NON-HERMITIAN ORTHOGONALITY
AND
MEROMORPHIC APPROXIMATION

By

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Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August, 2007

Nashville, Tennessee

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ACKNOWLEDGMENTS

I would like to express my gratitude to the people who helped me along the way in writing this thesis. I could not possible mention all of them.

First and foremost, I would like to thank my advisor, Professor Edward B. Saff, for his guidance, for all the seminars I had the privilege to attend, and for introducing me to the fields of Potential Theory and Orthogonal Polynomials. I am also deeply indebted to my co-advisor, Doctor Laurent Baratchart, for countless hours of discussions and the amazing experience of collaboration. Their passion for mathematics, deep results, and teaching skills set an example to follow.

Second, I would like to thank my committee members for their assistance and their time. In particular, I would like to express my gratitude to Professor Gieri Simonett for the excellent analysis course he taught during my first year, Professors Akram Aldroubi and Doug Hardin for the extremely interesting lectures they offered, and Professor Yanqin Fan from Vanderbilt Economics Department for being a member of my Ph.D. committee.

Throughout the five years I spent at Vanderbilt, I was honored to work with fellow graduate students Matthew Calef, Peter Hinow, Bogdan Nica, Iva Kozakova, Jan Spakula, Doctors Yuliya Babenko, Fumiko Futamura, Marcin Kozik, Casey Leonetti, Erwin Miña-Diaz, Alexey Muranov, Lin Shan, Dmitriy Sonkin, Tatyana Sorokina, Chris Stephens, and many others. I believe it was the most precious part of my studies at Vanderbilt University. I really enjoy their friendship and appreciate their support during the difficult times I had.

Finally, I thank my family and my fiancée Ana Carneiro for the care and support their provided through all this time.

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CHAPTER I

INTRODUCTION

This thesis deals with the asymptotic distribution of poles of *meromorphic approximants* to a class of Cauchy transforms of, in general, complex measures, as the number of poles grows large. This subject is motivated by the problems of inverse potential theory, where one wants to recover information about a measure from the knowledge of its potential [63]. More precisely, let μ be a compactly supported measure in the complex plane and U^μ be its logarithmic potential, i.e. the convolution $U^\mu = -\log|z| * \mu$. (Recall that $-\log|z|$ is a fundamental solution of the Laplace operator. In general, the potential of μ obtained by convoluting μ with the fundamental solution of any elliptic operator, not necessarily the Laplace operator, can be considered.) The inverse potential problem consists in locating the support of the measure μ, $S_\mu := \text{supp}(\mu)$, from the knowledge of U^μ outside of some neighborhood of S_μ. One of the suggested approaches is to consider an optimal (with respect to some criteria) discretization of U^μ. In other words, to approximate U^μ by a sequence of potentials $\{U^{\mu_n}\}_{n=1}^\infty$, where μ_n is a discrete measure with n point masses. Of course, the asymptotic behavior of μ_n depends on the chosen criteria of discretization. In particular, optimal discretization with respect to Sobolev-type norms (i.e. approximating the derivative) amounts to best rational approximation. Indeed, $\partial U^\sigma/\partial z$ is the Cauchy transform of a measure σ, which is a rational function if and only if σ is a discrete measure.

The subject of rational approximation has made substantial progress in the last several decades [54, 120, 75, 122, 123, 126, 58, 27, 28, 129, 19]. However, in many applications the potential of the measure is known only up to a harmonic function, which is the reason to consider meromorphic rather than rational approximants. Let f_μ be the Cauchy transform of μ, i.e.,

$$f_\mu(z) = \int (z-t)^{-1} d\mu(t).$$

A meromorphic approximant with n poles, say g_n, to f_μ in some fixed domain $D \supset S_\mu$ can be written as $g_n = h_n/q_n$, where h_n is a holomorphic function in D and q_n is a monic polynomial of degree n with all its zeros in D. Thus, the asymptotic behavior of the poles of meromorphic approximants g_n is the same as the asymptotic behavior of the zeros of polynomials q_n. Further, it was observed in [24, 25] that these polynomials are, under certain conditions, orthogonal with respect to some varying complex measures supported on S_μ. These facts link the subject of meromorphic approximation to the theory of non-Hermitian orthogonal polynomials. The latter is not only an interesting field on its own but also provides an efficient set of tools for the study of the asymptotic behavior of the poles of meromorphic approximants and the error of the approximation. However, the non-Hermitian character of orthogonality as well as the dependence upon varying weights constitute a substantial difficulty for the study of these polynomials. In the following sections we give a more extensive treatment of inverse source problems, meromorphic approximation, orthogonal polynomials, the connection among them, as well as provide an overview of known and obtained results.

2D "Crack" Detection

As was briefly discussed above, meromorphic (rational) approximation is motivated by the problem of locating the support of a measure from the knowledge of its potential. The following model is the main application that we bear in mind.

We consider a class of problems in which one needs to detect, from the boundary data, the presence and the location of cracks (fissures) in homogeneous media without destroying it. (For example, one can think of a two-dimensional planar region that contains a "crack", but the observer cannot see inside the region — only the boundary of the region is visible.) Boundary data can be obtained from either thermal, electric, acoustic, or elastic measurements. The existing methods for locating cracks can be divided into two groups: iterative and semi-explicit methods. Methods of the first group rely on the multiple iterative integration of the partial differential equations involved (cf. [114]). They are highly time consuming and very sensitive to the noise in the initial data and most importantly they are extremely sensitive to the initial guess. In other words, these methods based on algorithms converge only if the crack is in some sense already localized. On the other hand, semi-explicit methods provide such a localization and are computationally fast, but not fully constructive (cf. the reciprocity gap method [6], the factorization method [66], [34], [65], etc.). Computationally more simple constructive methods that do not depend on the initial guess would in many cases be much preferable. In 1999, my thesis advisor, E. B. Saff, working together with a team of researchers at the French federal research institute INRIA headed by L. Baratchart introduced a new technique that utilizes meromorphic approximants that can be constructed from the boundary data (cf. [20] and [23]). The meromorphic approximation technique provides a method that is cost-effective and does not depend on an initial guess.

The approach in [20] and [23] is based on the following physical interpretation. Consider a simply connected conductor D having oriented boundary Γ and filled with a homogeneous body except for a one-dimensional crack modeled by an oriented Jordan arc γ with distinct endpoints γ_0

and γ_1. We shall assume that Γ and γ are smooth enough to posses normals whenever necessary. This conductor is the subject of some physical experiment governed by the Laplace operator, such as being exposed to a heat source or to some electrical field for which the crack is acting like a perfect insulator. A general assumption is that a flux Φ (heat or current) is applied on the outer boundary of the conductor. When equilibrium is reached, the heat u (or the potential in the case of the electrostatics problem) satisfies a Neumann boundary value problem, which depends on the flux and the crack. In other words, the potential u is such that

$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \gamma, \\ \dfrac{\partial u}{\partial n_\Gamma} = \Phi & \text{on } \Gamma, \\ \dfrac{\partial u^\pm}{\partial n_\gamma^\pm} = 0 & \text{on } \gamma \setminus \{\gamma_0, \gamma_1\}, \end{cases} \tag{1.1}$$

where Δu is the Laplacian of u, n_Γ is the inner normal on Γ, and $n_\gamma^\pm$ are the one-sided inner normals on the crack. (Observe that the orientation of γ defines positive and negative regions in D. The positive (resp. negative) region lies on the right (resp. left) of γ while traversing γ in the positive direction.) Assuming that the flux Φ meets the compatibility condition $\int_\Gamma \Phi ds = 0$, where ds is the differential of arclength on Γ, and the boundary Γ together with the crack γ is piecewise $C^{1,\alpha}$ without cusps, a solution to the Neumann boundary problem (1.1) exists [23]. Moreover, this solution can be viewed as the real part of a function analytic on the conductor except across the crack. The boundary values of that function can be computed by the formula

$$f(\xi) := f(u;\xi) = u(\xi) + \int_{\xi_0}^{\xi} \Phi ds, \quad \xi \in \Gamma,$$

where ξ_0 is an arbitrary point on Γ. In another connection, the Cauchy integral formula implies that

$$f(z) = h(z) - \frac{1}{2\pi i}\int_\gamma \frac{(f^+ - f^-)(t)}{z-t} dt, \quad z \in D \setminus \gamma,$$

where h is analytic in D and $f^\pm$ are the nontangential boundary values of f on the positive and negative sides of γ.

The idea behind the suggested approach is the following. The function $f(u;\cdot)$ is approximated on Γ by meromorphic approximants. The crack is then localized through the asymptotic behavior of the poles of these approximants as their number increases. Unlike the semi-explicit methods, this approach can only locate the endpoints of the crack. However, it enjoys features that are not shared by other methods. First of all it is fully constructive. While in semi-explicit methods, one approximates the solution u of the Neumann boundary value problem (1.1) throughout D, the proposed approach requires knowledge of that solution only on the boundary. Second of all it demands, in general, only one numerical experiment (unless the crack lies on the level line of the equilibrium potential, in which case the direction of the flux has be changed). By comparison, the application of the factorization method is based on a large number of numerical experiments. Finally, the method of meromorphic approximants allows us to consider functions with branch points and polar singularities (modeled as a sum of a Cauchy transform and a rational function).

The latter case is of particular importance for the inverse EEG (ElectroEncephaloGraphy) problem. This problem consists in localizing epileptic foci in the brain from electrical data measured on the scalp. Initially, a simplified spherical model is studied. The head is assumed to be the unit ball in three dimensions, and after a number of technical steps dealing with different conductivity layers (for the scalp, skull, and brain) the problem is reduced to the following: given a flux and measured potential on the surface of the unit ball, find points inside of it such that the solution to the Laplace operator is equal to the potential on the unit sphere and its normal given by the flux. Cutting the unit ball into the parallel two-dimensional slices one can reduce the original problem to spotting the branch points of otherwise analytic (multi-valued) functions in the disk (cf. [21] and [18]). Note that such functions can be represented as Cauchy transforms of complex measures supported on the branch cuts. The latter can be an arbitrary curve or set of curves connecting the branch points. Thus, the techniques of meromorphic approximants can be applied to recover the endpoints of the cut, i.e. the wanted branch points.

Meromorphic Approximation

In the preceding section we described the idea of using meromorphic approximants to recover fissures in a homogeneous media. Now, we shall put this idea into a more rigorous mathematical framework.

Let μ be a Borel measure compactly supported in the unit disk, $\mathbb{D}$, of the complex plane. Further, let $\mathcal{F}$ be a function of the form

$$\mathcal{F}(z) := \mathcal{F}(\mu; R; z) = \int \frac{d\mu(t)}{z-t} + R(z), \tag{1.2}$$

where R is a rational function, holomorphic at infinity, and μ is a complex measure compactly and regularly supported on the real line. Thus, $\mathcal{F}(\mu; R; \cdot)$ is a sum of the Cauchy transform of the measure μ, which we denote by f_μ, and a rational function R of type $(m-1,m)$, i.e. the ratio of a polynomial of degree at most $m-1$ with a polynomial of degree at most m.

The meromorphic approximants that we consider are optimal (for a fixed number of poles in the unit disk) with respect to an L^p-norm on the unit circle, $\mathbb{T}$. When studying them, we assume that $S_\mu \subset (-1,1)$ and all poles of R lie in the open unit disk, so that $\mathcal{F}$ is indeed p-summable, in fact continuous, on $\mathbb{T}$. The asymptotic behavior is investigated as the number of poles grows large. In the case $p=\infty$, this type of approximant was introduced by V. M. Adamyan, D. Z. Arov, and M. G. Krein in their famous paper [1]. The latter was naturally generalized to L^p, $p\in[1,\infty)$, by L. Baratchart and F. Seyfert in [28] and independently by V. A. Prokhorov in [104]. It was shown that for any $n\in\mathbb{N}$ and $p\in[1,\infty]$ there exists (not necessarily unique) an optimal approximant $g_n = h_n/q_n$, where h_n is an analytic function in $\mathbb{D}$ with L^p boundary values on the unit circle and q_n is a monic polynomials whose zeros reside in $\mathbb{D}$. However, here we restrict ourselves to the case $p\in[2,\infty]$, since it was only for that range that the authors of [28] were able to express the error in terms of (generalized) singular vectors of a Hankel operator and subsequently obtain integral formulae for that error when the approximated function is represented as a Cauchy integral. These formulae are connected with non-Hermitian orthogonality and form the basis of the present approach. In fact, as will be shown later, the polynomials $\{q_n\}$ are such that

$$\int t^j q_n(t)\frac{q(t)w_n(t)}{\widetilde{q}_n^2(t)}d\mu(t)=0, \quad j=0,\dots,n-m-1, \tag{1.3}$$

where q is the denominator of R, $m=\deg(q)$, $\{w_n\}$ is some normal family of *outer functions* in $\mathbb{D}$, and $\widetilde{q}_n(z)=z^n\overline{q_n(1/\bar{z})}$ is the reciprocal polynomial of q_n.

The study of optimal meromorphic approximants in the above setting is quite recent. When $\mathcal{F}$ is a Markov function, i.e. $\mathcal{F}(\cdot)=\mathcal{F}(\mu;0;\cdot)$ and μ is a positive measure, best meromorphic approximants have been instrumental in [5] and [33] to construct *rational* approximants, and were studied *per se* by L. Baratchart, V. Prokhorov, and E. B. Saff in [27]. Using results from [5] to make connections with orthogonality, these authors prove (and give error rates for) the uniform convergence of such approximants, locally uniformly in $\overline{\mathbb{C}}\setminus E_\mu$, whenever $p\in[1,\infty]$ provided that μ satisfies the Szegő condition: $\log d\mu/dt\in L^1(E_\mu)$, where E_μ is the convex hull of $\mathrm{supp}(\mu)$. When $p=2$, the uniform convergence for any Markov function follows from the work [29] by L. Baratchart, H. Stahl, and F. Wielonsky. Apart from [28, Sec. 10], where the asymptotic behavior of the poles is established for any $p\in[2,\infty]$ when $\mathcal{F}$ has exactly two branch points and no pole, there are no other papers dealing with the convergence of best meromorphic approximants to functions of the form (1.2) in the case where μ is a *complex* measure.

It is important to note that the meromorphic approximation problem also has a conformally invariant formulation on Jordan domains with rectifiable boundary, to which the results of the present thesis transpose with obvious modifications if S_μ is contained in a closed hyperbolic geodesic arc rather than a segment. The procedure can be carried out with no difficulty using the construction of [23, Sec. 5].

When dealing with meromorphic approximants we restrict our attention to the unit disk. Since, the domain of analyticity of $\mathcal{F}(\mu; R; \cdot)$, $D_{\mathcal{F}}$, includes a neighborhood of infinity, there are other techniques of approximation that utilize this fact. So, here we shall treat one more type of approximants, the so-called, *multipoint (generalized) Padé approximants.*

Padé Approximation

A multipoint Padé approximant, Π_n, $n \in \mathbb{N}$, generally speaking, is a rational interpolant of type (n, n) where a set of $2n+1$ not necessarily distinct interpolation points has been prescribed, at least one of which is infinity. Moreover, we assume[1] that the interpolation points are conjugate-symmetric and chosen in such a manner that, as n gets large, they have a limit distribution whose support is contained in $D_{\mathcal{F}}$. Besides being interpolants, multipoint Padé approximants possess one more remarkable property. They turn out to be asymptotically best rational approximants [48, 51, 54, 55, 53, 52, 56, 57, 58, 122, 123, 59, 129, 10]. In other words,

$$\lim_{n\to\infty} \left(\frac{d_n(\mathcal{F}; K)}{\|\mathcal{F} - \Pi_n\|_K} \right)^{1/n} = 1,$$

where $\Pi_n(\cdot) = \Pi_n(\mathcal{F}; K; \cdot)$ is a multipoint Padé approximant and $d_n(\mathcal{F}; K) = \inf_r \|\mathcal{F} - r\|_K$ is the distance from $\mathcal{F}$ to the set of rational functions of type (n, n) in the Chebyshev (uniform) metric on a compact set[2] K. In this sense Padé approximants are also optimal approximants to $\mathcal{F}$ (see previous section).

One reason for treating jointly best meromorphic and Padé approximants is that essentially the same techniques can be used. These techniques rest on the orthogonality relations that the denominators of Π_n satisfy. Indeed, if we denote by $\Pi_n = p_n/q_n$ the n-th multipoint Padé approximant to $\mathcal{F}(\mu; R; \cdot)$ with respect to some interpolation set, then it is not hard to show that

$$\int t^j q_n(t) \frac{q(t)}{v_{2n}(t)} d\mu(t) = 0, \quad j = 0, \ldots, n-m-1, \tag{1.4}$$

[1] In some cases, these assumptions can be partially relaxed, as will be shown later.

[2] In fact, Padé approximants are optimal in rational approximation of not only analytic but also continuous functions (see [122, 123, 129]) and functions on unbounded sets (see [10]).

where v_{2n} is a polynomial with its zeros at finite interpolation points and the multiplicity of each zero is equal to the number of occurrences of that point in the interpolation set. A second reason is that when $p = 2$, best meromorphic approximation to $\mathcal{F}$ reduces to best *rational* approximation of fixed degree in L^2 of the circle. Such rational approximants turn out to be multipoint Padé approximants as well (although this time the interpolation points are not known *a priori*); hence the two theories make contact for $p = 2$.

Unlike the meromorphic case, there exists a vast amount of literature on Padé appoximants. To put our results into perspective, let us begin with an account of the existing results. When $\mathcal{F}$ is a Markov function (μ is a positive measure and $R \equiv 0$), the study of diagonal Padé approximants to $\mathcal{F}$ at infinity goes back to A. A. Markov who showed [76] that they converge uniformly to $\mathcal{F}$ on compact subsets of $\overline{\mathbb{C}} \setminus E_\mu$, where E_μ is the convex hull of S_μ. Later this work was extended to multipoint Padé approximants with conjugate-symmetric interpolation sets by A. A. Gonchar and G. López Lagomasino in [54]. A cornerstone of the theory is the close relationship between diagonal Padé (resp. multipoint Padé) approximants to Markov functions and orthogonal polynomials (recall (1.4)). Another generalization of Markov's result was obtained by A. A. Gonchar on adding polar singularities, i.e. including the rational function R. He proved in [50] that Padé approximants still converge to $\mathcal{F}$ locally uniformly in $\overline{\mathbb{C}} \setminus (S' \cup E_\mu)$, where S' is the set of poles of R, provided that μ is a positive measure with singular part supported on a set of logarithmic capacity zero. Subsequently, it was shown by E. A. Rakhmanov in [106] that weaker assumptions on μ can spoil the convergence, but at the same time, if the coefficients of R are real, then the locally uniform convergence holds for any positive μ. Although it is not a concern to us here, let us mention that one may also relax the assumption that S_μ be compact. In particular, Padé and multipoint Padé approximants to Cauchy transforms of positive measures supported in $[0, \infty]$ (such functions are said to be of *Stieltjes type*) were investigated by G. López Lagomasino in [68, 69].

It is an important point to emphasize that uniform convergence in [106] is spoiled by poles of Padé approximants accumulating in $D_\mathcal{F}$. It is customary in the literature to call such poles *spurious* (see [124, 125, 127, 16]). In particular, it was shown almost a century ago in [43] that poles of Padé approximants at infinity to

$$f(z) = \sqrt{(z - e_1) \cdot \ldots \cdot (z - e_4)} - z^2 + \frac{1}{2}(e_1 + \cdots + e_4)z$$

are dense in $\mathbb{C}$ under some conditions on e_1, e_2, e_3, and e_4. Such effects can be observed even for $\mathcal{F}(\mu; R; \cdot)$ with positive measures when S_μ is not connected. However, one type of convergence still holds in these cases, namely, the convergence *in capacity*. Exactly the same phenomenon (spurious poles) can be observed when μ is a complex measure. Thus, in this case it is more natural to speak of convergence in capacity than uniform convergence.

When uniform convergence of Padé approximants is obtained, it is, in fact, geometric. Nevertheless, the rate itself is known only up to a sub-geometric factor. This opens up a new direction of investigation in rational approximation: to determine the exact rate of convergence, in other words to show *strong* asymptotics for the error of approximation. In the case of positive measures and $R \equiv 0$ this question, among other things, was settled by G. López Lagomasino in [70] and [35]. Independently, H. Stahl presented in [128] an approach which was developed specially to target strong asymptotics only. Strong convergence of multipoint Padé approximants was obtained for the case $\mathcal{F}(\mu;0;\cdot)$ with μ having absolutely continuous part (with respect to the Lebesgue measure) in the Szegő class and conjugate-symmetric interpolation sets. Essentially the same conditions appeared in [35] but μ had no singular part. In the recent paper [60] A. A. Gonchar and S. P. Suetin showed the strong convergence of Padé approximants at infinity to $\mathcal{F}(\mu;R;\cdot)$ with $R \not\equiv 0$ and the measure μ having nonvanishing analytic derivative with respect to the normalized arcsine distribution on $[-1,1]$. One remarkable result of that paper is the following fact. Each pole of the rational function R attracts as many poles of the Padé approximants as its multiplicity and the latter distribute themselves asymptotically as the roots of unity. We shall show that this trait is shared not only by multipoint Padé approximants but also by meromorphic approximants.

Meanwhile H. Stahl opened up new perspectives in his path-breaking papers [120, 126], where he studied diagonal Padé approximants to (branches of) multi-valued functions which can be continued analytically without restriction except over a set of capacity zero (typical examples are functions with poles and branch points). By essentially representing the "main" singular part of the function as a Cauchy integral over a system of cuts of *minimal capacity* (a system having the *S-property*), and through a deep analysis of the zeros of non-Hermitian orthogonal polynomials on such systems of cuts, he established the asymptotic distribution of poles and subsequently the convergence in capacity of the Padé approximants on the complement of the cuts. For the future references we record that the system of cuts, say S, possesses the *S-property* if

$$\frac{\partial g(\cdot;\infty)}{\partial n_+}(t) = \frac{\partial g(\cdot;\infty)}{\partial n_-}(t) \tag{1.5}$$

quasi everywhere on S, where $g(z;\infty)$ is the *Green function with pole at infinity* for the domain $\mathbb{C}\setminus S$. In [58] this construction was generalized to certain carefully chosen multipoint Padé approximants by A. A. Gonchar and E. A. Rakhmanov, who used it to prove the sharpness of O. G. Parfenov's theorem (formerly Gonchar's conjecture) on the rate of approximation by rational functions over compact subsets of the domain of holomorphy, see [90][3]. Of course the true power of this method lies with the fact that it allows one to deal with measures supported on more general systems of arcs than a segment, which is beyond the scope of this thesis. However, since a segment is the simplest example of an arc of minimal logarithmic capacity connecting two points, the results we just mentioned apply, in particular, to functions of the form (1.2), where μ is a complex measure supported on a segment which is absolutely continuous there with continuous density that does not vanish outside a set of capacity zero. Combining the technique of minimal capacity contours and scalar Riemann-Hilbert boundary value problems, S. P. Suetin showed that there exists a subsequence of Padé approximants at infinity converging strongly to $\mathcal{F}(\mu; R; \cdot)$, $R \not\equiv 0$, where μ is supported on a system of disjoint analytic arcs having the S-property, and it is assumed that its Radon-Nikodym derivative is Hölder continuous. Observe that the derivative is taken with respect to $dt/\sqrt{p(t)}$, where p is a polynomial with simple zeros at the endpoints of arcs comprising S_μ. By different, operator-theoretic methods, combined with a well-known theorem of E. A. Rakhmanov on ratio asymptotics [107, 108], A. Magnus further showed that the diagonal Padé approximants to $\mathcal{F}$ converge *uniformly* on compact subsets of $\overline{\mathbb{C}} \setminus E_\mu$ when $R \equiv 0$ and $d\mu/dt$ is non-zero almost everywhere and has continuous argument [75].

To conclude this section a few words of comparison between meromorphic and Padé approximants are perhaps in order. On the one hand classical and multipoint Padé approximants were historically the first to be studied, becoming of great importance both in function theory and in number theory, and they offer *linear* computational schemes with remarkable properties that are exploited, *e.g.* for convergence acceleration in numerical analysis [15]. On the other hand, the phenomenon of spurious poles and the poor numerical stability when dealing with inaccurate data have impeded the use of such approximants for modeling and identification in the engineering sciences. In addition, given some criterion, the optimal or suboptimal choice of the interpolation points when their number is fixed (a crucial issue in practice) is a difficult nonconvex optimization problem which is still not well-understood. Indeed, most studies of the approximation rate are asymptotic in nature. In contrast, best meromorphic approximants on the circle have a guaranteed region of holomorphy (the complement of the closed unit disk in our case) and, by definition, they make in some sense best possible use of the freedom to choose the poles. Moreover, their optimality has important interpretations in control and identification of linear dynamical systems [14, 42, 92, 97, 17] and gives

[3] It is interesting to note that Parfenov's proof makes decisive use of the Adamyan-Arov-Krein theory; later, the result was carried over to multiply connected domains by V. A. Prokhorov in [103].

rise in specific cases to near-best *rational* approximation schemes [47, 90]. Their main drawback is that they are not so easy to compute, since, when $p = \infty$, the best approximation projection is generically not continuous with respect to the L^∞-norm, but only with respect to stronger norms [62, 96], while for $p < \infty$ one has to rely on a numerical search that may get trapped in local minima. In this respect, let us point out that the results of this thesis are valid, not only for best meromorphic approximants, but more generally for critical points of the L^p-criterion and thus also for *local* best approximants.

Orthogonal Polynomials

As was earlier emphasized and suggested by formulae (1.3) and (1.4), the study of rational and meromorphic approximants is closely linked to the theory of orthogonal polynomials. In this section we discuss some objectives of this theory.

Let μ be a Borel, generally complex, measure compactly supported in $\mathbb{C}$. A sequence of polynomials $\{q_n\}_{n\in\mathbb{N}}$ is called *orthogonal with respect to* μ if

$$\int t^j q_n(t)d\mu(t) = 0, \quad j = 0, \ldots, n-1. \tag{1.6}$$

Historically, the study of polynomials satisfying (1.6) originated in the setting of positive measures supported in $\mathbb{R}$. Hereafter, we shall call this case *Hermitian*. It is easy to see that Hermitian orthogonal polynomials have exact degree n and all their zeros belong to E_μ. For the non-Hermitian setting we shall describe only two cases. The first one is the case of complex measures having analytic Radon-Nikodym derivatives and supported on contours satisfying the S-property. The second, which is of primary interest to us, is the case of complex measures supported in $\mathbb{R}$, but having less regular derivatives. We would like to mention that it is also possible to consider *multiple* orthogonal polynomials. They are not of a concern to us here, but we shall reference some results in this direction whenever they are directly applicable to the subject of the discussion. One needs to realize that in the non-Hermitian setting, equations (1.6) neither define polynomials q_n uniquely nor yield the exact degree n. Further, a priori, nothing can be said about location of the zeros of q_n. There are other distinctions between Hermitian and non-Hermitian orthogonal polynomials that account for fairly different methods of study in each case (see [121]).

While discussing orthogonal polynomials, we shall concentrate on their asymptotic behavior in $\mathbb{C} \setminus S_\mu$. We shall distinguish so called *weak* (n-th root) and *strong* (Szegő) asymptotics of q_n. Polynomials q_n possess weak asymptotics if the sequence $\{|q_n(z)|^{1/n}\}$ converges locally uniformly in

$\mathbb{C} \setminus \mathcal{K}$, where $\mathcal{K}$ is the set of limit points of zeros of q_n. Weak asymptotics of q_n is ultimately related to the weak convergence of zeros of q_n. Indeed, let $\{\nu_n\}$ be a sequence of zero counting measures of $\{q_n\}$, i.e.

$$\nu_n := \nu_n(q_n) = \frac{1}{n} \sum_{z:\ q_n(z)=0} \delta_z, \tag{1.7}$$

where δ_z is the unit point mass distribution at z. Suppose that $\nu_n \xrightarrow{*} \nu$, where ν is some positive measure, then

$$|q_n(z)|^{1/n} = \exp\left\{\int \log|z-t| d\nu_n(t)\right\} \to \exp\left\{\int \log|z-t| d\nu(t)\right\}$$

locally uniformly in $\mathbb{C} \setminus \mathcal{K}$, where we assume q_n to be monic.

The study of weak asymptotics of polynomials q_n satisfying (1.6) goes back the to classical works of S. N. Bernstein [31, 32] and G. Szegő [132]. This theory naturally evolved to the larger setting of orthogonal polynomials with varying measures. In other words, to the case where polynomials q_n are such that

$$\int t^j q_n(t) d\mu_n(t) = 0, \quad j = 0, \ldots, n-1. \tag{1.8}$$

It is customary and most useful for the applications to consider varying weights of the form

$$d\mu_n(t) := \exp\left(-2nQ_n(t)\right) d\mu(t) \quad \text{and} \quad Q_n(z) \to Q(z), \quad \text{as} \quad n \to \infty. \tag{1.9}$$

It is known that in this case weak asymptotics can be described as follows:

$$|q_n(z)|^{1/n} \to \exp\left(-U^{\mu_w}(z)\right), \tag{1.10}$$

where U^{μ_w} is the *logarithmic potential* of μ_w, i.e. $U^{\mu_w}(z) = -\int \log|z-t| d\mu_w(t)$, and μ_w is the *weighted equilibrium distribution* in the presence of an *external field* $\mathrm{Re}(Q)$ (see [113]). Another traditional way of writing varying weights that naturally appears in meromorphic and multipoint Padé approximation is

$$d\mu_n(t) := \frac{d\mu(t)}{v_{2n}(t)}, \tag{1.11}$$

where each v_{2n} is a polynomial of degree at most $2n$. Even though it might seem that weights (1.9) and (1.11) are not related, it turns out that they are. Indeed, assume that the orthogonality relations (1.8) are Hermitian with μ_n of type (1.11). This is true whenever μ is a positive measure,

$S_\mu \subset \mathbb{R}$, and the polynomials v_{2n} have real coefficients. Suppose for convenience that $\deg(v_{2n}) = 2n$ and v_{2n} is positive on E_μ for each $n \in \mathbb{N}$. Then

$$v_{2n}(t) = \exp(\log|v_{2n}(t)|) = \exp\left(-2nU^{\sigma_{2n}}(t)\right), \quad t \in E_\mu,$$

where σ_{2n} is the zero counting measure of v_{2n}. Thus, the condition on the external fields Q_n in (1.9) is satisfied if and only if $\sigma_{2n} \xrightarrow{*} \sigma$, where σ is some probability measure supported in $\mathbb{C} \setminus E_\mu$.

Weak asymptotics of orthogonal polynomials in the Hermitian setting of varying measures were elaborated on in [54, 55, 109, 56, 77, 57, 78, 83, 59]. At the present moment various aspects of the theory can be found in the monographs [84], [113], [133], and [130].

The case of complex measures appears to be more difficult or at least less developed due to the non-Hermitian character of orthogonality. The first substantial contribution was made by J. Nuttall (see the review in [88]). His ideas were advanced by H. Stahl in [117] who later obtained formulae for weak asymptotics in [118, 119]. In turn, the methods used by H. Stahl were carefully studied and generalized by A. A. Gonchar and E. A. Rakhmanov in [58] for the case of varying weights. It was shown that relation (1.10) holds whenever $d\mu(t) = dt$ on a compact set S of positive *capacity*, Q_n are analytic and converge uniformly to Q in some neighborhood Ω of S, and S possesses the S-property

$$\frac{\partial\left(U^{\mu_w} + \mathrm{Re}(Q)\right)}{\partial n_+}(\zeta) = \frac{\partial\left(U^{\mu_w} + \mathrm{Re}(Q)\right)}{\partial n_-}(\zeta) \quad \text{quasi everywhere on} \quad \mathrm{supp}(\mu_w). \tag{1.12}$$

In contrast with previous works, the approach of Baratchart et al in [19] allows for the distribution of the complex measure μ to vanish on a large subset of E_μ, but the latter is assumed to be a subset of the real line. Specifically, it is required that the total variation measure $|\mu|$ has compact regular support and that it is not too thin, say, larger than a power of the radius on relative balls *of the support*. Although fairly general, these conditions could be further weakened, for instance down to the Λ-criterion introduced by H. Stahl and V. Totik in [130][4]. However, the most stringent assumption bears on the argument of μ, as it is required that the Radon-Nikodym derivative $d\mu/d|\mu|$ be of bounded variation on S_μ. This assumption, introduced in [67, 19], unlocks many difficulties and leads to the weak convergence of the poles in the case of complex measures.

We did not stress it earlier, but it can be seen from (1.10), that weak asymptotic behavior of orthogonal polynomials does not capture the specifics of the measure μ. In other words, two

[4]This depends on the corresponding generalization of the results in [19] to be found in [67].

sequences of polynomials orthogonal with respect to different measures whose supports coincide, have the same weak asymptotics. This is no longer true if one considers strong asymptotic behavior of orthogonal polynomials. A sequence $\{q_n\}$ is said to possess *strong asymptotics* if it satisfies

$$\frac{q_n(z)}{\Phi^n(z)} \to \mathcal{S}(\mu'; z) \tag{1.13}$$

where Φ is a normalizing factor that depends on S_μ and $\mathcal{S}(\mu'; \cdot)$ is the *Szegő function* of the Radon-Nikodym derivative of μ. Strong asymptotics provides finer understanding of the behavior of orthogonal polynomials and we are going to concentrate on this topic hereafter.

As for weak asymptotics, the Hermitian case has been thoroughly investigated [70, 7, 74, 73, 13, 110, 12, 8, 9]. For more detailed treatment see [134] and [128]. The methods employed extensively use *minimality* property[5]. This property no longer holds in the case of complex measures, so these methods are of little interest for us.

The case of complex measures was taken up by G. Baxter in [30] and by J. Nuttall and S. R. Singh in [87], who established strong asymptotics of non-Hermitian orthogonal polynomials on a segment for measures that are absolutely continuous with respect to the (logarithmic) equilibrium distribution of that segment, and whose density satisfies appropriate conditions expressing, in one form or another, that it is smoothly invertible. For instance, Baxter's condition is that $\log d\mu/d\mu_E$, when extended periodically, has an absolutely summable Fourier series, where $S_\mu = E$ is an interval and $d\mu_E$ is the normalized arcsine distribution on E[6]. Later, J. Nuttall was able to relax conditions on the Radon-Nikodym derivative of μ in [89], where $d\mu/d\mu_E$ was taken to be Hölder continuous on E. This approach was extended by S. P. Suetin in [131] to the case

$$d\mu(t) = \left. \frac{s(t)dt}{\sqrt{(t-e_1)\cdots(t-e_{2k})}} \right|_{t\in S},$$

where $S = \cup_{j=1}^k S_j$ is a system of analytic arcs S_j with endpoints e_{2j-1} and e_{2j} possessing the S-property, and s is a nonvanishing Hölder continuous function on S. An important extension of these results was obtained by A. I. Aptekarev in [10] (see also [11]) for the varying case (1.9). The major assumptions were that the functions Q_n satisfy

$$\|Q_n - Q\|_{\overline{\Omega}} = O\left(\frac{1}{n}\right),$$

[5]Monic orthogonal polynomial q_n has minimal $L^2(\mu)$ norm among all monic polynomials of degree no greater than n.

[6]In the case of a segment normalized arcsine distribution is the logarithmic equilibrium measure on that segment.

S_μ is a smooth Jordan arc of minimal capacity in the sense of (1.12), and the measure μ is such that $d\mu(t) = s(t)dt$, where s is an analytic nonvanishing function on S_μ. Two different approaches are used in [10]. One of them follows Nuttall-Suetin's line of argument where a Riemann boundary-value problem on a two-sheeted Riemann surface and a singular integral equation for the function of second kind are utilized. The other method is the matrix Riemann-Hilbert approach pioneered by P. Deift and his colleagues (see [64, 41, 38, 39, 37, 40, 36]).

Overview of the Results

In this thesis we deal with the convergence of meromorphic and multipoint Padé approximants to functions $\mathcal{F}(\mu; R; \cdot)$ defined in (1.2), where μ is a complex-valued measure compactly supported in $\mathbb{R}$ and $R \in \mathrm{R}_{m-1,m}$, where

$$\mathrm{R}_{k,n} := \{p_k/q_n : \ p_k \in \mathrm{P}_k, \ q_n \in \mathrm{M}_n\}$$

is the set of rational functions of type (k, n) with all their poles in the unit disk $\mathbb{D}$. Here P_n is the space of algebraic polynomials of degree at most n and M_n consists of monic polynomials of degree n with zeros in $\mathbb{D}$ only. Hereafter we shall denote by Q the denominator of R, assumed to be in the irreducible form, which is a monic polynomial with zeros in $\mathbb{D}$ of the form

$$Q(z) = \prod_{\eta \in S'} (z - \eta)^{m(\eta)}, \tag{1.14}$$

where S' is the set of poles of R and $m(\eta)$ stands for the multiplicity of $\eta \in S'$. Thus, $\mathcal{F}$ is a meromorphic function in $\mathbb{C} \setminus S_\mu$ with poles at each point of S'. We note that $\mathcal{F}$ does not reduce to a rational function since S_μ consists of infinitely many points, cf. [23, Sec. 5.1] for a detailed argument.

To simplify notation, it is convenient to formally rewrite the right-hand side of (1.2) as a single Cauchy integral. For this, we introduce for $\eta = x_\eta + iy_\eta \in \mathbb{C}$ the distribution $\Phi_\eta = \chi(x - x_\eta) \otimes \delta(y - y_\eta)$, where δ is the Dirac delta at 0, χ the characteristic function of the non-negative semi-axis, and $\otimes$ stands for the direct product of distributions. For each $k \in \mathbb{Z}_+$ (the set of nonnegative integers), the partial derivative $\partial_x^{k+1}\Phi_\eta$ is an analytic functional (although Φ_η itself is not), acting on any function h holomorphic in a neighborhood of η according to the rule

$$\left\langle \partial_x^{k+1}\Phi_\eta, h \right\rangle = h^{(k)}(\eta),$$

where $h^{(k)}$ indicates the k-th derivative. Therefore, if we define $\Delta_\eta^{(k)}$ to be $\partial_x^k \Phi_\eta / k!$, we can formally write

$$\int \frac{d\Delta_\eta^{(k)}(t)}{z-t} = \frac{1}{(z-\eta)^{k+1}},$$

and on rewriting $R(z)$ as

$$R(z) = \sum_{\eta \in S'} \sum_{k=0}^{m(\eta)-1} \frac{r_{\eta,k}}{(z-\eta)^{k+1}}, \quad r_{\eta,k} \in \mathbb{C},$$

we get

$$R(z) = \int \frac{d\mu'(t)}{z-t},$$

where μ' is given by

$$\mu' := \sum_{\eta \in S'} \sum_{k=0}^{m(\eta)-1} r_{\eta,k} \Delta_\eta^{(k)}, \quad \mathrm{supp}\,(\mu') = S'. \tag{1.15}$$

This way $\mathcal{F}$ can be put in the form

$$\mathcal{F}(z) = \int \frac{d\widetilde{\mu}(t)}{z-t}$$

with

$$\widetilde{\mu} := \mu + \mu', \qquad \widetilde{S}_\mu := \mathrm{supp}(\widetilde{\mu}) = S_\mu \cup S', \tag{1.16}$$

which makes for a convenient notation.

This thesis is organized as follows. The second chapter covers necessary material in potential theory, meromorphic and Padé approximation. In Chapter III we present results on convergence in capacity of the considered approximants. This yields uniform convergence on each compact set in $D_{\mathcal{F}}$, but up to a subsequence. The technique used is based on the weak convergence of orthogonal polynomials and is applicable to a fairly large class of measures. Chapter IV is devoted to exact rates of uniform convergence. We restrict the class of measures under consideration, but carry out more detailed analysis of asymptotic behavior of approximants and their poles. In Chapter V we show how developed methods can be applied in operator theory to estimating the multiplicity of singular numbers of a certain class of *Hankel* operators. In the Appendix we prove several potential-theoretic facts, that are of common knowledge, but the author was not able to ferret out appropriate references in the literature.

As we said earlier, there are essentially two methods targeting specifically the weak convergence of poles in the case of complex measures. One of them deals with measures supported on contours of minimal capacity consisting of analytic arcs. The second one was elaborated on in [19] and allows one to treat measures supported only in $\mathbb{R}$, but on regular sets. Using the last technique, we show that if the measure μ has a Radon-Nikodym derivative with an argument of bounded variation on supp(μ) and the support itself is a regular set with respect to the Dirichlet problem, the counting measures of poles of meromorphic approximants converge to the *Green equilibrium distribution* on S_μ. In particular, it is true that "almost all" poles of meromorphic approximants to $\mathcal{F}$ converge to the support of μ. ("Almost all" means that there exists a constant M independent of n such that for any neighborhood of S_μ only at most M poles lie outside of that neighborhood.) Moreover we prove that each pole of R attracts at least as many poles of the approximants as its multiplicity, and not much more. In fact, our hypotheses give rise to an explicit upper bound on the number of poles of the approximants that may lie outside a given neighborhood of the singular set of $\mathcal{F}$. Hence, on each compact subset K of $D_{\mathcal{F}}$, every sequence of approximants contains a subsequence that converges locally *uniformly* to $\mathcal{F}$ on $K \setminus K_0$, where K_0 consists of boundedly many (unknown) points. The same results are obtained for the multipoint Padé approximants associated to conjugate-symmetric interpolation sets having an asymptotic distribution σ. The only difference with the meromorphic case is that the counting measures of poles of the approximants converge to the *weighted equilibrium distribution* on S_μ in the presence of the external field $-U^\sigma$.

The next step we take is restricting the class of measures, but obtaining qualitatively better estimates of the convergence. In Chapter IV we assume that $S_\mu = E_\mu =: E$ and $d\mu/d\mu_E$ is nonvanishing and *Dini continuous* on E. Under these assumptions we deduce that each pole of R attracts exactly as many poles of approximants as its multiplicity. One fascinating feature of this convergence is that attracted poles of approximants distribute themselves asymptotically as the roots of unity. Originally this was shown in [60] only for Padé approximants at infinity and $d\mu/d\mu_E$ being analytic and nonvanishing on E. Moreover, the rest of the poles converges to E. This yields uniform convergence of approximants in $D_{\mathcal{F}}$ to $\mathcal{F}$ at a geometric rate and we provide the exact asymptotics of this convergence. The approach we take was suggested in [30], but we extended it to the varying case and enlarged the class of admissible measures.

CHAPTER II

PRELIMINARIES

In this chapter we provide necessary background for the upcoming results and explain considered types of approximation.

Potential Theory

As is typical in complex approximation, some answers can be given in terms of logarithmic potential theory, a brief account of which is sketched below for the convenience of the reader. We refer the reader to the monographs [71, 111, 113] for an extensive treatment.

The *logarithmic potential* and the *logarithmic energy* of a finite positive measure μ, compactly supported in $\mathbb{C}$, are defined by

$$U^\mu(z) := \int \log \frac{1}{|z-t|} d\mu(t), \qquad z \in \mathbb{C}, \tag{2.1}$$

and

$$I[\mu] := \int U^\mu(z) d\mu(z) = \int \int \log \frac{1}{|z-t|} d\mu(t) d\mu(z), \tag{2.2}$$

respectively. Clearly U^μ is a superharmonic function with values in $(-\infty, +\infty]$ which is not identically $+\infty$. In particular U^μ is lower semicontinuous, hence it attains its lower bound on $\operatorname{supp}(\mu)$ so that $I[\mu] \in (-\infty, +\infty]$.

Let now $E \subset \mathbb{C}$ be compact and $\Lambda(E)$ denote the set of all probability measures supported on E. If the logarithmic energy of every measure in $\Lambda(E)$ is infinite, we say that E is *polar*. Otherwise, there exists a unique $\mu_E \in \Lambda(E)$ that minimizes the logarithmic energy over all measures in $\Lambda(E)$. This measure is called the *equilibrium distribution* on E. It is characterized by the property that U^{μ_E} is constant on E except perhaps on a polar subset of E. The *logarithmic capacity*, or simply the capacity, of E is defined as

$$\operatorname{cap}(E) = \exp\left(-I[\mu_E]\right).$$

By definition, the capacity of an arbitrary subset of $\mathbb{C}$ is the *supremum* of the capacities of its compact subsets. We agree that the capacity of a polar set is zero.

A polar set has Lebesgue measure zero and a closed polar set is totally disconnected. A property is said to hold *quasi everywhere* (abbreviated "q.e.") if it holds everywhere except on a polar set.

Logarithmic potentials enjoy weak continuity and semi-continuity properties with respect to their defining measures that are worth recording. More precisely, if μ_n is a sequence of positive measures supported on a fixed compact set that converges weakly to μ, the *principle of descent* ([113, Thm. I.6.8]) asserts that

$$U^{\mu}(z) \leq \liminf_{n\to\infty} U^{\mu_n}(z), \qquad z \in \mathbb{C},$$

while the *lower envelope theorem* ([113, Thm. I.6.9]) says that

$$U^{\mu}(z) = \liminf_{n\to\infty} U^{\mu_n}(z), \qquad \text{for q.e. } z \in \mathbb{C}.$$

Another important concept is the *regularity* of a compact set. The outer boundary $\partial_e E$ of a nonpolar compact set E is said to be *regular* if the equilibrium potential U^{μ_E} is continuous on E, and then it is in fact continuous on $\mathbb{C}$. For Ω the unbounded connected component of the complement of E, let $g_\Omega(\cdot,\infty)$ be the Green function of Ω with pole at ∞, i.e. the unique function such that

(i) $g_\Omega(z,\infty)$ is a positive harmonic function in Ω, which is bounded when z stays away from ∞;

(ii) $g_\Omega(z,\infty) - \log|z|$ is bounded near ∞;

(iii) $\lim_{z\to\xi,\ z\in\Omega} g_\Omega(z,\infty) = 0$ for quasi every $\xi \in \partial\Omega$.

It is classical fact (cf. [111, Sec. 4.4]) that

$$U^{\mu_E}(z) = \log\left(\frac{1}{\operatorname{cap}(E)}\right) - g_\Omega(z,\infty), \quad z \in \Omega,$$

and if we set $g_\Omega(z,\infty) \equiv 0$ in $\mathbb{C} \setminus \overline{\Omega}$ one can see that $\partial_e E$ is regular if and only if $g_\Omega(z,\infty)$ is continuous on $\mathbb{C}$. Points of continuity of $g_\Omega(\cdot,\infty)$ on $\partial_e E$ are called *regular*, other points are called irregular; the latter form a polar set. If a boundary point $\xi \in \partial E$ is not in $\partial_e E$ but in the closure of some bounded component Ω' of $\overline{\mathbb{C}}$, it is called regular if and only if, after a conformal map Φ of $\overline{\mathbb{C}}$ sending some point of Ω' to infinity, the point $\Phi(\xi)$ is regular on $\partial_e\Phi(E)$. The definition does not depend on which conformal map is used, and if all points of ∂E are regular we say that E is regular.

Suppose that $\partial_e E$ consists of a finite number of piecewise analytic Jordan curves. Then the equilibrium distribution μ_E can be given explicitly in terms of the Green function of Ω (cf. [2, Thm.

2.2]), namely

$$d\mu_E = \frac{1}{2\pi}\frac{\partial}{\partial n} g_{\Omega_E}(\cdot, \infty) ds, \tag{2.3}$$

where the normal derivative is taken in the direction of the inner normal on $\partial_e E$, and ds is the arclength.

Throughout we use the concept of *balayage* of a measure ([113, Sec. II.4]). In particular, we make extensive use of the following fact. Let D be a domain (connected open set) with compact boundary ∂D whose complement has positive capacity, and μ be a finite Borel measure with compact support in D. Then there exists a unique Borel measure $\widehat{\mu}$ supported on ∂D, with total mass equal to that of μ: $\|\mu\| = \|\widehat{\mu}\|$, whose potential $U^{\widehat{\mu}}$ is bounded on ∂D and satisfies for some constant $c(\mu; D)$

$$U^{\widehat{\mu}}(z) = U^{\mu}(z) + c(\mu; D) \quad \text{for q.e.} \quad z \in \mathbb{C} \setminus D. \tag{2.4}$$

Necessarily then, we have that $c(\mu; D) = 0$ if D is bounded and $c(\mu; D) = \int g_D(t, \infty) d\mu(t)$ otherwise. Equality in (2.4) holds for all $z \in \mathbb{C} \setminus \overline{D}$ and also at all regular points of ∂D. The measure $\widehat{\mu}$ is called the balayage of μ onto ∂D. It has the property that

$$U^{\widehat{\mu}}(z) \leq U^{\mu}(z) + c(\mu; D) \quad \text{for every} \quad z \in \mathbb{C}, \tag{2.5}$$

and also that

$$\int h \, d\mu = \int h \, d\widehat{\mu} \tag{2.6}$$

for any function h which is harmonic in D and continuous in $\overline{D}$ (including at infinity if D is unbounded). From its defining properties $\widehat{\mu}$ has finite energy, therefore it cannot charge polar sets. Consequently, on solving the generalized Dirichlet problem [111, Thm. 4.1.5] for an arbitrary positive continuous function on ∂D, it follows from (2.6) that $\widehat{\mu}$ is positive if μ is positive; in particular the balayage of a probability measure is a probability measure. This entails a weak-continuity principle that we record for later use: *if μ_n is sequence of probability measures compactly supported in D converging weak* to some measure μ,and if there is a compact set $K \subset D$ such that $\mu_n(D \setminus K) \to 0$ as $n \to \infty$, then $\widehat{\mu}_n$ converges weak* to $\widehat{\mu}$.* Indeed, it is easily checked that μ is a probability measure supported on K and that any weak* limit point ν of $\widehat{\mu}_n$ is a probability measure supported on ∂D. Next, if we denote with superscripts 1 and 2 the restrictions to K and $D \setminus K$ respectively, we have that μ_n^1 converges weak* to μ and we see by the positive linearity of balayage that ν is a weak* limit point of $\widehat{\mu_n^1}$ since $\|\widehat{\mu_n^2}\| = \|\mu_n^2\| \to 0$. Then, applying (2.5) to μ_n^1 and using the principle of descent, we get that U^ν is bounded on ∂D since $U^{\mu_n^1}$ is uniformly bounded there; moreover, by the lower envelope theorem for μ_n^1, (2.4) is satisfied with ν in place of $\widehat{\mu}$.

The minimal energy problem can also be formulated for signed measures [113, Thm. VIII.1.4]. In particular for E_1, E_2 two disjoint compact sets of positive capacity, there exists a unique measure $\mu^* = \mu_1^* - \mu_2^*$, with $\mu_1^* \in \Lambda(E_1)$ and $\mu_2^* \in \Lambda(E_2)$, that minimizes the energy integral

$$I[\mu_1 - \mu_2] = \int \log \frac{1}{|z-t|} d(\mu_1 - \mu_2)(t) d(\mu_1 - \mu_2)(z), \quad \mu_j \in \Lambda(E_j), \quad j = 1, 2. \tag{2.7}$$

It can be proved ([113, Lemma VIII.1.8]) that $I[\mu^*]$ is positive and finite. The value

$$\mathrm{C}(E_1, E_2) = 1/I[\mu^*] \tag{2.8}$$

is called the *condenser capacity* of the pair (E_1, E_2). Further, it holds that $\mu_1 = \widehat{\mu}_2$ and $\mu_2 = \widehat{\mu}_1$ where $\widehat{\mu}_1$ (resp. $\widehat{\mu}_2$) indicates the balayage of μ_1 (resp. μ_2) onto $\partial(\overline{\mathbb{C}} \setminus E_2)$ (resp. $\partial(\overline{\mathbb{C}} \setminus E_1)$); this property in fact characterizes μ^*, see [113, Thm. VIII.2.6].

In analogy to the logarithmic case, one can define the *Green potential* and the *Green energy* of a positive measure μ supported in a domain D with compact non-polar boundary. The only difference is now that, in (2.1)—(2.2), the logarithmic kernel $\log(1/|z-t|)$ gets replaced by $g_D(z,t)$, the *Green function* for D with pole at $t \in D$. The latter is the unique function defined on D such that (cf. [111])

(i) $g_D(z,t)$ is a positive harmonic function in $D \setminus \{t\}$, which is bounded outside each neighborhood of t;

(ii) $g_D(z,t) - \log \dfrac{1}{|z-t|}$ is bounded near t;

(iii) $\lim_{z \to \xi} g_D(z,t) = 0$ for quasi every $\xi \in \partial D$.

Those points at which limit (iii) exists are precisely the regular points of ∂D [111, Thm. 4.4.9]. For example, it is easily checked that in the case of the unit disk we have

$$g_{\mathbb{D}}(z,t) = \log \left| \frac{1 - z\bar{t}}{z - t} \right|.$$

The Green potential relative to the domain D of a finite positive measure μ compactly supported in D is given by

$$U_G^\mu(z; D) = \int g_D(z,t)\, d\mu(t).$$

It can be re-expressed in terms of the logarithmic potentials of μ and of its balayage $\widehat{\mu}$ onto ∂D by the formula [113, Thm. II.4.7 and Thm. II.5.1]

$$U^{\widehat{\mu}-\mu}(z) = c(\mu; D) - U_G^{\mu}(z; D), \quad z \in D \tag{2.9}$$

where $c(\mu; D)$ was defined after equation (2.4). Moreover, (2.9) continues to hold at every regular point of ∂D; in particular, it holds q.e. on ∂D.

Exactly as in the logarithmic case, if E is a compact nonpolar subset of D, there exists a unique measure $\mu_{(E,\partial D)} \in \Lambda(E)$ which minimizes the Green energy among all measures in $\Lambda(E)$. This measure is called the *Green equilibrium distribution* on E relative to D. By (2.9) we have that

$$U_G^{\mu_{(E,\partial D)}}(z; D) = U^{\mu_{(E,\partial D)}}(z) - U^{\widehat{\mu}_{(E,\partial D)}}(z) + c(\mu_{(E,\partial D)}; D), \qquad z \in D, \quad \text{and q.e. } z \in \partial D,$$

where $\widehat{\mu}_{(E,\partial D)}$ is the balayage of $\mu_{(E,\partial D)}$ onto ∂D. In addition the Green equilibrium distribution satisfies

$$U_G^{\mu_{(E,\partial D)}}(z; D) = \frac{1}{\mathrm{C}(E, \partial D)}, \quad \text{for q.e.} \quad z \in E. \tag{2.10}$$

Moreover, equality in (2.10) holds at all regular points of E.

Green potentials have two important properties not shared by logarithmic potentials, namely they are positive and they transform naturally under conformal mappings since this is true of Green functions. In particular, one can define the Green equilibrium distribution with respect to D of a compact set E in $\overline{\mathbb{C}}$, because one can use conformal mapping to bring the situation back to the case where E is compact in $\mathbb{C}$. The same is true of the condenser capacity.

From the physical viewpoint, the equilibrium measure μ_E can be interpreted as the equilibrium distribution of a unit electric charge placed on the conductor E, and finding such a measure is called the electrostatics problem. A natural way of generalizing this problem is to add an external field acting on the charge. Specifically, let E be a closed subset of $\mathbb{C}$, and Q a lower semi-continuous function on E which is less than infinity on a nonpolar subset; if E is unbounded, one requires in addition that $Q(z) - \log|z|$ tends to infinity when $|z|$ tends to infinity in E. Then there exists a unique measure $\mu_w^E \in \Lambda(E)$, called the *weighted equilibrium distribution*, that minimizes the weighted energy integral

$$I_w[\mu] := \int\int \log\frac{1}{|z-t|} d\mu(t) d\mu(z) + 2\int Q(t) d\mu(t)$$

over all $\mu \in \Lambda(E)$. Moreover, there is a constant F_w ([113, Thm. I.1.3]) such that

$$\begin{aligned} U^{\mu_w^E}(z) + Q(z) &\geq F_w \qquad \text{for q.e.} \quad z \in E, \\ U^{\mu_w^E}(z) + Q(z) &\leq F_w \qquad \text{for every} \quad z \in \operatorname{supp}\left(\mu_w^E\right), \end{aligned}$$

F_w is called the *modified Robin constant* for Q and it is equal to

$$F_w = I_w\left[\mu_w^E\right] - \int Q(t) d\mu_w^E(t).$$

In few cases, equilibrium distributions and capacities can be derived explicitly. For instance, if $E = [a, b]$, the logarithmic equilibrium distribution is the arcsine distribution on E, i.e.

$$d\mu_E(t) = \frac{dt}{\pi\sqrt{(t-a)(b-t)}}, \quad t \in E, \quad \text{and} \quad \operatorname{cap}(E) = \frac{b-a}{4}. \tag{2.11}$$

Further, the Green equilibrium distribution for the condenser $(E, \mathbb{T})$ is also known, namely

$$d\mu_{(E,\mathbb{T})}(t) = \frac{(1-ab)dt}{2\mathcal{T}\sqrt{(t-a)(b-t)(1-at)(1-bt)}}, \quad t \in E, \tag{2.12}$$

where $\mathcal{T}$ denotes complete elliptic integral of the first kind

$$\mathcal{T} := F\left(\frac{\pi}{2}; k\right) = \int_{[0,1]} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{2.13}$$

with modulus $k := (b-a)/(1-ab)$ (cf. [82, Ch. VII]). The condenser capacity, in this case, is equal to

$$\mathrm{C}(E, \mathbb{T}) := \frac{2}{\pi}\frac{\mathcal{T}}{\mathcal{T}'}, \quad \mathcal{T}' := F\left(\frac{\pi}{2}; k'\right), \tag{2.14}$$

where $k' = \sqrt{1-k^2}$ is the conjugate modulus.

Harmonic Measures

Let, as usual, D be a domain with non-polar boundary ∂D. Further, let $a > 0$ and $t \in \partial D$. We define a *non-tangential region of approach* to t from inside D by setting

$$C(a, t) := \{z \in D \ : \ |z - t| < (1 + a)\operatorname{dist}(z, \partial D)\},$$

where $\operatorname{dist}(z, \partial D)$ is the Euclidean distance from z to ∂D. Let v be a complex-valued function in

D and denote by C_j, $j \in J$, connected components of $C(a,t)$ containing t in their closure. We say that v has non-tangential boundary values at t if $\lim_{z\to t, z\in C_j} v(z)$ exists for any $j \in J$. Moreover, we say that $v(z)$ has a *well-defined* boundary value at t if $\lim_{z\to t, z\in C(a,t)} v(z)$ exists.

The harmonic measure on D from a point $z \in D$, denoted by $\omega_D(z,\cdot)$, is the unique Borel measure on ∂D such that the equality

$$u(z) = \int u(t) d\omega_D(z,t)$$

holds for any function u harmonic in D with continuous well-defined boundary values on ∂D. Such a measure exists from any point in D, since ∂D is non-polar (cf. [111, Thm. 4.3.2]).

In the case where ∂D consists of sufficiently smooth Jordan arcs or curves the harmonic measure on D has a simple representation through the Green function. Recall that the *modulus of continuity* of a continuous function h defined on a compact set $E \subset \mathbb{C}$ is given by

$$\omega(h;\delta) := \sup_{|z-w|\leq\delta} |h(z) - h(w)|, \quad \delta \in [0, \mathrm{diam}(E)], \tag{2.15}$$

where $\mathrm{diam}(E) = \max\{|z-w| \ : \ z,w \in E\}$. Note that any modulus of continuity is a nondecreasing concave up function that is zero at zero. Also recall that a function h is said to be *Dini continuous* on E if

$$\int_{[0,\mathrm{diam}(E)]} \frac{\omega(h;t)}{t} dt < \infty. \tag{2.16}$$

A Jordan arc or curve is called *Dini-smooth* if it has a Dini continuous parametrization with non-vanishing derivative. In what follows we suppose that all arcs are oriented, so, we may distinguish *positive and negative "sides"* of them. Namely, let D' be a simply connected domain and Jordan arc γ be an oriented cross-cut of D. Then $D' = D'_+ \cup D'_-$, where D'_+ (resp. D'_-) is simply connected and lies from the left (resp. right) of γ. We define $\gamma^\pm := \overline{D'_\pm} \cap \{\gamma \setminus \{\gamma^1, \gamma^0\}\}$, where $\{\gamma^1,\gamma^0\}$ are the endpoints of γ. Note that in this definition $\gamma^\pm$ are open arcs.

Let now D be a domain in $\overline{\mathbb{C}}$ such that ∂D is either $\gamma_1 \cup \gamma_2$ ($\gamma_1 \cap \gamma_2 = \emptyset$) or γ_0, where each γ_j, $j = 0,1,2$, is a Dini-smooth Jordan arc or curve. We shall call such domain a DS_2-*domain*, i.e. DS_2-domain is a simply or doubly connected domain with Dini-smooth boundary. Then

$$d\omega_D(z,\cdot)|_{\gamma_j} = \begin{cases} \left.\dfrac{1}{2\pi}\left(\dfrac{\partial}{\partial n^+} g_D(\cdot,z) + \dfrac{\partial}{\partial n^-} g_D(\cdot,z)\right) ds\right|_{\gamma_j}, & \gamma_j \text{ is an arc}, \\ \left.\dfrac{1}{2\pi}\dfrac{\partial}{\partial n} g_D(\cdot,z) ds\right|_{\gamma_j}, & \gamma_j \text{ is a curve}, \end{cases} \tag{2.17}$$

where $n^{\pm}$ denote the inner normals from the positive and negative sides of an arc, n is the inner normal from a curve, and ds is the arclength. Unfortunately, the author could not locate an appropriate reference to a proof of (2.17) in the literature. Therefore, a proof is presented in Proposition A.3 of the Appendix. Let E be a compact non-polar set. Since $\mu_E = \omega_{\Omega_E}(\infty,\cdot)$ (cf. [111, Thm. 4.3.14]), an analogous representation to (2.3) takes place whenever Ω_E is DS_2-domain.

We define *one-sided harmonic measures* on a DS_2-domain D from z, denoted $\omega_D^{\pm}(z,\cdot)$, by the rule

$$d\omega_D^{\pm}(z,\cdot)\big|_{\gamma_j} = \begin{cases} \dfrac{1}{2\pi}\dfrac{\partial}{\partial n^{\pm}} g_D(\cdot,z)ds\Big|_{\gamma_j}, & \gamma_j \text{ is an arc,} \\ \dfrac{1}{2}d\omega_D(z,\cdot)\Big|_{\gamma_j}, & \gamma_j \text{ is a curve.} \end{cases} \tag{2.18}$$

Obviously we have that $\omega_D(z,\cdot) = \omega_D^{+}(z,\cdot) + \omega_D^{-}(z,\cdot)$. Finally, we would like to mention that Proposition A.3 also shows that $\omega_D(z,\cdot)$ is absolutely continuous with respect to the linear measure on ∂D.

Hardy Spaces and Meromorphic Functions

Let L_r^p stand for the space of p-summable functions on $\mathbb{T}_r := \{|z| = r\}$, $r > 0$, with the norm

$$\|h\|_{p,r}^p := \frac{1}{2\pi}\int_{\mathbb{T}} |h(r\xi)|^p |d\xi| < \infty, \quad \text{if} \quad p \in (0,\infty),$$
$$\|h\|_{\infty,r} := \text{ess.}\sup_{\xi\in\mathbb{T}} |h(r\xi)| < \infty, \quad \text{if} \quad p = \infty.$$

For simplicity we shall denote $\mathbb{T} := \mathbb{T}_1$, $L^p := L_1^p$, and $\|\cdot\|_p := \|\cdot\|_{p,1}$. The p-norm Hardy spaces, $p \in [1,\infty]$, of the open unit disk and the complement of the closed unit disk are defined as

$$H^p := \left\{ h \in \text{Hol}(\mathbb{D}) : \sup_{r<1} \|h\|_{p,r} < \infty \right\} \quad \text{and} \quad \bar{H}^p := \left\{ h \in \text{Hol}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}) : \sup_{r>1} \|h\|_{p,r} < \infty \right\},$$

respectively, where $\text{Hol}(D)$ is the space of holomorphic functions in a domain D. Denote by $\bar{H}_0^p$, $p \in [1,\infty]$, a subspace of $\bar{H}^p$ consisting of functions vanishing at infinity. Note that by the Fatou theorem any function from H^p or $\bar{H}^p$, $p \in [1,\infty]$, has an L^p-*trace* (non-tangential boundary values on $\mathbb{T}$). Under this correspondence H^p (resp. $\bar{H}^p$) is isometrically identified with a subspace of L^p functions whose Fourier coefficients with negative (resp. strictly positive) indices all vanish. This allows one to regard Hardy functions either as analytic functions of a complex variable $z \in \mathbb{D}$ or as functions on the circle of the variable $\xi \in \mathbb{T}$. The analytic extension to $\mathbb{D}$ is obtained from the trace on $\mathbb{T}$ through a Cauchy as well as a Poisson integral [112, Thm 17.11]. Moreover a Hardy function

h is determined, up to a purely imaginary constant, by its real part on $\mathbb{T}$, i.e.

$$h(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \mathrm{Re}(h(\xi)) \frac{|d\xi|}{2\pi} + i\mathrm{Im}(h(0)) =: \mathcal{RH}(\mathrm{Re}(h); z) + i\mathrm{Im}(h(0)), \quad z \in \mathbb{D}. \tag{2.19}$$

The integral on the right-hand side of (2.19) is called *Riesz-Herglotz transform* of $\mathrm{Re}(h)$. Moreover, $\mathcal{RH}(h; \cdot)$ is well-defined whenever h is integrable, generally complex-valued, function on $\mathbb{T}$.

From Parseval's theorem we have that $L^2 = H^2 \oplus \bar{H}_0^2$. Thus, we may define orthogonal projections $\mathcal{P}_+ : L^2 \to H^2$ (analytic) and $\mathcal{P}_- : L^2 \to \bar{H}_0^2$ (antianalytic). It is easy to see that

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h(\xi)}{\xi - z} d\xi = \begin{cases} \mathcal{P}_+(h)(z), & z \in \mathbb{D}, \\ -\mathcal{P}_-(h)(z), & z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}, \end{cases} \qquad h \in L^2.$$

The space of meromorphic functions of degree n in $\mathbb{D}$ is defined as

$$H_n^p := \mathrm{M}_n^{-1} H^p = \mathrm{B}_n^{-1} H^p, \tag{2.20}$$

where M_n stands for the set of monic polynomials of degree at most n with all zeros in $\mathbb{D}$ and B_n is the space of Blaschke products of degree at most n, i.e.

$$\mathrm{B}_n := \left\{ e^{ic} \prod_{j=1}^{k} \frac{z - z_j}{1 - \bar{z}_j z} : \ c \in \mathbb{R}, \ k \le n, \ z_j \in \mathbb{D}, \ j = 1, \dots, k \right\}.$$

Recall the well-known fact (cf. [46]) that any nonzero function in H^p can be uniquely factored as $h = jw$, where

$$w(z) = \exp\left\{ \frac{1}{2\pi} \int \frac{\xi + z}{\xi - z} \log |h(\xi)| |d\xi| \right\}$$

belongs to H^p and is called the *outer factor* of h, while j has modulus 1 a.e. on $\mathbb{T}$ and is called the *inner factor* of h. The latter may be further decompose as $j = bS$, where b is a Blaschke product that has the same zeros with the same multiplicities as h and thus assumes the form

$$b(z) = e^{ic} z^k \prod_j \frac{z_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z},$$

while

$$S(z) = \exp\left\{ -\int \frac{\xi + z}{\xi - z} d\nu(\xi) \right\}$$

is the *singular inner factor* associated with ν, a positive measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure. For simplicity, we often say that a function is outer (resp. inner) if it is equal to its outer (resp. inner) factor.

In another connection, it is possible to give a conformally invariant definition of Hardy spaces. Let $D \subset \overline{\mathbb{C}}$ be a domain. Then $h \in \text{Hol}(D)$ belongs to $H^p(D)$, $p \in [1, \infty)$, if and only if $|h|^p$ has a harmonic majorant in D. A Hardy space $H^\infty(D)$ consists of bounded analytic functions in D. The norm on $H^p(D)$ is defined as follows

$$\|h\|_p = \begin{cases} |u_h(z_0)|^{1/p}, & p \in [1, \infty), \\ \sup_{z \in D} |h(z)|, & p = \infty, \end{cases}$$

where u_h is the least harmonic majorant of $|h|^p$ in D and $z_0 \in D$. We remark that u_h is unique and the choice of z_0 is insignificant in a sense that different points give rise to the equivalent norms (cf. [45, Ch. 3] or [46, Ch. II]).

Let now D be a DS_2-domain. Before we define an outer function in $H^p(D)$, $p \in [1, \infty]$, we shall describe boundary behavior of functions in this class. It is explained in Proposition A.3 that there exists a conformal map φ from D to D', where D' is either the unit disk or an annulus (depending whether D is simply or doubly connected domain). Further, φ extends conformally to $\overline{D}^{\pm}$ with endpoints of arcs removed if ∂D contains arcs. Let $h \in H^p(D)$, define $h^* := h \circ \varphi^{-1}$. It is clear that $h^* \in H^p(D')$ by conformal invariance of Hardy spaces. It is a well-known fact (cf, for instance, [46, 44] and [115]) that h^* has boundary values almost everywhere on $\partial D'$ with respect to the Lebesgue measure there. Thus, h has boundary values almost everywhere on ∂D with respect to the Lebesgue measure and therefore with respect to the harmonic measure on D (recall that harmonic measures from different points are mutually absolutely continuous [111, Cor. 4.3.5]). Quite naturally, boundary values of h need not to be well-defined. Thus, we shall distinguish between boundary values on positive and negative sides of each arc in ∂D, h^+ and h^-, respectively. To treat all the cases simultaneously, we shall speak of $h^\pm$ all the time, understanding that they are equal to each other on curves in ∂D. It also known that $|h^*|^p$, $p \in [1, \infty)$, and $\log |h^*|$, $p \in [1, \infty]$, are integrable with respect to the harmonic measure $\omega_{D'}$ (cf. [44] and [115]). It follows from Proposition A.3 that

$$\omega_D(z, E) = \omega_{D'}(z', E'_+) + \omega_{D'}(z', E'_-),$$

where $z' = \varphi(z)$, $E'_\pm := \varphi^\pm(E)$, and $\varphi^\pm$ are continuous extensions of φ on the closure of positive and negative sides of each arc in ∂D. Thus, $|h^\pm|^p$ and $\log |h^\pm|$ are integrable with respect to ω_D. For refine treatment of boundary values of functions from Hardy spaces of doubly connected domains one may consult the Appendix of [26]. Now, we are ready to give a definition of an outer function in $H^p(D)$.

We say that $h \in H^p(D)$, $p \in [1, \infty]$, is an outer function in D if h has no zeros there and

$$\log|h(z)| = \int_{\partial D} \log|h^+(t)| d\omega_D^+(z,t) + \int_{\partial D} \log|h^-(t)| d\omega_D^-(z,t), \tag{2.21}$$

where $\omega_D^{\pm}$ are the one-sided harmonic measures on D. Clearly, equation (2.21) becomes

$$\log|h(z)| = \int_{\partial D} \log|h(t)| d\omega_D(z,t) \tag{2.22}$$

if $|h|$ has well-defined boundary values.

Meromorphic Approximation

The Best-$L^p(\mathbb{T})$ meromorphic approximation problem consists in the following.

$\mathbf{MA}(p)$: *Given $p \in [1,\infty]$, $f \in L^p$, and $n \in \mathbb{N}$, find $g_n \in H_n^p$ such that*

$$\|f - g_n\|_p = \inf_{g \in H_n^p} \|f - g\|_p. \tag{2.23}$$

This problem has a solution [28, Sec. 5]. It is known to be unique when $p = \infty$, provided that f belongs to the *Douglas algebra* $H^\infty + C(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of continuous functions on $\mathbb{T}$ [1]. In particular the solution to $\mathbf{MA}(\infty)$ is unique for $\mathcal{F}(\mu; R; \cdot)$, since the latter is analytic in some neighborhood of the unit circle. When $p < \infty$, a solution needs not be unique even if f is very smooth [28, Sec. 5]. Therefore, when making a statement about a sequence $\{g_n\}$ of solutions to $\mathbf{MA}(p)$, it is understood that a particular solution has been selected for each n and that the statement holds true regardless the selection. Also we shall restrict ourselves to the range $2 \leq p \leq \infty$ for these are the only values of p for which a concrete characterization of the solutions is known so far [28, Sec. 8]. We should note that when $p = 2$ the problem reduces to rational approximation. Indeed, since $L^2 = H^2 \oplus \bar{H}_0^2$ and $\mathcal{F}$ is the Cauchy transform of a measure supported in $\mathbb{D}$, it belongs to $\bar{H}_0^2$. Thus for any $g = (h + p_{n-1}/q_n) \in H_n^2$ with $h \in H^2$ and p_{n-1}/q_n a rational function in $\bar{H}_0^2$, we get by orthogonality

$$\|\mathcal{F} - g\|_2^2 = \|h\|_2^2 + \|\mathcal{F} - p_{n-1}/q_n\|_2^2.$$

Clearly then, for g to be a best approximant h must be zero, and we arrive at the following approximation problem:

Given $f \in \bar{H}_0^2$ *and* $n \in \mathbb{N}$, *minimize* $\|f - p_{n-1}/q_n\|_2$ *over* $p_{n-1} \in \mathrm{P}_{n-1}$ *and* $q_n \in \mathrm{M}_n$.

Moreover, thanks to Parseval's formula, p_{n-1}/q_n will be a best approximant to f not only from $\mathrm{R}_{n-1,n}$ but also from $\mathrm{R}_{n,n}$ since f vanishes at infinity. Despite this, we shall not distinguish $p = 2$ from the other cases but rather keep a unified operator approach.

As it is the case of interest here, we shall restrict our discussion to the situation where the approximated function is of the form (1.2), and accordingly we write $\mathcal{F}$ instead of f. The solution to $\mathbf{MA}(p)$ turns out to be exactly the functions of the form [28, Thm. 8.2]

$$g_n = \mathcal{F} - \frac{\mathcal{H}_\mathcal{F}(v_n)}{v_n} = \frac{\mathcal{T}_\mathcal{F}(v_n)}{v_n}, \tag{2.24}$$

where $\mathcal{H}_\mathcal{F}$ and $\mathcal{T}_\mathcal{F}$ are *Hankel* and *Toeplitz* operators with a symbol $\mathcal{F}$, respectively, i.e.

$$\begin{array}{ll} \mathcal{H}_\mathcal{F} : H^{p'} \to \bar{H}_0^2, \quad \mathcal{H}_\mathcal{F}(h) := \mathcal{P}_-(\mathcal{F}h), & \\ & \dfrac{1}{p} + \dfrac{1}{p'} = \dfrac{1}{2}. \\ \mathcal{T}_\mathcal{F} : H^{p'} \to H^2, \quad \mathcal{T}_\mathcal{F}(h) := \mathcal{P}_+(\mathcal{F}h), & \end{array}$$

Such a g_n is called a best meromorphic L^p-approximant of order n to $\mathcal{F}$. A function v_n is called a *singular vector associated to* g_n and has unit norm in $H^{p'}$. We point out that any singular vector can be factored as

$$v_n(z) = b_n(z) w_n(z), \quad z \in \mathbb{D}, \tag{2.25}$$

where $b_n \in \mathrm{B}_n \setminus \mathrm{B}_{n-1}$ and w_n is an outer function. Note that v_n is just a Blaschke product when $p = 2$, i.e. $w_n \equiv 1$ in this case. Moreover, we have that $\|\mathcal{F} - g_n\|_p = \sigma_n(\mathcal{H}_\mathcal{F})$, where $\sigma_n(\mathcal{H}_\mathcal{F})$ is the *n-th singular value* of $\mathcal{H}_f$, i.e

$$\sigma_n(\mathcal{H}_\mathcal{F}) := \inf\left\{ |||\mathcal{H}_\mathcal{F} - \Gamma||| : \ \Gamma : H^{p'} \to \bar{H}_0^2 \text{ a linear operator of rank } \leq n \right\}$$

and $|||\cdot|||$ stands for the operator norm; when $p = 2$ we assume in addition that Γ is weak* continuous.

Note that if $p = \infty$ then $p' = 2$, hence $\mathcal{H}_\mathcal{F}$ operates between Hilbert spaces, and since it is compact[7] the $\sigma_n(\mathcal{H}_\mathcal{F})$ are just the singular values of $\mathcal{H}_\mathcal{F}$, that is, the square-roots of the eigenvalues of $\mathcal{H}_\mathcal{F}^*\mathcal{H}_\mathcal{F}$ arranged in nonincreasing order; throughout $\mathcal{H}_\mathcal{F}^*$ indicates the adjoint of $\mathcal{H}_\mathcal{F}$. When $2 \leq p < \infty$, the usual eigenvector equation gets replaced by a *nonlinear* equation of Hammerstein

[7] This can be deduced from the fact that $\mathcal{F} \in C(\mathbb{T})$, see [97, Thm. I.5.5].

type. More precisely, to each singular value $\sigma_n(\mathcal{H}_\mathcal{F})$ there exists (at least one) singular vector $v_n \in H^{p'}$ of unit norm, whose inner factor is a Blaschke product of degree at most n, such that

$$\begin{aligned} &\mathcal{H}^*_\mathcal{F}\mathcal{H}_\mathcal{F}(v_n) = \sigma_n^2(\mathcal{H}_\mathcal{F})\mathcal{P}_+\left(|v_n|^{p'-2}v_n\right) && && \text{if } \ p>2, \\ &\mathcal{H}^*_\mathcal{F}\mathcal{H}_\mathcal{F}(v_n) = \mathcal{P}_+\left(|\mathcal{H}_\mathcal{F}(v_n)|^2 v_n\right) && \text{and} && \|\mathcal{H}_\mathcal{F}(v_n)\|_2 = \sigma_n(\mathcal{H}_\mathcal{F}) \quad \text{if } \ p=2. \end{aligned} \tag{2.26}$$

When $p \in [2,\infty)$ a best meromorphic approximant g_n is, in general, not unique, but it has exactly n poles; this implies that b_n in (2.25) has exact degree n for any n-th singular vector v_n. To the contrary, if $p=\infty$, then g_n is unique but may have less than n poles and b_n could have degree less than n. In the latter case the span of n-th singular vectors has dimension greater than 1, so that v_n is far from being unique up to a multiplicative constant, but it is remarkable that all the v_n give rise to the same g_n through formula (2.24); in fact there always exist n-th singular vectors for which b_n has exact degree n, but some of the poles may cancel with zeros of $\mathcal{T}_\mathcal{F}(v_n)$ in (2.24). Besides pointed peculiarities, the case $p=\infty$ has one more specific feature. Namely, the so-called *circularity property*, i.e.

$$|\mathcal{F}-g_n| = \|\mathcal{F}-g_n\|_\infty = \sigma_n(\mathcal{H}_\mathcal{F}) \quad \text{a.e. on} \quad \mathbb{T}. \tag{2.27}$$

The notion of a best approximant can be further weakened to the notion of a *critical point*. By definition, a function g_n is a critical point of order n in $\mathbf{MA}(p)$ if and only if it assumes the form

$$g_n = \mathcal{F} - \frac{\mathcal{H}_\mathcal{F}(v_n)}{v_n},$$

where v_n is a $H^{p'}$ function of unit norm (a Blaschke product if $p=2$) whose inner factor lies in $\mathrm{B}_n \setminus \mathrm{B}_{n-1}$, which is such that

$$\begin{aligned} &\mathcal{H}^*_\mathcal{F}\mathcal{H}_\mathcal{F}(v_n) = \sigma_n^2\mathcal{P}_+\left(|v_n|^{p'-2}v_n\right), \quad \sigma_n^2 \in \mathbb{R}, && \text{if } \ p>2, \\ &\mathcal{H}^*_\mathcal{F}\mathcal{H}_\mathcal{F}(v_n) = \mathcal{P}_+\left(|\mathcal{H}_\mathcal{F}(v_n)|^2 v_n\right) && \text{if } \ p=2. \end{aligned} \tag{2.28}$$

The difference with (2.26) is that here σ_n or $\|\mathcal{H}_\mathcal{F}(v_n)\|_2$ needs not be equal to $\sigma_n(\mathcal{H}_\mathcal{F})$. With a slight abuse of language, we will continue to say that v_n is a singular vector associated to g_n although σ_n may no longer be a singular value. Note that, as for best meromorphic approximants,

$$\sigma_n = \|\mathcal{F}-g_n\|_p = \|\mathcal{H}_\mathcal{F}(v_n)\|_p, \quad p \in [2,\infty] \tag{2.29}$$

and v_n is just a Blaschke product when $p = 2$. Thus it has an inner-outer factorization of type (2.25) where b_n has exact degree n and $w_n \equiv 1$ if $p = 2$.

A critical point g_n of order n may have less than n poles, even though we insisted in the definition that v_n has exactly n zeros. This happens when some zeros of v_n (which are, of course, zeros of b_n) cancel with some zeros of $\mathcal{T}_{\mathcal{F}}(v_n)$. In this case b_n can be further factorized as

$$b_n = b_{n_1} \cdot b_{n_2}, \tag{2.30}$$

where b_{n_1} is the greatest common inner divisor of $\mathcal{T}_{\mathcal{F}}(v_n)$ and v_n. Then ([28, Prop. 9.1])

$$\begin{aligned} \mathcal{H}_{\mathcal{F}}(v_n)(\xi) &= \sigma\bar{\xi}\overline{\left(b_{n_2} j_n w_n^{p'/2}\right)(\xi)} = \sigma_n \left(b_{n_2} j_n w_n^{p'/2}\right)^{\sigma}(\xi), \quad p > 2 \\ \mathcal{H}_{\mathcal{F}}(v_n)(\xi) &= \bar{\xi}\ \overline{(b_{n_2} u_n)(\xi)} = (b_{n_2} u_n)^{\sigma}(\xi), \quad p = 2, \end{aligned} \tag{2.31}$$

for a.e. $\xi \in \mathbb{T}$, where j_n is some inner function, $u_n \in H^2$, and $h^{\sigma}(z) := z^{-1}\overline{h(1/\bar{z})}$. Note that if $h \in H^2$ then $h^{\sigma} \in \bar{H}_0^2$ and vice versa. We remark that for the case$p = 2$ equation (2.31) is an interpolation condition saying that g_n interpolates $\mathcal{F}$ with order 2 at the reflection of its poles. The same can be said when $p \in (2, \infty]$, provided g_n is analytic at the reflection of its poles, but this is no longer automatic because it may no longer be rational.

Although their definition is a little technical, critical points are just those $g_n \in H_n^p$ for which the derivative of $\|\mathcal{F} - g_n\|_p$ with respect to $b_n \in B_n$ and $h \in H^p$ in the second part of factorization (2.20) does vanish. Beyond best approximants, the most important critical points are *local best approximant* [28, Prop. 9.3.]. By definition, a local best approximant is some $g_n \in H_n^p$ for which there exists $\delta > 0$ such that

$$g \in H_n^p \quad \text{and} \quad \|g - g_n\|_p \leq \delta \quad \text{imply} \quad \|\mathcal{F} - g_n\|_p \leq \|\mathcal{F} - g\|_p.$$

The reason why we introduce critical points is that all a numerical search can yield in general is a local best approximant, and we feel it is important that our results should apply to computable objects.

When $p \in [2, \infty)$, best and local best approximants have exactly n poles, counting multiplicities, hence they are *a fortiori* critical points of order n ([28, Prop. 9.2 and 9.3]); such critical points are called *irreducible*. For $p = \infty$ the critical points are just the best meromorphic approximants, so the notion is nothing new, but it may happen that a best approximant out of H_n^{∞} has less than n poles. However, each time the number of poles of g_n increases with n, it jumps to the maximum value n, in particular there exists a subsequence of natural numbers, say $\mathbb{N}_0 = \mathbb{N}_0(\mathcal{F})$, such that for each $n \in \mathbb{N}_0$ the best approximant g_n has exactly n poles in $\mathbb{D}$, *i.e.* it is irreducible ([28, p. 114]). Since the behavior of the poles of best approximants from H_n^∞ is entirely characterized by this subsequence, hereafter we say "a sequence of irreducible critical points of order n" to mean if $p = \infty$ that we pass to a subsequence if needed. Let $\{g_n\}$ be such a sequence. As usual, we denote by v_n an associated singular vector to g_n. According to (2.20), each g_n can be decomposed as

$$g_n = b_n^{-1} \cdot h_n,$$

where $h_n \in H^p$ and $b_n = b_{n_2}$, i.e. $b_{n_1} \equiv 1$ in (2.30). Moreover, we can write b_n as $q_n/\widetilde{q}_n$, where $q_n \in \mathrm{M}_n$ and $\widetilde{q}_n(t) := t^n \overline{q_n(1/\bar{t})}$ is the reciprocal polynomial of q_n. Arguing like in ([28, Sec. 10]), where R is not present, equation (2.31) implies easily the following orthogonality relations

$$\int t^k q_n(t) \frac{w_n(t)}{\widetilde{q}_n^2(t)} d\widetilde{\mu}(t) = 0, \quad k = 0, \ldots, n-1, \tag{2.32}$$

where w_n is the outer factor of v_n and $\widetilde{\mu}$ is given by (1.16). Upon rewriting (2.32) as

$$\int P_{n-1}(t) q_n(t) \frac{w_n(t)}{\widetilde{q}_n^2(t)} d\mu(t) + \sum_{\eta \in S'} \sum_{k=0}^{m(\eta)-1} \frac{r_{\eta,k}}{k!} \left(P_{n-1}(t) q_n(t) \frac{w_n(t)}{\widetilde{q}_n^2(t)} \right)^{(k)} \Bigg|_{t=\eta} = 0,$$

for all $P_{n-1} \in \mathrm{P}_{n-1}$ and taking P_{n-1} to be a multiple of Q, these relations yield for $n > m$

$$\int t^k q_n(t) Q(t) \frac{w_n(t)}{\widetilde{q}_n^2(t)} d\mu(t) = 0, \quad k = 0, \ldots, n-m-1 \tag{2.33}$$

where Q was defined in (1.14). When $p \in (2, \infty]$, the w_n are *a priori* holomorphic in the unit disk only, but due to the analyticity of $\mathcal{F}$ it is possible to say more. Since for $|z| > 1$

$$\begin{aligned} \mathcal{H}_{\mathcal{F}}(v_n)(z) &= \mathcal{P}_-(\mathcal{F}v_n)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(\mathcal{F}v_n)(\xi)}{z-\xi} d\xi \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \int_{\widetilde{S}_\mu} \frac{v_n(\xi)}{(z-\xi)(\xi-t)} d\widetilde{\mu}(t) d\xi = \int_{\widetilde{S}_\mu} \frac{v_n(t)}{z-t} d\widetilde{\mu}(t), \end{aligned} \tag{2.34}$$

we obtain from (2.31) that $\left(b_n j_n w_n^{p'/2}\right)^\sigma$ can be analytically extended to $D_{\mathcal{F}} = \overline{\mathbb{C}} \setminus \widetilde{S}_\mu$ by the formula

$$\sigma_n \left(b_n j_n w_n^{p'/2}\right)^\sigma (z) = \mathcal{H}_{\mathcal{F}}(v_n)(z) = \int \frac{v_n(t)}{z-t} d\widetilde{\mu}(t), \quad z \in D_{\mathcal{F}}. \tag{2.35}$$

Recall that $\sigma_n \neq 0$ since $\mathcal{F}$ is not rational. This, in particular, implies that $b_n j_n w_n^{p'/2}$ can be extended to a holomorphic function in $D_{\mathcal{F}}^*$, where K^* denotes, as before, the reflection of a set K across $\mathbb{T}$. This extension, in turn, implies that j_n must be a finite Blaschke product, because the zeros of an analytic function are isolated in the domain of analyticity. Multiplying (2.35) by $b_n j_n$ and applying $\mathcal{P}_-$, we get from the definition of "σ", since $w_n^{p'/2} \in H^2$ and $|b_n j_n| = 1$ on $\mathbb{T}$ that the left-hand side is

$$\sigma_n \mathcal{P}_- \left(b_n j_n \left(b_n j_n w_n^{p'/2} \right)^\sigma \right) = \sigma_n \mathcal{P}_- \left(\left(w_n^{p'/2} \right)^\sigma \right) = \sigma_n \left(w_n^{p'/2} \right)^\sigma$$

and computing the right-hand side as in (2.34) we get for $|z| > 1$

$$\begin{aligned} \sigma_n \left(j_n w_n^{p'/2} \right)^\sigma (z) &= \int \frac{b_n(t) v_n(t)}{z-t} d\widetilde{\mu}(t) = \int \frac{b_n^2(t) w_n(t)}{z-t} d\mu(t) \\ &+ \sum_{\eta \in S'} \sum_{k=0}^{m(\eta)-1} \frac{r_{\eta,k}}{k!} \left(\frac{b_n^2(t) w_n(t)}{z-t} \right)^{(k)} \Bigg|_{t=\eta} . \end{aligned} \tag{2.36}$$

This shows that $\left(w_n^{p'/2} \right)^\sigma$ has a meromorphic extension to $\overline{\mathbb{C}} \setminus S_\mu$ and may only have a pole of order at most $m(\eta)$ at each $\eta \in S'$, though possible cancellations may occur due to the zeros of $b_n j_n$. Equivalently, $w_n^{p'/2}$ can be meromorphically extended to $D_{\mathcal{F}}^*$ and may only have poles at the points $1/\bar{\eta}$, of respective multiplicities at most $m(\eta)$ for each $\eta \in S'$.

Multipoint Padé Approximation

Let a system of sets $\mathbf{A} := \{A_l\}_{l \in \mathbb{N}}$, $A_l := \{a_{j,l}\}_{j=1}^l \subset D_{\mathcal{F}}$, be given. To the each set A_l we associate a polynomial

$$v_l(t) := \prod_{a_{j,l} \neq \infty} (a_{j,l} - t). \tag{2.37}$$

The *diagonal multipoint Padé approximant* to $\mathcal{F}(\mu; R; \cdot)$ given by (1.2) is the unique rational function $\Pi_n = p_n/q_n$ where the polynomials p_n and q_n satisfy:

(i) $\deg p_n \leq n$, $\deg q_n \leq n$, and $q_n \not\equiv 0$;

(ii) $\left(q_n(z) \mathcal{F}(z) - p_n(z) \right) / v_{2n}(z)$ is analytic in $\overline{\mathbb{C}} \setminus E_\mu$;

(iii) $\left(q_n(z) \mathcal{F}(z) - p_n(z) \right) / v_{2n}(z) = O\left(1/z^{n+1} \right)$, as $z \to \infty$,

where E_μ is the convex hull of $S_\mu = \mathrm{supp}(\mu)$. A multipoint Padé approximant always exists since the conditions for p_n and q_n amount to solving a system of $2n+1$ homogeneous linear equations with $2n+2$ unknown coefficients, no solution of which can be such that $q_n \equiv 0$ (we may thus assume that q_n is monic); note that (iii) entails at least one interpolation condition at infinity. Polynomials p_n and q_n are not necessarily unique, but Π_n is. It is called *normal* if $\deg \Pi_n = n$. Diagonal Padé approximants to Markov functions are always normal, but nothing can be said *a priori* when the measure μ is complex. Clearly (ii) ensures that when Π_n is normal it interpolates $\mathcal{F}$ at the zeros of v_{2n}. Thus, we shall call the system $\mathbf{A}$ an *interpolation scheme*.

Let Γ_n be any closed Jordan curve that separates $\widetilde{S}_\mu$ and A_n and contains $\widetilde{S}_\mu$ in the bounded component of its complement, say $\mathrm{int}(\Gamma_n)$. Then properties (ii) and (iii) yield that

$$\int_{\Gamma_n} z^j q_n(z)\mathcal{F}(z)\frac{dz}{v_{2n}(z)} = 0, \quad j = 0, \ldots, n-1, \quad z \in \mathrm{int}(\Gamma_n).$$

This, in turn, implies

$$\int t^j q_n(t)Q(t)\frac{d\mu(t)}{v_{2n}(t)} = 0, \quad j = 0, \ldots, n-m-1, \tag{2.38}$$

by definition (1.2) of $\mathcal{F}$, the Fubini-Tonelli theorem, and the residue formula, where Q was defined in (1.14). Hence the denominators of the multipoint Padé approximants to $\mathcal{F}$ are polynomials satisfying a non-Hermitian orthogonality relation with respect to varying with n complex measures $Q(t)d\mu(t)/v_{2n}(t)$.

By the very definition, the behavior of multipoint Padé approximants depends on the choice of the interpolation scheme. In what follows, we shall deal with schemes having some admissibility properties. To describe these properties we need more notation.

We denote by $\varphi_i(\cdot)$ (resp. $\varphi_e(\cdot)$) the conformal mapping of the complement of $E_\mu = [a, b]$ to the interior (resp. exterior) of the unit circle such that $\varphi_i(\infty) = 0$ (resp. $\varphi_e(\infty) = \infty$) and $\varphi_i'(\infty) > 0$ (resp. $\varphi_e'(\infty) > 0$). These mappings are unique and given by

$$\varphi_i(z) := \frac{2z-b-a}{b-a} - \sqrt{\left(\frac{2z-b-a}{b-a}\right)^2 - 1}, \quad \varphi_e(z) = \frac{1}{\varphi_i(z)}, \quad z \notin E, \tag{2.39}$$

where we take a branch of the square root that is positive for positive values. It will be convenient to use also the following notation

$$\psi_n(z) := \mathrm{cap}^n(E)\varphi_e^n(z), \quad z \in \mathbb{C} \setminus E, \tag{2.40}$$

that is analytic non-vanishing in $\mathbb{C} \setminus E$ and $\psi_n(z) = z^n + O(1)$, as $z \to \infty$.

Let now $\mathbf{A} = \{A_l\}_{l\in\mathbb{N}}$ be an interpolation scheme and let C_l stand for $\varphi_i(A_l)$, i.e $C_l =$

$\{c_{j,l}\}_{j=1}^{l}$, $c_{j,l} = \varphi_i(a_{j,l})$. We say that $\mathbf{A}$ is an *admissible* interpolation scheme if the following holds: for each $l \in \mathbb{N}$ there exists a one-to-one correspondence $\Delta_l : C_l \to C_l$ such that

$$\sup_{l\in\mathbb{N}} \sum_{j=1}^{l} \frac{|\bar{c}_{j,l} - \Delta_l(c_{j,l})|}{(1-|c_{j,l}|)(1-|\Delta_l(c_{j,l})|)} < \infty \quad \text{and} \quad \lim_{l\to\infty} \sum_{j=1}^{l} (1-|c_{j,l}|) = \infty. \tag{2.41}$$

Roughly speaking, the first part of (2.41) says that $\{C_l\}$ forms a "nearly conjugate symmetric" sequence of sets. Indeed, it is obvious that if C_l is conjugate-symmetric then Δ_l can be defined as $\Delta(c_{j,l}) = \bar{c}_{j,l}$ and the corresponding sum in (2.41) is equal to zero. The second condition insures that the points $\{c_{j,l}\}$ do not approach the unit circle "too fast" (i.e points $\{a_{j,l}\}$ do not approach interval E too fast).

Most of the time we shall place stronger conditions on the admissible interpolation schemes. The interpolation scheme $\mathbf{A} = \{A_l\}_{l\in\mathbb{N}}$ is said to be *strongly admissible* if each set A_l is conjugate-symmetric, i.e. $A_l = \bar{A}_l$, the set of limit points of $\mathbf{A}$, indicated by $\mathcal{K}(\mathbf{A})$, is disjoint from $S' \cup E$, and if the counting measures of the points in A_l converge in the weak* topology, as $n \to \infty$, to some probability measure with finite logarithmic energy $\sigma = \sigma(\mathbf{A}) \in \Lambda(\mathcal{K}(\mathbf{A}))$, i.e.

$$\sigma_l := \frac{1}{l}\sum_{j=1}^{l} \delta_{a_{j,l}} \xrightarrow{*} \sigma.$$

We call such σ an *asymptotic distribution for* $\mathbf{A}$. Note that $\mathcal{K}(\mathbf{A})$ is not necessarily compact. If it is not compact, the finiteness of the logarithmic energy of σ is understood as follows. Since $\mathcal{K}(\mathbf{A})$ is closed and does not intersect $\widetilde{S}_\mu$, there exists $z_0 \in \mathbb{C}\setminus \cup_n A_n$ such that $z_0 \notin \mathcal{K}(\mathbf{A})$. Pick such a z_0 and set $M_{z_0}(z) := 1/(z - z_0)$. Then, all $M_{z_0}(A_n)$ are contained in some compact set and their counting measures converge weak* to $\sigma^\sharp$ such that $\sigma^\sharp(B) := \sigma(M_{z_0}^{-1}(B))$ for any Borel set $B \subset \mathbb{C}$. We say that $\mathbb{A}$ is admissible if $\sigma^\sharp$ has finite logarithmic energy. Obviously this definition does not depend on a particular choice of z_0. If all the interpolation points are assigned to infinity, i.e. $v_{2n} \equiv 1$ for all $n \in \mathbb{N}$, then Π_n is just the classical Padé approximant to $\mathcal{F}$.

CHAPTER III

WEAK ASYMPTOTICS

In this chapter we show convergence in capacity of meromorphic and multipoint Padé approximants to $\mathcal{F}(\mu; R; \cdot)$, where R is an arbitrary rational function vanishing at infinity and μ is in class **BVT** defined below.

Let μ be a complex Borel measure whose support $S := \text{supp}(\mu) \subset \mathbb{R}$ is compact and consists of infinitely many points. Further, denote by $|\mu|$ the total variation measure. Clearly μ is absolutely continuous with respect to $|\mu|$, and we shall assume that its Radon-Nikodym derivative (which is of unit modulus $|\mu|$-a.e.) is of bounded variation. In other words, μ is of the form

$$d\mu(t) = e^{i\Theta(t)} d|\mu|(t), \tag{3.1}$$

for some real-valued argument function $\Theta = \Theta_\mu$ such that[8]

$$V(\Theta, S) := \sup \left\{ \sum_{j=1}^{N} |\Theta(x_j) - \Theta(x_{j-1})| \right\} < \infty, \tag{3.2}$$

where the supremum is taken over all finite partitions $x_0 < x_1 < \ldots < x_N$ of S as N ranges over $\mathbb{N}$. For convenience, we extend the definition of Θ to the whole of $\mathbb{R}$ as follows. Let $E := [a, b]$ be the convex hull of S. It is easy to see that if we interpolate Θ linearly in each component of $E \setminus S$ and if we set $\Theta(x) := \lim_{t \to a,\, t \in S} \Theta(t)$ for $x < a$ and $\Theta(x) := \lim_{t \to b,\, t \in S} \Theta(t)$ for $x > b$ (the limits exist by (3.2)), the variation of Θ will remain the same. In other words, we may arrange things so that the extension of Θ, still denoted by Θ, satisfies $V(\Theta, S) = V(\Theta, \mathbb{R}) =: V(\Theta)$.

Now we are ready to define the class of measures under consideration. We say that a complex measure μ supported on $(-1, 1)$ belongs to the class **BVT** if

(1) μ *has an argument of bounded variation*;

(2) $\text{supp}(\mu)$ *is a regular set*;

(3) *there exist positive constants* c *and* L *such that, for any* $x \in \text{supp}(\mu)$ *and* $\delta \in (0, 1)$*, the total variation of* μ *satisfies* $|\mu|([x - \delta, x + \delta]) \geq c\delta^L$.

[8] Note that $e^{i\Theta}$ has bounded variation if and only if Θ can be chosen of bounded variation.

This chapter is organized as follows. In the next section we prove two theorems on zero distribution of orthogonal polynomials. These results lay the groundwork for the investigation of the behavior of poles of meromorphic and Padé approximants presented in the third and forth sections. These sections also contain results on convergence in capacity of the approximants to the approximated function.

Orthogonal Polynomials

The main results of this section are the following theorems on asymptotic behavior of zeros of polynomials orthogonal with respect to varying measures. In the first case the varying measures depend on the polynomials themselves, in the second case they are prescribed at the beginning.

To simplify the notation we shall denote throughout the rest of the text the support of the measure of orthogonality μ by S, the convex hull of S by E, and $S \cup S'$ by $\widetilde{S}$.

Theorem 3.1 *Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials of exact degree n with all zeros in $\mathbb{D}$ satisfying the orthogonality relations*

$$\int t^k q_n(t) \frac{\omega_n(t)}{\widetilde{q}_n^2(t)} d\widetilde{\mu}(t) = 0, \quad k = 0, \ldots, n-1, \tag{3.3}$$

where $\widetilde{\mu} = \mu + \mu'$ is given by (1.15) and (1.16) with $\mu \in \mathbf{BVT}$ and $S' \subset \mathbb{D}$, while $\mathcal{W} = \{\omega_n\}_{n=1}^{\infty}$ is a family of complex measurable functions on $S' \cup E$, whose moduli are uniformly bounded above and below by positive constants, and whose arguments are smooth with uniformly bounded derivatives on E. Then the counting measures of the zeros of $q_n(z) = \prod_{j=1}^{n}(z - \xi_{j,n})$, namely $\nu_n := (1/n)\sum_{j=1}^{n}\delta_{\xi_{j,n}}$, converge in the weak* sense to $\mu_{(S,\mathbb{T})}$, the Green equilibrium distribution on S relative to $\mathbb{D}$.

The above stated theorem is a direct generalization of Theorem 5.1 and Corollary 6.2 in [19]. The main difference is that here we add a distribution of the form (1.15) to the measure in (3.3). In the Ph.D. thesis of R. Küstner [67], an analog of Theorem 5.1 in [19] is given when the measure μ, instead of belonging to **BVT**, satisfies the so-called Λ-criterion introduced in [130, Sec. 4.2]:

$$\operatorname{cap}\left(\left\{t \in S : \limsup_{r\to 0} \frac{\operatorname{Log}(1/\mu[t-r,t+r])}{\operatorname{Log}(1/r)} < +\infty\right\}\right) = \operatorname{cap}(S).$$

Paralleling the arguments in [67], all the results in this section could be obtained under this weaker assumption, but the exposition would be heavier and we leave it to the interested reader to carry out the details.

Theorem 3.2 *Let $\widetilde{\mu}$ be as in Theorem 3.1 and $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials of degree at most n satisfying the weighted orthogonality relations*

$$\int t^j q_n(t)\frac{d\widetilde{\mu}(t)}{v_{2n}(t)} = 0, \quad j = 0, \dots, n-1, \tag{3.4}$$

where $\{v_{2n}\}_{n\in\mathbb{N}}$ is the sequence of monic polynomials associated via (2.37) to some strongly admissible interpolation scheme $\mathbf{A}$ with asymptotic distribution σ. Then the counting measures ν_n of the zeros of q_n converge in the weak sense to μ_w^S, the weighted equilibrium distribution on S with external field given by $-U^\sigma$.*

Note that, by Proposition A.2, the measure μ_w^S is nothing but the balayage $\widehat{\sigma}$ of σ onto S out of $\mathbb{C}\setminus S$. Let us stress also that, in general, the polynomials q_n satisfying (3.4) need not be unique up to a multiplicative constant nor have exact degree n. In the above theorem, it is understood that q_n is *any* sequence of such polynomials, and that ν_n is normalized by $1/n$ so that it may no longer be a probability measure. This is of no importance since, as will be shown later, the defect $n - \deg q_n$ is uniformly bounded.

The proofs of Theorems 3.1 and 3.2 rely on several auxiliary lemmas. To state them we need to introduce some more notation. For any $\xi \neq 0 \in \mathbb{C}$, we let $\mathrm{Arg}(\xi) \in (-\pi, \pi]$ be the principal branch of the argument and for $\xi = 0$ we set $\mathrm{Arg}(0) = \pi$. With this definition, $\mathrm{Arg}(\cdot)$ becomes a left continuous function on $\mathbb{R}$. Now, for any interval $[a, b] \subset \mathbb{R}$ we can define the angle in which this interval is seen at $\xi \in \mathbb{C}$ by

$$\mathrm{Angle}(\xi, [a, b]) := |\mathrm{Arg}(a - \xi) - \mathrm{Arg}(b - \xi)|.$$

Note that $0 \leq \mathrm{Angle}(\xi, [a, b]) \leq \pi$ and $\mathrm{Angle}(\xi, [a, b]) = \pi$ if and only if $\xi \in [a, b]$. We define additively this angle for a system of disjoint closed intervals: if $\{[a_j, b_j]\}_{j=1}^k$ is such a system, then the angle in which it is seen at ξ is defined by

$$\theta(\xi) := \sum_{j=1}^{k} \mathrm{Angle}(\xi, [a_j, b_j]). \tag{3.5}$$

The notation $\theta(\xi)$ does not reflect the dependency on the system of intervals, but the latter will always be made clear.

Lemma 3.3 *With the previous notation the following statements hold true:*

(a) Let ν be a positive measure which has infinitely many points in its support and assume the

latter is covered by finitely many disjoint intervals: $\operatorname{supp}(\nu) \subseteq \cup_{j=1}^{k}[a_j, b_j]$. *Let further* ψ *be a function of bounded variation on* $\operatorname{supp}(\nu)$. *If for some integer* l *we have*

$$\int P_{l-1}(t)e^{i\psi(t)}d\nu(t) = 0, \qquad \forall\, P_{l-1} \in \mathrm{P}_{l-1},$$

then

$$\sum_{j=1}^{k} V(\psi, [a_j, b_j]) \geq (l - k + 1)\pi. \tag{3.6}$$

(b) *Let* $[a, b] \subset (-1, 1)$ *and* $\xi \in \mathbb{D}$. *Define*

$$g(\xi, t) := \operatorname{Arg}(t - \xi) - 2\operatorname{Arg}(t - 1/\bar{\xi}), \tag{3.7}$$

where the term $2\operatorname{Arg}(t - 1/\bar{\xi})$ *is omitted if* $\xi = 0$. *Then*

$$V(g(\xi, \cdot), [a, b]) \leq \operatorname{Angle}(\xi, [a, b]). \tag{3.8}$$

(c) *Let* ψ *be a real function of bounded variation on an interval* $[a, b]$, $\{a_n(x)\}$ *a sequence of continuously differentiable real functions with uniformly bounded derivatives on* $[a, b]$, *and* q *a polynomial. Then there exists a polynomial* $T \neq 0$ *and a constant* $\beta \in (0, \pi/32)$ *such that*

$$\left|\operatorname{Arg}\left(e^{i(\psi(x)+a_n(x))}q(x)T(x)\right)\right| \leq \pi/2 - 2\beta \tag{3.9}$$

for all $x \in [a, b]$ *such that* $T(x)q(x) \neq 0$ *and all* n *from some infinite sequence* $\mathbb{N}_1 \subset \mathbb{N}$.

(d) *Assume* $I \subset (0, 1)$ *and* $\{q_n\}$ *is a sequence of polynomials of degree* m_n *whose roots* $\{\xi_{1,n}, \ldots, \xi_{m_n,n}\}$ *lie in* $\mathbb{D}$ *and satisfy*

$$\sum_{j=1}^{m_n}(\pi - \operatorname{Angle}(\xi_{j,n}, I)) \leq C,$$

where the constant C *is independent of* n. *Then, to every* $\epsilon > 0$ *there exists an integer* l *such that, for all* n *large enough, there is a polynomial* $T_{l,n}$ *of degree at most* l *satisfying:*

$$\left|\frac{\widetilde{q}_n(x)}{|\widetilde{q}_n(x)|} - T_{l,n}(x)\right| < \epsilon, \quad x \in I.$$

In particular, the argument of $T_{l,n}(x)/\widetilde{q}_n(x)$ *lies in the interval* $(-2\epsilon, 2\epsilon)$ *when* n *is large enough.*

Proof: (a) This assertion follows from the proof of Lemma 3.2 in [19] if we put $d_n = n$ there.

(b) When $\xi \notin I$, the proof of this statement is contained in that of Lemma 5.2 in [19]. In the remaining cases, one can see by inspection that (3.8) reduces to $0 \leq \pi$ when $\xi = b$ and to $\pi \leq \pi$ when $\xi \in [a, b)$.

(c) Observe that $\varphi(x) = \psi(x) + \mathrm{Arg}(q(x))$ is a real function of bounded variation on I. Therefore by Lemma 3.4 in [19], there exist a polynomial $T^* \neq 0$ and a constant $\beta^* \in (0, \pi/16)$ such that

$$\left|\mathrm{Arg}\left(e^{i\psi(x)}q(x)T^*(x)\right)\right| = \left|\mathrm{Arg}\left(e^{i\varphi(x)}T^*(x)\right)\right| \leq \pi/2 - 2\beta^* \quad \text{for} \quad x \in I, q(x)T^*(x) \neq 0.$$

For later use we also record that, by the very construction of T^* in the cited lemma, its zeros belong to I and are discontinuity points of φ.

Let N be such that $|a_n'(x)| \leq N$ for all $n \in \mathbb{N}$ and $x \in I$, where the superscript "prime" indicates the derivative. By Jackson's theorem (cf. *e.g.* [101]) there is a constant $C > 0$ and there are polynomials $\{T_{n,l}\}$ of degree at most l such that

$$\left|e^{-ia_n(x)} - T_{n,l}(x)\right| \leq \frac{CN}{l}.$$

Fix l so large that $CN/l \leq \beta^*/3$. Being bounded of bounded degree, the sequence $\{T_{n,l}\}$ has a subsequence converging uniformly on I to a polynomial T_l of degree at most l. Therefore, for some subsequence $\mathbb{N}_1$ we obtain

$$\left|1 - e^{ia_n(x)}T_l(x)\right| \leq \frac{\beta^*}{2}, \quad n \in \mathbb{N}_1,$$

which implies that

$$\left|\mathrm{Arg}\left(e^{ia_n(x)}T_l(x)\right)\right| \leq \frac{\beta^*\pi}{4} < \beta^*, \quad n \in \mathbb{N}_1.$$

Now inequality (3.9) follows by taking $T = T^*T_l$, $\beta = \beta^*/2$, and using that $|\mathrm{Arg}(\xi_1 + \xi_2)| \leq |\mathrm{Arg}(\xi_1)| + |\mathrm{Arg}(\xi_2)|$ for any $\xi_1, \xi_2 \in \mathbb{C}$.

(d) This is exactly what is proved in Lemma 5.4 of [19]. ■

Lemma 3.4 *Let* $q_n(z) = \prod_{j=1}^n (z - \xi_{j,n})$ *satisfy (3.3) for* $|\xi_{j,n}| < 1$ *for* $j = 1, \ldots, n$, *where* $\widetilde{\mu} = \mu + \mu'$ *is given by (1.15) and (1.16) with* $S \subset (-1, 1)$ *and* $S' \subset \mathbb{D}$, *while* ω_n *is a complex-valued measurable function on* $\widetilde{S}$. *Consider a covering of* S *by finitely many disjoint closed intervals:* $S \subseteq E_k :=$

$\bigcup_{j=1}^{k}[a_j, b_j]$. *Then*[9]

$$\sum_{j=1}^{n}(\pi - \theta(\xi_{j,n})) \le V(\Theta) + V(\arg(\omega_n), S) + \sum_{\eta\in S'} m(\eta)\theta(\eta) + (k+m-1)\pi, \tag{3.10}$$

where $\arg(\omega_n)$ *is any argument function for* ω_n *on* S *and* $m(\eta)$ *is the multiplicity of* η.

Proof: If ω_n has no argument function of bounded variation on S, there is nothing to prove. Otherwise, we pick one and extend it to the whole of $\mathbb{R}$ without increasing the variation, as explained in the beginning of this section. In particular, we get

$$V(\arg(\omega_n), S) = \sum_{j=1}^{m} V(\arg(\omega_n), [a_j, b_j]).$$

As in the case of (2.33), equation (3.3) yields

$$\int P_{n-m-1}(t) q_n(t) Q(t) \frac{\omega_n(t)}{\widetilde{q}_n^2(t)} d\mu(t) = 0, \tag{3.11}$$

where P_{n-m-1} is any polynomial in P_{n-m-1}.

Denote by $\psi_n(t)$ an argument function for $e^{i\Theta(t)} q_n(t) Q(t) \omega_n(t)/\widetilde{q}_n^2(t)$, say

$$\psi_n(t) = \Theta(t) + \arg(\omega_n(t)) + \sum_{\eta\in S'} \mathrm{Arg}(t-\eta) + \sum_{i=1}^{n}\left(\mathrm{Arg}(t-\xi_{i,n}) - 2\mathrm{Arg}(t-1/\bar{\xi}_{i,n})\right),$$

where it is understood that $\mathrm{Arg}(t-1/\bar{\xi}_{i,n})$ is omitted when $\xi_{i,n} = 0$. It is easy to see that ψ_n is of bounded variation. Then Lemma 3.3(a) with

$$\psi = \psi_n, \quad d\nu(t) = \left|\frac{q_n(t)Q(t)\omega_n(t)}{\widetilde{q}_n^2(t)}\right| d|\mu|(t), \quad \text{and} \quad l = n-m$$

implies that

$$\sum_{j=1}^{k} V(\psi_n, [a_j, b_j]) \ge (n-k-m+1)\pi.$$

So, we are left to show that

$$\sum_{j=1}^{k} V(\psi_n, [a_j, b_j]) \le V(\Theta) + \sum_{j=1}^{k} V(\arg(\omega_n), [a_j, b_j]) + \sum_{\eta\in S'} m(\eta)\theta(\eta) + \sum_{i=1}^{n}\theta(\xi_{i,n}).$$

[9] Note that the hypothesis $\mu \in \mathbf{BVT}$ is not required for this lemma to hold.

By the definition of ψ_n, we have

$$\begin{aligned}\sum_{j=1}^{k} V(\psi_n, [a_j, b_j]) \quad &\leq \quad \sum_{j=1}^{k} V(\Theta, [a_j, b_j]) + \sum_{j=1}^{k} V(\arg(\omega_n), [a_j, b_j]) \\ &+ \quad \sum_{j=1}^{k} \sum_{\eta \in S'} m(\eta) V(\mathrm{Arg}(\cdot - \eta), [a_j, b_j]) \\ &+ \quad \sum_{j=1}^{k} \sum_{i=1}^{n} V(g(\xi_{i,n}, \cdot), [a_j, b_j]),\end{aligned}$$

where $g(\xi, t)$ was defined in (3.7). The assertion of the lemma now follows from Lemma 3.3(b) and the fact that by the monotonicity of $V(\mathrm{Arg}(\cdot - \xi), [a, b]) = \mathrm{Angle}(\xi, [a, b])$. ■

Corollary 3.5 *Let* Θ *and* $\arg(\omega_n)$ *have bounded variation on* S*, with* $V(\arg(\omega_n), S) < C$ *where* C *is independent on* n*. Then, to each neighborhood* U *of* S*, there exists a constant* $N_U \in \mathbb{N}$ *such that each* q_n *has at most* N_U *zeros outside of* U *for* n *large enough.*

Proof: Since U is open, its intersection with $(-1, 1)$ is a countable union of intervals. By compactness, a finite number of them will cover S, say $\cup_{j=1}^{k}(a_j, b_j)$. Apply Lemma 3.4 to the closure of these intervals and observe that any zero of q_n which lies outside of U will contribute to the left-hand side of (3.10) by more than some positive fixed constant which depends only on U. Since the right-hand side of (3.10) does not depend on n and is finite we can have only finitely many such zeros. ■

Proof of Theorem 3.1: Hereafter we are going to use (3.11) rather than (3.3) and we set $q_n(z) = \prod_{j=1}^{n}(z - \xi_{j,n})$.

We start by observing that we may suppose $S \subset (0, 1)$. Indeed, if this is not the case, take a negative number w such that $-1 < w < a$, where $[a, b] = E$ denotes the convex hull of S. Then $M_w(S)$, the image of S under the Möbius transformation $M_w(z) := (z - w)/(1 - zw)$, is a subset of $(0, 1)$. Moreover, observe that the Green equilibrium measure is invariant under Möbius transformations, i.e., for any Borel set $E \subset M_w(S)$ we have that

$$\mu_{(M_w(S),\mathbb{T})}(E) = \mu_{(S,\mathbb{T})}(M_{-w}(E))$$

(M_{-w} is the inverse function of M_w). This implies that the weak* convergence of ν_n to $\mu_{(S,\mathbb{T})}$ is equivalent to that of ν_n^w to $\mu_{(M_w(S),\mathbb{T})}$, where ν_n^w is the counting measure of the images of the zeros of q_n under M_w. Now, if we let

$$\begin{aligned}\ell_n(\tau) &= q_n(M_{-w}(\tau))(1+w\tau)^n,\\ L(\tau) &= Q(M_{-w}(\tau))(1+w\tau)^m,\\ p_{n-m-1}(\tau) &= P_{n-m-1}(M_{-w}(\tau))(1+w\tau)^{n-m-1},\\ \omega_n^*(\tau) &= \omega_n(M_{-w}(\tau)),\end{aligned}$$

then ℓ_n is a polynomial of degree n with zeros at $M_w(\xi_{j,n})$, $j=1,\dots,n$. In addition, since $M_x(1/\bar{\xi}) = 1/\overline{M_x(\xi)}$, $x\in(-1,1)$, we have that

$$\widetilde{\ell}_n(\tau) = \widetilde{q}_n(M_{-w}(\tau))(1+w\tau)^n.$$

Analogously, L is a polynomial of degree m with zeros at $M_w(\eta)$, $\eta\in S'$, and p_{n-m-1} is an arbitrary polynomial of degree at most $n-m-1$. Let us show that ℓ_n satisfies orthogonality relations of type (3.11) for a new measure, supported this time in $(0,1)$, that still belongs to **BVT**.

With the above notation equation (3.11) implies

$$\begin{aligned}0 &= \int_S P_{n-m-1}(t)q_n(t)Q(t)\frac{\omega_n(t)}{\widetilde{q}_n^2(t)}e^{i\Theta(t)}d|\mu|(t)\\ &= \int_{M_w(S)}(P_{n-m-1}q_nQ)(M_{-w}(\tau))\frac{\omega_n(M_{-w}(\tau))}{\widetilde{q}_n^2(M_{-w}(\tau))}e^{i\Theta(M_{-w}(\tau))}d|\mu|(M_{-w}(\tau))\\ &= \int_{M_w(S)}p_{n-m-1}(\tau)L(\tau)\ell_n(\tau)\frac{\omega_n^*(\tau)}{\widetilde{\ell}_n^2(\tau)}e^{i\Theta^*(\tau)}d|\mu^*|(\tau),\end{aligned}$$

where $d|\mu^*|(\tau) = (1+w\tau)d|\mu|(M_{-w}(\tau))$ is a positive measure supported on $M_w(S)$, $\Theta^*(\tau) = \Theta(M_{-w}(\tau))$ is a function of bounded variation, and $\{\omega_n^*\}$ is a sequence of measurable functions whose moduli are uniformly bounded above and below, and whose arguments are smooth with uniformly bounded derivatives. Further, since Green functions are conformally invariant, $M_w(S)$ is regular so clearly $\mu^*\in$ **BVT**. This allows us to assume that $S\subset(0,1)$.

First we will suppose that *all zeros of the polynomials q_n lie outside some fixed neighborhood of zero.*

For each n denote by σ_n the counting measure of zeros of $\widetilde{q}_n$. By the assumption that we just made there exists a compact set K such that $0\notin K$ and $\text{supp}(\nu_n)\subset K$. Then $\text{supp}(\sigma_n)\subset K^*$ for all $n\in\mathbb{N}$, and K^* is also compact. Recall that operation $(\cdot)^*$, applied to a set, stands for the reflect of this set across the unit circle. By Helly's selection theorem (cf. [113, Thm. 0.1.3]) there exists a subsequence of natural numbers, $\mathbb{N}_1$, such that $\nu_n \stackrel{*}{\to} \nu$ for $n \in \mathbb{N}_1$, where $\stackrel{*}{\to}$ stands for weak* convergence. Denote by σ the reflection of ν across the unit circle, i.e. $d\sigma(t) = d\nu \circ (1/\bar{t})$. It is easy to check that $\sigma_n \stackrel{*}{\to} \sigma$. Observing that the assumptions on ω_n imply that the variation of its argument on E, thus *a fortiori* on S, is bounded independently of n, it follows from Corollary 3.5 that ν and σ are probability measures such that $\mathrm{supp}(\nu) \subset S \subset (0,1)$ and $\mathrm{supp}(\sigma) \subset S^* = S^{-1}$.

Claim: it is enough to show that the logarithmic potential of $\nu - \sigma$ is a constant q.e. on S. Indeed, let $U^{\nu-\sigma} = D_1$ q.e. on S, where

$$U^{\nu-\sigma} = \int \log \frac{1}{|z-t|} d(\nu - \sigma)(t).$$

Then, since U^σ is harmonic outside of S^{-1}, we have that U^ν is bounded quasi everywhere on S, hence everywhere by lower semi-continuity of potentials. Thus, ν has finite energy and by reflection so does σ. Moreover, for quasi every $z \in S^{-1}$, we have

$$\begin{aligned} U^{\nu-\sigma}(z) &= \int \log \frac{1}{|z-t|} d(\nu-\sigma)(t) = \int \log \frac{1}{|z-1/x|} d\left[(\nu-\sigma)\circ(1/x)\right] \\ &= \int \log \left| \frac{x/z}{x-1/z} \right| d(\sigma-\nu)(x) = \int \log|x/z| d(\sigma-\nu)(x) - U^{\nu-\sigma}(1/z) \\ &= \int \log|x| d(\sigma-\nu)(x) - D_1 =: D_2, \end{aligned} \tag{3.12}$$

where we used that $\nu - \sigma$ has total mass zero. Now, denote by $\widehat{\sigma}$ the balayage of σ onto S. Then

$$U^{\widehat{\sigma}}(t) = U^\sigma(t) + c(\sigma; \mathbb{C} \setminus S)$$

for quasi every t on S. Thus, as $(\nu - \widehat{\sigma})(\mathbb{C}) = 0$ and since ν and $\widehat{\sigma}$ have finite energy, we get

$$\begin{aligned} 0 &= \int D_1\, d(\nu-\widehat{\sigma})(z) = \int U^{\nu-\sigma}(z) d(\nu-\widehat{\sigma})(z) = \int U^{\nu-\widehat{\sigma}}(z) d(\nu-\widehat{\sigma})(z) \\ &= \int \log\frac{1}{|z-t|} d(\nu-\widehat{\sigma})(t) d(\nu-\widehat{\sigma})(z). \end{aligned}$$

But the energy of a signed measure is equal to zero if and only if the measure is zero ([113, Lemma I.1.8]), provided that this measure is the difference of two positive measures with finite energy; thus, $\nu = \widehat{\sigma}$. Using (3.12), we can obtain in a similar fashion that $\sigma = \widehat{\nu}$, where $\widehat{\nu}$ is the balayage of ν onto S^{-1}. Hence, we proved that $\sigma - \nu$ is the equilibrium signed measure for the condenser (S, S^{-1}) ([113, Thm. VIII.2.6]). Then Proposition A.1 points (b) and (e) ensures that $\nu = \mu_{(S,\mathbb{T})}$ is the Green equilibrium distribution relative to both $\mathbb{D}$ and $\mathbb{C} \setminus S^{-1}$. Since $\{\nu_n\}_{n\in\mathbb{N}_1}$ was an arbitrary subsequence, the whole sequence $\{\nu_n\}_{n\in\mathbb{N}}$ converges to $\mu_{(S,\mathbb{T})}$ in the weak* sense. *This proves the claim.*

Being left to prove that $U^{\nu-\sigma}$ is a constant q.e. on S, suppose to the contrary that this is not true. Then there exist nonpolar Borel subsets of S, say E_- and E_+, and two constants d and $\tau > 0$ such that

$$U^{\nu-\sigma}(x) \geq d + \tau, \quad x \in E_+, \quad U^{\nu-\sigma}(x) \leq d - 2\tau, \quad x \in E_-.$$

In this case we claim that there exists $y_0 \in \mathrm{supp}(\nu)$ such that

$$U^{\nu-\sigma}(y_0) > d. \tag{3.13}$$

Indeed, otherwise we would have that

$$U^{\nu}(x) \leq U^{\sigma}(x) + d, \quad x \in \mathrm{supp}(\nu). \tag{3.14}$$

Then the principle of domination ([113, Thm. II.3.2]) would yield that (3.14) is true for all $z \in \mathbb{C}$, but this would contradict the existence of E_+.

Since all σ_n are supported outside of the closed unit disk, the sequence of potentials $\{U^{\sigma_n}\}_{n\in\mathbb{N}_1}$ converges to U^σ uniformly on compact subsets of $\mathbb{D}$. This implies that for any given sequence of points $\{y_n\} \subset \mathbb{D}$ such that $y_n \to y_0$ as $n \to \infty$, $n \in \mathbb{N}_1$, we have

$$\lim_{n\to\infty,\ n\in\mathbb{N}_1} U^{\sigma_n}(y_n) = U^{\sigma}(y_0). \tag{3.15}$$

On the other hand all ν_n, $n \in \mathbb{N}_1$, have their support in $\mathbb{D}$. So, by applying the principle of descent for the above sequence $\{y_n\}$, we obtain

$$\liminf_{n\to\infty,\ n\in\mathbb{N}_1} U^{\nu_n}(y_n) \geq U^{\nu}(y_0). \tag{3.16}$$

Combining (3.13), (3.15), and (3.16) we get

$$\liminf_{n\to\infty\ n\in\mathbb{N}_1} U^{\nu_n-\sigma_n}(y_n) \geq U^{\nu-\sigma}(y_0) > d. \tag{3.17}$$

Since $\{y_n\}$ was an arbitrary sequence in $\mathbb{D}$ converging to y_0, we deduce from (3.17) that there exists $\rho > 0$ such that, for any $y \in [y_0 - 2\rho, y_0 + 2\rho]$ and $n \in \mathbb{N}_1$ large enough, the following inequality holds

$$U^{\nu_n-\sigma_n}(y) \geq d. \tag{3.18}$$

Since

$$U^{\nu_n-\sigma_n}(y) = \frac{1}{n}\log\left|\frac{1}{\alpha_n}\frac{\widetilde{q}_n(y)}{q_n(y)}\right|,$$

where $\alpha_n := \prod_{j=1}^n |\xi_{j,n}|$, inequality (3.18) can be rewritten as

$$\left|\alpha_n\frac{q_n(y)}{\widetilde{q}_n(y)}\right| \leq e^{-nd} \tag{3.19}$$

for any $y \in [y_0 - 2\rho, y_0 + 2\rho]$ and $n \in \mathbb{N}_1$ large enough.

Here we notice for later use that the above reasoning does not really require the polynomials to have exact degree n. Specifically, let $\{p_n\}$ be a sequence of monic polynomials of degree $d_n = n + o(n)$ where o denotes the Landau symbol "little oh". Moreover, suppose that the counting measures of their zeros normalized by $1/n$ rather than $1/d_n$ (so it may no longer be a probability measure) are supported on a fixed compact set of the complex plane. Call μ_n these measures and assume that they converge to ν in the weak* topology. In this case (3.16) and (3.17) still hold with ν_n replaced by μ_n, at the cost perhaps of dropping finitely many terms of $\mathbb{N}_1$ and making ρ smaller. Thus, we obtain that

$$\left|\alpha_n\frac{p_n(y)}{\widetilde{q}_n(y)}\right| \leq e^{-nd} \tag{3.20}$$

for any $y \in [y_0 - 2\rho, y_0 + 2\rho]$ and $n \in \mathbb{N}_1$ large enough.

In another connection, since $U^{\nu-\sigma}(x) \leq d - 2\tau$ on E_-, applying the lower envelope theorem gives us

$$\liminf_{n\to\infty,\, n\in\mathbb{N}_1} U^{\nu_n-\sigma_n}(x) = U^{\nu-\sigma}(x) \leq d - 2\tau, \quad \text{for q.e.} \quad x \in E_-. \tag{3.21}$$

Let Z be a finite system of points from $(-1, 1)$, to be specified later. Then by [3, 4] there exists $S_0 \subset S$ such that S_0 is regular, $\mathrm{cap}(E_- \cap S_0) > 0$ and $\mathrm{dist}(Z, S_0) > 0$, where $\mathrm{dist}(Z, S_0) := \min_{z\in Z} \mathrm{dist}(z, S_0)$. Put for simplicity $b_n(x) = q_n(x)/\widetilde{q}_n(x)$, which is a finite Blaschke product. Then

$$U^{\nu_n-\sigma_n}(x) = -\frac{1}{n}\log|\alpha_n b_n(x)|$$

and by (3.21), as $|\alpha_n| < 1$, there exist $\mathbb{N}_2 \subset \mathbb{N}_1$ and $x \in E_- \cap S_0$ such that

$$|b_n(x)| \geq |\alpha_n b_n(x)| \geq e^{-n(d-\tau)}$$

for any $n \in \mathbb{N}_2$. Let x_n be a point where $|b_n|$ reaches its maximum on S_0. Then

$$M_n := \max_{x \in S_0} |b_n(x)| = |b_n(x_n)| \geq |\alpha_n| M_n = |\alpha_n b_n(x_n)| \geq e^{-n(d-\tau)}. \tag{3.22}$$

Note that $M_n < 1$ and therefore $d - \tau$ is necessarily positive. For simplicity, we shall denote $\mathbb{D} \setminus S_0$ by D. Since the modulus of a Blaschke product is bounded by 1 in the unit disk and $\log |b_n|$ is a subharmonic function, the two-constant theorem ([111, Thm. 4.3.7]) yields the following pointwise estimate

$$\log |b_n(z)| \leq \omega_D(z, S_0) \log |M_n| \tag{3.23}$$

for any $z \in D$, where $\omega_D(z, S_0)$ is the harmonic measure on D (cf. [111, Sec. 4.3]). Combining the last two inequalities we get

$$|b_n(z)| \leq (M_n)^{\omega_D(z,S_0)} = M_n \left(\frac{1}{M_n}\right)^{1-\omega_D(z,S_0)} \leq M_n e^{n(d-\tau)(1-\omega_D(z,S_0))} \tag{3.24}$$

for $z \in D$, and obviously also when $z \in S_0$, where $\omega_D(\cdot, S_0) = 1$ for S_0 is regular. Moreover, by the regularity of S_0 again, it is known ([111, Thm. 4.3.4]) that for any $x \in S_0$

$$\lim_{z \to x} \omega_D(z, S_0) = 1$$

uniformly with respect to $x \in S_0$. Thus, for any $\delta > 0$ there exists $r(\delta) < \mathrm{dist}(S_0, \mathbb{T})$ such that for z satisfying $\mathrm{dist}(z, S_0) \leq r(\delta)$ we have

$$1 - \omega_D(z, S_0) \leq \frac{\delta}{d - \tau}.$$

This, together with (3.24), implies that for fixed δ, to be adjusted later, we have

$$|b_n(z)| \leq M_n e^{n\delta}, \quad |x_n - z| \leq r(\delta).$$

Note that b_n is analytic in $\mathbb{D}$, which, in particular, yields for $|z - x_n| < r(\delta)$

$$b_n'(z) = \frac{1}{2\pi i} \int_{|\xi - x_n| = r(\delta)} \frac{b_n(\xi)}{(\xi - z)^2} d\xi.$$

Thus, for any z such that $|z - x_n| \leq r(\delta)/2$ we get

$$|b_n'(z)| \leq \frac{1}{2\pi} \cdot \frac{4M_n e^{n\delta}}{r^2(\delta)} \cdot 2\pi r(\delta) = \frac{4M_n e^{n\delta}}{r(\delta)}.$$

Now, for any x such that

$$|x - x_n| \leq \frac{r(\delta)}{8e^{n\delta}} \tag{3.25}$$

the mean value theorem yields

$$|b_n(x) - b_n(x_n)| \leq \frac{4M_n e^{n\delta}}{r(\delta)} |x - x_n| = \frac{M_n}{2}.$$

Thus, for x satisfying (3.25) we have

$$|b_n(x)| \geq |b_n(x_n)| - |b_n(x) - b_n(x_n)| \geq M_n - \frac{M_n}{2} = \frac{M_n}{2}$$

and by (3.22)

$$|\alpha_n b_n(x)| \geq \frac{|\alpha_n| M_n}{2} \geq \frac{1}{2} e^{-n(d-\tau)}. \tag{3.26}$$

The estimates (3.20), (3.26), together with the relation (3.11) are the main ingredients in proving the claim that $U^{\nu-\sigma}$ is constant q.e. on S. To combine them we shall use a specific choice of P_{n-m-1} in (3.11).

First, we pick a polynomial T such that Lemma 3.3-(c) holds with $\psi = \Theta$, $a_n = \mathrm{Arg}(\omega_n)$, $q = Q$, and $[a, b] = E$ for $n \in \mathbb{N}_3 \subset \mathbb{N}_2$. We denote by k the degree of T. Second, for each $n \in \mathbb{N}_3$, we choose $T_{l,n}$ as in Lemma 3.3-(d) with $\epsilon = \delta/9$. Note that the use of Lemma 3.3-(d) is legitimate by Lemma 3.4 and our assumptions on S, φ and ω_n. Since all $T_{l,k}$ are bounded and have degree at most l, which does not depend on n, there exists $\mathbb{N}_4 \subset \mathbb{N}_3$ such that sequence $\{T_{l,n}\}_{n\in\mathbb{N}_4}$ converges uniformly to some polynomial T_l on E. In particular, we have that $\deg(T_l) \leq l$ and the argument of $T_l(x)/\widetilde{q}_n(x)$ lies in $(-\delta/4, \delta/4)$ for $n \in \mathbb{N}_4$ large enough. Denote by 2α the smallest even integer strictly greater than $2l + k + m$. As soon as n is large enough, since $y_0 \in \mathrm{supp}(\nu)$, there exist $\beta_{1,n}, \ldots, \beta_{2\alpha,n}$, zeros of q_n, lying in

$$O_\gamma([y_0 - \rho, y_0 + \rho]) := \{z \in \mathbb{C} : \ \mathrm{dist}(z, [y_0 - \rho, y_0 + \rho]) \leq \gamma/2\},$$

where γ, $0 < \gamma < \rho/2$, will be specified later. Define

$$P_n^*(z) = \frac{\overline{q_n(\overline{z})}T(z)T_l^2(z)}{\prod_{j=1}^{2\alpha}(z - \overline{\beta}_{j,n})}.$$

The polynomial P_n^* has degree $n - m - 1$ or $n - m - 2$, depending on the parity of $k + m$.

Denote by $I_n \subset (0, 1)$ the interval defined by (3.25). By comparing (3.19) with (3.26) and (3.22), it is clear that I_n and $[y_0 - 2\rho, y_0 + 2\rho]$ are disjoint when $n \in \mathbb{N}_4$ is large enough.

Now, we choose $\gamma = \gamma(\rho)$ so small that

$$\left|\sum_{j=1}^{2\alpha} \operatorname{Arg}\left(\frac{1}{x - \overline{\beta}_{j,n}}\right)\right| = \left|\sum_{j=1}^{2\alpha} \operatorname{Arg}\left(x - \overline{\beta}_{j,n}\right)\right| \leq \delta/2, \quad x \in \mathbb{R} \setminus [y_0 - 2\rho, y_0 + 2\rho].$$

Letting δ be such that $\delta < \beta$, the choices of T, T_l, and P_n^* together imply

$$\begin{aligned} & \left|\operatorname{Arg}\left(P_n^*(x)q_n(x)Q(x)\frac{\omega_n(x)}{\widetilde{q}_n^2(x)}e^{i\Theta(x)}\right)\right| \\ = & \left|\operatorname{Arg}\left(|q_n(x)|^2 \cdot \prod_{j=1}^{2\alpha} \frac{1}{(x - \overline{\beta}_{j,n})} \cdot \frac{T_l^2(x)}{\widetilde{q}_n^2(x)} \cdot T(x)Q(x)\omega_n(x)e^{i\Theta(x)}\right)\right| \\ \leq & \frac{\delta}{2} + \frac{\delta}{2} + \frac{\pi}{2} - 2\beta \leq \pi/2 - \delta, \end{aligned}$$

for $x \in E \setminus [y_0 - 2\rho, y_0 + 2\rho]$ except at points where T or q are equal to zero. This means that for such x

$$\begin{aligned} \operatorname{Re}\left(|\alpha_n|^2 P_n^*(x)q_n(x)Q(x)\frac{\omega_n(x)}{\widetilde{q}_n^2(x)}e^{i\Theta(x)}\right) &\geq \sin\delta \left|\alpha_n^2 P_n^*(x)q_n(x)Q(x)\frac{\omega_n(x)}{\widetilde{q}_n^2(x)}e^{i\Theta(x)}\right| \\ &\geq \sin\delta \left|\frac{\alpha_n^2 b_n^2(x)Q(x)T(x)T_l^2(x)\omega_n(x)}{\prod_{j=1}^{2\alpha}(x - \overline{\beta}_{j,n})}\right| \geq 0. \end{aligned}$$

Moreover, if $x \in S \setminus [y_0 - 2\rho, y_0 + 2\rho]$ satisfies (3.25), then by (3.22) and (3.26) the above quantity is bounded from below by

$$|T(x)Q(x)|\frac{\sin\delta\ \min_{x\in[a,b]}|T_l(x)|^2\ \inf_{n\in\mathbb{N}}\min_{x\in[a,b]}|\omega_n(x)|}{(\operatorname{diam}(S) + 2\rho)^{2\alpha}}e^{-2nd+2n\tau} = c_1|T(x)Q(x)|e^{-2nd+2n\tau},$$

where $\operatorname{diam}(S) := \max_{x,y\in S}|x - y|$ and

$$c_1 := \frac{\sin\delta\ \min_{x\in[a,b]}|T_l(x)|^2\ \inf_{n\in\mathbb{N}}\min_{x\in[a,b]}|\omega_n(x)|}{(\operatorname{diam}(S) + 2\rho)^{2\alpha}} > 0$$

by construction of T_l and the uniform boundedness of $\{|\omega_n|\}$ from below. Thus

$$\begin{aligned} & \mathrm{Re}\left(\int_{S\setminus[y_0-2\rho,y_0+2\rho]} |\alpha_n|^2 P_n^*(t) q_n(t) Q(t) \frac{\omega_n(t)}{\widetilde{q}_n^2(t)} e^{i\Theta(t)} d|\mu|(t)\right) \\ \geq \; & \sin\delta \int_{S\setminus[y_0-2\rho,y_0+2\rho]} \left|\alpha_n^2 P_n^*(t) q_n(t) Q(t) \frac{\omega_n(t)}{\widetilde{q}_n^2(t)} e^{i\Theta(t)}\right| d|\mu|(t) \\ \geq \; & c_1 e^{-2nd+2n\tau} \int_{S\cap I_n} |T(t)Q(t)| d|\mu|(t) \geq c_2 e^{-2nd+n(2\tau-L\delta)}. \end{aligned} \tag{3.27}$$

The last inequality is true by the following argument. Recall that x_n, the middle point of I_n, belongs to S_0, where $\mathrm{dist}(S_0, Z) > 0$ and Z is a finite system of points that we now choose to be the zeros of TQ in $(-1,1)$. Then TQ, which is independent of n, is uniformly bounded below on I_n and (3.27) follows from this and hypothesis **BVT** point (2). On the other hand (3.19) and (3.20), applied with $p_n = P_n^*$, yield that

$$\left|\int_{[y_0-2\rho,y_0+2\rho]} |\alpha_n|^2 P_n^*(t) q_n(t) Q(t) \frac{\omega_n(t)}{\widetilde{q}_n^2(t)} e^{i\Theta(t)} d|\mu|(t)\right| \leq c_3 e^{-2nd}, \tag{3.28}$$

where we used uniform boundedness of $\{|\omega_n|\}$ from above. Choosing δ so small that $2\tau - L\delta > 0$, which is possible, the bound in (3.27) becomes bigger than the bound in (3.28) for n large enough. But this is impossible, since by (3.11) the sum of these two integrals must be zero.

We just completed the case when all the zeros of polynomials q_n stay away from the point zero. Now we shall consider the general situation. Let $\epsilon > 0$ be such that $U := \mathbb{D} \setminus \{z : \; |z| \leq \epsilon\}$ is a neighborhood of S. Corollary 3.5 says that there exists a constant $N_U \in \mathbb{N}$ such that each q_n has at most N_U zeros outside of U, or equivalently, zeros which have modulus less than ϵ. In this case, from any sequence of natural numbers, we can extract a subsequence, say $\mathbb{N}_0$, such that for some number $N \leq N_U$, q_n has exactly N zeros outside of U for each $n \in \mathbb{N}_0$. Denote these zeros $\xi_{1,n}, \ldots, \xi_{N,n}$, and consider the polynomials

$$q_n^*(z) := \frac{\widetilde{q}_n(z)}{\prod_{j=1}^N (1 - z\bar{\xi}_{j,n})}, \quad n \in \mathbb{N}_0.$$

Then the sequence $\{q_n\}_{n\in\mathbb{N}_0}$ will satisfy the following weighted orthogonality relation:

$$\int P_{n-m-1}(t) q_n(t) Q(t) \frac{\omega_n^*(t)}{(q_n^*(t))^2} d\mu(t) = 0, \quad P_{n-m-1} \in \mathrm{P}_{n-m-1},$$

where

$$\omega_n^*(t) := \omega_n(t) \prod_{j=1}^{N} (1 - t\overline{\xi}_{j,n})^{-2}, \quad t \in E.$$

In what follows we are going to stress the modifications needed to adapt the previous proof to the present case. Let, as before, $\mathbb{N}_1 \subset \mathbb{N}_0$ be a subsequence of natural numbers such that $\nu_n \xrightarrow{*} \nu$, $n \in \mathbb{N}_1$. Because we only discarded a fixed number of zeros from $\widetilde{q}_n$ to obtain q_n^*, the counting measures σ_n^* of the zeros of q_n^* (normalized by $1/n$), again converge weak* to σ. Since $U^{\sigma_n^*}$ enjoys all the relevant properties of U^{σ_n}, inequalities (3.19) and (3.20) remain valid with $\widetilde{q}_n$ replaced by q_n^*.

Further, define $b_n(x)$ as $q_n(x)/q_n^*(x)$. In this case, b_n is no longer a Blaschke product, but rather a Blaschke product multiplied by the polynomial $\prod_{j=1}^{N}(z - \xi_{j,n})$. Then we get instead of (3.23) that

$$\log|b_n(z)| \leq \omega_D(z, S_0) \log|M_n| + (1 - \omega_D(z, S_0))N \log 2,$$

and (3.24) can be replaced by

$$|b_n(z)| \leq M_n e^{n(d-\tau+N\log 2/n)(1-\omega_D(z,S_0))} \leq M_n e^{n(d-\tau+1)(1-\omega_D(z,S_0))},$$

for n large enough. This yields an insignificant modification of $r(\delta)$ (we should make $1 - \omega_D(z, S_0)$ less than $\delta/(d - \tau + 1)$ rather than just $\delta/(d - \tau)$). Lemma 3.3(d) can be applied to polynomials q_n^* without change.

Therefore we are left to show that $\{\omega_n^*\}$ is uniformly bounded above and below on E, and that its arguments are smooth with uniformly bounded derivatives on E with respect to n. The uniform boundedness of $\{\omega_n^*\}$ easily follows from the estimates

$$\left(\frac{1}{2}\right)^{2N} \leq \prod_{j=1}^{N} \left|\frac{1}{(1 - t\overline{\xi}_{j,n})}\right|^2 \leq \left(\frac{1}{1-\epsilon}\right)^{2N}.$$

Since none of the $\xi_{j,n}$, $j = 1, \ldots, N$ can come close to E^{-1}, $\operatorname{Arg}(1 - t\overline{\xi}_{j,n})$ is a smooth function on E whose derivative $\operatorname{Im}\left(\xi_{j,n}/(1 - t\overline{\xi}_{j,n})\right)$ is uniformly bounded there independently of j and n. Then the rest of the assumptions on $\{\omega_n^*\}$ follows from the representation

$$\arg(\omega_n^*(t)) = \arg(\omega_n(t)) - 2\sum_{j=1}^{N} \operatorname{Arg}(1 - t\overline{\xi}_{j,n}).$$

This completes the proof of the theorem. ■

The proof of Theorem 3.2 runs parallel to the proof of Theorem 3.1. It relies on the following lemma.

Lemma 3.6 *Let* $q_n(z) = \prod_{j=1}^{d_n}(z-\xi_{j,n})$ *be a n-th orthogonal polynomial in the sense of (3.4). Then*

$$\sum_{j=1}^{d_n}(\pi - \theta(\xi_{j,n})) + (n-d_n)\pi \leq V(\Theta) + \sum_{\eta\in S'} m(\eta)\theta(\eta) + (k+m-1)\pi,$$

where $\theta(\cdot)$ *is the angle function defined in (3.5) for a system of intervals* $E_k = \cup_{j=1}^{k}[a_j, b_j]$ *which covers* S*, with* $E = [a_1, b_m]$ *the convex hull of* S*.*

Proof: The proof is the same as that of Lemma 3.4, observing that

$$V(\text{Arg}(v_{2n}), E_k) = 0,$$

since v_{2n} is real on S. ■

Note that Lemma 3.6, applied *e.g.* with $k=1$ and $E_1 = E$, implies that the defect $n - d_n$ is bounded above independently of n.

Corollary 3.7 *Let* U *be a neighborhood of* S*. Then there exists a constant* $N_U \in \mathbb{N}$ *such that each* q_n *has at most* N_U *zeros outside of* U *for* n *large enough.*

Proof of Theorem 3.2: Recall the notation $\mathbf{A} = \{A_{2n}\}$ for the strongly admissible interpolation scheme, and let σ_n be the counting measure of the points in A_{2n}. Set $q_n(z) = \prod_{j=1}^{d_n}(z-\xi_{j,n})$ and recall (see (2.38)) that it satisfies

$$\int_S P_{n-m-1}(t)q_n(t)Q(t)\frac{d\mu(t)}{v_{2n}(t)} = 0, \quad P_{n-m-1} \in \mathrm{P}_{n-m-1} \tag{3.29}$$

(the more general relation (3.4) is not used in this proof).

We claim that we may suppose $\mathbf{A}$ is contained in a compact set. Indeed, if this is not the case, we can pick a real number $x_0 \notin \mathcal{K}(\mathbf{A}) \cup S' \cup E$, where $\mathcal{K}(\mathbf{A})$ is, as usual, the set of limit points of $\mathbf{A}$. Consider the analytic automorphism of $\overline{\mathbb{C}}$ given by $M_{x_0}(z) := 1/(z-x_0)$, with inverse $M_{x_0}^{-1}(\tau) = x_0 + 1/\tau$. If we put $A_{2n}^\sharp := M_{x_0}(A_{2n})$, then $\mathbf{A}^\sharp = \{A_{2n}^\sharp\}$ is a strongly admissible interpolation scheme having asymptotic distribution $\sigma^\sharp$, with $\sigma^\sharp(B) = \sigma(M_{x_0}^{-1}(B))$ for any Borel set $B \subset \mathbb{C}$. Moreover, the choice of x_0 yields that $\mathcal{K}(\mathbf{A}^\sharp)$ is compact. Now, if we let

$$\begin{aligned} \ell_n(\tau) &= \tau^n q_n\left(M_{x_0}^{-1}(\tau)\right), \\ L(\tau) &= \tau^m Q\left(M_{x_0}^{-1}(\tau)\right), \end{aligned}$$

$$
\begin{aligned}
P^{\sharp}_{n-m-1}(\tau) &= \tau^{n-m-1}P_{n-m-1}\left(M^{-1}_{x_0}(\tau)\right), \\
v^{\sharp}_{2n}(\tau) &= \tau^{2n}v_{2n}\left(M^{-1}_{x_0}(\tau)\right),
\end{aligned}
$$

then ℓ_n is a polynomial of degree n with zeros at $M_{x_0}(\xi_{j,n})$, $j = 1, \ldots, d_n$, and a zero at the origin with multiplicity $n - d_n$. In addition, $v^{\sharp}_{2n}$ is a polynomial with a zero at each point of $A^{\sharp}_{2n}$, counting multiplicity. Thus, up to a multiplicative constant, $v^{\sharp}_{2n}$ is the polynomial associated with $A^{\sharp}_{2n}$ *via* (2.37). Analogously, L is a polynomial of degree m with a zero of multiplicity $m(\eta)$ at $M_{x_0}(\eta)$, $\eta \in S'$, and $P^{\sharp}_{n-m-1}$ is an arbitrary polynomial of degree at most $n - m - 1$. Making the substitution $t = M^{-1}_{x_0}(\tau)$ in (3.29), we get

$$
\int_{M_{x_0}(S)} P^{\sharp}_{n-m-1}(\tau)L(\tau)\ell_n(\tau)\frac{d\mu^{\sharp}(\tau)}{v^{\sharp}_{2n}(\tau)} = 0, \quad P^{\sharp}_{n-m-1} \in \mathrm{P}_{n-m-1},
$$

where $d\mu^{\sharp}(\tau) = \tau d\mu\left(M^{-1}_{x_0}(\tau)\right)$ is a complex measure with compact support $M_{x_0}(S) \subset \mathbb{R}$, having an argument of bounded variation and total variation measure $|\mu^{\sharp}| \in \mathbf{BVT}$. Note that τ is bounded away from zero on $\mathrm{supp}(\mu^{\sharp})$, since S is compact and therefore bounded away from infinity. Now, since Lemma 3.6 implies that $n - d_n$ is uniformly bounded above, the asymptotic distribution of the counting measures of zeros of ℓ_n is the same as the asymptotic distribution of the images of the counting measures of zeros of q_n under the map M_{x_0}. As the counting measures of the points in $A^{\sharp}_n$ converge weak$^{\sharp}$ to $\sigma^{\sharp}$, *the claim follows* from the fact that $\mu_w^{M_{x_0}(S)} = \widehat{\sigma^{\sharp}}$ since $\mu_w^S = \widehat{\sigma}$ by Proposition A.2 and since balayage is preserved under M_{x_0} (e.g. because harmonic functions are, *cf.* equation (2.6)). Hence we assume in the rest of the proof that $\mathbf{A}$ is contained in a compact set, say K_0, which is disjoint from $\widetilde{S}$ by the definition of strong admissibility.

Let Γ be a closed Jordan arc such that the bounded component of $\mathbb{C} \setminus \Gamma$, say D, contains $\widetilde{S}$ while the unbounded component, say D', contains K_0. Then $q_n = q_{n,1} \cdot q_{n,2}$, where

$$
q_{n,1}(z) = \prod_{\xi_{j,n} \in D}(z - \xi_{j,n})
$$

and

$$
q_{n,2}(z) = \prod_{\xi_{j,n} \notin D}(z - \xi_{j,n}).
$$

Corollary 3.7 assures that degrees of polynomials $q_{n,2}$ are uniformly bounded with respect to n, therefore the asymptotic distribution of the zeros of $q_{n,1}$ coincides with that of q_n. Denote by $\nu_{n,1}$ the zero counting measure of $q_{n,1}$ normalized with $1/n$. Since all $\nu_{n,1}$ are supported on a fixed compact set, Helly's selection theorem and Lemma 3.6 yield the existence of a subsequence $\mathbb{N}_1$ such that $\nu_{n,1} \xrightarrow{*} \nu$ for $n \in \mathbb{N}_1$ and some measure $\nu \in \Lambda(S)$; remember the defect is bounded which is why ν is a probability measure in spite of the normalization of $q_{n,1}$ with $1/n$.

Next, we observe it is enough to show that the logarithmic potential of $\nu - \sigma$ is constant q.e. on S. Indeed, the same argument as in the proof of Theorem 3.1 shows in this case that ν has finite logarithmic energy, and then we get from the characterization (2.4) of balayage that $\nu = \widehat{\sigma}$ $(= \mu_w^S)$, as desired.

Now, suppose that $U^{\nu-\sigma}$ is not a constant q.e. on S. Then there exist nonpolar Borel subsets of S, say E_- and E_+, and two constants d and $\tau > 0$ such that

$$U^{\nu-\sigma}(x) \geq d + \tau, \quad x \in E_+, \quad U^{\nu-\sigma}(x) \leq d - 2\tau, \quad x \in E_-.$$

Exactly as in Theorem 3.1, this leads to the existence of $y_0 \in \operatorname{supp}(\nu)$ and $\rho > 0$ such that

$$\left| \frac{q_{n,1}^2(y)}{v_{2n}(y)} \right| \leq e^{-2nd},$$

for any $y \in [y_0 - 2\rho, y_0 + 2\rho]$ and $n \in \mathbb{N}_1$ large enough. We also remark that the same bound holds if $\{q_n\}$ is replaced by a sequence of monic polynomials, say $\{u_n\}$, of degree $n + o(n)$, whose counting measures normalized with $1/n$ have asymptotic distribution ν. Moreover, in this case

$$\left| \frac{q_{n,1}(y)u_n(y)}{v_{2n}(y)} \right| \leq e^{-2nd} \tag{3.30}$$

for any $y \in [y_0 - 2\rho, y_0 + 2\rho]$ and $n \in \mathbb{N}_1$ large enough.

In another connection, arguing still as in the proof of Theorem 3.1, there exists $x \in E_- \cap S_0$ (where we will choose S_0 in exactly the same way as in that proof) such that

$$\left| \frac{q_{n,1}^2(x)}{v_{2n}(x)} \right| \geq e^{-2n(d-\tau)} \tag{3.31}$$

for $n \in \mathbb{N}_2 \subset \mathbb{N}_1$. Let x_n be a maximum place for $|q_{n,1}^2/v_{2n}|$ on S_0:

$$M_n := \max_{x \in S_0} \left| \frac{q_{n,1}^2(x)}{v_{2n}(x)} \right| = \left| \frac{q_{n,1}^2(x_n)}{v_{2n}(x_n)} \right|.$$

Since v_{2n} has no zeros on D, the function $\log |q_{n,1}^2/v_{2n}|$ is subharmonic there. Thus, by (3.31) and an obvious majorization, the two-constants theorem on $D \setminus S_0$ (compare (3.24)) yields

$$\left|\frac{q_{n,1}^2(z)}{v_{2n}(z)}\right| \leq M_n \exp\left\{2n\left(d-\tau+\log\frac{\operatorname{diam}(D)}{\operatorname{dist}(\Gamma,K_0)}\right)(1-\omega_{D\setminus S_0}(z,S_0))\right\}, \quad z\in D,$$

where $\omega_{D\setminus S_0}(\cdot,S_0)$ is the harmonic measure on $D\setminus S_0$. Note that $d-\tau+\log(\operatorname{diam}(D)/\operatorname{dist}(\Gamma,K_0))$ is necessarily positive otherwise $q_{n,1}^2/v_{2n}$ would be constant in D by the maximum principle, which is absurd. Then, following the arguments after (3.24), we get from the regularity of S_0 that for any $\delta>0$ there is $r(\delta)<\operatorname{dist}(S_0,\Gamma)$ such that

$$\left|\frac{q_{n,1}^2(x)}{v_{2n}(x)}\right| \geq \frac{1}{2}e^{-2n(d-\tau)} \tag{3.32}$$

for $x\in I_n$, where

$$I_n := \left[x_n-\frac{r(\delta)}{8e^{n\delta}}, x_n+\frac{r(\delta)}{8e^{n\delta}}\right].$$

Now, Lemma 3.3(c) guarantees that there exist a polynomial T of degree, say, k, and a number $\beta\in(0,\pi/32)$ such that

$$\left|\operatorname{Arg}\left(e^{i\Theta(t)}Q(t)T(t)\right)\right| \leq \frac{\pi}{2}-2\beta,$$

for all $t\in E$ such that $(TQ)(t)\neq 0$, where Θ is as in (3.1). Denote by 2α the smallest even integer strictly greater than $k+m+1$. Then, as in the proof of Theorem 3.1, there exist $\beta_{1,n},\dots,\beta_{2\alpha,n}$, zeros of $q_{n,1}$, so close to $[y_0-\rho,y_0+\rho]$ that

$$\left|\sum_{j=1}^{2\alpha}\operatorname{Arg}\left(\frac{1}{x-\bar\beta_{j,n}}\right)\right| = \left|\sum_{j=1}^{2\alpha}\operatorname{Arg}\left(x-\bar\beta_{j,n}\right)\right| \leq \beta, \quad x\in\mathbb{R}\setminus[y_0-2\rho,y_0+2\rho].$$

Define for $n\in\mathbb{N}_2$ sufficiently large

$$P_n^*(z) = \frac{\overline{q_n(\bar z)}T(z)(z-\zeta^*)}{\prod_{j=1}^{2\alpha}(z-\bar\beta_{j,n})},$$

where the factor $(z-\zeta^*)$ is present only if $\operatorname{Arg}(v_{2n}(t))\equiv\pi$ on E in which case ζ^* is a real zero of v_{2n} lying to the right of E (such a ζ^* exists for v_{2n} is positive at infinity and real on $\mathbb{R}$ by the conjugate-symmetry of A_n, while it does not vanish on E when n is large enough since $\mathcal{K}(\mathbf{A})\cap E=\emptyset$

by assumption). Then

$$\left|\mathrm{Arg}\left((P_n^* q_n Q)(x)\frac{e^{i\Theta(x)}}{v_{2n}(x)}\right)\right| = \left|\mathrm{Arg}\left(|q_n(x)|^2 \cdot \prod_{j=1}^{2\alpha} \frac{1}{(x-\bar{\beta}_{j,n})} \cdot \frac{(x-\zeta^*)(TQ)(x)e^{i\Theta(x)}}{v_{2n}(x)}\right)\right|$$
$$\leq \pi/2 - \delta, \tag{3.33}$$

for all $x \in I \setminus [y_0 - 2\rho, y_0 + 2\rho]$ except if $(TQ)(x) = 0$, where δ was chosen so small that $\delta < \beta$ and where $n \in \mathbb{N}_2$ is sufficiently large. Just like in the proof of Theorem 3.1, we fix S_0 to be at positive distance from the zeros of TQ, which is a polynomial independent of n.

Denote by $m_{n,1}$ the number of zeros of $q_{n,2}$ of modulus at least $2\max_{x\in S}|x|$, and put α_n for the inverse of their product. Let $m_{n,2} := \deg q_{n,2} - m_{n,1}$. Then, for $t \in S$, we have

$$(\mathrm{dist}(S,\Gamma))^{m_{n,2}} (1/2)^{m_{n,1}} \leq |\alpha_n q_{n,2}(t)| \leq \left(3\max_{x\in S}|x|\right)^{m_{n,2}} (3/2)^{m_{n,2}},$$

and since $m_{n,1} + m_{n,2} = \deg q_{n,2}$ is uniformly bounded with n so is $\{|\alpha_n q_{n,2}|\}$ bounded above and below on S. Therefore, using (3.33), (3.32), (3.30) with $u_n(z) = P_n^*(z)/\overline{q_{n,2}(\bar{z})}$, and reasoning as in the proof of Theorem 3.1, we obtain

$$\mathrm{Re}\left(\int_{S\setminus[y_0-2\rho,y_0+2\rho]} |\alpha_n|^2 P_n^*(t) q_n(t) Q(t) \frac{e^{i\Theta(t)}}{w_{2n}(t)} d|\mu|(t)\right) \geq c_1 e^{-2nd+n(2\tau-L\delta)} \tag{3.34}$$

and

$$\left|\int_{[y_0-2\rho,y_0+2\rho]} |\alpha_n|^2 P_n^*(t) q_n(t) Q(t) \frac{e^{i\Theta(t)}}{v_{2n}(t)} d|\mu|(t)\right| \leq c_2 e^{-2nd}. \tag{3.35}$$

This completes the proof, since δ can be taken such that $2\tau - L\delta > 0$ which would contradict the orthogonality relation (3.29) because, for n large enough, the integral in (3.34) is much bigger than in (3.35). ∎

Meromorphic Approximation

This section is devoted to the investigation of the asymptotic behavior of meromorphic approximants. We provide answers to the following three questions:

(a) *What is the asymptotic distribution of the poles of best-$L^p(\mathbb{T})$ meromorphic approximants to $\mathcal{F}$ as n tends to ∞?*

(b) *Do some of these poles converge to the polar singularities of $\mathcal{F}$?*

(c) *What can be said about the convergence of such approximants to $\mathcal{F}$?*

Answers to these questions are given by Theorems 3.8, 3.10, and 3.9, respectively.

Theorem 3.8 *Let $p \in [2,\infty]$, p' the conjugate exponent modulo 2, and $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of irreducible critical points of order n of $\mathbf{MA}(p)$ for $\mathcal{F}(\mu;R;\cdot)$, where $\mathcal{F}$ is given by (1.2) with $\mu \in$ **BVT** and $S' \subset \mathbb{D}$. Then the counting measures of the poles of g_n converge to $\mu_{(S,\mathbb{T})}$ in the weak* sense.*

The previous theorem gives one answer to question (a) raised above. The next one addresses question (c) by stating that approximants behave rather nicely toward the approximated function, namely they converge in capacity to $\mathcal{F}$ on $\mathbb{D}\setminus S$ and locally uniformly on $\mathbb{C}\setminus\overline{\mathbb{D}}$. Moreover, n-th root estimates for the error are provided. Recall that a sequence of functions $\{g_n\}$ converges in capacity to a function g on a compact set $K\subset\mathbb{C}$ if, for any $\epsilon>0$,

$$\operatorname{cap}\left(\{z\in K:\ |g_n(z)-g(z)|\geq\epsilon\}\right)\to 0, \quad \text{as} \quad n\to\infty.$$

We say that $\{g_n\}$ converges in capacity to g on a domain D if $\{g_n\}$ converges in capacity on compact subsets of D. Hereafter we denote by $\|.\|_K$ the *sup* norm on a set K.

Theorem 3.9 *Let $\{g_n\}_{n\in\mathbb{N}}$ and $\mathcal{F}(\mu;R;\cdot)$ be as in Theorem 3.8. Then $\{g_n\}$ converges to $\mathcal{F}$ in capacity on $\mathbb{D}\setminus S$; more precisely, for any compact $K\subset\mathbb{D}\setminus S$ there exist positive constants β and γ such that, for n large enough, one has*

$$\operatorname{cap}\left\{z\in K:\ |(\mathcal{F}-g_n)(z)|\geq e^{-\beta n}\right\}\leq e^{-\gamma n}.$$

In addition, in the case of rational approximation (i.e. when $p=2$)

$$\limsup_{n\to\infty}|(\mathcal{F}-g_n)(z)|^{1/2n}\leq\exp\left\{-\frac{1}{\mathrm{C}(S,\mathbb{T})}-U_G^{\mu_{(S,\mathbb{T})}}(1/\bar{z};\mathbb{D})\right\}$$

holds uniformly in $\overline{\mathbb{C}}\setminus\mathbb{D}$. In particular, $\{g_n\}$ converges uniformly in $\overline{\mathbb{C}}\setminus\mathbb{D}$.

The forthcoming theorem implies that each pole of $\mathcal{F}$ attracts at least as many poles of meromorphic approximants as its multiplicity and not much more. This is one possible answer to question (b).

Theorem 3.10 *Let $\{g_n\}_{n\in\mathbb{N}}$, $\mathcal{F}(\mu;R;\cdot)$, and μ be as in Theorem 3.8. Then for each $\eta\in S'\setminus S$*

there exist constants $\delta_\eta > 0$ *and* $C_\eta = C_\eta(\mathcal{F})$ *such that*

$$m(\eta) \leq \#\{S_n \cap B_\delta(\eta)\} \leq m(\eta) + C_\eta, \quad 0 < \delta \leq \delta_\eta, \tag{3.36}$$

for any $n \geq N(\eta, \delta)$*, where* $m(\eta)$ *is the multiplicity of* η*,* S_n *is the set of poles of* g_n*, and* $B_\delta(\eta) := \{|z - \eta| < \delta\}$.

In fact, the proof of the above theorem is constructive in the sense that we obtain an upper bound for C_η. Namely, let $E_k := \{[a_j, b_j]\}_{j=1}^k$ be any finite system of intervals covering S and let w_n be the outer function in the inner-outer factorization of the singular vector v_n associated to g_n for each $n \in \mathbb{N}$. Then

$$C_\eta \leq \frac{V(\Theta) + V_{\mathcal{W}} + (k-1)\pi + 2\sum_{\zeta \in S'} m(\zeta)\theta(\zeta)}{\pi - \theta(\eta)},$$

where

$$V_{\mathcal{W}} := \sup_{n \in \mathbb{N}} V(\mathrm{Arg}(w_n), S) \tag{3.37}$$

and $\theta(\cdot)$ is the angle function defined in (3.5) applied to E_k. We shall prove in due course that indeed $V_{\mathcal{W}} < +\infty$.

The next lemma is needed for the proof of Theorem 3.8. Recall that $\mathcal{H}$, a family of functions analytic in some fixed domain of the complex plane, is called *normal* if each sequence of functions from $\mathcal{H}$ contains a locally uniformly convergent subsequence.

Lemma 3.11 *Let* $p \in (2, \infty]$ *and* $\{g_n\}_{n \in \mathbb{N}}$ *be a sequence of irreducible critical points of order* n *of* $\mathbf{MA}(p)$ *for* $\mathcal{F}(\mu; R; \cdot)$ *given by (1.2), (1.15), and (1.16) with* $\widetilde{S} \subset \mathbb{D}$ *and* μ *satisfying*[10] *(3.1), (3.2). Further, let* v_n *be an associated singular vector to* g_n *with inner-outer factorization given by* $v_n = b_n \cdot w_n$ *for some Blaschke product* b_n *and* w_n *an outer function in* $H^{p'}$*. Then the families* $\mathcal{W} := \{w_n\}$ *and* $\mathcal{W}_{p'} := \left\{w_n^{p'/2}\right\}$ *are normal in* $\mathbb{D}$ *and* $D^*_{\mathcal{F}}$ *respectively, where* $D^*_{\mathcal{F}}$ *is the reflection of* $D_{\mathcal{F}}$ *across* $\mathbb{T}$*. Moreover, any limit point of* $\mathcal{W}$ *is zero free in* $\mathbb{D}$*.*

Proof: The main idea of the proof was given in [28, Thm. 10.1]. The necessary modification for the case of complex measures with argument of bounded variation were given in [23, Prop. 6.3]. Only a simple adjustment is needed in the present case where the approximated function may have polar singularities inside $\mathbb{D}$. Nevertheless, we present the whole proof for the ease of the reader.

[10] Note that we do not require the hypothesis $\mu \in \mathbf{BVT}$ to hold. It is sufficient to have a measure merely with an argument of bounded variation and infinitely many points in the support.

First we establish the normality of $\mathcal{W}_{p'}$. By a classical theorem of Montel it suffices to show that $\mathcal{W}_{p'}$ is locally uniformly bounded in $D^*_{\mathcal{F}}$. *We claim* it is enough to prove that

$$\mathcal{H} := \left\{ \left(j_n w_n^{p'/2} \right)(z) \overline{\left(Q^2 T\right)(1/\bar{z})} \right\}$$

is locally uniformly bounded in $D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}$, where j_n is the finite Blaschke product from (2.31) and T is a fixed polynomial with all zeros in S such that

$$\left| \operatorname{Arg} \left(e^{i\Theta(x)} T(x) \right) \right| \leq \frac{\pi}{2} - \beta, \quad x \in E, \tag{3.38}$$

for some $\beta \in (0, \pi/16)$, where Θ is given by (3.1). The existence of such a polynomial is guaranteed by Lemma 3.3-(c) (with $\psi = \Theta$ and $a_n = 0$) where it should be observed that, in the proof, the polynomial T_l is not needed since $a_n = 0$. Hence, as mentioned in that proof, the zeros of T belong to E and are discontinuity points of Θ; but since the latter was extended linearly on each component of $\mathbb{R} \setminus S$, the zeros of T actually belong to S, as announced. The family $\mathcal{H}$ is well defined, since $j_n w_n^{p'/2}$ extends analytically to $D^*_{\mathcal{F}}$ (see the discussion after equation (2.35)).

Now, the normality of $\mathcal{H}$ in $D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}$ is sufficient to establish the lemma, because $|\overline{(Q^2T)(1/\bar{z})}|$ is bounded away from zero on compact subsets of $D^*_{\mathcal{F}}$, since all zeros of Q^2T lie in $\widetilde{S}$, hence the sequence

$$\left\{ \left(j_n w_n^{p'/2} \right)(z) \right\}$$

will be locally uniformly bounded in $D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}$. Moreover, since j_n is a finite Blaschke product, we have

$$|j_n(z)| > 1, \quad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}},$$

and therefore $\mathcal{W}_{p'}$ will be locally uniformly bounded in $D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}$. But since each $w_n^{p'/2}$ is holomorphic in $D^*_{\mathcal{F}} \supset \overline{\mathbb{D}}$, the maximum principle yields that this family is, in fact, normal in $D^*_{\mathcal{F}}$. *This establishes the claim.*

So, it only remains to prove that $\mathcal{H}$ is locally uniformly bounded in $D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}$. This can be established in the following way.

Consider the family of traces of functions from $\mathcal{H}$ on $\mathbb{T}$:

$$\mathcal{H}_{\mathbb{T}} := \left\{ \left(j_n w_n^{p'/2} \right)(\xi) \overline{(Q^2T)(\xi)} \right\} \subset L^2(\mathbb{T}).$$

Each function of $\mathcal{H}_{\mathbb{T}}$ is a sum of its analytic and antianalytic parts $h_{n,1}(\xi)$ and $h_{n,2}(\xi)$, where

$$\begin{aligned} h_{n,1}(\xi) &= \mathcal{P}_+\left(\left(j_n w_n^{p'/2}\right)(\xi)\overline{(Q^2T)(\xi)}\right), \\ h_{n,2}(\xi) &= \mathcal{P}_-\left(\left(j_n w_n^{p'/2}\right)(\xi)\overline{(Q^2T)(\xi)}\right). \end{aligned}$$

We can regard $h_{n,2}$ as an analytic function in $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ that vanishes at infinity. Thus, upon writing

$$h_{n,2}(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\left(j_n w_n^{p'/2}\right)(\xi)\overline{(Q^2T)(\xi)}}{z-\xi}d\xi, \quad |z|>1, \tag{3.39}$$

we see that $\{h_{n,2}\}$ is locally uniformly bounded in $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$, since Q and $\mathbb{T}$ are fixed, $|j_n|\equiv 1$ on $\mathbb{T}$, and $\left\|w_n^{p'/2}\right\|_2 = 1$.

Now, multiplying the first line of (2.31) by $b_{n_2}Q^2T = b_nQ^2T$ and applying $\mathcal{P}_-$ yields

$$\begin{aligned} \mathcal{H}_{\mathcal{F}}(b_nv_nQ^2T)(\xi) &= \gamma_n^{1/2}\mathcal{P}_-\left(\bar{\xi}\,\overline{\left(j_n w_n^{p'/2}\right)(\xi)}\left(Q^2T\right)(\xi)\right) \\ &= \gamma_n^{1/2}\,\overline{\xi\mathcal{P}_+\left(\left(j_n w_n^{p'/2}\right)(\xi)\overline{(Q^2T)(\xi)}\right)} \\ &= \gamma_n^{1/2}\,\overline{\xi h_{n,1}(\xi)}, \quad \xi\in\mathbb{T}, \end{aligned} \tag{3.40}$$

where we used that

$$\mathcal{P}_-(h(\xi)) = \overline{\xi\mathcal{P}_+\left(\overline{\xi h(\xi)}\right)}, \quad h\in L^2(\mathbb{T}),$$

and

$$\mathcal{P}_-\left(\mathcal{P}_-(h)g\right) = \mathcal{P}_-(hg), \quad h\in L^2(\mathbb{T}),\ g\in H^2.$$

Following the computations preceding (2.35) we obtain

$$\mathcal{H}_{\mathcal{F}}(b_nv_nQ^2T)(\xi) = \int\frac{(b_nv_nQ^2T)(t)}{\xi-t}d\widetilde{\mu}(t) = \int\frac{(b_nv_nQ^2T)(t)}{\xi-t}d\mu(t), \quad \xi\in\mathbb{T},$$

where the second equality is valid since Q^2 has a zero at each $\eta\in S'$ of multiplicity greater than or equal to $m(\eta)+1$. Thus, using (3.40), we get an analytic extension of $h_{n,1}$ to $\overline{\mathbb{C}}\setminus S^{-1}$ by the formula

$$h_{n,1}(z) = \gamma_n^{-1/2}\int\frac{\overline{(b_nv_nQ^2T)(t)}}{1-tz}d\overline{\mu}(t). \tag{3.41}$$

Combining (3.39) and (3.41) we see that

$$\left(j_n w_n^{p'/2}\right)(z)\overline{(Q^2T)(1/\bar{z})} = h_{n,1}(z) + h_{n,2}(z), \quad z \in D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}.$$

So, for $\mathcal{H}$ to be normal, it remains to show that $\{h_{n,1}\}$ is uniformly bounded on compact subsets of $D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}$. Due to formula (3.41) this is actually true in $\overline{\mathbb{C}} \setminus S^{-1}$. Indeed, let $K \subset \overline{\mathbb{C}} \setminus S^{-1}$ be compact. Then for any $z \in K$ equation (3.41) yields the estimate

$$\begin{aligned}
|h_{n,1}(z)| &\leq \gamma_n^{-1/2} \int \frac{\left|b_n v_n Q^2 T\right|(t)}{|1-tz|} d|\mu|(t) \\
&\leq \frac{\gamma_n^{-1/2}}{\inf_{t\in S, z\in K}|1-tz|} \int \left|b_n v_n Q^2 T\right|(t) d|\mu|(t) \\
&\leq \frac{\gamma_n^{-1/2}(\sin\beta)^{-1}}{\inf_{t\in S, z\in K}|1-tz|} \left| \int \left|b_n^2 w_n Q^2 T\right|(t) e^{i(\Theta(t)+\operatorname{Arg} T(t))} d|\mu|(t) \right| \\
&= \frac{\gamma_n^{-1/2}(\sin\beta)^{-1}}{\inf_{t\in S, z\in K}|1-tz|} \left| \int \overline{\left(b_n w_n^{1/2} Q\right)(t)} \left(b_n w_n^{1/2} QT\right)(t) d\mu(t) \right|,
\end{aligned}$$

where we used (3.38). Note that $w_n^{1/2}$ is well-defined in $\mathbb{D}$ since w_n has no zeros there. For simplicity we shall denote

$$\gamma(K) := \frac{(\sin\beta)^{-1}}{\inf_{t\in S, z\in K}|1-tz|}.$$

We remark that this quantity is finite and depends solely on K and the measure μ.

Define $G_n(z) := \overline{\left(b_n Q w_n^{1/2}(\bar{z})\right)}$, $z \in \mathbb{D}$. Then $G_n \in H^{2p'}$, its trace has the same $L^{2p'}(\mathbb{T})$ norm as $\overline{\left(Q w_n^{1/2}\right)}$ and G_n coincides with $\overline{\left(b_n Q w_n^{1/2}\right)}$ on $(-1,1)$. Further, let $r < 1$, dependent on n, be such that $\mathbb{D}_r$ contains $\widetilde{S}$ and all zeros of $b_n j_n$. Then by the estimates above, the fact that Q vanishes on S', the Cauchy formula, equation (2.35), and Fubini-Tonelli's theorem we get for every $z \in K$

$$\begin{aligned}
|h_{n,1}(z)| &\leq \gamma(K) \left| \gamma_n^{-1/2} \int \left(G_n Q T w_n^{-1/2}\right)(t)\ v_n(t)\ d\widetilde{\mu}(t) \right| \\
&= \gamma(K) \left| \frac{\gamma_n^{-1/2}}{2\pi i} \int_{\widetilde{S}} \left(\int_{\mathbb{T}_r} \frac{\left(G_n Q T w_n^{-1/2}\right)(\xi) v_n(t)}{\xi - t}\ d\xi \right) d\widetilde{\mu}(t) \right| \\
&= \gamma(K) \left| \frac{1}{2\pi i} \int_{\mathbb{T}_r} \left(G_n Q T w_n^{-1/2}\right)(\xi) \overline{\left(b_n j_n w_n^{p'/2}\right)(1/\bar{\xi})}\ \frac{d\xi}{\xi} \right| \\
&= \gamma(K) \left| \frac{1}{2\pi i} \int_{\mathbb{T}_r} (G_n Q T)(\xi)\ \overline{(b_n j_n)(1/\bar{\xi})}\ \frac{\overline{w_n^{p'/2}(1/\bar{\xi})}}{w_n^{1/2}(\xi)}\ \frac{d\xi}{\xi} \right|.
\end{aligned} \tag{3.42}$$

The integrand

$$(G_nQT)(\xi)\ \overline{(b_nj_n)\left(1/\bar{\xi}\right)}\ \frac{\overline{w_n^{p'/2}\left(1/\bar{\xi}\right)}}{w_n^{1/2}(\xi)}\ \frac{1}{\xi}$$

is meromorphic in $\mathbb{D}\backslash S$. Moreover all of its poles, that are necessarily zeros of b_nj_n, are encapsulated both by $\mathbb{T}_r$ and $\mathbb{T}$. Despite possible zeros of w_n on the unit circle, the function

$$\frac{\overline{w_n^{p'/2}\left(1/\bar{\xi}\right)}}{w_n^{1/2}(\xi)}$$

extends continuously there, since $p' \geq 2$ and $w_n^{p'/2}$ is analytic on $\mathbb{T}$. Therefore, the integration contour $\mathbb{T}_r$ in the last integral of (3.42) can be replaced by $\mathbb{T}$ without changing the value of that integral. Thus, for any $z \in K$, we get by a straightforward majorization and Hölder's inequality that

$$\begin{aligned}
|h_{n,1}(z)| &\leq \gamma(K)\left|\frac{1}{2\pi}\int_{\mathbb{T}}(QT)(\xi)G_n(\xi)\ \frac{\overline{\left(b_nj_nw_n^{p'/2}\right)(\xi)}}{w_n^{1/2}(\xi)}\ d|\xi|\right| \\
&\leq \gamma(K)\ \|Q\|_\infty\ \|T\|_\infty\ \|G_n\|_2\ \|w_n^{(p'-1)/2}\|_2 \\
&\leq \gamma(K)\ \|Q\|_\infty\ \|T\|_\infty\ \|Qw_n^{1/2}\|_2\ \|w_n\|_{p'-1}^{(p'-1)/2} \leq \gamma(K)\ \|Q\|_\infty^2\ \|T\|_\infty.
\end{aligned}$$

This finishes the proof of the local uniform boundedness of $\mathcal{W}_{p'}$, since the above estimate does not depend on n, but only on the set K itself and the measure $\widetilde{\mu}$.

A classical result of Hurwitz states that each limit point of a normal family consisting of zero free function in the domain of the normality is zero free unless it is identically zero. Now, $\mathcal{W}_{p'}$ has no zero function as a limit point because these functions have unit $L^2(\mathbb{T})$ norm. Since each $w_n^{p'/2}$ is outer in $\mathbb{D}$, no limit point of $\mathcal{W}_{p'}$ can have zeros in $\mathbb{D}$. The corresponding result for $\mathcal{W}$ is then obvious. ■

Note that in the previous theorem the normality of $\mathcal{W}$ in $\mathbb{D}$ was clear beforehand by the Cauchy formula, since $\|w_n\|_{p'} = 1$, but the nonzeroing of every limit point was not.

Proof of Theorem 3.8: Let v_n be a singular vector associated to g_n with inner-outer factorization given by $v_n = b_n \cdot w_n$ for some Blaschke product $b_n = q_n/\widetilde{q}_n$ and some outer function $w_n \in H^{p'}$, $\|w_n\|_{p'} = 1$, where $w_n \equiv 1$ when $p = 2$. The poles of g_n are exactly the zeros of q_n. Moreover, $\{q_n\}$ is a sequence of polynomials satisfying the non-Hermitian weighted orthogonality relations (2.32). Thus, the assertion of the theorem will follow from Theorem 3.1 if $\mathcal{W} = \{w_n\}$ is uniformly bounded above and below on E, the convex hull of S, and if it is a family of functions whose arguments are smooth with uniformly bounded derivatives on E. In the case $p = 2$ this is trivial since each $w_n \equiv 1$. In the case $p \in (2, \infty]$ Lemma 3.11 says that $\mathcal{W}$ is a normal family. Thus, it is uniformly bounded above on E. Moreover, since all limit points of $\mathcal{W}$ are zero free in $\mathbb{D}$, this family is uniformly bounded below on E (in fact on any compact subset of $\mathbb{D}$). Further, the derivatives again form a normal family and so does the logarithmic derivative w_n'/w_n in $\mathbb{D}$. Since the imaginary part of the latter is equal to $d\arg(w_n)/dt$ on E, we see that the rest of conditions on $\mathcal{W}$ is satisfied. ∎

Before we prove Theorem 3.9 we need a simple observation that will be of further use later on.

Lemma 3.12 *Let A be a compact set such that $A \cap S = \emptyset$. Further, let $\{\mu_n\}$ and $\{\lambda_n\}$ be two sequences of positive measures such that* $\operatorname{supp}(\mu_n) \subset A$*, $\mu_n \stackrel{*}{\to} \mu$, $\widehat{S} = \bigcap_{n=1}^\infty \overline{\bigcup_{k=n}^\infty \operatorname{supp}(\lambda_k)}$ is bounded, and $\lambda_n \stackrel{*}{\to} \widehat{\mu}$, where $\mu \in \Lambda(A)$ and $\widehat{\mu}$ is the balayage of μ onto S relative to $\mathbb{C} \setminus S$. Then*

$$\limsup_{n\to\infty} \left\|\exp\left\{U^{\mu_n-\lambda_n}\right\}\right\|_S \leq \exp\{-c(\mu;\mathbb{C}\setminus S)\}$$

and for any $K \subset \mathbb{C} \setminus (\widehat{S} \cup A)$ holds

$$\lim_{n\to\infty} \left\|\exp\left\{U^{\lambda_n-\mu_n}\right\}\right\|_K = \left\|\exp\left\{c(\mu;\mathbb{C}\setminus S) - U_G^\mu(\cdot;\mathbb{C}\setminus S)\right\}\right\|_K.$$

Proof: Since S is regular, $U^{\widehat{\mu}}$ is continuous in $\mathbb{C}$. Thus, by the principle of descent ([113, Thm. I.6.8])

$$\liminf_{n\to\infty} U^{\lambda_n}(z) \geq U^{\widehat{\mu}}(z),$$

uniformly on compact subsets of $\mathbb{C}$. Further, it is readily verified that

$$\lim_{n\to\infty} U^{\mu_n}(z) = U^\mu(z) \quad \text{and} \quad \lim_{n\to\infty} U^{\lambda_n}(z) = U^{\widehat{\mu}}(z)$$

uniformly on compact subsets of $\mathbb{C}\setminus A$ and $\mathbb{C}\setminus\widehat{S}$, respectively. Therefore, since $A \cap S = \emptyset$, we derive

$$\limsup_{n\to\infty} U^{\mu_n-\lambda_n}(x) \leq U^{\mu-\widehat{\mu}}(x) = -c(\mu;\mathbb{C}\setminus S),$$

uniformly on S, where the last equality holds by the regularity of S. Analogously, we see that for

K as defined the limit

$$\lim_{n\to\infty} U^{\lambda_n-\mu_n}(z) = U^{\widehat{\mu}-\mu}(z) = c(\mu;\mathbb{C}\setminus S) - U_G^{\mu}(z;\mathbb{C}\setminus S)$$

holds uniformly on K. The assertions follow immediately from this. ■

Proof of Theorem 3.9: To prove the convergence in capacity we first establish an integral representation for the error $e_n = (\mathcal{F} - g_n)$. As usual we denote by $v_n = b_n \cdot w_n$ a singular vector associated to g_n, where b_n is a Blaschke product of degree n and w_n is an outer function. Since any critical point satisfies (see discussion before (2.28))

$$\mathcal{F} - g_n = \frac{\mathcal{H}_{\mathcal{F}}(v_n)}{v_n},$$

equation (2.35) allows us to write for $z \in \mathbb{D}\setminus\widetilde{S}$

$$e_n(z) = \frac{1}{v_n(z)}\int \frac{v_n(t)}{z-t} d\widetilde{\mu}(t) = \frac{\widetilde{q}_n(z)}{q_n(z)w_n(z)}\int \frac{q_n(t)}{z-t}\frac{w_n(t)}{\widetilde{q}_n(t)} d\widetilde{\mu}(t). \tag{3.43}$$

In another connection, the orthogonality relations (2.32) yield

$$\int \frac{\widetilde{q}_n(z)-\widetilde{q}_n(t)}{z-t}\, q_n(t)\, \frac{w_n(t)}{\widetilde{q}_n^2(t)} d\widetilde{\mu}(t) = 0.$$

This, in turn, implies that (3.43) can be rewritten as

$$e_n(z) = \frac{\widetilde{q}_n^2(z)}{q_n(z)w_n(z)}\int \frac{q_n(t)}{z-t}\frac{w_n(t)}{\widetilde{q}_n^2(t)} d\widetilde{\mu}(t), \quad z\in\mathbb{D}\setminus\widetilde{S}. \tag{3.44}$$

Let K be a compact subset of $\mathbb{D}\setminus S$ and let U an open neighborhood of K such that $U\subset\mathbb{D}$ and $U\cap S=\emptyset$. Denote by $\xi_{1,n},\ldots,\xi_{n,n}$ the zeros of q_n and define

$$q_{n,1}(z) = \prod_{\xi_{j,n}\in U}(z-\xi_{j,n}).$$

Corollary 3.5 implies that there exists $N\in\mathbb{N}$ such that $\deg q_{n,1}\le N$ for all $n\in\mathbb{N}$. Let $a<1$ be such that $S\subset\mathbb{D}_a$. Then by Lemma 3.4 the number of zeros of q_n in $D_a\setminus U$ tends to infinity with n. Thus, for n large enough and in any case bigger than m, we can pick m of them, say $\xi_{1,n},\ldots,\xi_{m,n}$ and define

$$l_{m,n}(z) := \prod_{j=1}^{m}(z-\xi_{j,n}).$$

Since the polynomial $q_{n,1}l_{m,n}$ has bounded degree, the asymptotic zero distribution of polynomials $q_n^\sharp := q_n/(q_{n,1}l_{m,n})$ is the same as that of q_n. Thus, by Theorem 3.1 and Proposition A.1, points (b)-(e), the counting measures $\nu_n^\sharp$ of $q_n^\sharp$ converge weak* to $\mu_{(S,\mathbb{T})}$, the Green equilibrium distribution on S relative to both $\mathbb{D}$ and $\mathbb{C}\setminus S^{-1}$. Using the orthogonality relations (2.32) and the fact that Q vanishes at the η with multiplicity $m(\eta)$, we can rewrite (3.44) as

$$e_n(z) = \frac{\widetilde{q}_n^2(z)}{(q_n^\sharp)^2(z)(q_{n,1}Q)(z)(l_{m,n}w_n)(z)}\int\frac{(q_n^\sharp)^2(t)(q_{n,1}Q)(t)(l_{m,n}w_n)(t)}{\widetilde{q}_n^2(t)}\frac{d\mu(t)}{z-t}, \quad z\in\mathbb{D}\setminus\widetilde{S}. \quad (3.45)$$

Lemma 3.11 says that $\{w_n\}$ is bounded above and below on compact subsets of $\mathbb{D}$. This and the choice of $l_{m,n}$ yield that

$$|e_n(z)| \leq \frac{c_1}{|q_{n,1}(z)Q(z)|}\left|\frac{\widetilde{q}_n(z)}{q_n^\sharp(z)}\right|^2\left\|\frac{q_n^\sharp}{\widetilde{q}_n}\right\|_S^2 \quad (3.46)$$

for any $z\in K$ and some positive $c_1<\infty$ which does not depend on n. We shall apply Lemma 3.12 with $A=\mathbb{T}$, $\lambda_n=\nu_n^\sharp$, and $\mu_n=\widetilde{\nu}_n$, where $\widetilde{\nu}_n$ is the balayage of ν_n onto the unit circle and ν_n is the counting measures of zeros of q_n. Because $\nu_n(\mathbb{D}\setminus S)\to 0$ by Corollary 3.5 the weak* limit of the balayage is the balayage of the weak* limit, hence Theorem 3.1 implies that $\mu=\widetilde{\mu_{(S,\mathbb{T})}}$ and Proposition A.1-(c) shows that $\widehat{\mu}=\mu_{(S,\mathbb{T})}$. Also notice that in this case

$$\begin{aligned} U^{\lambda_n-\mu_n}(z) &= U^{\nu_n^\sharp-\widetilde{\nu}_n}(z) = U^{\nu_n^\sharp}(z) - U^{\nu_n}(z) + U_G^{\nu_n}(z;\mathbb{D}) \\ &= \frac{1}{n}\log\left|\frac{1}{q_n^\sharp(z)}\right| - \frac{1}{n}\log\left|\frac{1}{q_n(z)}\right| + \int\log\left|\frac{1-z\bar{t}}{z-t}\right|d\nu_n(t) = \frac{1}{n}\log\left|\frac{\widetilde{q}_n(z)}{q_n^\sharp(z)}\right|, \quad z\in\mathbb{D}\setminus S, \end{aligned}$$

since $c(\nu_n;\mathbb{D})=0$ by the boundedness of $\mathbb{D}$. As $U_G^{\widetilde{\mu_{(S,\mathbb{T})}}}(z;\mathbb{C}\setminus S)$ is a non-constant positive harmonic function in $\mathbb{D}\setminus S$, it is strictly positive there so there exists $\beta>0$ such that

$$0 < 2\beta \leq \inf_{z\in K} U_G^{\widetilde{\mu_{(S,\mathbb{T})}}}(z;\mathbb{C}\setminus S).$$

Then Lemma 3.12, in particular, implies that for all n large enough

$$\left\|\frac{q_n^\sharp}{\widetilde{q}_n}\right\|_S \leq \exp\left\{-n\cdot c\left(\widetilde{\mu_{(S,\mathbb{T})}};\mathbb{C}\setminus S\right)+n\beta/2\right\},$$

and

$$\left|\frac{\widetilde{q}_n(z)}{q_n^\sharp(z)}\right| \leq \exp\left\{n\cdot c\left(\widetilde{\mu_{(S,\mathbb{T})}};\mathbb{C}\setminus S\right)-3n\beta/2\right\}, \quad z\in K.$$

Combining the last two estimates and (3.46) we obtain that

$$|e_n(z)| \leq \frac{c_1 e^{-2n\beta}}{|q_{n,1}(z)Q(z)|}$$

for any $z \in K$ and n large enough. Denote by K_n the following sets

$$K_n := \left\{ z \in K : \ |q_{n,1}(z)Q(z)| \leq e^{-n\beta} \right\}.$$

Since each K_n is the lemniscate of a monic polynomial of degree at most $m+N$ we get that ([111, Thm. 5.2.5])

$$\mathrm{cap}(K_n) \leq \exp\left\{ -n\frac{\beta}{m+N} \right\}.$$

But for $z \in K \setminus K_n$ we get

$$|e_n(z)| \leq c_1 e^{-n\beta}.$$

This establishes the main assertion of the theorem upon letting $\gamma := \beta/(m+k)$.

Let us finally fix $p = 2$. In this case $w_n \equiv 1$ for any $n \in \mathbb{N}$, $r_n(z)$ is defined everywhere outside of S, and has the representation

$$r_n(z) = \frac{\widetilde{q}_n^2(z)}{(q_n^\sharp)^2(z)Q_s(z)l_{s,n}(z)} \int \frac{(q_n^\sharp)^2(t)Q_s(t)l_{s,n}(t)}{\widetilde{q}_n^2(t)} \frac{d\mu(t)}{z-t}, \quad z \in \mathbb{C} \setminus S,$$

where $q_n^\sharp = q_n/l_{s,n}$ and we fixed s arbitrary zeros of q_n in $\mathbb{D}_a$ in order to define $l_{s,n}$. Let $b_{n,a}$ be the normalized Blaschke product with zeros at the zeros of $q_n^\sharp$ that are contained in $\overline{\mathbb{D}}_a$. Then by Corollary 3.5 the counting measures of the zeros of $b_{n,a}$, $\nu_{n,a}$, converge in the weak* topology to $\mu_{(S,\mathbb{T})}$. Further, for any $z \in K$, the reasoning that led us to (3.46) now gives

$$|r_n(z)| \leq c_2 \left| \frac{\widetilde{q}_n(z)}{q_n^\sharp(z)} \right|^2 \left\| \frac{q_n^\sharp}{\widetilde{q}_n} \right\|_S^2 \leq \frac{c_3}{|b_{n,a}(z)|^2} \left\| \frac{q_n^\sharp}{\widetilde{q}_n} \right\|_S^2 \tag{3.47}$$

for some c_2 and c_3 independent of n.

In another connection, the principle of descent and continuity of $U^{\mu_{(S,\mathbb{T})}}$ in $\mathbb{C}$ imply that

$$\limsup_{n\to\infty} \int \log|z-t| d\nu_{n,a}(t) \leq \int \log|z-t| d\mu_{(S,\mathbb{T})}(t) \tag{3.48}$$

holds uniformly in $\overline{\mathbb{D}}$. Further, continuity of $\log|1-\bar{t}z|$ for $z \in \overline{\mathbb{D}}$ and $t \in \overline{\mathbb{D}}_a$ yields

$$\lim_{n\to\infty} \int \log|1-\bar{t}z| d\nu_{n,a}(t) = \int \log|1-\bar{t}z| d\mu_{(S,\mathbb{T})}(t), \quad z \in \overline{\mathbb{D}}. \tag{3.49}$$

Clearly, the limit in (3.49) is uniform in $\overline{\mathbb{D}}$. Now, since for $z \in \mathbb{C} \setminus \mathbb{D}$ we have

$$|b_{n,a}(z)|^{-1/n} = \exp\left\{-U_G^{\nu_{n,a}}(1/\bar{z};\mathbb{D})\right\} = \exp\left\{\int \log\left|\frac{t-1/\bar{z}}{1-\bar{t}/\bar{z}}\right| d\nu_{n,a}(t)\right\},$$

equations (3.47), (3.48), (3.49) and Lemma 3.12 yield

$$\limsup_{n\to\infty} |r_n(z)|^{1/2n} \leq \exp\left\{-c(\widetilde{\mu_{(S,\mathbb{T})}}; \mathbb{C}\setminus S) - U_G^{\mu_{(S,\mathbb{T})}}(1/\bar{z};\mathbb{D})\right\}$$

uniformly in $\overline{\mathbb{C}} \setminus \mathbb{D}$. Using Proposition A.1-(e), we complete the proof of the theorem. ∎

Proof of Theorem 3.10: It was proved in Theorem 3.9 that $\{g_n\}$ converges in capacity on compact subsets of $\mathbb{D} \setminus S$ to $\mathcal{F}$. Then the lower bound in (3.36) follows from the classical Gonchar's lemma ([49, Lemma 1]). Indeed, suppose to the contrary that there exist $\delta^* > 0$ and a subsequence $\mathbb{N}_1 \subset \mathbb{N}$ such that

$$\#\{S_n \cap B_{\delta^*}(\eta)\} < m(\eta), \quad B_{\delta^*}(\eta) \cap \widetilde{S} = \{\eta\},$$

for $n \in \mathbb{N}_1$ and some fixed $\eta \in S' \setminus S$. Thus, $\{g_n\}_{n\in\mathbb{N}_1}$ is a sequence of meromorphic functions in $B_{\delta^*}(\eta)$ with at most $m(\eta)$ poles, which converges in capacity on $B_{\delta^*}(\eta)$ to the meromorphic function $F|_{B_{\delta^*}(\eta)}$ with exactly one pole of multiplicity $m(\eta)$ there. Then by Gonchar's lemma each g_n has exactly $m(\eta)$ poles in $B_\delta(\eta)$ and these poles converge to η. This contradiction proves the lower bound in (3.36).

Now, for each $\eta \in S'$ define $D_\eta := B_\delta(\eta)$, $\delta = \delta(\eta)$ to be chosen later. Moreover, assume that $D_\eta \cap S' = \{\eta\}$ and for each $\eta \in S' \setminus S$ assume further that $D_\eta \cap S = \emptyset$, which is always possible by the appropriate choice of δ.

Fix any $\eta \in S' \setminus S$. In general, the existence of C_η in (3.36) follows from Corollary 3.5 applied to $\mathbb{D} \setminus D_\eta$, but it is possible to say more. Let $E_k = \cup_{j=1}^k [a_j, b_j]$ cover S and let S_n stand for the set of all poles of g_n, $S_n = \{\xi_{1,n}, \ldots, \xi_{n,n}\}$. Then by Lemma 3.4 we have

$$\sum_{j=1}^n (\pi - \theta(\xi_{j,n})) \leq V(\Theta) + V_{\mathcal{W}} + (k+m-1)\pi + \sum_{\zeta\in S'} m(\zeta)\theta(\zeta), \tag{3.50}$$

where $\theta(\cdot)$ is the angle function defined in (3.5) for the system of intervals E_k and $V_{\mathcal{W}}$ was defined in (3.37). The finiteness of $V_{\mathcal{W}}$ was obtained in the proof of Theorem 3.8. Then for n large enough (3.50) yields

$$\#\{S_n \cap D_\eta\} - m(\eta) \leq \frac{V(\Theta) + V_{\mathcal{W}} + (k-1)\pi + \sum_{\zeta \in S'} m(\zeta)(\theta(\zeta) + \max_{\xi \in D_\zeta} \theta(\xi))}{\pi - \max_{\xi \in D_\eta} \theta(\xi)}.$$

Denote by C the right hand-side of the previous inequality. By a suitable choice of $\delta(\eta)$ and the continuity of $\theta(\cdot)$ in $\mathbb{D} \setminus [-1,1]$ we can make C such that

$$\lfloor C \rfloor \leq \frac{V(\Theta) + V_{\mathcal{W}} + (k-1)\pi + 2\sum_{\zeta \in S'} m(\zeta)\theta(\zeta)}{\pi - \theta(\eta)} < C,$$

where $\lfloor C \rfloor$ is the greatest integer smaller than C. Since C is a bound for an integer, we get that

$$\#\{S_n \cap D_\eta\} \leq m(\eta) + \frac{V(\Theta) + V_{\mathcal{W}} + (m-1)\pi + 2\sum_{\zeta \in S'} m(\zeta)\theta(\zeta)}{\pi - \theta(\eta)}.$$

■

Padé Approximation

In this section we describe the asymptotic behavior of diagonal multipoint Padé approximants to functions of type (1.2). We follow the same exposition as in the preceding section.

Theorem 3.13 *Let $\{\Pi_n\}_{n \in \mathbb{N}}$ be a sequence of diagonal multipoint Padé approximants, corresponding to some strongly admissible interpolation scheme* $\mathbf{A}$*, to $\mathcal{F}(\mu; R; \cdot)$ given by (1.2) with $\mu \in$* **BVT***. Then, the counting measures of the poles of Π_n converge to μ_w^S in the weak* sense.*

For classical Padé approximants, this theorem reduces to the following result.

Corollary 3.14 *Let $\{\Pi_n\}_{n \in \mathbb{N}}$ be a sequence of classical diagonal Padé approximants to $\mathcal{F}(\mu; R; \cdot)$ given by (1.2) with $\mu \in$* **BVT***. Then the counting measures of the poles of Π_n converge in the weak* sense to μ_S, the equilibrium distribution on S.*

The forthcoming theorem is of classical scope and stands as an analog to Theorem 3.9 for multipoint Padé approximants. It deals with their convergence in capacity to $\mathcal{F}$ and, in a way, it generalizes Theorem 6.1.6(a) in [130].

Theorem 3.15 *Let $\{\Pi_n\}_{n \in \mathbb{N}}$ and $\mathcal{F}(\mu; R; \cdot)$ be as in Theorem 3.13. Further, let σ stand for the asymptotic distribution of* $\mathbf{A}$*. Then $\{\Pi_n\}$ converges to $\mathcal{F}$ in capacity on $\mathbb{C} \setminus S$, and for any compact* *set $K \subset \mathbb{C} \setminus S$ there exist positive constants β and γ such that*

$$\mathrm{cap}\left\{z \in K : \ |(\mathcal{F} - \Pi_n)(z)| \geq e^{-\beta n}\right\} \leq e^{-\gamma n}.$$

Further, any $z \in \mathbb{C} \setminus (\widetilde{S} \cup \mathcal{K})$ we have

$$\limsup_{n\to\infty} |(\mathcal{F} - \Pi_n)(z)|^{1/2n} \leq \exp\{-U_G^\sigma(z; \mathbb{C} \setminus S)\}$$

and this inequality is uniform on compact subsets of $\mathbb{C} \setminus (\widetilde{S} \cup \mathcal{K} \cup \mathcal{K}(\mathbf{A}))$, where $U_G^\sigma(z; \mathbb{C} \setminus S)$ is the Green potential of σ relative to $\mathbb{C} \setminus S$, $\mathcal{K}$ is the set of limit points of the poles of Π_n, and $\mathcal{K}(\mathbf{A})$ is the set of limit points of $\mathbf{A}$.

Our last theorem asserts the convergence of some of the poles of the multipoint Padé approximants of $\mathcal{F}$ to its polar singularities. It parallels Theorem 3.10.

Theorem 3.16 *Let $\{\Pi_n\}_{n\in\mathbb{N}}$ and $\mathcal{F}(\mu; R; \cdot)$ be as in Theorem 3.13. Then for each $\eta \in S' \setminus S$ there exist constants $\delta_\eta > 0$ and $C_\eta = C_\eta(\mathcal{F})$ such that*

$$m(\eta) \leq \#\{S_n \cap B_\delta(\eta)\} \leq m(\eta) + C_\eta, \quad 0 < \delta \leq \delta_\eta,$$

for any $n \geq N(\eta, \delta)$, where $m(\eta)$ is the multiplicity of η, S_n is the set of poles of Π_n, and $B_\delta(\eta) := \{|z - \eta| < \delta\}$.

As in Theorem 3.10, we shall obtain in the course of the proof of the above result an upper bound for C_η. Namely if $E_k := \{[a_j, b_j]\}_{j=1}^k$ is a system of intervals covering S, then

$$C_\eta \leq \frac{V(\Theta) + (k-1)\pi + 2\sum_{\zeta\in S'} m(\zeta)\theta(\zeta)}{\pi - \theta(\eta)},$$

where $\theta(\cdot)$ is the angle function, defined in (3.5), applied to E_k.

Proof of Theorem 3.13: Denote by q_n the denominator of Π_n. As was explained in the preliminary chapter, q_n satisfies orthogonality relaions (2.38). Thus, we may apply Theorem 3.2, which yields the result. ∎

Proof of Theorem 3.15: Exactly as in the proof of Theorem 3.2 we can suppose that all the interpolation points are contained in some compact set K_0 disjoint from $\widetilde{S}$. By the Hermite interpolation formula, (*cf.* [130, Lemma 6.1.2, (1.23)]) the error $e_n := \mathcal{F} - \Pi_n$ has the following representation

$$e_n(z) = \frac{v_{2n}(z)}{q_n(z)} \int \frac{q_n(t)}{v_{2n}(t)} \frac{d\widetilde{\mu}(t)}{z-t}. \tag{3.51}$$

Let $K \subset \mathbb{C} \setminus S$ be compact and let U be a bounded open neighborhood of K not intersecting S. Further, let D be an open disk around the origin such that $(S \cup \overline{U}) \subset D$. Denote by $\xi_{1,n}, \dots, \xi_{d_n,n}$ zeros of q_n and define

$$\begin{aligned} q_{n,1}(z) &= \prod_{\xi_{j,n} \in U} (z - \xi_{j,n}) \\ q_{n,2}(z) &= \prod_{\xi_{j,n} \in \mathbb{C} \setminus \overline{D}} (z - \xi_{j,n}). \end{aligned}$$

Corollary 3.7 implies that there exist $m_1, m_2 \in \mathbb{N}$ independent of n such that $\deg q_{n,j} \leq m_j$ for any $n \in \mathbb{N}$ and $j = 1, 2$. Since at least $d_n - m_1 - m_2$ zeros lie in $D \setminus U$ we can fix m of them there, say $\xi_{1,n}, \dots, \xi_{m,n}$, for n large enough. Denote by

$$l_{m,n}(z) := \prod_{j=1}^{m} (z - \xi_{j,n}).$$

By Theorem 3.2, the counting measures of $q_n^\sharp := q_n/(q_{n,1} q_{n,2} l_{m,n})$, denoted by $\nu_n^\sharp$, converge weak* to $\widehat{\sigma}$, the balayage of σ onto S. Using the orthogonality relations (3.4) in the same way as we did to obtain (3.45), we can rewrite (3.51) as

$$e_n(z) = \frac{v_{2n}}{(q_n^\sharp)^2 (q_{n,1} Q)(|\alpha_n| q_{n,2}) l_{m,n}}(z) \int \frac{(q_n^\sharp)^2 (q_{n,1} Q)(|\alpha_n| q_{n,2}) l_{m,n}}{v_{2n}}(t) \frac{d\mu(t)}{z-t},$$

where α_n is the inverse of the product of the zeros of $q_{n,2}$. Then, the choice of $l_{m,n}$ and α_n yields that

$$|e_n(z)| \leq \frac{c_1}{|q_{n,1}(z) Q(z)|} \left| \frac{v_{2n}(z)}{(q_n^\sharp)^2(z)} \right| \left\| \frac{(q_n^\sharp)^2}{v_{2n}} \right\|_S \tag{3.52}$$

for all $z \in K$ and some $c_1 < \infty$ which does not depend on n. Now we may apply Lemma 3.12 with $A = K_0$, $\lambda_n = \nu_n^\sharp$, and $\mu_n = \sigma_n$, where σ_n is the counting measure of the zeros of v_{2n}. Notice that in this case

$$U^{\lambda_n - \mu_n}(z) = \frac{1}{2n} \log \left| \frac{v_{2n}(z)}{(q_n^\sharp)^2(z)} \right|.$$

By the regularity of S, the Green potential $U_G^\sigma(z; \mathbb{C} \setminus S)$ tends to zero whenever z tends to $\xi \in S$. Moreover, it is strictly positive and superharmonic in $\mathbb{C} \setminus S$. Thus, by the generalized minimum principle ([113, Thm. I.2.4]), there exists $\beta > 0$ such that

$$0 < 5\beta/2 \leq \inf_{z \in K} U_G^\sigma(z; \mathbb{C} \setminus S).$$

Then, Lemma 3.12 implies that for all n large enough

$$\left\| \frac{(q_n^\sharp)^2}{v_{2n}} \right\|_S \leq \exp\left\{-2n \cdot c(\sigma; \mathbb{C} \setminus S) + 2n\beta\right\}$$

and

$$\left| \frac{v_{2n}(z)}{(q_n^\sharp)^2(z)} \right| \leq \exp\left\{2n \cdot c(\sigma; \mathbb{C} \setminus S) - 4n\beta\right\}, \quad z \in K.$$

Combining the last two estimates and (3.52) we obtain for any $z \in K$ and n large that

$$|e_n(z)| \leq \frac{c_1 e^{-2n\beta}}{|q_{n,1}(z) Q(z)|}.$$

This yields the first assertion of the theorem exactly as in the proof of Theorem 3.9.

Now, let $K \subset \mathbb{C} \setminus (\widetilde{S} \cup \mathcal{K})$ be a compact set. Then for any $z \in K$

$$|e_n(z)| \leq c_3 \left| \frac{v_{2n}(z)}{(q_n^\sharp)^2(z)} \right| \left\| \frac{(q_n^\sharp)^2}{v_{2n}} \right\|_S$$

for some constant c_3 which is independent of n. The second assertion then follows upon applying Lemma 3.12. ∎

Proof of Theorem 3.16: The proof is exactly the same as the proof of Theorem 3.10. The only difference is that Lemma 3.6 states that

$$\sum_{j=1}^{d_n} (\pi - \theta(\xi_{j,n})) + (n - d_n)\pi \leq V(\Theta) + (k + m - 1)\pi + \sum_{\zeta \in S'} m(\zeta)\theta(\zeta),$$

and therefore

$$C_\eta \leq \frac{V(\Theta) + (k-1)\pi + 2\sum_{\zeta \in S'} m(\zeta)\theta(\zeta)}{\pi - \theta(\eta)}.$$

∎

Numerical Experiments

The Hankel operator $\mathcal{H}_{\mathcal{F}}$ with symbol $\mathcal{F} \in H^\infty + C(\mathbb{T})$ is of finite rank if and only if $\mathcal{F}$ is a rational function [91, Thm. 3.11]. In practice one can only compute with finite rank operators, due to the necessity of ordering the singular values, so a preliminary rational approximation to $\mathcal{F}$ is needed when the latter is not rational. One way to handle this problem is to truncate the Fourier series of $\mathcal{F}$ at some high order N. This provides us with a rational function $\mathcal{F}_N$ that approximates $\mathcal{F}$ in the Wiener norm which, in particular, dominates any L^p norm on the unit circle, $p \in [1,\infty]$. It was proved in [62] that the best approximation operator from H_n^∞ (mapping $\mathcal{F}$ to g_n according to (2.24)) is continuous in the Wiener norm provided $(n+1)$-st singular value of the Hankel operator is simple. It was shown in [22, Cor. 2] that the last assertion is satisfied for Hankel operators with symbols in some open dense subset of $H^\infty + C(\mathbb{T})$, and the same technique can be used to prove that it is also the case for the particular subclass (1.2). Thus, even though the simplicity of singular values cannot be asserted beforehand, it is generically true. When it prevails, one can approximates $\mathcal{F}_N$ instead of $\mathcal{F}$ and get a close approximation to g_n when N is large enough. This amounts to perform the singular value decomposition of $\mathcal{H}_{\mathcal{F}_N}$ (see [135, Ch. 16]). When $2 \leq p < \infty$ there is no difficulty with continuity issues, but the computation of g_n has to rely on a numerical search. To numerically construct rational approximants when $p = 2$, we used the above truncation technique together with the *Hyperion* software described in [61].

As to Padé approximants, we restricted ourselves to the classical case and we constructed their denominators by solving the orthogonality relations (3.4) with $v_{2n} \equiv 1$. Thus, finding these denominators amounts to solving a system of linear equations whose coefficients are obtained from the moments of the measure μ.

In the numerical experiments below we approximate function $\mathcal{F}$ given by the formula

$$\begin{aligned}\mathcal{F}(z) &= 7\int_{[-6/7,-1/8]}\frac{e^{it}dt}{z-t} - (3+i)\int_{[2/5,1/2]}\frac{1}{t-2i}\frac{dt}{z-t} + (2-4i)\int_{[2/3,7/8]}\frac{\ln(t)dt}{z-t} \\ &+ \frac{2}{(z+3/7-4i/7)^2} + \frac{6}{(z-5/9-3i/4)^3} + \frac{24}{(z+1/5+6i/7)^4}.\end{aligned}$$

On the figures the solid lines stand for the support of the measure, diamonds depict the polar singularities of $\mathcal{F}$, and circles denote the poles of the corresponding approximants. Note that the poles of $\mathcal{F}$ seem to attract the singularities first.

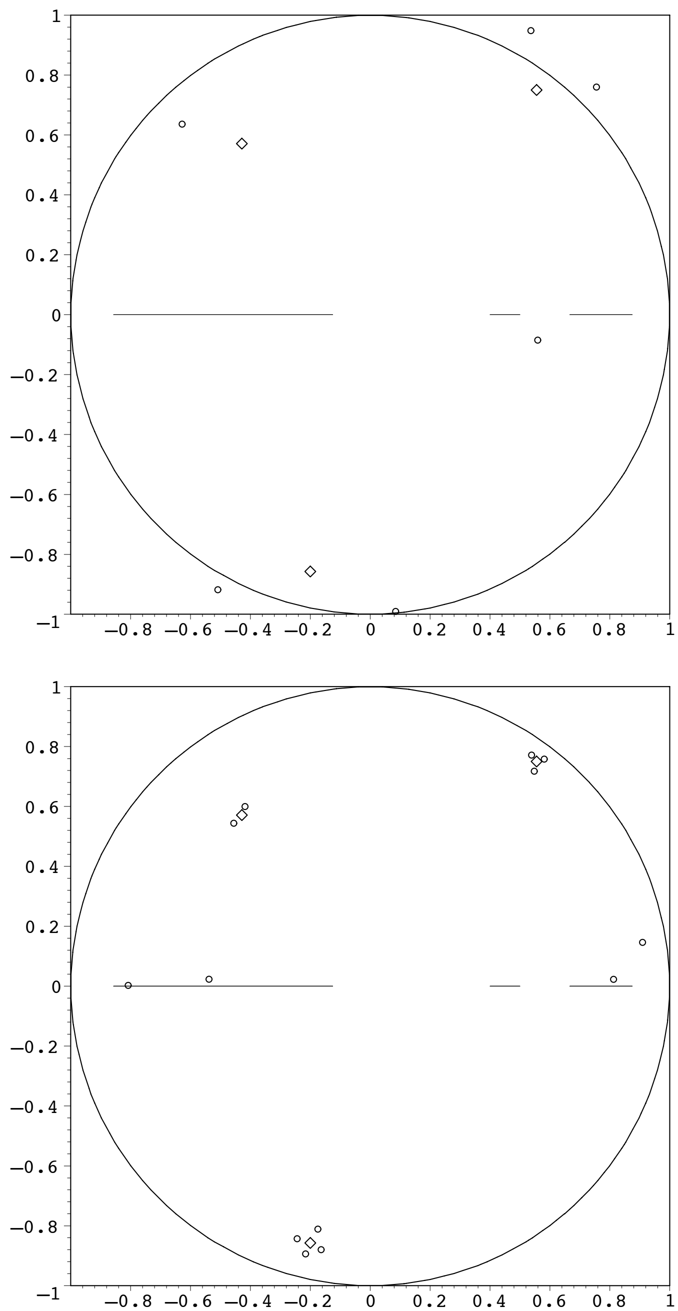

Figure 1: Padé approximants to $\mathcal{F}$ of degree 8 and 13

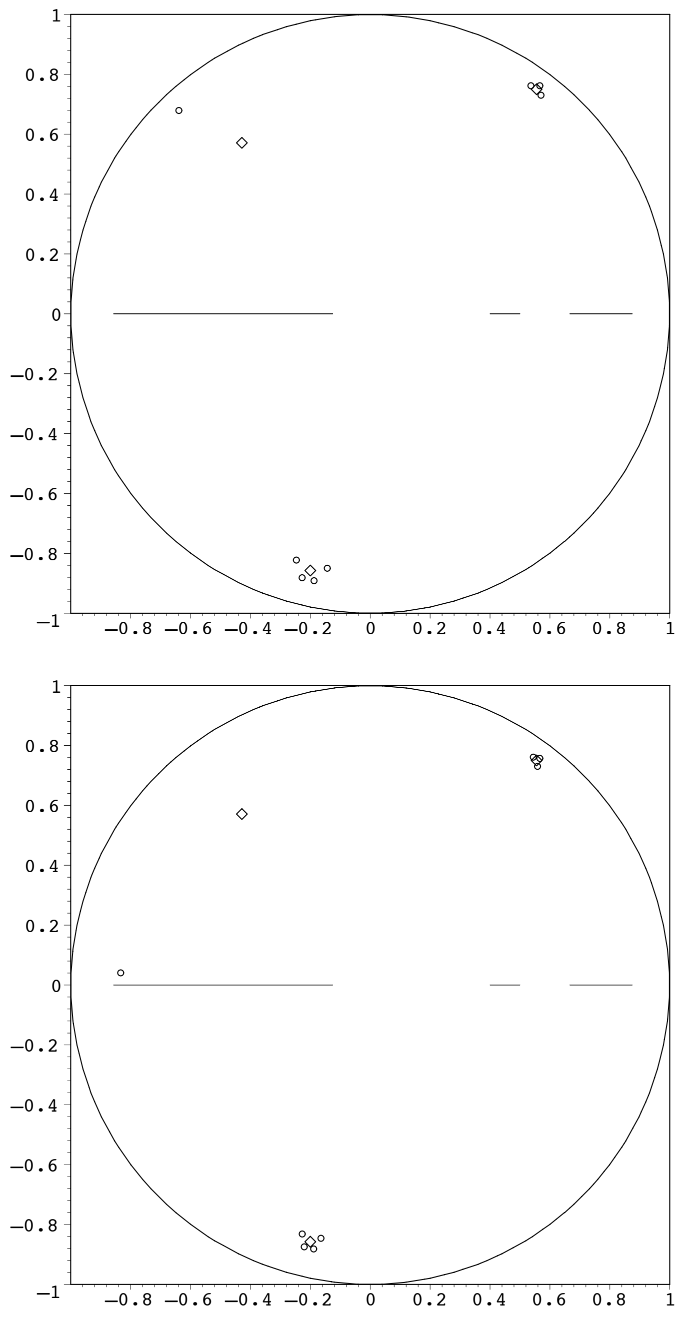

Figure 2: AAK (left) and rational (right) approximants to $\mathcal{F}$ of degree 8

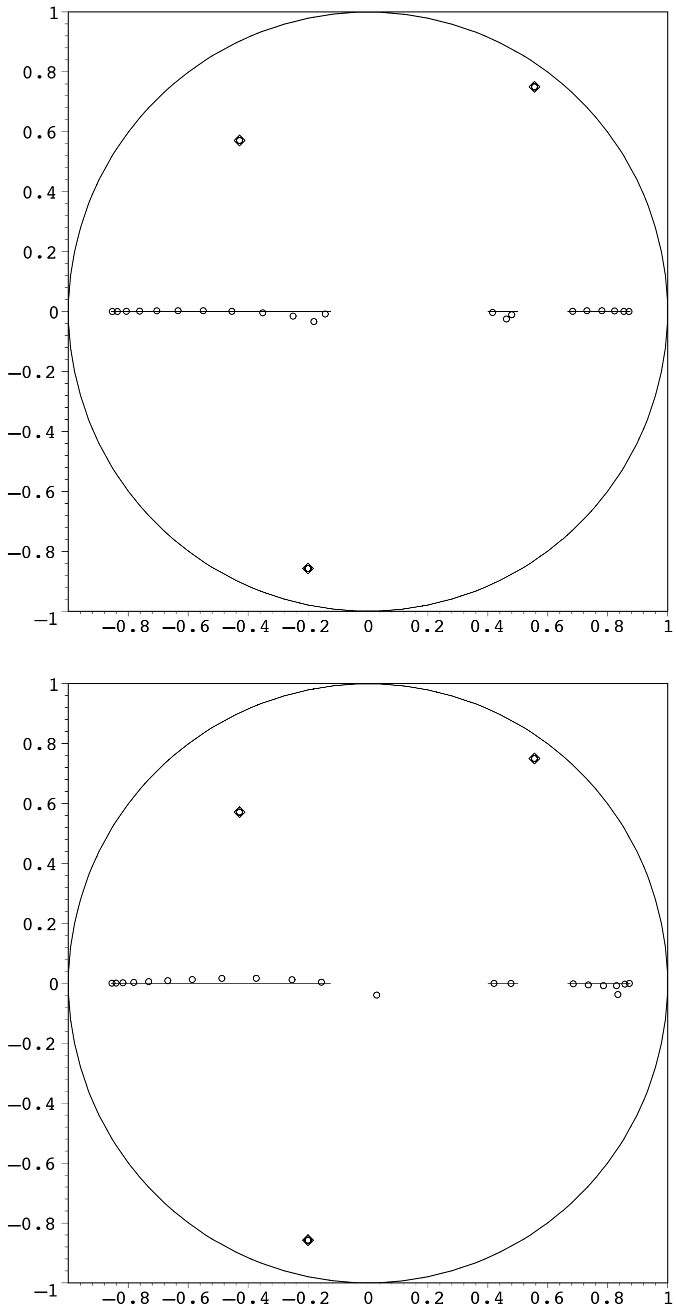

Figure 3: Padé (left) and AAK (right) approximants to $\mathcal{F}$ of degree 30

CHAPTER IV

STRONG ASYMPTOTICS

In this chapter we establish the uniform convergence and provide rates of this convergence of meromorphic and multipoint Padé approximants to functions of type $\mathcal{F}(\mu; R; \cdot)$ given by (1.2), where R is a rational function vanishing at infinity and μ has some special properties described below.

Let μ be a measure supported on an interval $E = [a, b]$. Denote by $\dot{\mu}$ the Radon-Nikodym derivative of μ with respect to μ_E, the logarithmic equilibrium distribution on E, i.e.,

$$d\mu(t) = \dot{\mu}(t) d\mu_E(t), \quad t \in E. \tag{4.1}$$

We shall consider measures such that $\dot{\mu} = \hat{s}s$, where $\hat{s}$ is a nonnegative functions on E that vanishes in a controllable manner and s is a Dini-continuous non-vanishing function on E possibly having an argument of bounded variation.

This chapter is organized as follows. In the next section we discuss a representation of continuous complex-valued functions on a segment as a product of traces of outer functions. This is needed to define the so-called Szegő function of a complex measure. The second section is devoted to the strong asymptotics for orthogonal symmetric trigonometric polynomials on the unit circle. This is the core of the method we use to analyze strong asymptotic behavior of polynomials orthogonal with varying measures on a segment. The main theorems on orthogonal polynomials are presented in the third section. In the last two sections we apply these results to meromorphic and Padé approximation of complex Cauchy transforms.

Szegő Functions

It was shown by G. Szegő (cf. [132, Ch. X]) that polynomials orthogonal on the unit circle with respect to a positive measure converge to a certain transformation of the Radon-Nikodym derivative of that measure, assuming that the latter is log-integrable. This convergence is a key ingredient in proving the exact rates of the uniform convergence of meromorphic and Padé approximants. The transformation of the measure of orthogonality is called now the *Szegő function* of this measure. Later, with the help of a conformal mapping, similar results were obtained on a segment. Initially, the Szegő function was defined only for positive measures, but recently it was shown (cf.

[60], [10], [131]) that it is also well-defined for complex analytic measures, where the boundary value problem approach was used. In this section we show that the Szegő function can be defined for more general classes of complex measures using harmonic analysis.

Szegő Function for the Unit Disk

Let S be a complex-valued Dini-continuous non-vanishing function on $\mathbb{T}$ having zero winding number[11] there. Recall that Dini-continuity means integrability of $\omega(S;x)/x$ around zero, where $\omega(S;\cdot)$ is the modulus of continuity of S. We define *interior and exterior Szegő functions* of S as

$$\exp\left(\int_{\mathbb{T}}\frac{\log S(\xi)}{\xi-\zeta}\frac{d\xi}{2\pi i}\right) =: \begin{cases} \mathcal{G}_i(S;\zeta), & \zeta\in\mathbb{D}, \\ \mathcal{G}_e(S;\zeta), & \zeta\in\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}, \end{cases} \tag{4.2}$$

with any continuous branch of $\log S$. The latter exists since S has zero winding number on $\mathbb{T}$. It follows immediately from [81, Sec. 35] and [46, Thm. III.1.3] that $\mathcal{G}_i(S;\cdot)$ and $\mathcal{G}_e(S;\cdot)$ are well-defined outer functions in $\mathbb{D}$ and $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$, respectively, have continuous boundary values on $\mathbb{T}$, and

$$S(\xi)=\mathcal{G}_i(S;\xi)\mathcal{G}_e^{-1}(S;\xi), \quad \xi\in\mathbb{T}. \tag{4.3}$$

Moreover, it is easy to see that

$$\mathcal{G}_e(S;\infty)=1 \quad\text{and}\quad \mathcal{G}_i(S;0)=\mathcal{G}(S), \tag{4.4}$$

where

$$\mathcal{G}(S):=\exp\left(\int_{\mathbb{T}}\log S(\xi)\frac{|d\xi|}{2\pi}\right) \tag{4.5}$$

is the *geometric mean* of S. We remark that it is a direct consequence of [81, Sec. 26] that if $\mathcal{G}_i$ and $\mathcal{G}_e$ are outer functions in $\mathbb{D}$ and $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$, respectively, have continuous traces on $\mathbb{T}$, and satisfy (4.3) and (4.4), then necessarily $\mathcal{G}_i(\cdot)=\mathcal{G}_i(S;\cdot)$ and $\mathcal{G}_e(\cdot)=\mathcal{G}_e(S;\cdot)$.

Now, let S be a *symmetric* function on the unit circle, i.e. $S(\xi)=S(1/\xi)$ for all $\xi\in\mathbb{T}$. If S is non-vanishing and Dini-continuous, it necessarily has zero winding number. Then $\mathcal{G}_i(S;\cdot)$ and $\mathcal{G}_e(S;\cdot)$ are well-defined,

$$\mathcal{G}_i(S;\zeta)=\mathcal{G}(S)\mathcal{G}_e^{-1}(S;1/\zeta), \quad \zeta\in\overline{\mathbb{D}}, \tag{4.6}$$

[11]The winding number of S is equal to $(2i\pi)^{-1}\int_{\mathbb{T}}T'(\xi)/T(\xi)d\xi$ where T is any trigonometric polynomial sufficiently close to S in the uniform norm.

and

$$\mathcal{G}(S)S(\xi) = \mathcal{G}_i(S;\xi)\mathcal{G}_i(S;1/\xi), \quad \xi \in \mathbb{T}. \tag{4.7}$$

It is also well known that if $S \in L^1$ is a nonnegative function such that $\log S \in L^1$, then the Szegő functions $\mathcal{G}_i(S;\cdot)$ and $\mathcal{G}_e^{-1}(S;\cdot)$ are outer in H^2 and $\bar{H}^2$, respectively,

$$\mathcal{G}_i(S;\zeta) = \mathcal{G}(S)\overline{\mathcal{G}_e^{-1}(S;1/\bar{\zeta})}, \quad \zeta \in \mathbb{D},$$

and

$$S(\xi) = \frac{|\mathcal{G}_i(S;\xi)|^2}{\mathcal{G}(S)} = \frac{\mathcal{G}(S)}{|\mathcal{G}_e(S;\xi)|^2}, \quad \text{a.e. on } \mathbb{T}.$$

It will be useful later on to examine one special case, namely, $S(\xi) = |(\xi-\zeta)(\xi-\bar{\zeta})|^{2\tau(\zeta)}$, $\zeta \in \mathbb{T}$. It is easy to verify (see [81, Section 85]) that

$$\mathcal{G}_i(S;\xi) = \left((\xi-\zeta)(\xi-\bar{\zeta})\right)^{\tau(\zeta)}.$$

Now, let $\arg(\xi-\zeta)$ (resp. $\arg(\xi-\bar{\zeta})$) be any continuous on $\mathbb{T}\setminus\{\zeta\}$ (resp. $\mathbb{T}\setminus\{\bar{\zeta}\}$) determination of the argument of $(\xi-\zeta)$ (resp. $(\xi-\bar{\zeta})$). It is then clear that

$$\mathcal{G}_i(S;\xi)\mathcal{G}_i^{-1}(S;1/\xi) = \exp\left(i\tau(\zeta)\arg((\xi-\zeta)(\xi-\bar{\zeta}))\right)$$

is a piecewise continuous function with jumps of magnitude $2|\sin(\tau(\zeta)\pi)|$ at ζ and $\bar{\zeta}$ if $\zeta \neq \pm 1$ and $2|\sin(2\tau(\pm1)\pi)|$ otherwise, where $\arg((\xi-\zeta)(\xi-\bar{\zeta})) = \arg(\xi-\zeta)+\arg(\xi-\bar{\zeta})$.

In another connection, continuity is merely sufficient for complex-valued S to have well-defined Szegő functions. Indeed, let S be a complex-valued function on $\mathbb{T}$ and θ_S be an argument for S, i.e., a measurable real-valued function on the unit circle such that $S(\xi) = |S(\xi)|\exp(i\theta_S(\xi))$ for a.e. $\xi \in \mathbb{T}$. We say that θ_S is *conjugate exp-*L^2 if both $\theta_S \in L^1$ and $\exp\left(|\widetilde{\theta}_S|\right) \in L^2$, where $\widetilde{\theta}_S$ is the conjugate function to θ_S (see Lemma 4.7). Not every S has a conjugate exp-L^2 argument, but if there is one it is unique up to the addition of an integral multiple of 2π. Indeed, if θ_1 and θ_2 are two such arguments and we put $\theta = \theta_1 - \theta_2$, then $\theta \in L^1$ assumes values in the multiples of 2π, and $\exp\left(|\widetilde{\theta}|\right) \in L^1$. In particular $\widetilde{\theta} \in L^1$, thus if we let $m := \int_{\mathbb{T}} \theta(\xi)|d\xi|/2\pi$ then $\widetilde{\theta} - i(\theta - m)$ is the boundary function of $\mathcal{RH}(\widetilde{\theta};\cdot)$, the Riesz-Herglotz transform of $\widetilde{\theta}$ defined in (2.19). As $\exp\left(\mathcal{RH}(\widetilde{\theta};\cdot)\right)$ is an outer function with modulus $\exp\left(\widetilde{\theta}\right)$ on $\mathbb{T}$, we deduce since it lies in L^1 that $\exp\left(\mathcal{RH}(\widetilde{\theta};\cdot)\right) \in H^1$. Because it is real-valued a.e. on $\mathbb{T}$, $\exp\left(\mathcal{RH}(\widetilde{\theta};\cdot)\right)$ is constant by the classical reflection principle [46, Ex. II.13] and consequently θ is also constant, as desired.

Now, if $S \in L^1$ has conjugate exp-L^2 argument θ_S and if moreover $\log|S| \in L^1$ then (4.5) with $\log S = \log|S| + i\theta_S$ is uniquely determined and defines the geometric mean of S. Moreover, for such S the interior and exterior Szegö functions given by (4.2) are also well-defined. It is easy to see that $\mathcal{G}_i(S;\cdot)$ does not depend on a particular choice of θ_S and

$$\mathcal{G}_i^2(S;\zeta)\mathcal{G}^{-1}(S) = \exp\left(\mathcal{RH}(\log|S| - \widetilde{\theta}_S;\zeta)\right), \quad \zeta \in \mathbb{D}.$$

Thus, it is an outer function in H^2. Observe also that

$$\mathcal{G}_e(S;\zeta)\mathcal{G}_i(S(1/\cdot);1/\zeta) = \mathcal{G}(S), \quad \zeta \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}, \tag{4.8}$$

so that $\mathcal{G}_e^{-1}(S;\cdot)$ is outer in $\bar{H}^2$. Now it is straightforward from (4.2) and the jump theorem [46, Ex. III.10] that (4.3) holds for such S, $\mathcal{G}_i(S;\cdot)$, $\mathcal{G}_e(S;\cdot)$ almost everywhere on $\mathbb{T}$.

When $S > 0$, we may choose $\theta_S \equiv 0$ and we recover the standard definition of Szegő functions. More important to us is the case where S is continuous and does not vanish on $\mathbb{T}$. Such a function has a well-defined winding number and if the winding number is zero then S has a continuous argument and therefore continuous logarithm.

Proposition 4.1 *Let S be a continuous non-vanishing complex-valued function on $\mathbb{T}$ having zero winding number. Then $\mathcal{G}_i(S;\cdot)$ and $\mathcal{G}_e(S;\cdot)$ given by (4.2) with continuous argument of S, are invertible in H^p and $\bar{H}^p$ respectively for all $p \in [1,\infty)$, and up to a multiplicative constant they are the only such functions for which (4.3) holds.*

Proof: Since θ_S is continuous, $\exp\left(|\widetilde{\theta}_S|\right)$ lies in L^p for $p \in [1,\infty)$ [46, Cor. 2.6]. In particular, θ_S is conjugate exp-L^2 and the previous considerations apply. Since $\mathcal{G}_i(S;\cdot)$ is outer with modulus $\sqrt{|\mathcal{G}(S)S|}\exp\left(-\widetilde{\theta}_S/2\right)$ on $\mathbb{T}$, and the latter function together with its inverse lies in L^p for $p \in [1,\infty)$, we get that $\mathcal{G}_i(S;\cdot)$ is invertible in H^p for such p. Because $S(1/\cdot)$ satisfies the same assumptions as S, we see from (4.8) that $\mathcal{G}_e(S;\cdot)$ is also invertible in every $\bar{H}^p$ for $p \in [1,\infty)$. Besides, if $F_1/F_2 = F_3/F_4$ for invertible $F_1, F_3 \in H^2$ and invertible $F_2, F_4 \in \bar{H}^2$, then $F_1/F_3 \in H^1$ must be equal to $F_2/F_4 \in \bar{H}^1$ and therefore is a constant. This establishes the uniqueness of (4.3). ■

Remarks: Beyond the case of non-vanishing continuous functions handled in Proposition 4.1, we can similarly address the factorization of invertible functions in $H^\infty + C(\mathbb{T})$; indeed, such functions assume the form [46, Ex. IX.4] $\xi^n g\exp(i(u+\widetilde{v}))$, where n is the winding number, g is invertible in H^∞, and $u, v \in C(\mathbb{T})$. If the winding number is zero, we may put $S = \exp(i(u+\widetilde{v}))$ which has a conjugate exp-L^p argument for all p such that $p \in [1, \infty)$ by [46, Cor. 2.6], and write the function as the product of $g\mathcal{G}_i(S; \cdot)$ that is outer in H^p and $1/\mathcal{G}_e(S; .)$ that is outer in $\bar{H}^p$.

Szegő Function for a Slit Plane

Let $E = [a, b] \subset \mathbb{R}$ be an interval and $\mathcal{J}$ be the Joukowski transformation mapping $\mathbb{D}$ (and $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$) conformally into $\overline{\mathbb{C}} \setminus E$, i.e.,

$$\mathcal{J}(\zeta) := \mathcal{J}(E; \zeta) = \frac{b-a}{4}\left(\zeta + \frac{1}{\zeta}\right) + \frac{b+a}{2}. \tag{4.9}$$

Recall that φ_i and φ_e, defined in (2.39), are the inverse function of $\mathcal{J}$ mapping $\overline{\mathbb{C}} \setminus E$ onto $\mathbb{D}$ and $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, respectively. Since $\mathcal{J}(\mathbb{T}) = E$ and $\mathcal{J}(\xi) = \mathcal{J}(1/\xi)$, $\xi \in \mathbb{T}$, φ_i, which is equal to $1/\varphi_e$, has well-defined continuous extension on both sides of E and $\varphi_i^+ = 1/\varphi_i^-$ on E with $\operatorname{Im}(\varphi_i^+) \geq 0$.

Now, let s be a non-vanishing Dini-continuous function on E. Then $S := s \circ \mathcal{J}$ is symmetric non-vanishing Dini-continuous function on $\mathbb{T}$ and therefore with well-defined geometric mean and exterior Szegő function. Thus, we define the *geometric mean* and *Szegő function* of s as

$$\mathcal{S}(s) := \mathcal{G}(S) \quad \text{and} \quad \mathcal{S}(s; z) := \mathcal{G}_e^{-1}(S; \varphi_e(z)), \quad z \in \overline{\mathbb{C}} \setminus E.$$

Furthermore, let h be a measurable function on E. Then h is summable with respect to μ_E if and only if $h \circ \mathcal{J}$ is summable on the circle, in which case it holds that

$$\sqrt{(z-a)(z-b)} \int \frac{h(t)}{z-t}\, d\mu_E(t) = \mathcal{RH}(h \circ \mathcal{J}; \varphi_i(z)) \quad z \in \overline{\mathbb{C}} \setminus E, \tag{4.10}$$

where the branch of the square root is chosen in such a way that $\sqrt{(z-a)(z-b)} = z + O(1)$ as $z \to \infty$. Indeed, it is a straightforward computation using the symmetry of $h \circ \mathcal{J}$ on $\mathbb{T}$ and (4.9) to obtain that

$$\begin{aligned}
\sqrt{(z-a)(z-b)} \int \frac{h(t)}{z-t} d\mu_E(t) &= \left(\frac{1}{\varphi_i(z)} - \varphi_i(z)\right) \int_{\mathbb{T}} \frac{(h \circ \mathcal{J})(\xi)}{(\varphi_i(z) - 1/\varphi_i(z)) - (\xi - 1/\xi)} \frac{|d\xi|}{2\pi} \\
&= \int_{\mathbb{T}} (h \circ \mathcal{J})(\xi) \left(\frac{\xi}{\xi - \varphi_i(z)} - \frac{\xi}{\xi - 1/\varphi_i(z)}\right) \frac{|d\xi|}{2\pi} \\
&= \mathcal{RH}(h \circ \mathcal{J}; \varphi_i(z)), \quad z \in \overline{\mathbb{C}} \setminus E.
\end{aligned}$$

Thus, we obtain the following representations:

$$\mathcal{S}(s) = \exp\left(\int \log s(t) d\mu_E(t)\right) \tag{4.11}$$

and

$$\mathcal{S}(s;z) = \exp\left(\frac{1}{2}\sqrt{(z-a)(z-b)}\int \frac{\log\left(s(t)/\mathcal{S}(s)\right)}{z-t} d\mu_E(t)\right), \tag{4.12}$$

where we take any continuous determination of $\log s$.

Recall that the conformal map φ_e extends continuously to both sides of E. Therefore, it follows from the corresponding properties of $\mathcal{G}_e(S;\cdot)$ that $\mathcal{S}(s;\cdot)$ is an outer function in $\overline{\mathbb{C}} \setminus E$, boundary values $\mathcal{S}^{\pm}(s;\cdot)$ are continuous on E, and

$$s(t) = \mathcal{S}(s)\mathcal{S}^{+}(s;t)\mathcal{S}^{-}(s;t), \quad t \in E. \tag{4.13}$$

As in the case of the circle, if $\mathcal{S}$ is a non-vanishing analytic function in $\overline{\mathbb{C}}\setminus E$ with continuous boundary values on both sides of E satisfying (4.13) and such that $\mathcal{S}(\infty) = 1$, then $\mathcal{S}(\cdot) = \mathcal{S}(s;\cdot)$. It is also well-known that if s is nonnegative and log-integrable on E, then $\mathcal{S}(s;\cdot)$ is an outer function in $H^2(\overline{\mathbb{C}} \setminus E)$, (4.13) holds almost everywhere on E, and

$$\mathcal{S}^{\pm}(s;t) = \overline{\mathcal{S}^{\mp}(s;t)} \quad \text{a.e. on} \quad E.$$

We conclude this subsection by showing the squared Szegő function of an algebraic polynomial v of degree at most k. Let $Z_k(v)$ consist of the zeros of v, repeated up to their multiplicities, and the point infinity with multiplicity $k - \deg(v)$; thus, $Z_k(v)$ has cardinality k. We assume that v has no zero on E, so that $\varphi_i(\zeta) \in \mathbb{D}$ for $\zeta \in Z_k(v)$. Define a function $\Psi_k(v;\cdot)$ on $\overline{\mathbb{C}}\setminus E$ by the rule

$$\Psi_k(v;z) := \prod_{\zeta \in Z_k(v)} \frac{\varphi_i(z) - \varphi_i(\zeta)}{1 - \varphi_i(z)\varphi_i(\zeta)}. \tag{4.14}$$

Clearly, Ψ_k extends continuously on E from above and below, and $\Psi_k^{+}(v;t)\Psi_k^{-}(v;t) = 1$ for $t \in E$ because $\varphi_i^{-} = 1/\varphi_i^{+}$. Note that $\Psi_k(1;z) = \varphi_i^k(z)$. Then it follows from the uniqueness of the Szegő function that

$$\mathcal{S}^2(v;z) = \mathcal{S}(v^2;z) = \frac{1}{\mathcal{S}(v)}\frac{v(z)\varphi_i^k(z)}{\Psi_k(v;z)}, \quad z \in \overline{\mathbb{C}} \setminus E, \tag{4.15}$$

and

$$\mathcal{S}(v) = v_0(\mathrm{cap}(E))^{\deg(v)} \prod_{\substack{\zeta \in Z_k(v) \\ \zeta \neq \infty}} \varphi_i^{-1}(\zeta), \tag{4.16}$$

where v_0 is the leading coefficient of v. As we shall see, the use of k instead of $\deg(v)$ in (4.15) is convenient when dealing with polynomials associated with interpolation schemes that may contain points at infinity.

Finally, we deduce from Proposition 4.1 the corresponding result on a segment.

Proposition 4.2 *Let s be a continuous non-vanishing function on E. Then the Szegő function of s, given by (4.12) for any continuous argument of s on E, is outer and invertible in $H^p(\overline{\mathbb{C}} \setminus E)$ for $p \in [1, \infty)$. In particular, $\mathcal{S}(s; \cdot)$ has non-tangential boundary values $\mathcal{S}^{\pm}(s; t)$ from above and below E for almost every $t \in E$ and the boundary functions $\mathcal{S}^{\pm}(s; \cdot)$ lie in $L^p(E, \mu_E)$ for all such p. Moreover, the factorization (4.13) takes place almost everywhere on E.*

Szegő Function for an Annulus and a Doubly Slit Plane

An analogue of the Szegő function for an annulus was introduced in [72] in the course of the investigation of the strong asymptotics for minimal Blaschke products. More precisely, let $\mathbb{A}_r = \{z \in \mathbb{C} : \ r < |z| < 1/r\}$, $r < 1$, then it was shown [72, Thm. 1.6] that for any function S that is continuous and strictly positive on $\mathbb{T}$ there exists a unique (up to a unimodular constant) function $\mathcal{D}_r(S; \cdot)$ such that

(i) $\mathcal{D}_r(S; \cdot)$ is an outer function in $\mathbb{A}_r$ and its winding number on $\mathbb{T}$ is equal to zero;

(ii) $\mathcal{D}_r(S; z)\overline{\mathcal{D}_r(S; 1/\bar{z})} = 1$, $z \in \mathbb{A}_r$;

(iii) $\mathcal{G}_S |\mathcal{D}_r(S; r\xi)|^2 = S(\xi)$, $\xi \in \mathbb{T}$, where $\mathcal{G}_S$ is the geometric mean of S.

We remark that (i) is not explicitly stated in [72, Thm. 1.6], but can be deduced easily. Indeed, $D_r(S; z) = \exp\{u(z) + iv(z)\}$, where u is the solution of the Dirichlet boundary value problem with data $(\log S(\cdot/r) - \log \mathcal{G}(S))/2$ on $\mathbb{T}_r$ and zero on $\mathbb{T}$. This, in fact, implies that u has well-defined conjugate harmonic function v in $\mathbb{A}_r$ ($\mathcal{D}_r(S; \cdot)$ has the winding number zero on any curve separating $\mathbb{T}_r$ and $\mathbb{T}_{1/r}$) and is an integral of its boundary values against the harmonic measure on $\mathbb{A}_r$ ($\mathcal{D}_r(S; \cdot)$ is an outer function).

The existence of a Szegő function for any DS_2-domain follows by the conformal equivalence of that domain to some annulus. In what follows we shall concentrate only on the case of a doubly slit plane, since this is our main interest.

Let $E = [a, b] \subset (-1, 1)$ be an interval and E^{-1} be its reflection across the unit circle. Define φ by the rule

$$\varphi(z) := \exp\left(2\pi\tau^2 \int_1^z \frac{dt}{\sqrt{(t-a)(t-b)(1-at)(1-bt)}}\right) \tag{4.17}$$

with the integration along any path in $\overline{\mathbb{C}} \setminus (E \cup E^{-1})$, where

$$\tau := \sqrt{\frac{1-ab}{\mathcal{T}}} \tag{4.18}$$

and $\mathcal{T}$ was defined in (2.13). Using (2.12) and [113, Thm. VIII.6.1] one can check that φ is the conformal map of $\overline{\mathbb{C}} \setminus (E \cup E^{-1})$ onto the annulus $\mathbb{A}_\rho$, where

$$\rho = \rho(E) = \varphi(b). \tag{4.19}$$

It can be readily verified that $\varphi(\mathbb{T}) = \mathbb{T}$, $\varphi(\bar{z}) = \overline{\varphi(z)}$, and $\varphi(1/z) = 1/\varphi(z)$, $z \in \overline{\mathbb{C}} \setminus (E \cup E^{-1})$. Thus, $\varphi(\mathbb{D} \setminus E) = \mathbb{A}_{\rho,1} := \{z : \ \rho < |z| < 1\}$ and the number $1/\rho$ is also known as a *modulus of the ring domain* $\mathbb{D} \setminus E$. Moreover, as it is shown in the Appendix, φ extends continuously on each side of E (resp. E^{-1}) and $\varphi^\pm(E) = \mathbb{T}_\rho^\pm$ (resp. $\varphi^\pm(E^{-1}) = \mathbb{T}_{1/\rho}^\pm$), where $\mathbb{T}_{1/\rho}^+$ and $\mathbb{T}_{1/\rho}^-$ are the upper and lower semicircles of $\mathbb{T}_{1/\rho}$.

Now, let s^+ and s^- be two continuous positive functions on E such that $s^+(a) = s^-(a)$ and $s^+(b) = s^-(b)$. We define a geometric mean of the pair (s^+, s^-) with respect to the condenser $(E, \mathbb{T})$ as

$$\mathcal{D}(s^+, s^-) := \exp\left(\frac{1}{2}\int \log(s^+ s^-)(t) d\mu_{(E,\mathbb{T})}(t)\right). \tag{4.20}$$

It can be checked that $\mathcal{D}(s^+, s^-) = \mathcal{G}_S$, where $S(\xi) = s^\pm(\varphi^{-1}(r\xi))$, $\xi \in \mathbb{T}^\pm$, respectively. For simplicity, we write $\mathcal{D}(s)$ if $s^+ = s^- = s$. Furthermore, the Szegő function of the pair (s^+, s^-) for the condenser $\overline{\mathbb{C}} \setminus (E \cup E^{-1})$ is defined by

$$\mathcal{D}(s^+, s^-; z) = \mathcal{D}_\rho(S; \varphi(z)). \tag{4.21}$$

It is an immediate consequence of the corresponding properties of the Szegő function for an annulus that $\mathcal{D}(s^+, s^-; \cdot)$ is an outer function in $H^p(\overline{\mathbb{C}} \setminus (E \cup E^{-1}))$, $p < \infty$, $(H^\infty(\overline{\mathbb{C}} \setminus (E \cup E^{-1}))$ if $s^\pm$ is a

Dini-continuous pair, i.e. S is Dini-continuous), has non-tangential boundary values on both sides of E and E^{-1} such that

$$|\mathcal{D}^{\pm}(s^+,s^-;t)|^2 = \begin{cases} s^{\pm}(t)/\mathcal{D}(s^+,s^-), & t\in E, \\ \mathcal{D}(s^+,s^-)/s^{\pm}(1/t), & t\in E^{-1}, \end{cases} \tag{4.22}$$

satisfies $\mathcal{D}(s^+,s^-;z)\overline{\mathcal{D}(s^+,s^-;1/\bar{z})} = 1$, and has winding number zero on any curve separating E from E^{-1}. We emphasize that $\mathcal{D}(s^+,s^-;\cdot)$ is unique up to a unimodular constant. It will be convenient for us to choose the normalization on each particular occasion.

Orthogonal Symmetric Trigonometric Polynomials on the Unit Circle

The approach that we take follows the path deliberated by G. Baxter in [30], where orthogonality relations for polynomials on a segment are translated into orthogonality relations for *symmetric trigonometric polynomials (STP)* on the unit circle. A monic STP of degree n is a function of the form

$$Q(\xi) = (\xi^n+\xi^{-n}) + \sum_{j=0}^{n-1} r_j(\xi^j+\xi^{-j}), \quad r_j\in\mathbb{C}.$$

Before we state the results of this section, we need some preparation to describe the varying weights that we shall use in the orthogonality relations for such trigonometric polynomials. Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of functions on $\mathbb{T}$ such that

$$0<m\leq|S_n(\xi)|\leq M<\infty, \quad \xi\in\mathbb{T}, \quad \text{and} \quad \omega(S_n;x)\leq\omega(x), \quad x\in[0,2], \tag{4.23}$$

for all $n\in\mathbb{N}$, where $\omega(S_n;\cdot)$ is a modulus of continuity of S_n and ω is a Dini-continuous function of the modulus continuity type. It follows immediately from the discussion in the preceding section that

$$F_n(\xi) := \mathcal{G}_i(S_n;\xi)\mathcal{G}_i^{-1}(S_n;1/\xi) \tag{4.24}$$

is a continuous function on $\mathbb{T}$ and $F_n(\pm1)=1$. Further, let $\{C_l\}$, $C_l=\{c_{j,l}\}_{j=1}^l\subset\mathbb{D}$, be a sequence of sets of points satisfying (2.41). We associate to each set C_l a polynomial V_l and two rational functions B_l and R_l given by

$$V_l(\xi) := \prod_{j=1}^l(1-c_{j,l}\xi), \quad B_l := \prod_{j=1}^l\frac{\xi-c_{j,l}}{1-c_{j,l}\xi}, \quad \text{and} \quad R_l(z) := \prod_{j=1}^l\frac{1-\bar{c}_{j,l}\xi}{1-c_{j,l}\xi}. \tag{4.25}$$

Notice that $|B_l(\xi)|=|R_l(\xi)|$ and $B_l(1/\xi)=1/B_l(\xi)$ for $\xi\in\mathbb{T}$. If C_l is a conjugate-symmetric set, then B_l is a Blaschke product and $R_l \equiv 1$. We also define a nonnegative function $\hat{S}$ on $\mathbb{T}$ by setting

$$\hat{S}(\xi) := \prod_{\zeta\in\mathbf{T}} |(\xi-\zeta)(\xi-\bar{\zeta})|^{\tau(\zeta)}, \tag{4.26}$$

where $\mathbf{T}$ is a conjugate-symmetric finite subset of $\mathbb{T}$ and the exponents $\tau(\zeta) \in [0, 1/2)$, $\zeta \in \mathbf{T}$, are such that

$$\tau(\zeta) = \tau(\bar{\zeta}) \quad \text{and} \quad \gamma := \max_{\zeta\in\mathbf{T}} \max_{n\in\mathbb{N}} \{\sin(\tau(\zeta)\pi)|F_n(\zeta)R_{2n+N}(\zeta)|\} < 1, \tag{4.27}$$

where $N \in \mathbb{Z}_+$ is fixed throughout the rest of the section. Finally, we set

$$W_n(\xi) := \hat{S}(\xi)S_n(\xi)V^{-1}_{2n+N}(\xi)V^{-1}_{2n+N}(1/\xi). \tag{4.28}$$

The following theorem is instrumental for the proofs of the results stated in the next section.

Theorem 4.3 *Let $\{Q_n\}_{n\in\mathbb{N}}$ be a sequence of STPs of degree at most n such that*

$$\int_{\mathbb{T}} \xi^j Q_n(\xi)W_n(\xi)\frac{|d\xi|}{2\pi} = 0, \quad j = 0, \ldots, n-1, \tag{4.29}$$

where W_n are given by (4.23)–(4.28). Then Q_n has exact degree n for all n large enough. Thus, Q_n can be normalized to be monic and under such a normalization possesses the following asymptotic behavior:

$$\zeta^{-n}Q_n(\zeta) = (1 + o(1))\mathcal{G}_e(W_n; \zeta), \tag{4.30}$$

where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$. Further, there exist functions H_n, analytic in $\mathbb{D}$, satisfying $\limsup_{n\to\infty} \|H_n\|_2 \leq \gamma/(1-\gamma)$ and $H_n = o(1)$ locally uniformly in $\mathbb{D}$ such that

$$Q_n(\xi) = (1 + H_n(\xi))\xi^n\mathcal{G}_e(W_n; \xi) + (1 + H_n(1/\xi))\xi^{-n}\mathcal{G}_e(W_n; 1/\xi), \quad \xi \in \mathbb{T}. \tag{4.31}$$

Apart from the asymptotic behavior of Q_n we are interested in the asymptotic properties of the Riesz-Herglotz transform of $Q_n^2 W_n$, defined (2.19), and, in particular, in its value at zero, i.e.,

$$\alpha_n := \mathcal{RH}(Q_nW_n; 0) = \int_{\mathbb{T}} Q_n^2(\xi)W_n(\xi)\frac{|d\xi|}{2\pi}. \tag{4.32}$$

The following theorem takes place.

Theorem 4.4 *Let W_n, Q_n, and H_n be as in Theorem 4.3 with Q_n monic. Then*

$$\alpha_n = (2 + o(1))\,\mathcal{G}(W_n). \tag{4.33}$$

and

$$\alpha_n^{-1}\mathcal{R}\mathcal{H}(Q_n^2 W_n; \zeta) = (1 + o(1))(1 + H_n(\zeta))\zeta^n Q_n(\zeta)\mathcal{G}_e^{-1}(W_n; 1/\zeta), \quad \zeta \in \overline{\mathbb{D}}, \tag{4.34}$$

where $o(1)$ holds uniformly on $\overline{\mathbb{D}}$.

Observe that by (4.30) we immediately get that $\alpha_n^{-1}\mathcal{R}\mathcal{H}(Q_n^2 W_n; \cdot) = 1 + o(1)$, where $o(1)$ holds locally uniformly in $\mathbb{D}$.

In the case of non-vanishing weights one more important result can be obtained.

Proposition 4.5 *Let Q_n and W_n be as in Theorem 4.3 with $\hat{S} \equiv 1$ and Q_n monic. Then*

$$\alpha_n^{-1} Q_n^2(\xi) W_n(\xi)|d\xi| \xrightarrow{*} |d\xi|. \tag{4.35}$$

Additional analyticity assumptions on the sequence $\{S_n\}_{n\in\mathbb{N}}$ allow one to improve asymptotic estimates of $\{H_n\}$.

Proposition 4.6 *Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of symmetric functions on $\mathbb{T}$ that extend analytically to $\mathbb{A}_r$, $r < 1$. Assume further that $\log S_n$ form a normal family in $\mathbb{A}_r$ and let V_{2n+N} be as in Theorem 4.3. For $\{Q_n\}_{n\in\mathbb{N}}$ a sequence of STPs of degree at most n satisfying (4.29) with $W_n(\xi) = S_n(\xi)V_{2n+N}^{-1}(\xi)V_{2n+N}^{-1}(1/\xi)$, we have in addition to the conclusions of Theorems 4.3 and 4.4 that $\|H_n\|_\infty = o(1)$.*

Before we proceed with proofs of the theorems above, we shall need several auxiliary lemmas.

Lemma 4.7 *Let $\{S_n\}_{n\in\mathbb{N}}$ be as in Theorem 4.3. Then the interior Szegő functions $\mathcal{G}_i(S_n; \cdot)$ form a uniformly equicontinuous family on $\overline{\mathbb{D}}$ and their moduli are uniformly bounded away from zero and infinity there.*

Proof: As $\omega(S_n; \cdot) \le \omega$, the family $\{S_n\}$ is uniformly equicontinuous on $\mathbb{T}$. Then by the Ascoli-Arzelà theorem any sequence of functions from this family contains a convergence subsequence. This, in particular, implies the existence of a constant M^*, independent of n, such that

$$|\arg(S_n; \xi) - \arg(S_n; 0)| \le \pi M^*$$

for any $\xi \in \mathbb{T}$, where $\arg(S_n; \cdot)$ is any continuous determination of the argument of S_n. The latter is well-defined since S_n has zero winding number. Now, let $t_0 \in (0, 2]$ be such that $\omega(t_0) \leq m/2$. Then for an arbitrary fixed $\xi_1 \in \mathbb{T}$ and all $n \in \mathbb{N}$ we have that

$$|S_n(\xi_1) - S_n(\xi)| \leq m/2 \tag{4.36}$$

for all $\xi \in \mathbb{T}$ such that $|\xi_1 - \xi| \leq t_0$. Since $|S_n(\xi_1)| \geq m$ for all $n \in \mathbb{N}$, we deduce from (4.36) that

$$|\arg(S_n; \xi_1) - \arg(S_n; \xi_2)| \leq \pi/3$$

whenever $|\xi_1 - \xi_2| \leq t_0$. This justifies the following equality:

$$\log S_n(\xi_1) - \log S_n(\xi_2) = \mathrm{Log}\left(1 + \frac{S_n(\xi_1) - S_n(\xi_2)}{S_n(\xi_2)}\right),$$

where Log indicates the principal branch of the logarithm. Therefore, using the inequality $|\mathrm{Log}(1 + z)| < 2|z|$ for $|z| < 1/2$, we have that

$$\omega(\log S_n; t) := \sup_{|\xi_1 - \xi_2| \leq t} |\log S_n(\xi_1) - \log S_n(\xi_2)| \leq \begin{cases} 2\omega(t)/m, & t \in (0, t_0), \\ 2(\pi M^* + \log M), & t \in [t_0, 2]. \end{cases} \tag{4.37}$$

By the assumptions on ω, this entails that $\{\log S_n\}$ is a uniformly equicontinuous family of Dini-continuous functions on $\mathbb{T}$. Let U_n be the harmonic extension of $\log S_n$ to the unit disk and let $\widetilde{U}_n$ be its conjugate function normalized at zero to be zero. By the Carleson-Jacobs theorem $\widetilde{U}_n$ is continuous on $\mathbb{T}$ and for $\delta < 1$ we get (see [46, Thm. III.1.3])

$$\begin{aligned} \omega(\widetilde{U}_n; \delta) &\leq C_1 \left(\int_{[0,\delta]} \frac{\omega(\log S_n; t)}{t} dt + \delta \int_{[\delta,2]} \frac{\omega(\log S_n; t)}{t^2} dt \right) \\ &\leq C_1 \left(\int_{[0,\sqrt{\delta}]} \frac{\omega(\log S_n; t)}{t} dt + \sqrt{\delta} \int_{[\sqrt{\delta},2]} \frac{\omega(\log S_n; t)}{t} dt \right), \end{aligned} \tag{4.38}$$

where C_1 is an absolute constant. Since $U_n = \log S_n$ on $\mathbb{T}$, we get from (4.37) and (4.38) that there exist positive continuous functions ω^* and $\widetilde{\omega}$ on $[0, 2]$ with $\omega^*(0) = \widetilde{\omega}(0) = 0$ such that

$$\begin{matrix} \omega(U_n; t) \leq \omega^*(t), \\ \omega(\widetilde{U}_n; t) \leq \widetilde{\omega}(t), \end{matrix} \quad t \in [0, 2], \quad n \in \mathbb{N}.$$

This implies uniform equicontinuity of $\{\widetilde{U}_n\}$ on $\mathbb{T}$. Moreover, by (4.37), we obtain

$$\|\widetilde{U}_n\|_\infty \leq \max_{t\in[0,2]} \omega(\widetilde{U}_n;t) \leq \max_{t\in[0,2]} \widetilde{\omega}(t) := \widetilde{M} < \infty, \quad n \in \mathbb{N},$$

where the first inequality is true because $\widetilde{U}_n$ vanishes on $[-1,1]$. Indeed, by the symmetry of $\log S_n$, we get that $U_n(\zeta) = U_n(\bar{\zeta})$, $\zeta \in \overline{\mathbb{D}}$. Thus, $\widetilde{U}_n(\zeta) = -\widetilde{U}_n(\bar{\zeta})$ there since $\widetilde{U}_n(0) = 0$.

It is an easy calculation to check that

$$\mathcal{G}_i^2(S_n;\zeta) = \mathcal{G}(S_n)\exp\left\{(U_n + i\widetilde{U}_n)(\zeta)\right\} =: \mathcal{G}(S_n)Y_n(\zeta), \quad \zeta \in \mathbb{D}.$$

Since the trace on $\mathbb{T}$ of Y_n equals to $\exp(U_n+i\widetilde{U}_n)$, we deduce that $\{Y_n\}$ is a family of outer functions in H^∞, continuous in $\overline{\mathbb{D}}$, satisfying

$$m\exp\left(-\widetilde{M}\right) \leq \exp\left(-\|\widetilde{U}_n\|_\infty\right)\min_{\varsigma\in\mathbb{T}}|S_n(\varsigma)| \leq |Y_n(\xi)| \leq \exp\left(\|\widetilde{U}_n\|_\infty\right)\|S_n\|_\infty \leq M\exp\left(\widetilde{M}\right),$$

for each $\xi \in \mathbb{T}$. Since $Y_n(0) = \mathcal{G}(S_n)$, the traces on $\mathbb{T}$ of the functions $\mathcal{G}_i(S_n;\cdot)$ form a uniformly equicontinuous family of functions whose moduli are uniformly bounded away from zero and infinity. The assertions of the lemma now follow from the maximum principle and the fact that, for uniformly bounded analytic functions on $\mathbb{D}$ with continuous boundary values, the equicontinuity of the latter on $\mathbb{T}$ implies the relative compactness in the uniform norm and therefore the equicontinuity on $\overline{\mathbb{D}}$.
■

Lemma 4.8 *Let $\{R_l\}_{l\in\mathbb{N}}$ be given by (4.25) and associated to a sequence of sets $\{C_l\}_{l\in\mathbb{N}} \subset \mathbb{D}$ satisfying (2.41). Then functions R_l form a uniformly equicontinuous sequence on $\mathbb{T}$ and their moduli are uniformly bounded away from zero and infinity there.*

Proof: By the first condition of (2.41), we get

$$\log|R_l(\xi)| = \sum_{j=1}^{l}\log\left|1 + \frac{\Delta_l(c_{j,l}) - \bar{c}_{j,l}}{1-\Delta_l(c_{j,l})\xi}\right| \leq \sum_{j=1}^{l}\frac{|\Delta_l(c_{j,l}) - \bar{c}_{j,l}|}{1-|\Delta_l(c_{j,l})|} \leq M_1, \quad \xi \in \mathbb{T}.$$

Thus, we have

$$\exp(-M_1) \leq |R_l^{-1}(1/\xi)| = |R_l(\xi)| \leq \exp(M_1), \quad \xi \in \mathbb{T}, \tag{4.39}$$

which establishes the uniform boundedness of $|R_l|$ on $\mathbb{T}$. Now, let $\delta > 0$ be given and $\xi_1, \xi_2 \in \mathbb{T}$ be such that $|\xi_1 - \xi_2| \leq \delta$. Then

$$\begin{aligned}\left|\frac{R_l(\xi_2)}{R_l(\xi_1)}\right| &\leq \exp\left(\sum_{j=1}^{l}\log\left|1+\frac{(\xi_1-\xi_2)(\bar{c}_{j,l}-\Delta_l(c_{j,l}))}{(1-\bar{c}_{j,l}\xi_1)(1-\Delta_l(c_{j,l})\xi_2)}\right|\right)\\ &\leq \exp\left(|\xi_1-\xi_2|\sum_{j=1}^{l}\frac{|\bar{c}_{j,l}-\Delta_l(c_{j,l})|}{(1-|c_{j,l}|)(1-|\Delta_l(c_{j,l})|)}\right) \leq \exp(M_2\delta) \qquad (4.40)\end{aligned}$$

by (2.41). Moreover, denoting by $\mathrm{Arg}(\zeta) \in (-\pi, \pi]$ the principal argument of $\zeta \in \mathbb{C} \setminus \{0\}$, we get since $|\mathrm{Arg}(1+z)| \leq |z|$ for $|z| < 1$ that

$$\begin{aligned}\left|\mathrm{Arg}\left(\frac{R_l(\xi_2)}{R_l(\xi_1)}\right)\right| &\leq \sum_{j=1}^{l}\left|\mathrm{Arg}\left(1+\frac{(\xi_1-\xi_2)(\bar{c}_{j,l}-\Delta_l(c_{j,l}))}{(1-\bar{c}_{j,l}\xi_1)(1-\Delta_l(c_{j,l})\xi_2)}\right)\right|\\ &\leq M_3|\xi_1-\xi_2|\sum_{j=1}^{l}\frac{|\bar{c}_{j,l}-\Delta_l(c_{j,l})|}{(1-|c_{j,l}|)(1-|\Delta_l(c_{j,l})|)} \leq M_4\delta, \qquad (4.41)\end{aligned}$$

where M_4 is some absolute constant. Combining (4.39), (4.40), and (4.41), we derive that

$$|R_l(\xi_1) - R_l(\xi_2)| \leq \exp(M_1)\left|1 - \frac{R_l(\xi_2)}{R_l(\xi_1)}\right| \leq M_5\delta,$$

where M_5 is an absolute constant. This finishes the proof of this lemma. ■

The following lemma is a technical step needed in the course of the proof of Theorem 4.3.

Lemma 4.9 *Let $\{F_n\}_{n\in\mathbb{N}}$ and $\{B_l\}_{l\in\mathbb{N}}$ be given by (4.24) and (4.25), respectively, where the sequence of sets $\{C_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$ satisfies (2.41). Let further $\hat{F}(\xi) := \xi^{-N}\mathcal{G}_i(\hat{S};\xi)\mathcal{G}_i^{-1}(\hat{S};1/\xi)$, $\xi \in \mathbb{T}$, with $N \in \mathbb{Z}_+$ fixed and $\hat{S}$ defined in (4.26) and (4.27). If $\{L_n\}_{n\in\mathbb{N}}$ is a sequence of functions of unit norm in H^2 satisfying*

$$L_n(0) - L_n(1/\xi) = \mathcal{P}_-\left(L_n B_{2n+N} F_n \hat{F}\right)(\xi), \qquad \xi \in \mathbb{T}\ , \quad n \in \mathbb{N}, \qquad (4.42)$$

then $L_n = L_n(0) + o(1)$, where $o(1)$ holds locally uniformly in $\mathbb{D}$, and

$$0 < 1 - \gamma \leq \liminf_{n\to\infty}|L_n(0)| \leq \limsup_{n\to\infty}|L_n(0)| \leq 1 + \gamma < \infty. \qquad (4.43)$$

Proof: Let $B_l^* := B_l/R_l$ be a Blaschke product with zeros C_l, where R_l was defined in (4.25). Then equations (4.42) can be written in the following form

$$L_n(0) - L_n(1/\xi) = \mathcal{H}_n\left(L_n B^*_{2n+N}\right)(\xi) \quad \xi \in \mathbb{T}, \quad n \in \mathbb{N}, \tag{4.44}$$

where $\mathcal{H}_n : H^2 \to \bar{H}^2_0$ is the Hankel operator with a symbol $F_n R_{2n+N}\hat{F}$.

First, we shall verify that the sequence $\{L_n B^*_{2n+N}\}$ weakly converges to zero in H^2. This is tantamount to show that functions $L_n B^*_{2n+N}$ converge to zero locally uniformly in $\mathbb{D}$. Notice that the second condition of (2.41) is necessary and sufficient for such a convergence of Blaschke products B^*_{2n+N} ([46, Thm. II.2.1]). Thus, the claim follows by the Cauchy integral formula and boundedness the L^2 norms of L_n.

Second, we claim that

$$\limsup_{n\to\infty} \|\mathcal{H}_n(L_n B^*_{2n+N})\|_2 \leq \gamma. \tag{4.45}$$

Indeed, it is enough to show that from any subsequence of $\mathbb{N}$ one can extract another subsequence along which (4.45) holds. By Lemmas 4.7 and 4.8, the families $\{F_n\}$ and $\{R_{2n+N}\}$ are bounded and uniformly equicontinuous; so, by the Ascoli-Arzelà theorem, we may assume that F_n and R_{2n+N} converge uniformly on $\mathbb{T}$ to some continuous functions F and R, respectively. Since

$$\|\mathcal{H}_n - \mathcal{H}_{FR\hat{F}}\| = \|\mathcal{H}_{(F_n R_{2n+N}\hat{F} - FR\hat{F})}\| \leq \|F_n R_{2n+N}\hat{F} - FR\hat{F}\|_\infty,$$

it is enough to prove (4.45) for $\mathcal{H}_{FR\hat{F}}$ instead of $\mathcal{H}_n$. We recall that $\hat{F}$ is a piecewise continuous function on $\mathbb{T}$ with the jump-type discontinuities at each point $\zeta \in \mathbf{T}$ of magnitude $2\sin(\tau(\zeta)\pi)$. Thus, by Power's theorem (cf. [86, Thm. 5.4.2]), the *essential norm* of the Hankel operator with piecewise continuous symbol $FR\hat{F}$ (*i.e.* its norm modulo compact operators) is equal to half the magnitude of the largest jump of the symbol, which does not exceed γ by (4.27). In other words, for any $\delta > 0$ there exists a compact operator $\mathcal{C}$ such that the norm of $(\mathcal{H}_{FR\hat{F}} - \mathcal{C})$ is less than $\gamma + \delta$. Then

$$\limsup_{n\to\infty} \|\mathcal{H}_{FR\hat{F}}(L_n B^*_{2n+N})\|_2 \leq \limsup_{n\to\infty}(\|(\mathcal{H}_{FR\hat{F}} - \mathcal{C})(L_n B^*_{2n+N})\|_2 + \|\mathcal{C}(L_n B^*_{2n+N})\|_2) \leq \gamma + \delta, \tag{4.46}$$

where the last inequality uses that $\|\mathcal{C}(L_n B^*_{2n+N})\|_2 \to 0$ by the compactness of $\mathcal{C}$. This proves (4.45) on letting $\delta \to 0$.

Finally, since any bounded operator preserves weak convergence, the described compactness argument implies that $\mathcal{H}_n(L_n B^*_{2n+N}) = o(1)$, where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}} \setminus \mathbb{D}$. This together with (4.44) yields locally uniform convergence to zero in $\mathbb{D}$ of $L_n - L_n(0)$. Further, combining (4.44) and (4.45), we see that $\limsup_{n\to\infty} \|L_n - L_n(0)\|_2 \leq \gamma$, and since $\|L_n\|_2 = 1$ we get $\limsup_{n\to\infty} |1 - |L_n(0)|| \leq \gamma$, which yields (4.43). ■

Lemma 4.10 *Let $\{B_l\}_{l\in\mathbb{N}}$ be given by (4.25) and associated to a sequence of sets $\{C_l\}_{l\in\mathbb{N}} \subset \mathbb{D}$ satisfying (2.41). Further, let $\{X_l\}_{l\in\mathbb{N}} \subset L^\infty$ be a compact family and $\{L_l\}_{l\in\mathbb{N}}$ be a sequence of functions of unit norm in H^2. Then for any fixed $N \in \mathbb{Z}_+$ the following limit holds locally uniformly in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$*

$$\lim_{n\to\infty} \zeta\mathcal{P}_- \left((1/\cdot)(B_{2n+N} L_n X_n)(1/\cdot)\right)(\zeta) = 0. \tag{4.47}$$

Proof: Let, as before, $B^*_l := B_l/R_l$ be a Blaschke product with zeros C_l (see (4.25)). Then functions $B_{2n+N}L_n$ have uniformly bounded norms in H^2 and converge to zero locally uniformly in $\mathbb{D}$, i.e., weakly converge to zero in L^2 (see Lemmas 4.8 and 4.9). Further, one can easily check that

$$\zeta\mathcal{P}_- \left((1/\cdot)(B_{2n+N} L_n A_n)(1/\cdot)\right)(\zeta) = \mathcal{P}_+(B_{2n+N}L_nX_n)(\zeta) = \mathcal{T}_{X_n}(B_{2n+N}L_n)(1/\zeta), \quad 1/\zeta \in \mathbb{D},$$

where $\mathcal{T}_{X_n}$ is a Toeplitz operator with symbol X_n. By a standard compactness argument, it is enough to show that functions $\mathcal{T}_X(B_{2n+N}L_n)$ converge to zero locally uniformly in $\mathbb{D}$, where X is any limit point of $\{X_n\}$. Since any bounded operator (the norm of $\mathcal{T}_X$ is dominated by $\|X\|_\infty$) preserves weak convergence, the latter follows. ■

Proof of Theorem 4.3: The orthogonality relations (4.29) and the symmetry of Q_nW_n yield that all the Fourier coefficients of this product whose indices range from $-(n-1)$ to $n-1$ are equal to zero. In other words,

$$Q_n(\xi)W_n(\xi) = G_n(\xi)\xi^n + G_n(1/\xi)\xi^{-n} \tag{4.48}$$

for some functions $G_n \in H^\infty$. In fact, G_n is a continuous function on $\mathbb{T}$. For simplicity, denote $\mathcal{G}_n(\xi) := \mathcal{G}_i(S_n;\xi)\mathcal{G}_i(\hat{S};\xi)$. Recall that the latter two Szegő functions are well-defined by the symmetry and Dini-continuity of S_n and log-integrability of $\hat{S}$. Then with the help of (4.7), we can write

$$W_n(\xi) = \mathcal{G}_i(W_n;\xi)\mathcal{G}_e^{-1}(W_n;\xi) = \mathcal{G}_n(\xi)V^{-1}_{2n+N}(\xi)\mathcal{G}_e^{-1}(W_n;\xi).$$

This, together with (4.48), implies

$$\xi^{-n}Q_n(\xi)\mathcal{G}_e^{-1}(W_n;\xi) = (G_nV_{2n+N}\mathcal{G}_n^{-1})(\xi) + \xi^N(G_nV_{2n+N}\mathcal{G}_n^{-1}B_{2n+N})(1/\xi)\mathcal{G}_n(1/\xi)\mathcal{G}_n^{-1}(\xi), \quad (4.49)$$

where B_{2n+N} is associated to C_{2n+N} via (4.25). Since $\tau(\zeta) \in [0, 1/2)$ for any $\zeta \in \mathbf{T}$ and G_nV_{2n+N} is continuous on $\mathbb{T}$, $\kappa_n := \|G_nV_{2n+N}\mathcal{G}_n^{-1}\|_2$ is finite for every $n \in \mathbb{N}$. Thus,

$$L_n(\xi) := \kappa_n^{-1}(G_nV_{2n+N}\mathcal{G}_n^{-1})(\xi) = \kappa_n^{-1}G_n(\xi)\mathcal{G}_i^{-1}(W_n;\xi)$$

is a sequence of functions of unit norm in H^2. Set $\hat{F}(\xi) := \xi^{-N}\mathcal{G}_i(\hat{S};\xi)\mathcal{G}_i^{-1}(\hat{S};1/\xi)$, $\xi \in \mathbb{T}$. Then (4.49) can be rewritten as

$$\kappa_n^{-1}\xi^{-n}Q_n(\xi)\mathcal{G}_e^{-1}(W_n;\xi) = L_n(\xi) + (L_nB_{2n+N}F_n\hat{F})(1/\xi), \quad (4.50)$$

where F_n was defined in (4.24). It is not hard to see that the left-hand side of (4.50) is an $\bar{H}^2$ function and the first summand on the right-hand side belongs to H^2. Thus

$$-\mathcal{P}_-(L_n(1/\xi)) = \mathcal{P}_-((L_nB_{2n+N}F_n\hat{F})(\xi)).$$

So, we can applying Lemma 4.9 to get

$$0 < 1-\gamma \le \liminf_{n\to\infty}\frac{|\lambda_n|}{\kappa_n} \le \limsup_{n\to\infty}\frac{|\lambda_n|}{\kappa_n} \le 1+\gamma < \infty, \quad (4.51)$$

$$\limsup_{n\to\infty}\left\|G_n\mathcal{G}_i^{-1}(W_n;\cdot) - \lambda_n\right\|_2 \le \frac{\gamma}{1-\gamma}\limsup_{n\to\infty}\frac{1}{|\lambda_n|}, \quad (4.52)$$

and

$$G_n\mathcal{G}_i^{-1}(W_n;\cdot) - \lambda_n = o(1), \quad (4.53)$$

where $o(1)$ holds locally uniformly in $\mathbb{D}$ and

$$\lambda_n := G_n(0)\mathcal{G}_i^{-1}(W_n;0) = G_n(0)\mathcal{G}^{-1}(W_n). \quad (4.54)$$

Again, since the left-hand side of (4.50) belongs to $\bar{H}^2$, we can write it as

$$\lambda_n^{-1}\xi^{-n}Q_n(\xi)\mathcal{G}_e^{-1}(W_n;\xi) = 1 + \frac{\kappa_n}{\lambda_n}\,\xi\,\mathcal{P}_-\left((1/\cdot)(L_nB_{2n+N}F_n\hat{F})(1/\cdot)\right)(\xi). \quad (4.55)$$

Thus, we obtain from (4.51), (4.55), and Lemma 4.10, applied with $X_n = F_n\hat{F}$, that

$$\frac{\mathcal{G}(W_n)}{G_n(0)}\frac{\zeta^{-n}Q_n(\zeta)}{\mathcal{G}_e(W_n;\zeta)} = 1 + o(1) \tag{4.56}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. By taking $\zeta = \infty$ in (4.56), we see that Q_n has necessarily the exact degree n for all n large enough. So, we may assume Q_n to be monic and again, by taking $\zeta = \infty$ in (4.56), we get

$$\frac{1}{\lambda_n} = \frac{\mathcal{G}(W_n)}{G_n(0)} = 1 + o(1). \tag{4.57}$$

Thus, it remains to show formula (4.31). Define $H_n(\zeta) := G_n(\zeta)\mathcal{G}_i^{-1}(W_n;\zeta) - 1, \quad \zeta \in \mathbb{D}$. Then $\limsup_{n\to\infty} \|H_n\|_2 \leq \gamma/(1-\gamma)$ and $H_n = o(1)$ locally uniformly in $\mathbb{D}$ by (4.52) and (4.57). From (4.48) and the symmetry of W_n, we obtain

$$\begin{aligned} Q_n(\xi) &= \xi^n G_n(\xi)W_n^{-1}(\xi) + \xi^{-n}G_n(1/\xi)W_n^{-1}(1/\xi) \\ &= \xi^n \mathcal{G}_e(W_n;\xi)\frac{G_n(\xi)}{\mathcal{G}_i(W_n;\xi)} + \xi^{-n}\mathcal{G}_e(W_n;1/\xi)\frac{G_n(1/\xi)}{\mathcal{G}_i(W_n;1/\xi)} \\ &= \xi^n\mathcal{G}_e(W_n;\xi)(1 + H_n(\xi)) + \xi^{-n}\mathcal{G}_e(W_n;1/\xi)(1 + H_n(1/\xi)), \end{aligned}$$

where we used (4.6) and the definition of H_n. This finishes the proof of the theorem. ■

Proof of Theorem 4.4: The limit in (4.33) follows from (4.57) and the following computation:

$$\alpha_n = 2\int \xi^n Q_n(\xi)W_n(\xi)\frac{|d\xi|}{2\pi} = 2\int \left(\xi^{2n}G_n(\xi) + G_n(1/\xi)\right)\frac{|d\xi|}{2\pi} = 2G_n(0), \tag{4.58}$$

where we used (4.48). To prove (4.34) observe that

$$\begin{aligned} \frac{1}{\alpha_n}\mathcal{RH}(Q_n^2W_n;\zeta) &= \frac{2}{\alpha_n}\int_{\mathbb{T}}\frac{Q_n^2(\xi)W_n(\xi)}{\xi-\zeta}\frac{d\xi}{2\pi i} - \frac{1}{\alpha_n}\int_{\mathbb{T}}\frac{Q_n^2(\xi)W_n(\xi)}{\xi}\frac{d\xi}{2\pi i} \\ &= \frac{2}{\alpha_n}\mathcal{P}_+\left(Q_n^2W_n\right)(\zeta) - 1. \end{aligned} \tag{4.59}$$

Now, it is a direct consequence of (4.48) that

$$Q_n^2(\xi)W_n(\xi) = \xi^nQ_n(\xi)G_n(\xi) + \xi^{-n}Q_n(\xi)G_n(1/\xi).$$

As the first summand on the right-hand side of the equation above is analytic and the second one is anti-analytic, it follows that

$$\mathcal{P}_+(Q_nW_n)(\zeta) = \zeta^n Q_n(\zeta)G_n(\zeta) + G_n(0), \quad \zeta \in \mathbb{D}. \tag{4.60}$$

Thus, combining (4.58), (4.59), and (4.60) we obtain

$$\begin{aligned} \alpha_n^{-1}\mathcal{RH}(Q_n^2W_n;\zeta) &= G_n^{-1}(0)\zeta^nQ_n(\zeta)G_n(\zeta) = G_n^{-1}(0)(1+H_n(\zeta))\zeta^nQ_n(\zeta)\mathcal{G}_i(W_n;\zeta) \\ &= (1+o(1))(1+H_n(\zeta))\zeta^nQ_n(\zeta)\mathcal{G}_e^{-1}(W_n;1/\zeta) \end{aligned}$$

locally uniformly in $\mathbb{D}$, where we used (4.6). Since $(1+H_n)\mathcal{G}_e(W_n;1/\cdot)$ extends continuously onto $\mathbb{T}$, the theorem follows. ■

Proof of Proposition 4.5: By the classical Weierstrass theorem it suffices to prove that

$$d_{j,n} := \frac{1}{\alpha_n}\int_{\mathbb{T}} \xi^j Q_n^2(\xi)W_n(\xi)\frac{|d\xi|}{2\pi} \to \int_{\mathbb{T}} \xi^j \frac{|d\xi|}{2\pi} = \delta_{0j}, \quad \text{as} \quad n \to \infty, \tag{4.61}$$

for any $j \in \mathbb{N}$, where δ_{kj} is the usual Kronecker symbol. With the use of (4.48) it is an easy computation to get

$$\begin{aligned} d_{j,n} &= \frac{1}{\alpha_n}\int_{\mathbb{T}} \xi^j \left(G_n(\xi)\xi^n + G_n(1/\xi)\xi^{-n}\right)^2 W_n^{-1}(\xi)\frac{|d\xi|}{2\pi} \\ &= \frac{\mathcal{G}(W_n)}{G_n(0)}\int_{\mathbb{T}} \xi^j(1+H_n(\xi)+H_n(1/\xi)+H_n(\xi)H_n(1/\xi))\frac{|d\xi|}{2\pi} \\ &\quad + \frac{\mathcal{G}(W_n)}{G_n(0)}\int_{\mathbb{T}}(\xi^j+\xi^{-j})(1+H_n(\xi))^2B_{2n+N}(\xi)F_n(\xi)\frac{|d\xi|}{2\pi}. \end{aligned}$$

Since $\|H_n\|_2 \to 0$ and $\mathcal{G}(W_n)/G_n(0) \to 1$ as $n \to \infty$, and $\|B_{2n+N}F_n\|_{\mathbb{T}}$ are uniformly bounded above, we derive that

$$d_{j,n} = (1+o(1))\left(\int_{\mathbb{T}} \xi^j\frac{|d\xi|}{2\pi} + \int_{\mathbb{T}}\frac{1}{\xi}(\xi^{j-1}+\xi^{-j-1})B_{2n+N}(\xi)F_n(\xi)\frac{|d\xi|}{2\pi}\right) + o(1).$$

As $\{(\xi^{j-1}+\xi^{-j-1})F_n(\xi)\}$ is still a compact family, equations (4.61) and respectively (4.35) follow from Lemma 4.10. ■

Proof of Proposition 4.6: Clearly, $\{S_n\}$ satisfies all the condition of Theorem 4.3. Thus, we can use all the results and notation of that theorem. As actual value of r does not matter, the boundary smoothness of analytic functions discussed below can always be achieved by increasing r but keeping it smaller than one.

Symmetry of $\log S_n$ and the decomposition of functions from Hardy spaces on multiply connected domains [44, Thm. 10.12] yield

$$\log S_n(\zeta) = T_n(\zeta) + T_n(1/\zeta), \quad \zeta \in \mathbb{A}_r, \tag{4.62}$$

where $T_n \in H^\infty(\mathbb{D}_{1/r})$. Formula (4.62) immediately implies

$$S_n(\zeta) = \exp(T_n(\zeta))\exp(T_n(1/\zeta)), \quad \zeta \in \mathbb{A}_r.$$

Thus, $\exp(T_n)$ is an analytic extension of $\mathcal{G}_n$ to $\mathbb{D}_{1/r}$ and consequently $\{F_n\}$ is a normal family in $\mathbb{A}_r$. Moreover, since $\hat{F} \equiv 1$, $\{L_n\}$ is, in fact, sequence of H^∞ functions. Now, observe that the use of the L^2 norm in Lemma 4.9 is induced by the L^2 character of $\{L_n\}$ and the following fact: any Hankel operator with continuous symbols is a compact operator from H^2 to $\bar{H}_0^2$. Let $F \in H^\infty(\mathbb{A}_r)$ and consider Hankel operator $\mathcal{H}_F$. As explained above, $F^* := \mathcal{P}_-(F) \in H^\infty(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_r)$. Then for any $h \in H^\infty$ we get

$$\begin{aligned}(\mathcal{H}_F h)(\zeta) &= \mathcal{P}_-(Fh)(\zeta) = \mathcal{P}_-(\mathcal{P}_-(F)h)(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(F^*h)(\xi)}{\zeta - \xi} d\xi \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \left(\frac{1}{2\pi i}\int_{\mathbb{T}_r} \frac{F^*(\eta)}{\xi - \eta} d\eta\right) \frac{h(\xi)}{\zeta - \xi} d\xi = \frac{1}{2\pi i}\int_{\mathbb{T}_r} \frac{(F^*h)(\eta)}{\zeta - \eta} d\eta, \quad \zeta \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}.\end{aligned}$$

In fact, $\mathcal{H}_F h$ is well-defined in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_r$ and $\|\mathcal{H}_F(h)\|_\infty \leq (1-r)^{-1}\|F^*\|_{\infty,r}\|h\|_\infty$. Thus, a Hankel operator with a symbol analytic in $\mathbb{A}_r$ is a continuous operator from H^∞ to $\bar{H}^\infty$. As clear now, we can use L^∞ norm in Lemma 4.9 and the assertion of this proposition follows. ■

Orthogonal Polynomials

The main result of this section provides asymptotic behavior of polynomials orthogonal with varying measures. In particular, it yields that all the zeros of orthogonal polynomials tend to the support. We start by describing the class of varying weights under consideration.

Fix $E = [a,b] \subset (-1,1)$ and let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of complex-valued functions on E. We shall assume that

$$0 < m \leq |s_n(t)| \leq M < \infty, \quad t \in E, \quad \text{and} \quad \omega(s_n; x) < \omega(x), \quad x \in [0, b-a], \tag{4.63}$$

for all $n \in \mathbb{N}$, where ω is a Dini-continuous function of the modulus continuity type on $[0, b-a]$. In other words, ω is a positive nondecreasing subadditive functions such that $\omega(0) = 0$ and $\int_{[0,b-a]}(\omega(t)/t)dt < \infty$. Let now $\mathbf{E} \subset E$ be a finite set containing the endpoints a, b. We put

$$\hat{s}(t) := |t-a|^{\tau(a)}|t-b|^{\tau(b)} \prod_{x \in \mathbf{E}\setminus\{a,b\}} |t-x|^{2\tau(x)}, \quad \tau(x) \in [0, 1/2), \ \ x \in \mathbf{E}, \tag{4.64}$$

Further, we shall assume that

$$\gamma := \max_{x \in \mathbf{E}} \max_{n \in \mathbb{N}} \left\{ \sin(\tau(x)\pi) \max\left\{ \left| \frac{\Psi^+_{2n+N}(x)\mathcal{S}^+(s_n;x)}{\mathcal{S}^-(s_n;x)} \right|, \left| \frac{\Psi^-_{2n+N}(x)\mathcal{S}^-(s_n;x)}{\mathcal{S}^+(s_n;x)} \right| \right\} \right\} < 1, \tag{4.65}$$

where $\Psi_{2n+N} = \Psi_{2n}(v_{2n+N};\cdot)$, polynomials v_l are associated to an admissible sequence of sets $\{A_l\}_{l \in \mathbb{N}}$ via (2.37), and $N \in \mathbb{Z}_+$ is fixed. Finally, we define a sequence of complex measures on E, $\{\nu_n\}_{n \in \mathbb{N}}$, by setting

$$\dot{\nu}_n(t) := \hat{s}(t)s_n(t)/v_{2n+N}(t), \quad t \in E, \tag{4.66}$$

where $\dot{\nu}_n$ is Radon-Nikodym derivative of ν_n with respect to μ_E.

It will be convenient for us to introduce a notion of N_δ-sequence for $\delta \in [0,1)$. A sequence of functions $\{h_n\}_{n \in \mathbb{N}}$ is called *N_δ-sequence* if the following properties hold:

(i) $\{h_n\} \subset \mathrm{Hol}(\overline{\mathbb{C}} \setminus E)$ and $h_n = o(1)$, where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}} \setminus E$;

(ii) $\int |h_n^\pm|^2 d\mu_E \leq c\delta/(1-\delta) + o(1)$, where constant c is independent of n.

Recall that $h_n^\pm$ stand for the boundary values of h_n from above and below on E.

In what follows we pick a branch of the square root such that $\sqrt{(z-a)(z-b)} = z + O(1)$ as $z \to \infty$.

The forthcoming theorem generalizes the results in [30] and [89] (see also [121, Thm. 3]) to a class of less smooth and varying measures. It provides an analog to [128, Thm. 1] for complex measures, and should be compared to [10, Thm. 2] where a sequence of convergent analytic weights was considered with prescribed convergence rate on the contours of minimal capacity.

Theorem 4.11 *Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of polynomials of degree at most n satisfying*

$$\int t^j u_n(t) d\nu_n(t) = 0, \quad j = 0, \ldots, n-1, \tag{4.67}$$

where $\{\nu_n\}$ is given by (4.63)–(4.66). Then each polynomial u_n has exact degree n for all n large *enough and therefore can be normalized to be monic. Under such a normalization we have*

$$\frac{u_n(z)}{\psi_n(z)} = \frac{1+o(1)}{\mathcal{S}(\dot{\nu}_n; z)}, \tag{4.68}$$

where ψ_n is given by (2.40), $o(1)$ holds locally uniformly in $\overline{\mathbb{C}} \setminus E$, and

$$u_n(t) = \left(\left[1 + h_n^-(t)\right] \frac{\psi_n^+(t)}{\mathcal{S}^+(\dot{\nu}_n; t)} + \left[1 + h_n^+(t)\right] \frac{\psi_n^-(t)}{\mathcal{S}^-(\dot{\nu}_n; t)} \right), \quad t \in E, \tag{4.69}$$

where $\{h_n\}$ is N_γ-sequence.

Before we proceed recall that function of second kind associated to u_n is defined by

$$R_n(z) := \int u_n(t) \frac{d\nu_n(t)}{z-t}, \quad z \in \overline{\mathbb{C}} \setminus E.$$

Moreover, we shall denote

$$\gamma_n := \int u_n^2(t) d\nu_n(t).$$

Then the next theorem takes place.

Theorem 4.12 *Let ν_n, u_n, and h_n be as in Theorem 4.11 with u_n monic. Then*

$$\gamma_n = (2 + o(1)) \mathrm{cap}^{2n}(E) \mathcal{S}(\dot{\nu}_n) \tag{4.70}$$

and

$$\sqrt{(z-a)(z-b)} \psi_n(z) R_n(z) = \gamma_n (1 + o(1))(1 + h_n(z)) \mathcal{S}(\dot{\nu}_n; z), \quad z \in \overline{\mathbb{C}} \setminus E, \tag{4.71}$$

where $o(1)$ holds uniformly in $\overline{\mathbb{C}}$.

As both sides of (4.71) are analytic in $\overline{\mathbb{C}} \setminus E$ and $o(1)$ holds uniformly in $\overline{\mathbb{C}}$, (4.71) can be extended to both sides of E.

In the case of non-vanishing weights we derive one more result that generalizes (4.71).

Proposition 4.13 *Let ν_n and u_n be as in Theorem 4.11 with u_n monic and $\hat{s} \equiv 1$. Then*

$$\gamma_n^{-1} u_n^2(t) d\nu_n(t) \xrightarrow{*} d\mu_E(t). \tag{4.72}$$

As one can see, asymptotic behavior of u_n and R_n on E depends on h_n. These functions form a normal family in $\overline{\mathbb{C}} \setminus E$, but under additional smoothness assumptions on s_n we can say more.

Proposition 4.14 *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of analytic functions in some fixed neighborhood of E. Assume that $\{\log s_n\}$ is a normal family in that neighborhood and let ν_n be as in Theorem 4.11 with $\hat{s}\equiv 1$ and s_n as described. If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence polynomials of degree at most n satisfying (4.67), then all the conclusions of Theorems 4.11 and 4.12 hold and $\|h_n^{\pm}\|_E = o(1)$.*

Proof of Theorem 4.11: Let $\mathcal{J}$ be the Joukowski transformation defined in (4.9) and φ_i and φ_e be defined in (2.39). Set

$$Q_n(\xi) := u_n\left(\mathcal{J}(\xi)\right)/\mathrm{cap}^n(E), \quad \xi\in\mathbb{T},$$

then Q_n is a symmetric trigonometric polynomial of degree at most n. Further, denote

$$V_{2n+N}(\xi) := \prod_{j=1}^{2n+N}\left(1-\varphi_i(a_{j,2n+N})\xi\right).$$

So that

$$v_{2n+N}(\mathcal{J}(\xi)) = \mathcal{S}(v_{2n+N})V_{2n+N}(\xi)V_{2n+N}(1/\xi) \quad \text{and} \quad \mathcal{S}(v_{2n+N}) = \prod_{a_{j,2n+N}\neq\infty}\frac{-\mathrm{cap}(E)}{\varphi_i(a_{j,2n+N})}.$$

Let now $\mathbf{T}$ be a conjugate-symmetric subset of $\mathbb{T}$ such that $\mathcal{J}(\mathbf{T}) = \mathbf{E}$. We set

$$\hat{S}_0\ \hat{S}(\cdot) := (\hat{s}\circ\mathcal{J})(\cdot), \quad \hat{S}_0 := (\mathrm{cap}(E))^{\sum_{\zeta\in\mathbf{T}}\tau(\zeta)}, \tag{4.73}$$

then

$$\tau(\zeta) = \tau(\bar{\zeta}) \quad \text{and} \quad \hat{S}(\xi) = \prod_{\zeta\in\mathbf{T}}|(\xi-\zeta)(\xi-\bar{\zeta})|^{\tau(\zeta)},$$

where $\tau(\zeta) := \tau(\mathcal{J}(\zeta))$. Finally, define $S_n(\xi) := s_n(\mathcal{J}(\xi))$, then

$$\omega(S_n;\delta) \le \omega(s_n;\mathrm{cap}(E)\delta) \le \omega\left(\mathrm{cap}(E)\delta\right), \quad \delta\in[0,2].$$

Upon the substitution $t = \mathcal{J}(\xi)$ consecutively with $\xi\in\mathbb{T}$, $\mathrm{Im}(\xi)\ge 0$, and $\xi\in\mathbb{T}$, $\mathrm{Im}(\xi)\le 0$, in (4.67), we see that (4.29) is satisfied with Q_n, $\hat{S}$, S_n, and V_{2n+N} as above. It is readily verified that

$$\frac{\mathcal{S}^{\pm}(s_n;x)}{\mathcal{S}^{\mp}(s_n;x)} = F_n(\varphi_i^{\pm}(x)) \quad \text{and} \quad |\Psi_{2n+N}^{\pm}(x)| = |R_{2n+N}(\varphi_i^{\pm}(x))|,$$

where $F_n(\xi) = \mathcal{G}_i(S_n;\xi)\mathcal{G}_i^{-1}(S_n;1/\xi)$ and $R_{2n+N}(\xi) = V_{2n+N}(1/\xi)/\overline{V_{2n+N}(\xi)}$. Thus, all the requirements of Theorem 4.3 are fulfilled and (4.30) and (4.31) take place. Since

$$\mathcal{G}_e(W_n;\zeta) = \mathcal{S}^{-1}(\dot\nu_n;z), \quad \zeta=\varphi_e(z), \quad z\in\overline{\mathbb{C}}\setminus E,$$

(4.68) follows. Set now

$$h_n(z) := H_n(\zeta), \quad \zeta=\varphi_i(z), \quad z\in\overline{\mathbb{C}}\setminus E,$$

where H_n was defined in Theorem 4.3. Then $h_n(z) = H_n(\varphi_i(z)) = o(1)$ locally uniformly in $\overline{\mathbb{C}}\setminus E$ and

$$\frac{1}{2}\int\left(|h_n^+(t)|^2+|h_n^-(t)|^2\right)d\mu_E(t) = \int_{\mathbb{T}}|H_n(\xi)|^2\frac{|d\xi|}{2\pi}$$

and therefore

$$\limsup_{n\to\infty}\|h_n^{\pm}\|_{L^2(d\mu_E)} \leq \sqrt{2}\frac{\gamma}{1-\gamma}.$$

So, $\{h_n\}$ is N_γ-sequence and asymptotic formula (4.69) follows from (4.31), which finishes the proof of the theorem. ■

Proof of Theorem 4.12: It is easy to check that

$$\gamma_n = \int u_n^2(t)d\nu_n(t) = \frac{\mathrm{cap}^{2n}(E)}{\hat{S}_0\mathcal{S}(v_{2n+N})}\int_{\mathbb{T}}Q_n^2(\xi)W_n(\xi)\frac{|d\xi|}{2\pi} = \frac{\mathrm{cap}^{2n}(E)}{\hat{S}_0\mathcal{S}(v_{2n+N})}\alpha_n,$$

where $\hat{S}_0$ was defined in (4.73). Thus, the limit in (4.70) follows from the corresponding limit in (4.33). Now, observe that orthogonality relations (4.67) yield that

$$R_n(z) = \int u_n(t)\frac{d\nu_n(t)}{z-t} = \frac{1}{u_n(z)}\int u_n^2(t)\frac{d\nu_n(t)}{z-t}, \quad z\in\overline{\mathbb{C}}\setminus E.$$

Therefore, we obtain from (4.10) that

$$\sqrt{(z-a)(z-b)}\psi_n(z)R_n(z) = \gamma_n\frac{\alpha_n^{-1}\mathcal{RH}(Q_n^2W_n;\zeta)}{\zeta^nQ_n(\zeta)},$$

where $\zeta=\varphi_i(z)$, $z\in\overline{\mathbb{C}}\setminus E$. Thus, it follows from (4.31) that

$$\begin{aligned}\sqrt{(z-a)(z-b)}\psi_n(z)R_n(z) &= \gamma_n(1+o(1))(1+H_n(\zeta))\mathcal{G}_e^{-1}(W_n;1/\zeta)\\ &= \gamma_n(1+o(1))(1+h_n(z))\mathcal{S}(\dot\nu_n;z),\end{aligned}$$

where $\zeta = \varphi_i(z)$, $z \in \overline{\mathbb{C}} \setminus E$, $o(1)$ holds uniformly in $\overline{\mathbb{C}}$ and $(1 + h_n^\pm)\mathcal{S}^\pm(\dot\nu_n; \cdot)$ are continuous on E as apparent from the proof of Theorem 4.3. ■

Proof of Proposition 4.13: Equations (4.72) can be easily deduced from the corresponding equations (4.35). ■

Proof of Proposition 4.14: Proceeding as in Theorem 4.11 we see that $\{\log S_n\}$ is a normal family in some fixed neighborhood of $\mathbb{T}$, where $S_n(\zeta) = s_n(\mathcal{J}(\zeta))$. Thus, we may apply Proposition 4.6 and the result follows. ■

Meromorphic Approximation

In the previous chapter we were able to deduce convergence in capacity of meromorphic approximants to approximated functions of the form $\mathcal{F}(\mu; R; \cdot)$. Moreover, on any compact set asymptotically free of poles of the approximants the convergence was, in fact, uniform. In this section we show locally uniform convergence everywhere in $\overline{\mathbb{C}} \setminus S$ under assumption that the Radon-Nikodym derivative of μ is Dini-continuous and vanishes in a controllable manner. Furthermore, we provide the rate of this convergence.

Recall that any irreducible critical point g_n (best meromorphic approximant is a particular case of an irreducible critical point) in meromorphic approximation problem to $\mathcal{F}$ is given by $\mathcal{T}_{\mathcal{F}}(v_n)/v_n$, where $\mathcal{T}_{\mathcal{F}}$ is the Toeplitz operator with symbol $\mathcal{F}$ and v_n is some singular vector of the Hankel operator $\mathcal{H}_{\mathcal{F}}$. In what follows we assume that singular vectors $v_n = b_n w_n$ are normalized in such a way that $w_n(0) > 0$. As we shall show, outer factors w_n form not merely a normal family, but a convergent sequence. For that, it will be of convenience to define for each $p \in (2, \infty]$ a function w by the rule

$$w(z) := \frac{\tau^{2/p'}}{[(1-az)(1-bz)]^{1/p'}}, \quad w(0) > 0, \quad z \in \overline{\mathbb{C}} \setminus E^{-1}, \tag{4.74}$$

where $1/p + 1/p' = 1/2$ and τ was defined in (4.18). Observe that

$$w^{p'/2}(z) = \frac{\tau}{\sqrt{(1-az)(1-bz)}} \quad \text{and} \quad \|w\|_{p'} = 1.$$

The normalization of singular vectors that was just introduced implies (it will be shown in the course of the proof of Theorem 4.15) that

$$\mathcal{S}\left(\dot\mu w_n q_{n,m}^2 / \widetilde{q}_n^2\right) = (1 + o(1)) \left|\mathcal{S}\left(\dot\mu w_n q_{n,m}^2 / \widetilde{q}_n^2\right)\right|,$$

where $q_{n,m} \in \mathrm{P}_m$ is a divisor of q_n ($b_n = q_n/\widetilde{q}_n$) and $\lim_{n\to\infty} q_{n,m}(z) = Q(z)$ locally uniformly in $\mathbb{C}$. Such polynomials exist whenever $\mu \in \mathbf{BVT}$ by Theorem 3.10. Then we can define $\mathcal{S}_n$ in such a way that $\mathcal{S}_n^2 = \mathcal{S}(\dot{\mu} w_n)$ and

$$\mathcal{S}_n/\mathcal{S}(\widetilde{q}_n/q_{n,m}) = (1+o(1))|\mathcal{S}_n/\mathcal{S}(\widetilde{q}_n/q_{n,m})|.$$

Due to the dependence of the varying weights in (2.33) on polynomials q_n themselves, we shall describe asymptotic behavior of the critical points in meromorphic approximation problem for $\mathcal{F}(\mu; R; \cdot)$ using Szegő functions for the condenser $\overline{\mathbb{C}} \setminus (E \cup E^{-1})$ (cf. (4.20)-(4.21)). Put

$$\mathcal{D}_n(z) := \mathcal{D}\left(|\mathcal{S}_n b \frac{\Psi_n^+}{\Psi^+} \mathcal{S}^+(\dot{\mu} w_n; \cdot)|^2, |\mathcal{S}_n b \frac{\Psi_n^-}{\Psi^-} \mathcal{S}^-(\dot{\mu} w_n; \cdot)|^2; z \right), \quad z \in \overline{\mathbb{C}} \setminus E,$$

where $\Psi_n(\cdot) := \Psi_n(\widetilde{q}_n; \cdot)$, $\Psi(\cdot) := \Psi_m(Q; \cdot)$, and $b := Q/\widetilde{Q}$. Since

$$\Psi^+(t)\Psi^-(t) = 1, \quad \Psi_n^+(t)\Psi_n^-(t) = 1 \quad \text{and} \quad \dot{\mu}(t) w_n(t) = \mathcal{S}_n^2 \mathcal{S}^+(\dot{\mu} w_n; t) \mathcal{S}^-(\dot{\mu} w_n; t), \quad t \in E,$$

it is easy to see that

$$\mathcal{D}\left(|\mathcal{S}_n b \frac{\Psi_n^+}{\Psi^+} \mathcal{S}^+(\dot{\mu} w_n; \cdot)|^2, |\mathcal{S}_n b \frac{\Psi_n^-}{\Psi^-} \mathcal{S}^-(\dot{\mu} w_n; \cdot)|^2 \right) = \mathcal{D}(|b^2 \dot{\mu} w_n|).$$

Therefore, it holds that

$$\sqrt{\mathcal{D}(|b^2 \dot{\mu} w_n|)} |\mathcal{D}_n^\pm(t)| = |\mathcal{S}_n b(t) \frac{\Psi_n^\pm(t)}{\Psi^\pm(t)} \mathcal{S}^\pm(\dot{\mu} w_n; t)|, \quad t \in E.$$

Further, it follows from the definition of a Szegő function for a condenser, Lemma 3.11, and properties of Ψ_n that the sequence $\{\mathcal{D}_n\}$ is a normal family in $\overline{\mathbb{C}} \setminus (E \cup E^{-1})$ and no limit point of this family has zeros.

Recall that the Szegő functions for a condenser are defined up to a multiplicative unitary constant only. Thus, we shall specify a choice of the face for $\mathcal{D}_n$. Fix an arbitrarily $t_0 \in (b, 1)$, we normalize $\mathcal{D}_n$ are normalized so that the ratios

$$\mathcal{S}_n \frac{b(t_0)\Psi_n(t_0)\mathcal{S}(\dot{\mu} w_n; t_0)}{\Psi(t_0)\mathcal{D}_n(t_0)}$$

are positive for each $n \in \mathbb{N}$. As apparent from the next theorem, the particular choice of t_0 is irrelevant.

Theorem 4.15 *Let $p \in (2, \infty]$ and $\{g_n\}$ be a sequence of irreducible critical points in meromorphic approximation problem for $\mathcal{F}(\mu; R; \cdot)$, given by (1.2), from H_n^p, where $R = P/Q$ has no poles on E and $\dot{\mu}$ is a product of a non-vanishing Dini-continuous function having an argument of bounded variation on E and $\hat{s}$, defined in (4.64) and associated with $\mathbf{E} = \{a, b\}$. Then*

$$\frac{b_n(z)}{\varphi^n(z)} = \frac{1 + o(1)}{\mathcal{D}_n(z)} \frac{b(z)}{\varphi^m(z)}, \tag{4.75}$$

where $o(1)$ holds locally uniformly in $D_{\mathcal{F}} \cap D_{\mathcal{F}}^*$ and φ was defined in (4.17). Further,

$$\sqrt{(1 - az)(1 - bz)}\ w_n^{p'/2}(z) = \tau + o(1), \tag{4.76}$$

where $o(1)$ holds locally uniformly in $D_{\mathcal{F}}^*$ and τ was defined in (4.18). Moreover, the following error estimates take place

$$\sigma_n = \|\mathcal{F} - g_n\|_p = \left(\frac{2\mathcal{D}(|b^2 \dot{\mu} w|)}{\tau} + o(1) \right) \rho^{2n}, \tag{4.77}$$

where σ_n is the critical value associated to g_n via (2.29) and ρ was defined in (4.19), and

$$(\mathcal{F} - g_n)(z) = \frac{2\mathcal{D}(|b^2 \dot{\mu} w|) + o(1)}{w(z)\sqrt{(z-a)(z-b)}} \left(\frac{\rho}{\varphi(z)} \right)^{2n} \frac{\mathcal{D}_n^2(z)}{b^2(z)}, \tag{4.78}$$

where $o(1)$ holds locally uniformly in $\overline{\mathbb{D}} \cap D_{\mathcal{F}}$.

We wish to remind that $w_n^{p'/2}$ may have a pole at each $1/\bar{\eta}$, $\eta \in S'$, of multiplicity at most $m(\eta)$ as evident from (2.36). This is the reason why asymptotic behavior in (4.76) is stated to hold in $D_{\mathcal{F}}^*$, but not in $\overline{\mathbb{C}} \setminus E^{-1}$.

We also remind the reader that in the case of rational approximation outer functions w_n are not present. This allows us to consider a slightly larger class of measures as in Theorem 4.15.

Theorem 4.16 *Let $\{g_n\}$ be a sequence of irreducible critical points in rational approximation problem ($p = 2$) for $\mathcal{F}(\mu; R; \cdot)$, given by (1.2), where $R = P/Q$ has no poles on E and $\dot{\mu}$ is a product of a non-vanishing Dini-continuous function having an argument of bounded variation on E and $\hat{s}$, defined in (4.64) and such that*

$$\max_{x \in \mathbf{E} \setminus \{a,b\}} \max_{n \in \mathbb{N}} \left\{ \sin(\tau(x)\pi) \exp\left(\frac{4r_2}{1 - r_1} V(\arg(s), E) \right) \max\left\{ \left| \frac{\mathcal{S}^+(s;x)}{\mathcal{S}^-(s;x)} \right|, \left| \frac{\mathcal{S}^-(s;x)}{\mathcal{S}^+(s;x)} \right| \right\} \right\} < 1 \tag{4.79}$$

and $\tau(a), \tau(b) \in [0, 1/2)$, *where* $V(\arg(s), E)$ *is the total variation of the argument of* s *on* E,

$$r_1 := \max_{\xi \in \mathbb{T}} |\varphi_i(\xi)|, \quad \textit{and} \quad r_2 := \max_{z \in \mathbb{D}_r} \left| \frac{\varphi_i'(1/z)}{z^2} \right|.$$

Then (4.75), (4.77), and (4.78) hold with $w \equiv 1$.

It follows from (4.75) that each q_n can be written as $u_{n-m} q_{n,m}$, where $q_{n,m} = (1 + o(1))Q$ in $\overline{\mathbb{C}} \setminus S'$ and u_{n-m} have no zeros on compact subsets of $\mathbb{C} \setminus E$ for all n large enough. Let $\eta_{1,n}, \ldots \eta_{m(\eta),n}$ be the zeros of b_n approaching $\eta \in S'$. Then the following holds.

Corollary 4.17 *For each* $\eta \in S'$ *and all* n *large enough there exists an arrangement of* $\eta_{1,n}, \ldots \eta_{m(\eta),n}$ *such that*

$$\eta_{k,n} = \eta + (h_{n,\eta} + o(1)) \left(\frac{\rho}{\varphi(\eta)} \right)^{2n/m(\eta)} \exp\left(\frac{2\pi k}{m(\eta)} i \right), \quad k = 1, \ldots, m(\eta), \tag{4.80}$$

where $\{|h_{n,\eta}|\}$ *is a bounded above sequence.*

We remark that the proof of Corollary 4.17 is an adaptation of arguments in [60, Thm. 3] to the present case.

Taking measures with non-vanishing derivatives we deduce addition information on the critical points g_n.

Proposition 4.18 *Let* $\{g_n\}$ *be as in Theorem 4.15 with* $\hat{s} \equiv 1$. *Then the following weak** *convergence takes place*

$$\sigma_n^{-1} b_n^2 w_n d\mu \xrightarrow{*} \tau \ d\mu_E. \tag{4.81}$$

The next two lemmas are needed for the proofs of Theorems 4.15 and 4.16.

Lemma 4.19 *Let* $\{g_n\}$ *be as in Theorem 4.15. Further, let* A_n *be the set of zeros of* $\widetilde{q}_n^2$. *Then* $\{A_n\}_{n \in \mathbb{N}}$ *is an admissible sequence of sets.*

Proof: Since $E \cap \mathbb{T} = \emptyset$, $r_1 := \max_{\xi \in \mathbb{T}} |\varphi_i(\xi)| < 1$. Thus, $A_n \subset \mathbb{D}_r$ for any $n \in \mathbb{N}$ and the second admissibility condition for $\{A_n\}$ follows.

To show the first admissibility condition we shall define $\Delta_n : \varphi_i(A_n) \to \varphi_i(A_n)$ as the identity map. Notice that $\varphi_i(1/z)$ is an analytic function in $\overline{\mathbb{C}} \setminus E^{-1} \supset \overline{\mathbb{D}}_{r_1}$. Thus, $|\varphi_i(1/z_1) - \varphi_i(1/z_2)| \leq r_2 |z_1 - z_2|$ for any $z_1, z_2 \in \mathbb{D}_{r_1}$, where $r_2 := \max_{z \in \mathbb{D}_{r_1}} |\varphi_i'(1/z)/z^2|$. This, in particular, implies that

$$
\begin{aligned}
2\sum_{j=1}^{n}\frac{|\varphi_i(1/\bar{\xi}_{j,n})-\overline{\varphi_i(1/\bar{\xi}_{j,n})}|}{(1-|\varphi_i(1/\bar{\xi}_{j,n})|)^2} &\leq \frac{2}{(1-r_1)^2}\sum_{j=1}^{n}\left|\varphi_i(1/\bar{\xi}_{j,n})-\varphi_i(1/\xi_{j,n})\right| \\
&\leq \frac{4r_2}{(1-r_1)^2}\sum_{j=1}^{n}|\mathrm{Im}(\xi_{j,n})| \leq \frac{4r_2}{(1-r_1)^2}\sum_{j=1}^{n}(\pi-\mathrm{Angle}(\xi_{j,n}))\,.
\end{aligned}
$$

It was shown in Lemma 3.4, under assumptions $\mu \in \mathbf{BVT}$, which are clearly satisfied, that the sums on the right-hand side of the equations above are bounded by $V(\arg(w_n\dot{\mu}))$. As $\{w_n\}$ is a normal family in $\mathbb{D}$, the sequence of sets $\{A_n\}$ is, indeed, admissible. ■

Lemma 4.20 *Let $\{g_n\}$ be as in Theorem 4.15. Then there exist constants $\hat{m}$, $\hat{M}$, and a Dini-continuous function of the modulus continuity type, $\hat{\omega}$, such that*

$$\omega(w_n;x)\leq\hat{\omega}(x),\quad x\in[0,b-a],\quad 0<\hat{m}\leq|w_n(t)|\leq\hat{M}<\infty,\quad t\in E,\quad n\in\mathbb{N}.$$

Proof: Recall that the sequence of outer functions $\{w_n\}$ forms a normal family in $\mathbb{D}$. Moreover any limit point of this family is zero free (cf. Lemma 3.11). By the classical theorem of Montel, normality is equivalent to the uniform boundedness on compact subsets. This shows the existence of $\hat{m}$ and $\hat{M}$. Moreover, there exists $M^\#$ such that $\|w_n\|_{\infty,\rho}\leq M^\#$ for all $n\in\mathbb{N}$, where $r<1$ and $E\subset\mathbb{D}_r$. Then by the Cauchy integral formula, we get

$$\omega(w_n;x)=\sup_{|t_1-t_2|\leq x}|w_n(t_1)-w_n(t_2)|=\left|\frac{1}{2\pi ri}\int\frac{w_n(\xi)(t_1-t_2)}{(\xi-t_1)(\xi-t_2)}d\xi\right|\leq\frac{M^\# x}{\mathrm{dist}^2(\mathbb{T}_r,E)}=:\hat{\omega}(x).$$

Obviously $\omega^*(t)$ is Dini-continuous on $[0,b-a]$ and is of modulus continuity type. ■

Proof of Theorem 4.15: Recall that the sequence $\{q_n\}$ satisfies orthogonality relations (2.33). Moreover, Lemma 4.19 shows that the zeros of $\widetilde{q}_n^2$ form an admissible sequence of set. This enables us to apply Theorem 4.11 with $\dot{\nu}_n:=q_{n+m,m}Q\dot{\mu}w_{n+m}/\widetilde{q}_{n+m}^2$. Indeed, let $\dot{\mu}=\hat{s}s$. Since

$$m:=\hat{m}\inf_E|s|>0\quad M:=\hat{M}\sup_E|s|<\infty,\ \text{ and }\ \omega(w_ns;x)\leq\hat{M}\omega(s;x)+\|s\|_E\hat{\omega}(x),$$

all the conditions of this theorem are fulfilled.

We start by proving (4.76). Denote then

$$\beta_n:=\int b_n^2(t)\frac{Q(t)}{q_{n,m}(t)}w_n(t)d\mu(t).$$

As explained in the introduction (see (2.36)), there exists a sequence of Blaschke products $\{j_n\}_{n\in\mathbb{N}}$ such that

$$\sigma_n\left(j_n w_n^{p'/2}\right)^\sigma(z)=\int\frac{b_n^2(t)w_n(t)}{z-t}d\mu(t)+\sum_{\eta\in S'}\sum_{k=0}^{m(\eta)-1}\frac{r_{\eta,k}}{k!}\left(\frac{b_n^2(t)w_n(t)}{z-t}\right)^{(k)}\Bigg|_{t=\eta}. \tag{4.82}$$

Denote the rational function on the right-hand side of (4.82) by E_n. We show that $\beta_n^{-1}E_n=o(1)$ in $\mathbb{C}\setminus S'$. To simplify the forthcoming argument, we shall adopt the following notation:

$$\frac{1}{\beta_n}E_n(z):=\sum_{\eta\in S'}\sum_{k=0}^{m(\eta)-1}\frac{r_{\eta,k}}{k!}\left(\frac{1}{\beta_n}\frac{b_n^2(t)w_n(t)}{z-t}\right)^{(k)}\Bigg|_{t=\eta}=:\sum_{\eta\in S'}\sum_{k=0}^{m(\eta)-1}E_n^{k,\eta}(z).$$

Fixed $\eta\in S'$ and $k\in\{0,\dots,m(\eta)-1\}$, then

$$E_n^{k,\eta}(z)=\sum_{j=0}^{k}\frac{r_{\eta,k}}{j!}\frac{1}{(z-\eta)^{k-j+1}}\left(\frac{1}{\beta_n}\frac{u_{n-m}^2(t)q_{n,m}(t)w_n(t)P(t)}{\widetilde{q}_n^2(t)}\frac{q_{n,m}(t)}{P(t)}\right)^{(j)}\Bigg|_{t=\eta}.$$

Recall that P is a polynomial that does not vanish at η. Thus, by the definition of $q_{n,m}$, we obtain

$$\lim_{n\to\infty}\left(\frac{q_{n,m}(t)}{P(t)}\right)^{(j)}\Bigg|_{t=\eta}=0 \tag{4.83}$$

for any $j=0,\dots,m(\eta)-1$. In another connection, error formula (3.45), applied with $q_{n,1}\equiv 1$, $l_{m,n}=q_{n,m}$, and $q_n^\sharp=u_{n-m}$, yields

$$(\mathcal{F}-g_n)(z)=\frac{q_{n,m}(z)}{b_n^2(z)w_n(z)Q(z)}\int\frac{b_n^2(t)w_n(t)Q(t)}{q_{n,m}(t)}\frac{d\mu(t)}{z-t}, \tag{4.84}$$

$z\in\mathbb{D}\cap D_{\mathcal{F}}$. Proceeding as in the proof of Theorem 4.12, we derive that

$$\sqrt{(z-a)(z-b)}\int\frac{b_n^2(t)w_n(t)Q(t)}{q_{n,m}(t)}\frac{d\mu(t)}{z-t}=\beta_n(1+o(1)), \tag{4.85}$$

where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}}\setminus E$. Denote by e_n the error $\mathcal{F}-g_n$ and observe that (4.84) and (4.85) imply

$$\frac{1}{\beta_n}\frac{u_{n-m}^2(z)q_{n,m}(z)(Qe_n)(z)}{\widetilde{q}_n^2(z)}=i_n(z),\quad z\in\mathbb{D}\setminus E, \tag{4.86}$$

where

$$i_n(z):=\frac{1}{w_n(z)}\frac{1}{\beta_n}\int\frac{b_n^2(t)w_n(t)Q(t)}{q_{n,m}(t)}\frac{d\mu(t)}{z-t},\quad z\in\mathbb{D}\setminus E,$$

is such that

$$\sqrt{(z-a)(z-b)}w_n(z)i_n(z) = 1 + o(1) \tag{4.87}$$

locally uniformly in $\mathbb{D}\setminus E$. In particular, it means that $\{i_n\}$ is a normal family in $\mathbb{D}\setminus E$, as well as $\left\{i_n^{(k)}\right\}$ for $k\in\mathbb{N}$. Now, since $(e_nQ)^{(j)}(\eta) = P^{(j)}(\eta)$ for any $j=0,\dots,m(\eta)-1$, we deduce from (4.86) that

$$\left(\frac{1}{\beta_n}\frac{u_{n-m}^2(t)q_{n,m}(t)w_n(t)P(t)}{\widetilde{q}_n^2(t)}\right)^{(j)}\Bigg|_{t=\eta} = \left|(w_n(z)i_n(z))^{(j)}\right|_{z=\eta}$$

for any $j=0,\dots,m(\eta)-1$. Then by the normality of $\{(w_ni_n)^{(j)}\}$, $j=0,\dots,m(\eta)-1$, in $\mathbb{D}\setminus E$ and (4.83) we see that $E_n^{k,\eta}$ is indeed $o(1)$ locally uniformly in $C\setminus S'$. Hence, (4.82) can be rewritten as

$$\frac{\sigma_n}{\bar{\beta}_n}\sqrt{(1-az)(1-bz)}\left(j_nw_n^{p'/2}\right)(z) = 1+o(1), \tag{4.88}$$

where $o(1)$ holds locally uniformly in $D^*_{\mathcal{F}}$ and we used (4.85). Recall that by definition $w_n^{p'/2}$ has unit L^2-norm. Therefore

$$\frac{\sigma_n}{|\beta_n|} = \left(\int_{\mathbb{T}}\frac{|1+o(1)|^2}{|(1-a\xi)(1-b\xi)|}\frac{|d\xi|}{2\pi}\right)^{1/2} = \frac{1}{\tau}+o(1) \tag{4.89}$$

by the definition of τ (see 4.18). Upon noticing that the right-hand side of (4.88) does not vanish for all n large enough, we deduce that necessarily $j_n\equiv 1$ for all such n. Moreover, since $w_n^{p'/2}(0)>0$ by the adopted normalization, we deduct from (4.88) that

$$\beta_n/|\beta_n| = 1+o(1), \tag{4.90}$$

and (4.76) follows.

As we established (4.76), we turn our attention to (4.75). Using (4.68) and (4.15) we get

$$1+o(1) = \frac{u_{n-m}(z)\mathcal{S}(\dot{\nu}_{n-m};z)}{\psi_{n-m}(z)}, \tag{4.91}$$

where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}}\setminus E$ and $q_n = u_{n-m}q_{n,m}$. Clearly, (4.91) yields

$$1+o(1) = \frac{u_{n-m}(z)}{\mathrm{cap}^{n-m}(E)}\mathcal{S}(w_n\dot{\mu};z)\mathcal{S}(\widetilde{q}_n/q_{n,m})\frac{q_{n,m}(z)}{\Psi_m(q_{n,m};z)}\frac{\Psi_n(z)}{\widetilde{q}_n(z)}$$

and successively

$$1+o(1) = \frac{b_n(z)}{\mathrm{cap}^{n-m}(E)}\mathcal{S}(w_n\dot{\mu};z)\mathcal{S}(\widetilde{q}_n/q_{n,m})\frac{\Psi_n(z)}{\Psi(z)}, \tag{4.92}$$

where $o(1)$ takes place locally uniformly in $D_{\mathcal{F}}$ and we used that $q_{n,m} = (1+o(1))Q$ on E and $\Psi/\Psi_m(q_{n,m};\cdot) = 1+o(1)$ in $D_{\mathcal{F}}$. This, in particular, implies that

$$\lambda_n y_n(z)\frac{b_n(z)\mathcal{D}_n(z)}{b(z)\varphi^{n-m}(z)} = 1+o(1),$$

where $o(1)$ holds locally uniformly in $D_{\mathcal{F}}$,

$$\lambda_n := \rho^{n-m}\frac{\sqrt{\mathcal{D}(|b^2\dot{\mu}w_n|)}}{\mathrm{cap}^{n-m}(E)}\frac{\mathcal{S}(\widetilde{q}_n/q_{n,m})}{\mathcal{S}_n}, \quad \text{and} \quad y_n(z) := \frac{\mathcal{S}_n\mathcal{S}(\dot{\mu}w_n;z)\Psi_n(z)}{\sqrt{\mathcal{D}(|b^2\dot{\mu}w_n|)}\ \mathcal{D}_n(z)}\frac{b(z)}{\Psi(z)}\left(\frac{\varphi(z)}{\rho}\right)^{n-m}.$$

Now, we shall show that $\lambda_n = 1+o(1)$ and $y_n(z) = 1+o(1)$, where $o(1)$ holds locally uniformly in $\overline{\mathbb{D}}$. Observe, that $|y_n^{\pm}(t)| = 1$, $t\in E$, by the very definition of y_n. Moreover, $\log y_n$ is an analytic function in some neighborhood of $\mathbb{T}$. Indeed, by (4.91), all the zeros of Ψ_n approach E^{-1} and the ratio b/Ψ is zero free; therefore y_n is a zero free function in some neighborhood of $\mathbb{T}$ (at least for n large enough). In fact, $|y_n|$ is zero free in some neighborhood of $\overline{\mathbb{D}}$. Further, the winding number of $\mathcal{S}(\dot{\mu}w_n;\cdot)\Psi_n(\cdot)$ on $\mathbb{T}$ is equal to $-n$, since this function is analytic in $\overline{\mathbb{C}}\setminus\mathbb{D}$ and has n zeros there. The ratio b/Ψ is meromorphic in $\overline{\mathbb{C}}\setminus\mathbb{D}$ with m poles there; thus, its winding number on $\mathbb{T}$ is equal to m. The winding number of $\mathcal{D}_n(\cdot)$ on $\mathbb{T}$ is equal to zero by the very definition of a Szegő function for a condenser. It follows from [82, Ch. VI] that φ has winding number one on $\mathbb{T}$. Thus, the increment of the argument of y_n on $\mathbb{T}$ is zero and $\log y_n$ is well-defined. In turn, this means that $\log|g_n|$ has well-defined harmonic conjugate in some neighborhood of $\mathbb{T}$. Now, since the inner normal derivative times arclength is the differential of the conjugate function, we get that

$$\int_{\mathbb{T}}\frac{\partial}{\partial n}\log|y_n|ds = 0.$$

Then by the virtue of [29, Lemma 4.7] we obtain

$$\inf_{\mathbb{T}}|y_n| \leq \sup_E|y_n^{\pm}| = 1 = \inf_E|y_n^{\pm}| \leq \sup_{\mathbb{T}}|y_n|. \tag{4.93}$$

In another connection, (4.92) and the definition of a Szegő function for a condenser yield

$$|y_n| = |\lambda_n|^{-1}(1+o(1)) \tag{4.94}$$

uniformly on $\mathbb{T}$. Combining (4.93) and (4.94), we get $|\lambda_n| = 1+o(1)$ and $|y_n| = 1+o(1)$, where $o(1)$ holds uniformly in $\overline{\mathbb{D}}$ (the latter asymptotic formula follows from the maximum principle for harmonic functions applied to $\log|y_n|$). This shows that $\{y_n\}$ is a normal family in $\mathbb{D}\setminus E$, and since $y_n(t_0) > 0$ for any $n \in \mathbb{N}$ by the normalization of Szegő functions $\mathcal{D}_n(\cdot)$, we derive $y_n(z) = 1 + o(1)$ uniformly in $\overline{\mathbb{D}}$. Now, it follows from (4.70) that

$$\beta_n = (2 + o(1))\mathrm{cap}^{2(n-m)}(E)\mathcal{S}(\dot{\nu}_{n-m}). \tag{4.95}$$

Thus, by (4.90) we get

$$\mathcal{S}(\dot{\nu}_{n-m})|\mathcal{S}(\dot{\nu}_{n-m})|^{-1} = 1 + o(1).$$

Since $\mathcal{S}(q_{n,m}Q) = (1 + o(1))\mathcal{S}(q^2_{n,m})$, the asymptotic formula above yields by the choice of $\mathcal{S}_n$ that

$$\mathcal{S}_n\mathcal{S}(q_{n,m}/\widetilde{q}_n)/|\mathcal{S}_n\mathcal{S}(q_{n,m}/\widetilde{q}_n)| = 1 + o(1).$$

Therefore, we obtain

$$\lambda_n = (1 + o(1))|\lambda_n| = 1 + o(1). \tag{4.96}$$

So, we deduce that

$$\frac{b_n(z)\mathcal{D}_n(z)}{b(z)\varphi^{n-m}(z)} = 1 + o(1),$$

where $o(1)$ holds uniformly in $\overline{\mathbb{D}} \cap D_{\mathcal{F}}$. Now, recall that $\overline{\mathcal{D}_n(1/\bar{z})} = 1/\mathcal{D}(z)$. Moreover, the same property holds for b_n, b, and φ. Thus,

$$\frac{b_n(z)\mathcal{D}_n(z)}{b(z)\varphi^{n-m}(z)} = \overline{\left(\frac{b(1/\bar{z})\varphi^{n-m}(1/\bar{z})}{b_n(1/\bar{z})\mathcal{D}_n(1/\bar{z})}\right)} = 1 + o(1),$$

where $o(1)$ holds locally uniformly in $D^*_{\mathcal{F}} \setminus \overline{\mathbb{D}}$. Since $b_n\mathcal{D}_n/b\varphi^{n-m}$ is analytic across $\mathbb{T}$, (4.75) holds.

It only remains to prove (4.77) and (4.78). Combining asymptotic formulae (4.95) and (4.96) with the definition of λ_n, we get

$$\begin{aligned} \beta_n &= (2 + o(1))\mathrm{cap}^{2(n-m)}(E)\mathcal{S}(\dot{\nu}_{n-m}) = (2 + o(1))\frac{\mathcal{D}(|b^2\dot{\mu}w|)\rho^{2(n-m)}}{\lambda_n^2} \\ &= (2 + o(1))\mathcal{D}(|b^2\dot{\mu}w|)\rho^{2(n-m)}. \end{aligned} \tag{4.97}$$

Hence, (4.77) follows. Further, (4.78) is deduced from (4.84), (4.85), (4.97), and (4.75). ■

Proof of Theorem 4.16: The conclusions of this theorem will follow upon showing that (4.65) holds with $v_{2n} = \widetilde{q}_n^2$. As observed in the proof of Theorem 4.11, we have that

$$|\Psi_n^{\pm}(x)|^2 = |R_{2n}(\varphi_i^{\pm}(x))|, \quad x \in E,$$

where $\Psi_n = \Psi_n(\widetilde{q}_n)$ and $R_{2n}(\zeta) = \prod_{j=1}^n \left((1-\varphi_i(1/\xi_{j,n})\zeta)/(1-\varphi_i(1/\bar{\xi}_{j,n})\zeta)\right)^2$. Thus, we need to show that

$$\max_n \|R_{2n}\|_{\mathbb{T}} \leq \exp\left(\frac{4r_2}{1-r_1}V(\arg(s);E)\right).$$

The latter follows from Lemmas 4.8, 4.19 and positivity of $\hat{s}$. ■

Before we prove Corollary 4.17, we need one auxiliary lemma.

Lemma 4.21 *Let $r = p/q$ be a rational function of degree d, $\zeta \in \mathbb{C}$, and $\delta > 0$. Assume further that p and q have no zeros in $\{z : |z-\zeta| \leq \delta\}$. Then for any $k < d$, $k \in \mathbb{N}$, there exists c_k independent of r such that*

$$\left|r^{(k)}(\zeta)/r(\zeta)\right| \leq C_k d^k. \tag{4.98}$$

Proof: Clearly, if T is a polynomial of degree at most d with no zeros in $\{z : |z-\zeta| \leq \delta\}$, then

$$\left|\frac{T^{(j)}(\zeta)}{T(\zeta)}\right| \leq \frac{d\cdot \ldots \cdot (d-j-1)}{\delta^j} \leq \left(\frac{d}{\delta}\right)^j, \tag{4.99}$$

$j = 1,\ldots,k$. Moreover, it is easy to verify that

$$\left|T(\zeta)\left(\frac{1}{T(\zeta)}\right)^{(j)}\right| = \left|\sum_{l=1}^{j}\sum_{\sum d_i = l}\prod_{\sum s_i d_i = j} c_{l,\{d_i\},\{s_i\}}\left(\frac{T^{(s_i)}(\zeta)}{T(\zeta)}\right)^{d_i}\right| \leq c_j^* d^j, \tag{4.100}$$

$j = 1,\ldots,k$, where coefficients $c_{l,\{d_i\},\{s_i\}}$ do not depend on T and we used (4.99). Then

$$\left|\frac{r^{(k)}(\zeta)}{r(\zeta)}\right| = \left|\sum_{j=1}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\frac{p^{(j)}(\zeta)}{p(\zeta)}q(\zeta)\left(\frac{1}{q(\zeta)}\right)^{(k-j)}\right| \leq \sum_{j=1}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\frac{c_{k-j}^*}{\delta^j}d^k = c_k d^k,$$

where we used (4.100). ■

Proof of Corollary 4.17: For each $\eta \in S'$ we decompose $q_{n,m}$ as

$$q_{n,m}(z) = u_{n,\eta}(z)q_{n,\eta}(z), \quad q_{n,\eta}(z) := \prod_{j=1}^{m(\eta)}(z-\eta_{j,n}).$$

Recall that $\eta_{j,n}$ are such that $\lim_{n\to\infty}\eta_{j,n}=\eta$ for all $j=1,\ldots,m(\eta)$. Then we derive from (4.86) for each $\eta\in S'$ the following:

$$E_{n,\eta}(z)q_{n,\eta}(z)=i_n(z),\quad z\in\mathbb{D}\setminus E, \tag{4.101}$$

where

$$E_{n,\eta}(z):=\frac{1}{\beta_n}\frac{u_{n-m}^2(z)u_{n,\eta}(z)(Qe_n)(z)}{\widetilde{q}_n^2(z)},\quad z\in\mathbb{D}\setminus E.$$

It is easy to check that

$$E_{n,\eta}^{(k)}(\eta)=\left(\frac{1}{\beta_n}\frac{u_{n-m}^2(z)u_{n,\eta}(z)P(z)}{\widetilde{q}_n^2(z)}\right)^{(k)}\Bigg|_{z=\eta} \tag{4.102}$$

for all $k=0,\ldots,m_n(\eta)-1$, where $R=P/Q$ and

$$E_{n,\eta}(\eta)=\left(\frac{1}{2}+o(1)\right)\frac{1}{\mathcal{D}\left(|b^2\dot{\mu}w|\right)}\left(\frac{\varphi(\eta)}{\rho}\right)^{2(n-m)}\frac{u_{n,\eta}(\eta)P(\eta)}{\mathcal{D}_n(\eta)\widetilde{q}_{n,m}^2(\eta)} \tag{4.103}$$

by (4.75) and (4.97). Since $E_{n,\eta}(\eta)q_{n,\eta}(\eta)\neq 0$ for all n large enough by (4.103), (4.76), (4.87), and (4.101), we derive for any $k=0,\ldots,m(\eta)-1$ that

$$\left(E_{n,\eta}(z)q_{n,\eta}(z)\right)^{(k)}\Big|_{z=\eta}=i_n^{(k)}(\eta). \tag{4.104}$$

Thus, we get for $k=0$ that

$$\chi_n^{m(\eta)}q_{n,\eta}(\eta)=-1,\quad \chi_n:=\left(-\frac{E_{n,\eta}(\eta)}{i_n(\eta)}\right)^{1/m(\eta)}. \tag{4.105}$$

Observe that χ_n tends to infinity geometrically fast as follows from (4.103) and the normality of $\{i_n\}$. Putting $k=1$ we obtain

$$\chi_n^{m(\eta)-1}q'_{n,\eta}(\eta)=\frac{1}{\chi_n}\left(\frac{i'_n(\eta)}{i_n(\eta)}-\frac{E'_{n,\eta}(\eta)}{E_{n,\eta}(\eta)}\right)=o(1) \tag{4.106}$$

since $E_{n,\eta}^{(j)}(\eta)$, $j=0,1$, coincides with the value at η of a rational function (see (4.102)) and therefore the second term on the right-hand side of (4.106) has polynomial order of growth by Lemma 4.21

while $1/\chi_n$ decays exponentially. Continuing by induction we get

$$\chi_n^{m(\eta)-k} q_{n,\eta}^{(k)}(\eta) = \frac{1}{\chi_n^k} \frac{i_n^{(k)}(\eta)}{i_n(\eta)} - \sum_{j=1}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \frac{E_{n,\eta}^{(j)}(\eta)}{E_{n,\eta}(\eta)} \frac{\chi_n^{m(\eta)-k+j} q_{n,m}^{(k-j)}(\eta)}{\chi_n^j} = o(1), \tag{4.107}$$

for any $k = 2, \ldots, m(\eta) - 1$. Equations (4.105) and (4.107) immediately imply

$$\prod_{j=1}^{m(\eta)} (z + \chi_n(\eta - \eta_{j,n})) = z^{m(\eta)} + \sum_{k=0}^{m(\eta)-1} \chi_n^{m(\eta)-j} q_{n,\eta}^{(j)}(\eta) z^j = z^{m(\eta)} - 1 + o(1). \tag{4.108}$$

In particular, this means that

$$\eta_{k,n} = \eta + \frac{1 + o(1)}{\chi_n} \exp\left(\frac{2\pi k}{m(\eta)} i\right). \tag{4.109}$$

By setting

$$h_{n,\eta} := \left(\frac{\varphi(\eta)}{\rho}\right)^{2n/m(\eta)} \frac{1}{\chi_n},$$

we see that equations (4.80) follow. The boundedness of $\{|h_{n,\eta}|\}$ is a consequence of (4.103) and (4.105). ■

Proof of Proposition 4.18: Formula (4.81) is a straightforward consequence of (4.72). ■

Padé Approximation

In this section, we turn our attention to multipoint Padé approximation. As before, we shall denote diagonal multipoint diagonal Padé approximant of order n by $\Pi_n = p_n/q_n$. We are interested in answers to the same questions that were raised in the preceding section for meromorphic approximants.

Theorem 4.22 *Let $\mathbf{A} = \{A_{2n}\}_{n\in\mathbb{N}}$ be a strongly admissible interpolation scheme and $\mathcal{F}(\mu; R; \cdot)$ be given by (1.2), where $R = P/Q$ has no poles on E and $\dot{\mu}$ is a product of a non-vanishing Dini-continuous function on E and $\hat{s}$, defined by (4.64) and (4.65) with v_{2n} associated to A_n via (2.37). Let further $\{\Pi_n\}_{n\in\mathbb{N}}$, $\Pi_n = p_n/q_n$, be a sequence of diagonal multipoint Padé approximants to $\mathcal{F}$ associated with $\mathbf{A}$. Then*

$$q_n^2(z) \frac{\Psi_{2n}(z)}{v_{2n}(z)} = \frac{\alpha_n}{\mathcal{S}(\dot{\mu})} \left(\frac{1}{2} + o(1)\right) \frac{\Psi^2(z)}{\mathcal{S}^2(\dot{\mu}; z)} \tag{4.110}$$

locally uniformly in $D_{\mathcal{F}}$*, where* $\Psi_{2n}(\cdot) = \Psi_{2n}(v_{2n};\cdot)$ *and* $\Psi(\cdot) = \Psi_m(Q;\cdot)$*. Moreover, we have that*

$$\alpha_n = (2+o(1))\text{cap}^{2(n-m)}(E)\mathcal{S}(\dot{\mu})\mathcal{S}\left(\frac{Q^2}{v_{2n}}\right). \tag{4.111}$$

Further, the following error estimate holds locally uniformly in $D_{\mathcal{F}}$*:*

$$(\mathcal{F}-\Pi_n)(z) = \mathcal{S}^2(\dot{\mu};z)\frac{2\mathcal{S}(\dot{\mu})+o(1)}{\sqrt{(z-a)(z-b)}}\frac{\Psi_{2n}(z)}{\Psi^2(z)}, \tag{4.112}$$

where the sign of the square root is chosen such that $\sqrt{(z-a)(z-b)} = z + O(1)$ *as* $z \to \infty$*.*

It follows from the preceding theorem that polynomials q_n can be written as $u_{n-m}q_{n,m}$, where u_{n-m} has no zeros on any compact $K \subset \mathbb{D} \setminus E$ and $q_{n,m} = (1+o(1))Q$ in $\overline{\mathbb{C}} \setminus S'$. Let $\eta_{1,n}, \dots \eta_{m(\eta),n}$ be the zeros of q_n approaching $\eta \in S'$. Then the following holds.

Corollary 4.23 *For each* $\eta \in S'$ *and all* n *large enough there exists an arrangement of* $\eta_{1,n}, \dots \eta_{m(\eta),n}$ *such that*

$$\eta_{k,n} = \eta + (h_{n,\eta}+o(1))\left(\Psi_{2n}(\eta)\right)^{1/m(\eta)}\exp\left(\frac{2\pi k}{m(\eta)}i\right), \quad k = 1,\dots,m(\eta), \tag{4.113}$$

where $\{|h_{n,\eta}|\}$ *is a bounded above sequence.*

Proof of Theorem 4.22: Recall (cf. (2.38)) that the denominators of Π_n satisfy the following non-Hermitian orthogonality relations

$$\int t^j q_n(t)Q(t)\frac{d\mu(t)}{v_{2n}(t)} = 0, \quad j = 0,\dots,n-m-1.$$

As shown in Theorem 3.16, polynomials q_n can be written as $u_{n-m}q_{n,m}$, where $q_{n,m} = (1+o(1))Q$ locally uniformly in $\mathbb{C} \setminus S'$. Thus, we may apply Theorems 4.11 and 4.12 with $\dot{\nu}_{n-m} := q_{n,m}Q\dot{\mu}/v_{2n}$. Define

$$\alpha_n := \int q_n^2(t)\frac{d\mu(t)}{v_{2n}(t)}.$$

Since Q has no poles on E, we get from Theorem 4.12 that

$$\alpha_n = (1+o(1))\int u_{n-m}^2(t)d\nu_{n-m}(t) = (2+o(1))\text{cap}^{2(n-m)}(E)\mathcal{S}(\dot{\mu})\mathcal{S}\left(\frac{Q^2}{v_{2n}}\right).$$

Further, from Theorem 4.11 and equation (4.15) we derive that

$$\begin{aligned} \frac{u_{n-m}^2(z)}{\psi_{2(n-m)}(z)} &= \frac{1+o(1)}{\mathcal{S}^2(\dot{\nu}_{n-m};z)} = \frac{1+o(1)}{\mathcal{S}^2(\dot{\mu};z)}\mathcal{S}^2\left(\frac{v_{2n}}{q_{n,m}Q};z\right) \\ &= \frac{1+o(1)}{\mathcal{S}^2(\dot{\mu};z)}\mathcal{S}\left(\frac{Q^2}{v_{2n}}\right)\frac{v_{2n}(z)}{\Psi_{2n}(z)}\frac{\Psi(z)\Psi_m(q_{n,m};z)}{Q(z)q_{n,m}(z)}\varphi_i^{2(n-m)}(z), \end{aligned} \tag{4.114}$$

where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}}\setminus E$. Since $q_{n,m}/Q = 1+o(1)$ and $\Psi_m(q_{n,m};\cdot)/\Psi = 1+o(1)$ locally uniformly in $D_{\mathcal{F}}$, (4.110) and (4.111) follow.

Proceeding as in the proof of Theorem 3.15, we see that

$$(\mathcal{F}-\Pi_n)(z) = \frac{v_{2n}(z)}{q_n^2(z)}\frac{q_{n,m}(z)}{Q(z)}\int u_{n-m}^2(t)\frac{d\nu_{n-m}(t)}{z-t}, \quad z\in D_{\mathcal{F}}. \tag{4.115}$$

Using the arguments from the proof of Theorem 4.12, we obtain

$$(\mathcal{F}-\Pi_n)(z) = \alpha_n\frac{v_{2n}(z)}{q_n^2(z)}\frac{1+o(1)}{\sqrt{(z-a)(z-b)}},$$

where $o(1)$ holds locally uniformly in $D_{\mathcal{F}}$. Now, (4.112) follows from (4.110), which finishes the proof of the theorem. ■

Proof of Corollary 4.23: Denote by e_n the error $\mathcal{F}-\Pi_n$ and observe that (4.115) implies

$$\frac{1}{\alpha_n}\frac{u_{n-m}^2(z)q_{n,m}(z)(Qe_n)(z)}{v_{2n}(z)} = i_n(z), \quad z\in\mathbb{C}\setminus E, \tag{4.116}$$

where

$$i_n(z) := \frac{1}{\alpha_n}\int\frac{q_n^2(t)}{v_{2n}(t)}\frac{Q(t)}{q_{n,m}(t)}\frac{d\mu(t)}{z-t}, \quad z\in\mathbb{D}\setminus E,$$

is such that

$$\sqrt{(z-a)(z-b)}\,i_n(z) = 1+o(1) \tag{4.117}$$

locally uniformly in $\overline{\mathbb{C}}\setminus E$. In particular, it means that $\{i_n\}$ is a normal family in $\overline{\mathbb{C}}\setminus E$, as well as $\left\{i_n^{(k)}\right\}$ for $k\in\mathbb{N}$.

Now, for each $\eta\in S'$ we decompose $q_{n,m}$ as

$$q_{n,m}(z) = u_{n,\eta}(z)q_{n,\eta}(z), \quad q_{n,\eta}(z) := \prod_{j=1}^{m(\eta)}(z-\eta_{j,n}).$$

Recall that $\eta_{j,n}$ are such that $\lim_{n\to\infty}\eta_{j,n}=\eta$ for all $j=1,\dots,m(\eta)$. Then we derive from (4.116) for each $\eta\in S'$ the following:

$$E_{n,\eta}(z)q_{n,\eta}(z)=i_n(z),\quad z\in\overline{\mathbb{C}}\setminus E, \tag{4.118}$$

where

$$E_{n,\eta}(z):=\frac{1}{\alpha_n}\frac{u_{n-m}^2(z)u_{n,\eta}(z)(Qe_n)(z)}{v_{2n}(z)},\quad z\in\overline{\mathbb{C}}\setminus E.$$

It is easy to check that

$$E_{n,\eta}^{(k)}(\eta)=\left(\frac{1}{\alpha_n}\frac{u_{n-m}^2(z)u_{n,\eta}(z)P(z)}{v_{2n}(z)}\right)^{(k)}\Bigg|_{z=\eta} \tag{4.119}$$

for all $k=0,\dots,m(\eta)-1$, where $R=P/Q$ and

$$E_{n,\eta}(\eta)=\left(\frac{1}{2}+o(1)\right)\left(\frac{\Psi}{Q}\right)^2(\eta)\frac{u_{n,\eta}(\eta)P(\eta)}{\mathcal{S}(\dot\mu)\mathcal{S}(\dot\mu;\eta)}\frac{1}{\Psi_{2n}(\eta)} \tag{4.120}$$

by (4.114) and (4.111). Since $E_{n,\eta}(\eta)q_{n,\eta}(\eta)\neq 0$ by (4.118) and (4.120), we derive for any $k=0,\dots,m(\eta)-1$ that

$$\left(E_{n,\eta}(z)q_{n,\eta}(z)\right)^{(k)}\Big|_{z=\eta}=i_n^{(k)}(\eta). \tag{4.121}$$

Thus, we get for $k=0$ that

$$\chi_n^{m(\eta)}q_{n,\eta}(\eta)=-1,\quad \chi_n:=\left(-\frac{E_{n,\eta}(\eta)}{i_n(\eta)}\right)^{1/m_n(\eta)}. \tag{4.122}$$

Observe that χ_n tends to infinity geometrically fast as follows from (4.120) and the normality of $\{i_n\}$. Putting $k=1$ we obtain

$$\chi_n^{m(\eta)-1}q'_{n,\eta}(\eta)=\frac{1}{\chi_n}\left(\frac{i'_n(\eta)}{i_n(\eta)}-\frac{E'_{n,\eta}(\eta)}{E_{n,\eta}(\eta)}\right)=o(1) \tag{4.123}$$

since $E_{n,\eta}^{(j)}(\eta)$, $j=0,1$, coincides with the value at η of a rational function (see (4.119)) and therefore the second term on the right-hand side of (4.123) has polynomial order of growth by Lemma 4.21 while $1/\chi_n$ decays exponentially. Continuing by induction we get

$$\chi_n^{m(\eta)-k}q_{n,\eta}^{(k)}(\eta)=\frac{1}{\chi_n^k}\frac{i_n^{(k)}(\eta)}{i_n(\eta)}-\sum_{j=1}^{k}\binom{k}{j}\frac{E_{n,\eta}^{(j)}(\eta)}{E_{n,\eta}(\eta)}\frac{\chi_n^{m(\eta)-k+j}q_{n,m}^{(k-j)}(\eta)}{\chi_n^j}=o(1), \tag{4.124}$$

for any $k = 2, \dots, m(\eta) - 1$. Equations (4.122) and (4.124) immediately imply

$$\prod_{j=1}^{m(\eta)} (z + \chi_n(\eta - \eta_{j,n})) = z^{m(\eta)} + \sum_{k=0}^{m(\eta)-1} \chi_n^{m(\eta)-j} q_{n,\eta}^{(j)}(\eta) z^j = z^{m(\eta)} - 1 + o(1). \tag{4.125}$$

In particular, this means that

$$\eta_{k,n} = \eta + \frac{1 + o(1)}{\chi_n} \exp\left(\frac{2\pi k}{m(\eta)} i\right). \tag{4.126}$$

By setting

$$h_{n,\eta} := (\Psi_{2n}(\eta))^{1/m(\eta)} \frac{1}{\chi_n},$$

we see that equations (4.113) follow. The boundedness of $\{|h_{n,\eta}|\}$ is a consequence of (4.126) and (4.120). ∎

Numerical Experiments

The Hankel operator $\mathcal{H}_f$ with symbol $f \in H^\infty + C(\mathbb{T})$ is of finite rank if and only if f is a rational function [91, Thm. 3.11]. In practice one can only compute with finite rank operators, due to the necessity of ordering the singular values, so a preliminary rational approximation to f is needed when the latter is not rational. One way to handle this problem is to truncate the Fourier series of f at some high order N. This provides us with a rational function f_N that approximates f in the Wiener norm which, in particular, dominates any L^p norm on the unit circle, $p \in [1, \infty]$. It was proved in [62] that the best approximation operator from H_n^∞ (mapping f to g_n according to (2.24)) is continuous in the Wiener norm provided $(n+1)$-st singular value of the Hankel operator is simple. It was shown in [22, Cor. 2] that the last assertion is satisfied for Hankel operators with symbols in some open dense subset of $H^\infty + C(\mathbb{T})$, and the same technique can be used to prove that it is also the case for the particular subclass (1.2). Thus, even though the simplicity of singular values cannot be asserted beforehand, it is generically true. When it prevails, one can approximates f_N instead of f and get a close approximation to g_n when N is large enough. This amounts to perform the singular value decomposition of $\mathcal{H}_{f_N}$ (see [135, Ch. 16]).

As to Padé approximants, we restricted ourselves to the classical case and we constructed their denominators by solving the orthogonality relations (2.38) with $w_{2n} \equiv 1$. Thus, finding these denominators amounts to solving a system of linear equations whose coefficients are obtained from the moments of the measure μ.

In the numerical experiments below we approximate function f given by the formula

$$\mathcal{F}(z) = 7\int_{[-0.7,0]} \frac{e^{it}}{z-t}\frac{dt}{\sqrt{(t+0.7)(0.4-t)}} + \int_{[0,0.4]} \frac{it+1}{z-t}\frac{dt}{\sqrt{(t+0.7)(0.4-t)}} + \frac{1}{5!(z-0.7-0.2i)^6}.$$

On the figures the solid line stands for the support of the measure and circles denote the poles of the corresponding approximants.

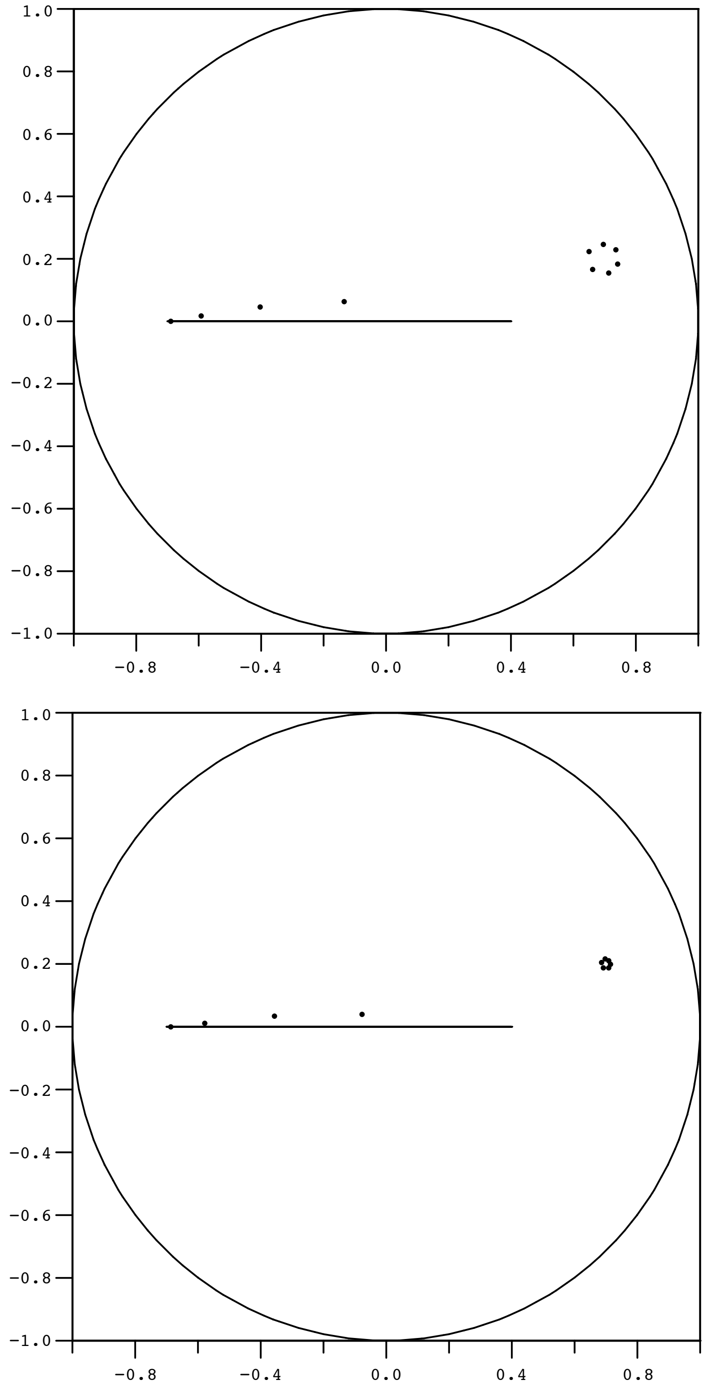

Figure 4: Poles of Padé (left) and AAK (right) approximants of degree 10 to $\mathcal{F}$.

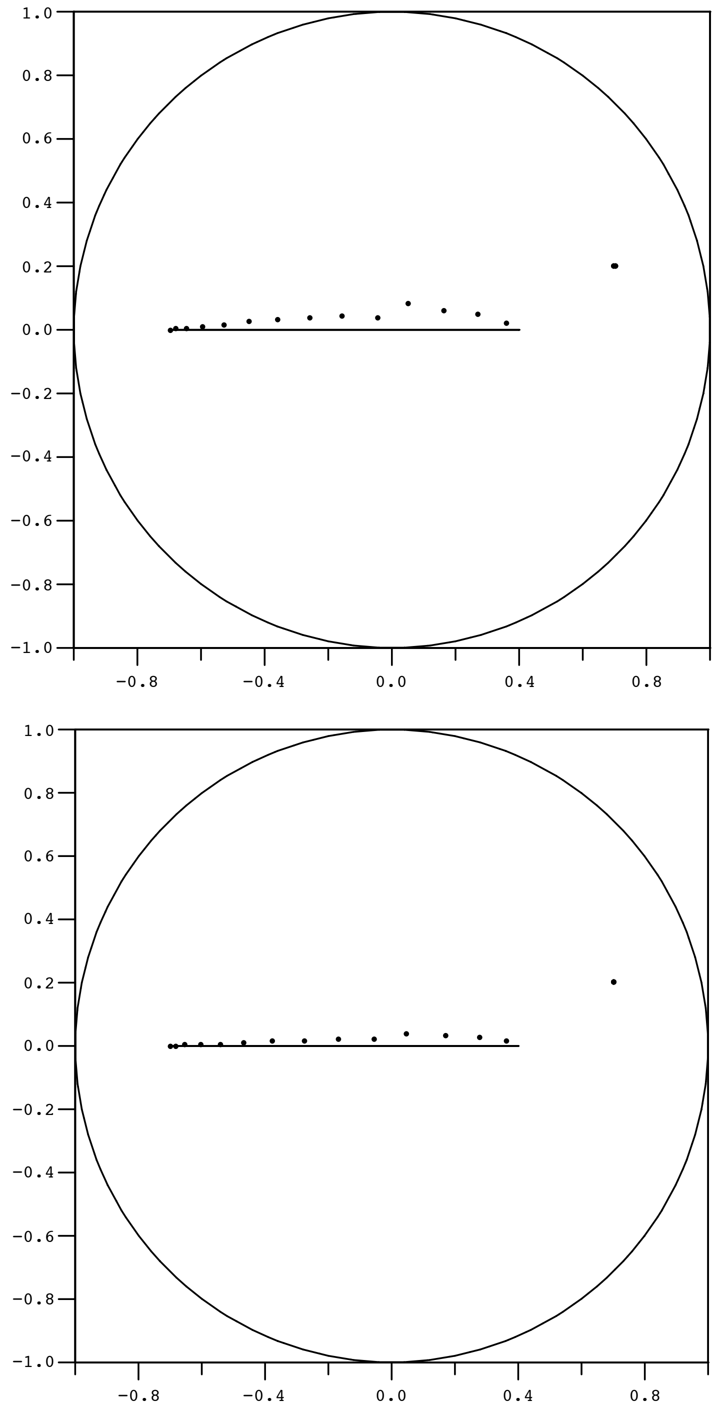

Figure 5: Poles of Padé (left) and AAK (right) approximants of degree 20 to $\mathcal{F}$.

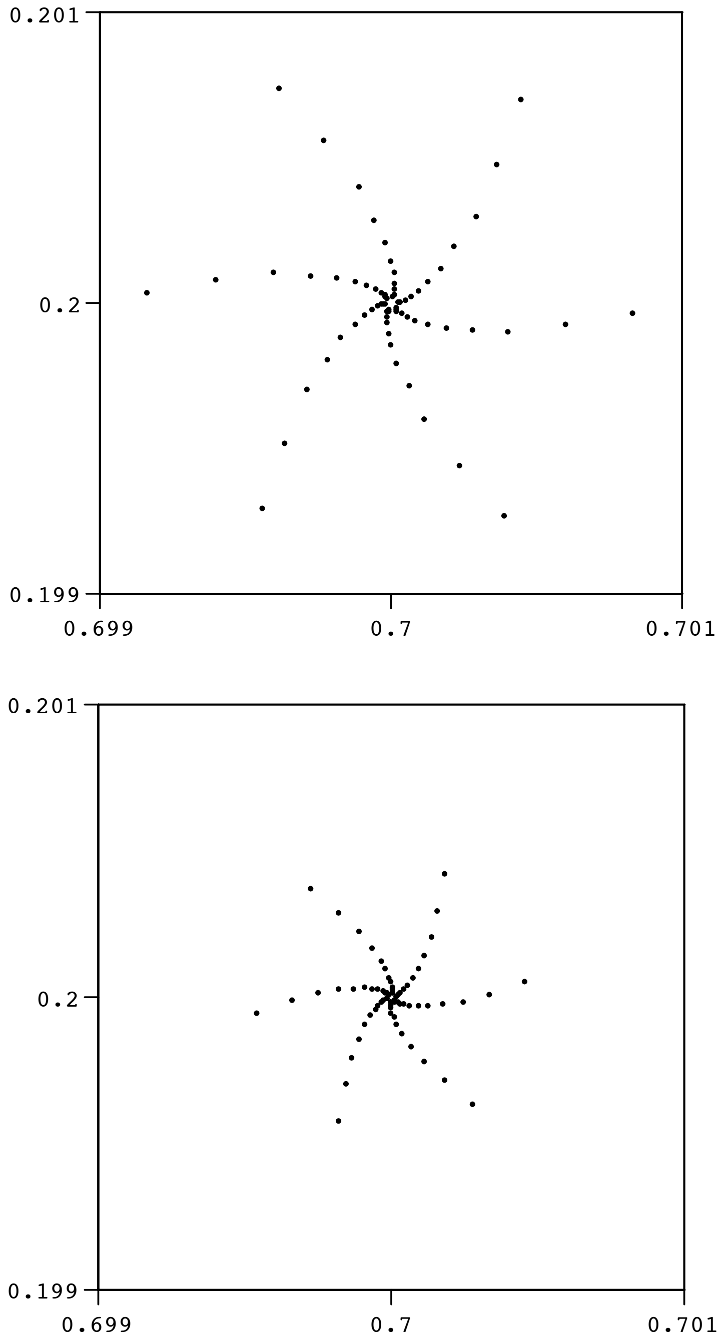

Figure 6: Poles of Padé (left) and AAK (right) approximants of degrees 21-33 to $\mathcal{F}$ lying in an neighborhood of the polar singularity.

CHAPTER V

ON MULTIPLICITY OF SINGULAR VALUES OF CERTAIN HANKEL OPERATORS

The modest objective of the present chapter is to prove that the multiplicities of the singular values of Hankel operators whose symbol is the Cauchy transform of a complex measure with argument of bounded variation is bounded in terms of that variation. If moreover the measure is sufficiently nonvanishing, the singular values are asymptotically simple. This also entails results on the conjugate Hankel operator as an anti-linear operator. The methods employed rely on the techniques developed in Chapter III, which is the reason to present forthcoming results in this thesis.

In the most general setting a Hankel operator is an operator acting on ℓ_2 given in the canonical basis by a matrix of the form $\{\alpha_{j+k}\}_{j,k\geq 0}$ with $\alpha_j \in \mathbb{C}$. Such a definition admits numerous realizations which, in turn, imply a wide range of applications of Hankel operators. In particular, they appeared to be an extremely important class of operators in approximation theory. The elaboration of the properties of Hankel operators from the approximation view point initiated with the celebrated AAK-Theory that showed the link between meromorphic approximation of L^∞ functions and singular numbers of the corresponding Hankel operators ([1], see also [97, Ch. 4]). Later, this theory was generalized to L^p functions on the unit circle, $2 \leq p \leq \infty$ ([28], [104], and [80]), and to more general domains of approximation (see [102]). Moreover, these methods turned out to be instrumental for investigating the degree of rational approximation of analytic functions (see [98], [90], and [105]) and helped to describe classes of analytic functions in the disk (Besov spaces) in terms of the rate of rational approximation (see [94], [95], [93], and [116]). In another connection, Hankel operators also play a significant role in operator theory. In particular, G. Pisier [99] (see also [97, Thm. 15.3.1]) showed that there are polynomially bounded operators on a Hilbert space that are not similar to a contraction by using Hankel operators techniques. Further, it was shown that geometric problems in the theory of stationary Gaussian processes can be reduced to the question of describing those bounded linear operators on a Hilbert space that are unitary equivalent to Hankel operators. A program of N. K. Nikolskii to characterize such bounded linear operators in spectral terms (see [85]) was successfully completed in the self-adjoint case (see [79]). In the course of the proof it was shown that the absolute value of difference of the multiplicities of symmetric eigenvalues of a Hankel operator (self-adjoint or not) is bounded by one. Nevertheless the question of the boundedness of the multiplicities themselves remained open.

Statements of the Results

Throughout this section the capital letters H and K shall be reserved for the notation of Hilbert spaces. Further, $\mathcal{L}(H;K)$ will stand for the space of linear operators from H to K.

It will be convenient for us to use the following formal definition of Hankel operators acting on the Hardy class H^2. Let $f \in L^\infty(\mathbb{T})$. The *Hankel operator with symbol* f, denoted by $\mathcal{H}_f \in \mathcal{L}(H^2; \bar{H}_0^2)$, is defined by the rule

$$\mathcal{H}_f(h) := \mathcal{P}_-(fh),$$

where $\mathcal{P}_-$ is the antianalytic projection, i.e. the projection of $L^2(\mathbb{T})$ onto $\bar{H}_0^2$.

For $n \in \mathbb{Z}_+$, the *n-th singular number* of the operator $\mathcal{H}_f$ is defined as

$$s_n(\mathcal{H}_f) := \inf\left\{\|\mathcal{H}_f - \mathcal{O}\| : \ \mathcal{O} : H^2 \to \bar{H}_0^2 \text{ a linear operator of rank } \le n\right\},$$

where $\|\cdot\|$ stands for the operator norm between two Hilbert spaces. Clearly $\{s_n(\mathcal{H}_f)\}_{n\in\mathbb{N}}$ is nonincreasing sequence. By $s_\infty(\mathcal{H}_f)$ we shall denote the distance from $\mathcal{H}_f$ to compact operators, i.e.

$$s_\infty(\mathcal{H}_f) = \lim_{n\to\infty} s_n(\mathcal{H}_f).$$

By the well-known theory of E. Schmidt (cf. [86, Vol. I, Thm. 7.1.1]), s is a singular number of a compact operator $\mathcal{O} \in \mathcal{L}(H;K)$ if and only if s^2 is an eigenvalue of the operator $\mathcal{O}^*\mathcal{O}$, where $\mathcal{O}^*$ is the *adjoint* operator to $\mathcal{O}$.

Although $\{s_n(\mathcal{H}_f)\}$ is nonincreasing, it is not necessarily strictly decreasing. Let $\mu_n(\mathcal{H}_f)$ stand for the multiplicity of $s_n(\mathcal{H}_f)$, i.e. $\mu_n(\mathcal{H}_f)$ is an integer such that there exist constants $k, j \in \mathbb{Z}_+$ for which $\mu_n(\mathcal{H}_f) = j - k - 1$ and

$$s_k(\mathcal{H}_f) > s_{k+1}(\mathcal{H}_f) = \ldots = s_n(\mathcal{H}_f) = \ldots = s_{j-1}(\mathcal{H}_f) > s_j(\mathcal{H}_f).$$

The main goal of this chapter is to investigate the behavior of the sequence $\{\mu_n(\mathcal{H}_f)\}$ for Hankel operators whose symbol assumes some special form.

Theorem 5.1 *Let $\mathcal{F}(\mu; R; \cdot)$ be of the form (1.2), where the measure μ satisfies (3.1) and (3.2) with $S = \mathrm{supp}(\mu) \subset (-1,1)$, while $R = P/Q$ is a rational function with no poles on $\mathbb{T}$. Then the sequence of multiplicities of singular values of the Hankel operator $\mathcal{H}_\mathcal{F}$, $\{\mu_n(\mathcal{H}_\mathcal{F})\}_{n\in\mathbb{Z}_+}$, is uniformly* *bounded. More precisely, the following upper bound holds for any $n \in \mathbb{Z}_+$:*

$$\mu_n(\mathcal{H}_\mathcal{F}) \leq \frac{2}{\pi}\left(V(\Theta) + V(\arg(Q;\cdot);[a,b]) + \pi m + V_\mathcal{W}\right) + N_\mathcal{W} + 1, \tag{5.1}$$

where $[a,b]$ is the convex hull of S, $m = \deg Q$,

$$N_\mathcal{W} := \max_{n\in\mathbb{Z}_+} \#\{\xi \in \mathbb{T}: \ w_n(\xi) = 0\}, \tag{5.2}$$

$$V_\mathcal{W} := \sup_{n\in\mathbb{Z}_+} V(\arg(w_n;\cdot);[a,b]), \tag{5.3}$$

and w_n is the outer factor of a singular vector v_n with exactly n zeros in $\mathbb{D}$ associated to g_n, the best meromorphic approximant to $\mathcal{F}$ of order n given by the AAK-Theory[12].

The finiteness of the constants $N_\mathcal{W}$ and $V_\mathcal{W}$ will be shown during the proof the theorem.

It is worth noting that in the case where $\mathcal{F}$ is just a *Markov function*, i.e. the Cauchy transform of a positive measure supported on the real line, all the singular values of the corresponding Hankel operator are simple (see [27]). This phenomenon is due to the positivity of the measure and cannot be expected to hold in the complex case. Nevertheless, in the case where $\mathcal{F}$ is the Cauchy transform of a complex measure supported on an interval that has a Dini continuous nonvanishing Radon-Nykodim derivative with respect to the logarithmic equilibrium distribution on this interval, it is possible to deduce more detailed information on the sequence of outer factors $\{w_n\}$, which, in turn, can be used to show that singular values of the corresponding Hankel operator are asymptotically simple.

Theorem 5.2 *Let $\mathcal{F}(\mu;R;\cdot)$ be given by (1.2), where the measure $\mu \in \mathbf{BND}_0$ and $R = P/Q$ is a rational function with no poles on $\mathbb{T}$. Then $\mu_n(\mathcal{H}_\mathcal{F}) = 1$ for all n large enough.*

Proofs

The proofs of the theorems rely on one known result that is significant on its own. For the ease of the reader we present it below. However, before we state this theorem we need to introduce several additional concepts. Let, as before, $f \in L^\infty(\mathbb{T})$. Recall that the *Toeplitz operator with symbol f*, $\mathcal{T}_f \in \mathcal{L}(H^2)$, is defined as

$$\mathcal{T}_f(h) := \mathcal{P}_+(fh).$$

It is easy to see that

$$\mathcal{T}_f + \mathcal{H}_f = \mathcal{M}_f,$$

[12] Solution of the meromorphic approximation problem for $p = \infty$.

where $\mathcal{M}_f : H^2 \to L^2(\mathbb{T})$ is the operator of multiplication by f. Recall also that an operator $\mathcal{O} \in \mathcal{L}(H)$ is called *Fredholm* if it is invertible modulo compact operators. The *index* of a Fredholm operator $\mathcal{O}$ is defined by

$$\operatorname{ind}(\mathcal{O}) := \dim\ker(\mathcal{O}) - \dim\ker(\mathcal{O}^*).$$

The *essential spectrum*, $\sigma_{\rm e}(\mathcal{O})$, of a bounded operator $\mathcal{O}$ is, by definition,

$$\sigma_{\rm e}(\mathcal{O}) := \{z \in \mathbb{C} : \ \mathcal{O} - z\mathcal{I} \ \text{ is not Fredholm}\}.$$

The next notion that we need is the notion of the *winding number with respect to the origin* of a continuous function on $\mathbb{T}$. Let $u \in C(\mathbb{T})$ and let $\arg(u;\cdot)$ be any branch of the argument of u. Assume further that u does not vanish on $\mathbb{T}$. Then the winding number of u with respect to the origin is defined by

$$\operatorname{wind}(u) := \frac{1}{2\pi}\left[\arg(u;2\pi) - \arg(u;0)\right].$$

Clearly $\operatorname{wind}(u)$ does not depend on the choice of the branch of the argument of u.

In general, let u be an *invertible* function in $H^\infty + C(\mathbb{T})$, i.e. $1/u \in H^\infty + C(\mathbb{T})$. Denote also by u the harmonic extension of u into $\mathbb{D}$. Then it is known (see [97, Thm. 3.3.5]) that there exists $r_0 \in (0,1)$ such that $|u|$ is bounded away from zero on the annulus $\{z : r_0 < |z| < 1\}$ and functions $u_r(\xi) := u(r\xi)$, $\xi \in \mathbb{T}$, have the same winding number for any $r \in (r_0, 1)$. Thus, for any invertible function u in $H^\infty + C(\mathbb{T})$ we define the winding number as

$$\operatorname{wind}(u) := \operatorname{wind}(u_r), \quad r \in (r_0, 1).$$

Now we can describe the essential spectrum of a Toeplitz operator (see [97, Thm. 3.3.8]).

Theorem P *Let $u \in H^\infty + C(\mathbb{T})$. Then for any $z_0 \notin \sigma_{\rm e}(\mathcal{T}_u)$*

$$\operatorname{ind}(\mathcal{T}_u - z_0\mathcal{I}) = -\operatorname{wind}(u - z_0). \tag{5.4}$$

Moreover, if u is a continuous function then $\sigma_{\rm e}(\mathcal{T}_u) = u(\mathbb{T})$.

For the upcoming proofs lets recall one more concept, namely, the *angle in which an interval is seen at a point.* For any $\xi \neq 0 \in \mathbb{C}$, we let $\operatorname{Arg}(\xi) \in (-\pi, \pi]$ be the principal branch of the argument and for $\xi = 0$ we set $\operatorname{Arg}(0) = \pi$. Under such a definition, $\operatorname{Arg}(\cdot)$ becomes a left continuous function on $\mathbb{R}$. Now, for any interval $[a,b] \subset \mathbb{R}$ we define the angle in which this interval is seen at $\xi \in \mathbb{C}$ by

$$\mathrm{Angle}(\xi,[a,b]) := |\mathrm{Arg}(a-\xi) - \mathrm{Arg}(b-\xi)|.$$

It is easy to see that for any $\xi \notin \mathbb{D}$ and any $[a,b] \subset (-1,1)$ there holds

$$\mathrm{Angle}(\xi,[a,b]) \leq \pi/2.$$

Proof of Theorem 5.1: Fix an arbitrary $n \in \mathbb{Z}_+$. Without loss of generality we may assume that $s_{n-1}(\mathcal{H}_\mathcal{F}) > s_n(\mathcal{H}_\mathcal{F})$. Denote by g_n the best meromorphic approximant to $\mathcal{F}$ on $\mathbb{T}$ out of H^∞_n (recall that g_n is unique by the compactness of $\mathcal{H}_\mathcal{F}$). Then, the by circularity property (2.27), the function

$$u_n := s_n(\mathcal{H}_\mathcal{F})^{-1}(\mathcal{F} - g_n)$$

is unimodular almost everywhere on the unit circle. It is known [97, Thm. 4.1.7] that in this case

$$\dim\ker(\mathcal{T}_{u_n}) = 2n + \mu_n(\mathcal{H}_\mathcal{F}).$$

It is also known [97, Thm. 3.1.4] that either $\ker(\mathcal{T}_u) = \{0\}$ or $\ker(\mathcal{T}_u^*) = \{0\}$ for any nonzero function from $L^\infty(\mathbb{T})$. Thus,

$$\mathrm{ind}(\mathcal{T}_{u_n}) = \dim\ker(\mathcal{T}_{u_n}) = 2n + \mu_n(\mathcal{H}_\mathcal{F}). \tag{5.5}$$

Therefore, upon showing that u_n is a continuous and nonvanishing function on $\mathbb{T}$, we will obtain from (5.5) and (5.4) that

$$\mu_n(\mathcal{H}_\mathcal{F}) = -2n - \mathrm{wind}(u_n). \tag{5.6}$$

Indeed, in this case zero does not belong to $\sigma_{\mathrm{e}}(\mathbb{T}_{u_n}) = u(\mathbb{T})$ and we may apply Theorem P. To show continuity of u_n recall that by the AAK Theorem there exists a singular vector $v_n \in H^2$ with the inner-outer factorization

$$v_n = b_n w_n, \tag{5.7}$$

where b_n is a Blaschke product of exact degree n and w_n is an outer function, such that

$$u_n = \frac{\mathcal{H}_\mathcal{F}(v_n)}{v_n}. \tag{5.8}$$

Moreover, it is known (see e.g. [28, Sec. 8 and 9]) that $\mathcal{H}_{\mathcal{F}}(v_n)$ has the following representation

$$\mathcal{H}_{\mathcal{F}}(v_n)(z) = \frac{s_n(\mathcal{H}_F)}{z}\,\overline{(j_n b_n w_n)\left(\frac{1}{\bar{z}}\right)}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}, \tag{5.9}$$

where j_n is some inner function.

In another connection, by the definition of Hankel operators and (1.2) we have that

$$\begin{aligned} \mathcal{H}_{\mathcal{F}}(v_n)(z) &= \mathcal{P}_{-}(\mathcal{F}v_n)(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\mathcal{F}(\xi)v_n(\xi)}{z-\xi}d\xi \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}}\int \frac{v_n(\xi)}{(z-\xi)(\xi-x)}d\mu(x)d\xi + \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{R(\xi)v_n(\xi)}{z-\xi}d\xi \\ &= \int \frac{v_n(x)}{z-x}d\mu(x) + \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(\xi)v_n(\xi)}{z-\xi}\frac{d\xi}{Q(\xi)}, \quad z \in \mathbb{C}\setminus\overline{\mathbb{D}}, \end{aligned} \tag{5.10}$$

where $R = P/Q$. Note that the second integral in (5.10) is, in fact, a rational function with denominator Q by the Cauchy integral formula. Combining (5.9) and (5.10) we get that

$$(j_n b_n w_n)(z) = s_n(\mathcal{H}_{\mathcal{F}})^{-1}\left(\int \frac{\overline{v_n(x)}}{1-xz}\overline{d\mu(x)} + \overline{\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(\xi)v_n(\xi)}{1-\xi\bar{z}}\frac{d\xi}{Q(\xi)}}\right), \quad z \in \mathbb{D}. \tag{5.11}$$

Observe that the right-hand side of (5.11) is well-defined for $z \in D^*_{\mathcal{F}}$, where $D^*_{\mathcal{F}}$ is the reflection across the unit circle of $D_{\mathcal{F}}$, the domain of analyticity of $\mathcal{F}$. In other words, equation (5.11) provides an analytic continuation of the product $j_n b_n w_n$ outside of the unit disk. In particular, this means that j_n is a finite Blaschke product and the number of zeros of w_n on $\mathbb{T}$ is finite. Let $\{\zeta_{j,n}\}$ be the set of zeros of w_n on $\mathbb{T}$. Then w_n can be written as

$$w_n(z) = w_n^{\#}(z)P_n(z), \quad P_n(z) := \prod_j (z-\zeta_{j,n}), \tag{5.12}$$

where $w_n^{\#}$ is an analytic and zero-free function in some neighborhood of $\overline{\mathbb{D}}$. Then (5.8) with the help of (5.7), (5.9), and (5.12) yields

$$\begin{aligned} u_n(\xi) &= \frac{s_n(\mathcal{H}_{\mathcal{F}})}{\xi}\frac{\overline{(j_n b_n w_n^{\#})(\xi)}}{(b_n w_n^{\#})(\xi)}\prod_j\left(-\frac{1}{\zeta_{j,n}\xi}\right) \\ &= \frac{s_n(\mathcal{H}_{\mathcal{F}})}{\xi(j_n b_n^2)(\xi)}\frac{\overline{w_n^{\#}(\xi)}}{w_n^{\#}(\xi)}\prod_j\left(-\frac{1}{\zeta_{j,n}\xi}\right), \quad \xi \in \mathbb{T}. \end{aligned} \tag{5.13}$$

Equation (5.13) shows that u_n is a continuous nonvanishing function on $\mathbb{T}$ which, in turn, validates equation (5.6).

Denote by Q_n the numerator of the Blaschke product j_n. Then we obtain from (5.13) that

$$\mathrm{wind}(u_n) = -1 - \deg(P_n) - \deg(Q_n) - 2n, \tag{5.14}$$

since $w_n^\#$ is zero-free and analytic in some neighborhood of $\overline{\mathbb{D}}$. Combining (5.14) and (5.6) we get that

$$\mu_n(\mathcal{H}_\mathcal{F}) = \deg(Q_n) + \deg(P_n) + 1 \leq \deg(Q_n) + N_\mathcal{W} + 1, \tag{5.15}$$

where $N_\mathcal{W}$ was defined in (5.2). Thus, to prove (5.1) it remains to show that

$$k_n := \deg(Q_n) \leq \frac{2}{\pi}\left(V(\Theta) + V(\arg(Q;\cdot);[a,b]) + m\pi + V_\mathcal{W}\right) \tag{5.16}$$

and that the constants $N_\mathcal{W}$ and $V_\mathcal{W}$ are finite. Recall that $V_\mathcal{W}$ was defined in (5.3).

It was shown in Lemma 3.11 that the sequence $\{w_m\}_{m\in\mathbb{Z}_+}$ forms a normal family in $D_\mathcal{F}^*$. Moreover, the zero function is not a limit point of this family, since $\|w_m\|_2 = 1$ for each $m \in \mathbb{Z}_+$. This proves the finiteness of $N_\mathcal{W}$.

Now, recall that j_n can be represented as $j_n = Q_n/\widetilde{Q}_n$, where $\widetilde{p}(z) = z^k\overline{p(1/\bar{z})}$, $k = \deg(p)$, for any polynomial p. Similarly we can write $b_n = q_n/\widetilde{q}_n$, where q_n is a monic polynomial with all zeros in $\mathbb{D}$ and of exact degree n. Let $z_0 \in \mathbb{D}$ be such that $(q_nQ_n)(z_0) = 0$. Then we deduce from (5.11) that

$$\int \frac{v_n(x)}{1 - x\bar{z}_0}d\mu(x) + \frac{1}{2\pi i}\int_\mathbb{T} \frac{P(\xi)v_n(\xi)}{1 - \xi\bar{z}_0}\frac{d\xi}{Q(\xi)} = 0. \tag{5.17}$$

By taking linear combinations of equation (5.17) with different roots of q_n and Q_n we obtain that

$$\int \frac{p(x)v_n(x)}{\widetilde{q}_n(x)\widetilde{Q}_n(x)}d\mu(x) + \frac{1}{2\pi i}\int_\mathbb{T} \frac{p(\xi)P(\xi)v_n(\xi)}{\widetilde{q}_n(\xi)\widetilde{Q}_n(\xi)}\frac{d\xi}{Q(\xi)} = 0 \tag{5.18}$$

for any polynomial p of degree at most $n + k_n - 1$. It can be readily verified that equation (5.18) and the Cauchy integral theorem imply the following orthogonality relations

$$\int \frac{x^jQ(x)v_n(x)}{\widetilde{q}_n(x)\widetilde{Q}_n(x)}d\mu(x) = 0, \quad j = 0,\ldots,n + k_n - m - 1. \tag{5.19}$$

By using the inner-outer factorization (5.7) we can rewrite (5.19) in the form

$$\int x^jq_n(x)\frac{Q(x)w_n(x)}{\widetilde{q}_n^2(x)\widetilde{Q}_n(x)}d\mu(x) = 0, \quad j = 0,\ldots,n + k_n - m - 1. \tag{5.20}$$

Then the bound (3.6) of Lemma 3.3-(a) applied with $k = 1$ together with orthogonality relations (5.20) yields

$$(n + k_n - m)\pi \leq V\left(\arg\left(\frac{q_n(x)Q(x)w_n(x)}{\widetilde{q}_n^2(x)\widetilde{Q}_n(x)};\cdot\right) + \Theta; [a,b]\right), \tag{5.21}$$

where $[a, b]$ is the convex hull of the measure μ. It follows from the normality of the family $\{w_m\}_{m\in\mathbb{Z}_+}$ in $D^*_{\mathcal{F}}$ that the sequence $\{V(w_m; [a, b])\}_{m\in\mathbb{Z}_+}$ is uniformly bounded, i.e. $V_{\mathcal{W}}$ is finite. Therefore by (5.21) and the sublinearity of $V(\cdot; [a, b])$ we obtain

$$\begin{aligned}(n + k_n)\pi \quad &\leq \quad V(\Theta) + V(\arg(Q;\cdot); [a,b]) + m\pi + V_{\mathcal{W}} \\ &\quad +V\left(\arg\left(\frac{q_n}{\widetilde{q}_n^2};\cdot\right); [a,b]\right) + V\left(\arg\left(\widetilde{Q}_n;\cdot\right); [a,b]\right). \end{aligned} \tag{5.22}$$

Write $q_n(z) = \prod_{j=1}^n (z - \xi_{j,n})$. It was shown in [19, Lemma 5.2] that

$$V\left(\arg\left(\frac{q_n}{\widetilde{q}_n^2};\cdot\right), [a,b]\right) \leq \sum_{j=1}^n \text{Angle}(\xi_{j,n}, [a,b]). \tag{5.23}$$

By writing the polynomial Q_n in the form $Q_n(z) = \prod_{j=1}^{k_n}(z - \eta_{j,n})$ we obtain from the monotonicity of $\text{Angle}(\cdot, [a, b])$ that

$$\begin{aligned} V\left(\arg\left(\widetilde{Q}_n;\cdot\right); [a,b]\right) \quad &\leq \quad \sum_{j=1}^{k_n} V\left(\arg(\cdot - 1/\bar{\eta}_{j,n}), [a,b]\right) \\ &= \quad \sum_{j=1}^{k_n} \text{Angle}(1/\bar{\eta}_{j,n}, [a,b]) \leq \frac{k_n\pi}{2}, \end{aligned} \tag{5.24}$$

since $\eta_{j,n} \in \mathbb{D}$ for all $j = 1, \ldots, k_n$. Combining (5.22), (5.23), and (5.24) we get that

$$\sum_{j=1}^n (\pi - \text{Angle}(\xi_{j,n}, [a,b])) + \frac{k_n\pi}{2} \leq V(\Theta) + V(\arg(Q;\cdot); [a,b]) + m\pi + V_{\mathcal{W}}. \tag{5.25}$$

The last inequality proves (5.16) and therefore the assertion of the theorem. ■

Proof of Theorem 5.2: Let Λ be a subsequence of natural numbers defined by the rule

$$\Lambda := \{n \in \mathbb{N} : \ s_{n-1}(\mathcal{H}_{\mathcal{F}}) > s_n(\mathcal{H}_{\mathcal{F}})\},$$

where $s_n(\mathcal{H}_{\mathcal{F}})$ stands, as before, for the n-th singular value of the Hankel operator with symbol $\mathcal{F}$. It is obvious that we may apply the preceding theorem for a measure of the form (**??**). Namely, it can be deduced from equations (5.12) and (5.15) that

$$\mu_n(\mathcal{H}_{\mathcal{F}}) \leq \deg(Q_n) + \#\left\{\xi \in \mathbb{T} : \ w_n(\xi) = 0\right\} + 1, \quad n \in \Lambda, \tag{5.26}$$

where $j_n = Q_n/\widetilde{Q}_n$ and w_n were defined in (5.7)-(5.9), with v_n being a singular vector with exactly n poles associated to the best meromorphic approximant to $\mathcal{F}$ of order n.

As shown in Theorem 4.15 that in the case where the measure μ belongs to $\mathbf{BND}_0$ the sequence $\{j_n w_n\}$ is not only a normal family in $D^*_{\mathcal{F}}$, where S' is the set of poles of R, but also is locally uniformly convergent to the function

$$w(z) = \frac{\tau}{\sqrt{(1-az)(1-bz)}},$$

where τ is some positive constant defined in (4.18). This, in particular, means that

$$\lim_{n\to\infty} \#\left\{\xi \in \mathbb{T} : \ w_n(\xi) = 0\right\} = 0$$

and for n large enough $j_n \equiv 1$. Combining equation (5.26) with these two observations the theorem follows. ■

APPENDIX

For the reader's convenience, we formulate three propositions that are of particular use to us. The first two are potential-theoretic in nature and the third one deals with representation of harmonic functions.

On Balayage of Equilibrium Measures

These two propositions are of common knowledge for the potential-theoretic community but the author encountered some difficulties in his attempts to location appropriate references in the literature.

Proposition A.1 *Let $E \subset \mathbb{D}$ be a compact set of positive capacity not containing 0 with connected complement, and E^* stand for its reflection across the unit circle, i.e. $E^* := \{z \in \mathbb{C} : \ 1/\bar{z} \in E\}$. Further, let $\mu \in \Lambda(E)$ and $\sigma \in \Lambda(E^*)$ solve the signed energy problem for the condenser (E, E^*). Then, we have that*

(a) σ is reflected from μ across the unit circle, i.e. $\sigma(B) = \mu(B^*)$ for any Borel set B, and likewise μ is reflected from σ;

(b) μ is the Green equilibrium distribution on E relative to $\overline{\mathbb{C}} \setminus E^*$ and σ is the Green equilibrium distribution on E^* relative to $\overline{\mathbb{C}} \setminus E$;

(c) $\widetilde{\mu} = \widetilde{\sigma}$, where $\widetilde{\lambda}$ denotes the balayage of the measure λ onto the unit circle. Moreover, the balayage of $\widetilde{\mu}$ onto E is μ and the balayage of $\widetilde{\mu}$ onto E^* is σ;

(d) $\widetilde{\mu}$ is the Green equilibrium distribution on $\mathbb{T}$ relative to both $\overline{\mathbb{C}} \setminus E$ and $\overline{\mathbb{C}} \setminus E^*$;

(e) μ is the Green equilibrium distribution on E relative to $\mathbb{D}$ and σ is the Green equilibrium distribution on E^* relative to $\mathbb{C} \setminus \overline{\mathbb{D}}$. Further,

$$U_G^{\widetilde{\mu}}(z; \mathbb{C} \setminus E) = \frac{1}{\mathrm{C}(E, \mathbb{T})}, \quad z \in \mathbb{C} \setminus \mathbb{D}, \quad \textit{and} \quad U_G^{\widetilde{\mu}}(z; \mathbb{C} \setminus E^*) = \frac{1}{\mathrm{C}(E^*, \mathbb{T})}, \quad z \in \overline{\mathbb{D}},$$

where in fact

$$\mathrm{C}(E, \mathbb{T}) = \mathrm{C}(E^*, \mathbb{T}).$$

Proof: (a) First of all we observe that E^* is compact since E does not contain 0. Then, if we denote by μ' and σ' respectively the reflections of μ and σ across $\mathbb{T}$, and likewise by z' the reflection of the point $z \in \mathbb{C}$, we get on changing variables in (2.7)

$$\begin{aligned} I[\mu-\sigma] &= \int \log \frac{|z't'|}{|t'-z'|} d(\mu'-\sigma')(t')d(\mu'-\sigma')(z') \\ &= I[\sigma'-\mu'] + \int \log|z'| d(\sigma'-\mu')(t')d(\sigma'-\mu')(z') + \int \log|t'| d(\sigma'-\mu')(t')d(\sigma'-\mu')(z') \\ &= I[\sigma'-\mu'], \end{aligned} \tag{A.1}$$

where the last equality uses that $\log|z|$ is bounded on $E \cup E^*$ together with Fubini's theorem and the fact that $\sigma' - \mu'$ has total mass zero. Since $\sigma' \in \Lambda(E)$ and $\mu' \in \Lambda(E^*)$ we get that $\mu = \sigma'$ and $\sigma = \mu'$ by uniqueness of the solution to the signed energy problem on (E, E^*).

(b) This follows immediately from [113, Cor. VIII.2.7] as applied to both $\mu - \sigma$ and $\sigma - \mu$.

(c) Let $z_0 \in \mathbb{D}$ and δ_{z_0} be the Dirac measure with a unit mass at z_0. Then, by the properties of balayage (see 2.4), we get for each $z \in \mathbb{T}$ since the latter is regular

$$U^{\widetilde{\delta}_{z_0}}(z) = -\log|z-z_0| = -\log|z-1/\bar{z}_0| - \log|z_0| = U^{\widetilde{\delta}_{1/\bar{z}_0}}(z) - g_{\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}}(1/\bar{z}_0, \infty) - \log|z_0|.$$

Therefore the potentials of $\widetilde{\delta}_{z_0}$ and $\widetilde{\delta}_{1/\bar{z}_0}$ differ by a constant on their common support $\mathbb{T}$, and since these two measures have equal mass and finite energy we get from the unicity theorem [113, Thm. II.4.6] that $\widetilde{\delta}_{z_0} = \widetilde{\delta}_{1/\bar{z}_0}$. Using the linearity of balayage [113, Eq. II.4.13] and the definition of σ, we now obtain

$$\widetilde{\mu} = \int \widetilde{\delta}_t d\mu(t) = \int \widetilde{\delta}_{1/\bar{t}} d\mu(t) = \int \widetilde{\delta}_t d\sigma(t) = \widetilde{\sigma}.$$

Since balayage onto ∂E from outside $\overline{\mathbb{D}}$ can be done in two steps (we first balayage onto $\mathbb{T}$ and then onto ∂E), we get on denoting this balayage with a " ˆ " that

$$\widehat{\widetilde{\mu}} = \widehat{\widetilde{\sigma}} = \widehat{\sigma} = \mu,$$

where the last equality follows from the properties of equilibrium signed measures mentioned after (2.7). The equality $\widehat{\widetilde{\sigma}} = \sigma$ is obtained by the same argument.

(d) By *(c)* the pairs $(\widetilde{\mu}, \mu)$ and $(\widetilde{\mu}, \sigma)$ satisfy the mutual balayage property and therefore solve the signed energy problem for the condensers $(\mathbb{T}, E)$ and $(\mathbb{T}, E^*)$, respectively. The assertion now follows from [113, Cor. VIII.2.7].

(e) We have that

$$U_G^\mu(z; \mathbb{D}) = U^\mu(z) - U^{\widetilde{\mu}}(z) = \int g_{\overline{\mathbb{C}}\setminus E}(t, \infty) d\widetilde{\mu}(t), \qquad \text{q.e. } z \in E,$$

where the first equality is a general property of balayage onto the boundary of a bounded domain (in our case D since we balayage out μ onto $\mathbb{T}$) [113, Thm. II.5.1] and the second equality follows from (c) and (2.4) applied with $D = \overline{C} \setminus E$ and $\widetilde{\mu}$ in place of μ. Then, since μ has finite energy and $U_G^{\mu}(z;\mathbb{D})$ is constant q.e. on E, we deduce that μ is the Green equilibrium measure on E relative to $\mathbb{D}$ [113, Thm. II.5.12] and moreover that

$$\frac{1}{\mathrm{C}(E,\mathbb{T})} = \int g_{\overline{\mathbb{C}}\setminus E}(t,\infty)d\widetilde{\mu}(t). \tag{A.2}$$

Now, using again (c) and the general relation between Green potentials and balayage [113, Thm. II.5.1], this time for the measure $\widetilde{\mu}$ over the unbounded domain $\mathbb{C} \setminus E$, we get for any $z \in \mathbb{C} \setminus \mathbb{D}$

$$U_G^{\widehat{\mu}}(z;\mathbb{C}\setminus E) = U^{\widetilde{\mu}}(z) - U^{\mu}(z) + \int g_{\overline{\mathbb{C}}\setminus E}(t,\infty)d\widetilde{\mu}(t) = \frac{1}{\mathrm{C}(E,\mathbb{T})}$$

where the last equality comes from (A.2) and the fact that $U^{\widetilde{\mu}}$ and U^{μ} coincide on $\mathbb{C} \setminus \mathbb{D}$ by the regularity of $\mathbb{T}$. The assertions on σ and on $U_G^{\widehat{\mu}}(z;\mathbb{C}\setminus E^*)$ are proved in the same manner. Finally, it follows from the proof of (d) that

$$\frac{1}{\mathrm{C}(E,\mathbb{T})} = I[\mu - \widetilde{\mu}], \qquad \frac{1}{\mathrm{C}(E^*,\mathbb{T})} = I[\sigma - \widetilde{\mu}],$$

and since $\widetilde{\mu}$ is its own reflection across $\mathbb{T}$ we get from (a) and a computation similar to (A.1) that these two quantities indeed coincide. ■

Proposition A.2 *Let E be a compact set of positive capacity and Ω be the unbounded component of $\overline{\mathbb{C}} \setminus E$. Let further σ be a probability measure such that* $\mathrm{supp}(\sigma) \subset \Omega$. *Then, if $\widehat{\sigma}$ denotes the balayage of σ onto $\partial\Omega$, it holds that*

$$\mu_w^E = \widehat{\sigma} \quad \textit{and} \quad F_w = \int g_{\mathbb{C}\setminus E}(z,\infty)d\sigma(t),$$

where μ_w^E is the weighted equilibrium distribution on E for the external field $-U^{\sigma}$ and F_w is the modified Robin constant. Moreover, if E is regular, then $\mathrm{supp}(\mu_w^E) = \partial\Omega = \partial_e E$, *the so-called exterior boundary of E.*

Proof: This is a consequence of [113, Thm IV.1.10(g)]. ■

On Harmonic Measures on DS_2-Domains

Harmonic measures are the widely studied subject and representation of the form (2.17) are well-known. However, it is usual to assume that the boundary consists of a finite number of Jordan analytic curves (cf. [45, Thm. 1.6.4]). In the following proposition we show validness of representation (2.17) with more relax assumptions on the smoothness of the boundary and allow arcs to be part of it.

Proposition A.3 *Let D be DS_2-domain such that $\partial D = \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are disjoint oriented Dini-smooth Jordan arcs. Then*

$$d\omega_D(z,\cdot) = \frac{1}{2\pi}\left(\frac{\partial}{\partial n^+}g_D(\cdot,z) + \frac{\partial}{\partial n^-}g_D(\cdot,z)\right)ds, \tag{A.3}$$

where $n^\pm$ are inward normals on both γ_1 and γ_2 determined by their orientation and ds is the arc length on ∂D.

Remark: Representation (2.17) is stated to hold for any DS_2-domain. It will be clear from the forthcoming proof that the rest of the cases follow by obvious simplifications.

Proof: Since γ_1 is Dini-smooth, it is rectifiable. This, in turn, means that every point of γ_1 is *accessible*. Thus, every *prime end* of γ_1 is a single point and consequently the set of prime ends consists of the endpoints of γ_1, say γ_1^0 and γ_1^1, and positive and negative sides γ_1^+ and γ_1^-.

Let φ_1 be a conformal map of $D_1 := \overline{\mathbb{C}} \setminus \gamma_1$ into $\mathbb{D}$. Then by the Continuity and Prime End Theorems ([100, Sec 2.1]) φ_1 extends to a homeomorphism of $\overline{D_1}^\pm$ (recall that $\overline{D_1}^\pm := D_1 \cup \gamma_1^+ \cup \gamma_1^- \cup \{\gamma_1^0, \gamma_1^1\}$) into $\overline{\mathbb{D}}$. Moreover, by [100, Thm. 3.9]

$$w \to \frac{(\varphi_1^{-1})'(w)}{(w-\varphi_1(\gamma_1^0))(w-\varphi_1(\gamma_1^1))} \tag{A.4}$$

extends continuously to $\overline{\mathbb{D}} \to \mathbb{C}\setminus\{0\}$. It is clear that $\gamma_3 := \varphi_1(\gamma_2) \subset \mathbb{D}$ is a Dini-smooth arc. It was shown in [23, Prop. 4.2] that there exists a conformal map φ_2 from $D_2 := \mathbb{D}\setminus\gamma_3$ into $\mathbb{A}_{1,r}$, for some $r > 1$ such that $\mathbb{T}$ is mapped into $\mathbb{T}_r$ and $\gamma_3^+ \cup \gamma_3^- \cup \{\gamma_3^0, \gamma_3^1\}$ is mapped into $\mathbb{T}$. Further, this map extends to a homeomorphism of $\overline{D_2}^\pm$ into $\overline{\mathbb{A}}_{1,r}$ and

$$z' \to \frac{(\varphi_2^{-1})'(z')}{(z'-\varphi_2(\gamma_3^0))(z'-\varphi_2(\gamma_3^1))} \tag{A.5}$$

extends continuously to $\overline{\mathbb{A}}_{1,r} \to \overline{\mathbb{D}}$.

Denote $\varphi := \varphi_2 \circ \varphi_1$. By what proceeds, φ is a conformal map of D into $\mathbb{A}_{1,r}$ such that $\gamma_1^+ \cup \gamma_1^- \cup \{\gamma_1^0, \gamma_1^1\}$ is mapped into $\mathbb{T}_r$ and $\gamma_2^+ \cup \gamma_2^- \cup \{\gamma_2^0, \gamma_2^1\}$ is mapped into $\mathbb{T}$. Moreover, it extends to a homeomorphism of $\overline{D}^{\pm}$ into $\overline{\mathbb{A}}_{1,r}$ and by (A.4) and (A.5) we have that

$$\begin{aligned} z' &\to \frac{(\varphi^{-1})'(z')}{(z'-\varphi(\gamma_1^0))(z'-\varphi(\gamma_1^1))(z'-\varphi(\gamma_2^0))(z'-\varphi(\gamma_2^1))} \\ z &\to \varphi'(z)(\varphi(z)-\varphi(\gamma_1^0))(\varphi(z)-\varphi(\gamma_1^1))(\varphi(z)-\varphi(\gamma_2^0))(\varphi(z)-\varphi(\gamma_2^0)) \end{aligned} \tag{A.6}$$

extend continuously to $\overline{\mathbb{A}}_{1,r} \to \mathbb{C}\setminus\{0\}$ and $\overline{D}^{\pm} \to \mathbb{C}\setminus\{0\}$, respectively. Thus, $(\varphi^+)'$ (resp. $(\varphi^-)'$) is well-defined on $\gamma_1^+ \cup \gamma_2^+$, (resp. $\gamma_1^- \cup \gamma_2^-$), where φ^+ (resp. φ^-) is a continuous extensions of φ to $\overline{\gamma_1^+ \cup \gamma_2^+}$ (resp. $\overline{\gamma_1^- \cup \gamma_2^-}$).

Let $t \in \gamma_j \setminus \{\gamma_j^0, \gamma_j^1\}$, $j = 1,2$. Since φ^{-1} is conformal up to $\{\mathbb{T}\cup\mathbb{T}_r\}\setminus\{\varphi(\gamma_1^0), \varphi(\gamma_1^1), \varphi(\gamma_2^0), \varphi(\gamma_2^1)\}$ by (A.6) and since the inner normal relative to $\mathbb{A}_{1,r}$ at $\xi \in \mathbb{T}$ is ξ and at $\xi \in \mathbb{T}_r$ is $-\xi/|\xi|$, we have that

$$n^{\pm}(t) = (-1)^{j-1}\frac{(\varphi^{-1})'\circ\varphi^{\pm}(t)}{|(\varphi^{-1})'\circ\varphi^{\pm}(t)|}\,\frac{\varphi^{\pm}(t)}{|\varphi^{\pm}(t)|} = (-1)^{j-1}\frac{|(\varphi^{\pm})'(t)|}{(\varphi^{\pm})'(t)}\,\frac{\xi}{|\xi|},$$

where $t = \varphi^{-1}(\xi)$. Let g_D and g be the Green functions for D and $\mathbb{A}_{1,r}$, respectively. It is readily verified that

$$g_D(z,w) = g(z',w'),$$

where $z' = \varphi(z)$ and $w' = \varphi(w)$, $z,w \in D$. Therefore we obtain that

$$\begin{aligned} \nabla g_D(z,w)\ \cdot\ n^{\pm}(t) &= (-2)^{j-1}\mathrm{Re}\left(\frac{|(\varphi^{\pm})'(t)|}{(\varphi^{\pm})'(t)}\,\frac{\xi}{|\xi|}\,\frac{\partial g_D(z,w)}{\partial z}\right) \\ &= (-2)^{j-1}\mathrm{Re}\left(\frac{|(\varphi^{\pm})'(t)|}{(\varphi^{\pm})'(t)}\,\frac{\xi}{|\xi|}\,\frac{\partial g(z',w')}{\partial z'}\,\varphi'(z)\right) \\ &= |(\varphi^{\pm})'(t)|\,(\nabla g(z',w')\ \cdot\ n(\xi)+ \\ &\quad +(-2)^{j-1}\mathrm{Re}\left(\frac{\xi}{|\xi|}\,\frac{\partial g(z',w')}{\partial z'}\,\frac{(\varphi)'(z)-(\varphi^{\pm})'(t)}{(\varphi^{\pm})'(t)}\right)\Bigg), \end{aligned} \tag{A.7}$$

where $n(\xi)$ is the inner normal at ξ and $\partial u/\partial z := (\partial u/\partial x - i\partial u/\partial y)/2$, $z = x+iy$. It is easily checked that $h = \partial u/\partial z$ is analytic when u is harmonic. Let now $z \to t$ from either positive or negative side of γ_j depending on which inner normal we consider. Since ∂D and $\partial\mathbb{A}_{1,r}$ are regular sets with respect to the Dirichlet problem, $g_D(\cdot,w) \equiv 0$ and $g(\cdot,w') \equiv 0$ on ∂D and $\partial\mathbb{A}_{1,r}$ respectively. Thus, they can be continued across by the reflection principle for harmonic functions. This, in particular, implies that $\partial g_D(z,w)/\partial z$ and $\partial g(z',w')/\partial z'$ have well-defined limits when $z \to t$ and $z' \to \xi$, respectively.

This, continuous extensions (A.6), and (A.7) yield that

$$\frac{1}{|(\varphi^{\pm})'(t)|}\ \frac{\partial}{\partial n^{\pm}} g_D(t,w) = \frac{\partial}{\partial n} g(\xi, w'). \tag{A.8}$$

Now, let u be a harmonic function in D with continuous well-defined boundary values on ∂D, then $u^* := u \circ \varphi^{-1}$ is continuous in $\overline{\mathbb{A}}_{1,r}$ and harmonic in $\mathbb{A}_{1,r}$. Since the boundary of $\mathbb{A}_{1,r}$ consists of disjoint analytic Jordan curves, we may use [45, Thm. 1.6.4] and (A.8) to obtain

$$\begin{aligned} u(w) &= u^*(w') = \frac{1}{2\pi}\int_{\mathbb{T}\cup\mathbb{T}_r} u^*(\xi)\frac{\partial}{\partial n} g(\xi,w')d|\xi| \\ &= \frac{1}{2\pi}\int_{\varphi^+(\gamma_1\cup\gamma_2)} \frac{u^*(\varphi^+(t))}{|(\varphi^+)'(t)|}\ \frac{\partial}{\partial n^+} g_D(t,w)\ |d\varphi^+(t)| \\ &+ \frac{1}{2\pi}\int_{\varphi^-(\gamma_1\cup\gamma_2)} \frac{u^*(\varphi^-(t))}{|(\varphi^-)'(t)|}\ \frac{\partial}{\partial n^-} g_D(t,w)\ |d\varphi^-(t)| \\ &= \frac{1}{2\pi}\int_{\partial D} u(t)\left(\frac{\partial}{\partial n^+} g_D(t,w) + \frac{\partial}{\partial n^-} g_D(t,w)\right) ds. \end{aligned} \tag{A.9}$$

Since the equalities above hold for any such u and do not depend on the choice of w, (A.3) follows.

■

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