On the fine structure of graphs avoiding certain complete bipartite minors

By

Ryan Solava

Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August 31st, 2019

Nashville, Tennessee

Approved:

Mark Ellingham, Ph.D.

Akram Aldroubi, Ph.D.

Paul Edelman, Ph.D.

Mike Mihalik, Ph.D.

Jeremy Spinrad, Ph.D.

TABLE OF CONTENTS

LIST OF FIGURES

Page

Chapter 1

Introduction

In this dissertation, we present new results in the area of graph minors. In particular, we focus on characterizing families of graphs that avoid certain small minors. We begin in Chapter 1 by providing some definitions and presenting some known results. In Chapter 2, further definitions and known results are presented. The technical details of many of these results are used in later chapters. Chapter 3 focuses on the family of 3-connected $K_{2,5}$-minor-free graphs. We define a family of graphs and show they are 3-connected, $K_{2,5}$-minor-free graphs, prove a new result on fan expansions with general applications, and present a program for characterizing the family of 3-connected, $K_{2,5}$-minor-free graphs. In Chapter 4, we present a detailed description of the families of 3- and 4-connected $K_{2,t}$-minor-free graphs for general, fixed t. Finally, in Chapter 5, possible future directions for related research are presented. All of the new results in Chapters 3 and 4 are joint work with my advisor Mark Ellingham.

1.1 Definitions and notation

We will first lay the ground work with some definitions. For any definitions not stated here, we direct the reader to [32] for more details. Unless stated otherwise, all graphs we discuss here are finite and simple, meaning there are no loops or multiple edges. Given a graph G, we denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, it is said to be *induced* if $vw \in E(G)$ and $v, w \in V(H)$ implies $vw \in E(H)$, i.e., all edges on $V(H)$ in G are included in H. For a given vertex in a graph G the *neighborhood of v in G*, denoted $N_G(v)$, is the set of all vertices adjacent to v in G. The *closed neighborhood of v in G*, denoted $N_G[v]$, is defined to be $N_G(v) \cup \{v\}$, i.e., all vertices adjacent to v and v itself. We will simply write $N(v)$ or $N[v]$ when the graph is clear from context.

A *path* of a graph G is a sequence of distinct vertices $v_0, v_1, \ldots v_k$ such that $v_i v_{i+1} \in E(G)$ for $0 \leq i \leq k-1$. A *cycle* is a sequence of vertices $v_0 v_1 \ldots v_k$ such that $v_0 = v_k$, no other vertices are repeated, and $v_i v_{i+1} \in E(G)$ for $0 \leq i \leq k-1$.

We will now define and provide notation for a few specific graphs. The graph C_n is the cycle on n vertices, and P_n is the path on n vertices. The *complete graph on n vertices*, denoted K_n, is the graph on the vertices $\{v_1, v_2, \ldots v_n\}$ with edge set $E = \{v_i v_j | 1 \leq i, j, \leq n, i \neq j\}$, i.e., all possible edges. A *t-clique* in a graph G is a subgraph isomorphic to K_t in G. The *complete bipartite graph*, denoted $K_{r,s}$, is the graph on the vertices $\{x_1, x_2, \ldots x_r, y_1, y_2, \ldots y_s\}$ with edge set $E = \{x_i y_j | 1 \leq r, 1 \leq s\}$. In other words, the vertices are partitioned into sets X and Y, with $|X| = r$ and $|S| = s$, only edges between the sets X and Y are allowed, and all such edges are included. A generalization of the complete bipartite graph is the *complete multipartite graph*, $K_{r_1, r_2, \ldots r_k}$, with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$, where $V_1, V_2, \ldots V_k$ are pairwise disjoint and edge set $E = \{xy | x \in V_i, y \in V_j, i \neq j\}$. The *wheel graph*, W_n consists of a cycle on $n-1$ vertices and one additional vertex adjacent to all vertices of the cycle. The edges of this cycle in W_n are called *rim edges* and all other edges are called *spoke edges*. Examples of these graphs specifically, K_5, $K_{3,3}$, and W_9 are shown in Figure 1.1

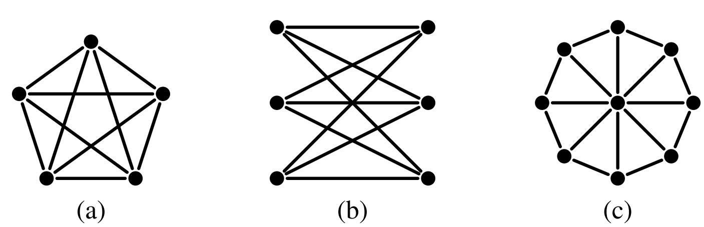

Figure 1.1: Examples of various graphs: (a) the complete graph K_5, (b) the complete bipartite graph $K_{3,3}$, (c) The wheel on 9 vertices W_9.

We now present a definition of an operation that forms a new graph by combining two graphs.

Definition 1.1. Given two disjoint graphs G_1 and G_2, let $\{v_1, v_2, \ldots v_k\}$ and $\{w_1, w_2, \ldots w_k\}$ be the vertices of k-cliques in G_1 and G_2. Then a *k-clique-sum* or simply a k-sum of G_1 and G_2 is formed by identifying the vertices $\{v_i, w_i\}$ for each $1 \leq i \leq k$, and then deleting any number of edges from the resulting k-clique on the identified vertices.

Another definition that builds a graph from two other graphs is presented below.

Definition 1.2. The *lexicographic product* of two graphs G and H, denoted $G[H]$ is the graph with vertex set $V(G) \times V(H)$, where there is an edge between (g_1, h_1) and (g_2, h_2) if and only if $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

The graphs $C_n[K_2]$ and $P_n[K_2]$ will be of particular interest to us.

1.2 Connectivity and relative connectivity

A graph G is *connected* if for every pair of vertices of G, there is a path in G between those vertices. A cutset of G is a set of vertices S of G such that $G - S$ is not connected. A cutset of size k is called a *k-cut*. A graph G is called *k-connected* if G has at least $k + 1$ vertices and every cutset of G has size at least k. The connectivity $\kappa(G)$ of a graph G is the largest k such that G is k-connected. A *component* of a graph is a maximal connected subgraph.

Connectivity is way of measuring how hard it is to disconnect a graph; the larger the connectivity, the harder it is to disconnect. Often, adding a connectivity condition to a family makes the family easier to work with or describe, and so many results presented here include a connectivity condition.

Given a subgraph H of a graph G, an *H-bridge* or a *bridge of H* in G is a component C of $G - V(H)$ together with all edges with one endpoint in C and the other endpoint in H, or else it is an edge of $E(G) - E(H)$ with both ends in H.

Relative connectivity is a recently introduced weakening of k-connectivity that will be useful to us in Chapter 4. Given a graph G and a non-empty set $S \subseteq V(G)$, G is said to be *k-connected relative to S* or *(k, S)-connected* if for every set $T \subseteq V(G)$ with $|T| < k$ every component of $G - T$ contains at least one vertex of S. This is a weaker condition than k-connectivity, since there are allowed to be cutsets with size less than k. This definition was introduced in [3], but similar concepts were explored elsewhere such as in [27] and [33].

Lemma 2.1 in [3] states a number of useful results for (k, S)-connectivity. Here we present only the parts of the lemma that we will use later in this document. Part (1) gives an alternative and perhaps more intuitive definition of being (k, S)-connected.

Lemma 1.3.

(1) A graph G is (k, S)-connected if and only if for every vertex $v \notin S$, there are k-disjoint paths from v to S in G.

(2) Let G be a (k, S)-connected graph with a subgraph H with $S \subseteq V(H)$. If H is k'-connected for $0 \leq k' \leq k$, then G is k'-connected. Furthermore, if $H \cong K_k$, then G is k-connected.

(3) Suppose G is k-connected and $S \subseteq V(G)$ with $|S| \geq k$. Let H be the union of zero or more S-bridge in G along with any number of edges with both ends in S (not necessarily edges of G. Then H is (k, S)-connected.

1.3 Graph minors

This dissertation focuses on results that characterize families of graph that avoid certain small minors, often with connectivity restrictions. We begin this section by presenting two equivalent definitions of a graph minor and then several known results.

Given an edge $e = v_1v_2$ of a graph G, *contracting* the edge v_1v_2 results in a new graph G', obtained by deleting v_1 and v_2, and then adding a new vertex v with neighborhood $N(v) = N(v_1) \cup N(v_2)$. We denote this new vertex of G' by $[e]$ or $[v_1v_2]$. Denote the graph obtained from contracting the edge e in G by G/e, and denote the graph obtained by deleting the edge e from G by $G\backslash e$. A graph H is a *minor* of G, if a graph isomorphic to H can be obtained from G by a sequence of edge deletions, edge contractions, and vertex deletions. Note that if v_1 and v_2 have a common neighbor x, the edge v_1x and v_2x correspond to only one edge in the graph G' obtained from contracting v_1v_2, since we are working only with simple graphs. Some define edge contraction, and therefore minors, in a context that allows for multiple edges, which results in a slightly altered definition. For us though, our graphs are simple and multiple edges are ignored.

At times, it will be useful to consider the inverse operation to contracting an edge. Given a vertex v of G, *splitting* v is the operation that forms a new graph G' by deleting v and adding two new adjacent vertex v_1 and v_2 such that $N(v_1) \cup N(v_2) = N(v) \cup \{v_1, v_2\}$. For a given vertex, this operation is not usually uniquely defined as it depends on the choice of neighborhoods for v_1 and v_2. A graph G can be obtained from a graph G' by contracting an edge if and only if G' can be obtained from G splitting a vertex. In this sense, edge contraction and vertex splitting are inverse operations.

An equivalent way to define that a graph H is a minor of a graph G follows. For each vertex v of H, let S_v be a corresponding subset of $V(G)$. This collection $\{S_v\}_{v \in V(H)}$ forms a *model* of H in G if the following conditions hold. The sets are pairwise disjoint, each set S_v induces a connected subgraph of G, and for every edge v_1v_2 in H, there is an edge x_1x_2 in G where $x_1 \in S_{v_1}$ and $x_2 \in S_{v_2}$. We say that H is a *minor* of G if there is a model of G in H. The set S_v is called the *branch set* of v in the given model. A graph G is called *H-minor-free* if it does not have H as a minor.

The following well-known lemma will prove useful for working with k-connected graphs that avoid a particular H-minor.

Lemma 1.4. *Suppose G' is obtained from a k-connected graph G by splitting a vertex v of G into v_1 and v_2 in G', such that v_1 and v_2 both have degree at least k. Then G' is k-connected.*

Proof. Suppose that there exists a cutset S in G' such that $|S| < k$. We consider the cases when zero, one, or both of v_1 and v_2 are in S, and in each case find a cutset of size at most $k-1$ in G, which provides a contradiction.

If neither v_1 nor v_2 is in S, v_1 and v_2 are in the same component of $G' - S$, since they are adjacent. Thus, S is still a cutset of G.

If both v_1 and v_2 are in S and $v = [v_1v_2]$, let $S' = S \backslash \{v_1, v_2\} \cup \{v\}$. Then $G - S'$ is the same graph as $G' - S$, so S' is a cutset of G with $|S'| < |S| < k$.

Suppose that exactly one of v_1 and v_2 is in S. Without loss of generality, $v_1 \in S$ and $v_2 \notin S$. Now $G' - S$ is disconnected, so it has at least two components C_1 and C_2; suppose $v_2 \in C_2$. Then v_2 has degree at least k in G', and so v_2 has a neighbor outside of S. Thus, C_2 contains a vertex other than v_2, and $S' = S\backslash\{v_1, v_2\} \cup \{v\}$ separates that vertex from C_1 in G. G. Again $|S'| < |S| < k$.

So in all cases, if G' has such a cutset, so does G. Since G is k-connected, G' is k-connected.

□

The following lemma presents another condition under which k-connectedness is preserved.

Lemma 1.5 (See West, [32] pg. 145)**.** *Suppose G' is obtained from a k-connected graph G by adding a vertex v and edges to v such that the degree of v in G' is at least k. Then G' is k-connected.*

In a k-connected graph G, an edge e of G is said to be *k-contractible* if G/e is k-connected. The following well-known result of Tutte discusses the existence of a 3-contractible edge in 3-connected graphs.

Lemma 1.6 (Tutte, [29])**.** *Every 3-connected graph other than K_4 has a 3-contractible edge.*

We now present another lemma that will prove useful. As stated this lemma appeared in [10], but it was originally proven simultaneously by Seymour in [25] in the context of matroids and by Negami in [18] in the context of multigraphs. These versions of the lemma are often referred to as Seymour's Splitter Theorem.

Lemma 1.7. *Given a 3-connected graph G with a 3-connected minor H, there is a sequence of graphs $H = G_0, G_1 \ldots G_{j-1}, G_j = G$ such that each G_i is 3-connected and is obtained from G_{i+1} by contracting or deleting a single edge.*

The statements of this lemma in [25] and [18] require that H is either not a wheel or H is the largest wheel that is a minor of G. This is because in these contexts, edge contraction is allowed to create multiple edges. In particular, consider W_{n-1} as a minor of W_n. When multiple edges are allowed, W_{n-1} is obtained from W_n by contracting a rim edge e of W_n, which results in a multiple edge, and the deleting one of the multiple edges. The graph W_n/e is not 3-connected, however, due to the multiple edge and their definition of 3-connectivity. In our context, which does not allow multiple edges, contracting any rim edge of W_n gives W_{n-1} directly. For this reason, we do not need to include the condition that H is not a wheel in our statement of this lemma. More details of this can be found in [10].

Graph minors are graph substructures that are more general than subgraphs. They have connections to many areas within graph theory including graph embedding and graph coloring.

A graph G is called *planar* if it can be embedded in the plane, that is drawn in the plane so that edges do not cross. The following characterizes planar graphs in terms of excluded minors.

Theorem 1.8 (Wagner, [30]). *A graph G is planar if and only if it is K_5-minor-free and $K_{3,3}$-minor-free.*

In the case when the graph is 3-connected, the characterization simplifies to the following.

Theorem 1.9 (Wagner, [30]). *For any 3-connected graph G other than K_5, G is planar if and only if it is $K_{3,3}$-minor-free.*

These well-known results are an early instance of characterizing useful families of graphs by excluding certain minors. These results are very specific instances of a much more general result, formally known as Wagner's Conjecture. A family of graphs $\mathcal{G}$ is called *minor closed* if $G \in \mathcal{G}$ implies $H \in \mathcal{G}$ for any minor H of G, i.e., it is closed under the operation of taking minors.

Conjecture 1.10 (Wagner, [31]). *Every minor-closed family of graphs can be characterized by excluding a finite set of minors.*

Note that the planar graphs, and in fact the family of graphs that are embeddable on any given surface, is minor closed. In this way, Wagner's Theorem is one specific case of this conjecture. More recently, this conjecture was proven true by the Robertson-Seymour Graph Minors Project, a massive project spanning around twenty publications. To state their main result in full, we need a few definitions.

A *quasi-ordering*, also called a *preordering*, of a set S is a binary relation $\leq$ such that $x \leq x \forall x \in X$ ($\leq$ is reflexive), and $x \leq y$ and $y \leq z$ implies $x \leq z$ ($\leq$ is transitive). A quasi-ordering is a weaker notion then a partial order, as it drops the condition of antisymmetry, that is the condition that for no two distinct elements x and y of S do we have $x \leq y$ and $y \leq x$. A quasi-ordering on S is called a *well-quasi-ordering* if for every infinite sequence of elements of S, $x_1, x_2, \ldots$, there exist a pair of elements x_i and x_j with $i < j$ such that $x_i \leq x_j$. With this definition we can state the Robertson-Seymour Graph Minors Theorem.

Theorem 1.11 (Robertson and Seymour, [24])**.** *The minor relation is a well-quasi-ordering on every minor-closed class of graphs.*

A well-quasi-ordering contains no infinite antichains (a set of incomparable elements). If a family of graphs cannot be characterized by excluding a finite set of minors, then the set of minor-minimal graphs not in the family must be an infinite antichain under the minor relation. Thus, Wagner's Conjecture follows immediately from Theorem 1.11.

As part of their Graph Minors Project, Robertson and Seymour describe families of graph that avoid a planar graph as a minor. The graph $K_{2,t}$ is planar, and so this is particularly relevant to the work presented here. The following definition is needed to state their result.

Definition 1.12. The *treewidth* of a graph G is the smallest k such that G is obtained from graphs $G_1, G_2, \ldots G_m$ each on at most k vertices by clique-sums.

The treewidth of a graph quantifies how treelike a graph is. The following theorem shows how avoiding a planar graph as a minor relates to treewidth.

Theorem 1.13 (Robertson and Seymour [22])**.** *A minor-closed family of graphs has bounded treewidth if and only if it excludes a planar graph as a minor.*

In particular, this shows that the family of graphs that avoid $K_{2,t}$ as a minor for a given t has bounded treewidth.

1.4 Graph coloring and Hadwiger's Conjecture

In this section, we will introduce the notion of graph coloring, and discuss briefly how it relates to graph minors. A *k-coloring* (also called a *k-vertex-coloring*) of a graph G is a function ϕ :

$V(G) \to \{1, 2, \dots k\}$ such that $\phi(x) \neq \phi(y)$ if $xy \in E(G)$. In other words, it assigns each vertex of G one of k colors such that adjacent vertices receive different colors. A graph G is *k-colorable* if it has a k-coloring. The *chromatic number* of G, $\chi(G)$ is the smallest number k such that G has a k-coloring. Graph coloring is a challenging and well-studied area of graph theory. Determining if a graph is k-colorable, and in particular finding the chromatic number, is computationally hard; determining whether a graph is k-colorable is NP-complete for each fixed $k \geq 3$ [15]. (For more on NP-completeness see [26]). Thus, finding bounds on the chromatic number of a class of graphs is of interest. One of the best-known results in graph theory is the Four Color Theorem, which relates graph coloring to planarity.

Theorem 1.14 (Appel and Haken, [1])**.** *Every planar graph is 4-colorable.*

As demonstrated by the Four Color Theorem there is a deep connection between the topological properties of a graph and its chromatic number. Earlier we discussed how Wagner's Theorem (Theorem 1.8), connects graph minors to graph embeddings. One might expect then, that there is a direct connection between graph minors and graph coloring. The following conjecture addresses exactly this.

Conjecture 1.15 (Hadwiger's Conjecture)**.** *If G does not have a t-coloring, then G has a K_{t+1} minor.*

It was conjectured in 1943 by Hadwiger, and in the paper where it was introduced, he proved it for the case when $t = 3$ [14]. The case when $t = 4$ states that if G has no 4-coloring, it has a K_5-minor, so it particular it is not planar. Thus, this particular case of the conjecture is more general than the Four Color Theorem, and the overall conjecture is a far-reaching generalization of the Four Color Theorem. Wagner proved that the case when $t = 4$ is in fact equivalent to the Four Color Theorem [30], so we know now with the proof of that theorem that it holds. No additional progress was made on this conjecture until 1993 when Robertson, Seymour, and Thomas proved that it holds for $t = 5$ [21]. In particular, they showed that the case $t = 5$ follows from the Four Color Theorem.

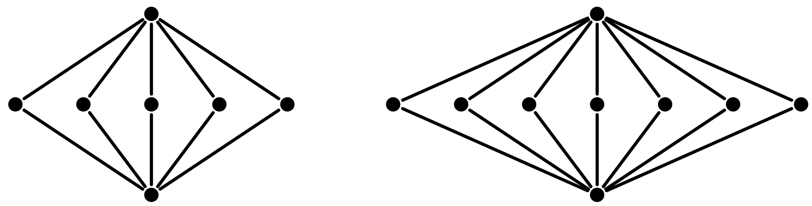

Figure 1.2: Examples of complete bipartite graphs: (a) $K_{2,5}$ and (b) $K_{2,7}$.

1.5 Complete multipartite minors and Hamiltonicity

For most graphs H, the family that arises from avoiding H as a minor is not well understood. An exact characterization of that family would be the ultimate goal, but for most H, this is not known. The complete bipartite graphs are a natural choice for graphs to avoid as minors. As observed in Theorem 1.9, graphs with high enough connectivity that avoid $K_{3,3}$ as a minor are all planar. As we will discuss later, avoiding $K_{2,t}$ minors has implications for Hamiltonicity, which makes this a particularly interesting property.

Before discussing $K_{2,t}$ minors and Hamiltonicity, we will discuss $K_{1,t}$-minor free graphs. Before we can state a result on these families, we need the following definition. *Subdivision* of an edge $e = uv$ in a graph G is the operation that replaces the edge e with two new edges ux and xv, where x is a new vertex. A *subdivision* of a graph G is a graph obtained from G by subdividing edges. A subdivision can be thought of as a restricted type of vertex split.

Theorem 1.16 (Robertson and Seymour, [23])**.** *Given a positive integer t there exists a positive integer N_t, such that every component of a $K_{1,t}$-minor-free graph is a subdivision of a graph on at most N_t vertices.*

Since subdividing an edge results in a degree 2 vertex, a corollary of this result is that there are finitely many 3-connected, $K_{1,t}$-minor-free graphs for any given t.

We now turn our attention to $K_{2,t}$-minor-free graphs. In the rest of this chapter, we present several known results for $K_{2,t}$-minor-free graphs. The new results presented in Chapters 3 and 4 focus on $K_{2,t}$-minor-free graphs.

A cycle that contains all vertices of G is called a *Hamilton cycle*. A graph that has a Hamilton cycle is *Hamiltonian*. A slightly weaker structure that is often of interest is a *Hamilton path*, a path containing every vertex of a graph. Determining if a graph has a Hamilton cycle is an NP-complete problem, as is determining if a graph has a Hamilton path [15]. It is of interest then to find classes of graph that are Hamiltonian, i.e., sufficient conditions for the existence of a Hamilton cycle. As mentioned, Hamiltonicity can be connected to avoiding certain complete bipartite graphs as minors.

We will first consider graphs without $K_{2,3}$ minors. A graph G is called *outerplanar* if it can be embedded in the plane such that there is one face that contains all the vertices of G.

Theorem 1.17. *A graph is outerplanar if and only if it is $K_{2,3}$- and K_4-minor-free.*

This result follows from Theorem 1.8. Analogously to the planar case and Theorem 1.9, for graphs with sufficient connectivity, essentially only $K_{2,3}$ needs to be avoided.

Theorem 1.18. *A 2-connected graph G other than K_4 is outerplanar if and only if it is $K_{2,3}$-minor-free.*

A corollary to this fact is that all 2-connected, $K_{2,3}$-minor-free graphs are Hamiltonian.

We need to make one note before we continue. Let H be a graph with a minor H'. Then any graph that has an H-minor has an H'-minor, and therefore, any graph that is H'-minor-free is H-minor-free. Thus, the class of graphs that are $K_{2,t-1}$-minor-free is a subset of the class of graphs that are $K_{2,t}$-minor-free. Contracting any edge of $K_{2,t}$ gives the graph $K_{1,1,t-1}$, which can also be obtained by adding an edge to $K_{2,t-1}$. The $K_{2,t-1}$-minor-free graphs are a subset of the $K_{1,1,t-1}$-minor-free graphs, which in turn are a subset of the $K_{2,t}$-minor-free graphs. For this reason, $K_{1,1,t}$-minor-free graphs are closely related to $K_{2,t}$-minor-free graphs, and so results of this form are included in this section as well.

Avoiding complete bipartite minors is closely related to Hamiltonicity and the existence of other spanning substructures. The following theorem connects $K_{2,t}$-minor-freeness to the existence of a large cycle.

Theorem 1.19 (Chen et al., [4])**.** *Every 2-connected, $K_{2,t}$-minor free graph on n vertices contains a cycle of length at least $\frac{n}{t-1}$.*

Since all 2-connected, $K_{2,3}$-minor-free graphs are Hamiltonian, it is natural to consider the Hamiltonicity of the $K_{2,4}$-minor-free graphs. To this end, a full characterization of the $K_{2,4}$-minor-free graphs was presented in [11].

Definition 1.20. For $n \geq 3$, and $0 \leq r, s \leq n-3$, let $G_{n,r,s}$ be the graph with vertices $v_1, v_2, \ldots v_n$ with edges $v_i v_{i+1}$ for $1 \leq i \leq n-1$, $v_1 v_{n-i}$ for $1 \leq i \leq r$ and $v_n v_{1+j}$ for $1 \leq j \leq s$. Define $G^+_{n,r,s}$ to be $G_{n,r,s}$ with the additional edge $v_1 v_n$.

Let $\mathcal{G}$ be the class of graphs isomorphic to one of the graphs in the following set.

$$\{G^+_{n,1,n-3}, G^+_{n,n-3,1} | n \geq 4\} \cup \{G^+_{n,r,s}, G_{n,r,s} | n \geq 5, 2 \leq r, s \leq n-3, r+s = n-1 \text{ or } n-2\}$$

An example of a graph in $\mathcal{G}$ is presented in Figure 1.3.

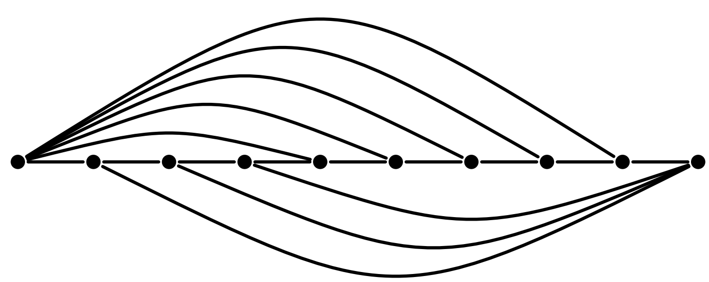

Figure 1.3: The graph $G_{10,5,3}$.

Theorem 1.21. *For a 3-connected graph G, G is $K_{2,4}$-minor-free if and only if $G \in \mathcal{G}$ or G is isomorphic to one of nine exceptional graphs on at most eight vertices.*

From this characterization, they were able to show the following Hamiltonicity result.

Theorem 1.22 (Ellingham et al., [10]). *Every 3-connected $K_{2,4}$-minor-free graph is Hamiltonian. Furthermore, every 2-connected $K_{2,4}$-minor-free graph has a Hamilton path.*

Avoiding a $K_{2,5}$-minor is not sufficient to be Hamiltonian, even with a 3-connectivity condition. Adding a planarity condition as well suffices though.

Theorem 1.23 (Ellingham et al., [11]). *Every 3-connected, planar, $K_{2,5}$-minor-free graph is Hamiltonian.*

Note that the planarity condition cannot be dropped in this theorem, i.e., there exist 3-connected, non-planar, non-Hamiltonian, $K_{2,5}$-minor-free graphs. Theorem 1.22 can be expanded to $K_{1,1,5}$-minor-free graphs with one exception, the Herschel graph, depicted in Figure 1.4.

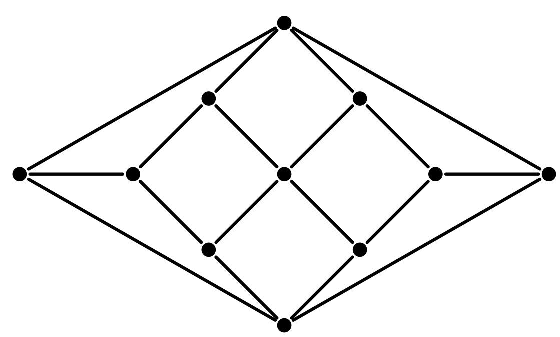

Figure 1.4: The Herschel graph, which is not Hamiltonian

Theorem 1.24 (Ellingham, Gaslowitz, O'Connell, and Royle, see [19]). *Every 3-connected, planar, $K_{1,1,5}$-minor-free graph is Hamiltonian, except the Herschel graph.*

As discussed earlier, the $K_{1,1,5}$-minor-free graphs are a superset of the $K_{2,5}$-minor-free graphs, so this is a generalization of the previous result. There is an infinite family of 3-connected, planar, $K_{2,6}$-minor-free graphs that are not Hamiltonian described in [19]. A conjecture due to Ellingham, Marshall, and Royle is that all 3-connected, planar $K_{2,6}$-minor-free graphs on sufficiently many vertices (at least 16) are Hamiltonian except the graphs of this small family, with only 40 members on any given number of vertices.

Families of $K_{2,t}$-minor-free graphs have been studied in contexts other than Hamiltonicity as well. Chudnovsky, Reed, and Seymour studied the edge density of $K_{t,2}$-minor-free graphs, and proved the following bound.

Theorem 1.25 (Chudnovsky, Reed, and Seymour [6])**.** *For a given $t \geq 2$ and a $K_{2,t}$-minor-free graph G, the following holds:*

$$|E(G)| \leq \frac{1}{2}(t+1)(|V(G)|-1)$$

This result is an improvement of an earlier result by Myers [17], which proved the same result but requiring $t > 10^{29}$.

There has also been some work on Hamiltonicity and the existence of spanning substructures in $K_{3,t}$-minor-free graphs. One early result along these lines is Tutte's result that the 4-connected, planar graphs are Hamiltonian; since K_5 is also Hamiltonian it follows that all 4-connected, $K_{3,3}$-minor-free graphs are Hamiltonian [28]. Another more recent such result is the following.

Theorem 1.26 (Chen, Yu, and Zang, [5])**.** *There exists a function $\alpha(t)$ and constant β that does not depend on t, such that every 3-connected, $K_{3,t}$-minor-free graph on n vertices has a cycle of length at least $\alpha(t)n^{\beta}$.*

A spanning tree of G is a tree subgraph of G that contains every vertex of G. A Hamilton path is a spanning tree with maximum degree 2. The existence of spanning trees with low degree is related to but weaker than the existence of Hamilton paths. Ota and Ozeki showed in [20] that for $t \geq 4$ every 3-connected, $K_{3,t}$-minor free graph contains a spanning tree with maximum degree at most $(t-1)$ when t is even, and at most t when t is odd, and these bounds are sharp. This also holds for $t = 3$ due to a result of Barnette, which is stated in terms of planar graphs [2].

Chapter 2

Fans and strips

In this chapter, we first examine several known results on characterizing $K_{2,t}$-minor-free graphs for small t. Then we will consider a general result that provides a rough description of every $K_{2,t}$-minor-free graphs for any given t. These results are closely related to the new work presented in this dissertation, and several technical definitions and lemmas presented here will be used throughout the document. In particular, the definitions of fans are strips will be important to us.

Throughout this dissertation we will assume that a model of $K_{2,t}$ in a graph has a particular type of structure. Label the vertices of $K_{2,t}$ with $r_1, r_2, x_1, x_2, \ldots x_t$, where each r_i has degree t and each x_i has degree 2. Let $(R_1, R_2, T_1, T_2, \ldots T_t)$, be a model of $K_{2,t}$ in a graph G, where the set R_i corresponds to r_i and the set T_i corresponds to the vertex x_i. We may assume that each T_i contains a single vertex for each $1 \leq i \leq t$ by the well-known following observation.

Observation 2.1. *Suppose H is a minor of a graph G. Suppose X is an independent set of vertices in G with maximum degree 2. Then there is a model of H in G such that the branch set corresponding to each $x_i \in X$ consists of a single vertex.*

Proof. Suppose we have a graph G with a minor of H. Fix some model of H in G. Let S_v be the branch set for the vertex v in the model of H in G. Now suppose S_x contains more than one vertex for some $x \in X$. If x has degree 0 or 1 in H, we can clearly take S_x to contain only a single vertex without changing any other branch set. Suppose then that $d_H(x) = 2$. Let y_1 and y_2 be the neighbors of x in H. Note that $y_1, y_2 \notin X$, since X is an independent set. In G there is a vertex $x_1 \in S_x$ adjacent to a vertex of S_{y_1} and a vertex $x_2 \in S_x$ adjacent to a vertex of S_{y_2}. The branch set S_x is connected, so there is a path P from x_1 to x_2 in S_x. Take $S'_x = \{x_1\}$ and $S'_{y_2} = S_{y_2} \cup P(x_1, x_2]$. Then $\{x_1\}$ is adjacent to a vertex of both S_{y_1} and S'_{y_2}, and S'_x and S'_{y_2} are connected. Thus, if we let $S'_v = S_v$ for $v \neq y_2, x$, then $\{S'_v | v \in V(H)\}$ is a model for H in G

with $|S'_x| = 1$. Since $y_2 \notin X$, the branch sets for all other vertices of X are unchanged. Thus, we can find a model for H in G such that for the branch set for every vertex of X consist of a single vertex. □

In fact this observation holds when the vertices of X have degree at most 3 in G, but the observation as stated is sufficient here. Thus, throughout this document, when we consider a model of a $K_{2,t}$ minor, $(R_1, R_2, T_1, T_2, \ldots T_t)$, we will assume each T_i contains a single vertex and let $T = T_1 \cup T_2 \cup \cdots \cup T_t$. We use the notation $(R_1, R_2; T)$ to refer to such a $K_{2.t}$ model.

2.1 Fans and characterizing $K_{1,1,4}$-minor-free graphs

With Theorem 1.21, we saw a characterization of all $K_{2,4}$-minor-free graphs. O'Connell gave a complete characterization of the 3-connected $K_{1,1,4}$-minor-free graphs in [19]. Much of the work done there involves particular induced subgraphs called fans. We present the definition of a fan, some related definitions, and a few results before stating her main result.

Definition 2.2. An induced subgraph F of G on the vertices $r, v_1, v_2, \ldots v_k$ with edge set $E(F) = \{rv_i | 1 \leq i \leq k\} \cup \{v_i v_{i+1} | 1 \leq i \leq k-1\}$ such that $d_G(v_i) = 3$ for $1 \leq i \leq k$, and $d_G(r) \geq 4$ is called a *k-fan*. We call r the *rivet* vertex of F, and each v_i is called an *rim vertex* of F. The $v_i v_{i+1}$ edges are called *rim edges* and the $v_i r$ edges are called *spoke edges*. The path $v_1 v_2 \ldots v_k$ is called the *rim path* of F. The vertices v_1 and v_k each have exactly one neighbor outside of F. The path $v_0 v_1 v_2 \ldots v_k v_{k+1}$ is called the *extended rim* of F, where v_0 and v_{k+1} are the neighbors of v_1 and v_k outside of F.

We will use the term *k^+-fan* to refer to a k'-fan for any $k' \geq k$. For a graph G with fan F, contracting any rim edge of F is called *collapsing* the fan. Note that graph obtained from G by contracting any edge of F is independent of the choice of edge, up to isomorphism. In particular, if F is a k-fan with $k \geq 2$, the result is the graph obtained from G by replacing F with a $(k-1)$-fan (assuming the degree of the rivet vertex remains at least 4). Thus, the operation of collapsing a fan is well defined.

Given a fan F in a graph G, the operation of *expanding* F is the split of any rim vertex v_i of F subject to the following conditions. We know that v_i is adjacent to the rivet r of F and v_{i-1} and v_{i+1}. If v_i' and v_i'' are the vertices v_i splits into, then $N(v_i') = \{v_{i-1}, r, v_i''\}$ and $N(v_i'') = \{v_{i+1}, r, v_i''\}$.

Note that expanding a k-fan F in G results in a graph obtained from G by replacing F with a $(k+1)$-fan. The resulting graph does not depend on the choice of rim vertex to split, up to isomorphism. Thus, the fan expansion operation is well defined.

The following lemma implies that collapsing a 2^+-fan preserves 3-connectivity.

Lemma 2.3 (see O'Connell, [19])**.** *Given a 3-connected graph G with a fan F, then every rim edge of F is a 3-contractible edge of G.*

The following observation states that fan expansions preserve 3-connectivity. It follows from Lemma 1.4.

Observation 2.4. *If G' is obtained from a 3-connected graph G by expanding a fan F, then G' is 3-connected as well.*

The next lemma states that expanding a fan F preserves the property of being M-minor-free for certain graphs M when F is large enough.

Lemma 2.5 (O'Connell, [19])**.** *Let G be a graph with 5^+-fan F, and let M be a complete multipartite graph on at least six vertices with no degree 1 vertices (i.e., M is not $K_{1,t}$ for any t). Then if G has an M minor, the graph obtained from G by collapsing F also has an M minor.*

We in fact need a slightly stronger result, which we state here. This proof is adapted from a proof of this claim in a more specific scenario (expanding 3^+-fans in specific graphs, and with $t = 6$), found in [10].

Lemma 2.6. *Let G be a $K_{2,t}$-minor-free graph for some $t \geq 4$ with a 3^+-fan F. Then the graph G' obtained by expanding the fan F to a fan F' is $K_{2,t}$-minor-free as well.*

Proof. It suffices to prove the result when F is a $(k-1)$-fan and F' is a k-fan, for some $k \geq 4$; the result then follows by induction.

Suppose G' has a $K_{2,t}$-minor with model $(R_1, R_2; T)$. Let $P = v_1v_2 \dots v_k$ be the rim path of F', let $v_0v_1v_2 \dots v_kv_{k+1}$ be its extended rim path, and let r be its rivet vertex. We cannot have $R_j \subseteq V(P)$ for $j = 1$ or 2, since each connected subgraph of P is adjacent to at most three vertices in G', and $t \geq 4$. Then for each $v_i \in R_j \cup T$, there exists a path $P_j(v_i)$ from v_i to a vertex of R_j, all of whose internal vertices are in $R_j \cap V(P)$, and where the other end is v_0, v_{k+1} or r.

Suppose there is some $v_i \notin R_1 \cup R_2 \cup T$, $1 \leq i \leq k$. Then there is an edge e of P with one end in $R_1 \cup R_2 \cup T$. Contracting e gives a graph isomorphic to G, and preserves the existence of a $K_{2,t}$-minor, which is a contradiction. So we have $V(P) \subseteq R_1 \cup R_2 \cup T$.

Suppose v_i and v_{i+1} are both in R_j, for some $1 \leq i \leq k-1$. Then contracting v_iv_{i+1} gives a graph isomorphic to G and preserves the existence of a $K_{2,t}$-minor. Thus, there are no two consecutive vertices of P that belong to the same R_j.

First, we consider the case when $r \notin R_1 \cup R_2$. If there is $v_a \in T$, then $P_j(v_a) = v_av_{a-1} \dots v_1v_0$, and $P_{3-j}(v_a) = v_av_{a+1} \dots v_kv_{k+1}$ for $j = 1$ or 2. If $a > 2$, there are two consecutive R_j vertices in P, and if $a < k-1$, there are two consecutive R_{3-j} vertices in P. Since $k \geq 4$, at least one of $a > 2$ or $a < k-1$ holds, so there are either two consecutive R_j vertices in P or two consecutive R_{3-j} vertices in P, which is a contradiction. If there is no vertex of T in P, then every vertex of P is in R_j for some j. There are no two consecutive vertices of P in the same R_j, so $v_1 \in R_j, v_2 \in R_{3-j}$, and $v_3 \in R_j$. However, now there is no path from v_2 to v_0 or v_{k+1} all of whose internal vertices are in $R_{3-j} \cap V(P)$. This is a contradiction.

Thus, we may assume without loss of generality that $r \in R_1$. Suppose there is a vertex $v_i \in R_1$ for some $2 \leq i \leq k-1$. The vertex v_i is adjacent to no vertices outside of F', and all vertices of T in F' are adjacent to r. Thus, $(R_1 - \{v_i\}, R_2; T)$ is a model of $K_{2,t}$ such that $v_i \in V(P)$ is not in $R_1 \cup R_2 \cup T$, which is a contradiction. Thus, we may assume $v_i \notin R_1$ for all $2 \leq i \leq k-1$. We now will consider cases based on the number of vertices in $|T \cap V(P)|$.

Suppose $|T \cap V(P)| \geq 2$. Let v_i be the first vertex of P in T and let v_j be the last vertex of P in T. The path $P_2(v_i)$ must be $v_0v_1v_2 \dots v_i$ and the path $P_2(v_j)$ must be $v_jv_{j+1} \dots v_kv_{k+1}$. There are no two consecutive R_2 vertices, so $i \leq 2$ and $j \geq k-1 \geq 3$. If $v_1 \in R_2$ then $v_0 \in R_1$, and

$(R_1, R_2 - \{v_1\}; (T - \{v_2\}) \cup \{v_1\})$ is a model of $K_{2,t}$ in G' where $v_2 \notin R_1 \cup R_2 \cup T$, which is a contradiction. Therefore, we may assume $v_1 \in T$ and $i = 1$. But now since $v_2 \notin R_1$ from above, we have $v_2 \in R_2 \cup T$. Since $v_1 \in T$ and $v_j \in T$ with $j \geq 3$, there is no path $P_2(v_2)$, a contradiction.

Suppose $|T \cap V(P)| = 1$. Let v_i be the vertex in T. Without loss of generality $P_2(v_i) = v_0v_1 \dots v_i$. There are no two consecutive vertices of R_2 in P. Thus, $i \leq 2$. Consider vertex v_3. It is not in T, and not in R_1, so it must be in R_2. Then $P_2(v_3) = v_3v_4 \dots v_{k+1}$. However, then there are two consecutive vertices in R_2, a contradiction.

Suppose $|T \cap V(P)| = 0$. There are no consecutive vertices of P in R_2, and all vertices of P must be in R_1 or R_2. Thus, there must be a sequence of vertices $v_i \in R_1$, $v_{i+1} \in R_2$, $v_{i+2} \in R_1$ for some $1 \leq i \leq k-2$. However, this give a vertex $v_i \in R_1$ for some i between 2 and $k-1$, a contradiction. □

Using computer results and the previously-stated facts about fans, O'Connell proves the following characterization of the 3-connected, $K_{1,1,4}$-minor-free graphs.

Theorem 2.7 (O'Connell, [19])**.** *There exists a set of base graphs $\mathcal{B}$ on at most 10 vertices such that a graph on at least 13 vertices is 3-connected and $K_{1,1,4}$-minor-free if and only if it is obtained from a graph in $\mathcal{B}$ by fan expansions.*

In fact, O'Connell gives an exact description of the base graphs in $\mathcal{B}$, but we omit those details here. This characterization allows her to prove the following counting result.

Theorem 2.8 (O'Connell, [19])**.** *For $n \geq 13$, there are exactly $\left\lfloor \frac{23n}{2} \right\rfloor - 39$ non-isomorphic, 3-connected, $K_{1,1,4}$-minor-free graphs on n vertices.*

2.2 $\mathcal{Q}$-sequences and characterizing 4-connected, $K_{2,5}$-minor-free graphs

We now turn our attention to 4-connected $K_{2,5}$-minor-free graphs. Ellingham and Gaslowitz gave a complete characterization of these graphs, which is presented in [13]. They characterize them in terms of $\mathcal{Q}$-sequences, which provide a way of constructing ring-like graphs out of small pieces.

Define an *X-, I-, Δ^+-, Δ^-, or Q-piece* to be a copy of the corresponding graph in Figure 2.1.

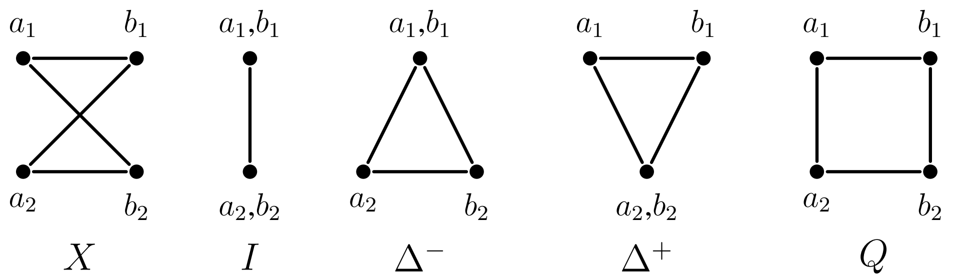

Figure 2.1: The five types of pieces that are used to build $\mathcal{Q}$-sequence graphs.

Definition 2.9. A *$\mathcal{Q}$-sequence* is a cyclic sequence of X-, I-, Δ^+-, Δ^--, and Q-pieces such that:

(1) Each I- or Q-piece appears between two X-pieces.

(2) There are no two consecutive Δ^+-pieces and no two consecutive Δ^--pieces.

Given a cyclic sequence of pieces $(P_0, P_1, \dots P_{m-1})$, refer to each vertex a_i or b_i in piece P_j as a_i^j. The index of a piece will always be considered only up to congruence modulo m. A *$\mathcal{Q}$-sequence graph* is a graph that can be built from a $\mathcal{Q}$-sequence in the following way. For each j, with $0 \leq j \leq m-2$, identify b_1^j with a_1^{j+1} and b_2^j with a_2^{j+1}. Then either identify b_1^{m-1} with a_1^0 and b_2^{m-1} with a_2^0, or identify b_1^{m-1} with a_2^0 and b_1^{m-1} with a_2^0. If the last and first pieces are Δ^+ and Δ^- (in either order) then we must identify b_1^{m-1} with a_1^0 and b_2^{m-1} with a_2^0; if the last and first pieces are both Δ^+ or both Δ^- then we must identify b_1^{m-1} with a_2^0 and b_2^{m-1} with a_1^0.

It is worth noting that the definitions of $\mathcal{Q}$-sequences and $\mathcal{Q}$-sequence graphs presented here are equivalent to but differ slightly from the ones presented in [13]. In particular, there they consider Δ^+ and Δ^- both as a single Δ piece, and allow for attaching pieces in different orientations. The definition as presented here is more analogous to the one we present for $\mathcal{A}$-sequences in Chapter 3. Also, with this definition, it is clear that $\mathcal{Q}$-sequence graph are built from (2,2)-terminal graphs, a concept introduced in Chapter 3.

The following theorem provides a characterization of the 4-connected $K_{2,5}$-minor-free graphs.

Theorem 2.10 (Ellingham and Gaslowitz, see [13])**.** *For a graph G on at least 9 vertices, the following are equivalent:*

(1) G is 4-connected and $K_{2,5}$-minor-free.

(2) G is a $\mathcal{Q}$-sequence graph.

(3) G is a 4-connected minor of $C_n[K_2]$ for some n.

As part of the proof of this result, they also prove the following, which will be useful to us throughout this document. This result also appears without proof in [6].

Lemma 2.11 (see [13])**.** *$C_n[K_2]$ is $K_{2,5}$-minor-free.*

The characterization described in Theorem 2.10 also provides exact and asymptotic counting results on the number of these graphs.

In Chapter 3, we present results towards a characterization of the family of 3-connected $K_{2,5}$-minor-free graphs. This family contains both the 4-connected $K_{2,5}$-minor-free graphs and the 3-connected $K_{1,1,4}$-minor-free graphs.

2.3 Strips and a rough description of $K_{2,t}$-minor-free graphs

Most of the families of graphs considered thus far in this chapter avoided $K_{2,t}$ minors for some small, specific t ($3 \leq t \leq 5$). We will now discuss a rough description of the $K_{2,t}$-minor-free graphs for any given t proven by Guoli Ding in [8]. The results in Chapter 4 refine these results in order to give a characterization of the $K_{2,t}$-minor-free graphs for any given t.

The characterizations of the $K_{2,4}$-minor-free graphs, the 3-connected, $K_{1,1,4}$-minor-free graphs, and the 4-connected $K_{2,5}$-minor-free graphs presented earlier can be thought of as refinements of Ding's result for particular small t. Before we can state the result, a few definitions are needed.

Let G be a graph with a Hamilton cycle C, which we call the *reference cycle* of G. Any edge of G not in C is a *chord*. Two chords ab and cd with distinct endpoints *cross* if the endpoints appear in the order a, c, b, d in C. A graph G is said to be a *type-I* graph if every chord crosses at most one other chord and for any two crossing chords ab and cd, either ac and bd are both edges of C or ad and bc are both edges of C.

Let H be a type-I graph with reference cycle C. Suppose there exist edges $ab, cd \in E(C)$ such that all chords of G go between the two paths of $C - \{ab, cd\}$. If ab and cd do not share an endpoint, the $H - \{ab, cd\}$ is called a *strip* with *corners* $a, b, c,$ and d. If ab and cd do share a vertex, say $b = c$, then the graph $H - \{ab, cd\}$ is called a *weak fan* with *corners* $a, b,$ and d and *center* or *rivet* b.

It is worth noting that for a weak fan F, if we demand that each vertex other than the corners has degree at least 3 and that the rivet vertex has degree at least 4 (perhaps when considered as part of a larger graph containing F), then F is in fact a fan along with its extended rim in terms of our earlier definition of a fan.

Observation 2.12. *Every strip is a minor of $P_n[K_2]$. Moreover, given a strip S with corners $a, b, c,$ and d obtained from a graph H by deleting edges of the reference cycle ab and cd, then there are disjoint paths' from a to c and from b to d or from a to d and from b to c.*

For a graph G, the operation of *adding* a weak fan or strip to G identifies the corners of a fan or a strip to distinct vertices of G. If more than one weak fan or strip is added to G, then any two of the corners of added weak fans and strips are identified with distinct vertices unless one of them is the center of a weak fan and the other is either the center of another weak fan or a corner of a strip.

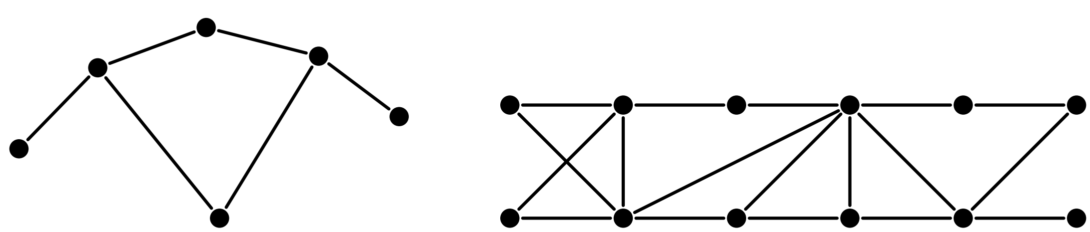

Figure 2.2: Examples of a weak fan (on the left) and a strip (on the right).

Ding defines a slight variation of the 2-sum, which we call the modified 2-sum. Given two graphs G_1 and G_2, let z_i be a vertex of G_i that is incident with exactly two edges x_iz_i and y_iz_i.. Let $G'_i = G_i - z_i$. Then the *modified 2-sum* of G_1 and G_2 is obtained by identifying x_1 with x_2 and y_1 with y_2 in $G'_1 \cup G'_2$. Note that this is similar to the usual 2-sum, as two vertices from G_1 are identified with two vertices of G_2.

Theorem 2.13. *For a given t, there exists a finite set of base graphs $\mathcal{B}_t$, such that every 2-connected, $K_{2,t}$-minor-free graph G is obtained by a sequence of zero or more modified 2-sum operations from graphs $G_1, G_2, \ldots G_k$, where each G_i is obtained from a graph $B_i \in \mathcal{B}_t$ by adding strips and weak fans.*

Note that if G is obtained from a modified 2-sum of two graphs, it contains a 2-cut, and so is not 3-connected. Any graph G with a fan has a degree 3 vertex and so is not 4-connected. The corners of a strip form a 4-cut, so any graph with a strip with at least one non-corner vertex is not 5-connected. With this in mind, Theorem 2.13 gives the following descriptions when G is 3-, 4-, or 5-connected.

Corollary 2.14. *For a given t, there exists a finite set of base graphs $\mathcal{B}_t$ such that for every $K_{2,t}$-minor-free graph G:*

- *When G is 3-connected, G is obtained by adding fans and strips to a graph $B \in \mathcal{B}_t$.*
- *When G is 4-connected, G is obtained by adding strips to a graph $B \in \mathcal{B}_t$.*
- *When G is 5-connected, $G \in \mathcal{B}_t$.*

It is worth noting that this description is not a characterization. It provides necessary conditions for a graph to be k-connected and $K_{2,t}$-minor-free, but does not show that they are sufficient conditions as well. In Chapter 4, we provide a refinement of these descriptions for the 3- and 4-connected cases. This then gives a characterization of the 3- or 4-connected, $K_{2,t}$-minor-free graphs for any given t up to a specific set of base graphs.

Chapter 3

Towards a characterization of the 3-connected $K_{2,5}$-minor-free graphs

In this chapter, we present a number of new results that move towards a characterization of the 3-connected, $K_{2,5}$-minor-free graphs, and the planar, 3-connected, $K_{2,5}$-minor-free graphs. We conjecture that a graph is 3-connected and $K_{2,5}$-minor-free if and only if it is obtained from one of a finite set of base graphs by expanding fans, or is an $\mathcal{A}$-sequence graph, which will be defined later, and have a similar conjecture for the planar case. We begin by defining and proving some results on these $\mathcal{A}$-sequence graphs. In particular, we show that they are 3-connected and $K_{2,5}$-minor-free. Several of the definitions and results here are critical for our work in Chapter 4 on general $K_{2,t}$-minor-free graphs. Then we prove a new result on the topic of fan expansions. This result, Proposition 3.17, provides a strategy for proving a characterization of the 3-connected, $K_{2,5}$-minor-free graphs. Also, it has applications outside of characterizing this particular family; it applies when attempting to show that any family of 3-connected graphs defined by excluded minors is obtained by fan expansions. This chapter concludes with a discussion of the conjectured characterizations of these families, the computational evidence for them, and what further work might be used to give complete characterizations.

3.1 (2,2)-terminal graphs and $\mathcal{A}$-sequence graphs

We begin this section by defining a set of partially labelled graphs and a concatenation operation on them that gives a monoid structure. Recall that a *semigroup* is a set along with an associative binary operation, and a *monoid* is a semigroup with an identity. Let a *(2,2)-terminal graph* be an unlabelled graph except for (not necessarily distinct) vertices identified as $a_1(G), a_2(G), b_1(G)$, and $b_2(G)$. When it is clear from context we may denote these vertices as simply a_1, a_2, b_1 and b_2 and indicate which vertex is $a_1(G)$ by labelling it a_1, and so on. Given (2,2)-terminal graphs G_1 and G_2 let the concatenation G_1G_2 be the graph obtained by identifying $b_1(G_1)$ with $a_1(G_2)$ and

$b_2(G_1)$ with $a_2(G_2)$, and setting $a_1(G_1G_2) = a_1(G_1)$, $a_2(G_1G_2) = a_2(G_1)$, $b_1(G_1G_2) = b_1(G_2)$, and $b_2(G_1G_2) = b_2(G_2)$. This gives us a new (2,2)-terminal graph. This concatenation operation is shown in Figure 3.1

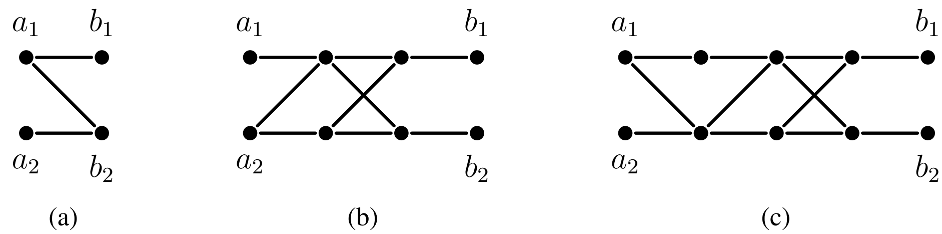

Figure 3.1: An example of concatenation of (2,2)-terminal graphs: (a) G_1 (b) G_2 (c) G_1G_2.

The operation of concatenation is associative, so the (2,2)-terminal graphs up to label-preserving isomorphism along with this concatenation operation form a semigroup. Let E be the (2,2)-terminal graph consisting of two vertices, one labelled a_1 and b_1 and the other labelled a_2 and b_2, and no edges. For any (2,2)-terminal graph G, $EG = GE = G$, so E is the identity of the semigroup. Thus, the (2,2)-terminal graphs along with the operation of concatenation form a monoid. The concept of (2,2)-terminal graphs will be used extensively in Chapter 4. It will also be used here to define $\mathcal{A}$-sequence graphs.

We now present a family of graphs, the $\mathcal{A}$-sequence graphs, and then show that this family is 3-connected and $K_{2,5}$-minor-free.

Define an *X-, I-, A^+-, A^--, A^*-, $Y_{0,0}$-, $Z_{0,0}$-, and T-piece* to be a copy of the (2,2)-terminal graph pictured in Figure 3.2. Certain vertices of $Y_{0.0}$ and $Z_{0,0}$ in this figure are marked as p_1 and p_2 to assist in the definition of $Y_{i,j}$ and $Z_{i,j}$ for general i and j below; these are not formal labels. An *A-type piece* is either an A^+-, A^--, or A^*-piece.

Define a *$Y_{i,0}$-piece* (or *$Z_{i,0}$-piece*) for $i > 0$ to be the graph obtained by adding vertices $v_1, v_2, \dots v_i$ to the (2,2)-terminal graph $Y_{0,0}$ (respectively $Z_{0,0}$) so that $a_1v_1v_2 \dots v_ia_2$ is a path, each v_k for $1 \leq k \leq i$ is adjacent to p_1 (as marked in Figure 3.2), and there are no additional edges.

Define a *$Y_{i,j}$-piece* (or *$Z_{i,j}$-piece*) for $j > 0$ to be the graph obtained by adding the vertices

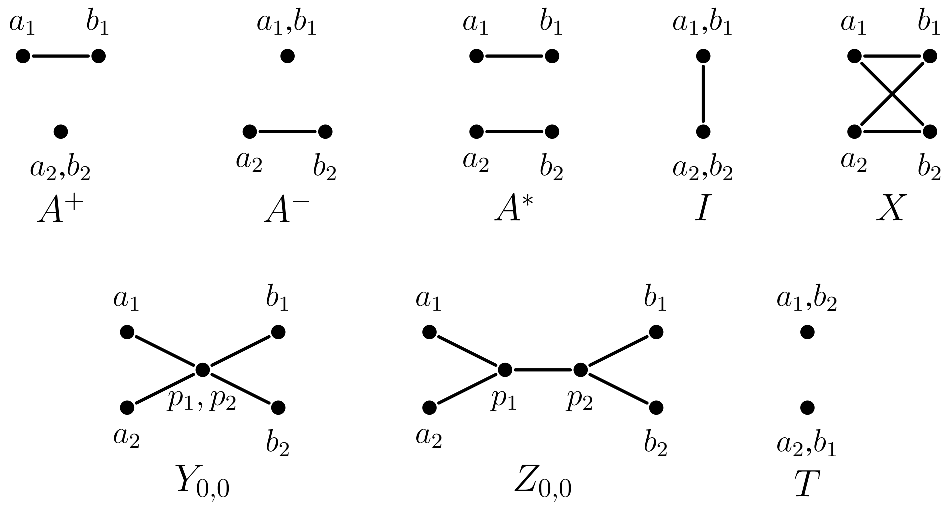

Figure 3.2: Some of the pieces that are used to build $\mathcal{A}$-sequence graphs.

$w_1, w_2, \dots w_k$ to the (2,2)-terminal graph $Y_{i,0}$ (respectively $Z_{i,0}$) so that $b_1 w_1 w_2 \dots w_j b_2$ is a path, each w_k for $1 \leq k \leq j$ is adjacent to p_2 (as marked in Figure 3.2), and there are no additional edges. Examples of $Y_{i,j}$ and $Z_{i,j}$ are shown in Figure 3.3.

Definition 3.1. An $\mathcal{A}$-sequence is a cyclic sequence of X-, I-, A^+-, A^--, A^*-, $Y_{i,j}$-, and $Z_{i,j}$-pieces $(P_0, P_1, \dots P_{m-1}), m \geq 1$ with the following properties.

(1) There are no two consecutive A-pieces.

(2) No A-type piece immediately precedes a $Y_{0,j}$ or $Z_{0,j}$-piece, and no A-type piece immediately follows a $Y_{i,0}$- or $Z_{i,0}$-piece.

(3) There is at most one total $Y_{i,j}$- or $Z_{i,j}$-piece.

(4) There are no two consecutive I-pieces.

(5) There is at most one T-piece, and it occurs next to an I-piece, but not between two I-pieces.

Given a $\mathcal{A}$-sequence $(P_0, P_1, \dots P_{m-1})$, we will consider the index of a piece up to congruence modulo m, unless otherwise noted. We will use the notation a_i^j and b_i^j to represent a_i or b_j in the piece P_j. Given a (2,2)-terminal graph G, let $\gamma(G)$ be the graph obtained by identifying $a_1(G)$ with $b_1(G)$ and $a_2(G)$ with $b_2(G)$ in G and dropping the labelling. We say that a graph G is an

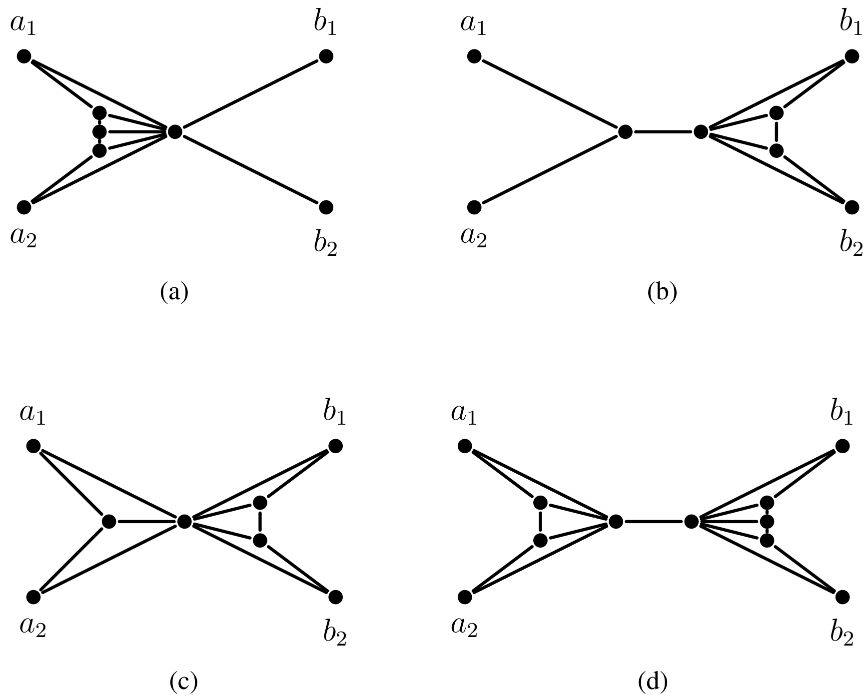

Figure 3.3: Some examples of $Y_{i,j}$ and $Z_{i,j}$ for various i and j: (a) $Y_{3,0}$, (b) $Z_{0,2}$, (c) $Y_{1,2}$, and (d) $Z_{2,3}$.

$\mathcal{A}$-sequence graph if it is equal to $\gamma(P_0P_1P_2\ldots P_{m-1})$ where $P_0, P_1, \ldots P_{m-1}$ is an $\mathcal{A}$-sequence. This gives a unique graph for each $\mathcal{A}$-sequence.

We can also describe how to create this graph from a $\mathcal{A}$-sequence without using the language of (2,2)-terminal graphs. Start with the piece P_0. Then for each piece P_j, with $1 \leq j \leq m-1$ add the piece P_j to the graph by identifying a_1^j with b_1^{j-1} and a_2^j with b_2^{j-1}. Then after adding the final piece, P_{m-1} identify b_1^{m-1} and b_2^{m-1} with a_1^0 and a_2^0.

The definition of $\mathcal{A}$-sequences above is analogous to the definition of $\mathcal{Q}$-sequence graphs. While this notion of (2,2)-terminal graphs was not used in that definition, it could have been.

The conditions (1)-(6) in the definition of an $\mathcal{A}$-sequence either prevent the inclusion of redundant edges or help to ensure that the resulting graph is 3-connected.

The following identities are useful when working with $\mathcal{A}$-sequence graphs. We state them using the terminology of (2,2)-terminal graphs.

Observation 3.2. *The following equalities hold in the monoid of (2,2)-terminal graphs.*

(1) $I^2 = II = I$.

(2) $A^+A^- = A^-A^+ = A^*$.

The piece T is somewhat strange, as it does not add any edges to the graph. However, it switches the 'top' and 'bottom' of the graph, effectively adding a twist to the strip-like structure of the graph, hence the name T-piece. For a (2,2)-terminal graph G, let $\widetilde{G}$ be the (2,2)-terminal graph obtained from G by swapping labels a_1 and a_2 and also swapping labels b_1 and b_2. If we join the pieces in a horizontal sequence, $\widetilde{G}$ can be thought of as a vertical reflection of G. For any (2,2)-terminal graph, $\widetilde{G} = TGT$. Note also that $TT = E$. These two facts give rules that allows us to move a T-piece past other pieces. In particular, $TG = \widetilde{G}T$ and $GT = T\widetilde{G}$. The effects of T on the pieces used in $\mathcal{A}$-sequence graphs are summarized in the following observation.

Observation 3.3. *The following equalities hold in the monoid of (2,2)-terminal graphs.*

(1) $T^2 = TT = E$.

(2) $TG = \widetilde{G}T$ *and* $GT = T\widetilde{G}$. *In particular cases this gives the following:*

(a) $TA^+ = A^-T$.

(b) $TA^- = A^+T$.

(c) $TA^* = A^*T$.

(d) $TI = IT$.

(e) $TX = XT = X$.

(f) $TY_{i,j} = Y_{i.j}T$.

(g) $TZ_{i,j} = Z_{i,j}T$.

For two (2,2)-terminal graphs S_1 and S_2, we will say $S_1 \sim S_2$ if $\gamma(S_1) \cong \gamma(S_2)$. With this notation, we can write a few more simplifying rules.

Observation 3.4. *(1)* $P_0P_1 \dots P_{m-1} \sim P_1P_2 \dots P_{m-1}P_0$, *i.e., cyclic shifts of sequences give isomorphic graphs.*

(2) $P_0P_1\dots P_{m-1} \sim TP_0P1\dots P_{m-1}T = TP_0TTP_1T\dots TP_{m-1}T = \widetilde{P}_0\widetilde{P}_1\dots\widetilde{P}_{m-1}$, *i.e., vertical reflections of sequences give isomorphic graphs.*

We now show that the $\mathcal{A}$-sequence graphs are $K_{2,5}$-minor-free.

Lemma 3.5. *Every $\mathcal{A}$-sequence graph is $K_{2,5}$-minor-free.*

Proof. Let G be an $\mathcal{A}$-sequence graph with $\mathcal{A}$-sequence $(P_0, P_1, \dots P_{m-1})$. Without loss of generality, we will take P_{m-1} to be the $Y_{i,j}$- or $Z_{i,j}$-piece if there is one, and $i \geq j$.

Assume first that if $(P_0, P_1, \dots P_{m-1})$ contains a $Y_{i,j}$- or $Z_{i,j}$-piece, then $i, j \leq 1$. Every $\mathcal{A}$-sequence graph G with a $Y_{0,0}$- $Y_{1,0}$-, $Y_{1,1}$-, $Z_{0,0}$-, or $Z_{1,0}$-piece can have that piece replaced by $Z_{1,1}$ to get G'. Then G' is a valid $\mathcal{A}$-sequence graph and G is a minor of G', so if G' is $K_{2,5}$-minor-free, so is G.

So assume that if G contains an $Y_{i,j}$- or $Z_{i,j}$-piece it is $Z_{1,1}$. To show that such a G is $K_{2,5}$-minor-free, we will show that every G is a minor of $C_n[K_2]$ for some n. For any n, $C_n[K_2]$ is $K_{2,5}$-minor-free by Lemma 2.11, so this suffices. We will then show that every $\mathcal{A}$-sequence graph is a minor of a graph obtained from such a G by expanding 3^+-fans, which preserves $K_{2,5}$-minor-freeness by Lemma 2.6.

Let $k = 2$ if $P_{m-1} = Z_{1,1}$, and let $k = 0$ otherwise. We will show that G is a minor of $C_{2m+k}[K_2]$. Label the vertices of $C_{2m+k}[K_2]$, such that one copy of C_{2m+k} has vertices labelled $c_0, c_1, \dots c_{2m+k-1}$ and the other is labelled $d_0, d_1, \dots d_{2m+k-1}$. Then for each piece P_i, perform the following operations depending on the type of piece.

If P_i is an X-piece: Delete edges $c_{2i}d_{2i}$ and $c_{2i+1}d_{2i+1}$.

If P_i is an A^*-piece: Delete edges $c_{2i}d_{2i}$, $c_{2i+1}d_{2i}$, $c_{2i}d_{2i+1}$, and $c_{2i+1}d_{2i+1}$.

If P_i is an A^+-piece: Delete edges $c_{2i}d_{2i}$, $c_{2i+1}d_{2i}$, $c_{2i}d_{2i+1}$, and $c_{2i+1}d_{2i+1}$. Contract edge $d_{2i}d_{2i+1}$.

If P_i is an A^--piece: Delete edges $c_{2i}d_{2i}$, $c_{2i+1}d_{2i}$, $c_{2i}d_{2i+1}$, and $c_{2i+1}d_{2i+1}$. Contract edge $c_{2i}c_{2i+1}$.

If P_i is an I-piece: Contract edges $c_{2i}c_{2i+1}$ and $d_{2i}d_{2i+1}$.

If P_i is a T-piece: Delete edges $c_{2i}d_{2i}$, $c_{2i+1}d_{2i+1}$, $c_{2i}c_{2i+1}$, and $d_{2i}d_{2i+1}$. Contract edges $c_{2i}d_{2i+1}$ and $c_{2i+1}d_{2i}$.

If P_i is a $Z_{1,1}$-piece: Note that $i = m - 1$ and $k = 2$ here. Delete edges $c_{2i}d_{2i}$, $c_{2i+1}c_{2i+2}$, $c_{2i+1}d_{2i+2}$, $c_{2i+2}d_{2i+1}$, and $c_{2i+3}d_{2i+3}$.

Then for each i between 0 and $m-2$, delete edges $c_{2i+1}d_{2i+2}$ and $c_{2i+2}d_{2i+1}$ and contract edges $c_{2i+1}c_{2i+2}$ and $d_{2i+1}d_{2i+2}$. Then delete edges $c_{2m+k-1}d_0$ and c_0d_{2m+k-1} and contract edges $c_{2m+k-1}c_0$ and $d_{2m+k-1}d_0$.

The resulting graph is G, so G is a minor of $C_n[K_2]$ for some n. Thus, G is $K_{2,5}$-minor-free.

Now it remains to be shown that G is $K_{2,5}$-minor-free if it contains a $Y_{i,j}$- or $Z_{i,j}$-piece with i, j, or both greater than or equal to 2. Suppose that $P_{m-1} = Y_{i,0}$ for $i \geq 2$. Consider the graph $\gamma(P_0P_1P_2 \dots P_{m-2}A^*Y_{1,0})$. The vertices a_1 and a_2 in the $Y_{1,0}$ piece have degree 3, so $a_1v_1a_2$ is the rim path of a 3-fan with rivet vertex $p_1 = p_2$, where $p_1 = p_2$ and v_1 are as defined in the definition of the $Y_{1,0}$-piece. Call this fan F. The graph $\gamma(P_0P_1 \dots P_{m-2}A^*Y_{i,0})$ is obtained from $\gamma(P_0P_1 \dots P_{m-2}A^*Y_{1,0})$ by expanding F. By the above reasoning, $\gamma(P_0P_1 \dots P_{m-2}A^*Y_{1,0})$ is $K_{2,5}$-minor-free, so by Lemma 2.6, $\gamma(P_0P_1 \dots P_{m-2}A^*Y_{i,0})$ is $K_{2,5}$-minor-free as well. The graph G is a minor of the graph $\gamma(P_0P_1 \dots P_{m-2}A^*Y_{i,0})$, so G is $K_{2,5}$-minor-free. Similar reasoning covers the cases $P_{m-1} = Y_{i,j}$ with $i, j \geq 1$ and $i \geq 2$, and $P_{m-1} = Z_{i,j}$ by considering fan expansions in the following graphs: $\gamma(P_0P_1 \dots P_{m-2}A^*Y_{1,1}A^*)$, $\gamma(P_0P_1 \dots P_{m-2}A^*Z_{1,0})$, $\gamma(P_0P_1 \dots P_{m-2}A^*Z_{1,1}A^*)$, and . $\square$

It will be useful to restrict our attention to $\mathcal{A}$-sequence graphs without large fans. Given an $\mathcal{A}$-sequence graph G with a given $\mathcal{A}$-sequence representation, let the *reduction* of G, denoted $\overline{G}$, be the $\mathcal{A}$-sequence graph where each IA^+IA^+I sequence is replaced with IA^+I, each IA^-IA^-I sequence is replaced with IA^-I, and every $Y_{i,j}$ or $Z_{i,j}$ piece is replaced with a $Y_{i',j'}$ or $Z_{i',j'}$-piece where $i' = 1$ if $i \geq 1$, and $i' = 0$ if $i' = 0$ and $j' = 1$ if $j \geq 1$ and $j' = 0$ if $j = 0$. Then the *reduction size* of G, denoted $r(G)$, is $|V(\overline{G})|$. If $\overline{G} = G$ we say that G is *reduced*.

Lemma 3.6. *Every $\mathcal{A}$-sequence graph with reduction size at least 7 is 3-connected.*

Proof. We will proceed by induction on $r(G)$. Note that if $\overline{G}$ is 3-connected, G is as well, by Observation 2.4, since G is obtained from $\overline{G}$ by expanding fans.

Thus, it suffices to prove the statement for all reduced graphs on $n \geq 7$ vertices. The cases of $n = 7, 8, 9, 10,$ and 11 were checked via computer. Take as the base cases for the induction $n = 10, 11$. Every reduced $\mathcal{A}$-sequence graph with $|V(G)| \geq 12$ has minimum degree 3. This claim was checked by checking all possible sequences of pieces to which an individual vertex in such a graph could belong by hand. Thus, it suffices to show that G is obtained from a 3-connected graph by a vertex split by Lemma 1.4.

We claim that for a reduced $\mathcal{A}$-sequence graph G on at least 12 vertices there is a rail edge e such that G/e is an $\mathcal{A}$-sequence graph and $r(G/e) \geq |V(G)| - 2$. Proving this claim is sufficient to complete the proof because then G has minimum degree at least 3 and is obtained by a vertex split from the graph G/e, which is 3-connected by induction.

Let $G = \gamma(G_1 G_2 \ldots G_m)$. The sequence $G_1, G_2, \ldots G_m$ must contain an X, IA^*I, IA^+I, or IA^-I subsequence, by the following reasoning. Suppose we have a $\mathcal{A}$-sequence without any of the subsequences X, IA^*I, IA^+I, or IA^-I. Then every IA^*, IA^+, or IA^- sequence is followed by a $Y_{i,j}$- or a $Z_{i,j}$-piece, and every A^*I, A^+I, or A^-I sequence is preceded by a $Y_{i,j}$- or $Z_{i,j}$-piece. Every A-type piece is followed and preceded by an I-, $Y_{i,j}$-, or $Z_{i,j}$-piece, since there are no X-pieces. Thus, the sequence consists of at most two A-type pieces, at most one $Y_{i,j}$- or $Z_{i,j}$-piece, and some number of I-pieces that do not contribute vertices. Therefore, the graph corresponding to such a sequence has at most 10 vertices, but $|V(G)| \geq 12$.

Suppose that G has an X-piece G_i or G has an IA^*I subsequence, where $G_i = A^*$ in that sequence. Then taking $e = a_2^i b_2^i$, we have that $G/e = \gamma(G_1 G_2 \ldots G_{i-j} IA^+I G_{i+j} \ldots G_m)$, where $j = 1$ if $G_i = X$ and $j = 2$ if $G_i = A^*$. After replacing any occurrences of II with I in this sequence, we have an $\mathcal{A}$-sequence representation for G/e. If G/e has reduction size less than $|V(G)| - 2$, G_i is both preceded by IA^+I or IA^+ and followed by IA^+I or A^+I. In any event, we can instead chose $e = a_1^i b_1^i$, and obtain $r(G/e) = |V(G)| - 1$.

Suppose then that G has an IA^+I or IA^-I subsequence. Without loss of generality, assume a IA^+I subsequence, and let $G_i = A^+$ in that sequence. Then let $e = a_1^i b_1^i$. Then $\gamma(G_1G_2 \ldots G_{i-2}IG_{i+2} \ldots G_m)$ is an $\mathcal{A}$-sequence for G/e. We have $r(G/e) \geq |V(G)| - 2$, since in the worst case, G_i may be both preceded by an IA^-I and followed by an IA^-I subsequence, but this gives $r(G/e) = |V(G/e)| - 1 = |V(G)| - 2$. □

This result is sharp in that there is an infinite family of $\mathcal{A}$-sequence graphs with reduction size 6 that are not 3-connected. The graph $G_0 = \gamma(IA^+IA^*IA^-IA^*)$ has a 2-cut, and every graph G such that $\overline{G} = G_0$ has a 2-cut as well. An example of such a G is depicted in Figure 3.4.

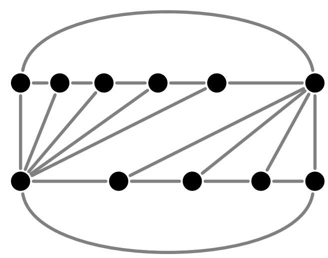

Figure 3.4: An $\mathcal{A}$-sequence graph with reduction size 6 and a 2-cut.

3.2 Expanding fans

In this section we present a number of results on fans in graphs, leading to a proof of Proposition 3.17. This new result gives a condition on the number of fan edges in a graph G, such that if G/e has this property for every 3-contractible edge e, then G has this property as well. We know that expanding 3^+-fans does not introduce a $K_{2,t}$-minor for any $t \geq 4$ by Lemma 2.6. Proposition 3.17 gives a condition under which we can verify that only fan expansions occur, giving us a way to characterize the 3-connected, $K_{2,5}$-minor-free graphs.

This result has potential additional applications. By Lemma 2.5, expanding 4^+-fans in M-minor-free graphs for complete multipartite graphs M such as $K_{1,1,t}$ preserves M-minor-freeness. It may be true that expanding 3^+-fans in M-minor-free graphs for such M does not introduce M

minors as well. Since Proposition 3.17 does not depend on the specific minor we avoid, it may apply for $K_{1,1,t}$ minors or other complete multipartite minors.

The literature contains other instances where fans must be dealt with when considering minors in 3-connected graphs. For instance, in [9], Ding and Liu use 3^+-connected graphs that exclude two configurations, one of which is a 3-fan, as part of characterizing 3-connected, (Cube$/e$)-minor-free graphs. Our Proposition 3.17 can be restated in language similar to their key result, [9, Theorem 1.2]. In some results of Costalonga [7], generalized versions of fans play an important role in finding forests or even spanning trees consisting of 3-contractible edges.

Throughout this section, we will be concerned about how edge contraction can change the degree of a vertex. This observation will be helpful to us.

Observation 3.7. *Contracting an edge $e = ab$ has the following effects on the degrees of vertices.*

(1) $d_{G/e}([ab]) = d_G(a) + d_G(b) - 2 - |N(a) \cap N(b)|$, so $[ab]$ has lower degree than a only if the neighborhood of b is a subset of the closed neighborhood of a.

(2) Any vertex v which is adjacent to both a and b in G has its degree decrease by 1 when e is contracted: $d_{G/e}(v) = d_G(v) - 1$.

(3) All other vertices have the same degree in G and G/e.

The following notes a few basic facts about fans.

Observation 3.8. *Let G be a 3-connected graph, and let F be a fan in G with vertices labelled as in Definition 2.2.*

(1) Of the rim vertices v_1 and v_k are the only ones which have any neighbors outside of F. Furthermore, if G is not a wheel, those neighbors must be distinct, i.e., $v_0 \neq v_{k+1}$ to have 3-connectivity,

(2) If G is not a wheel, the rivet vertex must have a neighbor outside of the extended rim, $v_0v_1, \ldots v_kv_{k+1}$, otherwise $\{v_0, v_{k+1}\}$ is a 2-cut.

(3) No two adjacent rim vertices have a common neighbor other than r.

The following is a convenient way to check if a vertex split is a fan expansion.

Observation 3.9. *Suppose G' is obtained from G by splitting a rim vertex v_i of a fan with rivet r into v_i' and v_i''. Then if v_i' and v_i'' each have degree 3 in G' and each is adjacent to r in G', G' is obtained from G by a fan expansion.*

We now define a natural notion for a fan to be maximal in a graph G.

Definition 3.10. We say that a fan F is *maximal* if it is not a subgraph of any other fan of G.

With this definition in place, we can make the following observation about maximal fans.

Observation 3.11. *In a 3-connected graph that is not a wheel, given a rivet vertex, r and rim vertex v of a fan, there is a unique maximal fan F containing both of these vertices.*

Proof. Consider the largest connected subgraph P of G containing v that consists of degree 3 vertices adjacent to r. This is uniquely defined and each vertex of P has at degree at most 2 in P, since r is not in P. Thus, P is either a path or cycle. However, if P is a cycle, there are no neighbors of P other than r, so r is a 1-cut of G, unless r and P are all of G, i.e., we have a wheel. So we may assume P is a path. Thus, P along with r and all edges from r to the vertices of P forms a fan F. Furthermore, for any rim vertex v' in any fan containing both v and r, there is a path of degree 3 vertices adjacent to r from v to v'. Thus, v' must be in P. So F contains all possible rim vertices, and so is the unique maximal fan. □

The wheel W_n has overlapping maximal fans, since the subgraph P described in the proof above is in fact a cycle for any fan in W_n; it is the rim cycle of W_n. Any path on $n-2$ vertices that is a subgraph of that cycle is the rim path of a maximal fan in W_n. Wheels are also exceptions for several of the following results.

Definition 3.12. A *contractible triangle* of a graph G is a K_3 subgraph of G such that each vertex has degree 3 in G.

These subgraphs are called contractible triangles because if G is 3-connected with a contractible triangle T, the graph obtained from G by replacing T with a single vertex x adjacent to the neighbors of the vertices of T outside of T, i.e., by contracting T to a single vertex, is a 3-connected graph.

Now we state a result on how maximal fans and contractible triangles must be disjoint.

Lemma 3.13. *For a 3-connected graph G that is not a wheel, each contractible triangle and each maximal fan of G is edge-disjoint from every other contractible triangle and every other maximal fan.*

Furthermore, each contractible triangle is vertex-disjoint from every maximal 2^+-fan and every other contractible triangle, and if two maximal 2^+-fans share a vertex, that vertex is the rivet of each of the fans.

Proof. Suppose there are two distinct subgraphs F_1 and F_2 that share an edge $e = xy$ such that each of F_1 and F_2 is either a maximal fan or a contractible triangle.

Suppose first that x and y each have degree 3 in G. Then consider the set $\{r_1, r_2\}$ where r_i is either the rivet vertex or third vertex of F_i depending on whether F_i is a fan or contractible triangle. The vertices r_1 and r_2 are common neighbors of x and y. We claim that $r_1 \neq r_2$. If F_1 and F_2 are both contractible triangles then $r_1 = r_2$ means $F_1 = F_2$. If F_1 and F_2 are both maximal fans, then x (and y as well) must be a rim vertex of both fans, and so $r_1 = r_2$ implies $F_1 = F_2$ by Observation 3.11. Otherwise, without loss of generality, F_1 is a maximal fan and F_2 is a contractible triangle. This means, $d_G(r_1) \geq 4$ and $d_G(r_2) = 3$, so $r_1 \neq r_2$. Thus, r_1 and r_2 are distinct vertices. Since x and y each have degree 3, they are only adjacent to each other and r_1 and r_2. Since $d_G(r_1) \geq 4$, there is at least one vertex outside of $F_1 \cup F_2$. Therefore, $\{r_1, r_2\}$ is a 2-cut of G, which is a contradiction.

Thus, without loss of generality, $d_G(y) \geq 4$. Thus, F_1 and F_2 are both fans, x is a rim vertex of each of them, and y is the rivet of each of them. However, by Observation 3.11, this means $F_1 = F_2$.

Thus, no two distinct such subgraphs share an edge.

If a contractible triangle with vertices a, b, c shared vertex a with either a maximal 2^+-fan or another contractible triangle, then either edge ab or ac must be an edge of that fan or contractible triangle. So this cannot occur.

Suppose two maximal 2^+-fans F_1 and F_2 share a vertex v. If v is a rim vertex of F_1 it is a rim vertex of F_2. The vertex v has at most one incident edge outside of F_1, and F_1 and F_2 are edge-disjoint. Thus, v has at most one incident edge in F_2, which is a contradiction. Thus, v must be the rivet vertex of both v_1 and v_2. □

The following result states loosely that contracting a rim edge of a fan preserves the contractible triangles and maximal fans of a graph.

Lemma 3.14. *For a 3-connected graph G that is not a wheel, let $F_1, F_2, \dots F_k$ be the set of maximal 2^+-fans of G, let $T_1, T_2, \dots T_l$ be the set of contractible triangles of G and let e be a rim edge of a 3^+-fan F_i. Then the set of maximal 2^+-fans of G/e is exactly $F_1, \dots, F_{i-1}, F_{i+1}, \dots F_k$ and the set of contractible triangles of G/e is exactly $T_1, \dots, T_l$, excepting F_i/e which is either a maximal 2^+-fan or a contractible triangle of G/e.*

Proof. First, note that F_i/e is either a 2^+-fan or a contractible triangle in G/e. The rivet of F_i, which we call r_i, had its degree reduced by 1, so if it had degree at least 5 in G, then F_i/e remains a fan in G/e. It is still a maximal fan, since the endpoints of e had no common neighbors, and so there are no new degree 3 vertices other than $[e]$ in G/e. Otherwise, r_i has degree 4 in G, and thus degree 3 in G/e. The subgraph F_i/e consists of r_i and its two adjacent degree three neighbors, so F_i/e is a contractible triangle. Thus, F_i/e is either a fan or contractible triangle. For each contractible triangle T_j in G, all three vertices remain degree 3, and all remain adjacent to each other, so T_j remains a contractible triangle in G/e.

Now consider a maximal 2^+-fan F_j of G where $j \neq i$. Let r_j be its rivet vertex. We want to show first that it is a maximal fan in G/e. Each rim vertex of F_j remains degree 3, adjacent to the rivet of F_j, and adjacent to its two rim neighbors. The rivet of F_j has its degree unchanged unless it is equal to r_i. If r_i is a rivet of another fan F_j, then it has degree at least 5 in G, and so has degree at least 4 in G/e. Thus, F_j remains a 2^+-fan in G/e. Now we want to show that this fan is still maximal in G/e. If F_j is not maximal then, there is some degree 3 vertex s adjacent to r_j and a rim vertex v of F_j in G/e, but in G, the vertex s either is not degree 3 in G or not adjacent to v and r_j in G. However, r and v have no new neighbors in G/e. The only possible degree 3 vertex of G/e that does not have degree 3 in G is r_i. However, if r_i has degree 3 in G/e then as discussed earlier, F_i/e is a contractible triangle. Every contractible triangle is vertex-disjoint from each 2^+-fan by Lemma 3.13, so r_i is not s, and s does not exist. Thus, F_j is a maximal fan of G/e.

Now, suppose J is a maximal 2^+-fan or contractible triangle that is distinct from $F_1, \ldots, F_i/e$, $F_{i+1}, \ldots F_k$ and $T_1, T_2, \ldots T_l$. Then J is edge-disjoint from each F_j and each T_j by Lemma 3.13. The subgraph J cannot contain $[e]$ by Lemma 3.13 since $[e]$ has degree 3, F_i/e contains $[e]$, and F_i/e is a contractible triangle or a maximal 2^+-fan. □

The following observation enumerates the possible ways that contracting an edge can create a new 3^+-fan.

Observation 3.15. *Suppose G is a 3-connected graph with no 3^+-fan and a contractible edge $e = ab$. Then if G/e has a 3^+-fan F with extended rim path $v_0 v_1 \ldots v_k v_{k+1}$, and rivet r, one of the following holds:*

(1) $[e] \notin V(F)$. Then $[e]$ is either v_0 or v_{k+1}. Furthermore, v_1 or v_k, the neighbor of $[e]$ in F, has degree 4 in G and is adjacent to both a and b in G.

(2) $[e]$ is the rivet of F.

(3) $[e]$ is a rim vertex of F.

Proof. The vertex $[e]$ must be either in F or adjacent to a vertex in F, since otherwise F would still be a fan in G. Suppose $[e]$ is not in F, and the conditions of (1) above do not hold. Then all rim vertices of F must have degree 3 in G. The rivet of F has degree at least 4 in G/e so it must have degree at least 4 in G. The edges between the vertices of F are unaffected, so G contains a 3^+-fan. Thus, either $[e]$ is not in F and (1) holds, or $[e]$ is in F and is either a rivet or rim vertex. □

We need one additional lemma before we can prove Proposition 3.17. If $e = ab$ is an edge in a graph G, we say that an edge $f' = xy$ in G *corresponds* to an edge f in G/e if x and y are distinct from the endpoints of $[e]$ and $f' = f$, or $x = a$ and $f = [e]y$.

Lemma 3.16. *Given a 3-connected graph G and a 3-contractible edge e, the following two statements hold:*

(1) For any 3^+-fan F of G/e there exists an edge f' of G corresponding to some rim edge f of F, such that G/f' is 3-connected.

(2) Furthermore, if $[e]$ is not the rivet of F and f_1 is a rim edge of F not having $[e]$ as an endpoint, then we may take $f' = f = f_1$.

Proof. Suppose G is a wheel. Then e must be a rim edge of G, and G/e is a wheel as well. Since f is a rim edge of a fan in G/e, f is a rim edge of the wheel G/e. Every edge of G corresponding to a rim edge of G/e is a rim edge of the wheel G. Each of those edges is 3-contractible in G, so we are finished.

Let $e = ab$. Let F be an arbitrary 3^+-fan of G/e, and let f_1 be an arbitrary rim edge of F. Let the rim path of F be $v_1v_2 \ldots v_k$ and let r be the rivet of F. Let $f_1 = v_iv_{i+1}$. For every edge f' of G corresponding to a fan edge f of F, G/f' is a split of $[e]$ in $G/e/f$. Now $G/e/f$ is 3-connected, so G/f' is 3-connected as long as a and b (or the vertices that correspond to them) have degree 3 in G/f' . We will consider two cases, when $[e]$ is not an endpoint of f_1 and when $[e]$ is.

Suppose first that $[e]$ is not an endpoint of f_1. Then there is a unique edge in G corresponding to f_1, so we can simply call it f_1. Suppose G/f_1 is not 3-connected, then one of a or b has degree less than 3 in G/f_1. Without lose of generality take it to be a. Then a is adjacent to $[f_1] = [v_iv_{i+1}]$ in G/f_1, adjacent to both v_i, v_{i+1} in G, and has degree 3 in G. In G/e, f_1 is a rim edge of F. Both v_i and v_{i+1} are adjacent to $[e]$ so $[e]$ is the rivet of F. Thus, at this point we have proved (2): if $[e]$ is not the rivet of F, then G/f_1 is 3-connected.

Now G/f_1 is not 3-connected, and $[e]$ is the rivet of F. Since F is a 3^+-fan, it has another fan edge $f_2 \neq f_1$. Since $[e]$ is the rivet of F, $[e]$ is not an endpoint of f_2, so by the same reasoning, one of a and b must have degree 3 in G and be adjacent to both endpoints of f_2. Now a is adjacent to v_i, v_{i+1}, and b in G and f_2 includes at least one vertex distinct from these. Therefore b must be adjacent to both endpoints of f_2 and have degree 3 in G. However, then $[e]$ is only adjacent to rim vertices of F in G/e, which gives a 2-cut $\{v_1, v_k\}$ and a contradiction to the 3-connectedness of

G/e.

Now consider the case when $[e]$ is an endpoint of f_1. Without loss of generality let $[e] = [ab] = v_{i+1}$, so v_i is the other endpoint of f_1. Consider a, b, and v_i as vertices of G. Note that the edges av_i and bv_i (if they are edges of G) both correspond to the fan edge f, so both are candidates for f'. If one of a and b, say b, is not adjacent to v_i in G or has degree at least 4 in G, then let $f' = av_i$. Then b has degree at least 3 in G/f'. The neighbors of v_i in G/e are r, $[e]$, and another vertex v_{i-1} that may or may not be a rim vertex of F. In G, v_i is adjacent to both r and v_{i-1}, so in G/f' $[av_i]$ is adjacent to b, r, and v_{i-1} in G/f', and has degree at least 3. Thus, G/f' is 3-connected.

So we may assume a and b both are adjacent to v_i in G and have degree 3 in G. The fan F is a 3^+-fan of G so it has another fan edge f_2. If $[e]$ is not an endpoint of f_2, then G/f_2 is 3-connected by statement (2). So $[e]$ is an endpoint of f_2, which must be $v_{i+1}v_{i+2}$. If a and b are not both adjacent to v_{i+2}, we are finished by the above reasoning. So a and b each have degree 3 in G and are adjacent to exactly v_i, v_{i+2} and each other. This means that $v_{i+1} = [ab]$ has degree 2 in G/e, contradicting 3-connectedness of G/e.

Thus, there is an edge f' in G, corresponding to a fan edge f of F, such that G/f' is 3-connected, and (1) holds. □

We can now state and prove the main result of this section.

Proposition 3.17. *Suppose G is 3-connected and that for every 3-contractible edge e of G, G/e has at least three rim edges of 3^+-fans. Then G has at least three rim edges of 3^+-fans.*

Proof. Suppose G is a wheel, W_n. Then G/e is a wheel as well, specifically W_{n-1}. If G/e has at least three rim edges in 3^+-fans, $n-1 \geq 4$. Thus, $G = W_n$ for $n \geq 5$ has at least three rim edges of 3^+-fans. So we may assume G is not a wheel.

All fans referred to in this proof are taken to be maximal fans.

Suppose G is not a wheel, G/e has a least three rim edges of 3^+-fans for each 3-contractible edge e of G, and that G does not have at least three rim edges in 3^+-fans. The graph G cannot have a 3^+-fan since if f is a rim edge of a 3^+-fan of G, then G/f will have at least 3 rim edges of 3^+-fans by assumption. By Lemma 3.14, each rim edge of a 3^+-fan of G/f is a rim edge of a 3^+-fan in G. So we may assume G has no 3^+-fan.

By Lemma 1.6, G has a 3-contractible edge $e = ab$. By assumption G/e has a 3^+-fan F. By Lemma 3.16, there is a rim edge f of F, and an edge f' corresponding to f in G such that G/f' is 3-connected.

By assumption, G/f' has a 3^+-fan K. Let $v_0v_1v_2 \dots v_kv_{k+1}$ be the extended rim path of F, and let r be the rivet vertex. Similarly, let $w_0w_1w_2 \dots w_lw_{l+1}$ be the extended rim path of K, and let s be the rivet vertex. The edge $f = v_iv_{i+1}$ for some i with $1 \leq i \leq k-1$. The graph G contains no 3^+-fan, but both G/e and G/f' do, so we will consider the cases from Observation 3.15. There are three possible positions of $[ab]$ relative to F, and for each of those cases, there are three possible positions of $[v_iv_{i+1}]$ relative to K. Note that since f is a rim edge of F, the only time $[ab]$ could be an endpoint of f is if $[ab]$ is a rim vertex of F. We discuss this possibility in Case 3. For Cases 1 and 2, when $[ab]$ is not an endpoint of f, there is a unique edge f' in G corresponding to f, which we just regard as f.

Case 1: Suppose $[ab]$ is not in F. By Observation 3.15(1), we may assume $[ab] = v_0$. Furthermore, in G, v_1 is adjacent to both a and b and v_1 has degree 4 in G.

By Lemma 3.16(2), we may take f to be any rim edge of F. In particular we chose f to be the rim edge v_1v_2 of F. Since F is a 3^+-fan v_3 is another rim neighbor of v_2. Note that $[v_1v_2]$ has degree 4 in G/f, since it is adjacent to exactly a, b, r, and v_3.

Case 1.1: Suppose $[v_1v_2]$ is not in K, then by Observation 3.15(1), we may assume $[v_1v_2] = w_0$. Furthermore, in G, w_1 is adjacent to both v_1 and v_2 and w_1 has degree 4 in G.

The vertex v_2 is not adjacent to $[ab]$ in G/e, so v_2 is adjacent to neither a nor b. Thus, r must be w_1, since no other vertex is adjacent to both v_1 and v_2 in G. Thus, r is a rim vertex of K in G/v_1v_2. The vertex w_1 has neighbors $[v_1v_2] = w_0, w_2,$ and s in G/f. The vertex v_3 is adjacent to $r = w_1$ in G/f, so it must be either w_2 or s. Moreover, v_3 has degree 3 in G/e, and is not adjacent to $[ab]$, so it has degree 3 in G. It still has degree 3 in G/f. However, it cannot be s since it has degree 3. In G/f, w_2 is not adjacent to $[v_1v_2]$, but v_3 is, so $w_2 \neq v_3$. This is a contradiction.

Case 1.2: The vertex $[v_1v_2]$ is the rivet of K.

The vertex $[v_1v_2]$ has degree at least 4 in G/f, so its neighbors must be exactly v_3, r, a, and b. The rim path has at least three vertices, since K is a 3^+-fan, so at most one of v_3, r, a, and b is not in the rim of K. The vertex v_3 is adjacent to r, and a is adjacent to b in G/f, so all of v_3, r, a, and b are either rim vertices or adjacent to a rim vertex. Since none of them are the rivet, that means all of them are in the extended rim path. However, by Observation 3.8(2), $[v_1v_2]$ has a neighbor outside of the extended rim path of K in G/f, so this is a contradiction.

Case 1.3: The vertex $[v_1v_2]$ is a rim vertex of K.

The vertex $[v_1v_2]$ has degree 4 in G/f. Thus, it cannot be a rim vertex of K.

Case 2: The vertex $[ab]$ is the rivet of F.

If there is another 3^+-fan F' in G/e, then either $[ab]$ is not in F' and we can apply Case 1 above, or $[ab]$ is the rivet of F', since maximal fans can only share rivet vertices by Lemma 3.13. If $[ab]$ is the rivet of F', since F and F' are edge-disjoint by Lemma 3.13, $[ab]$ has degree at least 6 in G/e. Thus, we may assume that either F is a 4^+-fan or there are two 3^+-fans with rivet $[ab]$; in either case $[ab]$ has degree at least 5 in G/e.

Case 2.1: Suppose $[v_iv_{i+1}]$ is not in K, then by Observation 3.15, we can assume that $[v_iv_{i+1}] = w_0$. Furthermore, in G, w_1 is adjacent to both v_i and v_{i+1} and w_1 has degree 4 in G.

The vertex w_1 must be either a or b, since v_i and v_{i+1} have only $[ab]$ as a common neighbor in G/e. Without loss of generality, take w_1 to be a, and so a is a rim vertex of K. The vertex $a = w_1$ has neighbors $[v_iv_{i+1}], w_2$, and s in G/f. Thus, b is either s or w_2 in G/f. If b is s, then $e = ab$ is a spoke edge of K, and $G/f/e$ is not 3-connected. However, $G/f/e = G/e/f$ which is 3-connected, so this is a contradiction.

Suppose, on the other hand, that $b = w_2$. Then b is adjacent to exactly a, w_3, and s. Then, in G/e, $[ab]$ is adjacent to exactly v_i, v_{i+1}, s, and w_3. This contradicts the fact that $[ab]$ has degree at least 5 in G/e.

Case 2.2: The vertex $[v_iv_{i+1}]$ is the rivet of K, i.e., $[v_iv_{i+1}] = s$.

In $G/[v_iv_{i+1}]$, $[v_iv_{i+1}]$ has degree exactly 4, since the only possible neighbors are a, b, v_{i-1}, and

v_{i+1}. Thus, at least one of a or b is a rim vertex of K and has degree 3 in G/f. In fact, both a and b must be rim vertices of K, since otherwise $[v_iv_{i+1}]$ would have no neighbor outside the extended rim path of K. Therefore $[ab]$ has degree exactly 4 in G/e, which contradicts the fact that $[ab]$ has degree at least 5 in G/e.

Case 2.3: The vertex $[v_iv_{i+1}]$ is a rim vertex of K.

Each rim vertex of K including $[v_iv_{i+1}]$ is adjacent to at most one of a and b in G/f, since otherwise they would have degree 2 in $G/f/e$. Without loss of generality, assume that $[v_iv_{i+1}]$ is not adjacent to b. Then, since v_i and v_{i+1} have degree 3 in G/e and neither is adjacent to b, they must each have degree 3 in G as well. The vertices v_i and v_{i+1} are both adjacent to a in G, so if a is the rivet of K, the split of $[v_iv_{i+1}]$ is a fan expansion by Observation 3.9, and G has a 3^+-fan. Therefore, we may assume a is not the rivet of K.

The vertex $[v_iv_{i+1}]$ has neighbors v_{i-1}, v_{i+2}, and a in G/f. By Observation 3.8, v_{i-1} is not adjacent to v_{i+2} in G/e, so they are not adjacent in G/f. One of v_{i-1} and v_{i+2} is the rivet s of K, since a is not, and it is adjacent to a rim neighbor of $[v_iv_{i+1}]$. Without loss of generality, assume $s = v_{i-1}$.Thus, a is a rim vertex of K, v_{i-1} is the rivet vertex of K, and v_{i+2} is not a vertex of K. The vertex b is not the rivet of K, so the neighbors of a in G/f are b, $[v_iv_{i+1}]$ and s. Therefore, b is not a rim vertex of K, since otherwise $[ab]$ would have degree at most 4 in G/e, but this contradicts the fact that $[ab]$ has degree at least 5 in G/e.

Case 3: The vertex $[ab]$ is a rim vertex of F.

Suppose $[ab]$ is an endpoint of f. The graph G/e has either at least two 3^+-fans or F is a 4^+-fan. If F is a 4^+-fan, than there is a rim edge of F that is not incident with $[ab]$. In that case, take f to be that edge; by Lemma 3.16(2) $f = f'$ is 3-contractible in G, . Suppose that F is not a 4^+-fan. Then there is another 3^+-fan F' in G/e. Then, $[ab]$ has degree 3 in G/e, since it is a rim vertex of F, so it is not the rivet of F'. It is not a rim vertex of F', either. Thus, we can apply Case 1 above, taking f to be some fan edge of F'. Thus, we may assume that $[ab]$ is not an endpoint of f in G/e, and we may assume F is a 4^+-fan. By Lemma 3.16(2), we can choose f such that $[ab]$ is one of v_{i-1} or v_{i+2}. So without loss of generality $[ab]v_iv_{i+1}$ is part of the rim path of F.

Now $[ab]$ is not an endpoint of f, so there is a unique edge f' corresponding to f in G. We will just denote this edge as f.

Case 3.1: Suppose $[v_iv_{i+1}]$ is not in K; then we may assume that $[v_iv_{i+1}] = w_0$. Furthermore, in G, w_1 is adjacent to both v_i and v_{i+1} and w_1 has degree 4 in G.

The fan K has at least three rim vertices, so $w_1w_2w_3$ is part of the rim path of K.

Since $[ab]$ is distinct from both v_i and v_{i+1}, w_1 is adjacent to both v_i and v_{i+1} in G/e. Thus, w_1 must be r, the rivet vertex of F. Since $[ab]$ is a rim vertex of F, $w_1 \neq [ab]$. The vertex w_1 has degree 4 in G, and so has degree at most 4 in G/e. However, we assumed F is a 4^+-fan, so w_1 by Observation 3.8(2) has degree at least 5 in G/e, which is a contradiction.

Case 3.2: Suppose $[v_iv_{i+1}]$ is the rivet of K.

Since we assumed that $[ab]v_iv_{i+1}$ is part of the rim path of F, we have that v_{i+1} is adjacent to neither a nor b in G. In G/e, v_{i+1} is adjacent to exactly r, v_i, and v_{i+2}, and v_i is adjacent to exactly $v_{i+1}, r,$ and $[ab]$. Since $[v_iv_{i+1}]$ is the rivet of K, it must have degree at least 4 in G/f. Thus, $[v_iv_{i+1}]$ is adjacent to exactly $r, v_{i+2}, a,$ and b in G/e. Therefore, $[v_iv_{i+1}]$ is adjacent to a vertex x outside the extended rim path of K. Since K is exactly a 3-fan, all neighbors of $[v_iv_{i+1}]$ other than x are rim vertices of K. The vertex r is adjacent to either a or b, since r is adjacent to $[ab]$. Thus, r is a rim vertex of K. Vertices a, b are not adjacent to v_{i+2} in G/f or G since $[ab]$ is not adjacent to v_{i+2} in G/e, but three of the vertices form the rim path of K, so they must be a, b, r and r is a rim vertex of K. However, r is the rivet of the 4^+-fan F in G/e, so it has degree at least 5 in G/e by Observation 3.8(2). In G, r has degree at least 5, so r has degree at least 4 in G/f. Thus, it is not a rim vertex of K, which is a contradiction.

Case 3.3: Suppose $[v_iv_{i+1}]$ is a rim vertex of K.

The neighbors of $[v_iv_{i+1}]$ in G/f include r and v_{i+2} and exactly one of a and b, since $[v_iv_{i+1}]$ has degree 3 in G/f. Without loss of generality, take $[v_iv_{i+1}]$ to be adjacent to a and not b in G/f and thus in G as well. Therefore, $a = v_{i-1}$ and v_i is adjacent to a in G while v_{i+1} is not. Thus, one of a, r, and v_{i+2} is s, the rivet of K. We will show that this is r, i.e., F and K have the same rivet vertex. By Observation 3.8(2), r has degree at least 5 in G/e, since F is a 4^+-fan. Thus, r

has degree at least 5 in G/e, degree at least 5 in G, and thus degree at least 4 in G/f. Therefore, it is not a rim vertex of K. If one of v_{i+2} or a is the rivet vertex of K, then the other must be a rim neighbor of $[v_iv_{i+1}]$, since $[v_iv_{i+1}]$ has a least one rim neighbor. However, v_{i+2} and $[ab]$ are not adjacent in G/e, so v_{i+2} and a are not adjacent in G or G/f. Thus, neither a nor v_{i+2} is the rivet of K, since the rivet of K is adjacent to the rim vertices of K. Thus, $r = s$.

Now we will show that G has a 3^+-fan. In G/e, v_i and v_{i+1} have degree 3. The vertex v_{i+1} is not adjacent to $[ab]$, so it has degree 3 in G as well. The vertex $[v_iv_{i+1}]$ is not adjacent to b in G/f, so v_i is not adjacent to b in G. Thus, v_i has degree 3 in G. The vertices v_i and v_{i+1} are both adjacent to $r = s$ in G. Thus, by Observation 3.9 applied to K and $[v_iv_{i+1}]$ the split of $[v_iv_{i+1}]$ that creates G is a fan expansion. Therefore, G has a 3^+-fan (in fact, even a 4^+-fan), which is a contradiction. □

3.3 Conjectures and computational evidence

In this section we state some conjectures regarding the structure of general and planar 3-connected, $K_{2,5}$-minor-free graphs. We discuss computational evidence for these and computational strategies for proving them. Proposition 3.17 plays an important role in these strategies.

Conjecture 3.18. *There exist positive integers c_1, c_2 and a finite set of base graphs $\mathcal{B}$ such that a graph G on at least c_1 vertices is 3-connected and $K_{2,5}$-minor-free if and only if it is either a $\mathcal{A}$-sequence graph with $r(G) \geq c_2$ or is obtained from a graph in $\mathcal{B}$ by expanding 3^+-fans.*

The conjecture for the planar case is analogous except that only planar $\mathcal{A}$-sequence graphs are included. The planar $\mathcal{A}$-sequence graphs are those with $\mathcal{A}$-sequence representations without X-pieces.

Conjecture 3.19. *There exist positive integers d_1, d_2 and a finite set of base graphs $\mathcal{C}$ such that a graph G on at least d_1 vertices is planar, 3-connected, and $K_{2,5}$-minor-free if and only if it is either a planar $\mathcal{A}$-sequence graph with $r(G) \geq d_2$ or is obtained from a graph in $\mathcal{C}$ by expanding 3^+-fans.*

Although we have not been able to prove these conjectures, there is some computational evidence that they are true, and the results presented in this chapter form part of a possible strategy for proving them.

Let $\mathcal{G}_n$ be the set of 3-connected, $K_{2,5}$-minor-free graphs on n vertices, and let $\mathcal{H}_n$ be the set of planar, 3-connected, $K_{2,5}$-minor-free graphs on n vertices.

The family of graphs in $\mathcal{G}_n$ were generated by Zachary Gaslowitz from the family $\mathcal{G}_{n-1}$ in the following way, up to $n = 16$. Consider a graph $G \in \mathcal{G}_n$. Every 3-connected graph has a 3-contractible edge by Lemma 1.6, so G has a 3-contractible edge e. The graph G/e is 3-connected and $K_{2,5}$-minor-free, so $G/e \in G_{n-1}$. Thus, every graph in $\mathcal{G}_n$ was obtained from a graph in $\mathcal{G}_{n-1}$ by a vertex split.

Let $\mathcal{S}_n$ be the set of all graphs obtained from a graph in $\mathcal{G}_{n-1}$ by splitting a vertex v into v_1 and v_2 such that v_1 and v_2 have degree at least 3. The set $\mathcal{S}_n$ certainly contains all 3-connected splits of graphs in $\mathcal{G}_{n-1}$, since v_1 and v_2 must have degree at least 3 for the split to be 3-connected. By Lemma 1.4, every graph in $\mathcal{S}_n$ is 3-connected. Thus, $\mathcal{S}_n$ is exactly the set of all 3-connected splits of graphs in $\mathcal{G}_{n-1}$. Isomorphic copies of graphs were removed using Brendan McKay's `nauty`, a framework that can put graphs in a canonical form, and thus filter out isomorphic copies [16]. Then $\mathcal{G}_n$ is obtained from $\mathcal{S}_n$ by filtering out all graphs that have $K_{2,5}$ as a minor using Zachary Gaslowitz's minor checker, `canary` [12].

In fact, when generating all splits of graphs in $\mathcal{G}_{n-1}$, we need only consider splits where v_1 and v_2 share at most four neighbors, since otherwise, v_1, v_2 and five of their shared neighbors give $K_{2,5}$ as a subgraph.

Gaslowitz also generated all planar, 3-connected, $K_{2,5}$-minor-free graphs on n vertices for n up to 18, in a similar fashion, but at each step also filtered out the non-planar graphs by using `canary` to check for a $K_{3,3}$ minor. Since planarity is closed under taking minors, we need only consider splits of graphs in $\mathcal{H}_{n-1}$ to obtain the graphs in $\mathcal{H}$.

We generated all $\mathcal{A}$-sequence graphs on up to 16 vertices, and the planar $\mathcal{A}$-sequence graphs on up to 18 vertices, using `planarg` from the `nauty` package to check for planarity [16]. Then we used this to analyze the graphs in $\mathcal{G}_n$ and $\mathcal{H}_n$ in more detail.

Our computer results gave the following findings for the general 3-connected, $K_{2,5}$-minor-free graphs. Every graph in $\mathcal{G}_{15}$ and $\mathcal{G}_{16}$ that is not an $\mathcal{A}$-sequence graph is obtained from a graph in $\mathcal{G}_{14}$ by expanding fans (potentially only 1-fans). What this suggests is that all base graphs are obtained by expanding small fans to 3^+-fans in graphs on at most 14 vertices.

We expect that for sufficiently large reduction size, all splits of $\mathcal{A}$-sequence graphs are $\mathcal{A}$-sequence graphs. Computational results show that this holds for all planar $\mathcal{A}$-sequence graphs G with $r(G) = n$ and $|V(G)| = n$ for $n = 13, 14, 15, 16, 17, 18$. It does not however hold for $n = 12$. Proving such a result holds for all planar $\mathcal{A}$-sequence graphs G with $r(G) \geq 13$ appears to be quite feasible, using an analytical proof for $r(G) \geq n_0$ and computer results for $13 \leq r(G) < n_0$, where n_0 is probably about 19. Suppose we also had a computational result showing that for some n all graph $G \in \mathcal{H}_n$ either are $\mathcal{A}$-sequence graphs with $r(G) \geq 13$ or have at least three edges in 3^+-fans. Through the use of Proposition 3.17, these two ingredients would allow us to provide a characterization of the planar, 3-connected, $K_{2,5}$-minor-free graphs and verify Conjecture 3.19.

Unfortunately, this does not seem to be possible in a direct way. There are planar $\mathcal{A}$-sequence graphs of reduction size less than 13, without at least three edges in 3^+-fans, on up to 37 vertices. For example, $\gamma((IA^+IA^+IA^+IA^-IA^-IA^-IA^-)^6IA^-)$ has this property. Checking all 3-connected planar $K_{2,5}$-minor-free graphs on up to 37 vertices is well beyond what is computationally possible.

For the planar case, there are only 137 graphs in $\mathcal{H}_{18}$ that are not $\mathcal{A}$-sequence graphs and do not have at least three edges in 3^+-fans, out of 1,308,508 total graphs in $\mathcal{H}_{18}$. We suspect that for some n about 21, $\mathcal{H}_n$ contains no graphs that are not $\mathcal{A}$-sequence graphs and do not have at least three edges in 3^+-fans.

A strategy that may solve this issue is by further analyzing the $\mathcal{A}$-sequence graphs with reduction size less than 13 in more detail, and describing the splits that can give non-$\mathcal{A}$-sequence graphs. If we could do this, the size of the graphs that must be examined computationally could be brought down to something more manageable, perhaps about 19 or 22.

Chapter 4

Refining the description of $K_{2,t}$-minor-free graphs

As discussed in Chapter 1, Guoli Ding gave a rough description of all 2-connected $K_{2,t}$-minor-free graphs. Here in this section we look to refine that description and give a characterization for the 3- and 4-connected cases. Recall that every 3-connected, $K_{2,t}$-minor-free graph is obtained by attaching strips and fans to a finite set of base graphs. Furthermore, every 4-connected $K_{2,t}$-minor-free graph is obtained by attaching strips to a finite set of base graphs. First, we present a few definitions relevant to the fine structure of these strips in the 3- and 4-connected cases. In fact, the strips are closely related to the $\mathcal{Q}$-sequence and $\mathcal{A}$-sequence graphs. We then show that the fine structure we define exactly describes the strips in 3- and 4-connected graphs. We discussed in Chapter 1 that expanding 3^+-fans preserves 3-connectedness and $K_{2,t}$-minor-freeness. Towards the end of this chapter, we will prove similar results for strips. These results then will give us a characterization for the 3- and 4-connected $K_{2,t}$-minor-free graphs.

4.1 Fine structure of strips

We want to discuss the fine structure of strips in 3- and 4-connected graphs. We will consider strips as (2,2)-terminal graphs. Given a strip S with corners $a, b, c,$ and d, where the corners occur in the order a, b, c, d in the reference cycle C, and edges ab and cd were deleted in C to form S, let $a_1 = a$, $a_2 = b$, $b_1 = d$, and $b_2 = c$. We call the paths from a_1 to b_1 and from a_2 to b_2 in C that remain in S the *rail paths* of S.

Lemma 4.1. *For every strip or weak fan G, $G \cong G_1 G_2 \ldots G_m$ where $G_i = I, A^+, A^-,$ or X.*

Proof. Let G be a strip with rail paths P from a_1 to b_1 and Q from a_2 to b_2. We will proceed by induction on the number of chords of G.

For the initial step, suppose that G has no chords. Then G consists only of P and Q. Thus, $G = (A^+)^k (A^-)^l$, where k is the length of P and l is the length of Q.

Suppose G has a chord $e = xy$ that does not cross any other chord of G, with $x \in V(P)$ and $y \in V(Q)$. Let $P_1 = P[a_1, x]$, $Q_1 = Q[a_2, y]$, $P_2 = P[x, b_1]$, and $Q_2 = Q[y, b_2]$. Then the graph $G - \{e\}$ is equal to $G_1 \cup G_2$, where G_1 is the graph induced on the vertices $V(P_1) \cup V(Q_1)$ and G_2 is the graph induced on the vertices $V(P_2) \cup V(Q_2)$. Note that the edge a_1a_2 is not in G by the definition of a strip, so it is not the case that $x = a_1$ and $y = a_2$. The edges a_1a_2 and xy are not in G_1, and the cycle C_1 consisting of P_1 followed by xy, followed by Q_1 traversed from y to b_1 followed by a_1b_1 is a Hamilton cycle in $H_1 = G_1 \cup \{a_1b_1, xy\}$. Consider C_1 as the reference cycle of H_1. No chord of H_1 crosses two other chords of H_1, since such a chord would cross two chords in G as well. Furthermore, for any two crossing chords of H_1, $f_1 = v_1w_1$ and $f_2 = v_2w_2$ without loss of generality, $v_1v_2, w_1w_2 \in C_1$. Thus, H_1 is a type-I graph. By symmetric reasoning, H_2 is a type-I graph as well. Thus, G_1 and G_2 are strips or weak fans, and so by the inductive hypothesis, can be written as a sequence of I, A^+, A^-, or X. Furthermore, $G = G_1IG_2$, so the statement holds for G as well.

We now assume that every chord of G crosses another chord. Suppose there is a pair of crossing chords $e_1 = x_1y_2$ and $e_2 = x_2y_1$, where x_1, x_2 occur in that order on P, and y_1, y_2 occur in that order on Q. Then by the definition of a strip, without loss of generality, x_1 and x_2 are adjacent in P and y_1 and y_2 are adjacent in Q. Let $P_1 = P[a_1, x_1]$, $Q_1 = Q[a_2, y_1]$, $P_2 = P[x_2, b_1]$, and $Q_2 = Q[y_2, b_2]$. There are no other edges of G that cross e_1 or e_2, so $G - \{e_1, e_2, x_1x_2, y_1y_2\}$, is equal to $G_1 \cup G_2$ where G_1 is the graph induced on the vertices $V(P_1) \cup V(Q_1)$ and G_2 is the subgraph induced by the vertices $V(P_2) \cup V(Q_2)$. Note that neither x_1y_1 nor x_2y_2 is an edge of G, since otherwise that edge would cross no other chords of G, which contradicts our earlier assumption. If $x_1 = a_1$ and $y_1 = a_2$, then $G_1 = E$. Recall that E is simply the (2,2)-terminal graph on two vertices with no edges. Similarly, if $x_2 = a_2$ and $y_2 = b_2$, then $G_2 = E$. By analogous reasoning to the above case, G_1 and G_2 are each either a strip, a weak fan, or simply E. Thus, by the inductive hypothesis, G_1 and G_2 are either are E or can be written as a sequence of I-,A^+-, A^--, or X-pieces. Furthermore, $G = G_1XG_2$, and so the statement holds for G as well. $\square$

We now refine this structure for the 3-connected case.

Definition 4.2. A *linear $\mathcal{A}$-sequence* is a (non-cyclic) sequence $P_1, P_2, \dots P_m$ of I, A^+, A^-, A^* and X-pieces such that the following hold:

(1) There are no two consecutive A-type pieces.

(2) There are no two consecutive I-pieces.

A strip that is generated from a linear $\mathcal{A}$-sequence is called an $\mathcal{A}$*-sequence strip*.

Note that we require $\mathcal{A}$-sequence strips to be strips themselves. For an $\mathcal{A}$-sequence strip S, this in particular means that the edges a_1a_2 and b_1b_2 are not in S, and so the linear $\mathcal{A}$-sequence for S does not start or end with an I. Furthermore, the vertices a_1, a_2, b_1, and b_2 must be distinct, and so the sequence for S cannot consist of only A^+ and I or only of A^- and I. These are the only times when a linear $\mathcal{A}$-sequence does not create a valid strip.

A linear $\mathcal{A}$-sequence is essentially just a non-cyclic $\mathcal{A}$-sequence with no T-, Y-type, or Z-type pieces. In a 3-connected graph, every strip is in fact an $\mathcal{A}$-sequence strip, as shown by the following result.

Lemma 4.3. *If S is a strip such that each non-corner piece has degree at least 3, then S is an $\mathcal{A}$-sequence strip.*

Proof. Let S be such a strip. Then by Lemma 4.1, there exist $G_1, G_2, \dots G_m$ such that $S = G_1G_2 \dots G_m$, $G_i = I, A^+, A^-$, or X. As we noted in Observation 3.2, $II = I$ and $A^+A^- = A^-A^+ = A^*$. So replace each II subsequence with I and each A^+A^- or A^-A^+ subsequence with A^*.

So we may assume $S = G_1G_2 \dots G_m$, where $G_i = I, A^+, A^-, A^*$, or X and there are no two consecutive I-pieces and no A^+A^- or A^-A^+ subsequence. Any A^+A^+, A^+A^*, A^-A^-, A^-A^*, A^*A^+, A^*A^- or A^*A^- subsequence results in a vertex that is only in those two pieces, and with degree 2 overall. Thus, there are no consecutive A-type pieces. Thus, G_1, G_2, $\dots G_m$ is a linear $\mathcal{A}$-sequence and S is an $\mathcal{A}$-sequence strip. □

Given a 3-connected graph G, G has minimum degree 3. If S is a strip in G, then $d_G(v) = d_S(v)$ for every non-corner vertex v. Thus, every non-corner vertex of S has minimum degree 3 in

G, and so G must be an $\mathcal{A}$-sequence strip. As we defined for $\mathcal{A}$-sequence graph in Chapter 3, it will be useful to have a notion of the size of an $\mathcal{A}$-sequence strip that ignores large fans. Given an $\mathcal{A}$-sequence strip $S = G_1G_2 \dots G_m$, define the *reduction* of S, denoted $\overline{S}$, to be the $\mathcal{A}$-sequence strip $\overline{S} = H_1H_2 \dots H_{m'}$, where every IA^+IA^+I in $G_1G_2 \dots G_m$ is replaced with IA^+I and IA^-IA^-I in $G_1G_2 \dots G_m$ is replaced with IA^-I. Then define the *reduction size* of S, denoted $r(S)$, to be the number of vertices in the reduction of S. Note that this completely analogous to the definition of reduction size for $\mathcal{A}$-sequence graphs.

We now further refine the structure of strips for the 4-connected case.

Definition 4.4. A *linear $\mathcal{Q}$-sequence* is a (non-cyclic) sequence $P_1, P_2, \dots P_m$ of I-, Δ^+-, Δ^--, Q- and X-pieces such that the following hold:

(1) If $P_i = I$, then $G_{i-1} = X$ if $i > 1$ and $G_{i+1} = X$ if $i < m$. In other words, any piece consecutive with an I-piece is an X-piece.

(2) If $P_i = Q$, then $G_{i-1} = X$ if $i > 1$ and $G_{i+1} = X$ if $i < m$. In other words, any piece consecutive with a Q-piece is an X-piece.

(3) There are no two consecutive Δ^+-pieces.

(4) There are no two consecutive Δ^--pieces.

If a strip G' can be obtained from $G = P_1P_2 \dots P_m$ where $P_1, P_2, \dots P_m$ is a linear $\mathcal{Q}$-sequence by removing the edges $a_1^1a_2^1$ or $b_1^mb_2^m$ if they are in G, then G' is called a *$\mathcal{Q}$-sequence strip*.

Note that as for $\mathcal{A}$-sequence strips above, we require that $\mathcal{Q}$-sequence strips are strips themselves. Since the edges a_1a_2 and b_1b_2 are always deleted, that is not a concern in this case. We do require that a_1, a_2, b_1, and b_2 must be distinct. Thus, the sequences consisting of a single I-, Δ^+-, or Δ^--piece do not give valid $\mathcal{Q}$-sequence strips. All other sequences do, however.

$\mathcal{Q}$-sequence strips are essentially just graphs built from non-cyclic $\mathcal{Q}$-sequences with possibly some alterations to the first and last piece. We will now show that in a 4-connected graph, every strip is in fact a $\mathcal{Q}$-sequence strip.

Lemma 4.5. *If S is a strip such that each non-corner piece has minimum degree 4, then S is a $\mathcal{Q}$-sequence strip.*

Proof. Let S be such a strip. By Lemma 4.3, S is an $\mathcal{A}$-sequence strip. Thus, $S = G_1G_2 \dots G_m$, where $G_i = I, A^+, A^-, A^*$ or X, and there are no two consecutive I-pieces and no two consecutive A-type pieces.

If G_i is an A-type piece and $i \neq 1$ then $a_1^i a_2^i \in E(S)$, since otherwise a_1^i or a_2^i will have degree less than 4 in S. Similarly, if G_i is an A-type piece and $i \neq m$, then $b_1^i b_2^i \in E(G)$. Thus, if we replace each A^+ with Δ^+, each A^- with Δ^-, and each A^* with Q and let $G'_1, G'_2, \dots, G'_{m'}$ be the resulting sequence, S differs from $G'_1G'_2 \dots G'_{m'}$ only by the possible removal of the edges $a_1^1a_2^1$ or $b_1^{m'}b_2^{m'}$.

We have $G'_iI = IG'_i = G'_i$ when G'_i is any Δ-type piece or Q-piece, so we may assume every piece consecutive with an I is an X. Each of the sequences QQ, $Q\Delta^+$, $Q\Delta^-$, Δ^+Q, $\Delta^+\Delta^+$, Δ^-Q, and $\Delta^-\Delta^-$ results in a non-corner vertex in only those two pieces and with degree 3 in G. Thus, every piece consecutive with a Q is an X, and there is no $\Delta^+\Delta^+$ or $\Delta^-\Delta^-$ subsequence. Thus, there is a linear $\mathcal{Q}$-sequence $G_1, G_2, \dots G_m$ such that S differs from $G_1G_2 \dots G_m$ only by the possible removal of the edges $a_1^1a_2^1$ or $b_1^mb_2^m$. Therefore, S is a $\mathcal{Q}$-sequence strip. □

4.2 Crossed and uncrossed strips

Given a $K_{2,t}$-minor-free graph with a sufficiently long strip S, we want to show now that G', the graph obtained by replacing S with any strip, is $K_{2,t}$-minor-free. Ding's result gives a necessary condition for a graph to be 3-connected and $K_{2,t}$-minor-free; we will give a necessary and sufficient condition. However, we need to make a distinction between strips with an X-piece and strips without one.

Definition 4.6. Given a strip S, we can let $S = G_1G_2 \dots G_m$ where each G_i is a A^+, A^-, I, or X, from the decomposition in Lemma 4.1. We say S is *crossed* if least one $G_i = X$. Otherwise, we say that the strip is *uncrossed* .

A strip is crossed exactly when it has a pair of crossing chords.

We will construct graphs that show that in general you cannot replace an uncrossed strip with a crossed strip without introducing a $K_{2,t}$-minor. Lemma 4.7 gives a construction that shows for that for any sufficiently large t replacing an uncrossed strip with a crossed strip does not in general preserve $K_{2,t}$-minor-freeness. Figure 4.1 shows an example of this construction. Lemma 4.8 gives a construction for 4-connected graphs as well. The proofs of these lemmas are postponed until after the proof of Proposition 4.31, as many lemmas used for that theorem will be used in the construction of these graphs.

Lemma 4.7. *For any $t \geq 16$, there exists a graph G that is $K_{2,t}$-minor-free and has an uncrossed strip S, but the graph G' obtained by replacing S by any crossed strip S' has a $K_{2,t}$-minor.*

While Lemma 4.7 shows that replacing an uncrossed strip with a crossed one cannot be done in general, the examples presented are only 2-connected. Perhaps then, in the higher connectivity cases we are considering, there are no such examples. The following result shows that in fact there are examples that are 4-connected.

Lemma 4.8. *For any $t \geq 420$, there exists a 4-connected, $K_{2,t}$-minor-free graph G such that G contains an uncrossed strip S, but the graph G' obtained by replacing S with a crossed strip S' has a $K_{2,t}$-minor.*

Since there are 4-connected examples, there are certainly 3-connected examples as well. It is worth noting that there do exist 3-connected examples that are not 4-connected, but the construction is not presented here. They are of a similar form, and do not require t to be as large.

4.3 Expanding strips preserves $K_{2,t}$-minor-freeness

Now that the importance of the distinction between uncrossed and crossed strips is clear, we want to show that if G is a $K_{2,t}$-minor-free graph with a sufficiently large crossed strip S, then G' obtained from G by replacing S with *any* strip S' is also $K_{2,t}$-minor-free. Furthermore, if G is $K_{2,t}$-minor-free with a sufficiently large uncrossed strip S, then G' obtained from G by replacing S with any *uncrossed* strip S' is also $K_{2,t}$-minor-free. In all the following, we will suppose that G' is a graph with a strip S' and a $K_{2,t}$-minor with model $(R_1', R_2'; T')$. We aim to show then, that every graph G obtained by replacing S' with any strip S (or any crossed strip if S' is crossed) with sufficient reduction size has a $K_{2,t}$-minor.

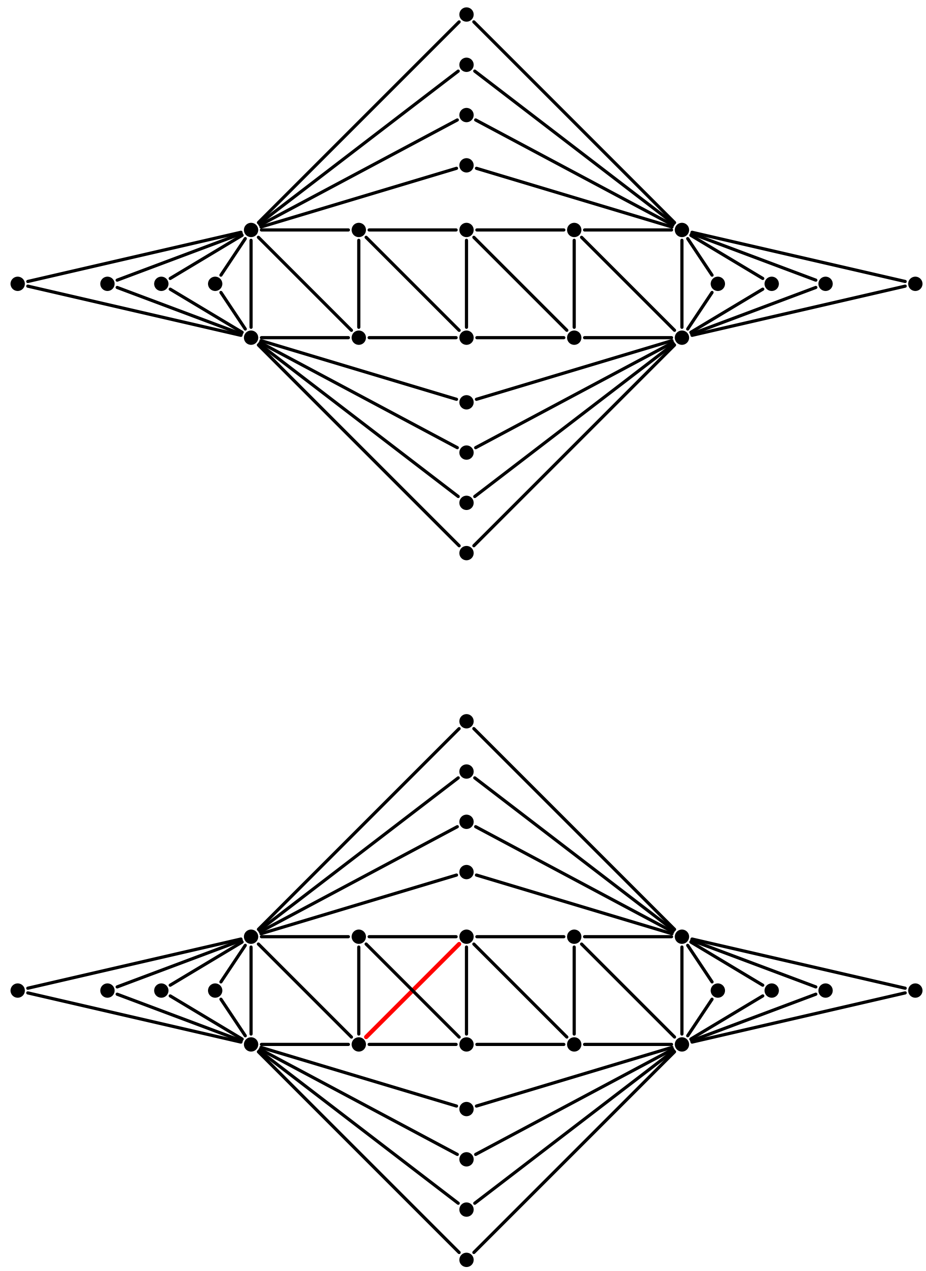

Figure 4.1: A graph that is $K_{2,16}$-minor-free, but when an X is added it has a $K_{2,16}$-minor.

We will now consider G' and S' as in the above framework, but to simplify notation we will temporarily replace G', S', R_1', R_2', and T' by G, S, R_1, R_2, and T respectively. The reader should note that when used in the proof of Proposition 4.31, G and S in Lemmas 4.9 through 4.17 will play the role of G' and S'. In order to analyze a $K_{2,t}$-minor in G' and construct a corresponding minor in G, we will abstract out the properties that a strip S contributes to the $K_{2,t}$-minor. In particular, we will consider the following four properties for S: (1) whether S is uncrossed or crossed, (2) for each corner, whether it is in R_1, R_2, or neither, (3) if two corners x, y are both in R_i, whether there is a path between them in $R_i \cap S$, and (4) the number of vertices x in $V(S) \cap T$ such that there are edges r_1x and r_2x in S where $r_1 \in R_1$ and $r_2 \in R_2$. We denote the set of vertices x in $V(S) \cap T$ such that there are edges r_1x and r_2x in S by $\tau(S)$, and so this last value is denoted by $|\tau(S)|$.

In a series of lemmas, we will prove an upper bound on $|\tau(S)|$ for each possible configuration of (1), (2) and (3). Then we will provide constructions that show that these bounds can be achieved for any strip S of sufficient length. This will then give us a $K_{2,t}$-minor in G as desired.

For Lemmas 4.9 through 4.17, we take S to be a strip (not necessarily an $\mathcal{A}$-sequence strip) in a graph G that has a $K_{2,t}$ model $(R_1, R_2; T)$. We will treat S as a (2,2)-terminal graph with corners labelled a_1, a_2, b_1, b_2. We will also take $S = G_1G_2 \dots G_m$, where each G_i is X, A^+, A^-, or I, using the decomposition from Lemma 4.1, throughout without explicit mention. We remind the reader that a strip contains neither a_1a_2 nor b_1b_2 as an edge.

Lemma 4.9. *Let G be a graph with a model of a $K_{2,t}$ minor $(R_1, R_2; T)$. For any strip S in G, $|T \cap V(S)| \leq 4$.*

Proof. Every strip S is a minor of $P_n[K_2]$, and so the graph H obtained by adding the edges of a K_4 to the corners of S is a minor of $C_n[K_2]$. By Lemma 2.11, $C_n[K_2]$ is $K_{2,5}$-minor-free, so H is $K_{2,5}$-minor-free as well. While R_1 and R_2 are connected in G, the sets $R_1 \cap V(S)$ and $R_2 \cap V(S)$ may not be connected in S. However, only the corner vertices of S have neighbors outside of S in G, so the sets $R_1 \cap V(H) = R_1 \cap V(S)$ and $R_2 \cap V(H) = R_2 \cap V(S)$ are connected in H. Suppose then that S contains 5 or more vertices of T, then $(R_1 \cap V(H), R_2 \cap V(H); T \cap V(H))$ is a model of a $K_{2,t}$-minor in H for $t \geq 5$. Thus, $|T \cap V(S)| \leq 4$. □

Lemma 4.10. *If $|\tau(S)| > 0$, there is at least one corner of S in each of R_1 and R_2; moreover, every component of $S[R_1]$ or $S[R_2]$ contains a corner of S.*

Proof. Suppose there is a vertex $x \in \tau(S)$. Suppose for a contradiction, that no corner vertex of S is in R_1. There is some neighbor $r_1 \in R_1$ of x in S. Thus, R_1 is contained in S. There are no corner vertices in R_1, and all non-corners of S have all their incident edges in S. Thus, $T \subseteq V(S)$ and $|T \cap V(S)| = t$. However, by Lemma 4.9, $|T \cap V(S)| \leq 4$, and $t \geq 5$, which is a contradiction. Therefore, a corner vertex of S is in R_1, and by the same reasoning, there is a corner vertex of S in R_2. The branch sets R_1 and R_2 are connected in G, and only the corner vertices have edges not in S. Thus, every component of $S[R_1]$ or $S[R_2]$ contains a corner of S. $\square$

The next lemma will be used extensively in the proofs of the upper bounds on $\tau(S)$. It describes the structure of the graph obtained after removing a path between two corners on opposite sides of a strip. The resulting subgraph is a union of disjoint paths and every corner occurring in this subgraph is an end of one of the paths, so we may add edges (which are not necessarily edges of G) to join these paths into a single path between the other two corners. In order to emphasize that the edges of this path may not be in G we refer to this path as a *virtual path*.

Lemma 4.11. *Suppose G is a strip. Then if P is a path in G from a_i to b_j, where $i, j \in \{1, 2\}$, then $G - V(P)$ is a (possibly empty) subgraph of a virtual path from a_{3-i} to b_{3-j}.*

Proof. Let $S = G_1 G_2 \ldots G_m$ be the decomposition of G into pieces as in Lemma 4.1.

For each G_k, a_1^k or $a_2^k \in V(P)$ and b_1^k or $b_2^k \in V(P)$ since deleting $\{a_1^k, a_2^k\}$ destroys all paths that go from $\{a_1, a_2\}$ to $\{b_1, b_2\}$, as does deleting $\{b_1^k, b_2^k\}$. Let $x_1 = a_{3-i}$, choose x_k to be some a_l^k such that a_{3-l}^k is in $V(P)$ for $2 \leq k \leq m$, and let $x_{m+1} = b_{3-j}$. The choice of x_k is not unique if both $a_1^k, a_2^k \in V(P)$. Also, the vertices x_k and x_{k+1} may not be distinct. The set of x_k vertices contains the set of vertices of $G - V(P)$.

For $1 \leq k \leq m$, let Q_k be the edge $x_k x_{k+1}$ if $x_k \neq x_{k+1}$, and the single vertex $x_k = x_{k+1}$ otherwise. Each $G_k - V(P)$ is a subgraph of Q_k (they may not be equal since x_k or x_{k+1} may be in $V(P)$ and the edge $x_k x_{k+1}$ may not be in G_k). Each edge of G is in some G_k, so each edge of

$G - V(P)$ is in some Q_k. Let $Q = Q_1 \cup Q_2 \cup Q_m$. The graph $G - V(P)$ is a subgraph of Q, and the graph Q is a virtual path from a_{3-i} to b_{3-j}. □

For $i \in \{R_1, R_2\}$, we define a *R_i-path* to be a path all of whose vertices belong to R_i. A *long R_i-path in S* is a R_i-path in S from a_j to b_k where $j, k \in \{1, 2\}$.

Lemma 4.12. *If S has a long R_1-path and a long R_2-path, then $|T \cap V(S)| = 0$ and $|\tau(S)| = 0$.*

Proof. Let P_1 be the long R_1-path, without loss of generality from a_1 to b_i. Let P_2 be the long R_2-path. Then by Lemma 4.11, $P_2 \subseteq S - V(P_1) \subseteq Q_2$ where Q_2 is a virtual path from a_2 to b_{3-i}. Thus, $P_2 = Q_2$ and $V(S) - V(P_1) = V(P_2)$. Thus, $V(S) = V(P_1) \cup V(P_2) \subseteq R_1 \cup R_2$, and $T \cap V(S) = \emptyset$. □

Lemma 4.13. *If there is a long R_1-path in S, then $|\tau(S)| \leq 2$.*

Proof. Let P_1 be the long R_1-path, without loss of generality from a_1 to b_i. By Lemma 4.11, $S - V(P_1)$ is a subgraph of a virtual path Q_2 from a_2 to b_{3-i}. Each component of $S[R_2]$ is a subpath of Q_2 containing a_2 or b_{3-i}. There are at most two such components, and each component is adjacent to at most one vertex of $\tau(S)$. Thus, $|\tau(S)| \leq 2$. □

Lemma 4.14. *If there is a long R_1-path in S, and $a_1, a_2 \notin R_2$, then $|\tau(S)| \leq 1$.*

Proof. Let P_1 be the long R_1-path, without loss of generality from a_1 to b_i. Then $S - V(P_1)$ is a subgraph of a virtual path Q_2 from a_1 or a_2 to b_{3-i}. Since $a_1, a_2 \notin R_2$, the subgraph $S[R_2]$ has at most one component, and this component can be adjacent to at most one vertex of $\tau(S)$. Thus, there is at most one vertex of $\tau(S)$ in Q_2, and $|\tau(S)| \leq 1$. □

Lemma 4.15. *If $b_1, b_2 \notin R_1$ and $a_1, a_2 \notin R_2$, then $|\tau(S)| \leq 2$.*

Proof. Suppose $|\tau(S)| \geq 3$. Then by Lemma 4.10, $a_i \in R_1$ and $b_j \in R_2$.

We have $S = G_1 G_2 \ldots G_m$. Let H be the graph $\gamma(G_1 G_2 \ldots G_m IXIXIXIXI)$. Call these new pieces $H_{m+1}, H_{m+2}, \ldots H_{m+8}, H_{m+9}$. This graph is a minor of $C_n[K_2]$, so by Lemma 2.11, H is $K_{2,5}$-minor-free. Let $R_1' = (R_1 \cap V(S)) \cup V(H_{m+3})$, let $T' = \tau(S) \cup V(H_{m+5})$, and let

$R_2' = (R_2 \cap V(S)) \cup V(H_{m+7})$. Then $(R_1', R_2'; T')$ is a model of a $K_{2,t}$ minor in H for $t \geq 5$, which is a contradiction. □

Lemma 4.16. *If $b_1, b_2 \notin R_1, R_2$, then $|\tau(S)| \leq 2$. Furthermore, if S is an uncrossed strip $|\tau(S)| \leq 1$.*

Proof. Without loss of generality, we may assume $a_1 \in R_1$ and $a_2 \in R_2$, since otherwise by Lemma 4.10, $|\tau(S)| = 0$. Let x be a vertex of $\tau(S)$, and let G_i be the first piece containing x. Take x to be the the vertex of $\tau(S)$ such that this value i is maximized. Then every vertex of $\tau(S)$ occurs in $G_1 G_2 \dots G_i$. Moreover, x must be b_1^i or b_2^i, and must not be one of a_1^i or a_2^i, since otherwise it would occur in G_{i-1} as well. For each piece $G_{i'}, i' < i$, one of $a_1^{i'}, a_2^{i'} \in R_1$ and the other is in R_2. Similarly, one of $b_1^{i'}, b_2^{i'} \in R_1$ and the other is in R_2. Furthermore, in piece G_i, one of a_1^i, a_2^i is in R_1 and the other is in R_2. Thus, there is no T vertex in $G_{i'}$ for $i' < i$, and at most two T vertices in G_i. Therefore, $|\tau(S)| \leq 2$. Furthermore, if there are two T vertices in G_i, a_1^i and a_2^i are each adjacent to b_1^i and b_2^i, and so $G_i = X$ and S is crossed. Thus, if S is uncrossed $|\tau(S)| \leq 1$. □

Lemma 4.17. *If $b_1, b_2 \notin R_1$, then $|\tau(S)| \leq 3$.*

Proof. One of b_1 or b_2 is in R_2, since otherwise $|\tau(S)| \leq 2$ by Claim 4.16. Let i be the smallest value such that G_i contains a vertex of $\tau(S)$. There are at most two vertices of T in G_i, since at most two new vertices first appear in any given piece. Suppose G_i contains two vertices of $\tau(S)$. Then $b_1^i, b_2^i \in T$. Then let H be the strip $G_{i+1} G_{i+2} \dots G_m$. In H, $a_1, a_2, b_1, b_2 \notin R_1$, so by Claim 4.10, $|\tau(H)| = 0$. Thus, $|\tau(S)| \leq 2$.

So assume instead that G_i contains only one vertex of T. Again consider the strip $H = G_{i+1} G_{i+2} \dots G_m$. One corner of a_1, a_2 of H is in T. If the other is in R_2, then again by Claim 4.10, $\tau(H) = 0$. Thus, we may assume a_1 and a_2 of H are not in R_2. Thus, by Claim 4.15, $|\tau(H)| \leq 2$, and $|\tau(S)| \leq 3$. □

Lemma 4.18. *Suppose $a_1, b_2 \in R_1$ and $a_2, b_1 \in R_2$ and S is an uncrossed strip. Then there is not both a long R_1-path and long R_2-path in S.*

Proof. Suppose that there exist such paths. One of a_1^i or a_2^i must be in R_1 for each i and the other must be in R_2. Similarly, one of b_1^i or b_2^i must be in R_1 for each i and the other must be in R_2. Let i be the smallest value such that b_2^i is in R_1. Such an i exists since $b_2 = b_2^m \in R_1$. Then $a_1^i = b_1^{i-1} \in R_1$ by minimality of i, since otherwise $b_2^{i-1} \in R_1$. So $a_2^i \in R_2$, $b_1^i \in R_2$, and edges $a_2^i b_1^i$ and $a_1^i b_2^i$ are in G_i. Thus, $G_i = X$ and S is crossed. □

We will show now that bounds given in Lemmas 4.9 - 4.18 can be achieved in general. Define a *partial model* $(R_1, R_2; T)$ in a strip S to be a model of $K_{2,|T|}$ in S, except that R_1 and R_2 are not required to be connected, but every component of $S[R_i]$ contains a corner of S. Given a sufficiently large strip S and whether a_1, a_2, b_1, b_2 are in R_1, R_2, we look to provide a construction of a partial model $(R_1, R_2; T)$ in S. Furthermore, we may require the existence of other paths within $S[R_1]$ and $S[R_2]$. In the following lemmas, when we specify branch sets, we sometimes give subgraphs rather than simply vertex sets for ease of notation. If we state that a branch set is equal to a subgraph H, we mean that the branch set is $V(H)$. The first lemma provides the construction for when there are two long paths in S.

Lemma 4.19. *For any strip S, given $a_1, b_1 \in R_1$ and $a_2, b_2 \in R_2$, we can construct a partial model $(R_1, R_2; T)$ in S such that there is a long R_1-path and long R_2-path in S. If S is a crossed strip, then given $a_1, b_2 \in R_1$ and $a_2, b_1 \in R_2$, we can construct a partial model $(R_1, R_2; T)$ in S such that there is a long R_1-path and long R_2-path in S.*

Proof. By Observation 2.12, there exist disjoint paths P_1 and P_2 from a_1 to b_1 and from a_2 to b_2. Let $R_1 = P_1$, $R_2 = P_2$, and $T = \emptyset$. See Figure 4.2(a).

If S is crossed, let G_i be an X-piece in S. Let R_1 be the rail path from a_1 to a_1^i along with the rail path from b_2^i to b_2. Let R_2 be the rail path from a_2 to a_2^i along with the rail path from b_1^i to b_1. See Figure 4.2(b). □

The following lemmas give information on $\mathcal{A}$-sequence strips with sufficient reduction size. Recall that for an $\mathcal{A}$-sequence strip S, the reduction size of S is the number of vertices in the graph generated by the linear $\mathcal{A}$-sequence obtained from the $\mathcal{A}$-sequence of S by replacing all IA^+IA^+I

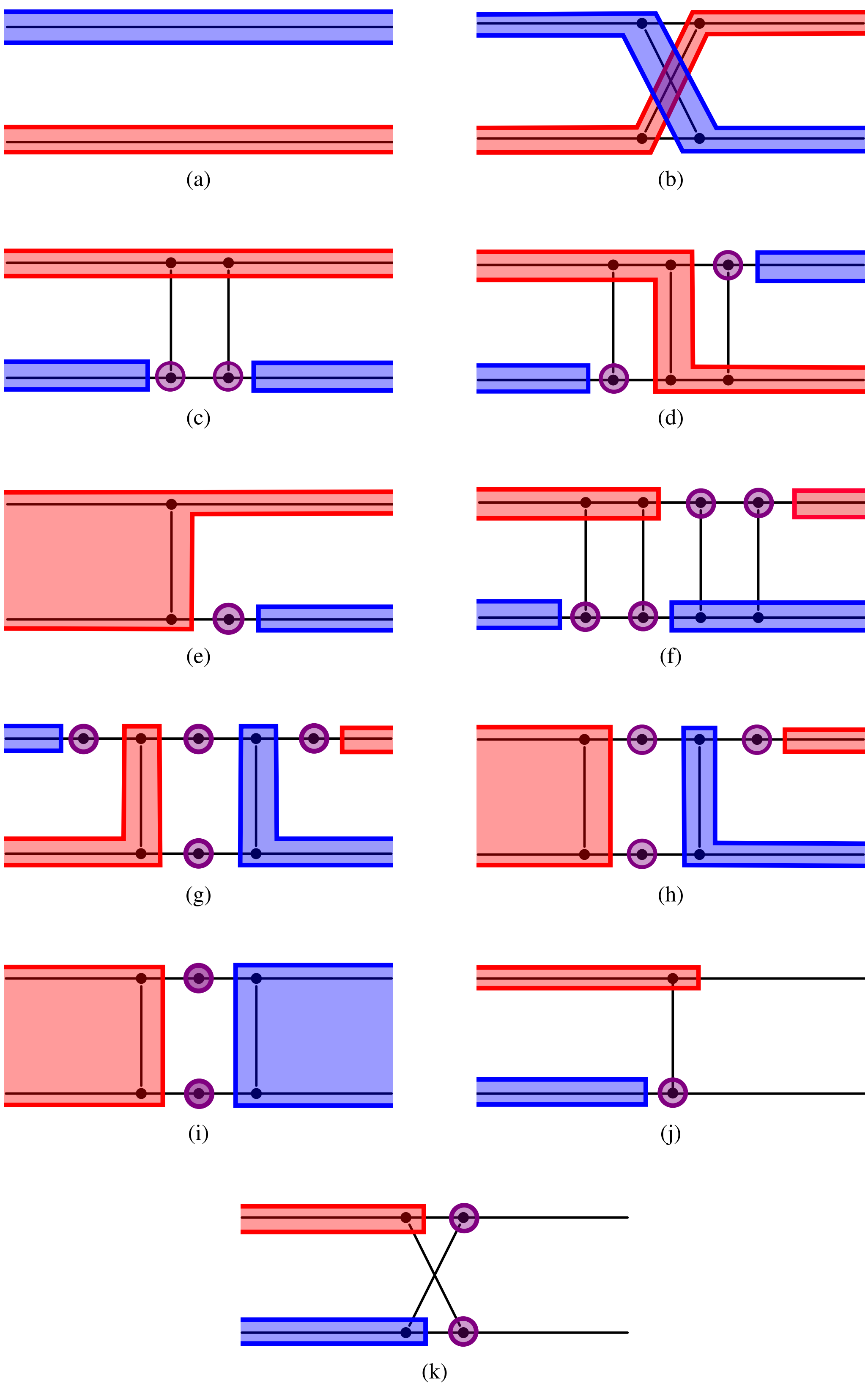

Figure 4.2: The constructions given in Lemmas: (a) and (b) 4.19, (c) 4.22, (d) 4.23, (e) 4.24, (f) 4.25, (g) 4.26, (h) 4.27, (i) 4.28, (j) 4.29, (k) 4.30.

subsequences with IA^+I and all IA^-IA^-I subsequences with IA^-I. The reduction size of S is denoted $r(S)$.

Lemma 4.20. *(1) Let $S = G_1G_2\dots G_m$ be a graph represented by the linear $\mathcal{A}$-sequence $G_1, G_2, \dots G_m$. If $r(S) \geq 7$, there exists an i such that b_1^i and b_2^i are both distinct from a_1 and a_2. Furthermore, if we let i be the smallest value such that b_1^i and b_2^i are distinct from a_1 and a_2, then $r(G_1G_2\dots G_i) \leq 7$.*

(2) Let $S = G_1G_2\dots G_m$ be a graph represented by the linear $\mathcal{A}$-sequence $G_1, G_2, \dots G_m$. If $r(S) \geq 8$, there exists an i such that there is a non-rail edge $e = xy$ in G_i where x and y are both distinct from a_1 and a_2. Furthermore, if we let i be the smallest value such that there is a non-rail edge $e = xy$ in G_i where x and y are both distinct from a_1 and a_2, then $r(G_1G_2\dots G_i) \leq 8$.

(3) Let $S = G_1G_2\dots G_mH_1H_2\dots H_k$, be a graph represented by the linear $\mathcal{A}$-sequence $G_1, G_2, \dots, G_m, H_1, H_2, \dots H_k$. If we let $S_1 = G_1G_2\dots G_m$ and $S_2 = H_1H_2\dots H_k$, then $r(S) = r(S_1S_2) \leq r(S_1) + r(S_2) - 2$.

Proof. For the proofs of (1) and (2), let $\overline{S} = L_1L_2\dots L_k$. If we can find a j such that b_1^j and b_2^j are both distinct from a_1 and a_2 in H with $V(L_1L_2\dots L_j) \leq 6$, there is a corresponding value i such that b_1^i and b_2^i are both distinct from a_1 and a_2 in G and $\overline{G_1G_2\dots G_i} = L_1L_2\dots L_j$. Thus, it suffices to find such a j in $\overline{S}$ to prove (1). Similarly, to prove (2) it suffices to find a j in $\overline{S}$ such that there is a non-rail edge $e = xy$ in L_j where x and y are both distinct from a_1 and a_2, and $|V(L_1L_2\dots L_j)| \leq 8$.

If there is a subsequence $L_1L_2\dots L_a$ with fewer vertices than $\overline{S}$, then $a < k$, that is, there is at least one more piece after L_a. We will use this fact without explicit mention throughout the proofs of (1) and (2).

Proof of (1). If $L_1 = X$ or A^*, then $j = 1$, and $|V(L_1)| \leq 4$, as required.

Suppose then $L_1 = A^+$ or A^-; without loss of generality take $L_1 = A^+$. If the sequence $L_1L_2\dots L_m$ starts with A^+IX, A^+IA^-, A^+IA^* or A^+X, then $j \leq 3$, and $|V(L_1\dots L_j)| \leq 5$, as required. The only remaining possibility is that sequence begins A^+IA^+. If this sequence is followed by one of the subsequence IX, X, IA^-, or IA^*, each of the sequences A^+IA^+IX, A^+IA^+X, $A^+IA^+IA^-$, $A^+IA^+IA^*$, give $|V(L_1L_2 \dots L_j)| \leq 6$ as required. Otherwise we have the sequence $A^+IA^+IA^+$ which must be followed by X. Thus, $j = 6$, and $|V(L_1L_2 \dots L_6)| \leq 7$ as required.

If $L_1 = I$, then apply the above reasoning to $L_2L_3 \dots L_k$, to obtain the required result, since $|V(L_1L_2 \dots L_j)| = |V(L_2L_3 \dots L_j)|$.

Proof of (2). As in the proof for (1) above, we can assume $L_1 \neq I$, since the I-piece does not add any vertices.

Suppose $L_1 = A^*$. Then $L_2 = I$ or X, and $|V(L_1L_2 \dots L_j)| \leq 6$

Suppose $L_1 = X$. If $L_2 = I$ or X, then $j = 2$, and $|V(L_1L_2)| \leq 6$, as required. If $L_2 = A^+, A^-$, or A^*, then L_3 must be an I or X piece, $j = 3$, and $|V(L_1L_2L_3)| \leq 8$.

Now, without loss of generality, we may assume that $L_1 = A^+$. It must be followed by an X or I. If $L_2 = X$, then $j = 2$, and $|V(L_1L_2)| \leq 5$, and we are done. Thus, we may assume $L_2 = I$. If $L_3 = X$, then $j = 3$, and $|V(L_1L_2L_3)| \leq 5$, as required. If $L_3 = A^+$ or A^*, then L_4 is either an X- or I-piece. In either case, if $L_3 = A^*$ then $j = 4$ and $|V(L_1L_2L_3L_4)| \leq 7$. Thus, we may assume $L_3 = A^+$. If $L_4 = X$, then $j = 4$, and $|V(L_1L_2L_3L_4)| \leq 6$. Therefore we can assume $L_4 = I$. If $L_5 = X$, then $j = 5$, and $|V(L_1L_2L_3L_4L_5)| \leq 6$. If $L_5 = A^*$ or A^-, then $L_6 = X$ or I, and we have $j = 6$, and $|V(L_1L_2 \dots L_6)| \leq 8$. If $L_5 = A^+$, then $L_6 = X$, $j = 6$, and $|V(L_1L_2 \dots L_6)| \leq 7$.

Proof of (3). If there is a subsequence IA^+IA^+I or IA^-IA^-I in $G_1G_2 \dots G_m$ or $H_1H_2 \dots H_k$, that subsequence appears in the sequence $G_1G_2 \dots G_mH_1H_2 \dots H_k$ as well. Thus, $\bar{S}_1\bar{S}_2$ has no more vertices then $\overline{S_1S_2}$. There are $|V(\overline{S_1})| + |V(\overline{S_2})| - 2 = r(S_1) + r(S_2) - 2$ vertices in $\bar{S}_1\bar{S}_2$, so $r(S_1S_2) \leq r(S_1) + r(S_2) - 2$. □

From these lemmas we can give a sufficient reduction size on a strip S, such that every such strip contains five vertex-disjoint non-rail edges with endpoints that are also disjoint from the corners.

Lemma 4.21. *Given an $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, there exist five vertex-disjoint edges with endpoints that are disjoint from the corners of S. Furthermore, we can take these five edges $e_1 = x_1y_1$, $e_2 = x_2y_2$, $e_3 = x_3y_3$, $e_4 = x_4y_4$, and $e_5 = x_5y_5$, such that x_1, x_2, x_3, x_4, x_5 occur in that order in one rail path of S, and y_1, y_2, y_3, y_4, y_5 occur in that order in the other rail path of S.*

Proof. Let S be such a strip. Let i be the smallest positive integer such that G_i contains an edge $e_1 = x_1y_1$ such that x_1 and y_1 are both distinct from a_1 and a_2. Then $r(G_1G_2 \ldots G_i) \leq 8$ by Lemma 4.20(2). Then by Lemma 4.20(3), we have $r(G_{i+1}G_{i+2} \ldots G_m) \geq r(S) - r(G_1G_2 \ldots G_i) + 2 \geq 31$. Let j be the smallest positive integer greater than i such that G_j contains a non-rail edge, $e_2 = x_2y_2$, where x_2 and y_2 are both distinct from x_1 and y_1. Then $r(G_{i+1}G_{i+2} \ldots G_j) \leq 8$ by Lemma 4.20(2), and $r(G_jG_{j+1} \ldots G_m) \geq 25$.

By repeated use of this reasoning we can find such edges e_3, e_4, and e_5, such that the reduction size of the remaining graph is at least 19, 13, and 7 respectively. The endpoints of e_5 must be distinct from the corners b_1 and b_2, by Lemma 4.20(1). □

The bound on $r(S)$ in the above lemma may not be best possible, but improving that bound would not meaningfully change the final conclusions of this chapter.

For most of the remaining constructions, Lemmas 4.22 - 4.29, we will use the existence of these five edges $e_1 = x_1y_1$, $e_2 = x_2y_2$, $e_3 = x_3y_3$, $e_4 = x_4y_4$, and $e_5 = x_5y_5$, given by Lemma 4.21. In order to apply Lemma 4.21, we will assume $r(S) \geq 37$. We do not always require all five of those edges, and so in fact this bound could be reduced in some cases, but for simplicity, we assume $r(S) \geq 37$ for those lemmas. Furthermore, throughout those proofs, we will let $P = p_1p_2 \ldots p_k$ and $Q = q_1q_2 \ldots q_l$ be the rail paths of S, where $a_1 = p_1$, $b_1 = p_k$, $a_2 = q_1$, and $b_2 = q_l$, each $x_i \in P$ and each $y_i \in Q$. The next three lemmas provide the constructions for when there is one long path in S.

Lemma 4.22. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given $a_1, b_1 \in R_1$ and $a_2, b_2 \in R_2$, there exists a partial model $(R_1, R_2; T)$ such that $|\tau(S)| = 2$ and there is a long R_1-path in S.*

Proof. Let $R_1 = P$, let $T = \{y_1, y_2\}$, and let $R_2 = Q[q_1, y_1) \cup Q(y_2, q_l]$. The vertices of T, y_1 and

y_2, each have a neighbor in R_1 and R_2. See Figure 4.2(c). □

Lemma 4.23. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given $a_1, b_2 \in R_1$ and $a_2, b_1 \in R_2$, there exists a partial model $(R_1, R_2; T)$, such that $|\tau(S)| = 2$ and there is a long R_1-path in S.*

Proof. Let $T = \{x_3, y_1\}$, $R_1 = P[p_1, x_2] \cup Q[y_2, q_l]$, and $R_2 = Q[q_1, y_1) \cup P(x_3, p_k]$. See Figure 4.2(d). □

Lemma 4.24. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given a_1 and/or a_2 is in R_1, $b_1 \in R_1$, and $b_2 \in R_2$, there exists a partial model $(R_1, R_2; T)$ in S such that $|\tau(S)| = 1$ and there exist paths in $S[R_1]$ between each element of $\{a_1, a_2\} \cap R_1$ and b_1.*

Proof. Let $T = y_2$. Let $R_2 = Q(y_2, q_l]$. Let $P[x_1, p_k] \cup Q[y_1, y_2) \subseteq R_1$. Then if $a_1 \in R_1$, add $P[p_1, x_1]$ to R_1, and if $a_2 \in R_1$, add $Q[q_1, y_1]$ to R_1. The necessary paths in R_1 exist. The vertex y_2 has neighbors in R_1 and in R_2. See Figure 4.2(e). □

The following six lemmas give the constructions for when there are no long paths in S. They are presented mostly in decreasing order of $\tau(S)$.

Lemma 4.25. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given $a_1, b_1 \in R_1$ and $a_2, b_2 \in R_2$, there exists a partial model of $(R_1, R_2; T)$ in S such that $|\tau(S)| = 4$.*

Proof. Let $T = \{y_1, y_2, x_3, x_4\}$. Let $R_1 = P[x_1, x_3) \cup P(x_4, p_k]$, and let $R_2 = Q[q_1, y_1) \cup Q(y_2, q_l]$. Each vertex of T has a neighbor in R_1 and in R_2. See Figure 4.2(f). □

Note that the following lemma is where all five of x_1, x_2, x_3, x_4, and x_5 are used.

Lemma 4.26. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given $a_1, b_2 \in R_1$ and $a_2, b_1 \in R_2$, there exists a partial model of $(R_1, R_2; T)$ in S such that $|\tau(S)| = 4$.*

Proof. Let $T = \{x_1, x_3, x_5, y_3\}$, let $R_1 = P[p_1, x_1) \cup P(x_3, x_5) \cup Q(y_3, q_l]$, and let $R_2 = Q[q_1, y_3) \cup P(x_1, x_3) \cup P(x_5, p_k]$. Then each T vertex has a neighbor in R_1 and R_2. See Figure 4.2(g). □

Lemma 4.27. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given a_1 and/or a_2 is in R_1, $b_1 \in R_1$, and $b_2 \in R_2$, there exists a partial model $(R_1, R_2; T)$ in S such that $|\tau(S)| = 3$. Furthermore, this construction includes a path from a_1 to a_2 in $S[R_1]$ if both are in R_1.*

Proof. Let $T = \{x_2, y_2, x_4\}$, and let $R_2 = P(x_2, x_4) \cup Q(y_2, q_l]$. Let $P[x_1, x_2) \cup Q[y_1, y_2) \cup P(x_4, p_k] \subseteq R_1$. Then if $a_1 \in R_1$ add $P[a_1, x_1]$ to R_1, and if $a_2 \in R_2$ add $Q[a_2, y_1]$ to R_1. Then each vertex of T has a neighbor in R_1 and R_2. See Figure 4.2(h). □

Lemma 4.28. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given a_1 and/or a_2 is in R_1 and b_1 and/or b_2 is in R_2, there exists a partial model $(R_1, R_2; T)$ in S such that $|\tau(S)| = 2$. Furthermore, if a_1 and a_2 are in R_1, then the construction includes a path from a_1 to a_2 in $S[R_1]$. Similarly, if b_1 and b_2 are both in R_2, the construction includes a path from b_1 to b_2 in $S[R_2]$.*

Proof. Let $T = \{x_2, y_2\}$. Let $P[x_1, x_2) \cup Q[y_1, y_2) \subseteq R_1$ and let $P(x_2, x_3] \cup Q(y_2, y_3] \subseteq R_2$. Then if $a_1 \in R_1$ add $P[p_1, x_1]$ to R_1 and if $a_2 \in R_1$ add $Q[q_1, y_1]$ to R_1. Similarly for R_2, if $b_1 \in R_2$ add $P[x_3, p_k]$ to R_2, and if $b_2 \in R_2$ add $Q[y_3, q_l]$ to R_2. Each vertex of T has a neighbor in R_1 and R_2, and the required paths exist. See Figure 4.2(i). □

Lemma 4.29. *For any $\mathcal{A}$-sequence strip S with $r(S) \geq 37$, given $a_1 \in R_1$ and $a_2 \in R_2$, there exists a partial model $(R_1, R_2; T)$ in S such that $|\tau(S)| = 1$.*

Proof. Let $\{q_j\} = T$, let $R_1 = P[p_1, x_1]$, and let $R_2 = Q[q_1, y_1)$. Then q_j has neighbors in R_1 and R_2. See Figure 4.2(j). □

It is worth noting that the following lemma requires an additional condition that none of the other construction lemmas require, namely that there is an X-piece that does not include b_1 or b_2. In the case when the only X-piece includes one of those corners, there can be difficulties if one or both of those vertices are in T. We will be careful to consider this possibility when this lemma is applied in the proof of Proposition 4.31.

Lemma 4.30. *Let S be a crossed $\mathcal{A}$-sequence strip $S = G_1 G_2 \ldots G_m$, such that $G_i = X$ for some i where $b_1^i \neq b_1, b_2$ and $b_2^i \neq b_1, b_2$. Then given $a_1 \in R_1$ and $a_2 \in R_2$, there exists a partial* *model $(R_1, R_2; T)$ in S such that $|\tau(S)| = 2$. When the only X-piece in S contains b_1 or b_2, the construction still exists except that b_1 or b_2 or both may be assigned to $\tau(S)$.*

Proof. We have $G_i = X$ and $b_1^i \neq b_1, b_2$ and $b_2^i \neq b_1, b_2$. Let $T = \{b_1^i, b_2^i\}$. Let $R_1 = P[p_1, a_1^i]$ and $R_2 = Q[q_1, a_2^i]$. Then each vertex of T is adjacent to $a_1^i \in R_1$ and $a_2^i \in R_2$. See Figure 4.2(k). □

We can now prove the result on replacing strips and $K_{2,t}$-minor-freeness. Note that Lemmas 4.9 through 4.17 do not require S to be $\mathcal{A}$-sequence strips. This allows the replacing strip, S' below, to not necessarily be an $\mathcal{A}$-sequence strip. Lemmas 4.19 - 4.30 do generally require S to be an $\mathcal{A}$-sequence strip, so the replaced strip, S below, must be an $\mathcal{A}$-sequence strip.

Proposition 4.31. *Fix $t \geq 5$, and let G be a $K_{2,t}$-minor-free graph with an $\mathcal{A}$-sequence strip S having reduction size at least 37. If S is crossed then every graph G' obtained by replacing S with any strip is $K_{2,t}$-minor-free. If S is uncrossed then every graph G' obtained by replacing S with any uncrossed strip is $K_{2,t}$-minor-free.*

Proof. Suppose G is a $K_{2,t}$-minor-free graph with an $\mathcal{A}$-sequence strip S having reduction size at least 37. Let G' be the graph obtained from G by replacing S with a strip S'. Assume S is crossed if S' is crossed.

Suppose G' has a $K_{2,t}$ minor with model $(R'_1, R'_2; T')$. Let $C = \{a_1, a_2, b_1, b_2\}$ be the set of corners of S' and S. Let $G_0 = (G - (V(S) - C)) = (G' - (V(S') - C))$, i.e., G minus the non-corner vertices of S. We will provide a model $(R_1, R_2; T)$ of $K_{2,t}$ in G. This model will be the same as R'_1, R'_2, T' outside of S. For the most part, it will also be unchanged on the corners of S as well, with exception of vertices of T' that neither have edges to both R'_1 and R'_2 in S' nor edges to both R'_1 and R'_2 outside of S'. Each such vertex x will essentially be moved to the interior of S, and the corresponding corner will be moved to R_i where R_i is the set R_1 or R_2 such that x has an edge to R_i outside of S. To do this, we slightly modify G', S' and the model $(R'_1, R'_2; T')$ to obtain G'', S'', and $(R''_1, R''_2; T'')$. The details of this follow.

Let B be the set of vertices x such that $x \in T' \cap C$, and the only edges from x to $R_{i(x)}$ are not in S' and the only edges from x to $R_{3-i(x)}$ are in S', where $i(x) \in \{1, 2\}$. Consider the strip S''

obtained from S' in the following way. For each $x \in B$, add a new vertex z_x that is adjacent to x and all neighbors of x in S'. Then delete all edges $xv \in E(S')$, so that x is adjacent only to z_x in S''. The graph S'' may not be an $\mathcal{A}$-sequence strip (S' was not even require to be such a strip), but it is a strip. Let G'' be the graph obtained from G' by replacing S' with S''. Let $X_1 = \{x \in B | i(x) = 1\}$ and $X_2 = \{x \in B | i(x) = 2\}$. Then let $R_1'' = R_1' \cup X_1$ and $R_2'' = R_2' \cup X_2$, and $T'' = T'$. Then $(R_1'', R_2''; T'')$ is a model of $K_{2,t}$ in G''. For every vertex $x \in T'' \cap V(S'')$ such that the only edges from x to R_i are in S'' for at least one of $i = 1$ or $i = 2$, we have $x \in \tau(S'')$.

It suffices now to find a partial model $(R_1^*, R_2^*; T^*)$ in S such that the vertices of C are assigned as in G'', except perhaps a vertex of $\tau(S'') \cap C$ may not appear in T^*, such that if there is a path between two corners of C in $S''[R_i]$ then there is a path between those corners in $S[R_i^*]$, and such that $|\tau(S)| = |\tau(S'')|$. Such a partial model gives a model $(R_1, R_2; T)$ of $K_{2,t}$ in G, where $R_1 = R_1^* \cup (R_1'' \cap V(G_0))$, $R_2 = R_2^* \cup (R_2'' \cap V(G_0))$, and $T = (T'' - \tau(S'')) \cup T^*$.

We will consider cases based first on $|R_2'' \cap C|$ and then on $|R_1'' \cap C|$, where we assume $|R_1'' \cap C| \geq |R_2'' \cap C|$.

Case 1: $|R_2'' \cap C| = 0$.

Then $\tau(S'') = \emptyset$ by Lemma 4.10. If $R_1'' \cap C$ is non-empty, assigning all non-corner vertices of S to R_1 gives a partial model $(R_1^*, R_2^*; T^*)$ in S as required. When $|R_1'' \cap C| = 0$, taking R_1^*, R_2^*, and T^* to not include any non-corner vertices of S gives a partial model in S.

Case 2: $|R_2'' \cap C| = 1$.

Then $|R_1'' \cap C| = 1, 2$, or 3.

Case 2.1: $|R_1'' \cap C| = 1$.

Suppose first that one of a_1 or a_2 is in R_1'' or R_2'', and one of b_1 or b_2 is in R_1'' or R_2'', that is there is a corner of R_1'' or R_2'' on each side. Then by Lemma 4.15, $|\tau(S'')| \leq 2$. By Lemma 4.28, there exists a partial model (R_1^*, R_2^*, T^*) in S such that $|\tau(S)| = 2$.

Thus, we may assume that one of a_1 and a_2 is in R_1'' or R_2'' and the other of a_1 and a_2 is in the other of R_1'' and R_2''. Without loss of generality, $a_1 \in R_1''$ and $a_2 \in R_2''$. By Lemma 4.16, $|\tau(S'')| \leq 2$. If $\tau(S'') \leq 1$, there exists a partial model (R_1^*, R_2^*, T^*) in S such that $|\tau(S)| = 1$ by

Lemma 4.29.

If $|\tau(S'')| = 2$, then S'' must be crossed by Lemma 4.16. Thus, S is crossed as well. If there is an X in S that does not include b_1 or b_2, then by Lemma 4.30, there is a partial model $(R_1^*, R_2^*; T^*)$ in S with $|\tau(S)| = 2$. If $b_1, b_2 \notin T'' - \tau(S'')$, then by Lemma 4.30, there is a partial model $(R_1^*, R_2^*; T^*)$ in S with $|\tau(S)| = 2$ but that may assign b_1 or b_2 or both to T. Since $b_1, b_2 \notin R_1'', R_2''$ and $b_1, b_2 \notin T'' = \tau(S'')$, we can put them in T^* and still have a $K_{2,t}$-minor in G.

If one of b_1 or b_2, say b_i, is in $T'' - \tau(S'')$, then it must have edges to R_1'' and R_2'' in G_0. Then assign b_i to R_1^*, and by Lemma 4.27, we can construct a partial model $(R_1^*, R_2^*; T^*)$ in S such that $|\tau(S)| = 3$, giving a $K_{2,t}$-minor in G. If both b_1 and b_2 are in $T'' - \tau(S'')$, then each of b_1 and b_2 has an edge to R_1'' and R_2'' in G_0. Assign b_1 to R_1^* and b_2 to R_2^* in S. Then by Lemma 4.25 or Lemma 4.26, there is a partial model $(R_1^*, R_2^*; T^*)$ in S such that $\tau(S) = 4$, giving a $K_{2,t}$-minor in G.

Case 2.2: $|R_1'' \cap C| \geq 2$.

Then if $|R_1'' \cap C| = 2$ and $a_1, a_2 \in R_1''$ or $b_1, b_2 \in R_1''$, i.e., both R_1'' corners are on the same side, $|\tau(S'')| \leq 2$ by Lemma 4.15. Then by Lemma 4.28, there exists a partial model $(R_1^*, R_2^*; T^*)$ in S such that $|\tau(S)| = 2$, and such that there is a path between the two R_1^* corners in $S[R_1^*]$.

Thus, we may assume $a_1 \in R_1'', a_2 \in R_2''$. If there is no long R_1''-path in S'', then $|\tau(S'')| \leq 3$ by Lemma 4.17, and by Lemma 4.27, there exists a partial model $(R_1^*, R_2^*; T^*)$ in S such that $|\tau(S)| = 3$. If there is a long R_1''-path in S'', $|\tau(S'')| \leq 1$ by Lemma 4.14, and by Lemma 4.24, there exists a partial model $(R_1^*, R_2^*; T^*)$ in S such that $|\tau(S)| = 1$ and any paths between the corners in $S''[R_1'']$ correspond to paths between those corners in $S[R_1^*]$.

Case 3: $|R_2'' \cap C| = 2$.

Then $|R_1'' \cap C| = 2$ as well. If a_1 and a_2 are both in R_1'' or both in R_2'', then $|\tau(S'')| \leq 2$ by Lemma 4.15, and by Lemma 4.28, there exists a partial model $(R_1^*, R_2^*; T^*)$ in S such that $|\tau(S)| = 2$, and there are paths between the corners of R_1^* in $S[R_1^*]$ and between the corners of R_2^* in $S[R_2^*]$. Thus, we may assume that one of $a_1, a_2 \in R_1''$ and the other is in R_2'', and the same for b_1 and b_2. Without loss of generality $a_1 \in R_1''$ and $a_2 \in R_2''$.

We will now consider cases based on the number of long paths in S''.

Case 3.1: There are a long R_1''-path and a long R_2''-path in S''.

Then by Lemma 4.12, $\tau(S'') = 0$.

Suppose $b_1 \in R_2''$ and $b_2 \in R_1''$. Then by Lemma 4.18, S'' is crossed, so S is crossed as well by assumption. Then by Lemma 4.19, there exists a partial model of $(R_1^*, R_2^*; T^*)$ in S such that there exist an long R_1^*-path and long R_2^*-path in S.

Otherwise, we have $b_1 \in R_1''$ and $b_2 \in R_2''$. Then by Lemma 4.19, there exists a partial model $(R_1^*, R_2^*; T^*)$ in S such that there is a long R_1^*-path and long R_2^*-path in S.

Case 3.2: There is one long R_i''-path in S''.

By Lemma 4.13, $|\tau(S'')| \leq 2$. By Lemmas 4.22 and 4.23, there exists a partial model $(R_1^*, R_2^*; T^*)$ in S such that there is the required long path in S, and $|\tau(S)| = 2$.

Case 3.3: There is no long R_1''-path or long R_2''-path in S''.

By Lemma 4.9, $|\tau(S'')| \leq 4$. By Lemmas 4.25 and Lemmas 4.26, there there exists a partial model $(R_1^*, R_2^*; T^*)$ in S such that $|\tau(S)| = 4$. □

4.4 Replacing strips and connectivity

Now that we have shown that replacing strips does not create a $K_{2,t}$-minor, we want to show that, as long as we are careful about the type of strip we are replacing with, replacing strips preserves 3- or 4-connectivity.

In the following proof we use the notion of a *modified $\mathcal{Q}$-sequence representation* for $\mathcal{Q}$-sequence strips; since we do not have edges a_1a_2, b_1b_2 we are allowed to use $Q_0, \Delta_0^+, \Delta_0^-$ as the first piece and $Q_1, \Delta_1^+, \Delta_1^-$ as the last piece, where $P_0 = P - a_1a_2$ and $P_1 = P - b_1b_2$ for $P = Q, \Delta^+, \Delta^-$. The same adjacency rules for P apply to P_0 and P_1.

The following observation is required.

Observation 4.32. *Let S be a strip, regarded as a (2,2)-terminal graph in the usual way and decomposed into one of more I, A^+, A^-, X-pieces as in Lemma 4.3. If the first piece is X, both a_1 and a_2 have degree 2 in S, and otherwise at least one of a_1 or a_2 has degree 1 in S. A similar* *observation applies to the last piece and* b_1, b_2.

Proof. If the first piece is not X, it is A^+, A^-, or A^* and gives a degree 1 vertex. □

Lemma 4.33. *Let* $k = 3$ *or 4, and let* G *be* k*-connected with a strip* S *with* $r(S) \geq 9$*. Let* S' *be a* $\mathcal{Q}$*-sequence strip with at least 9 vertices if* $k = 4$*, and let* S' *be an* $\mathcal{A}$*-sequence strip with* $r(S') \geq 7$ *if* $k = 3$*. Let* G' *be the graph obtained from* G *by replacing* S *with* S'*, possibly with a twist (i.e., replacing* S *by* $S'T$*). Then if each corner of* S' *has degree at least* k *in* G'*,* G' *is* k*-connected.*

Proof. Let C be the set of corners of S', let int(S) be $V(S) - C$, and let int(S') be $V(S') - C$. Let $H = G' - \text{int}(S') = G - \text{int}(S)$, and let $\text{int}(H) = V(H) - C$. The graph H is the union of zero or more C-bridges of G, so H is (k, C)-connected by Lemma 1.3. Adding the edges of a K_4 to C in S' gives a k-connected graph by Theorem 2.10 and Lemma 3.6. Thus, S' is (k, C)-connected as well.

We will use a_1, a_2, b_1, b_2 to refer to the corners of S' in the usual way. Note that this does not agree with the labelling in S if S' is added with a twist. We will also consider H as a (2,2)-terminal graph with vertices labelled in the same way as in S', but for some purposes we will need to swap labels a_1 with b_1 and a_2 with b_2; we write H^R for this modified version of H. We use P_i to denote the rail path in S' from a_i to b_i.

Assume for a contradiction that G' is not k-connected. Let K be a cutset of G' with $|K| < k$. Both $S' - K$ and $H - K$ must be disconnected, since if either is connected, all vertices of $C - K$ lie in the same component, and every component of the other contains a vertex of $C - K$ since it is (k, C)-connected. For any Q-sequence strip or $\mathcal{A}$-sequence strip, removing any number of the corners does not disconnect the strip. Thus, at least one vertex of K belongs to int(S').

Now we prove some facts about the structure of H.

Claim 4.34. *There are* $k - 2$ *disjoint paths in* H *from* $\{a_1, a_2\}$ *to* $\{b_1, b_2\}$*.*

Proof. Since $r(S) \geq 9$, there is a 2-cut $\{u, v\}$ which separates $\{a_1, a_2\}$ from $\{b_1, b_2\}$ and also separates two non-corner vertices y, z. This fact was confirmed by checking all small cases by hand. There are still $k-2$ internally disjoint paths from y to z in $G' - \{u, v\}$, which shows that there are $k-2$ vertex-disjoint paths in H from $\{a_1, a_2\}$ to $\{b_1, b_2\}$. □

Claim 4.35. *If $k = 3$ then either (1) H is connected, or (2) there exists a $c \in C$ such that H has two components c and $H - c$, where $H - c$ is $(3, C - \{c\})$-connected.*

Proof. Suppose that $|V(H)| = 4$. Then applying Observation 4.32, there are at least two edges of H with at least one end in $\{a_1, a_2\}$. Thus, H has at least two edges, so it is one of (1) connected, (2) a 3-vertex component and an isolated vertex, or (3) a 2-edge matching. In cases (a) and (b) the claim holds, so assume we have (3). By Claim 4.34 we know that H has at least one edge from $\{a_1, a_2\}$ to $\{b_1, b_2\}$, so H must consist of two edges $a_1 b_i$ and $a_2 b_{3-i}$. But then H is just A^* or A^*T, so H^R is A^* or TA^*. Let $S' = L_1 L_2 \ldots L_m$ be the $\mathcal{A}$-sequence representation of S'. By Observation 4.32, $L_1 = L_m = X$. Thus, $G' = \gamma(H^R L_1 L_2 \ldots L_m)$ is an $\mathcal{A}$-sequence graph. The reduction size of G' as an $\mathcal{A}$-sequence graph is equal to the reduction size of S' as an $\mathcal{A}$-sequence strip. Thus, $r(G') \geq 7$ and G' is 3-connected, contradicting our assumption by Lemma 3.6.

Suppose then that $|V(H)| \geq 5$. Since H is $(3, C)$-connected, from every $v \in V(H) - C$ there are paths in H to a set of corners $C_v \subseteq C$ with $|C_v| \geq 3$. If H is not connected then for some $c \in C$ we must have $C_v = C - \{c\}$ for all $v \in V(H) - C$. The result follows. □

Claim 4.36. *If $k = 4$ then $|V(H)| \geq 5$ and H is connected.*

Proof. Suppose that $|V(H)| = 4$. Applying Observation 4.32, there are at least three edges of H with at least one end in $\{a_1, a_2\}$. By Claim 4.34 there are two independent edges from $\{a_1, a_2\}$ to $\{b_1, b_2\}$. Suppose some vertex has degree 1 in H, say $d_H(a_1) = 1$. Then $d_H(a_2) \leq 2$. So $d_{S'}(a_1) \geq 3$ and $d_{S'}(a_2) \geq 2$, which contradicts Observation 4.32. Thus, every vertex of H has degree at least 2 and H has a perfect matching, $a_1 b_i$, $a_2 b_{3-i}$. This means that $H = X, IX, XI, IXI, Q, QT, \Delta^+\Delta^-, \Delta^+\Delta^- T, \Delta^-\Delta^+$, or $\Delta^-\Delta^+ T$. Thus, $H^R = X, XI, IX, IXI, Q, TQ, \Delta^-\Delta^+, T$ or $T\Delta^+\Delta^-$. Let $S' = L_1 L_2 \ldots L_m$ be the modified $\mathcal{Q}$-sequence representation of S'. If H^R is Q or TQ then, by Observation 4.32, $L_1 = L_m = X$. In the case where H^R contains Δ^+ and Δ^-, then by Observation 4.32, $L_1 \neq Q_0$. Possibly $L_1 = X$. Otherwise, $L_1 = \Delta_0^+$ or Δ_0^-, and then L_1

and the last piece of H^R must be aligned when they are joined so that a_1, a_2 have degree at least 4. To make this alignment we can use T^α where $\alpha \in \{0, 1\}$. A similar alignment may be necessary after L_m. In any case, we have $G' = \gamma(H^R T^\alpha L_1 L_2 \ldots L_m T^\beta)$ for some $\alpha, \beta \in \{0, 1\}$. It is not hard to verify that this graph G' is a $\mathcal{Q}$-sequence graph on at least 9 vertices, and thus by Theorem 2.10, G' is 4-connected, contradicting our assumption.

So $|V(H)| \geq 5$, and since H is $(4, C)$-connected, from every $v \in V(H) - C$ there are paths in H to all four corners, so H is connected. □

We will now consider several cases based on the position of the vertices of K. We know that $|\text{int}(S') \cap K| \geq 1$ for every cutset K with $|K| < k$, and that $|\text{int}(S') \cap K| + |C \cap K| + |\text{int}(H) \cap K| \leq k - 1$.

Case 1: Suppose first that $\text{int}(S') \cap K = \{x\}$.

Case 1.1: Suppose that $|C \cap K| = 0$. Then $S' - x$ is disconnected, and x must be the only neighbor of some corner vertex, without loss of generality a_1, and $S' - \{x, a_1\}$ is connected. Since a_1 has degree at least k in G', there must be another vertex in the same component as a_1. Thus, $K' = (K - \{x\}) \cup \{a_1\}$ is a cutset of G', which contradicts $|\text{int}(S') \cap K'| \geq 1$.

Case 1.2: Suppose that $|C \cap K| = 1$, say $C \cap K = \{a_1\}$.

If $x \in P_1$, then $P_2 - K = P_2$ is connected. If every vertex in $P_1 - K$ has a neighbor in P_2, then $S' - K$ is connected, which is a contradiciton. The only situation in which every vertex of $P_1 - K$ does not have a neighbor in P_2 is when b_1 is a degree 1 vertex of S'; even then, $S' - K$ will be connected unless x is the rail neighbor of b_1. In that case, since b_1 has degree at least k there is another vertex in the same component of $G' - K$ as b_1, and so $K' = (K - \{x\}) \cup \{b_1\}$ is a cutset of G', which contradicts $|\text{int}(S') \cap K'| \geq 1$.

Thus, we may assume $x \in P_2$. Consider the path $P_2[a_2, x)$. Each vertex of this path must have only a_1 as its neighbor in P_1, since otherwise $S' - K$ is connected. Thus, if x' is the rail neighbor of a_2, $K' = (K - \{x\}) \cup \{x'\}$ remains a cutset for G'. The vertex a_2 has degree at least k, so there is another vertex in the same component of $G' - K'$ as a_2. $S' - K' - a_2$ is connected, so $K'' = (K' - \{x'\}) \cup \{a_2\}$ is a cutset for G, which is a contradicts $|\text{int}(S') \cap K''| \geq 1$.

Case 1.3: Suppose that $|C \cap K| = 2$. Then we must have $k = 4$. By Claim 4.36, $|V(H)| \geq 5$, so there are vertices in $V(H) - C$. Since H is $(4, C)$-connected, each such vertex has paths in H to both elements of $C - K$, so $H - K$ is connected, a contradiction.

Case 2: Suppose that $|\text{int}(S') \cap K| = 2$.

Case 2.1: Suppose that $|V(H) \cap K| = 0$, so that $K = \text{int}(S') \cap K$.

Since $H - K$ must be disconnected, by Claims 4.35 and 4.36 we see that $k = 3$ and H consists of two components, namely some $c \in C$ and $H_0 = H - c$, which $(3, C_0)$-connected, where $C_0 = C - \{c\}$. Then $G' = H_0 \cup S'$. Suppose without loss of generality that $c = a_1$. We must have $d_{S'}(a_1) \geq 3$, so by Observation 4.32 we get $d_{S'}(a_2) = 1$. Let y be the rail neighbor of a_2 in S'.

If $y \in K$ then because $d_{G'}(a_2) \geq k$, $K' = (K - \{y\}) \cup \{a_2\}$ is still a cutset in G' to which we may apply Case 1 to obtain a contradiction. So we assume $y \notin K$.

Let U be the (2,2)-terminal graph that is isomorphic to the path $v_1v_2v_3v_4$, with $a_1(U) = v_1, a_2(U) = v_2, b_1(U) = v_4$, and $b_2(U) = v_3$. Note that $A^-UA^- = Z_{0,0}$, $XUA^- = Z_{1,0}$, $A^-UX = Z_{0,1}$, and $XUX = Z_{1,1}$.

Let $S' = L_1L_2 \dots L_m$ be the $\mathcal{A}$-sequence representation of S', then we must have $L_1 = A^-$ and $L_2 = I$. We also know that $L_m \neq I$ and by Observation 4.32, some b_j has degree $d_{S'}(b_j) \geq 2$. If $j = 1$, let $G^* = \gamma(UL_3L_4 \dots L_m)$, and if $j = 2$, let $G^* = \gamma(UL_3L_4 \dots L_mT)$; then G^* is an $\mathcal{A}$-sequence graph with $r(G^*) = r(S') \geq 7$. Thus, it is 3-connected by Lemma 3.6. Now $S' - a_2$ is a union of $\{b_1, b_2, y\}$-bridges of G^*, and hence is $(3, \{b_1, b_2, y\})$-connected. For each $v \in V(S' - a_2) - \{b_1, b_2, y\}$ we can find three paths in S_2 disjoint except at v from v to $\{b_1, b_2, a_2\}$ (extending each path to y using ya_2), and for $v = y$ we can also find three such paths (using ya_2, $P_2[y, b_2]$, and $ya_1 \cup P_1$). Thus, S' is $(3, \{a_2, b_1, b_2\})$-connected, i.e., $(3, C_0)$-connected.

Now $G' = H_0 \cup S'$ where H_0 is connected and, since S' is $(3, C_0)$-connected and $|K| = 2$, each vertex of $S' - K$ has a path in $S' - K$ to a vertex of $C_0 \subseteq V(H_0)$. Thus, $G' - K$ is connected, which is a contradiction.

Case 2.2: Suppose that $|V(H) \cap K| = 1$. Then $K = (\text{int}(S') \cap K) \cup \{z\}$, where $z \in V(H)$ and $k = 4$.

Suppose that $z \in C \cap K$. By Claim 4.36, H is connected and $|V(H)| \geq 5$. Since H is $(4, C)$-connected, every $v \in V(H) - C \neq \emptyset$ has paths in $H - K = H - z$ to all vertices of $C - \{z\}$, so $H - K$ is connected, which is a contradiction.

Therefore, $z \in \text{int}(H) \cap K$. Let $G_1 = G - z$, which is 3-connected, and let $G'_1 = G' - z$, which is obtained by replacing S in G_1 by S' or $S'T$. Let $K_1 = K - \{z\}$. Then since $G'_1 - K_1 = G' - K$, K_1 is a 2-cut in G_1, and we may apply Case 2.1 above to G_1, S, S', G'_1, and K_1 to obtain a contradiction.

Case 3: Suppose that $|\text{int}(S') \cap K| = 3$. Then we must have $k = 4$ and $V(H) \cap K = \emptyset$. By Claim 4.36, $H - K = H$ is connected, which is a contradiction. □

4.5 Proofs of Lemmas 4.7 and 4.8

We can now prove Lemmas 4.7 and 4.8. Those lemmas are restated here for convenience.

Lemma 4.7. *For any $t \geq 16$, there exists a graph G that is $K_{2,t}$-minor-free and has an uncrossed strip S, but the graph G' obtained by replacing S by any crossed strip S' has a $K_{2,t}$-minor.*

Proof. We construct such a G. Let S be a crossed strip with corners a, b, c, d, where they occur in the order a, b, c, d in the reference cycle of S and the edges ab and cd are deleted in the reference cycle to form S. Then let V_{ab}, V_{bc}, V_{cd}, and V_{ad} be sets of vertices that are pairwise disjoint and disjoint from S. For each vertex $v \in V_{ab}$, let $N(v) = \{a, b\}$. For each vertex in V_{bc}, V_{cd}, or V_{ad}, define their neighborhoods analogously. Furthermore, we require that $|V_{ab}|, |V_{bc}|, |V_{cd}|, |V_{ad}| \geq 4$.

Let $t = |V_{ab} \cup V_{bc} \cup V_{cd} \cup V_{ad}|$. Then we claim any G with this structure is $K_{2,t}$-minor-free. Suppose for a contradiction that G has a $K_{2,t}$-minor with model $(R_1, R_2; T)$. By Lemma 4.9, S contains at most 4 vertices of T.

There must be at least $t-4$ vertices of T in $V_{ab} \cup V_{bc} \cup V_{cd} \cup V_{ad}$. Since $t = |V_{ab} \cup V_{bc} \cup V_{cd} \cup V_{ad}|$, there are at most 4 vertices of $V_{ab} \cup V_{bc} \cup V_{cd} \cup V_{ad}$ not in T. At least three of V_{ab}, V_{bc} ,V_{cd}, and V_{ad} contain a vertex of T, since each V_{xy} contains at least 4 vertices. Each vertex of V_{xy} is only adjacent to x and y. Thus, for at least 3 of the pairs $\{a, b\}, \{b, c\}, \{c, d\}$, and $\{a, d\}$, at least one vertex is in R_1 and the other is in R_2. Without loss of generality, we may assume that $a, c \in R_1$

and $b, d \in R_2$. Thus, there must be a path from a to c in $S[R_1]$ and a path from b to d in $S[R_2]$. However, this is not possible by Lemma 4.18 since S is uncrossed.

Now we want to show that the graph G' obtained by replacing S with any crossed strip S' has a $K_{2,t}$-minor. Let $T = V_{ab} \cup V_{bc} \cup V_{cd} \cup V_{ad}$. There exist disjoint paths P_1, P_2 from a to c and from b to d in S by Lemma 4.19, since S is crossed. Let $R_1 = P_1$ and $R_2 = P_2$. Then R_1 and R_2 are connected, and each vertex of T is adjacent to a vertex of R_1 and of R_2. Thus, G' contains a $K_{2,t}$-minor. □

The examples presented below are effectively the same as the examples in Lemma 4.7 but adapted to be 4-connected. The construction below requires $t \geq 636$, but this could perhaps be reduced.

Lemma 4.8. *For any $t \geq 636$, there exists a 4-connected, $K_{2,t}$-minor-free graph G such that G contains an uncrossed strip S, but the graph G' obtained by replacing S with a crossed strip S' has a $K_{2,t}$-minor.*

Proof. We will construct G having the following general structure. Each vertex of G will be in one of the sets W, X, Y, Z, WX, XY, YZ, or WZ, which we describe in the following. Take four K_3's with vertex sets $W = \{w_1, w_2, w_3\}$, $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, and $Z = \{z_1, z_2, z_3\}$. Take S_0 to be the uncrossed $\mathcal{Q}$-sequence strip $S_0 = (A^-IA^+I)^3A^-IA^+$ with corners w_1, x_1, y_1, and z_1 such that $a_1 = w_1$, $a_2 = x_1$, $b_1 = y_1$ and $b_2 = z_1$; the rail paths go from w_1 to z_1 and x_1 to y_1. Let WX be a non-empty set of vertices such that each $v \in WX$ has degree at least four and all neighbors of v are in either W or X. Note that v has at least one neighbor in W and X. Define sets XY, YZ, and WZ analogously, and let $t = |WX| + |XY| + |YZ| + |WZ|$.

If we replace S_0 with any crossed $\mathcal{Q}$-sequence strip S' to obtain a graph G', G' contains a $K_{2,t}$-minor. Take $T = WX \cup XY \cup YZ \cup WZ$, take R_1 to be W, Y, the top rail of S' before the X-piece, and the bottom rail of S' after the X-piece, and take R_2 to be X, Z and the bottom rail of S' before the X-piece, and the top rail of S' after the X-piece. Since R_1 and R_2 are connected and each vertex of T is adjacent to a vertex of W or X and a vertex of Y or Z, we have $(R_1, R_2; T)$ as a model of $K_{2,t}$ in G'. Moreover, if S' is a $\mathcal{Q}$-sequence strip on at least 9 vertices, G' is 4-connected if G is 4-connected by Lemma 4.33.

Now we will describe some further conditions on WX, XY, YZ, and WZ and show that they guarantee G is 4-connected and prevent G from having a $K_{2,t}$-minor. For each A, B, $A \neq B$ denoting W, X, Y, or Z, let X_iY (or YX_i) denote the set of vertices v in G with $N(v) = \{x_i\} \cup Y$. Then we require that $|W_iX|$, $|WX_i|$, $|X_iY|$, $|XY_i|$, $|Y_iZ|$, $|YZ_i|$, $|W_iZ|$, $|WZ_i| \geq 13$, for all $i \in \{1, 2, 3\}$. Similarly, for each A, B, $A \neq B$ denoting W, X, Y, or Z, we let $A_{ij}B_{kl}$ denote the set of vertices v in G with $N(v) = \{a_i, a_j, b_k, b_l\}$. We require that $|W_{ij}X_{kl}|$, $|X_{ij}Y_{kl}|$, $|Y_{ij}Z_{kl}|$, $|W_{ij}Z_{kl}| \geq 9$ for $i, j, k, l \in \{1, 2, 3\}$ with $i \neq j$ and $l \neq k$.

We assume that G has no vertices other than those already described. Therefore, the smallest value of t for our construction is $t = 8 \times 3 \times 13 + 4 \times 3 \times 3 \times 9 = 636$.

We want to show that such a graph G as described above is 4-connected. To do so, we define a graph G_0, as follows. Let G_0, be as G above, except in G_0 the sets WX, XY, YZ, and WZ are as follows. Let $WX = WX_1 \cup WX_2 \cup WX_3 \cup W_1X \cup W_2X \cup W_3X$ and $|WX_i| = |W_iX| = 1$ for $i \in \{1, 2, 3\}$. In other words, WX consists of exactly 6 vertices three of which are adjacent to all vertices of W and one x_i, and the other three of which are adjacent to all vertices of X and one of w_i. Let XY, YZ, and WY be defined analogously. This single graph G_0 was checked to be 4-connected by computer. Every graph G as described above is obtained from G_0 by adding vertices of degree at least 4 to G. Thus, by Lemma 1.5, G is 4-connected.

Now we will show these conditions prevent a $K_{2,t}$-minor in G. Suppose for a contradiction, that G has a $K_{2,t}$-minor with model $(R_1, R_2; T)$. By Lemma 4.9, there are at most 4 vertices of T in S_0. There are 8 vertices of T in $(W \cup X \cup Y \cup Z) - V(S_0)$, so at most 12 vertices of T are in $W \cup X \cup Y \cup Z \cup V(S_0)$. Therefore, there are at most 12 vertices in $WX \cup XY \cup YZ \cup WZ$ that are not in T.

Since $|W_iX| \geq 13$, there is at least one vertex in $W_iX \cap T$. This holds for each of the sets $W_iX, WX_i, X_iY, XY_i, Y_iZ, YZ_i, W_iZ,$ and WZ_i for $i \in \{1, 2, 3\}$.

Suppose that $W \subseteq R_1$. Since $WX_i \cap T$ is not empty, $x_i \in R_2$. Thus, $X \subseteq R_2$. By analogous reasoning $Y \subseteq R_1$ and $Z \subseteq R_2$. Thus, R_1 must contain a path from w_1 to y_1, and the only such paths that do not intersect X and Z are contained in S_0. Therefore, S_0 contains a long R_1-path from $w_1 = a_1$ to $y_1 = b_2$, and by analogous reasoning, a long R_2-path from $x_1 = a_2$ to $z_1 = b_1$. However, by Lemma 4.18 two such long paths do not exist, since S_0 is uncrossed.

Therefore, we may assume there is a vertex $w_i \in W$ not in R_1. Then R_1 must contain at least one vertex of X since $|W_iX \cap T| \geq 1$. Moreover, $A \cap R_i \neq \emptyset$ for $A = W, X, Y, Z$ and $i \in \{1, 2\}$ by symmetry.

Thus, there are at most 4 vertices of T in $W, X, Y,$ or Z, so at most 8 vertices of T are in $W, X, Y, Z,$ or S_0, and all but at most 8 vertices of $WX, XY, YZ,$ and WZ are in T. Since $|W_{ij}X_{kl}| \geq 9$, there must be a vertex of T in $W_{ij}X_{kl}$ for $i, j, k, l \in \{1, 2, 3\}$ with $i \neq j$ and $l \neq k$. This holds for $X_{ij}Y_{kl}, Y_{ij}Z_{kl}$, and $W_{ij}Z_{kl}$ as well for $i, j, k, l \in \{1, 2, 3\}$ with $i \neq j$ and $l \neq k$.

Suppose there is a vertex $w_i \in W \cap T$. From the above we also have $w_j \in W \cap R_1$ and $w_h \in W \cap R_2$. We must have $|R_s \cap X| \leq 1$ for either $s = 1$ or $s = 2$, say $|R_1 \cap X| \leq 1$. Then there are $x_k, x_l \notin R_1$ with $k \neq l$. However, this contradicts the fact that $W_{ij}X_{kl}$ contains a vertex of T.

Thus, we may assume there is no vertex of T in W, X, Y, or Z. In particular, no corner of S_0 is in T. Therefore, there are at most 4 vertices of WX, XY, YZ, and WZ not in T. Since each of $W, X, Y,$ and Z contain a vertex of R_1, there exist paths P_1, P_2 and P_3 in $G[R_1]$ between three of the distinct pairs from among $W, X, Y,$ and Z, each of which intersects $W \cup X \cup Y \cup Z$ only at its ends. Each such path that uses a vertex of WX must have the form w_ivx_j, where $v \in WX$ belongs to no other such path; similar statements hold for XY, YZ, and WZ. Let Q_1, Q_2, and Q_3 be similar such paths for R_2. Suppose that the path P_1 is not contained in S_0. Then P_1 must contain a vertex of WX, XY, YZ, or WZ. The same is true of each path P_i and Q_i. Thus, at least two of $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are contained in S_0.

Suppose exactly two of those paths are contained in S_0. Then four of those paths contain distinct vertices of WX, XY, YZ or WZ, and so S_0 must contain at least four vertices of T. Since the corners of S_0 are not in T, $|\tau(S_0)| \geq 4$. If there is a long R_1-path in S_0, $|\tau(S_0)| \leq 2$ by Lemma

4.13, which is a contradiction. Thus, we may assume there is no long R_i-path for $i = 1, 2$. Without loss of generality, $w_1, x_1 \in R_1$, and $y_1, z_1 \notin R_1$. Therefore by Lemma 4.17, $|\tau(S_0)| \leq 2$, which is again a contradiction.

Suppose then that three or more of $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are contained in S_0. Then without loss of generality, P_1 and P_2 are contained in S_0. Thus, at least three corners of S_0 are in R_1, and Q_1, Q_2, Q_3 are not contained in S_0. Each of Q_1, Q_2, and Q_3 contain distinct vertices of $WX \cap XY \cap YZ \cap WZ$, which are thus not in T. Therefore, S_0 must contain at least three vertices of T. However, since three of the given paths are contained in S_0, P_3 must be contained in S_0 as well, and all corners of S_0 are in R_1. By Lemma 4.10, $|\tau(S_0)| = 0$, which is a contradiction. □

Given a graph G as constructed in the above proof, replacing S_0 with any uncrossed $\mathcal{Q}$-sequence strip S on at least 9 vertices gives us another such example. Let H be such a graph. By Lemma 4.33, H remains 4-connected. The above reasoning that showed G is $K_{2,t}$-minor-free also applies to H, since S is uncrossed as well; the particular structure of S_0 was not used. As noted earlier, we can also construct 3-connected examples, with a smaller value of t than required for the 4-connected examples.

The examples constructed above also show that replacing an uncrossed strip with a twisted uncrossed strip, i.e. $S'T$ for some uncrossed strip S', can introduce a $K_{2,t}$ minor.

4.6 Characterizing 3- and 4-connected $K_{2,t}$-minor-free graphs

Define the operation of *expanding* an $\mathcal{A}$-sequence strip S in G to be the operation that replaces S with an $\mathcal{A}$-sequence strip S' such that $r(S') \geq r(S)$, $|V(S') > |V(S)|$, and S' is uncrossed if S is uncrossed. Similarly, define the operation of *expanding* an $\mathcal{Q}$-sequence strip S in G to be the operation that replaces S with a $\mathcal{Q}$-sequence strip S' such that $r(S') \geq r(S)$, $|V(S')| > |V(S)|$, and S' is uncrossed if S is uncrossed.

Theorem 4.37. *For a given t, there exists N_t and a finite set of graphs $\mathcal{B}_t$, such that a graph G on at least N_t vertices is 3-connected and $K_{2,t}$-minor-free if and only if it is obtained from a graph in B_t by expanding 3^+-fans and expanding $\mathcal{A}$-sequence strips with reduction size at least 37.*

Proof. Let N_t be the smallest integer such that every 3-connected, $K_{2,t}$-minor-free graph on N_t or more vertices contains a 4^+-fan or a strip S with $r(S) \geq 38$. Such an N_t exists by Corollary 2.14, since each 3-connected, $K_{2,t}$-minor-free graph is obtained from one of a finite number of base graphs by attaching strips and fans. The base graphs are of finite size, so each such graph contains a finite number of fans and strips. Every sufficiently large $\mathcal{A}$-sequence strip S either has $r(S) \geq 38$ or a 4^+-fan. Let $\mathcal{B}_t$ be the set of all 3-connected, $K_{2,t}$-minor-free graphs on fewer than N_t vertices that contain a 3^+-fan or an $\mathcal{A}$-sequence strip S with $r(S) \geq 37$.

Let $\mathcal{S}$ be the set of all graphs obtained from a graph in $\mathcal{B}_t$ by expanding 3^+-fans and $\mathcal{A}$-sequence strips with $r(S) \geq 37$. We want to show that $\mathcal{S}$ is exactly the set of 3-connected, $K_{2,t}$-minor-free graphs on at least N_t vertices. By Observation 2.4 and Lemma 2.6, expanding 3^+-fans preserves 3-connectivity and the property of being $K_{2,t}$-minor-free. Expanding $\mathcal{A}$-sequence strips preserves 3-connectivity and the property of being $K_{2,t}$-minor-free by Proposition 4.31 and Lemma 4.33. Thus, every graph in $\mathcal{S}$ is 3-connected and $K_{2,t}$-minor-free.

We will now show the following claim by induction on n: if G is an n-vertex, 3-connected, $K_{2,t}$-minor-free graph and $n \geq N_t$ then $G \in \mathcal{S}$. Take the base case to be $n \leq N_t - 1$, for which the statement is vacuously true. Suppose then that G is a 3-connected, $K_{2,t}$-minor-free graph on $n \geq N_t$ vertices. The graph G must contain either a 4^+-fan or an $\mathcal{A}$-sequence strip S with $r(S) \geq 38$, by the definition of N_t. Suppose G contains a 4^+-fan F. Then the graph G' obtained by collapsing the fan F to a fan F' is 3-connected and $K_{2,t}$-minor-free by Lemmas 2.3 and 2.6. Note that F' is a 3^+-fan. By the inductive hypothesis G' is either in $\mathcal{S}$ or $\mathcal{B}_t$. Thus, G is obtained from a graph in either $\mathcal{S}$ or $\mathcal{B}_t$ by a 3^+-fan expansion, so it is itself in $\mathcal{S}$. Suppose then that G has no 4^+-fan, so G has an $\mathcal{A}$-sequence strip S with $r(S) \geq 38$. Consider the graph G' obtained by replacing S with a strip S' such that $r(S') \geq 37$, $|V(S')| < |V(S)|$, and S' is uncrossed if S is. Such an S' always exists; we can just take the sequence $A^+I(A^-IA^+I)^{12}A^-$, By Proposition 4.31 and Lemma 4.33, G' is 3-connected and $K_{2,t}$-minor-free. Furthermore, $|V(G')| < |V(G)|$ so by the inductive hypothesis either $G' \in \mathcal{S}$ or $G' \in \mathcal{B}_t$. Thus, G is obtained from a graph in either $\mathcal{S}$ or $\mathcal{B}_t$ by expanding an $\mathcal{A}$-sequence strip with $r(S) \geq 37$, so it is itself in $\mathcal{S}$. $\square$

Theorem 4.38. *For a given t, there exists M_t and a finite set of graphs $\mathcal{C}_t$, such that a graph G on at least M_t vertices is 4-connected and $K_{2,t}$-minor-free if and only if it is obtained from a graph in C_t by expanding $\mathcal{Q}$-sequence strips on at least 37 vertices.*

Proof. For a given t, let $M_t = N_t$ and let $\mathcal{C}_t$ be the set of all 4-connected graphs in $\mathcal{B}_t$, from Theorem 4.37 above. Let $\mathcal{S}$ be the set of all graphs obtained from a graph in $\mathcal{C}_t$ by expanding $\mathcal{Q}$-sequence strips on at least 37 vertices. We want to show that $\mathcal{S}$ is exactly the set of 4-connected, $K_{2,t}$-minor-free graphs on at least M_t vertices.

By Proposition 4.31 and Lemma 4.33, every graph in $\mathcal{S}$ is 4-connected and $K_{2,t}$-minor-free. Thus, it remains to be shown that $\mathcal{S}$ contains all 4-connected $K_{2,t}$-minor-free graphs on at least M_t vertices

Suppose G is a 4-connected and $K_{2,t}$-minor-free graph on $n \geq M_t$ vertices not in $\mathcal{S}$ and take G to be minimal in n. Then by the definition of N_t, G contains a 4^+-fan or an $\mathcal{A}$-sequence strip S with $r(S) \geq 38$. Since G is 4-connected, it does not contain a fan. Thus, it contains an $\mathcal{A}$-sequence strip S with $r(S) \geq 38$. By Lemma 4.5, S is in fact a $\mathcal{Q}$-sequence strip. Since $r(S) = |V(S)|$ for $\mathcal{Q}$-sequence strips, $|V(S)| \geq 38$. Consider the graph G' obtained by replacing S with a $\mathcal{Q}$-sequence strip S', such that $|V(S')| \geq 37$, $|V(S) < |V(S)|$, and S' is uncrossed if S is. By Proposition 4.31 and Lemma 4.33, G' is 4-connected and $K_{2,t}$-minor-free. Furthermore, $|V(G')| < |V(G)|$, so by minimality, $G' \in \mathcal{S}$ or $G' \in \mathcal{C}_t$. Thus, G is obtained from a graph in either $\mathcal{S}$ or $\mathcal{C}_t$ by expanding a $\mathcal{Q}$-sequence strip with $|V(S)| \geq 37$. Thus, G is in $\mathcal{S}$. □

Chapter 5

Future work

In this chapter, we will consider a number of problems related to the new results presented in this document.

Ellingham and Gaslowitz [13] gave a generating function for the non-isomorphic n-vertex $\mathcal{Q}$-sequence graphs, and showed that the number is asymptotic to $\alpha_4^n/(2n)$, where α_4 is the largest root of $f(x) = 1 - x + x^2 - 2x^3 - x^4 + x^5$ and $\alpha_4 \approx 1.86$. We have worked out the generating functions for the number $\mathcal{Q}$-sequence strips, which is asymptotic to $c_4\alpha_4^n$ for some constant c_4 and the number of $\mathcal{A}$-sequence strips, which is asymptotic to $c_3\alpha_3^n$ for some constant c_3, where $\alpha_3 \approx 2.8$. We expect that the variability in 3- and 4-connected, $K_{2,t}$-minor-free graphs will come mainly from their strips because there are very few non-isomorphic ways to expand fans. Based on the counting results for $\mathcal{Q}$- and $\mathcal{A}$-sequence strips, we can make some conjectures.

Conjecture 5.1. *For each t there exist $\gamma_{4,t}$ and $\beta_{4,t}$ such that there are asymptotically $\gamma_{4,t}n^{\beta_{4,t}}\alpha_4^n$ non-isomorphic 4-connected $K_{2,t}$-minor-free graphs on n vertices, where $\alpha_4 \approx 1.86$. Similarly, for each t there exist $\gamma_{3,t}$ and $\beta_{3,t}$ such that there are asymptotically $\gamma_{3,t}n^{\beta_{3,t}}\alpha_3^n$ non-isomorphic 3-connected $K_{2,t}$-minor-free graphs on n vertices, where $\alpha_3 \approx 2.8$.*

The value of $\gamma_{k,t}$ depends on the specifics of the base graphs, so we do not expect to be able to describe this precisely. The value $\beta_{k,t}$ is related to the maximum number of strips in a 3- or 4-connected $K_{2,t}$-minor-free graph. Bounds on the number of strips already exist, as we discuss below, but we would hope to find significantly improved bounds.

Guoli Ding's description of the $K_{2,t}$-minor-free graphs does give an upper bound on the number of strips in these graphs. Since the base graphs are finite, there must be some finite number of strips for any given t. However, the bounds obtainable from Ding's results come from Ramsey-type results and thus are unwieldily large. We expect that the actual value for the number of strips in these graphs is much smaller, and that is why we hope to find such an improved bound.

Problem 5.2. *Find explicit improved upper and lower bounds, depending on t, on the number of strips, or the number of crossed strips, in a 3- or 4-connected, $K_{2,t}$-minor-free graph.*

As discussed in Chapter 1, the class of $K_{2,t}$-minor-free graphs is a subset of the class of $K_{2,t+1}$-minor-free graphs. Furthermore, the class of $K_{1,1,t}$-minor-free graphs lies between those two families; that is the class of $K_{2,t}$-minor-free graphs is a subset of the class of $K_{1,1,t}$-minor-free graph, which is a subset of the class of $K_{2,t+1}$-minor-free graphs. It would interesting to investigate the families of $K_{1,1,t}$-minor-free graphs and how they relate to the $K_{2,t}$-minor-free graphs. A number of the lemmas presented in Chapter 4 either apply to $K_{1,1,t}$ minors as well or could be easily adapted to that case.

Assuming that $\beta_{3,t+1} > \beta_{3,t}$, which seems reasonable to expect, the number of 3-connected $K_{2,t+1}$-minor-free graphs that are not $K_{2,t}$-minor-free on n vertices grows exponentially with n. In other words, there is an exponential increase when going from $K_{2,t}$-minor-free to $K_{2,t+1}$-minor-free. O'Connell [19] has some results that allow us to compare number of $K_{2,t}$-minor-free and $K_{1,1,t}$-minor-free graphs for small t. For $t = 4$, the number of n-vertex, 3-connected, $K_{1,1,t}$-minor-free graphs that are not $K_{2,t}$-minor-free only grows polynomially with n. And the proportion of n-vertex, planar, 3-connected, $K_{1,1,5}$-minor-free graphs that are not $K_{2,5}$-minor-free goes to 0 as $n \to \infty$. This leads us to the following conjecture.

Conjecture 5.3. *The number of 3-connected $K_{1,1,t}$-minor-free graphs on n vertices, $b_t(n)$, is asymptotically equal to the number of 3-connected, $K_{2,t}$-minor-free graphs on n vertices, $a_t(n)$.*

If this conjecture is true, a further problem would be to describe the difference between these two numbers by estimating $(b_t(n) - a_t(n))/a_t(n)$.

One clear area of future work is to finish the characterization of the 3-connected, $K_{2,5}$-minor-free graphs, or the planar, 3-connected, $K_{2,5}$-minor-free graphs, as discussed at the end of Chapter 3; recall the conjectures stated there.

Conjecture 3.18. *There exist positive integers c_1, c_2 and a finite set of base graphs $\mathcal{B}$ such that a graph G on at least c_1 vertices is 3-connected and $K_{2,5}$-minor-free if and only if it is either a $\mathcal{A}$-sequence graph with $r(G) \geq c_2$ or is obtained from a graph in $\mathcal{B}$ by expanding 3^+-fans.*

Conjecture 3.19. *There exist positive integers d_1, d_2 and a finite set of base graphs $\mathcal{C}$ such that a graph G on at least d_1 vertices is planar, 3-connected, and $K_{2,5}$-minor-free if and only if it is either a planar $\mathcal{A}$-sequence graph with $r(G) \geq d_2$ or is obtained from a graph in $\mathcal{C}$ by expanding 3^+-fans.*

Assuming the characterization can be completed in the general case, it would be interesting to examine a slightly larger family of graphs.

Problem 5.4. *Characterize the 3- or 4-connected, $K_{1,1,5}$-minor-free graphs.*

This problem, in particular in the 3-connected case, may prove challenging, since the 3-connected $K_{2,5}$-minor-free graphs have themselves been complex and computationally difficult. For this reason, it may be more manageable to add a planarity condition to these graphs.

Problem 5.5. *Characterize the planar, 3- or 4-connected $K_{1,1,5}$-minor-free graphs.*

This family of graphs should be easier to handle, and therefore may be a more reasonable place to start.

Another related area of work that would be interesting to pursue is examining how Hamiltonicity relates to $K_{2,t}$-minor-freeness. In particular, examining the planar graphs that are $K_{2,6}$-minor-free but not Hamiltonian could be of interest. As discussed in Chapter 1, there is a conjecture on the characterization of all 3-connected, planar $K_{2,6}$-minor-free graphs that are not Hamiltonian.

BIBLIOGRAPHY

[1] K. Appel and W. Haken. Every planar map is four colorable. *Bulletin of the American Mathematical Society*, 82(5):711–712, 1976.

[2] David Barnette. Trees in polyhedral graphs. *Canadian Journal of Mathematics*, 18:731–736, 1966.

[3] D. P. Biebighauser and M. N. Ellingham. One-way infinite 2-walks in planar graphs. *Journal of Graph Theory*, 88:110–130, 2018.

[4] G. Chen, L. Sheppardson, X. Yu, and W. Zang. The circumference of a graph with no $K_{3,t}$-minor. *Journal of Combinatorial Theory, Series B*, 96(6):822–845, 2006.

[5] G. Chen, X. Yu, and W. Zang. The circumference of a graph with no $K_{3,t}$-minor, II. *Journal of Combinatorial Theory, Series B*, 102(6):1211–1240, 2012.

[6] M. Chudnovsky, B. Reed, and P. Seymour. The edge-density for $K_{2,t}$ minors. *Journal of Combinatorial Theory, Series B*, 101:18–46, 2011.

[7] J.P. Costalonga. Contractible edges in 3-connected graphs that preserve a minor. *Journal of Combinatorial Theory, Series B*, to appear, available online 14 May 2019, available as arXiv:1507.06006.

[8] G. Ding. Graphs without large $K_{2,n}$-minors. *arXiv:1702.01355*, 5 February 2017.

[9] G. Ding and C. Liu. A chain theorem for 3^+-connected graphs. *SIAM Journal on Discrete Mathematics*, 26(1):102–113, 2012.

[10] M. N. Ellingham, E. A. Marshall, K. Ozeki, and S. Tsuchiya. A characterization of $K_{2,4}$-minor-free graphs. *SIAM Journal on Discrete Mathematics*, 30(2):955–975, 2016.

[11] M. N. Ellingham, E. A. Marshall, K. Ozeki, and S. Tsuchiya. Hamiltonicity of planar graphs with a forbidden minor. *Journal of Graph Theory*, 90(4):459–483, 2019.

[12] J.Z. Gaslowitz. canary. `https://github.com/JZacharyG/canary`, 2017.

[13] J.Z. Gaslowitz. *Characterizations of Graphs without Certain Small Minors*. PhD thesis, Vanderbilt University, 2018.

[14] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88(2):133–142, 1943.

[15] R.M. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer, 1972.

[16] B.D. McKay and A. Piperno. Practical graph isomorphism, II. *Journal of Symbolic Computation*, 60:94–112, 2014.

[17] J.S. Myers. The extremal function for unbalanced bipartite minors. *Discrete Mathematics*, 271(1-3):209–222, 2003.

[18] S. Negami. A characterization of 3-connected graphs containing a given graph. *Journal of Combinatorial Theory, Series B*, 32:69–74, 1982.

[19] K. O'Connell. *Characterisation and Hamiltonicity of $K_{1,1,t}$-minor-free graphs: A Fan-based Approach*. PhD thesis, Vanderbilt University, 2018.

[20] K. Ota and K. Ozeki. Spanning trees in 3-connected $K_{3,t}$-minor-free graphs. *Journal of Combinatorial Theory, Series B*, 102(5):1179–1188, 2012.

[21] N. Robertson, P. Seymour, and R. Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica*, 13(3):279–361, 1993.

[22] N. Robertson and P. D. Seymour. Graph minors. III. Planar tree-width. *Journal of Combinatorial Theory, Series B*, 36:49–64, 1984.

[23] N. Robertson and P. D. Seymour. Graph minors a survey. *Surveys in Combinatorics*, 103:153–171, 1985.

[24] N. Robertson and P.D. Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004.

[25] P. D. Seymour. Decomposition of regular matroids. *Journal of Combinatorial Theory, Series B*, 28(3):305–359, 1980.

[26] M. Sipser. *Introduction to the Theory of Computation, Second Edition*. Thomson Course Technology Boston, 2006.

[27] R. Thomas, X. Yu, and W. Zang. Hamilton paths in toroidal graphs. *Journal of Combinatorial Theory, Series B*, 94(2):214–236, 2005.

[28] W. T. Tutte. A theorem on planar graphs. *Transactions of the American Mathematical Society*, 82(1):99–116, 1956.

[29] W.T. Tutte. A theory of 3-connected graphs. *Indagationes Mathematicae*, 23:441–455, 1961.

[30] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Mathematische Annalen*, 114:570–590, 1937.

[31] Klaus Wagner. Graphentheorie: BI-Hochschultaschenbücher. 1970.

[32] D. B. West. *Introduction to Graph Theory*. Prentice Hall, Upper Saddle River, NJ, 1996.

[33] X. Yu. Infinite paths in planar graphs I: Graphs with radial nets. *Journal of Graph Theory*, 47(2):147–162, 2004.