ON SOME COMPLEXITY PROBLEMS IN FINITE ALGEBRAS

By

Marcin Kozik

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Approved by:
Professor Ralph McKenzie
Professor Constantine Tsinakis
Professor Alexander Ol'shanskiĭ
Professor Mark Sapir
Professor Alan Peters

Dedicated to my wife, Regina, with my everlasting love.

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Table of contents

Page

I Introduction

It is an old and interesting question to characterize how intrinsically "complex" a finite algebra can be. An even more interesting problem is to define a "complexity" function that would measure this property. In this work we present three well studied ways of measuring complexity of algebras and give examples of algebras that are complex according to these measures.

The first measure is a computational complexity of a membership problem for the variety a given algebra generates. More precisely the complexity of a finite algebra $\mathbf{A}$ is the computational complexity of the following problem:

INPUT	a finite algebra $\mathbf{B}$
PROBLEM	decide if $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$.

This problem is very closely related with a universal membership problem:

INPUT	a pair of finite algebras $(\mathbf{B}, \mathbf{A})$
PROBLEM	decide if $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$.

Jan Kalicki showed in [Kal52] that both of these problems are decidable. Clifford Bergman and Giorra Slutzki presented in [BS00] an algorithm that solves the universal membership problem (and so a membership problem for any given algebra) in 2-EXPTIME. This algorithm establishes the best known upper bound on the complexity of these problems. We are interested in exploring how complex this problem can be, by constructing algebras with membership problems of high complexity. Unfortunately for all the algebras with a finite base of equations the membership problem is solvable in polynomial time – trivial in a computational complexity sense. This excludes, as possible candidates for algebras with high–complexity membership problems, algebras such as finite groups [OP64], finite associative rings [Kru73] and [L'v73], finite Lie rings [BO75], finite lattices [McK70] and many other structures. Nethertheless Zoltán Székely in [Szé02] produced an example of a finite algebra with NP-complete membership problem. This construction was refined to a groupoid by the author in [Koz], and to a semigroup with NP–hard membership problem by Marcel Jackson and Ralph McKenzie in [JM]. Thus a known lower bound on the complexity of a membership problem as well as on the complexity of a universal membership problem is NP. In this paper we produce a finite algebra that generates a variety with PSPACE–complete membership problem and another algebra generating a variety with EXPSPACE–hard membership problem. These examples imply that a membership problem for certain algebras can be as complex as EXPSPACE, and that the universal membership problem is EXPSPACE–hard.

Another complexity measure of algebras was introduced by George McNulty and Zoltán Székely. This measure of an algebra $\mathbf{A}$ is a function $\beta_{\mathbf{A}}$ defined in the following way: $\beta_{\mathbf{A}}(k)$ is the minimal n such that for every

$$\mathbf{B} \notin \mathrm{HSP}(\mathbf{A}) \text{ such that } |\mathbf{B}| \leq k$$

there is an identity of size smaller than n that holds in $\mathbf{A}$ and fails in $\mathbf{B}$. In his paper [Szé02] Zoltán Székely presented an algebra with sublinear growth of the $\beta_{\mathbf{A}}$ function. During the Logic Colloquium 2004 held in Torino, Italy, Vera Vértesi in [KV] presented examples of algebras with β function growing faster than any given polynomial. In this paper we present an algebra with the $\beta_{\mathbf{A}}$ function growing at least doubly exponentially. We wish to remark that by trivial considerations the β function cannot grow faster then a triple exponential function.

The third complexity measure is the rate of growth of the function $\gamma_{\mathbf{A}}$. We define the function $\gamma_{\mathbf{A}}$ by saying that $\gamma_{\mathbf{A}}(k)$ is the minimal n such that for every

$$\mathbf{B} \in \mathrm{HSP}(\mathbf{A}) \text{ such that } |\mathbf{B}| \leq k$$

there is $\mathbf{C}$ and an onto homomorphism h such that

$$\mathbf{A}^n \geq \mathbf{C} \xrightarrow{h} \mathbf{B}.$$

This function was introduced by Paweł Idziak during the conference on Structural Theory of Automata, Semigroups and Universal Algebra (a NATO Advanced Study Institute) held at the Université de Montréal July 7 – 18, 2003. Examples of algebras with $\gamma_{\mathbf{A}}$ growing faster then any polynomial were not known. In this paper, we present an algebra with $\gamma_{\mathbf{A}}$ function growing at least exponentially.

To obtain such results we adapt a construction invented by Ralph McKenzie. In [McK96b] and [McK96a], Ralph McKenzie introduced a construction of an algebra $\mathbf{A}(\mathbf{T})$ and proved a number of results on the residual behavior of finitely generated varieties. In [McK96c] he modified $\mathbf{A}(\mathbf{T})$ and answered a long–standing open question, posed by Alfred Tarski, by proving that the property of having a finite base of equations is not decidable for finite algebras. The construction he introduced was adopted and reused many times. It contributed to various results on the behavior of residual bounds of finite algebras. One of the results following Ralph McKenzie's construction was a paper of Ross Willard [Wil97], where the author showed that the original construction of $\mathbf{A}(\mathbf{T})$ is sufficient to answer Tarski's question. To obtain algebras of high complexities we modify Ralph McKenzie's construction. While doing so, we use a framework presented by Ross Willard in [Wil97].

II Preliminaries

For an introduction to the concepts used in this paper we invite the reader to consult the book of Stanley Burris and H.P. Sankappanavar [BS81]. For a more exhausting overview see the book by Ralph McKenzie, Geogre McNulty and Walter Taylor [RNMT87]. We follow the notation of these works in all the basic concepts. We introduce a new notation for dealing with operations of the algebra: for a k-ary operation $F(x_0, \ldots, x_{k-1})$ we put

$$F_{(i)}(y, x_0, \ldots, x_{k-2}) = F(x_0, \ldots, x_{i-1}, y, x_i, \ldots, x_{k-2}).$$

For an algebra $\mathbf{A}$ we say that $A' \subseteq A$ is absorbing for $\mathbf{A}$ if for any basic operation $F(x_0, \ldots, x_{k-1})$ and every i we have

$$F_{(i)}(a, a_0, \ldots, a_{k-2}) \in A' \text{ whenever } a \in A'.$$

Moreover we define a word of length n in alphabet $\mathcal{L}$ to be a member of the set $\mathcal{L}^n$. For each such word $w \in \mathcal{L}^n$ we define prefixes of w as follows: $w_{[m]}$ is a prefix of w of length m for any $0 \leq m \leq n$. We use a lexicographical order on words and denote it by $<$, eventually we define a notion of $w + i$ for i being a natural number to be the ith successor of w. Finally we define a function φ from the set of words on $\{0, 1\}$ to natural numbers to return the position in order $<$ among words of the same length, or equivalently the number with binary representation equal to the word.

III The structure of the algebra

In this section we adapt Ralph McKenzie's construction presented in [McK96a] using Ross Willard's approach presented in [Wil97]. We substitute Willard's 0 with $\perp$, and weaken some of his assumptions (as we note below). The algebra $\mathbf{A}$ with which we work is described in the following way.

The universe of $\mathbf{A}$ is the set

$$A = \{\perp\} \cup X_0 \cup X_1 \cup Y,$$

a disjoint union of four finite sets. We put $X = X_1 \cup X_0$. Together with the set A there is a bijection between X_0 and X_1 denoted by $x \to \tilde{x}$. We define two auxiliary functions, $\delta : X \cup \{\perp\} \to X \cup \{\perp\}$ and $\nu : A \to A$ by

$$\delta(a) = \begin{cases} \tilde{a} & \text{if } a \in X_0 \\ b & \text{if } a = \tilde{b} \in X_1 \\ \perp & \text{if } a = \perp, \end{cases}$$

and

$$\nu(a) = \begin{cases} b & \text{if } a = \tilde{b} \in X_1 \\ a & \text{otherwise.} \end{cases}$$

We follow Ross Willard's convention and denote by A_0 the set $A \setminus X_1$ consisting of all "unbarred" elements of A. The set $\mathcal{F}$ of operations of $\mathbf{A}$ is divided into seven finite, disjoint groups

$$\mathcal{F} = \{\perp, \wedge\} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \{J, J'\}.$$

The operations are subject to the following restrictions, which are taken directly from [Wil97].

Condition 1. The nullary symbol $\perp$ is interpreted by itself, and $\wedge$ is a binary operation which makes A a flat meet-semilattice with $\perp$ being the bottom element. More precisely for any $a, b \in A$ we have

$$a \wedge b = \begin{cases} a & \text{if } a = b, \\ \perp & \text{otherwise.} \end{cases}$$

Ross Willard's next condition is split into two parts. In most of the constructions we require only the first part of this condition.

Condition 2. For each operation $F(\bar{x}) \in \mathcal{A}$ of arity k we require the following:

- $k > 0$,
- $\text{range}(F) \subseteq X \cup \{\perp\}$,
- A_0 is a subuniverse of $\langle A, F \rangle$,
- for every $i = 0, \dots, k-1$ and all $a \in X$, and all $\bar{e}$ in A we have

$$F_{(i)}(\perp, \bar{e}) = \perp$$
$$F_{(i)}(\delta(a), \bar{e}) = \delta(F_{(i)}(a, \bar{e})).$$

Note that each of the operations of the set $\mathcal{A}$ is uniquely determined by its restriction to A_0. We will often define an operation of $\mathcal{A}$ on A_0 and leave a full (and usually cumbersome) definition to the reader.

Ross Willard's Condition 2 requires, in addition, the so called injectivity in A_0, which is presented in the following condition.

Condition 2a. Under the assumptions of Condition 2, given $b, c, \bar{d}$ in A_0 we have

$$F_{(i)}(b, \bar{d}) = F_{(i)}(c, \bar{d}) \neq \bot \to b = c.$$

Condition 3. For each member $S(\bar{x})$ of $\mathcal{B}$ we denote its arity by $k+3$, and require the following:

- $k > 0$,
- for every $\bar{a} \in A^k$, either

$$\mathbf{A} \models S(\bar{a}, x, y, z) \approx (x \wedge y) \vee (x \wedge z)$$

  or

$$\mathbf{A} \models S(\bar{a}, x, y, z) \approx \bot,$$

- if $\bar{a} \in A^k$ and some $a_i = \bot$, then

$$\mathbf{A} \models S(\bar{a}, x, y, z) \approx \bot.$$

Included in $\mathcal{B}$ is the operation $S_2(u, v, x, y, z)$ defined by

$$S_2(u, v, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u = \delta(v) \in X \\ \bot & \text{otherwise.} \end{cases}$$

Condition 4. For each k-ary member $T(x_0, \ldots, x_{k-1})$ of $\mathcal{C}$ there exist an $\mathcal{A} \cup \{\wedge\}$-term $t(x_0, \ldots, x_{k-1})$, a conjunction $\Sigma(\bar{x})$ of equations between pairs of these variables and an (arbitrary) k-ary predicate $\Phi(\bar{x})$ on A, satisfying the following:

- some member of $\mathcal{A}$ appears in $t(\bar{x})$ (hence $\mathrm{range}(t(\bar{x})) \subseteq X \cup \{\bot\}$);
- if $I = \{x_i | \, x_i \text{ occurs in } t(x_0, \ldots, x_{k-1})\}$, then for every $j \in \{0, \ldots, k-1\}$ there exists $i \in I$ such that $\Sigma(\bar{x}) \vdash x_i \approx x_j$;

- the following hold:
$$\mathbf{A} \models \big(t(\bar{x}) \not\approx \bot \,\&\, \Sigma(\bar{x})\big) \to \Phi(\bar{x})$$
and
$$\mathbf{A} \models \big(t(\bar{x}) \not\approx \bot \,\&\, \Phi(\bar{x})\big) \to \&_{i=0}^{k-1} x_i \not\approx \bot;$$

- the operation is defined by
$$T(\bar{x}) = \begin{cases} t(\bar{x}) & \text{if } \Phi(\bar{x}) \text{ and } \Sigma(\bar{x}), \\ \delta(t(\bar{x})) & \text{if } \Phi(\bar{x}) \text{ but not } \Sigma(\bar{x}), \\ \bot & \text{otherwise.} \end{cases}$$

Conditions 1, 2 and 4 allow the following construction of one more term associated with the operations of the set $\mathcal{C}$. The construction is presented in [Wil97] as Condition 10 and our proof is a carbon copy of Ross Willard's reasoning.

Proposition III.1. *For each $T(\bar{x}) \in \mathcal{C}$ and a corresponding term $t(\bar{x})$ as above, there exists a term $\hat{t}(\bar{x})$ in the language of $\mathcal{A} \cup \{\wedge\}$ such that*
$$\mathbf{A} \models \hat{t}(\bar{x}) \approx T(\bar{x}) \wedge t(\bar{x}).$$

Proof. Fix an arbitrary $T(\bar{x}) \in \mathcal{C}$ of arity k and find corresponding $\Sigma(\bar{x})$ and $t(\bar{x})$. For simplicity assume that the variables occurring in $t(\bar{x})$ are precisely $x_0, \dots, x_{l-1}$. For each $i < l$, define $V_i = \{j |\, \Sigma(\bar{x}) \vdash x_i \approx x_j\}$ and $\hat{x}_i = \bigwedge_{j \in V_i} x_j$ and finally
$$\hat{t}(\bar{x}) = t(\hat{x}_0, \dots, \hat{x}_{l-1}, x_l, \dots, x_{k-1}).$$

Since, by Conditions 1 and 2, all the operations of $\mathcal{A} \cup \{\wedge\}$ are monotone, we get
$$\mathbf{A} \models \hat{t}(\bar{x}) \leq t(\bar{x}).$$

Since the element $\bot$ is absorbing for all the operations of $\mathcal{A} \cup \{\wedge\}$, if $\hat{t}(\bar{a}) \neq \bot$ for some $\bar{a} \in A^k$ then $t(\bar{a}) \neq \bot$ and $\Sigma(\bar{a})$ holds. Condition 4 implies that in such a case $\Phi(\bar{a})$ holds as well and we obtain $T(\bar{a}) = t(\bar{a}) = \hat{t}(\bar{a})$. This proves
$$\mathbf{A} \models \hat{t}(\bar{x}) \leq T(\bar{x}) \wedge t(\bar{x}).$$

On the other hand if $T(\bar{a}) = t(\bar{a}) \neq \bot$ then $\Sigma(\bar{a})$ holds and so $T(\bar{a}) = t(\bar{a}) = \hat{t}(\bar{a})$ as required. □

Condition 5. The set $\mathcal{D}$ consists of nullary operations only and $\mathcal{E} \subseteq \{v\}$.

Condition 6. The operations $J(x,y,z)$ and $J'(x,y,z)$ are defined in the following way:

$$J(x,y,z) = \begin{cases} x & \text{if } x = \delta(y) = z \in X, \\ x \wedge y & \text{otherwise,} \end{cases}$$

$$J'(x,y,z) = \begin{cases} x & \text{if } x = \delta(y) \in X, \\ x \wedge y \wedge z & \text{otherwise.} \end{cases}$$

In the following sections, we work with algebras of the described above kind i.e. satisfying Conditions 1-6, except possibly Condition 2a. We remark that all the basic operations of $\mathbf{A}$, except possibly operations of the set $\mathcal{B}$ and the operations $J(x,y,z)$ and $J'(x,y,z)$, are $\perp$–absorbing. Moreover the following corollary is true.

Corollary III.2. *All the operation of* $\mathbf{A}$ *are monotone with respect to the order introduced on the set* A *by the flat semilattice operation* $\wedge$.

IV The structure of the subdirectly irreducible algebras in the variety generated by A

For an algebra $\mathbf{A}$ complying with Conditions 1 to 6, with possible exception for Condition 2a, we describe the structure of the subdirectly irreducible algebras in the variety $\mathbf{A}$ generates. Before we start, we need some auxiliary definitions.

Definition IV.1. For algebras $\mathbf{C}$ and $\mathbf{D}$ with $\mathbf{D} \in \mathcal{V}(\mathbf{C})$, we define the dimension of $\mathbf{D}$ with respect to $\mathbf{C}$, written $\dim_{\mathbf{C}}(\mathbf{D})$, to be the least cardinal κ such that $\mathbf{D} \in \mathrm{HS}(\mathbf{C}^X)$, for some set X with $|X| = \kappa$. We say that $\mathbf{S}$ is a large s.i. in $\mathcal{V}(\mathbf{C})$ if $\mathbf{S} \in \mathcal{V}(\mathbf{C})$ and $\dim_{\mathbf{C}}(\mathbf{S}) > 1$.

In the remaining part of this section we fix an arbitrary finite, large s.i. $\mathbf{S}$ in $\mathcal{V}(\mathbf{A})$. We pick $\mathbf{B}$ and θ such that $\mathbf{S} \cong \mathbf{B}/\theta$ and $\mathbf{B} \subseteq \mathbf{A}^I$, where $1 < |I| = \dim_{\mathbf{A}}(\mathbf{S}) < \omega$. We write $\bar{\theta}$ for the unique cover of θ in $\mathbf{Con}(\mathbf{B})$.

The structure of $\mathbf{A}$ allows us to introduce the notion of support. For an element $a \in B$ we define $\mathrm{supp}(a) = \{s \in I |\ a(s) \neq \perp\}$. An element of full support is an element $a \in B$ such that $\mathrm{supp}(a) = I$. The following lemma is the copy of Lemmas 6.4 and 6.5 from [McK96b]. In [McK96b], Ralph McKenzie proves the result in a more restrictive setting, but the proof itself depends only on the properties of the algebra $\mathbf{A}$ that are guaranteed by Corollary III.2.

Lemma IV.2. *There exist elements $p, q \in B$ and $s_0 \in I$ satisfying the following:*

1. *For any $a, b \in B$, the pair $(a, b) \in \theta$ iff for every polynomial $f(x)$ of* **B***,*

$$f(a) = p \leftrightarrow f(b) = p.$$

2. *The entire interval bounded by $\perp$ and p in the semilattice $\langle A(T), \wedge\rangle^I$ is included in B and* $\mathrm{supp}(p) = I$.

3. *$q < p, (q, p) \in \bar{\theta}$, and $q(s_0) = 0$ while $q(s) = p(s)$ for all $s \in I \setminus \{s_0\}$.*

4. *For all $a, b \in B$ we have $p\theta a \to p = a$ while*

$$a < p, b < p, (a, b) \in \bar{\theta} \to (a, b) \in \theta$$

   In particular $a < p$ implies $(a, a \wedge q) \in \theta$.

5. *Let $f(x)$ be any polynomial operation of* **B***. If $a < p$ and $f(a) = p$, then also $f(q) = p$.*

For the reminder of the paper, we will fix elements $p, q \in B$ and $s_0 \in I$ satisfying the properties above. The following proposition is a consequence of Condition 3 imposed on the set $\mathcal{B}$, and is a version of a result presented by McKenzie in [McK96b].

Proposition IV.3. *For any $k+3$-ary $S(\bar{v}, x, y, z) \in \mathcal{B}$ and any $\bar{a} \in B^k$, there exists $s \in I$ such that*

$$\mathbf{A} \models S(\bar{a}(s), x, y, z) \approx \perp.$$

Proof. Assume otherwise and fix $S(\bar{v}, x, y, z) \in \mathcal{B}$ and $\bar{a} \in B^k$ such that

$$\mathbf{B} \models S(\bar{a}, x, y, z) \approx (x \wedge y) \vee (x \wedge z).$$

Let $b \in B$ be an element such that for some s_1, $s_1 \neq s_0$ we have $b(s_1) = \perp$ and $b(s) = p(s)$ for $s \neq s_1$. Lemma IV.2 guarantees the existence of such an element and implies that $(b, b \wedge q) \in \theta$. But then

$$p = S(\bar{a}, p, b, q)\, \theta\, S(\bar{a}, p, b \wedge q, q) = q,$$

a contradiction. □

The following proposition is a copy of a part of Proposition 6.6 in [McK96b] and is a consequence of the conditions imposed on **A**.

Proposition IV.4. *The following are true in* $\mathbf{B}$*.*

- *There are no* $b_0, b_1 \in B$ *such that* $\perp \neq b_1(s) = \delta(b_0(s))$ *for all* $s \in I$*.*
- *If* $b \in B$ *and* $\nu(b(s)) = \nu(p(s))$ *for all* $s \in I$*, then* $b = p$*.*

Proof. To prove the first fact, we apply Proposition IV.3 to the elements $b_0, b_1 \in B$ and the operation $S_2(u, v, x, y, z) \in \mathcal{B}$. For the second fact, assume that $b \in B$ and $\nu(b(s)) = \nu(p(s))$ for all $s \in I$, and set

$$K = \{s \in I \mid b(s) = \delta(p(s))\}.$$

We have $K \neq I$ (given by the first part of the proposition), and $K \neq \emptyset$ assuming $b \neq p$. We consider two cases. If $s_0 \in K$ then we choose $s_1 \in I \setminus K$, and fix $c \in B$ such that $c(s_1) = \perp$ and $c(s) = p(s)$ for $s \neq s_1$. The existence of such an element is guaranteed by Statement 2 of Lemma IV.2. We get $J(p, b, c) = J(p, b, p) = p$ and $J(p, b, q) = q$, which contradicts Statement 5 of Lemma IV.2. On the other hand, if $s_0 \notin K$ we choose $s_1 \in K$, define c as above, and the same contradiction is obtained by $J'(p, b, c) = p$, while $J'(p, b, q) = q$. □

The following definition is a version of a definition given by Ralph McKenzie in [McK96a] tailored to Ross Willard's approach.

Definition IV.5. Let B_1 be the smallest subset of the set B such that $p \in B_1$ and for any $F(\bar{x}) \in \mathcal{A}$ if $F(\bar{a}) \in B_1$ for some $\bar{a}$ in B then $a_i \in B_1$ for all i.

The next corollary is an immediate consequence of the fact that, by Condition 2, all the operations of $\mathcal{A}$ are $\perp$–absorbing, and that $\operatorname{supp}(p) = I$.

Corollary IV.6. *For any* $b \in B_1$ *we have* $\operatorname{supp}(b) = I$*.*

The following proposition is once again a copy of a result of McKenzie, and generalizes the second part of Proposition IV.4.

Proposition IV.7. *For any* $a \in B_1$ *and* $b \in B$ *if* $\nu(a(s)) = \nu(b(s))$ *for any* $s \in I$*, then* $a = b$*.*

Proof. We proceed by the way of contradiction. Among the pairs of elements that contradict the claim choose $a \in B_1$ and $b \in B$ that require a minimal number of applications of operations of $\mathcal{A}$ to obtain p from a. Since, by our assumption, $a \neq b$ and $\nu(a(s)) = \nu(b(s))$ for any $s \in I$, we fix $s_1 \in I$ such that $a(s_1) = \delta(b(s_1))$. By Proposition IV.4, we know that $a \neq p$. There is an operation $F(x, \bar{y}) \in \mathcal{A}$, and $\bar{c}$ in B such that it takes one less application of operations from $\mathcal{A}$ to obtain p from $F_{(i)}(a, \bar{c})$. Then, by Condition 2, we get $F_{(i)}(b(s_1), \bar{c}(s_1)) = \delta\big(F_{(i)}(a(s_1), \bar{c}(s_1))\big)$ and $\nu\big(F_{(i)}(b(s), \bar{c}(s))\big) = \nu\big(F_{(i)}(a(s), \bar{c}(s)\big)$ for any $s \in I$ – a contradiction to the minimality of a. □

We are going to define a complexity measure of a term in the language of $\mathbf{A}$ in an unusual way.

Definition IV.8. The complexity measure of a term in the language of $\mathbf{A}$, is defined recursively by

$$\begin{aligned} \operatorname{depth}(x) &= 1 && \text{for any variable symbol } x, \\ \operatorname{depth}\big(F(\bar{g}(\bar{x})\big) &= \max_i\big(\operatorname{depth}(g_i(\bar{x}))\big) + 1 && \text{for any } F \notin \mathcal{C}. \end{aligned}$$

Having defined the measure for any term without symbols in $\mathcal{C}$, we can define it for any term in the language of $\mathbf{A}$ recursively by

$$\operatorname{depth}\big(T(\bar{g}(\bar{x})\big) = \max_i\big(\operatorname{depth}(g_i(\bar{x}))\big) + \operatorname{depth}(\hat{t}(\bar{y})) + 1 \quad \text{for any } T \in \mathcal{C}.$$

The following corollary is a straightforward consequence of the definition and we omit its proof.

Corollary IV.9. *For any term $f(\bar{y})$ and any tuple of terms $\bar{g}(\bar{x})$ in the language of $\mathbf{A}$ we have*

$$\operatorname{depth}\big(f(\bar{g}(\bar{x}))\big) \leq \operatorname{depth}(f(\bar{y})) + \max_i\big(\operatorname{depth}(g_i(\bar{x}))\big).$$

We define the depth of a polynomial to be the smallest depth of a term defining this polynomial. Note that if a polynomial is given by a term, then its depth is bounded from above by the depth of this term.

The following proposition, together with its proof, is a variation of the third item of Lemma 5.7 in [McK96a].

Proposition IV.10. *The set B_1 is identical with the set of all $a \in B$ such that $f(a) = p$ for some non-constant polynomial $f(x)$ of algebra $\mathbf{B}$.*

Proof. Obviously the set B_1 is smaller. To prove the converse inequality we follow Ralph McKenzie's example and show that for every non-constant polynomial $f(x)$ of $\mathbf{B}$ we have $f(B \setminus B_1) \subseteq B \setminus B_1$. Assume that $f(x)$ is a non-constant polynomial, $a \in B \setminus B_1$ and $f(a) \in B_1$ and that for any non-constant polynomial $g(x)$ of depth smaller then the depth of $f(x)$ we have $g(B \setminus B_1) \subseteq B \setminus B_1$. The depth of the polynomial $f(x)$ is certainly greater then one. We consider cases depending on the structure of the polynomial $f(x)$.

If $f(x) = g_0(x) \wedge g_1(x)$ then both $g_0(x)$ and $g_1(x)$ are of smaller depth. Moreover, since $\operatorname{supp}(f(a)) = I$ we get $g_0(a) = g_1(a) = f(a)$ and the minimality of $f(x)$ implies that $g_0(x)$ and $g_1(x)$ are constant, and so is $f(x)$ – a contradiction.

If $f(x) = F(\bar{g}(x))$ for some $F(\bar{y}) \in \mathcal{A}$, then $g_i(a) \in B_1$ for all i. The minimality of the depth of $f(x)$ implies that all $g_i(x)$ are constant, and hence so is $f(x)$ – a contradiction.

The polynomial $f(x)$ cannot be of the form $S(\bar{g}(x))$ for some $S(\bar{y}) \in \mathcal{B}$, since Proposition IV.3 implies that for any $\bar{b}$ in B we have $\operatorname{supp}(S(\bar{b})) \neq I$.

If $f(x) = T(\bar{g}(x))$ for some $T(\bar{y}) \in \mathcal{C}$ we consider the polynomial $\hat{f}(x)$ defined to be equal to $\hat{t}(\bar{g}(x))$. Since $\operatorname{supp}(f(a)) = I$, Condition 4 implies that $\nu\big(t(\bar{g}(a))(s)\big) = \nu(f(a)(s))$ for any $s \in I$, so by Proposition IV.7 we get $t(\bar{g}(a)) = f(a)$. This, together with Proposition III.1 implies that $\hat{f}(a) = f(a)$. Since, by Corollary IV.9, the depth of $\hat{f}(x)$ is smaller then the depth of $f(x)$ we get that $\hat{f}(x)$ is constant. Then, using the fact that $\operatorname{supp}(f(a)) = I$ and $\hat{f}(x) \leq f(x)$ we obtain $f(x) = \hat{f}(x)$, which is constant – contradiction.

If $f(x) = \nu(g(x))$ then instantly, by Proposition IV.7, we obtain $f(a) = g(a)$ and that $g(x)$ and $f(x)$ are constant, which is a contradiction.

If $f(x) = J(g_0(x), g_1(x), g_2(x))$ then Condition 6 implies that $f(a) \leq g_0(a)$ and since $\operatorname{supp}(f(a)) = I$ we get $f(a) = g_0(a)$, so $g_0(x)$ is constant and equal to $f(a)$. The fact that $\operatorname{supp}(f(a)) = I$ implies as well that $\nu\big(g_0(a)(s)\big) = \nu\big(g_1(a)(s)\big)$, and so by Proposition IV.7 we get $g_1(a) = g_0(a) = f(a)$ which implies that $g_1(x)$ is constant, and so is $f(x)$. The case of $f(x) = J'(g_0(x), g_1(x), g_2(x))$ is similar.

We conclude the proof with the remark that all the remaining operations of $\mathbf{A}$ are nullary.

□

The statements of Lemma IV.2 imply that the set of all elements of B that do not map to p under unary non-constant polynomials is an equivalence class of θ. Moreover, for any $b, c \in B$, such that $b \neq c$ and $b \,\theta\, c$ we get $b \,\theta\, c \,\theta\, b \wedge c \notin B_1$. The following corollary summarizes our results concerning the structure of $\mathbf{S}$ and $\mathbf{B}$.

Corollary IV.11. *The following hold for* $\mathbf{S}$ *and* $\mathbf{B}$

1. *The underlying set of* $\mathbf{B}$ *decomposes as a disjoint union* $B = B_\perp \cup B_1$.
2. *For any* $b \in B_1$ *we get* $\operatorname{supp}(b) = I$ *and* $b/\theta = \{b\}$ *and* $B_\perp$ *is one equivalence class of the congruence* θ.
3. *The operation* $\wedge$ *makes* S *a flat semilattice, with the image of* $B_\perp$ *being the bottom element.*
4. *All the operations of the set* $\mathcal{B}$ *are constant and equal to the bottom element of* $\mathbf{S}$.
5. *For any operation* $T(\bar{x}) \in \mathcal{C}$ *we get*

$$\mathbf{S} \models T(\bar{x}) \approx \hat{t}(\bar{x}).$$

6. We have

$$\mathbf{S} \models J(x,y,z) \approx x \wedge y,$$

and similarly

$$\mathbf{S} \models J'(x,y,z) \approx x \wedge y \wedge z.$$

Moreover if the operation ν is present in the algebra $\mathbf{A}$ in the set $\mathcal{E}$ we can infer more.

Corollary IV.12. *If the operation ν is present in the algebra $\mathbf{A}$, then*

- *the set $B_1 \subseteq A_0^I$; and*
- $\mathbf{S} \models \nu(x) \approx x$.

V Sequentiability of the elements of A and B

The ultimate goal of this paper is to present a number of algebras that are examples of fast growth of different 'complexity measures'. All the examples presented in this paper will satisfy Conditions 1-6. Moreover all of them will have common structure. In this section we describe the structural properties that are common and prove facts implied by this structure.

We define the set $\mathcal{M}$ to be

$$\mathcal{M} = \{L, H, R\}.$$

In most cases this set is not to be a part of an algebra $\mathbf{A}$; rather it will describe the common structure that different elements of A have. We define a relation of sequentiability $\prec$ on $\mathcal{M}$ by stating its only instances $L \prec L \prec H \prec R \prec R$ and extend it pointwise to any cartesian power of $\mathcal{M}$. We say that a subset of $\mathcal{M}$ or $\mathcal{M}^I$ is sequentiable if and only if all its elements can be arranged in a sequence such that $a_0 \prec \cdots \prec a_l$.

Each algebra $\mathbf{A}$ comes equipped with a function $\pi : A \setminus \{\bot\} \to \mathcal{M}$ such that

$$\pi(a) = \pi(\delta(a)), \text{ for any } a \in X.$$

This implies that $\pi(a) = \pi(\nu(a))$, for $a \neq \bot$. We proceed to impose further conditions on the operations of $\mathbf{A}$.

Condition 7. The set $\mathcal{A}$ is a disjoint union of three sets — $\mathcal{A}_{\prec}$, $\mathcal{A}_{=}$, and $\mathcal{A}_{\succ}$ — and any operation of $\mathcal{A}$ is ternary. Moreover we require the following conditions:

- for any $F(x,y,z) \in \mathcal{A}$ we have $F(a,b,c) = \perp$ unless

$$\pi(a) = \pi(c) \text{ and } \pi(b) = \pi\big(F(a,b,c)\big);$$

- for any $F(x,y,z) \in \mathcal{A}_{\prec}$ we have $F(a,b,c) = \perp$ unless

$$\pi(a) \prec \pi(b) \text{ and } a,b \in Y;$$

- for any $F(x,y,z) \in \mathcal{A}_{=}$ we have $F(a,b,c) = \perp$ unless $\pi(a) = \pi(b)$;
- for any $F(x,y,z) \in \mathcal{A}_{\succ}$ we have $F(a,b,c) = \perp$ unless

$$\pi(a) \succ \pi(b) \text{ and } a,b \in Y.$$

Further conditions have to be imposed on the set $\mathcal{C}$.

Condition 8. For any operation $F(x,y,z) \in \mathcal{A}_{\prec}$ there exist operations $T_1(v,x,y,z)$ and $T_2(v,x,y,z)$ in $\mathcal{C}$ defined by $\Sigma_1(v,x,y,z) = \{v \approx x\}$ and $\Sigma_2(v,x,y,z) = \{v \approx y\}$, the predicate $\Phi_1(v,x,y,z)$ equal to "$\pi(v) \prec \pi(y)$ and $v \in Y$", the predicate $\Phi_2(v,x,y,z)$ equal to "$\pi(x) \prec \pi(v)$ and $v \in Y$" and

$$t_1(v,x,y,z) = t_2(v,x,y,z) = F(x,y,z).$$

Condition 9. For any operation $F(x,y,z) \in \mathcal{A}_{\succ}$, there exist operations $T_1(v,x,y,z)$ and $T_2(v,x,y,z)$ in $\mathcal{C}$ defined by $\Sigma_1(v,x,y,z) = \{v \approx x\}$ and $\Sigma_2(v,x,y,z) = \{v \approx y\}$, the predicate $\Phi_1(v,x,y,z)$ equal to "$\pi(v) \succ \pi(y)$ and $v \in Y$", the predicate $\Phi_2(v,x,y,z)$ equal to "$\pi(x) \succ \pi(v)$ and $v \in Y$" and

$$t_1(v,x,y,z) = t_2(v,x,y,z) = F(x,y,z).$$

It is easy to see that Condition 7 implies that the operations defined in Conditions 8 and 9 comply with the requirements of Condition 4. These conditions allow us to derive some consequences on the structure of $\mathbf{B}$.

We extend the marking function π to the function $\pi : B_1 \to \mathcal{M}^I$ in a natural way. Note that, by Corollary IV.11, all the elements of the set B_1 are of full support, so this extension is well defined. We present first a trivial corollary.

Corollary V.1. *For any $a,b,c \in B_1$ if $\mathbf{S} \models F(a/\theta, b/\theta, c/\theta) \neq \perp$, then*

- $F(x,y,z) \in \mathcal{A}_{\prec}$ *implies* $\pi(c) = \pi(a) \prec \pi(b) = \pi(F(a,b,c))$;
- $F(x,y,z) \in \mathcal{A}_{=}$ *implies* $\pi(c) = \pi(a) = \pi(b) = \pi(F(a,b,c))$;
- $F(x,y,z) \in \mathcal{A}_{\succ}$ *implies* $\pi(c) = \pi(a) \succ \pi(b) = \pi(F(a,b,c))$.

This brings us to the following propositions:

Proposition V.2. *If $|\pi(B_1)| > 1$, then for any $a \in B_1$ there exists an element $a' \in B_1 \cap Y^I$ such that $\pi(a) = \pi(a')$.*

Proof. If $|\pi(B_1)| > 1$ then the definition of the set B_1 implies that for any element $a \in B_1$ there exist elements $a_0, a_1, a_2 \in B_1$ and $F(x,y,z) \in \mathcal{A}_{\prec} \cup \mathcal{A}_{\succ}$ such that $F(a_0,a_1,a_2) \in B_1$ and $\pi(a) = \pi(a_0)$ or $\pi(a) = \pi(a_1)$. □

Proposition V.3. *If $|\pi(B_1)| > 1$ then for any $a, a' \in B_1 \cap Y^I$, if $\pi(a) = \pi(a')$ then $a = a'$.*

Proof. Our assumption implies that there exist $b_0, b_1, b_2 \in B_1$ and $F(x,y,z) \in \mathcal{A}_{\prec} \cup \mathcal{A}_{\succ}$ such that $F(b_0,b_1,b_2) \in B_1$ and $\pi(a) = \pi(b_0)$ (or $\pi(a) = \pi(b_1)$). Comparing the elements $F(b_0,b_1,b_2)$ and $T_1(a,b_0,b_1,b_2)$ (or $T_2(a,b_0,b_1,b_2)$) for the operations of set $\mathcal{C}$ defined for $F(x,y,z)$ we deduce that $a = b_0$. By the same token $a' = b_0$ and the proposition is proved. □

Proposition V.4. *The set $\pi(B_1)$ is sequentiable.*

Proof. The case when $|\pi(B_1)| = 1$ is trivial. If it's not the case then we prove that for any set $B^* \subseteq B_1$ such that $\pi(B^*)$ is sequentiable, and any elements $a, b, c \in B$, and operation $F(x,y,z) \in \mathcal{A}$ if $F(a,b,c) \in B^*$ then the set $\pi(B^* \cup \{a,b,c\})$ is sequentiable. We consider cases depending on $F(x,y,z)$.

The case when $F(x,y,z) \in \mathcal{A}_{=}$ is trivial since Condition 7 implies that

$$\pi(a) = \pi(b) = \pi(c) = \pi\big(F(a,b,c)\big) \in \pi(B^*).$$

Now, assume that the set $\pi(B^*)$ can be arranged in the sequence $a_0 \prec \cdots \prec a_l$. And that for some $F(x,y,z) \in \mathcal{A}_{\prec}$, and $a, b, c \in B$ we have $F(a,b,c) \in B^*$. By reasoning as in the previous case we get

$$\pi(a) = \pi(c) \prec \pi(b) = \pi\big(F(a,b,c)\big) \in \pi(B^*).$$

If $\pi(b) = a_0$ we are done, otherwise we have some i such that $\pi(b) = a_i$. Assume, for a contradiction, that $\pi(a) \neq a_{i-1}$. Then there is an element $a' \in B_1 \cap Y^I$ such that $\pi(a') = a_{i-1}$.

We consider $T_1(a', a, b, c)$ for $T_1(v, x, y, z)$ constructed in Condition 8 for $F(x, y, z)$. We infer that $\Phi(a'(s), a(s), b(s), c(s))$ holds for all $s \in I$. On the other hand $a' \neq a$, since $\pi(a) \neq \pi(a')$. By the definition of the operations of the set $\mathcal{C}$ we get that the elements $F(a, b, c)$ and $T(a', a, b, c)$ contradict Proposition IV.7. The case of $F(x, y, z) \in \mathcal{A}_{\succ}$ is an alphabetical variant of this one and the proposition is proved. □

The following condition has crucial consequences on the structure of the algebra **B**.

Condition 10. There exists an operation $S_1(v, x, y, z) \in \mathcal{B}$ such that

$$S_1(v, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } \pi(v) \in \{L, R\}, \\ \bot & \text{otherwise.} \end{cases}$$

This condition together with Proposition V.4 implies the following corollary.

Corollary V.5. *The following are true in* **B**

- *for any $b \in B_1$ there is $s \in I$ such that $\pi(b(s)) = H$,*
- *for any $b, b' \in B_1$ either $\pi(b(s)) = \pi(b'(s))$ for any $s \in I$, or*

$$\pi(b(s)) = H \leftrightarrow \pi(b'(s)) \neq H \textit{ for all } s \in I,$$

VI An algebra with at least exponential growth of the γ function

In this section we present an algebra **A** with at least exponential growth of the $\gamma_{\mathbf{A}}$ function. The algebra we are going to define will comply with Conditions 1-10, but not with Condition 2a. The universe of the algebra **A** is a disjoint union of Y, X_0, X_1 and $\{\bot\}$, where

$$Y = \{Y^{L,0}, Y^{H,0}, Y^{R,0}, Y^{L,1}, Y^{H,1}, Y^{R,1}\},$$

$$X_0 = \{W^{L,0}, W^{H,0}, W^{R,0}, W^{L,1}, W^{H,1}, W^{R,1}\},$$

and

$$X_1 = \{\tilde{W}^{L,0}, \tilde{W}^{H,0}, \tilde{W}^{R,0}, \tilde{W}^{L,1}, \tilde{W}^{H,1}, \tilde{W}^{R,1}\}.$$

The operations δ and ν are defined in the natural way. The operation π returns the first superscript of the element, and we introduce an operation τ which returns the superscript from the set $\{0,1\}$. Note that the results of π and τ together with membership in one of the sets Y, X_0 and X_1 determine the element completely.

The operations $\perp$ and $\wedge$ make A a flat semilattice. We define two operations of the set $\mathcal{A}_{\prec}$ on the set A_0 and leave for the reader the extension of these operations to A in accordance with Condition 2. For $a, b, c \in A_0$ we put

$$F^{\min}(a,b,c) = \begin{cases} W^{\pi(b),\min\{\tau(b),\tau(c)\}} & \text{if } a,b \in Y,\ c \in X_0 \text{ and } \pi(c) = \pi(a) \prec \pi(b), \\ \perp & \text{otherwise,} \end{cases}$$

$$F^{\min}_{\text{compl}}(a,b,c) = \begin{cases} W^{\pi(b),\min\{(1-\tau(b)),\tau(c)\}} & \text{if } a,b \in Y,\ c \in X_0,\ \pi(c) = \pi(a) \prec \pi(b), \\ \perp & \text{otherwise.} \end{cases}$$

The set $\mathcal{B}$ will consist of $S_2(u,v,x,y,z)$ as defined in Condition 3, $S_1(v,x,y,z)$ as defined in Condition 10, and one extra operation $S^*(v,x,y,z)$ defined to be

$$S^*(v,x,y,z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } \tau(v) = 0 \\ \perp & \text{otherwise.} \end{cases}$$

The set $\mathcal{C}$ consists of four functions defined for $F^{\min}(x,y,z)$ and $F^{\min}_{\text{compl}}(x,y,z)$ according to Condition 8. The sets $\mathcal{D}$ and $\mathcal{E}$ are empty, and the operations $J(x,y,z)$ and $J'(x,y,z)$ are present in the algebra.

We leave it as an exercise for the reader to prove the algebra just defined complies with Conditions 1-10, but does not comply with Condition 2a. We begin with some trivial results on the algebra $\mathbf{A}$.

Corollary VI.1. *The following are true in* $\mathbf{A}$

- *the only terms in $\mathcal{A}_{\prec}$ that are not constantly equal to $\perp$ are the ones with a non-trivial subterm appearing only as a last argument of each operation*
- *for two $\mathcal{A}_{\prec}$ terms $r'(\bar{x})$ and $r''(\bar{x})$ such that one is obtained from the other by substituting some of the appearances of $F^{\min}(x,y,z)$ with $F^{\min}_{\text{compl}}(x,y,z)$ or vice versa we have for all $\bar{f}$*

  $$r'(\bar{f}) \neq \perp \text{ if and only if } r''(\bar{f}) \neq \perp,$$

- *moreover for any pair of such terms, and for each $\bar{f}$ such that $r'(\bar{f}) \neq \perp$ we have $\pi(r'(\bar{f})) = \pi(r'(\bar{f}))$.*

We define a special family of terms in the language of $\mathbf{A}$ in a recursive way. Each term is determined by a number n and a word w of length n over an alphabet consisting of 0 and

1 (with ε – an empty word). Terms are defined in the following way

$$f^0_\varepsilon(x_0, y) = y$$
$$f^n_w(x_0, \ldots, x_n, y) = \begin{cases} F^{\min}(x_{n-1}, x_n, f^{n-1}_{w_{[n-1]}}(x_0, \ldots, x_{n-1}, y)) & \text{if } w(n-1) = 0, \\ F^{\min}_{\text{compl}}(x_{n-1}, x_n, f^{n-1}_{w_{[n-1]}}(x_0, \ldots, x_{n-1}, y)) & \text{if } w(n-1) = 1. \end{cases}$$

This definition implies the following corollary.

Corollary VI.2. *For any n and for any word w of length n if $f^n_w(\bar{a}, b) \neq \bot$ for some $\bar{a}, b$ in A then*

- *we have $f^n_w(\bar{a}, b) \in X_0$ if and only if $b \in X_0$ (similarly for X_1), and*
- *we have $\pi(f^n_w(\bar{a}, b)) = \pi(a_n)$, and*
- *finally we have $\tau(f^n_w(\bar{a}, b)) = 1$ if and only if $\tau(b) = 1$ and for any $0 < j \leq n$ we have $w(j) + \tau(a_j) = 1$.*

We define another family of terms by putting $f^n(\bar{x}) = f^n_{0^n}(\bar{x})$. With each of the terms $f^n(\bar{x})$ we associate an algebra in the language of $\mathbf{A}$, denoted by $\mathbf{S}_n$. The universe of this algebra consist of the elements Ω and $\Lambda_0, \ldots, \Lambda_n$, $\Sigma_1, \ldots, \Sigma_n$ and the bottom element denoted by $\bot$. The operations of the algebra $\mathbf{S}_n$ are the following. The operation $\bot$ is evaluated as itself, and $\wedge$ makes $\mathbf{S}_n$ a flat semilattice with bottom equal to $\bot$. The operation $F^{\min}(x, y, z)$ is defined in the following way.

$$F^{\min}(\Lambda_0, \Lambda_1, \Omega) = \Sigma_1$$
$$F^{\min}(\Lambda_{i-1}, \Lambda_i, \Sigma_{i-1}) = \Sigma_i \text{ for } 1 < i \leq n,$$

while all the other applications of $F^{\min}(x, y, z)$ are evaluated to $\bot$. Operation $F^{\min}_{\text{compl}}(x, y, z)$ and all operations of the set $\mathcal{B}$ are constantly equal to $\bot$. For each $T(v, x, y, z) \in \mathcal{C}$ we have $T(v, x, y, z) = \hat{t}(v, x, y, z)$ which means that all $\mathcal{C}$ operations associated with $F^{\min}_{\text{compl}}(x, y, z)$ are constantly $\bot$ and that for those associated with $F^{\min}(x, y, z)$ we have

$$\mathbf{S}_n \models T_1(v, x, y, z) \approx F^{\min}(v \wedge x, y, z),$$
$$\mathbf{S}_n \models T_2(v, x, y, z) \approx F^{\min}(x, v \wedge y, z).$$

In the algebra $\mathbf{S}_n$ we also have

$$\mathbf{S}_n \models J(x,y,z) \approx x \wedge y,$$
$$\mathbf{S}_n \models J'(x,y,z) \approx x \wedge y \wedge z.$$

First, we are going to show that $\mathbf{S}_n \in \mathrm{HSP}(\mathbf{A})$ for any $n > 1$. To do so, we fix $n > 1$ and choose the elements ω and $\lambda_0, \ldots, \lambda_n$ in $\mathbf{A}^{2^n}$. We choose ω such that $\omega \in X_0^{2^n}$ and $\tau(\omega(i)) = 1$ for all i, and such that $\pi(\omega(0)) = H$ and $\pi(\omega(i)) = L$ for $i > 0$. We choose $\lambda_i \in Y^{2^n}$ such that

$$\pi(\lambda_j(i)) = \begin{cases} L & \text{if } j < i \\ H & \text{if } j = i \\ R & \text{if } j > i, \end{cases}$$

and such that elements $\tau(\lambda_1), \ldots, \tau(\lambda_n)$ together with their complements in $\{0,1\}^{2^n}$ generate, under the operation of taking minimum, all the atoms of $\{0,1\}^{2^n}$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^{2^n}$ generated by the elements $\omega, \lambda_0, \ldots, \lambda_n$. We denote by B_0 the set of all elements of $\mathbf{B}$ of support smaller then 2^n.

Note that

$$\pi(\omega) = \pi(\lambda_0) \prec \cdots \prec \pi(\lambda_n).$$

Further, a quick examination of definitions implies that, for any $i \leq n$ and any word w in $\{0,1\}$ of length i, the support $\mathrm{supp}(f^i_w(\lambda_0, \ldots, \lambda_i, \omega)) = 2^n$. Then by Corollary VI.2, we get $\pi(f^i_w(\lambda_0, \ldots, \lambda_i, \omega)) = \pi(\lambda_i)$. Moreover by our choice of the τ images of λ_i and ω, via Corollary VI.2, for any word w in $\{0,1\}$ of length $i \leq n$ we get $\tau(f^i_w(\lambda_0, \ldots, \lambda_i, \omega)) \neq 0^{2^n}$. To see this, we remark that

$$\tau(f^i_w(\lambda_0, \ldots, \lambda_i, \omega)) = \min\big(\{\tau(\lambda_j) |\, w(j) = 0,\, j \leq i\} \cup \{\tau(\lambda_j)^c |\, w(j) = 1,\, j \leq i\}\big),$$

where the minimum is taken pointwise.

Claim. Under these assumptions,

$$B = B_0 \cup \{\lambda_0, \ldots, \lambda_n\} \cup \{f^i_w(\lambda_0, \ldots, \lambda_i, \omega) \,|\, \text{ for any } i \leq n \text{ and any word } w\}$$

Proof. Let's denote the right hand side of the equality above by B^*, and the last factor of this side by B^+. Certainly $B^* \subseteq B$, and all the generators of B are in B^*. It remains to prove that B^* is closed with respect to the basic operations of $\mathbf{A}$.

The set B^* is certainly closed with respect to $\perp$ and $\wedge$. For any operation $F(x,y,z)$ of the set $\mathcal{A}$ if one of the arguments is in B_0 the result of the operation is there as well. If, on the other hand, $\mathrm{supp}\big(F(a,b,c)\big) = 2^n$ then certainly $a = \lambda_i$ and $b = \lambda_{i+1}$ for some i and $c \in B^+$. Moreover, we have $\pi(c) = \pi(\lambda_i)$ and hence $c = f^i_w(\lambda_0, \dots, \lambda_i, \omega)$ for some w of length i. It follows that $F(a,b,c) = f^{i+1}_{w1}(\lambda_0, \dots, \lambda_{i+1}, \omega) \in B^+$ or $F(a,b,c) = f^{i+1}_{w0}(\lambda_0, \dots, \lambda_{i+1}, \omega) \in B^+$.

The image of the operation $S_2(u,v,x,y,z) \in \mathcal{B}$ is included in B_0, since no element of full support in B^* has a barred element in the range. As for the operation $S_1(v,x,y,z) \in \mathcal{B}$, its image is again in B_0, since, according to the remarks above, each element of full support is mapped by the function π to $\pi(\lambda_i)$, for some i. Finally, the image of $S^*(v,x,y,z)$ is in B_0 by our choice of the τ images of λ_i and ω.

It is easy to see that for any $T(v,x,y,z) \in \mathcal{C}$ and for any elements $a,b,c,d \in B^*$ we have $T(a,b,c,d) \in B_0 \cup \{F^{\min}(b,c,d), F^{\min}_{\mathrm{compl}}(b,c,d)\}$. Thus this case reduces to the case of operations from the set $\mathcal{A}$.

The case of operation $J(x,y,z)$ and $J'(x,y,z)$ is trivial, since $J(x,y,z) \leq x$, and similarly $J'(x,y,z) \leq x$. □

Set

$$B_1 = \{\lambda_0, \dots, \lambda_n\} \cup \{f^i(\lambda_0, \cdots, \lambda_i, \omega) \mid \text{for any } i \leq n\}$$

and define an equivalence relation θ on B by $x\,\theta\,y$ if and only if $x = y$ or $x, y \in B \setminus B_1$.

Claim. The relation θ is a congruence of the algebra $\mathbf{B}$.

Proof. We begin by proving that for all the basic operations of the algebra $\mathbf{A}$ (except for $J(x,y,z)$) the set $B \setminus B_1$ is absorbing. The set is obviously absorbing for $\wedge$ and $\perp$. To consider operations from $\mathcal{A}$, we remark first that, by our choice of τ coordinates of λ_i, for any $i,j \leq n$, if $f^i_w(\lambda_0, \dots, \lambda_i, \omega) = f^j_{w'}(\lambda_0, \dots, \lambda_j, \omega)$ then $i = j$ (since $\pi(f^i_w(\lambda_0, \dots, \lambda_i, \omega)) = \pi(\lambda_i)$, and similarly for j) and $w = w'$ (by Corollary VI.2 and the fact that for no element a of full support we have $\pi(a) = 0^{2^n}$).

Now assume that for some $F(x,y,z) \in \mathcal{A}$ we have $F(a,b,c) \in B_1$. Obviously then $a = \lambda_i$ and $b = \lambda_{i+1}$ and $c \in B^+$. By our definition of B_1 we get $F(a,b,c) = F(\lambda_i, \lambda_{i+1}, c) = f^j(\lambda_0, \dots, \lambda_j, \omega)$ for some j. It follows from the considerations at the beginning of the proof that $F(x,y,z) = F^{\min}(x,y,z)$ and $c = f^i(\lambda_0, \dots, \lambda_i, \omega)$ and $j = i+1$. Thus the set is absorbing for all the operations of $\mathcal{A}$. It is also absorbing for all the operations of $\mathcal{B}$ by range consideration from the proof of previous claim. With regard to the operation $T(v,x,y,z) \in \mathcal{C}$ if $T(a,b,c,d) \in B_1$, then $a \in B_1$ and $T(a,b,c,d) = F(b,c,d) \in B_1$. So $b,c,d \in B_1$ and this case reduces to the case of $\mathcal{A}$.

Finally for $J(a,b,c) \in B_1$ we need $a = b \in B_1$ and then $J(a,a,z) = a$. Similarly, for

$J'(a,b,c) \in B_1$ we need $a = b = c \in B_1$ so θ is respected by those two operations.The proof of the theorem is now complete. □

We leave it as an easy exercise to the reader to prove that $\mathbf{B}/\theta$ is isomorphic to $\mathbf{S}_n$, thereby proving that $\mathbf{S}_n \in \mathrm{HSP}(\mathbf{A})$ for any $n > 1$.

To obtain a lower bound on the function $\gamma_{\mathbf{A}}$ we will prove that if $\mathbf{S}_n$ is isomorphic to $\mathbf{B}/\theta$ for some θ and some $\mathbf{B} \leq \mathbf{A}^I$, then $|I| \geq 2^n$. We fix $n > 1$ set $\mathbf{S} = \mathbf{S}_n$ and consider $\mathbf{B} \leq \mathbf{A}^I$ such that $\mathbf{S}$ is isomorphic to $\mathbf{B}/\theta$ and $|I| = \dim_{\mathbf{A}}(\mathbf{S})$. Corollary IV.11 describes the structure of the congruence θ on $\mathbf{B}$. We denote by lower case Greek letters the elements of $\mathbf{B}$ that have a one–element θ-class, denoted by the respective capital letters. The element p is identified with σ_n. By Corollary V.1,

$$\pi(\omega) = \pi(\lambda_0) \prec \pi(\sigma_1) = \pi(\lambda_1) \prec \cdots \prec \pi(\lambda_n) = \pi(p)$$

and by Corollary VI.1,

$$\mathrm{supp}\big(f_w^n(\lambda_0, \ldots, \lambda_n, \omega)\big) = I.$$

Consider the subset of $\{0,1\}^I$ consisting of the elements $\tau\big(f_w^n(\lambda_0, \ldots, \lambda_n, \omega)\big)$ for all w. Note that in view of the definition of the operation $S^*(v,x,y,z)$ and Proposition IV.3, every element of this set has 1 in its range. On the other hand, the definition of $f_w^n(x_0, \ldots, x_n, y)$ implies, via Corollary VI.2, that for different words we obtain disjoint subsets of I mapped to 1 by $\tau\big(f_w^n(\lambda_0, \ldots, \lambda_n, \omega)\big)$. This proves that the set I is of cardinality at least 2^n.

VII Computational structure

In this section we work with a non-deterministic Turing machine $\mathbf{T}$ and construct, for such a machine, an algebra $\mathbf{C}(\mathbf{T})$. This construction is a modification of a construction introduced by Ralph McKenzie in [McK96a]. The algebra we construct does not have any structure imposed by Conditions 1-10. The algebra $\mathbf{C}(\mathbf{T})$ will be a building block for the algebras we shall eventually construct.

We denote the tape alphabet of the machine $\mathbf{T}$ by $\mathcal{L}$, and list its states as $0, \ldots, l$. The machine is a set of five-tuples $iabLj$ or $iabRj$ where $0 \leq i, j \leq l$ and $\{a, b\} \subseteq \mathcal{L}$. The tuples are understood as machine instructions in the following sense: $iabLj$ means — in the state i if reading a write b, move the head left and change state to j.

A configuration of dimension k of a Turing machine $\mathbf{T}$ is a triple $\langle t, h, s\rangle$ where $t \in \mathcal{L}^k$ and $0 \leq h < k$ and $0 \leq s \leq l$. We introduce a number of relations on the universe of configurations of dimension k. We say that $\langle t', h', s'\rangle \lhd^c_{iabDj} \langle t'', h'', s''\rangle$ (where $0 \leq i, j < l$, $a, b \in \mathcal{L}$ and D is R or L) if the instruction $iabDj$ executed by a machine $\mathbf{T}$ working on a tape t' reading $t'(h')$ in state s' will produce a tape t'' with the head of the machine reading c on the position h'' in state s''. More precisely:

- $i = s'$ and $j = s''$;
- if $D = R$ then $h'' = h' + 1$, if $D = L$ then $h'' = h' - 1$ and $0 \leq h', h'' < k$;
- $t'(m) = t''(m)$ for m different then h'; and
- $t'(h') = a$, $t''(h') = b$ and $t''(h'') = c$.

Note that if two configurations are in one of the relations mentioned above, then the two configurations together determine the relation; moreover a relation together with any configuration determines the other configuration.

The universe of the algebra $\mathbf{C}(\mathbf{T})$ will be a disjoint union of the set

$$\mathcal{M} = \{L, H, R\},$$

together with

$$V = \{L_{(i,a),b}, H_{(i,a),a}, R_{(i,a),b} | \text{ for } \{a, b\} \subseteq \mathcal{L} \text{ and } 0 \leq i \leq l\}$$

and one extra element $\perp$. We introduce the map ρ from $V \cup \mathcal{M}$ onto $\mathcal{M}$ that returns the main symbol of an element of the algebra $\mathbf{C}(\mathbf{T})$.

For any machine instruction $iabLj$ and any $c \in \mathcal{L}$, we introduce an operation of the algebra defined as follows:

$$\begin{aligned}
F^c_{iabLj}(x, y, z) &= L_{(j,c),a'} && \text{if } x = y = L, z = L_{(i,a),a'} \text{ for some } a' \in \mathcal{L} \\
&= H_{(j,c),c} && \text{if } x = L, y = H, z = L_{(i,a),c} \\
&= R_{(j,c),b} && \text{if } x = H, y = R, z = H_{(i,a),a} \\
&= R_{(j,c),a'} && \text{if } x = y = R, z = R_{(i,a),a'} \text{ for some } a' \in \mathcal{L} \\
&= \perp && \text{otherwise.}
\end{aligned}$$

Note that for such an instruction we have $F^c_{iabLj}(f, g, h) = \perp$, unless

$$\rho(h) = \rho(f) \prec \rho(g) = \rho(F^c_{iabLj}(x, y, z)).$$

For an instruction $iabRj$ and any $c \in \mathcal{L}$, we put

$$\begin{aligned} F^c_{iabRj}(x,y,z) &= L_{(j,c),a'} && \text{if } x = y = L, z = L_{(i,a),a'} \text{ for some } a' \in \mathcal{L} \\ &= L_{(j,c),b} && \text{if } x = H, y = L, z = H_{(i,a),a} \\ &= H_{(j,c),c} && \text{if } x = R, y = H, z = R_{(i,a),c} \\ &= R_{(j,c),a'} && \text{if } x = y = R, z = R_{(i,a),a'} \text{ for some } a' \in \mathcal{L} \\ &= \bot && \text{otherwise.} \end{aligned}$$

Similarly we have $F^c_{iabRj}(f,g,h) = \bot$ unless

$$\rho(h) = \rho(f) \succ \rho(g) = \rho(F^c_{iabRj}(x,y,z)).$$

These are all the operations of the algebra $\mathbf{C}(\mathbf{T})$. In the following corollary we present a number of basic consequences of the definitions of the operations of $\mathbf{C}(\mathbf{T})$.

Corollary VII.1. *The following facts are true in* $\mathbf{C}(\mathbf{T})$.

1. *The element $\bot$ is absorbing for all the operations of* $\mathbf{C}(\mathbf{T})$.
2. *For any operation $F(x,y,z)$ of $\mathbf{C}(\mathbf{T})$, for any elements a, a', b, c of* $\mathbf{C}(\mathbf{T})$,

$$F(a,b,c) = F(a',b,c) \neq \bot \text{ implies } a = a';$$
$$F(a,b,c) = F(a,b',c) \neq \bot \text{ implies } b = b'; \text{ and}$$
$$F(a,b,c) = F(a,b,c') \neq \bot \text{ implies } c = c'.$$

   Moreover for any $F(x,y,z)$ of $\mathbf{C}(\mathbf{T})$, there are unique a, a', b, b' in $\mathbf{C}(\mathbf{T})$ such that

$$F(H,a,b) \neq \bot \text{ and } F(a',H,b') \neq \bot.$$

3. *For any non-trivial term $r(\bar{x})$ for which there exists a tuple of elements $\bar{a}$ of $\mathbf{C}(\mathbf{T})$ such that $r(\bar{a}) \neq \bot$, we have $r(\bar{x}) = F(x_i, x_j, r'(\bar{x}))$ for some operation $F(x,y,z)$, some i and j, and some term $r'(\bar{x})$ and $r'(\bar{a}) \neq \bot$.*
4. *For any non-trivial term $r(\bar{x})$ in which each x_i appears, there exists j such that*

$$r(\bar{a}) \neq \bot \text{ implies } \big(a_j \in V \text{ and } a_i \in \mathcal{M} \text{ for } i \neq j\big).$$

Proof. The first two claims of this corollary follow directly from the definitions of the operations of $\mathbf{C}(\mathbf{T})$. Note the similarity between item 2 and Condition 2a from Section III.

Item 3 is straightforward by the domain considerations, and item 4 is an easy consequence of it. ☐

We define an injective map between configurations of length k and the elements of V^k in the following way

$$\Psi_k\langle t,h,s\rangle(i) = \begin{cases} L_{(s,t(h)),t(i)} & \text{if } i<h \\ H_{(s,t(h)),t(h)} & \text{if } i=h \\ R_{(s,t(h)),t(i)} & \text{if } i>h. \end{cases}$$

Next, we define a number of auxiliary elements of $\mathcal{M}^k$

$$\delta_j^k(i) = \begin{cases} L & \text{if } i<j \\ H & \text{if } i=j \\ R & \text{if } i>j. \end{cases}$$

For so defined elements δ_j^k, we have $\delta_0^k \succ \cdots \succ \delta_{k-1}^k$. Note that an element a of $\mathbf{C}(\mathbf{T})^k$ is in the range of the function Ψ_k if and only if $\pi(a) = \pi(\delta_i^k)$, for some i, and there are $j \le l$ and $b \in \mathcal{L}$ such that for every $s < k$ we have

$$a(s) \in \{L_{(j,b),c}, H_{(j,b),b}, R_{(j,b),c} \mid \text{for } c \in \mathcal{L}\}$$

We proceed to prove a lemma.

Lemma VII.2. *For any two configurations* $\langle t',h',s'\rangle$ *and* $\langle t'',h'',s''\rangle$*,*

$$\langle t',h',s'\rangle \lhd^c_{iabDj} \langle t'',h'',s''\rangle$$

if and only if

$$\Psi_k\langle t'',h'',s''\rangle = F^c_{iabDj}(\delta^k_{h'}, \delta^k_{h''}, \Psi_k\langle t',h',s'\rangle).$$

Proof. By symmetry, we may assume without loss of generality that $D = L$. We begin by proving the "if" direction. The definition of the operations on $\mathbf{C}(\mathbf{T})$ immediately implies that $i = s'$ and $j = s''$. On coordinate h'' we have

$$\Psi_k\langle t'',h'',s''\rangle(h'') = H_{(s'',t''(h'')),t''(h'')},$$

which implies that $\delta^k_{h'}(h'') = L$, $\delta^k_{h''}(h'') = H$,

$$\Psi_k\langle t',h',s'\rangle(h'') = L_{(s',t'(h'')),t''(h'')}$$

and $t''(h'') = c$. Focusing on coordinate $h''+1$, we see that $\delta^k_{h''}(h''+1) = R$, and since

$$F^c_{iabDj}(\delta^k_{h'}, \delta^k_{h''}, \Psi_k\langle t', h', s'\rangle)(h''+1) \neq \bot,$$

we infer that $\delta^k_{h'}(h''+1) = H$ so $h' = h''+1$. Now

$$\Psi_k\langle t'', h'', s''\rangle(h') = R_{(t''(h''),s''),t''(h')}$$

so that

$$F^c_{iabDj}(H, R, \Psi_k\langle t', h', s'\rangle(h')) = R_{(t''(h''),s''),t''(h')}$$

and $t'(h') = a$ and $t''(h') = b$. Focusing on the remaining coordinates we see that $t'(m) = t''(m)$ for $m \neq h'$ and one direction of the implication is proved.

We prove the other direction coordinatewise. We consider cases with respect to the relation between a coordinate and h'

- If $m < h'-1$, then

$$\begin{aligned} F^c_{iabDj}(\delta^k_{h'}, \delta^k_{h'-1}, \Psi_k\langle t', h', s'\rangle)(m) &= \\ &= F^c_{iabDj}(L, L, L_{(i,a),t'(m)}) \\ &= L_{(j,c),t'(m)} = \Psi_k\langle t'', h'', s''\rangle(m). \end{aligned}$$

- If $m = h'-1$, then

$$\begin{aligned} F^c_{iabDj}(\delta^k_{h'}, \delta^k_{h'-1}, \Psi_k\langle t', h', s'\rangle)(m) &= \\ &= F^c_{iabDj}(L, H, L_{(i,a),c}) \\ &= H_{(j,c),c} = \Psi_k\langle t'', h'', s''\rangle(m). \end{aligned}$$

- If $m = h'$, then

$$\begin{aligned} F^c_{iabDj}(\delta^k_{h'}, \delta^k_{h'-1}, \Psi_k\langle t', h', s'\rangle)(m) &= \\ &= F^c_{iabDj}(H, R, H_{(i,a),a}) \\ &= R_{(j,c),b} = \Psi_k\langle t'', h'', s''\rangle(m). \end{aligned}$$

- if $m > h'$, then

$$\begin{aligned} F^c_{iabDj}(\delta^k_{h'}, \delta^k_{h'-1}, \Psi_k\langle t', h', s'\rangle)(m) = \\ &= F^c_{iabDj}(R, R, R_{(i,a),t'(m)}) \\ &= R_{(j,c),t'(m)} = \Psi_k\langle t'', h'', s''\rangle(m). \end{aligned}$$

This completes the proof of the lemma. □

We next prove a more interesting lemma.

Lemma VII.3. *For any f', f'' in $\mathbf{C}(\mathbf{T})^k$ and any basic operations $F^c_I(x, y, z)$ of the algebra, if $f''(m) \neq \perp$ for all $0 \leq m < k$ and*

$$F^c_I(\delta^k_{l'}, \delta^k_{l''}, f') = f''$$

for some $0 \leq l', l'' < k$; then there exist $t', t'' \in \mathcal{L}^k$ and $0 \leq s', s'' \leq l$ such that

$$f' = \Psi_k\langle t', l', s'\rangle \text{ and } f'' = \Psi_k\langle t'', l'', s''\rangle.$$

Proof. The fact that $f''(m) \neq \perp$, for all $0 \leq m < k$, immediately implies that $\rho(\delta^k_{l'}) = \rho(f')$ and $\rho(\delta^k_{l''}) = \rho(f'')$. If $I = iabDj$, then we immediately get

$$f'(m) \in \{L_{(i,a),d}, H_{(i,a),a}, R_{(i,a),d} |\ d \in \mathcal{L}\}$$

and

$$f''(m) \in \{L_{(j,c),d}, H_{(j,c),c}, R_{(j,c),d} |\ d \in \mathcal{L}\}.$$

This implies the existence of appropriate t' and t'' in $\mathcal{L}^k$. Putting $s' = i$ and $s'' = j$, we obtain the required configurations and the lemma is proved. □

We define a "computation" of the Turing machine $\mathbf{T}$ to be a sequence of configurations $\langle t_m, h_m, s_m\rangle$ such that for any m

$$\langle t_m, h_m, s_m\rangle \lhd^{c_m}_{I_m} \langle t_{m+1}, h_{m+1}, s_{m+1}\rangle$$

for some c_m in $\mathcal{L}$ and I_m instruction of $\mathbf{T}$. Note that such a sequence is a formal way of describing computations of a Turing machine on a bounded tape. For any computation we introduce a computation term recursively. If a computation comp of dimension k has length one, we put

$$r_{\text{comp}}(x_0, \ldots, x_{k-1}, y) = y.$$

If two computations comp′ and comp″ are such that the last element of comp′ is $\langle t', h', s'\rangle$, the first element of comp″ is $\langle t'', h'', s''\rangle$ and

$$\langle t', h', s'\rangle \lhd^{c}_{iabDj} \langle t'', h'', s''\rangle,$$

we put

$$r_{(\mathrm{comp}' \circ \mathrm{comp}'')}(\bar{x}, y) = r_{\mathrm{comp}''}\big(\bar{x}, F^{c}_{iabDj}(x_{h'}, x_{h''}, r_{\mathrm{comp}'}(\bar{x}, y))\big).$$

We call the first element of the computation an "initial" configuration, and a last element a "terminal" configuration. Note that Lemma VII.2 has a counterpart dealing with computation terms:

Corollary VII.4. *For any two configurations $\langle t', h', s'\rangle$ and $\langle t'', h'', s''\rangle$ of dimension k, $\langle t', h', s'\rangle$ is an initial and $\langle t'', h'', s''\rangle$ a terminal configuration of a computation* comp *if and only if for the computation term $r(\bar{x}, y)$ corresponding to the computation* comp*, we have*

$$\Psi_k\langle t'', h'', s''\rangle = r(\delta^k_0, \ldots, \delta^k_{k-1}, \Psi_k\langle t', h', s'\rangle).$$

We define an additional construction connected with computations of machine $\mathbf{T}$, namely the notion of a computation algebra.

Definition VII.5. A computation algebra of dimension k for a machine $\mathbf{T}$ is an algebra $\mathbf{C}$ in the signature of $\mathbf{C}(\mathbf{T})$ whose underlying set C consists of $\perp$, elements $\Delta^k_0, \ldots, \Delta^k_{k-1}$ and an arbitrary collection of configurations of $\mathbf{T}$ of dimension k. The operations of the algebra are defined according to the following scheme:

$$F^{c}_{iabDj}(\Delta^k_{h'}, \Delta^k_{h''}, \langle t', h', s'\rangle) = \langle t'', h'', s''\rangle$$

if and only if

$$\langle t', h', s'\rangle \lhd^{c}_{iabDj} \langle t'', h'', s''\rangle,$$

and $\langle t', h', s'\rangle, \langle t'', h'', s''\rangle \in C$. Thus, all the other applications of operations of $\mathbf{C}(\mathbf{T})$ are equal to $\perp$.

For any k we introduce a function Ψ'_k defined on any computation algebra of dimension k take away $\perp$ into $\mathbf{C}(\mathbf{T})^k$ by

$$\Psi'_k(a) = \begin{cases} \Psi_k(a) & \text{if } a \text{ is a configuration of } \mathbf{T}, \\ \delta^k_i & \text{if } a = \Delta^k_i. \end{cases}$$

Now, for any computation algebra $\mathbf{C}$ of dimension k we let C_1^* be the Ψ'_k image of $C \setminus \{\bot\}$. In view of Lemmas VII.2 and VII.3, we have for any $a, b, c \in C$,

$$F(a,b,c) \neq \bot \text{ if and only if } F(\Psi'_k(a), \Psi'_k(b), \Psi'_k(c)) \in C_1^*,$$

and in such a case

$$\Psi'_k(F(a,b,c)) = F(\Psi'_k(a), \Psi'_k(b), \Psi'_k(c)).$$

Let $\mathbf{C}^*$ denote the subalgebra of $\mathbf{C}(\mathbf{T})^k$ generated by C_1^* and let C_0^* be the set of elements of C^* of support different then k. We define C^+ by

$$C^+ = \left\{ \Psi_k\langle t_n, h_n, i_n\rangle \mid \text{s.t. } \langle t_0, h_0, i_0\rangle \lhd_{I_0}^{c_0} \cdots \lhd_{I_{n-1}}^{c_{n-1}} \langle t_n, h_n, i_n\rangle;\ \langle t_0, h_0, i_0\rangle \in C \right\} \setminus C_1^*.$$

By Lemma VII.2 we get $C^+ \subseteq C^*$ and by Lemma VII.3 we obtain

$$C^* = C_0^* \cup C_1^* \cup C^+.$$

We define an equivalence relation θ on $\mathbf{C}^*$ by $a\theta b$ if and only if $a = b$ or $a, b \notin C_1^*$.

Corollary VII.6. *The following conditions are equivalent:*

1. *The relation θ is a congruence.*
2. *The set $C_0^* \cup C^+$ is absorbing for $\mathbf{C}^*$.*
3. *There is no sequence of configurations of dimension k such that*

$$\langle t_0, h_0, i_0\rangle \lhd_{I_0}^{c_0} \cdots \lhd_{I_{n-1}}^{c_{n-1}} \langle t_n, h_n, i_n\rangle,$$

$\langle t_0, h_0, i_0\rangle, \langle t_n, h_n, i_n\rangle \in C$ *and* $\langle t_j, h_j, i_j\rangle \notin C$ *for some* j.

We leave the proof of this corollary as an exercise for the reader and introduce a more general definition.

Definition VII.7. Let A' be a subset of the universe of an algebra $\mathbf{A}$ and let $a \in A'$. We call an element a a root of A', if for any $b \in A' \setminus \{a\}$ there exists a non-trivial (not equal to a single variable) term $r(x_0, \dots, x_{n-1})$ (where each of x_j appears in this term) of the algebra $\mathbf{A}$, and elements $a_0, \dots, a_{n-1} \in A'$ such that

$$r(a_0, \dots, a_{n-1}) = a$$

and there is j such that $a_j = b$.

We call a computation algebra rooted if its underlying set take away $\perp$ is rooted. Note that a rooted computation algebra has a very simple intuitive meaning. Namely it consists of configurations of some fixed dimension such that there exists a configuration that can be reached from any other configuration via a computation of the machine $\mathbf{T}$. We remark that for a dimension greater than two, every rooted computation algebra has at least five elements, such that two of them are configurations. Here is an interesting property of rooted computation algebras.

Theorem VII.8. *If two rooted computation algebras of dimensions greater than two are isomorphic, then they are equal.*

We will prove the theorem in a way that will also give us an efficient (working in polynomial time) algorithm for deciding if a given algebra in the signature of $\mathbf{C}(\mathbf{T})$ is isomorphic to a rooted computation algebra of dimension greater then two and, if so, construct this computation algebra.

Proof. For any non-trivial, rooted algebra in a signature of $\mathbf{C}(\mathbf{T})$, we present a construction that produces a unique computation algebra isomorphic with a given algebra or fails when a given algebra is not isomorphic to any computation algebra. We start with an arbitrary algebra $\mathbf{D}$ in the signature of $\mathbf{C}(\mathbf{T})$. It is easy to check whether $\mathbf{D}$ has an absorbing element, and if $\mathbf{D} \setminus \{\perp\}$ is rooted. If so, we denote a root of $D \setminus \{\perp\}$ by f_r. Our next step is to divide the universe of the algebra $\mathbf{D} \setminus \{\perp\}$ into two disjoint sets D_1 and D_2 such that for any $F(x, y, z)$ we have

$$F(a, b, c) \neq \perp \text{ implies } \big(a, b \in D_1 \text{ and } c, F(a, b, c) \in D_2\big).$$

If the algebra $\mathbf{D}$ is isomorphic to a computation algebra, the set D_1 corresponds to the set of Δ_i^k and the set D_2 corresponds to the set of configurations. Moreover D_1 and D_2 are unique in this case. If the algebra $\mathbf{D}$ does not allow such a decomposition then it is not a computation algebra. The cardinality of the set D_1 is the only possible choice for the dimension of $\mathbf{D}$.

The next step is to define a function e from D_2 into integers. The function is defined recursively as follows:

$$e(f_r) = 0$$
$$e(f) = \begin{cases} l+1 & \text{if } e(F^c_{iabLj}(g, h, f)) = l, \\ l-1 & \text{if } e(F^c_{iabRj}(g, h, f)) = l. \end{cases}$$

Note that the image of the function is an interval in the integers. If the algebra $\mathbf{D}$ is isomorphic to a computation algebra and the element f_r corresponds to $\langle t_r, h_r, i_r\rangle$, then function e for each configuration $\langle t, h, i\rangle$ returns $h - h_r$. If for an algebra $\mathbf{D}$ such a function is not well defined, then the algebra is not isomorphic to any computation algebra. We modify the function e by adding to it a smallest constant function that makes all the values non-negative. We denote such obtained function by e'.

We extend the definition of e' to the set D_1 in the following way:

$$e'(f) = \begin{cases} l & \text{if } F(f, g, h) \neq \bot \text{ and } e'(h) = l, \\ l & \text{if } F(g, f, h) \neq \bot \text{ and } e'(F(g, f, h)) = l. \end{cases}$$

If the algebra $\mathbf{D}$ is isomorphic to a computation algebra then the function e' on D_1 fully determines the sequence $\Delta_0^k, \ldots, \Delta_{k-1}^k$. If the algebra $\mathbf{D}$ does not allow such a function then it is not isomorphic to any computation algebra. We have proved that any two isomorphic rooted computation algebras have the same dimension. Moreover a homomorphism has to fix Δ_i^k and preserve the same head position (since it is recognized by e').

Now we construct a candidate for a root, corresponding to f_r, of a computation algebra that may be isomorphic to $\mathbf{D}$. We denote this candidate by $\langle t_r, h_r, i_r\rangle$, and for any $0 \leq s < k$ we find a sequence of elements $f_0, \ldots, f_{n-1} \in D_2$ such that $f_{n-1} = f_r$ and $e'(f_0) = s$ and $e'(f_m) \neq s$ for $m \neq 0$, and such that for any m there are g_m, h_m and $F(x, y, z)$ such that $F(g_m, h_m, f_m) = f_{m+1}$. For $s \neq e'(f_r)$ let $F^c_{iabDj}(x, y, z)$ be such that $F^c_{iabDj}(g_0, h_0, f_0) = f_1$. We construct a tape by putting $t_r(s) = b$. If $s = e'(f_r)$ then since dimension of $\mathbf{D}$ is greater than two we get $F^c_{iabDj}(\Delta_m^k, \Delta_s^k, g) = f_r$ for some $g \in D_2$ and then we put $t_r(s) = c$. If the algebra $\mathbf{D}$ is isomorphic to a computation algebra then such a tape has to be equal to t'_r where $\langle t'_r, h'_r, i'_r\rangle$ is the element sent to f_r. Thus we fully determine the root element of the algebra. If $\mathbf{D}$ does not allow such a construction then it is not isomorphic to any computation algebra. We constructed the root of the algebra, and the elements Δ_i^k. Since the algebra is rooted, and for any configuration $\langle t, h, s\rangle$ and any $\lhd^c_{iabDj}$ if $x \lhd^c_{iabDj} \langle t, h, s\rangle$ then x is unique we reconstruct the whole computation algebra isomorphic to $\mathbf{D}$, or prove that such an algebra does not exist. Moreover, since from the structure there is a unique way of reconstructing the computation algebra we prove that two isomorphic rooted computation algebras are equal. □

We finish this section with a useful corollary.

Corollary VII.9. *Any non-trivial rooted subset C_1^* of $\mathbf{C}(\mathbf{T})^k$, such that*

$$\{\delta_0^k, \ldots, \delta_{k-1}^k\} = C_1^* \cap \mathcal{M}^k$$

and C_1^* consists of elements of full support only, is a Ψ'_k image of $C \setminus \{\perp\}$ for some computation algebra $\mathbf{C}$.

A proof of this corollary is an obvious application of Lemmas VII.2 and VII.3 together with the definition of Ψ'_k.

VIII An algebra with a PSPACE-complete membership problem

In this section, for an arbitrary Turing machine $\mathbf{T}$, we construct a corresponding to it algebra that is similar to Ralph McKenzie's $\mathbf{A}(\mathbf{T})$. Throughout this section we work with a machine $\mathbf{T}$ and for simplicity sake we require that the halting state of $\mathbf{T}$ is the state 0. We will use the algebra $\mathbf{C}(\mathbf{T})$ as constructed in Section VII and define a new algebra $\mathbf{P}(\mathbf{T})$ (or simply $\mathbf{P}$). The universe of $\mathbf{P}$ is a disjoint union of $\mathcal{M}$, $\{\perp\}$ and two copies of the set V from the definition of $\mathbf{C}(\mathbf{T})$. Formally

$$P = \mathcal{M} \cup V \cup \tilde{V} \cup \{\perp\}.$$

In accordance with Ross Willard's construction, we put $Y = \mathcal{M}$, $X_0 = V$, $X_1 = \tilde{V}$ and define δ and ν in a natural way. Note that the set A_0 of the algebra $\mathbf{P}(\mathbf{T})$ is the universe of $\mathbf{C}(\mathbf{T})$.

The set of basic operations of $\mathbf{P}$ includes $\wedge$ and $\perp$, which make P a flat semilattice with bottom element $\perp$. We consider all the operations of $\mathbf{C}(\mathbf{T})$ as operations defined on A_0. There is a unique extension of each such operation on P that complies with the requirements imposed on the operations of the set $\mathcal{A}$. These are the only members of the set $\mathcal{A}$.

We define a function π from $P \setminus \{\perp\}$ onto $\mathcal{M}$ to return the main symbol of the element of the algebra. Note that the definition of the operations of $\mathbf{C}(\mathbf{T})$ implies that each of the operations in $\mathcal{A}$ is either in $\mathcal{A}_{\prec}$ or in $\mathcal{A}_{\succ}$ and $\mathcal{A}_{=}$ is empty.

The set $\mathcal{B}$ consists of $S_2(u, v, x, y, z)$ as defined in Condition 3, $S_1(v, x, y, z)$ as defined in Condition 10 and one extra operation $S^*(v, x, y, z)$ defined by

$$S^*(v, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } \nu(v) \in \{L_{(0,a),b}, H_{(0,a),a}, R_{(0,a),b} |\ a, b \in \mathcal{L}\} \\ \perp & \text{otherwise.} \end{cases}$$

The set $\mathcal{C}$ consists of all the operations defined according to Conditions 8 and 9 for the operations of $\mathcal{A}_{\prec} \cup \mathcal{A}_{\succ}$. The set $\mathcal{D}$ is empty and ν is the member of the set $\mathcal{E}$, moreover the operations $J(x, y, z)$ and $J'(x, y, z)$ are present in the algebra.

We leave it as an easy exercise for the reader to check that the algebra $\mathbf{P}$ complies with Conditions 1-10 including Condition 2a.

We proceed to characterize the large subdirectly irreducible algebras in the variety generated by $\mathbf{P}(\mathbf{T})$. We claim that the algebra $\mathbf{S}$ is a large s.i. in the variety generated by $\mathbf{P}(\mathbf{T})$ if and only if all of the following conditions hold

1. The operations $\wedge$ and $\perp$ make S a flat semillatice with $\perp$ as its bottom element.

2. All the operations of $\mathcal{B}$ are constantly equal to $\perp$.

3. For any $T(\bar{x}) \in \mathcal{C}$ and corresponding $\hat{t}(\bar{x})$, we have

$$\mathbf{S} \models T(\bar{x}) \approx \hat{t}(\bar{x}).$$

4. We have

$$\mathbf{S} \models \nu(x) \approx x.$$

5. We have

$$\mathbf{S} \models J(x, y, z) \approx x \wedge y.$$

6. We have

$$\mathbf{S} \models J'(x, y, z) \approx x \wedge y \wedge z.$$

7. The $\mathcal{A}$ reduct of the algebra $\mathbf{S}$ is isomorphic to a non–trivial, rooted computation algebra such that

   (a) for any computation

   $$\langle t_0, h_0, i_0 \rangle \lhd_{I_0}^{c_0} \cdots \lhd_{I_k}^{c_k} \langle t_{k+1}, h_{k+1}, i_{k+1} \rangle,$$

   if $\langle t_0, h_0, i_0 \rangle$ and $\langle t_{k+1}, h_{k+1}, i_{k+1} \rangle$ are in the computation algebra, then so are $\langle t_j, h_j, i_j \rangle$ for all j;

   (b) there is no computation

   $$\langle t_0, h_0, i_0 \rangle \lhd_{I_0}^{c_0} \cdots \lhd_{I_k}^{c_k} \langle t_{h+1}, h_{k+1}, 0 \rangle$$

   such that $\langle t_0, h_0, i_0 \rangle$ is inside the computation algebra.

First we show that an algebra that satisfies conditions 1 to 7 is, in fact, in the variety generated by $\mathbf{P}(\mathbf{T})$. To do so, we fix a non-trivial rooted computation algebra $\mathbf{D}$ and denote by B_1 the Ψ'_k image of $D \setminus \{\bot\}$ in $\mathbf{C}(\mathbf{T})^k$. Note that $B_1 \subseteq A_0^k$. We denote the the subalgebra of $\mathbf{P}^k$ generated from B_1 by $\mathbf{B}$ and the set of elements of B of support different than k by B_0 and establish the following claim.

Claim. In the algebra $\mathbf{B}$,

$$B = B_0 \cup B_1 \cup B^*$$

where

$$B^* = \{\Psi_k(\alpha_{m+1})| \text{ s.t. } \alpha_0 \lhd_{I_0}^{c_0} \cdots \lhd_{I_m}^{c_m} \alpha_{m+1} \text{ for some } \alpha_0 \in D\}.$$

Proof. By Lemma VII.2, the right hand side is a subset of B; so it suffices to prove that it is closed under all basic operations of $\mathbf{P}(\mathbf{T})$. It is obviously closed under $\wedge$ and $\bot$. The set B_0 is absorbing for all the operations of $\mathcal{A}$. Lemmas VII.2 and VII.3 imply that all the images of operations from $\mathcal{A}$ that are not in B_0 are in $B_1 \cup B^*$. Thus the right hand side is closed with respect to the operations of $\mathcal{A}$.

There are no elements of full support in the right hand side with a barred element in the range, so the range of $S_2(u, v, x, y, z)$ is contained in B_0. There are no elements of full support with range mapped by π into $\{L, R\}$, hence the range of $S_1(v, x, y, z)$ is included in B_0 as well. Condition 7b implies that the range of $S^*(v, x, y, z)$ is fully in B_0 as well.

As for the elements of $\mathcal{C}$, since δ_i^k are the only elements of the right hand side in Y^k, we obtain for every $T(v, x, y, z)$ (and a corresponding $F(x, y, z) \in \mathcal{A}$) that $T(a, b, c, d) \in B_0 \cup \{F(b, c, d)\}$. This reduces the reasoning to the case of the operations from $\mathcal{A}$, which has already been handled.

For any $a \in B_1 \cup B^*$, we have $\nu(a) = a$. Moreover $\nu(B_0) \subseteq B_0$ and the right hand side is closed with respect to $\nu(x)$.

The case of operations $J(x, y, z)$ and $J'(x, y, z)$ is trivial since we have $J(x, y, z) \leq x$ and $J'(x, y, z) \leq x$. The claim is proved. □

Before we proceed we need to establish an additional claim.

Claim. For all the basic operations of $\mathbf{B}$ other than $J(x, y, z)$, the set $B \setminus B_1$ is absorbing.

Proof. The set is clearly absorbing for $\wedge$ and $\bot$. Condition 7a, via Lemmas VII.2 and VII.3, implies that it is absorbing for all the operations of the set $\mathcal{A}$. By the proof of the previous claim we know that range of all the operations of the set $\mathcal{B}$ is in B_0. Likewise the case of the operations of the set $\mathcal{C}$ reduces to $\mathcal{A}$. As seen in the previous proof, $\nu(x)$ preserves $B_0 \cup B^*$. Since no element of full support has a member of X_1 in its range we get the required property for $J'(x, y, z)$. □

We define an equivalence relation θ on B by declaring that $a\theta b$ if and only if $a = b$ or $\{a, b\} \cap B_1 = \emptyset$. By the last claim, the relation θ is respected by all the basic operations of the algebra $\mathbf{B}$ except possibly $J(x, y, z)$. On the other hand, if $J(a, b, c) \in B_1$ then $a = b \in B_1$ and further $\mathbf{B} \models J(a, a, z) \approx a$. Thus the relation θ is a congruence of $\mathbf{B}$ and it is a routine exercise to check that $\mathbf{B}/\theta$ is isomorphic to a required subdirectly irreducible algebra obtained from $\mathbf{D}$.

To complete the description of large subdirectly irreducible algebras in the variety, we need to show that an arbitrary large s.i. satisfies conditions 1 to 7. We fix an arbitrary large s.i. $\mathbf{S}$. Find $\mathbf{B} \leq \mathbf{A}^k$ such that $\mathbf{S}$ is isomorphic to $\mathbf{B}/\theta$ and $k = \dim_{\mathbf{A}}(\mathbf{S})$. Corollarys IV.11 and IV.12 describe the structure of the congruence θ on $\mathbf{B}$ and instantly imply conditions 1 to 6. We can assume that $\mathbf{B}$ is generated by B_1 as defined in Definition IV.5. Moreover by Corollary IV.12 and the domain-range consideration for the operations of the set $\mathcal{A}$, we imply that $B_1 \subseteq \mathcal{M}^k \cup X_0^k$. Note that the definition of B_1 with respect to p states that p is a root for B_1. To obtain condition 7 we need a following claim

Claim. We have for the algebra $\mathbf{B}$,

$$B \cap \mathcal{M}^k = \{\delta_0^k, \ldots, \delta_{k-1}^k\}$$

Proof. Let $k' \leq k$ denote the number of elements in $B \cap \mathcal{M}^k$, which we denote by $\zeta_0, \ldots, \zeta_{k'-1}$. For a proof by contradiction, we suppose that m_0 and m_1 denote coordinates such that for some ζ_m we have $\zeta_m(m_0) = \zeta_m(m_1) = H$. Since the π images of ζ_i are sequentiable, we infer that for any i we have $\zeta_i(m_0) = \zeta_i(m_1)$. Since $\zeta_m \in B_1$, there is an operation of the set $\mathcal{A}$ such that $F(\zeta_m, \zeta_i, f) = g \in B_1$ or $F(\zeta_i, \zeta_m, f) = g \in B_1$, for some $f \in B_1 \cap X^k$ and some i. By Corollary VII.1, we get that $g(m_0) = g(m_1)$ and $p(m_0) = p(m_1)$. Since all the operations of the set $\mathcal{A}$ are injective (in the sense of Corollary VII.1) it follows that for any element $h \in B_1$ we have $h(m_0) = h(m_1)$. Since B_1 generates the algebra $\mathbf{B}$ the same property holds for all the elements of B. This fact contradicts the assumption that $k = \dim_{\mathbf{A}}(\mathbf{S})$.

The other possibility of falsifying our claim is to assume that there exists a coordinate m_0 such that $\zeta_i(m_0) \neq H$ for all i. By Propositions V.4 and V.2, we infer that either $\zeta_i(m_0) = L$ for all i, or $\zeta_i(m_0) = R$ for all i. Suppose, without loss of generality, the first case. Let us denote by B^* the set of all the elements of $\mathbf{B}$ that are of full support and are generated from B_1 by applying the operations from $\mathcal{A}$. Note that, by the non-triviality of B_1 and the definition of operations of $\mathbf{C}(\mathbf{T})$, for each $f \in B^*$ there exists a and i such that

$$\{f(m) \mid 0 \leq m < k\} \subseteq \{L_{(i,a),b}, H_{(i,a),a}, R_{(i,a),b} | \; b \in \mathcal{L}\}.$$

We certainly have $p = L_{(a,i),c}$ for some $a, c \in \mathcal{L}$ and $i \leq l$. By the fact that B_1 is rooted, for any $f \in B^*$ there are $b \in \mathcal{L}$ and $i \leq l$ such that $f(m_0) = L_{(b,j),c}$. Since $B_1 \subseteq A_0^k$ we immediately infer that $B^* \subseteq A_0^k$ and so for any $f, g \in B^*$ if $f(m_0) \neq g(m_0)$ then $f(i) \neq g(i)$ for all i.

Let B_0^* be the subset of B such that

$$B_0^* = \{f \in B \,|\, \text{such that } f(i) = \bot \text{ for some } i \neq m_0\}.$$

Our subclaim is that $B = B_1 \cup B^* \cup B_0^*$. It suffices to prove that the right hand side is closed with respect to all the basic operations of $\mathbf{B}$, and to do so we consider cases.

Consider $e \wedge f$ for some members e, f of the right hand side of the equality. If $e \in B_0^*$ or $f \in B_0^*$ then $e \wedge f \in B_0^*$ as well. On the other hand, if $e, f \in B_1 \cup B^*$ and $e \neq f$ then there is i such that $e(i) \wedge f(i) = \bot$. The structure of the elements of $B_1 \cup B^*$ implies that if $e(m_0) \wedge f(m_0) = \bot$ then $e(i) \wedge f(i) = \bot$ for all i.

For an operation $F(x, y, z) \in \mathcal{A}$ if one of the arguments is in B_0^* then so is the result. We assume that $e, f, g \in B_1 \cup B^*$ and $F(e, f, g) \notin B_1 \cup B^*$, but then (by definition of B^*) there is i such that $F(e, f, g)(i) = \bot$. Once again an analysis of the definitions of the operations of $\mathbf{C}(\mathbf{T})$ implies that if $F(e, f, g)(m_0) = \bot$ then $F(e, f, g) = \bot^k$.

We consider the operations of the set $\mathcal{B}$. For $S_2(u, v, x, y, z)$, if u and v are evaluated to elements of B_0^* then the result of the operation is in B_0^* as well. On the other hand, for any $e, f \in B_1 \cup B^*$ we have $\mathbf{B} \models S_2(e, f, x, y, z) \approx \bot^k$ since no member of B^* has a barred element in the range. For any $d \in B_1 \cup B^*$, we have $\pi(d(m_0)) = L$, so, by Proposition IV.3, there exists $s \neq m_0$ such that $\pi(d(s)) = H$ and $\mathbf{P} \models S_1(d(s), x, y, z) \approx \bot$. On the other hand, if the first argument is in B_0^* then so is the result. The same applies to $S^*(v, x, y, z)$, so it suffices to consider $d \in B_1 \cup B^*$. Then, the structure of the elements of $B_1 \cup B^*$ and Proposition IV.3 imply that $\mathbf{B} \models S^*(d, x, y, z) \approx \bot^k$.

For $T(v, x, y, z) \in \mathcal{C}$ if one of the arguments is in B_0^* then so is the result. Consider then $e, f, g, h \in B_1 \cup B^*$. If $T(e, f, g, h)$ is of full support then $T(e, f, g, h) = F(f, g, h) \in B^*$. Otherwise, if $T(e, f, g, h)(m_0) = \bot$ then $T(e, f, g, h) = \bot^k$.

The operation $\nu(x)$ is equal to the identity on B^* and $\bot$ is its absorbing element. If one of the first two arguments of $J(x, y, z)$ is evaluated in B_0^* then the result of the operation is in B_0^* as well. On the other hand, for any $e, f \in B_1 \cup B^*$ we have $\mathbf{B} \models J(e, f, x) \approx e \wedge f$; this case was handled while considering the operation $\wedge$. Similarly for $J'(x, y, z)$ and $e, f \in B_1 \cup B^*$ we have $\mathbf{B} \models J'(e, f, x) \approx e \wedge f \wedge x$..

The fact that $B = B_1 \cup B_0^* \cup B^*$ implies that there exist no elements $e, f \in B$ such that f is not congruent modulo θ to g and that $f(i) = g(i)$ for all $i \neq m_0$ – this contradicts the assumption that $k = \dim_{\mathbf{A}}(\mathbf{S})$. Thus the claim is proved. $\square$

Having established the claim, we immediately obtain, via Corollary VII.9, a non-trivial, rooted computation algebra mapped by Ψ'_k onto B_1. Condition 7a is the consequence of Lemma VII.2 and Corollary VII.6. Condition 7b is implicit, via Lemma VII.2, from Proposition IV.3. Thus our characterization of large s.i. algebras in the variety generated by $\mathbf{P}(\mathbf{T})$ is complete.

We present a NPSPACE algorithm deciding for a given algebra $\mathbf{A}$ if $\mathbf{A} \notin \mathrm{HSP}(\mathbf{P}(\mathbf{T}))$, since by [Sav70] NPSPACE=PSPACE we have also a PSPACE algorithm deciding the same question. Since PSPACE is a deterministic class we can decide in the same class its negation. The algorithm has as input a finite algebra $\mathbf{A}$ in the signature of $\mathbf{P}(\mathbf{T})$.

1. Decompose, in polynomial time, the algebra $\mathbf{A}$ into its subdirectly irreducible factors using Demel's polynomial time algorithm from [Dem82].

2. For each factor $\mathbf{S}$,

   - check if $\mathbf{S}$ is one of a finite number of small s.i. algebras

     YES proceed to next factor

     NO proceed to next step

   - check if $\mathbf{S}$ complies with conditions 1 to 6

     YES proceed to next step

     NO STOP, answer $\mathbf{A} \notin \mathrm{HSP}(\mathbf{P}(\mathbf{T}))$

   - construct the computation algebra for the $\mathcal{A}$ reduct of $\mathbf{S}$ (as in a proof of Theorem VII.8) and, using non-determinicity, check whether there is a sequence of computations that contradicts conditions 7a and 7b

     YES STOP, answer $\mathbf{A} \notin \mathrm{HSP}(\mathbf{P}(\mathbf{T}))$

     NO proceed to the next factor

3. The algorithm produces no answer.

To construct an algebra with a PSPACE-complete membership problem, start with a Turing machine $\mathbf{T}$ that solves a PSPACE-complete problem with an input w on a bounded tape equal to $w \circ \lambda^{C|w|^n}$, where λ denotes an empty position on a tape, for some constants C and n. Modify the machine $\mathbf{T}$ to obtain $\mathbf{T}'$ with following properties:

- There is a new selected letter η in the tape alphabet of the machine $\mathbf{T}'$ and a new constant C'.

- For any input, a computation of the machine $\mathbf{T}'$ consists of two parts. During the initial part, the machine starts the computation on η, moves left until it finds next

η, and then begins the main part of the computation which is basically a simulation of $\mathbf{T}$. The states used for the initial part cannot be used for the main part of the computation.

- If the original machine $\mathbf{T}$ answers yes on input w, then the machine $\mathbf{T}'$, working on a bounded tape consisting of $\eta \circ w \circ \lambda^{C'|w|^n} \circ \eta$, will never reach state 0.
- If the original machine $\mathbf{T}$ answers no on input w, then the machine $\mathbf{T}'$, reaches state 0 on a bounded tape $\eta \circ w \circ \lambda^{C'|w|^n} \circ \eta$.

After constructing $\mathbf{T}'$, we produce an algebra $\mathbf{P}(\mathbf{T}')$. We present a reduction that translates the question answered by $\mathbf{T}$ into a problem of membership for some algebra.

1. Take the input w for the machine $\mathbf{T}$.
2. Produce a word $\eta \circ w \circ \lambda^{C'|w|^n} \circ \eta$.
3. Simulate the computation of the machine $\mathbf{T}'$ until it reads the whole new input and produce a computation algebra for this computation.
4. Construct an algebra complying with conditions 1 to 7 whose $\mathcal{A}$ reduct is the computation algebra constructed in the previous step.

It is obvious that the algebra constructed by such a procedure is in $\mathrm{HSP}(\mathbf{P}(\mathbf{T}'))$ if and only if the machine $\mathbf{T}$ answers yes on w. This gives us a finite algebra that generates a variety with PSPACE-complete membership problem.

IX A finite algebra with EXPSPACE–hard membership problem, and doubly exponential β function.

We are ready to begin the final construction in this paper. We construct an algebra $\mathbf{E}(\mathbf{T})$ for a given Turing machine $\mathbf{T}$ working on states $0, \ldots, l$. We add to the set of states one state–like number -1. The zero state is the halting state of machine $\mathbf{T}$ and state 1 is its initial state. A set $\mathcal{L}'$ is a tape alphabet of machine $\mathbf{T}$, with one selected element λ denoting the blank position on tape. There is an extra letter–like symbol η and $\mathcal{L} = \mathcal{L}' \cup \{\eta\}$. We fix a machine $\mathbf{T}$ and denote $\mathbf{E}(\mathbf{T})$ by $\mathbf{E}$.

The universe of the algebra $\mathbf{E}$ is the union of three pairwise disjoin sets

$$Y = \{Y^{L,0}, Y^{L,1}, Y^{H,0}, Y^{H,1}, Y^{R,0}, Y^{R,1}\},$$

a set

$$X_0 = \{L^{Z,a}_{(i,b),c}, H^{Z,a}_{(i,b),b}, R^{Z,a}_{(i,b),c} | \text{ where } a, b, c \in \mathcal{L},\ Z \in \{L, H, R\} \text{ and } -1 \le i \le l\},$$

a copy of X_0 denoted by X_1

$$X_1 = \{\tilde{L}^{Z,a}_{(i,b),c}, \tilde{H}^{Z,a}_{(i,b),b}, \tilde{R}^{Z,a}_{(i,b),c} | \text{ where } a, b, c \in \mathcal{L},\ Z \in \{L, H, R\} \text{ and } -1 \le i \le l\}$$

and an extra element $\perp$. We define two sets X'_0 and X'_1, subsets of (respectively) X_0 and X_1 to consist of the elements of those sets for which $i = -1$.

We define δ and ν in the natural way and in accordance with Willard's construction. We define a function π to return the first element of the superscript (Z in the definition of the sets X_0 and X_1) and use two auxiliary functions

- a map τ from E into $\{0, 1\} \cup \mathcal{L}$ defined to return the second superscript, and
- a function ρ from X into $\mathcal{M}$ set to return the main symbol of the element.

The basic operations of the algebra $\mathbf{E}(\mathbf{T})$ are the following. The operations $\wedge$ and $\perp$ make E a flat semilattice in accordance with Condition 1. The elements of sets $\mathcal{A}$ and $\mathcal{B}$ we will introduce gradually. The elements of the set $\mathcal{C}$ are the operations defined in Conditions 8 and 9 for the operations of $\mathcal{A}_{\prec} \cup \mathcal{A}_{\succ}$. The operations $J(x, y, z)$ and $J'(x, y, z)$ defined in Condition 6 are present.

We proceed to define a number of basic operations of the set $\mathcal{A}_{\prec}$. These operations are responsible for generating an exponentially long tape for the computation of machine $\mathbf{T}$. The tape is generated using the techniques which were employed in Section VI to produce exponentially growing $\gamma_{\mathbf{A}}$ function. While generating elements for "tape" we use the element η as a marker for applications of operations of certain kind. For each element $a \in \mathcal{L}' \setminus \{\lambda\}$ we define two operations of $\mathbf{E}(\mathbf{T})$, $F^a(x, y, z) \in \mathcal{A}_{\prec}$ and $F^a_{\text{compl}}(x, y, z) \in \mathcal{A}_{\prec}$. We define these operations on the set A_0 to extend it to the whole universe of the algebra $\mathbf{E}(\mathbf{T})$ later.

- For any $a \in \mathcal{L}' \setminus \{\lambda\}$ and any $f, g, h \in A_0$ the following are equivalent:
  - $F^a(f, g, h) \neq \perp$
  - $F^a_{\text{compl}}(f, g, h) \neq \perp$

– both of the following hold

  * we have $\pi(h) = \pi(f) \prec \pi(g)$ and $h \in X_0'$ and $f, g \in Y$
  * if $\pi(h) = H$ then $\tau(h) = a$

- if $F^a\big(Y^{Z_0,i_0}, Y^{Z_1,i_1}, W^{Z_0,d}_{(-1,b),c}\big) \neq \bot$ for some $W, Z_0, Z_1 \in \{L, H, R\}$ and $i_0, i_1 \in \{0,1\}$ and $b, c, d \in \mathcal{L}$ then it's defined according to the following rule

$$F^a\big(Y^{Z_0,i_0}, Y^{Z_1,i_1}, W^{Z_0,d}_{(-1,b),c}\big) = \begin{cases} H^{Z_1,d}_{(-1,a),a} & \text{if } W = H \text{ and } i_0 = 0 \\ R^{Z_1,d}_{(-1,a),c} & \text{if } W = H \text{ and } i_0 = 1 \\ W^{Z_1,d}_{(-1,a),c} & \text{else.} \end{cases}$$

- if $F^a_{\text{compl}}\big(Y^{Z_0,i_0}, Y^{Z_1,i_1}, W^{Z_0,d}_{(-1,b),c}\big) \neq \bot$ for some $W, Z_0, Z_1 \in \{L, H, R\}$ and $i_0, i_1 \in \{0,1\}$ and $b, c, d \in \mathcal{L}$ then it's defined according to the following rule

$$F^a_{\text{compl}}\big(Y^{Z_0,i_0}, Y^{Z_1,i_1}, W^{Z_0,d}_{(-1,b),c}\big) = \begin{cases} L^{Z_1,d}_{(-1,\eta),c} & \text{if } W = H \text{ and } i_0 = 0 \\ H^{Z_1,d}_{(-1,\eta),\eta} & \text{if } W = H \text{ and } i_0 = 1 \\ W^{Z_1,d}_{(-1,\eta),c} & \text{else.} \end{cases}$$

Each of these operations extends from A_0 to E in the usual unique way. We introduce one extra operation of $\mathcal{A}_\prec$ by putting for elements of A_0

$$G^\lambda\big(Y^{Z_0,1}, Y^{Z_1,i}, H^{Z_0,a}_{(-1,\lambda),\lambda}\big) = H^{Z_1,a}_{(-1,\lambda),\lambda}$$

whenever $Z_0 \prec Z_1$ and $Z_0 = H$ implies $a = \lambda$, and putting all the other applications of $G^\lambda(x,y,z)$ to be equal to $\bot$. This operation as well uniquely extends to the whole universe of $\mathbf{E}(\mathbf{T})$. These are the only elements of $\mathcal{A}_\prec$. We advise the reader to seek similarities between operations just defined and the example of the algebra that was introduced in Section VI. Closer examination of the definitions of these operations leads to the following corollary.

Corollary IX.1. *The following are true in* $\mathbf{E}(\mathbf{T})$

- *the only terms in $\mathcal{A}_\prec$ that are not constantly equal to $\bot$ are the ones with a non-trivial subterm appearing only as a last argument of each operation*

- *for two $\mathcal{A}_\prec$ terms $r'(\bar{x})$ and $r''(\bar{x})$ such that one is obtained from the other by substituting some of the appearances of $F^a(x,y,z)$ with $F^a_{\text{compl}}(x,y,z)$ or vice versa we have for all $\bar{f}$*

$$r'(\bar{f}) \neq \bot \textit{ if and only if } r''(\bar{f}) \neq \bot,$$

- *moreover for any pair of such terms, and for each $\bar{f}$ such that $r'(\bar{f}) \neq \bot$ we have $\pi(r'(f)) = \pi(r''(f))$ and $\tau(r'(f)) = \tau(r''(f))$.*

This corollary is to be compared with Corollary VI.1. We proceed to define further operations of the algebra $\mathbf{E}(\mathbf{T})$. There are no operations in the set $\mathcal{A}_{\succ}$. The elements of $\mathcal{A}_{=}$ are responsible for simulating computations of machine $\mathbf{T}$ on a tape generated by the operations from the set $\mathcal{A}_{\prec}$. To define such operations we use the algebra $\mathbf{C}(\mathbf{T})$ defined in Section VII and then for every $F_{\mathbf{C}(\mathbf{T})}(x,y,z)$ operation of this algebra we introduce an operation $F(x,y,z)$ of $\mathbf{E}(\mathbf{T})$ on A_0.

- For any $f,g,h \in A_0$ we have $F(f,g,h) \neq \bot$ only if there are $Z \in \mathcal{M}$ and $a \in \mathcal{L}$ such that
$$f,g \in \{L^{Z,a}_{(-1,b),c}, H^{Z,a}_{(-1,b),b}, R^{Z,a}_{(-1,b),c} | \text{ where } b,c \in \mathcal{L}\}$$
and
$$h \in \{L^{Z,a}_{(i,b),c}, H^{Z,a}_{(i,b),b}, R^{Z,a}_{(i,b),c} | \text{ where } b,c \in \mathcal{L} \setminus \{\eta\},\ 0 \leq i \leq l\}$$
- for all the arguments complying with the previous condition we have
$$F(W^{Z,a}_{(-1,b_w),c_w}, U^{Z,a}_{(-1,b_u),c_u}, V^{Z,a}_{(i,b_v),c_v}) = \\ = \begin{cases} f^{Z,a} & \text{if } f = F_{\mathbf{C}(\mathbf{T})}(W,U,V_{(i,b_v),c_v}) \neq \bot, \\ \bot & \text{else.} \end{cases}$$

All of these operations are uniquely extended to the whole universe of $\mathbf{E}(\mathbf{T})$ in accordance with Condition 2 and create a set $\mathcal{A}'_{=}$. We define a number of maps between certain subsets of E and $C(\mathbf{T})$. We use these maps to model a computation of machine $\mathbf{T}$ in certain subalgebras of cartesian powers of $\mathbf{E}(\mathbf{T})$. For fixed $Z \in \mathcal{M}$ and $a \in \mathcal{L}$ we introduce a map from the set

$$X^{Z,a} = \left\{\tilde{W}^{Z,a}_{(i,b),c}, W^{Z,a}_{(i,b),c} \,\middle|\, \text{either } i=-1 \text{ or } [b \neq \eta \text{ and } c \neq \eta]\right\} \cup \{\bot\}$$

onto $C(\mathbf{T})$ by putting $\Phi^{Z,a}(\bot) = \bot$ and

$$\Phi^{Z,a}\left(W^{Z,a}_{(i,b),c}\right) = \begin{cases} W & \text{if } i=-1, \\ W_{(i,b),c} & \text{if } i \neq -1. \end{cases}$$

Note that every such map is a homomorphism of a $\mathcal{A}'_{=}$ reduct of $X^{Z,a}$ onto $C(\mathbf{T})$ and that it is injective on $X^{Z,a} \cap (X_0 \setminus X')$ and on $X^{Z,a} \cap (X_1 \setminus X')$. We define a final member of the set

$\mathcal{A}_=$ by defining an operation on A_0 that's once more uniquely extend to the whole universe

$$C_1(x,y,z) = \begin{cases} W^{Z,a}_{(1,b),c} & \text{if } x=y=z=W^{Z,a}_{(-1,b),c} \text{ for } a \in \mathcal{L},\ b,c \in \mathcal{L}';\ Z,W \in \mathcal{M} \\ \perp & \text{else.} \end{cases}$$

The operation $C_1(x,y,z)$ is responsible for generating a starting tape for computations of $\mathbf{T}$. This is the last operations of the set $\mathcal{A}$.

In the set $\mathcal{B}$ there is an operation $S_2(u,v,x,y,z)$ as defined in Condition 3 and $S_1(v,x,y,z)$ as defined in Condition 10 for π given above. Moreover we introduce into $\mathcal{B}$ an operation $S^*(v,x,y,z)$ defined to be

$$S^*(v,x,y,z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } \rho(v) \in \{L,R\}, \\ \perp & \text{otherwise.} \end{cases}$$

Finally we define last member of $\mathcal{B}$ denoted by $S^{**}(v,x,y,z)$ to be

$$S^{**}(v,x,y,z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } \nu(v) \in \{W^{Z,c}_{(0,a),b} |\ a,b,c \in \mathcal{L};\ W,Z \in \mathcal{M}\}, \\ \perp & \text{otherwise.} \end{cases}$$

We leave it as an exercise for the reader to check that this algebra $\mathbf{E}(\mathbf{T})$ complies with Conditions 1-10 with an exception for Condition 2a.

Now we are ready to characterize the variety generated by $\mathbf{E}(\mathbf{T})$. We define a special family of terms in the language of $\mathbf{E}(\mathbf{T})$ recursively. Each term is determined by two words of the same length: w' over an alphabet consisting of 0 and 1 and w over $\mathcal{L}' \setminus \{\lambda\}$. Terms are defined in the following way. For the empty word ε we have

$$f^{\varepsilon}_{\varepsilon}(x_{-1},x_0,y) = G^{\lambda}(x_{-1},x_0,y)$$

and for w,w' of length n

$$f^{w}_{w'}(x_{-1},\dots,x_n,y) = \\ = \begin{cases} F^{w(n-1)}(x_{n-1},x_n,f^{w_{[n-1]}}_{w'_{[n-1]}}(x_{-1},\dots,x_{n-1},y)) & \text{if } w'(n-1)=0, \\ F^{w(n-1)}_{\text{compl}}(x_{n-1},x_n,f^{w_{[n-1]}}_{w'_{[n-1]}}(x_{-1},\dots,x_{n-1},y)) & \text{if } w'(n-1)=1. \end{cases}$$

We define another family of terms by putting for every word w of length n over the alphabet $\mathcal{L}' \setminus \{\lambda\}$; $f^w(\bar{x},y) = f^w_{0^n}(\bar{x},y)$. We list a number of results about the structure of terms of $\mathbf{E}(\mathbf{T})$ and it's correlations with computations of machine $\mathbf{T}$.

Proposition IX.2. *For any word w in $\mathcal{L}' \setminus \{\lambda\}$ and for any two words w', w'' in $\{0,1\}$ such that $|w| = |w'| = |w''| = n$ and for any $a_{-1}, \ldots, a_n, b \in E$ the following conditions are equivalent*

- $f^w_{w'}(a_{-1}, \ldots, a_n, b) \neq \bot$,
- $f^w_{w''}(a_{-1}, \ldots, a_n, b) \neq \bot$ *and*
- *all of the following are true*
  - $\tau(a_{-1}) = 1$,
  - $a_i \in Y$ *for any* i, *and*

$$b \in \{H^{Z,a}_{(-1,\lambda),\lambda}, \tilde{H}^{Z,a}_{(-1,\lambda),\lambda} \mid \textit{for some } Z \in \mathcal{M} \textit{ and } a \in \mathcal{L}\},$$

  - $\pi(b) = \pi(a_{-1}) \prec \cdots \prec \pi(a_n)$ *and if* $\pi(a_i) = H$ *for* $i < n$ *then either* $i = -1$ *and* $\tau(b) = \lambda$ *or* $i \geq 0$ *and* $\tau(b) = w(i)$.

Proof. The equivalence of the first two conditions is trivial by Corollary IX.1. It is also easy to see that a violation of any subcondition in the third condition leads to $f^w_{w'}(a_{-1}, \ldots, a_n, b) = \bot$. It remains to prove that if the third condition holds then so do the other two. We prove this fact by induction on n. For $n = 0$ we have $G^\lambda(Y^{Z_0,1}, Y^{Z_1,i}, H^{Z_0,a}_{(-1,\lambda),\lambda}) \neq \bot$ (for some $Z_0 \prec Z_1 \in \mathcal{M}$ and $a \in \mathcal{L}$) or similarly with a last argument in X_1. Assume now that the implication holds for $n-1$ and proceed to prove it for n. For simplicity we consider a case with $w'(n-1) = 0$, then

$$f^w_{w'}(a_{-1}, \ldots, a_n, b) = F^{w(n-1)}\big(a_{n-1}, a_n, f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1}, \ldots, a_{n-1}, b)\big).$$

By inductive assumption $f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1}, \ldots, a_{n-1}, b) \neq \bot$ and this implies that we have

$$\pi(f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1}, \ldots, a_{n-1}, b)) = \pi(a_{n-1}).$$

By parts of the third condition $a_n \in Y$, $a_{n-1} \prec a_n$ and if $\pi(a_{n-1}) = H$ then $\tau(b) = w(n-1)$. The proposition is proved by the remark that

$$\tau(b) = \tau\big(f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1}, \ldots, a_{n-1}, b)\big). \qquad \square$$

Note that this proposition implies that for any two words w in $\mathcal{L}' \setminus \{\lambda\}$ and w' in $\{0,1\}$, any sequentiable sequence of elements of $\mathcal{M}$ of length $n+2$ and for any element $c \in \{0,1\}^n$

we can find $a_{-1}, \ldots, a_n, b \in E$ such that $f^w_{w'}(a_{-1}, \ldots, a_n, b) \neq \perp$ and the π images of the elements $a_{-1}, \ldots, a_n$ are equal to the given sequence and such that $\tau(a_i) = c(i)$ for any $0 \leq i < n$.

Proposition IX.3. *For any $w \in \mathcal{L}^n$ and $w' \in \{0,1\}^n$ and for any $a_{-1}, \ldots, a_n, b \in E$ we denote by w'' the word $\tau(a_0) \cdots \tau(a_{n-1})$ and let m denote the size of a maximal common prefix of w' and w''. If*

$$f^w_{w'}(a_{-1}, \ldots, a_n, b) \neq \perp$$

then

$$\rho(f^w_{w'}(a_{-1}, \ldots, a_n, b)) = \begin{cases} L & \text{if } m < n \text{ and } w'(m) = 1, \\ H & \text{if } m = n, \\ R & \text{if } m < n \text{ and } w'(m) = 0. \end{cases}$$

Moreover,

$$\nu(f^w_{w'}(a_{-1}, \ldots, a_n, b)) =$$
$$= \begin{cases} W^{Z,a}_{(-1,c),\lambda} & \text{if } m = 0, \\ W^{Z,a}_{(-1,c),w(m-1)} & \text{if } m \neq 0 \text{ and } w'(m-1) = 0, \\ W^{Z,a}_{(-1,c),\eta} & \text{if } m \neq 0 \text{ and } w'(m-1) = 1, \end{cases}$$

for some $W, Z \in \mathcal{M}$ and $a, c \in \mathcal{L}$.

Note that in the convention introduced in Section II one can restate the results of the proposition in the following way:

$$\rho(f^w_{w'}(a_{-1}, \ldots, a_n, b)) = \begin{cases} L & \text{if } w'' < w', \\ H & \text{if } w'' = w', \\ R & \text{if } w'' > w'. \end{cases}$$

and, this implies that if $w' + 1$ denotes the successor of w' in the order $<$, we have

$$\rho(f^w_{w'}(a_{-1}, \ldots, a_n, b)) \succ \rho(f^w_{w'+1}(a_{-1}, \ldots, a_{n-1}, b))$$

whenever $f^w_{w'}(a_{-1}, \ldots, a_n, b) \neq \perp$.

Proof. We prove the first claim of the proposition by induction on n. Note that by the definition of $G^\lambda(x, y, z)$ the case of $n = 0$ is trivial. Assume that the proposition holds for $n - 1$, and fix $a_{-1}, \ldots, a_n, b \in E$ and words w and w' such that $f^w_{w'}(a_{-1}, \ldots, a_n, b) \neq \perp$. Let

w'' denotes the word $\tau(a_0)\cdots\tau(a_{n-1})$ and m denotes the length of maximal common prefix of w' and w''. If

$$\rho\big(f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1},\dots,a_{n-1},b)\big)\in\{L,R\}$$

then by the definition of the operations we imply that

$$\rho(f^{w}_{w'}(a_{-1},\dots,a_n,b))=\rho\big(f^{w_{[n-1]}}_{w'_{[n-1]}}(a_0,\dots,a_{n-1},b)\big)$$

and by inductive assumption the claim holds. If, on the other hand,

$$\rho\big(f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1},\dots,a_{n-1},b)\big)=H \text{ then } m\geq n-2.$$

One should consider four cases for four possible evaluations of $w'(n-1)$ and $w''(n-1)$ in $\{0,1\}$. By symmetry without loss of generality we assume that $w'(n-1)=0$ and so

$$f^{w}_{w'}(a_{-1},\dots,a_n,b)=F^{w(n-1)}\big(a_{n-1},a_n,f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1},\dots,a_{n-1},b)\big).$$

If $\tau(a_{n-1})=0$ then $\rho(f^{w}_{w'}(a_{-1},\dots,a_n,b))=H$. On the other hand if $\tau(a_{n-1})=1$ then $\rho(f^{w}_{w'}(a_{-1},\dots,a_n,b))=R$ as claimed.

To prove the second claim of the proposition we remark that $\nu(f^{\varepsilon}_{\varepsilon}(a_{-1},a_0,b))=H^{Z,a}_{(-1,\lambda),\lambda}$ for some $Z\in\mathcal{M}$ and $a\in\mathcal{L}$ so the base case is trivial. Assume that claim holds for $n-1$ and proceed to prove it for n. Define m, w, w' and w'' as in the previous part of the proof. If $m<n$ then $\rho(f^{w}_{w'}(a_{-1},\dots,a_n,b))\in\{L,R\}$ and, by definition of the operations last lower subscripts of $f^{w}_{w'}(a_{-1},\dots,a_n,b)$ and $f^{w_{[n-1]}}_{w'_{[n-1]}}(a_{-1},\dots,a_{n-1},b)$ are equal, so, using the induction assumption, the claim holds. If, on the other hand, $\rho(f^{w}_{w'}(a_{-1},\dots,a_n,b))=H$ then $m=n$ if $w'(n-1)=0$ then

$$\nu(f^{w}_{w'}(a_{-1},\dots,a_n,b))=H^{Z,a}_{(-1,w(n-1)),w(n-1)}$$

On the other hand if $w'(n-1)=1$ then

$$\nu(f^{w}_{w'}(a_{-1},\dots,a_n,b))=H^{Z,a}_{(-1,\eta),\eta} \qquad \square$$

Having the proposition proved we list the first correspondence between defined above terms and elements of cartesian powers of $\mathbf{C}(\mathbf{T})$ used to model computations of the machine $\mathbf{T}$. We recall that the elements δ^k_i of $\mathbf{C}(\mathbf{T})^k$ are defined in Section VII on page 23 and immediately obtain the following corollary

Corollary IX.4. *Whenever* $f^w_{w'}(a_{-1},\ldots,a_n,b) \neq \perp$ *then*

$$\rho\big(f^w_{w'}(a_{-1},\ldots,a_n,b)\big) = \delta^{2^n}_{\varphi(w')}(j),$$

where $\varphi(\tau(a_0)\cdots\tau(a_{n-1})) = j$.

A starting tape for a computation of the machine $\mathbf{T}$ which we model is of special form. With each word w we associate another word denoted by e^{-w} which, under special circumstances, is a starting tape for the machine $\mathbf{T}$. The word e^{-w} is constructed in the following way. If $|w| = n$ then

$$e^{-w} = w(n-1)w(n-2)w(n-3)^2\ldots w(0)^{2^{n-2}}\lambda^{2^{n-1}}.$$

The new word e^{-w} is constructed in such a way that for any $i \in \{0,1\}^n$ for j the length of the maximal prefix of i consisting only of 0 we have $e^{-w}(\varphi(i)) = w(j-1)$ if $j \neq 0$ and $e^{-w}(\varphi(i)) = \lambda$ if $j = 0$.

The following proposition defines a relation between some terms of the algebra $\mathbf{E}(\mathbf{T})$ and computations of the machine $\mathbf{T}$. Using this correspondence we are able to distinguish configurations on which our machine reaches the halting state. Note that due to the definition of $S^{**}(v,x,y,z)$ the halting state of $\mathbf{T}$ is recognized in the algebra $\mathbf{E}(\mathbf{T})$.

Proposition IX.5. *If* $f^w(a_{-1},\ldots,a_n,b) \neq \perp$ *for some elements* $a_{-1},\ldots,a_n,b \in E$ *and word* $w \in (\mathcal{L}' \setminus \{\lambda\})^n$ *then for any computation of dimension* 2^n *of the form*

$$\langle e^{-w},0,1\rangle \lhd^{c_1}_{I_1}\cdots\lhd^{c_k}_{I_k}\langle t_k,h_k,s_k\rangle$$

and corresponding with it term $t(\bar{x},y)$, *putting* $\sigma^w_{w'} = f^w_{w'}(a_{-1},\ldots,a_n,b)$, *and*

$$t(\sigma^w_{0^n},\ldots,\sigma^w_{1^n},C_1(\sigma^w_{0^n},\sigma^w_{0^n},\sigma^w_{0^n})) = c$$

we have

- $c \neq \perp$,
- $\pi(c) = \pi(a_n)$,
- $\tau(c) = \tau(b)$, *and*
- $\Phi^{\pi(c),\tau(c)}(c) = \Psi_{2^n}\langle t_k,h_k,s_k\rangle(j)$ *where* $\varphi(\tau(a_0)\cdots\tau(a_{n-1})) = j$
- $c \in X_0$ *if and only if* $b \in X_0$ *and similarly for* X_1.

Proof. The last item of the proposition is trivial by the structure of considered terms. We prove the remaining claims by induction on k. A first step is to show the claim for $c = C_1(\sigma^w_{0^n}, \sigma^w_{0^n}, \sigma^w_{0^n})$. By Proposition IX.3, and the definition of the operation $C_1(x, y, z)$ we get $c \neq \bot$ and $\pi(c) = \pi(\sigma^w_{0^n}) = \pi(a_{n-1})$. It is trivial that $\tau(c) = \tau(b)$. If $\Phi^{\pi(c),\tau(c)}(c) = W_{(1,d),e}$ then we get $d = w(n-1)$ and by Proposition IX.3 and the remarks about e^{-w} we have $e = e^{-w}(j)$. Finally by Corollary IX.4 we get $\rho(c) = \delta^{2^m}_0(j)$ and the base case is proved.

Next we assume that the proposition is true for computations of length k. For a computation of length $k+1$ and corresponding to it term $t(\bar{x}, y)$ we have by induction assumption

$$c = F^{c_{k+1}}_{I_{k+1}}(\sigma^w_i, \sigma^w_{i'}, c')$$

where $\Phi^{\pi(c'),\tau(c')}(c') = \Psi_{2^n}\langle t_k, h_k, s_k\rangle(j)$ and

$$\langle t_k, h_k, s_k\rangle \lhd^{c_{k+1}}_{I_{k+1}} \langle t_{k+1}, h_{k+1}, s_{k+1}\rangle.$$

By the induction assumption we get

$$\pi(\sigma^w_i) = \pi(\sigma^w_{i'}) = \pi(c') = \pi(a_{n-1}),$$

and similarly

$$\tau(\sigma^w_i) = \tau(\sigma^w_{i'}) = \tau(c') = \tau(b).$$

And since by Lemma VII.2 we have

$$\begin{aligned}\mathbf{C}(\mathbf{T}) \models F^{c_{k+1}}_{I_{k+1}}\big(\delta^{2^m}_{\varphi(i)}(j), \delta^{2^m}_{\varphi(i')}(j), \Psi_{2^n}\langle t_k, h_k, s_k\rangle(j)\big) &= \\ = \Psi_{2^n}\langle t_{k+1}, h_{k+1}, s_{k+1}\rangle(j) &\neq \bot\end{aligned}$$

then, using Corollary IX.4 we get $c \neq \bot$ and the rest follows trivially. □

Given this proposition we can characterize some of the identities that hold in $\mathbf{E}(\mathbf{T})$. The first group of identities consists of equalities between computation terms (as defined in Section VII on page 25) with the same terminal configurations and with initial configurations of special form. Let $r_0(\bar{x}, y)$ and $r_1(\bar{x}, y)$ be computation terms for two computations of dimension 2^n such that for some w of length n we have

$$\langle e^{-w}, 0, 1\rangle \lhd^{c^i_1}_{I^i_1} \cdots \lhd^{c^i_{k_i}}_{I^i_{k_i}} \langle t^i_{k_i}, h^i_{k_i}, s^i_{k_i}\rangle$$

for $r_i(\bar{x}, y)$ and that $\langle t^0_{k_0}, h^0_{k_0}, s^0_{k_0}\rangle = \langle t^1_{k_1}, h^1_{k_1}, s^1_{k_1}\rangle$. Note that by Proposition IX.5 we have

$$\begin{aligned}\mathbf{E}(\mathbf{T}) \models r_0\big(f^w_{0^n}(\bar{x}, y), \dots, f^w_{1^n}(\bar{x}, y), C_1(f^w_{0^n}(\bar{x}, y), f^w_{0^n}(\bar{x}, y), f^w_{0^n}(\bar{x}, y))\big) \approx \\ \approx r_1\big(f^w_{0^n}(\bar{x}, y), \dots, f^w_{1^n}(\bar{x}, y), C_1(f^w_{0^n}(\bar{x}, y), f^w_{0^n}(\bar{x}, y), f^w_{0^n}(\bar{x}, y))\big)\end{aligned}$$

since the result is uniquely determined by $\pi(x_n)$, $\tau(y)$, the last configuration of a computation and a membership of b in X_0. We denote the set consisting of all such identities by $\mathbf{\Sigma}'$. To define one more set of identities on $\mathbf{E}(\mathbf{T})$ we put

$$\begin{aligned}\mathbf{\Sigma} =& \mathbf{\Sigma}' \cup \{x \wedge x \approx x, J(x, x, y) \approx x, J'(x, x, x) \approx x\} \cup \\ &\cup \{T_1(x, x, y, z) \approx F(x, y, z) \approx T_2(y, x, y, z) \mid \text{for corresponding } F \in \mathcal{A}_\prec\}.\end{aligned}$$

We proceed to define a subalgebra of $\mathbf{E}(\mathbf{T})^{2^n}$, which, under certain conditions will allow us to generate subdirectly irreducible algebras in the variety $\mathrm{HSP}(\mathbf{E}(\mathbf{T}))$. Such an algebra is constructed for any word over an alphabet $\mathcal{L}' \setminus \{\lambda\}$. For any word w in $\mathcal{L}' \setminus \{\lambda\}$ of length n we define an algebra $\mathbf{B}(w)$ in the following way. We put $\omega \in E^{2^n}$,

$$\omega(i) = \begin{cases} H^{H,\lambda}_{(-1,\lambda),\lambda} & \text{if } \varphi(i) = 0 \\ H^{L,w(\varphi(i)-1)}_{(-1,\lambda),\lambda} & \text{if } 0 < \varphi(i) \leq n \\ H^{L,w(n-1)}_{(-1,\lambda),\lambda} & \text{if } n < \varphi(i), \end{cases}$$

and define elements $\lambda_{-1}, \dots, \lambda_n \in Y^{2^n}$ such that

$$\pi(\lambda_j(i)) = \begin{cases} R & \text{if } j+1 > \varphi(i), \\ H & \text{if } j+1 = \varphi(i), \\ L & \text{if } j+1 < \varphi(i) \end{cases}$$

and such that

$$\tau(\lambda_j(i)) = \begin{cases} 1 & \text{if } j = -1 \text{ or } j = n \\ i(j) & \text{else.} \end{cases}$$

We denote by $\mathbf{B}(w)$ the subalgebra of $\mathbf{E}(\mathbf{T})^{2^n}$ generated by ω and λ_j's. We need to characterize some elements of $\mathbf{B}(w)$. Note that $\pi(\omega) = \pi(\lambda_{-1}) \prec \cdots \prec \pi(\lambda_n)$, and moreover by Proposition IX.2 we have

$$\mathrm{supp}(f^{w[m]}_{w'}(\lambda_{-1}, \dots, \lambda_m, \omega)) = 2^n$$

and we denote these elements by $\sigma_{w'}^{w_{[m]}}$. We list a number of easy facts about these elements.

Corollary IX.6. *The following are true in* $\mathbf{B}(w)$

- *we have* $\tau(\sigma_{w'}^{w_{[k]}}) = \tau(\sigma_{w''}^{w_{[m]}})$ *for any* $k, m \leq n$ *and* w', w'' *such that* $|w'| = k$ *and* $|w''| = m$,
- *we have*
$$\pi(\sigma_{w'}^{w_{[k]}}) = \pi(\sigma_{w''}^{w_{[m]}}) \text{ if and only if } k = m,$$
for any w', w'' *such that* $|w'| = k$ *and* $|w''| = m$,
- *we have*
$$\rho(\sigma_{w'}^{w_{[m]}}(i)) = \begin{cases} L & \text{if } i_{[m]} < w', \\ H & \text{if } i_{[m]} = w', \\ R & \text{if } i_{[m]} > w' \end{cases}$$
in particular
$$\rho(\sigma_{w'}^{w_{[m]}}) \succ \rho(\sigma_{w'+1}^{w_{[m]}}) \text{ and } \rho(\sigma_{w'}^{w_{[m]}})(i) = \delta_{\varphi(w')}^{2^m}(\varphi(i_{[m]})),$$
- *we have* $\sigma_{w'}^{w_{[m]}}(i) = W_{(-1,b),c}^{Z,a}$, *and if* j *denotes the length of maximal common prefix of* i *and* w' *then*
$$c = \begin{cases} \lambda & \text{if } j = 0, \\ \eta & \text{if } j \neq 0 \text{ and } w'(j-1) = 1, \\ w(j) & \text{if } j \neq 0 \text{ and } w'(j-1) = 0. \end{cases}$$
- *if* $r'(\bar{x}, y)$ *is equal to*
$$r\big(f_{0^m}^{w_{[m]}}(\bar{x}, y), \ldots, f_{1^m}^{w_{[m]}}(\bar{x}, y), C_1(f_{0^m}^{w_{[m]}}(\bar{x}, y), f_{0^m}^{w_{[m]}}(\bar{x}, y), f_{0^m}^{w_{[m]}}(\bar{x}, y))\big)$$
where $r(\bar{v}, z)$ *is a computation term for some computation*
$$\langle e^{-w_{[m]}}, 0, 1\rangle \lhd_{I_1}^{c_1} \cdots \lhd_{I_k}^{c_k} \langle t_k, h_k, s_k\rangle$$
of dimension 2^m *then putting* $c = r'(\lambda_{-1}, \ldots, \lambda_m, \omega)$ *we have*
$$\Phi^{\pi(c(i)),\tau(c(i))}(c(i)) = \Psi_{2^m}\langle t_k, h_k, s_k\rangle(\varphi(i_{[m]})).$$

Proof. The first two statements are obvious from the definition of the operations of the set $\mathcal{A}_{\prec}$. The third and fourth statements are implied by Proposition IX.3 and the fact that

$\tau(\lambda_j(i)) = i(j)$. The fifth statement is a straightforward consequence of Proposition IX.5, and the definitions of the λ_j's. □

We proceed to a proof of the main proposition in this section. This proposition directly connects computations of the Turing machine $\mathbf{T}$ starting from the configuration $\langle e^{-w}, 0, 1\rangle$ with the elements of full support in the algebra $\mathbf{B}(w)$. We define a notion of degree of a term. We put the degree of a term equal to the depth of the term tree of the term.

Proposition IX.7. *For any term $t(\bar{x}, y)$ in the language of $\mathbf{E}(\mathbf{T})$ if*

$$\operatorname{supp}(t(\lambda_{-1}, \ldots, \lambda_n, \omega)) = 2^n,$$

then either there is a subterm of $t(\bar{x}, y)$ of the form

$$S^{**}(s_0(\bar{x}, y), s_1(\bar{x}, y), s_2(\bar{x}, y), s_3(\bar{x}, y))$$

such that $\operatorname{supp}(s_0(\lambda_{-1}, \ldots, \lambda_n, \omega)) = 2^n$ *or there is a term $t'(\bar{x}, y)$ such that*

- $\deg(t'(\bar{x}, y)) \leq \deg(t(\bar{x}, y))$,
- $\mathbf{\Sigma} \vdash t'(\bar{x}, y) \approx t(\bar{x}, y)$,
- *the term $t'(\bar{x}, y)$ is equal to one of the following*
    1. *x_i for some i*
    2. *y or $f^{w[m]}_{w'}(x_{-1}, \ldots, x_m, y)$ for some $m \leq n$ and w' of length m*
    3. *the term*

$$r\big(f^{w[m]}_{0^m}(\bar{x}, y), \ldots, f^{w[m]}_{1^m}(\bar{x}, y), C_1(f^{w[m]}_{0^m}(\bar{x}, y), f^{w[m]}_{0^m}(\bar{x}, y), f^{w[m]}_{0^m}(\bar{x}, y))\big)$$

where $r(\bar{v}, z)$ is a computation term for some computation of dimension 2^m starting at $\langle e^{-w[m]}, 0, 1\rangle$.

Proof. Note first, that the image of $\lambda_{-1}, \ldots, \lambda_n, \omega$ under the elements of first class are in Y^{2^n}, elements of the second one in $(X')^{2^n}$ and the elements of the third one in $(X \setminus X')^{2^n}$. This allows us to recognize the structure of operations by domain–range considerations.

We suppose, for a contradiction, that the proposition fails and find a term $t(\bar{x}, y)$ such that it satisfies the assumption of the proposition and fails to comply with the statements, and is of minimal depth among such. We consider cases with respect to the structure of $t(\bar{x}, y)$. Note that if $t(\bar{x}, y) = F(\bar{t}(\bar{x}, y))$ and one of the $t_i(\bar{x}, y)$ has a subterm of the form

$S^{**}(s_0(\bar{x},y), s_1(\bar{x},y), s_2(\bar{x},y), s_3(\bar{x},y))$ such that $\mathrm{supp}(s_0(\lambda_{-1},\ldots,\lambda_n,\omega)) = 2^n$ then so does $t(\bar{x},y)$. Since we assumed that $t(\bar{x},y)$ is a counterexample to the proposition such a behavior is impossible. Thus $t(\bar{x},y)$ cannot be such, and it suffices to consider cases such that for every proper subterm $s(\bar{x},y)$ of $t(\bar{x},y)$ if

$$\mathrm{supp}(s(\lambda_{-1},\ldots,\lambda_n,\omega)) = 2^n$$

then $s(\bar{x},y)$ has a corresponding primed term.

If $t(\bar{x},y) = t_0(\bar{x},y) \wedge t_1(\bar{x},y)$ for some terms $t_0(\bar{x},y)$ and $t_1(\bar{x},y)$, then those terms are of smaller depth. They comply with the assumptions and so there are $t'_0(\bar{x},y)$ and $t'_1(\bar{x},y)$ complying with statements. Since $\mathrm{supp}(t(\bar{\lambda},\omega)) = 2^n$ we have $\mathrm{supp}(t'_0(\bar{\lambda},\omega) \wedge t'_1(\bar{\lambda},\omega)) = 2^n$ as well. This implies that the terms are of the same kind, and in the case of two first kinds it is easy to see, via Corollary IX.6, that $t'_0(\bar{x},y) = t'_1(\bar{x},y)$. If the two terms are of third kind, then by Corollary IX.6 we imply that the final configurations of their computations are identical and so $\mathbf{\Sigma}' \vdash t'_0(\bar{x},y) \approx t'_1(\bar{x},y)$. Thus in all of the cases we have

$$\mathbf{\Sigma} \vdash t(\bar{x},y) \approx t'_0(\bar{x},y) \wedge t'_1(\bar{x},y) \approx t'_0(\bar{x},y)$$

which is a contradiction.

If $t(\bar{x},y) = F(t_0(\bar{x},y), t_1(\bar{x},y), t_2(\bar{x},y))$ for some $F \in \mathcal{A}_{\prec}$ then reasoning as in the previous case, and considering the ranges of subsets we get $t'_0(\bar{x},y) = x_j$, $t'_1(\bar{x},y) = x_k$ and $t'_2(\bar{x},y)$ is of the second kind. By Corollary IX.6 we immediately imply that $k = j+1$ and $t'_2(\bar{x},y) = f^{w_{[j]}}_{w'}(x_{-1},\ldots,x_j,y)$ if $j \geq 0$ or y if $j = -1$. We know that $\pi(\lambda_j(i)) = H$ if and only if $j+1 = \varphi(i)$. Then $\tau(\omega(i)) = w(\varphi(i)-1)$ if $\varphi(i) \geq 1$ and $\tau(\omega(0)) = \lambda$. We infer that for $j \geq 0$ we have $F(x,y,z) \in \{F^{w(j)}(x,y,z), F^{w(j)}_{\mathrm{compl}}(x,y,z)\}$ and so

$$\mathbf{\Sigma} \vdash t(\bar{x},y) \approx f^{w_{[j+1]}}_{w'a}(x_{-1},\ldots,x_j,y),$$

for some $a \in \{0,1\}$ which is a contradiction to the choice of $t(\bar{x},y)$. For $j = -1$ the reasoning is simpler and we leave it for the reader.

If $t(\bar{x},y) = C_1(t_0(\bar{x},y), t_1(\bar{x},y), t_2(\bar{x},y))$ then by a definition of the operation and the range consideration we have that

$$t'_0(\bar{x},y) = t'_1(\bar{x},y) = t'_2(\bar{x},y) = f^{w_{[m]}}_{w'}(x_{-1},\ldots,x_m,y).$$

If $w' \neq 0^m$ then, via Corollary IX.6, there is a coordinate on which the last subscript of $f^{w_{[m]}}_{w'}(\lambda_{-1},\ldots,\lambda_m,\omega)$ is equal to η. In such a case $t(\lambda_{-1},\ldots,\lambda_m,\omega)$ is not of full support – a contradiction. If, on the other hand, $w' = 0^m$ then, modulo $\mathbf{\Sigma}$, the term $t(\bar{x}, y)$ is equal to

$$C_1\big(f_{0^m}^{w_{[m]}}(x_{-1}, \dots, x_m, y), f_{0^m}^{w_{[m]}}(x_{-1}, \dots, x_m, y), f_{0^m}^{w_{[m]}}(x_{-1}, \dots, x_m, y)\big),$$

and this implies that, modulo $\mathbf{\Sigma}$, we have $t(\bar{x}, y)$ as a computation term for computation of length 0 – a contradiction to our choice of $t(\bar{x}, y)$.

If $t(\bar{x}, y) = F_I^c(t_0(\bar{x}, y), t_1(\bar{x}, y), t_2(\bar{x}, y))$ for some operation I of the Turing machine $\mathbf{T}$ and letter c, by domain considerations we imply that $t_0'(\bar{x}, y) = f_{w'}^{w_{[m]}}(x_{-1}, \dots, x_m, y)$ and $t_1'(\bar{x}, y) = f_{w''}^{w_{[m]}}(x_{-1}, \dots, x_m, y)$ are of the second kind and $t_2'(\bar{x}, y)$ is of the third kind for a common dimension m . We denote the final configuration of a computation corresponding to $t_2'(\bar{x}, y)$ by $\langle t, h, s\rangle$ and then via Corollary IX.6 and Corollary IX.4 we get

$$\mathbf{C}(\mathbf{T})^{2^m} \models \operatorname{supp}\big(F_I^c(\delta_{\varphi(w')}^{2^m}, \delta_{\varphi(w'')}^{2^m}, \Psi_{2^m}\langle t, h, s\rangle))\big) = 2^m,$$

and by Lemma VII.2 together with the definition of the operations of $\mathcal{A}'_=$ we infer that modulo $\mathbf{\Sigma}$ we have $t(\bar{x}, y)$ equal to a computation term for a computation one step longer then a computation for $t_2'(\bar{x}, y)$ – a contradiction.

The top most operation of the term $t(\bar{x}, y)$ cannot be from the set $\mathcal{B} \setminus \{S^{**}(v, x, y, z)\}$. It cannot be equal to $S_2(u, v, x, y, z)$ since no primed term applied to $\lambda_{-1}, \dots, \lambda_n, \omega$ has a barred element in range. It cannot be equal to $S_1(v, x, y, z)$ since, via Corollary IX.4, we have that for any image of $\lambda_{-1}, \dots, \lambda_n, \omega$ via a primed term we have a coordinate that is mapped by π to H. For the same reasons we have (possibly different) coordinate mapped to H by ρ and so the top most operation cannot be equal to $S^*(v, x, y, z)$. If on the other hand

$$t(\bar{x}, y) = S^{**}(s_0(\bar{x}, y), s_1(\bar{x}, y), s_2(\bar{x}, y), s_3(\bar{x}, y))$$

then we immediately obtain $\operatorname{supp}(s_0(\lambda_{-1}, \dots, \lambda_n, \omega)) = 2^n$ which gives a contradiction to the choice of $t(\bar{x}, y)$.

For $t(\bar{x}, y) = T_1(t_0(\bar{x}, y), t_1(\bar{x}, y), t_2(\bar{x}, y), t_3(\bar{x}, y))$, by range considerations we have that $t_0'(\bar{x}, y), t_1'(\bar{x}, y), t_2'(\bar{x}, y)$ are of first kind and by the definition of λ_i we get that $t_0'(\bar{x}, y) = t_1'(\bar{x}, y)$, thus we get

$$\mathbf{\Sigma} \vdash t(\bar{x}, y) \approx F(t_1'(\bar{x}, y), t_2'(\bar{x}, y), t_3'(\bar{x}, y)),$$

for appropriate $F(x, y, z)$ and we have reduced this case to the case of the operations from $\mathcal{A}_\prec$ which gives us a contradiction. The reasoning for the operations $T_2(v, x, y, z)$ is an alphabetical variant of this one.

If $t(\bar{x}, y) = J(t_0(\bar{x}, y), t_1(\bar{x}, y), t_2(\bar{x}, y))$ then since there are no barred elements in the ranges of the operations $t_0'(\bar{x}, y)$ and $t_1'(\bar{x}, y)$ we have that $t_0'(\bar{\lambda}, \omega) = t_1'(\bar{\lambda}, \omega)$ and reasoning as in a case of the operation $\wedge$ we obtain $\mathbf{\Sigma} \vdash t_0'(\bar{x}, y) \approx t_1'(\bar{x}, y)$ and so we got

$$\mathbf{\Sigma} \vdash t(\bar{x}, y) \approx J(t_0'(\bar{x}, y), t_0'(\bar{x}, y), t_2(\bar{x}, y)) \approx t_0'(\bar{x}, y)$$

which is a contradiction. We finish our proof of the proposition with a remark that the case of the operation $J'(x, y, z)$ is very similar. □

We are ready to define a special class of subdirectly irreducible algebras. We construct an algebra for each word in the alphabet $\mathcal{L}' \setminus \{\lambda\}$. The membership of such an algebra in the variety $\mathrm{HSP}(\mathbf{E}(\mathbf{T}))$ implies certain conditions on the computations of the machine $\mathbf{T}$ on appropriate inputs. We construct the algebras in such a way that $\mathbf{\Sigma}$ holds for every one of them. For each word w of length n in $\mathcal{L}' \setminus \{\lambda\}$ we define an algebra in the language of $\mathbf{E}(\mathbf{T})$, denoted by $\mathbf{S}_w$. The universe of this algebra consist of the elements Ω and $\Lambda_{-1}, \ldots, \Lambda_n$, $\Sigma_0, \ldots, \Sigma_n$ and the bottom element denoted by $\perp$. The operations of the algebra $\mathbf{S}_w$ are the following. The operation $\perp$ is evaluated as itself, and $\wedge$ makes $\mathbf{S}_w$ a flat semilattice with bottom equal to $\perp$. For the operations of $\mathcal{A}_{\prec}$ we have

$$\begin{aligned} G^{\lambda}(\Lambda_{-1}, \Lambda_0, \Omega) &= \Sigma_0 \\ F^{w(i)}(\Lambda_i, \Lambda_{i+1}, \Sigma_i) &= \Sigma_{i+1} \text{ for } 0 \leq i < n, \end{aligned}$$

and all other applications of the operations from $\mathcal{A}_{\prec}$ are equal to $\perp$. All the operations of sets $\mathcal{A}_{=}$ and $\mathcal{B}$ are constantly equal to $\perp$. Each $T(v, x, y, z) = \hat{t}(v, x, y, z)$ and in the algebra $\mathbf{S}_w$ we also have

$$\begin{aligned} \mathbf{S}_w &\models J(x, y, z) \approx x \wedge y \text{ and} \\ \mathbf{S}_w &\models J'(x, y, z) \approx x \wedge y \wedge z. \end{aligned}$$

The following theorem is the main element of our reasoning.

Theorem IX.8. *The algebra $\mathbf{S}_w$ is* **not** *in* $\mathrm{HSP}(\mathbf{E}(\mathbf{T}))$ *if and only if the machine* $\mathbf{T}$ *halts on some configuration $\langle e^{-w_{[m]}}, 0, 1\rangle$ for some prefix $w_{[m]}$ of w of length m.*

First we show that if the machine $\mathbf{T}$ does not halt on any $\langle e^{-w_{[m]}}, 0, 1\rangle$ then $\mathbf{S}_w$ is in the variety generated by $\mathbf{E}(\mathbf{T})$. To do so we fix an arbitrary such w of length n and construct for it $\mathbf{B}(w)$ generated by $\lambda_{-1}, \ldots, \lambda_n, \omega$ in the way described above. Note that Corollary IX.6 and Proposition IX.7 fully describe all the elements of full support of $B(w)$ generated by the operations of $\mathbf{E}(\mathbf{T})$ take away $S^{**}(v, x, y, z)$. On the other hand $S^{**}(v, x, y, z)$ does not generate any elements of full support in $\mathbf{B}(w)$ since by the fact that the machine $\mathbf{T}$ does not halt on $\langle e^{-w_{[m]}}, 0, 1\rangle$ we infer that every term of the third kind (in Proposition IX.7) corresponds to a computation with terminal configuration for a state of a machine different then the halting state.

We define a subset of $B(w)$ by putting

$$B_1 = \{\lambda_{-1}, \ldots, \lambda_n, \omega\} \cup \{\sigma_{0^m}^{w_{[m]}} \mid \text{for } m \leq n\}$$

and claim that the set $B(w) \setminus B_1$ is absorbing for all the operations of the algebra $\mathbf{B}$ except for possibly $J(x, y, z)$.

This set is certainly absorbing for $\wedge$ and $\perp$. For $F(x, y, z) \in \mathcal{A}_{\prec}$ if $F(a, b, c) \in B_1$ then $\rho(c) \in \{R, H\}^{2^n}$ and by range considerations we get $c \in B_1$ by Corollary IX.6. It trivially is absorbing for all the operations of $\mathcal{A}_{=}$ and $\mathcal{B}$ since their ranges are disjoint with B_1. The case of the operations from the set $\mathcal{C}$ reduces to the one of the operations from the set $\mathcal{A}$ in the same way as in the proof of Proposition IX.7. If $J'(a, b, c) \in B_1$ then $a \in B_1$ and then $b \in B_1$ so eventually $a = b = c \in B_1$.

We proceed to the definition of a congruence on $\mathbf{B}(w)$. We define a congruence θ by putting $a\theta b$ if and only if $a = b$ or $\{a, b\} \cap B_1 = \emptyset$. To see that θ is a congruence we note that for all the basic operations of the algebra $\mathbf{E}(\mathbf{T})$ except for $J(x, y, z)$ the set $B(w) \setminus B_1$ is absorbing. For $J(x, y, z)$ and a, b, c to get $J(a, b, c) \in B_1$ we need $a = b \in B_1$ but then $J(a, a, c) = a$ for any c. This implies that θ is a congruence and we leave it for the reader to verify that $\mathbf{B}(w)/\theta$ is the required algebra $\mathbf{S}_w$. Thus we proved that such defined $\mathbf{S}_w$ is in the variety generated by $\mathbf{E}(\mathbf{T})$.

Assume now, that for some w of length n there is a computation of dimension m such that machine $\mathbf{T}$ halts on configuration $\langle e^{-w_{[m]}}, 0, 1\rangle$. Let $r(\bar{z}, v)$ denote computation term for this computation and we put $r'(\bar{x}, y)$ equal to

$$r\big(f_{0^m}^{w_{[m]}}(\bar{x}, y), \ldots, f_{1^m}^{w_{[m]}}(\bar{x}, y), C_1(f_{0^m}^{w_{[m]}}(\bar{x}, y), f_{0^m}^{w_{[m]}}(\bar{x}, y), f_{0^m}^{w_{[m]}}(\bar{x}, y))\big).$$

Note that by Proposition IX.5 we get $r'(\bar{a}, b) \neq \perp$ if $f^{w_{[m]}}(\bar{a}, b) \neq \perp$. Moreover by the same proposition in such cases the results correspond to configurations with $\mathbf{T}$ in a halting state, so

$$\mathbf{E}(\mathbf{T}) \models S^{**}\big(r'(\bar{x}, y), f^{w_{[m]}}(\bar{x}, y), f^{w_{[m]}}(\bar{x}, y), f^{w_{[m]}}(\bar{x}, y)\big) \approx f^{w_{[m]}}(\bar{x}, y),$$

and this equation fails in $\mathbf{S}_w$ witnessing a fact that $\mathbf{S}_w \notin \mathrm{HSP}(\mathbf{E}(\mathbf{T}))$. This finishes a proof of theorem.

Next, we show that the term used to prove the theorem above is one of the smallest such terms. We need two lemmas before we are ready to prove the theorem.

Lemma IX.9. *For any term $t(\bar{x}, y)$ such that*

$$\mathbf{S}_w \models t(\Lambda_{-1}, \ldots, \Lambda_{|w|}, \Omega) \neq \bot$$

there is term $t'(\bar{x}, y)$ such that

- $\deg(t'(\bar{x}, y)) \leq \deg(t(\bar{x}, y))$,
- $\mathbf{\Sigma} \vdash t'(\bar{x}, y) \approx t(\bar{x}, y)$,
- *the term $t'(\bar{x}, y)$ is equal to one of the following*
  1. *x_j for some j*
  2. *y or $f^{w[m]}_{w'}(x_{-1}, \ldots, x_m, y)$ for some $m \leq |w|$ and w' of length m.*

Proof. We prove this lemma by a standard for this paper reasoning. We suppose, for a contradiction, that there is a term $t(\bar{x}, y)$ such that the lemma fails for $t(\bar{x}, y)$. We can also assume that $t(\bar{x}, y)$ is of the minimal degree among such terms. We consider cases with respect to the structure of $t(\bar{x}, y)$.

If $t(\bar{x}, y) = t_0(\bar{x}, y) \wedge t_1(\bar{x}, y)$ then $t_0(\bar{x}, y)$ and $t_1(\bar{x}, y)$ comply with assumptions of the theorem and we get appropriate $t'_0(\bar{x}, y)$ and $t'_1(\bar{x}, y)$. By range considerations in $\mathbf{S}_w$ we infer that $t'_0(\bar{x}, y) = t'_1(\bar{x}, y)$ and so $\mathbf{\Sigma} \vdash t(\bar{x}, y) \approx t'_0(\bar{x}, y)$ which is a contradiction to our choice of $t(\bar{x}, y)$.

If $t(\bar{x}, y) = F(t_0(\bar{x}, y), t_1(\bar{x}, y), t_2(\bar{x}, y))$ for some $F(x, y, z) \in \mathcal{A}$ by range considerations we instantly imply that $F(x, y, z) \in \mathcal{A}_{\prec}$ and that $t'_0(\bar{x}, y) = x_j$ and $t'_1(\bar{x}, y) = x_{j+1}$ and that

$$t'_2(\bar{x}, y) = f^{w[j]}(x_{-1}, \ldots, x_j, y).$$

The definitions of the operations of $\mathbf{S}_w$ imply that $F(x, y, z) = F^{w(j)}(x, y, z)$ if $j \geq 0$ or $F(x, y, z) = G^{\lambda}(x, y, z)$ if $j = -1$. This implies that

$$\mathbf{\Sigma} \vdash t(\bar{x}, y) \approx f^{w[j+1]}(x_{-1}, \ldots, x_{j+1}, y),$$

which is a contradiction to our choice of $t(\bar{x}, y)$.

All the operations of the set $\mathcal{B}$ are constantly equal to $\bot$, so the top operation of the term $t(\bar{x}, y)$ cannot be from the set $\mathcal{B}$. If $t(\bar{x}, y) = T(t_0(\bar{x}, y), t_1(\bar{x}, y), t_2(\bar{x}, y), t_3(\bar{x}, y))$ then by domain considerations we get $t'_0(\bar{x}, y) = t'_1(\bar{x}, y)$ (or $t'_0(\bar{x}, y) = t'_2(\bar{x}, y)$) and by the appropriate identity from $\mathbf{\Sigma}$ we have

$$\mathbf{\Sigma} \vdash t(\bar{x}, y) \approx F\big(t'_1(\bar{x}, y), t'_2(\bar{x}, y), t'_3(\bar{x}, y)\big)$$

for corresponding $F(x, y, z) \in \mathcal{A}$ (or similarly in second case). And we can apply the reasoning as for the case of operations from $\mathcal{A}$ which gives us a contradiction to the choice of $t(\bar{x}, y)$.

Finally, if $t(\bar{x}, y) = J(t_1(\bar{x}, y), t_2(\bar{x}, y), t_3(\bar{x}, y))$ then, by definition of $J(x, y, z)$ and range considerations we have $t'_0(\bar{x}, y) = t'_1(\bar{x}, y)$ and so a contradiction is a fact that

$$\mathbf{\Sigma} \vdash t'_0(\bar{x}, y) \approx J\big(t'_0(\bar{x}, y), t'_0(\bar{x}, y), t'_1(\bar{x}, y)\big) \approx t'_0(\bar{x}, y).$$

The situation for $J'(x, y, z)$ is similar and this finished the proof. □

A similar reasoning in $\mathbf{E}(\mathbf{T})$ gives us the following lemma. We leave the proof of this lemma as a routine exercise for the reader.

Lemma IX.10. *For any term $t(\bar{x}, y)$ such that for some j we have*

$$\mathbf{E}(\mathbf{T}) \models x_j \neq \bot \ \rightarrow \ t(\bar{x}, y) \neq \bot$$

we have

$$\mathbf{\Sigma} \vdash t(\bar{x}, y) \approx x_j.$$

To measure the depth of the identities that hold in $\mathbf{E}(\mathbf{T})$ and fail in certain $\mathbf{S}_w$ we need one more auxiliary function. For an arbitrary word w in $\mathcal{L}' \setminus \{\lambda\}$ of length n let $\varpi(w)$ denotes the minimal length of a computation among all halting computations staring from $\langle e^{-w_{[m]}}, 0, 1\rangle$ (for any $m \leq n$) whenever such a computation exists.

Theorem IX.11. *There is a constant C such that if $\mathbf{S}_w \notin \mathrm{HSP}(\mathbf{E}(\mathbf{T}))$ then for any two terms $r_0(\bar{v})$ and $r_1(\bar{v})$ such that*

$$\mathbf{E}(\mathbf{T}) \models r_0(\bar{v}) \approx r_1(\bar{v})$$

and

$$\mathbf{S}_w \models r_0(\bar{v}) \not\approx r_1(\bar{v})$$

we have $\deg(r_0(\bar{v})) + \deg(r_1(\bar{v})) \geq \varpi(w) - C|w|$.

Proof. Fix an arbitrary word w of length n and an arbitrary pair of terms as in the statement of theorem. Assumptions imply that for one of the terms, say for $r_0(\bar{v})$ there is $\bar{a}$ in $\mathbf{S}_w$ such that $r_1(\bar{a}) \neq r_0(\bar{a}) \neq \bot$. Since the algebra $\mathbf{S}_w$ is generated by the elements $\Lambda_{-1}, \ldots, \Lambda_n, \Omega$, then, by substituting variables of $r_0(\bar{v})$ and $r_1(\bar{v})$ with appropriate terms we obtain $t_0(\bar{x}, y)$ and $t_1(\bar{x}, y)$ such that:

- $t_0(\Lambda_{-1}, \ldots, \Lambda_n, \Omega) = r_0(\bar{a})$,
- $t_1(\Lambda_{-1}, \ldots, \Lambda_n, \Omega) = r_1(\bar{a})$ and
- $\mathbf{E}(\mathbf{T}) \models t_0(\bar{x}, y) \approx t_1(\bar{x}, y)$ and such that
- $\deg(t_i(\bar{x}, y)) \leq \deg(r_i(\bar{v})) + n + 1$.

Having such defined $t_0(\bar{x}, y)$ we obtain via Lemma IX.9 the term $t_0'(\bar{x}, y)$. Since $\mathbf{E}(\mathbf{T}) \models \mathbf{\Sigma}$ and $\mathbf{S}_w \models \mathbf{\Sigma}$ the term $t_0'(\bar{x}, y)$ satisfies the same identities as $t_0(\bar{x}, y)$ in both algebras. By Lemma IX.10 we infer that the term $t_0'(\bar{x}, y)$ cannot be equal to one of its variables (since it would contradict the inequality in $\mathbf{S}_w$). So for some $m \leq n$ we have $t'(\bar{x}, y) = f^{w_{[m]}}(x_{-1}, \ldots, x_m, y)$. The fact that

$$\mathbf{E}(\mathbf{T}) \models f^{w_{[m]}}(x_{-1}, \ldots, x_m, y) \approx t_1(\bar{x}, y)$$

implies that the term $t_1(\bar{x}, y)$ complies with the assumptions of Proposition IX.7 for the algebra $\mathbf{B}(w)$. If, by the Proposition IX.7 we obtain a primed term corresponding to $t_1(\bar{x}, y)$ then, by domain considerations in $\mathbf{E}(\mathbf{T})$ it cannot be of first, or third kind. If it is of the second kind then, by Corollary IX.6 we have $t_1'(\bar{x}, y) = f^{w_{[m]}}(x_{-1}, \ldots, x_m, y)$ and this is a contradiction with the inequality of terms in $\mathbf{S}_w$.

Thus the operation $S^{**}(v, x, y, z)$ appears in the term $t_1(\bar{x}, y)$ in such a way that

$$S^{**}(s_0(\bar{x}, y), s_0(\bar{x}, y), s_0(\bar{x}, y), s_0(\bar{x}, y))$$

is a subterm of $t_1(\bar{x}, y)$ and that $s_0(\bar{x}, y)$ complies with assumptions of the same proposition. Choosing a minimal such $s_0(\bar{x}, y)$, we obtain $s_0'(\bar{x}, y)$ which have to be of the one of three kinds and – by domain range consideration we imply that it is of the third kind for some halting computation of $\mathbf{T}$. Thus the term $s_0'(\bar{x}, y)$ is of depth not smaller then $\varpi(w)$ and the theorem is proved. □

To produce an example of a finite algebra generating a variety with EXPSPACE–hard membership problem one has to modify a Turing machine solving EXPSPACE–complete problem in exponential space to a machine that answers the same questions on the words e^{-w}. It is easy to construct such a machine and we leave for the reader.

It is an easy exercise to produce a Turing machines that on some configurations $\langle e^{-w}, 0, 1\rangle$ for w of arbitrary length n require 2^{2^n} steps to halt in state 0. The $\beta_{\mathbf{E}(\mathbf{T})}$ function of $\mathbf{E}(\mathbf{T})$ for such a machine grows doubly exponentially.

References

[BO75] Yu. A. Bahturin and A. Yu. Ol'shanskiĭ, *Identical relations in finite lie rings*, Matematicheskiĭ Sbornik. Novaya Seriya **96(138)** (1975), no. 4, 543–559, 645 (Russian).

[BS81] Stanley Burris and H.P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, New York-Berlin, 1981.

[BS00] Clifford Bergman and Giora Slutzki, *Complexity of some problems concerning varieties and quasi-varieties of algebras*, SIAM Journal on Computing **30** (2000), no. 2, 359–382.

[Dem82] J. Demel, *Fast algorithms for finding a subdirect decomposition and interesting congruences of finite algebras*, Kybernetika **2** (1982), 121–130.

[JM] Marcel Jackson and Ralph McKenzie, *Interpreting graph colourability in finite semigroups*, Submited for publication.

[Kal52] Jan Kalicki, *On comparison of finite algebras*, Proceedings of the American Mathematical Society **3** (1952), 46–40.

[Koz] Marcin Kozik, *The subdirectly irreducible algebras in the variety generated by graph algebras*, In preparation.

[Kru73] Robert L. Kruse, *Identities satisfied by a finite ring*, Journal of Algebra **26** (1973), 298–318.

[KV] Gábor Kun and Vera Vértesi, *The membership problem in finite flat hypergraph algebras*, Submited for publication.

[L'v73] I. V. L'vov, *Varieties of associative rings. I, II*, Algebra i Logika **11** (1973), 269–297,667–688 (Russian).

[McK70] Ralph McKenzie, *Equational bases for lattice theories*, Mathematica Scandinavica **27** (1970), 24–38.

[McK96a] ______, *The residual bound of a finite algebra is not computable*, International Journal of Algebra and Computation **6** (1996), no. 1, 29–48.

[McK96b] ______, *The residual bounds of finite algebras*, International Journal of Algebra and Computation **6** (1996), no. 1, 1–28.

[McK96c] ———, *Tarski's finite basis problem is undecidable*, International Journal of Algebra and Computation **6** (1996), no. 1, 49–104.

[OP64] Sheila Oates and M.B. Powell, *Identical relations in finite groups*, Journal of Algebra **1** (1964), 11–39.

[RNMT87] George F. McNulty Ralph N. McKenzie and Walter F. Taylor, *Algebras, lattices, varieties. Vol. I*, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987.

[Sav70] Walter J. Savitch, *Relationships between nondeterministic and deterministic tape complexities*, Journal of Computer and System Sciences **4** (1970), 177–192.

[Szé02] Zoltán Székely, *Computational complexity of the finite algebra membership problem for varieties*, International Journal of Algebra and Computation **12** (2002), no. 6, 811–823.

[Wil97] Ross Willard, *Tarski's finite basis problem via* $\mathbf{A}(\mathbf{T})$, Transactions of the American Mathematical Society **349** (1997), no. 7, 2755–2774.