NTF	1	0.270233	3.76295

Both the improved RBDO methods proposed in chapter 3 and the robust design methods discussed in chapter 4 were used to solve the problem. First the results obtained for the RBDO techniques will be shown and then for robust design.

5.5.2 Reliability based design optimization

The optimization was done by considering variables in Tables 5.2 and 5.3 as the design and random variables respectively. Initially the tolerance was set at 10^{-6} which was one order of magnitude smaller than the smallest standard deviation. 100 initial designs were first obtained using optimosymmetric latin hypercube sampling method. When the optimization was executed with these initial designs, different results were observed. Some of these results are shown in Table 5.5.

Table 5.5: RBDO solution with tolerance =10^{-6}

Solution No.	*Tether Length (mm)*	*Vent Size (Scaling)*	*Twist Shaft Level (N)*	*Knee Bolster Stiffness (Scaling)*	*Inflator Output (Scaling)*	*Pre-tension Spool (mm)*	*Pre-tension Firing Time (mS)*	*Column Stroke (mm)*	*AirBag Size (Scaling)*	*Mean Star Rating*
1	300.0	1.366	4250.0	1.000	1.400	55.0	5.00	50.05	1.0139	5.347
2	175.0	1.366	2500.0	1.000	1.400	55.0	5.00	25.04	1.0262	5.248
3	300.0	1.366	6000.0	1.000	1.400	55.0	5.00	75.00	1.0642	5.179
4	175.0	1.366	4250.0	1.500	1.400	55.0	10.00	25.13	1.0697	5.238
5	175.0	1.366	2500.0	1.000	1.400	55.0	5.00	75.00	1.0930	5.327
6	175.0	1.366	6000.0	1.000	1.400	55.0	5.00	75.00	1.1516	5.144
7	300.0	1.366	2500.0	1.000	1.400	55.0	5.00	50.04	1.0000	5.365

Later the tolerance was decreased to 10^{-12} and it was found that there were only three solutions to which the optimization converged each time. These solutions are listed in Table 5.6. On further study it was found that even large changes in a design variable brought about very little change in the objective, this necessitated the very low tolerance value.

Table 5.6: RBDO solution with tolerance =10^{-12}

Solution No.	*Tether Length (mm)*	*Vent Size (Scaling)*	*Twist Shaft Level (N)*	*Knee Bolster Stiffness (Scaling)*	*Inflator Output (Scaling)*	*Pre-tension Spool (mm)*	*Pre-tension Firing Time (mS)*	*Column Stroke (mm)*	*Air Bag Size (Scaling)*	*Mean Star Rating*
1	300.0	1.366	3169.1	1.301	1.400	55.0	5.00	75.00	1.0000	5.393
2	175.0	1.366	3328.2	1.286	1.400	55.0	5.00	75.00	1.1060	5.328
3	175.0	1.366	3330.5	1.500	1.400	55.0	10.00	75.00	1.1110	5.323

The solutions are graphically compared. Using the obtained solution as the mean values of the design variables, 1000 samples are generated and the corresponding HIC and Chest G values are calculated. The HIC vs Chest G graphs for the three solutions are given in Fig 5.7 (a), 5.7 (b) and 5.7 (c) respectively. The star rating contours are drawn to easily observe how the solution ranks on the safety scale.

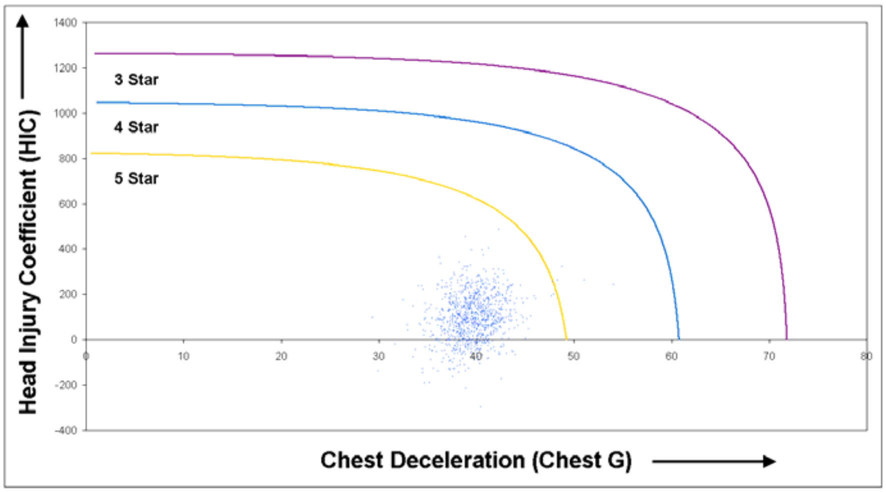

Figure 5.1 (a): Plot of the first solution from Table 5.6

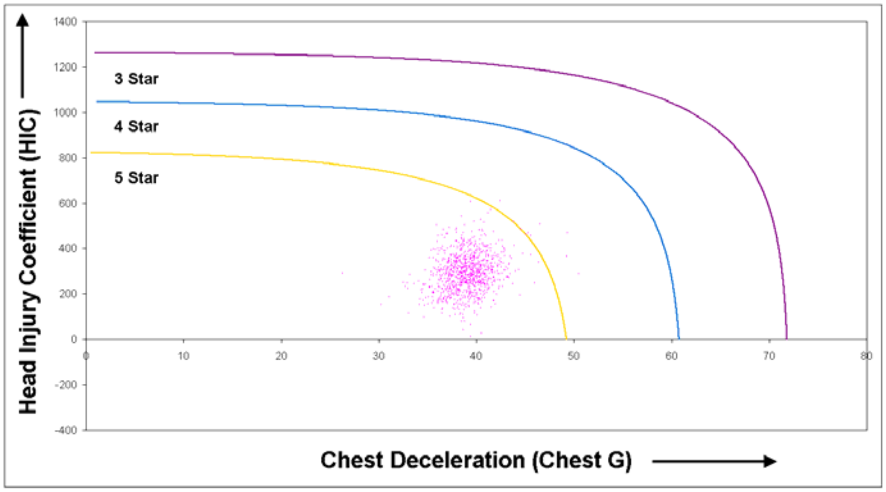

Figure 5.1 (b): Plot of the second solution from Table 5.6

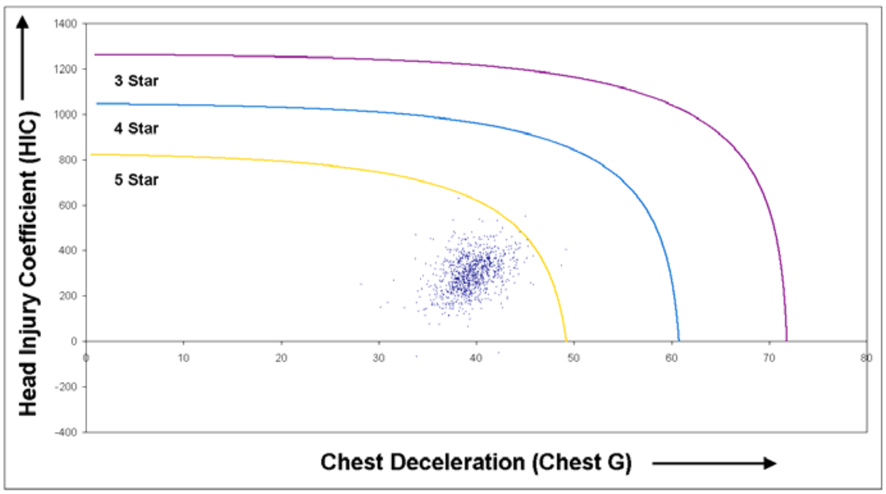

Figure 5.1 (c): Plot of the third solution from Table 5.6

It can be seen that quite a few samples for the first solution and two samples for the third solution have a negative value for HIC. This is not possible as HIC is not defined for negative values. This could be because most of the design variables converged to either one of the bounds and the response surfaces were not accurate in this region. Therefore, the original bounds are narrowed by a value of 2σ to get the new design bounds, as shown in Table 5.7.

Table 5.7: Design bounds

Variable	*Variable Range*		*Std. Dev. Range*		*Design Bounds*	
	LB	*UB*	*LB*	*UB*	*LB*	*UB*
Tether Length (mm)	175.0	300.0	8.750	15.00	1.93E+02	2.70E+02
Vent Area Scale	1.366	3.534	0.0683	0.1767	1.50E+00	3.18E+00
Twist Shaft Level (N)	2500	6000	125.0	300.0	2.75E+03	5.40E+03
Knee Bolster Stiffness Scaling	1.000	1.500	0.050	0.075	1.10E+00	1.35E+00
Inflation Output	1.000	1.400	0.050	0.07	1.10E+00	1.26E+00
PreTension Spool (mm)	25.00	55.00	1.250	2.75	2.75E+01	4.95E+01
PreTension Firing Time (s)	0.005	0.010	0.00025	0.0005	5.50E-03	9.00E-03

The optimization was then performed within the above design bounds and the result turned out to be infeasible. It was found that a target reliability of 97% could not be satisfied as the design space was getting too constrained. The target reliability was therefore reduced to 90% and the corresponding reliability index reduced to 1.28 from the previous value of 1.96. The new results after performing the sequential optimization and reliability analysis are given in Table 5.8. The graphical plots for the solutions obtained in Table 5.8 do not show any negative values for HIC.

Table 5.8. Results within the Design Bounds

Solution No.	*Tether Length (mm)*	*Vent Size (Scaling)*	*Twist Shaft Level (N)*	*Knee Bolster Stiffness (Scaling)*	*Inflator Output (Scaling)*	*Pre-tension Spool (mm)*	*Pre-tension Firing Time (ms)*	*Column Stroke (mm)*	*AirBag Size (Scaling)*
Design 1: **Star rating = 5.27**									
1	270.00	1.50	3244.0	1.21	1.26	49.50	5.50	67.50	1.10
Design 2: **Star rating = 5.25**									
2	193.00	1.50	3266.0	1.20	1.26	49.50	5.50	67.50	1.10

The plots of samples from the two solutions in Table 5.8 are shown in Fig 5.2 (a) and (b).

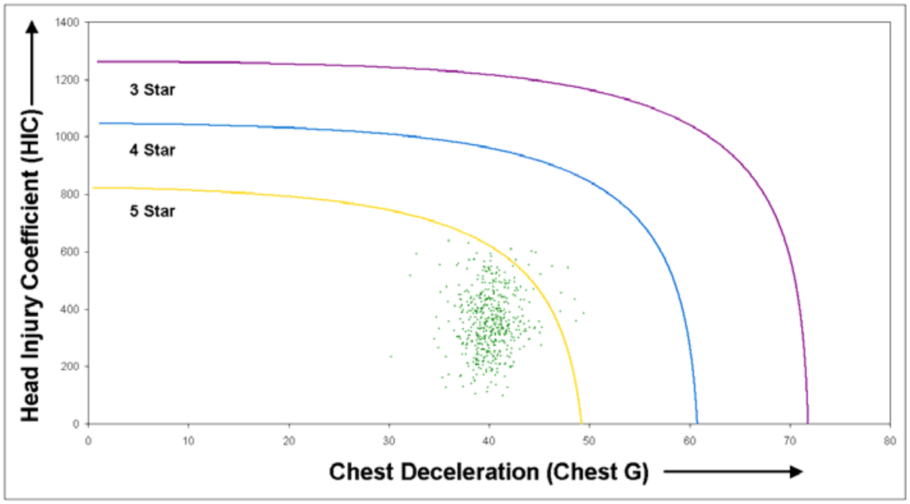

Figure 5.2(a): Plot of the first solution from Table 5.8

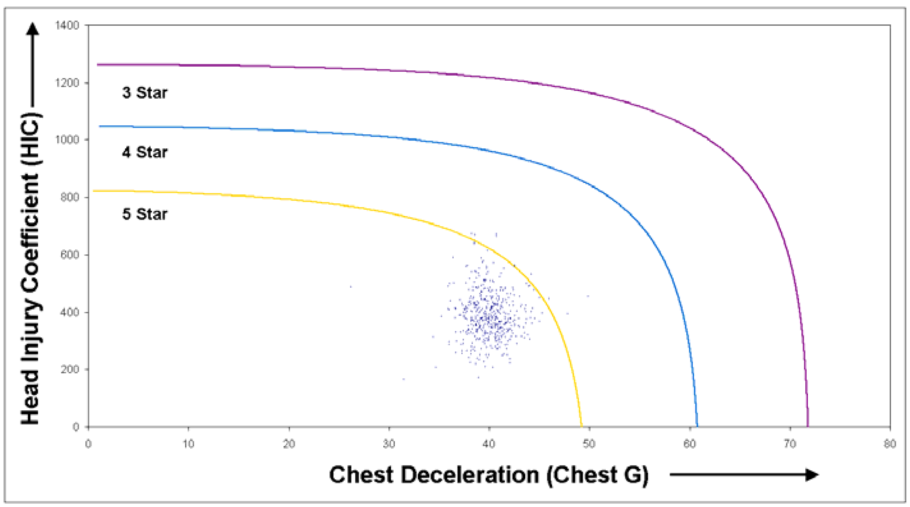

Figure 5.2(b): Plot of second solution from Table 5.8

5.5.3 Robust Design

Optimization was done using both the weighted sum and percentile formulations. For the weighted sum approach the original bounds, as given in Table 5.2 and the same initial design points (OSLH samples), as used in the calculation of the RBDO solution in Table 5.5, were employed. The standard deviations were assumed to be constant. The target reliability was set to 97%.

Table 5.9. Results using weighted sum method for robust design

w	*Tether Length (mm)*	*Vent Size (Scaling)*	*Twist Shaft Level (N)*	*Knee Bolster Stiffness (Scaling)*	*Inflator Output (Scaling)*	*Pre-tension Spool (mm)*	*Pre-tension Firing Time (ms)*	*Column Stroke (mm)*	*AirBag Size (Scaling)*	Star Rating	Function Count
0.90	0.175	1.679	5037.2	1.356	1.191	0.054	0.0088	71.43	1.052	5.178	147320

The variance in the above formulation is calculated using the first order method discussed in the previous chapter. For the percentile formulation, the optimization was performed within the original bounds given in Table 5.5. The results are shown in Table 5.10. The 90th percentile value of the total injury probability is the objective that is being minimized. Again the standard deviations are constants and the target reliability of limit states is set to 97%.

Table 5.10: Results for percentile formulation (original bounds)

S.No.	*Tether Length (mm)*	*Vent Size (Scaling)*	*Twist Shaft Level (N)*	*Knee Bolster Stiffness (Scaling)*	*Inflator Output (Scaling)*	*Pre-tension Spool (mm)*	*Pre-tension Firing Time (ms)*	*Column Stroke (mm)*	*AirBag Size (Scaling)*	Star Rating	Function Count
1	0.175	1.366	4212.8	1.459	1.378	0.0550	0.0069	57.34	1.157	5.275	681523
2	0.300	1.366	4131.3	1.500	1.081	0.0550	0.0050	74.99	1.031	5.343	420968
3	0.300	1.366	4655.2	1.412	1.262	0.0549	0.0075	75.00	1.141	5.293	453130

Next, the above procedure is repeated for design bounds given in Table 5.7 and keeping the standard deviations constant. The results are shown in Table 5.11.

Table 5.11: Results for percentile formulation (design bounds)

S.No.	*Tether Length (mm)*	*Vent Size (Scaling)*	*Twist Shaft Level (N)*	*Knee Bolster Stiffness (Scaling)*	*Inflator Output (Scaling)*	*Pre-tension Spool (mm)*	*Pre-tension Firing Time (ms)*	*Column Stroke (mm)*	*AirBag Size (Scaling)*	Star Rating	Function Count
1	0.270	1.503	5300.9	1.350	1.260	0.0495	0.0055	67.50	1.100	5.141	342183
2	0.270	2.552	2934.4	1.349	1.259	0.0489	0.0089	67.23	1.100	5.234	13721885

<u>Discussion</u>: The problem above is highly nonlinear and reliability assessment requires a large number of function evaluations. Problems were faced in the convergence of the optimizer for the robust design formulations. Also the design space seems to get very constrained with the inclusion of the second objective, leading to infeasible results.

5.6 Summary

The efficient use of RBDO and robust design methodologies for the crash safety problem resulted in an increase in the star ratings from the base design. Due to the high nonlinearity and multimodal nature of the objective function several design solutions were obtained. It was found that the response surfaces were inaccurate near the upper or lower bounds of the limit state; hence design bounds were created. The resulting optimal design had a five star rating for more than 90% of the samples.

CHAPTER VI

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

Decoupling reliability analysis and optimization iterations helps to achieve computational efficiency in RBDO. In this thesis the following extensions for the PMA-based decoupled method are developed:

1. Inclusion of standard deviations as design parameters in RBRD.
2. Inclusion of simulation-based reliability assessment to replace FORM-based reliability assessment when highly nonlinear functions are involved. Simulation-based RBDO can meet the target reliability conditions more accurately for the nonlinear limit states, whereas the FORM-based methods need less computational effort. In this context the single loop methods need much less computational effort.
3. Solutions of reliability–based robust design (RBRD) problems with the weighted sum and percentile formulations and extending the decoupled and single loop formulations for the same.

6.2 Future Work

Researchers are pursuing several directions to improve RBDO and robust design methods.

(1) Inclusion of system-level reliability requirements in design optimization and robust design: Zou and Mahadevan's (2004) decoupled method based on direct reliability analysis appears useful in this regard. In this method the Taylor series expansion is applied directly to the probability of failure function. The sensitivities are found using Monte Carlo simulation, similar to Eq. 3.2. This method is also capable of identifying the inactive limit states to improve the computational efficiency. This approach needs to be investigated for application to robust design.

(2) Use of efficient meta-modeling techniques to reduce the computational requirements: These techniques are coupled with global sensitivity analysis methodologies to give the robust design solution. A considerable amount of work is being done to link these decompositions to the metamodels using basis functions (Chen et al, 2004).

(3) Inclusion of model uncertainty: With the increasing use of response surfaces in RBDO, it has become imperative to verify and validate these models. Assumptions and approximations made during the modeling stage may induce errors when predicting from these models. Therefore there is a need to develop computational methods to quantify physical, informational, and model uncertainties in complex engineering systems and incorporate them in design. The concepts being used to address these issues include Bayesian statistics and networks (Zhang and Mahadevan, 2003; Rebba, 2002), Markov Chain Monte Carlo methods, hypothesis testing etc. Many times extrapolations are made using these models; therefore there is a need to estimate the confidence in model predictions, and include this information in design decisions.

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