SPECTRAL MORPHISMS, K-THEORY, AND STABLE RANKS

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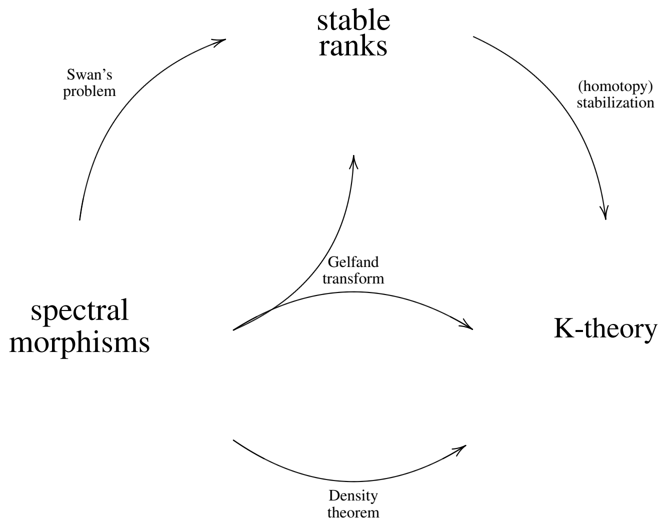
INTRODUCTION

The main theme of this thesis is the interplay between spectral morphisms, K-theory, and stable ranks. Schematically, the inter-connections are captured by the following diagram:



A unital morphism $\phi : A \to B$ between two algebras is said to be spectral if it is spectrum-preserving, that is, $\mathrm{sp}_B\big(\phi(a)\big) = \mathrm{sp}_A(a)$ for all $a \in A$. Typically, the algebras A will be invested with a topology, and the morphism ϕ is furthemore required to be continuous.

The importance of spectral morphisms was firmly established by the Gelfand transform. More recently, it has been used in connection with K-theory (via the Density Theorem) and stable ranks (Swan's problem). In turn, K-theory and stable ranks are connected by stabilization phenomena. This is not surprising, due to the interpretation of stable ranks as noncommutative notions of dimension.

The natural setting for K-theory is that of Banach algebras (Taylor [83], Blackadar [9]). Also, stable ranks are defined in the context of Banach algebras (Rieffel [71]). The larger world of good Fréchet algebras turns out, however, to be the most convenient for our purposes. K-theory is painlessly extended to this context (cf. Bost [11]). In situations pertaining to stable ranks, even the goodness assumption suffices.

Fréchet algebras appear naturally in Noncommutative Geometry (Connes [16]). A motivating example for stepping outside of the Banach context is $C^\infty(M)$, the Fréchet algebra of smooth functions on a compact manifold M. The Fréchet structure on $C^\infty(M)$ is given by the norms $\|f\|_k = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_\infty$, defined using local charts on M. The natural topology on $C^\infty(M)$ is thus finer than the one inherited from $C(M)$. Moreover, the subalgebra $C^\infty(M)$ is dense in $C(M)$, and every element in $C^\infty(M)$ that is invertible in $C(M)$ is actually invertible in $C^\infty(M)$. In particular, it follows that $C^\infty(M)$ has an open group of invertibles, i.e., it is a good Fréchet algebra.

The opening chapter reviews classical facts around the Gelfand transform which can be viewed as premonitions of some of the topics explored in this thesis.

Chapter 2 is a brief introduction to smooth subalgebras. Unlike the Gelfand context, and in line with the commutative example provided by $C^\infty(M)$, we assume the subalgebras to be dense; more generally, spectral morphisms are almost always assumed to have dense image as well. We also review the noncommutative analogue of the the smooth inclusion $C^\infty(M) \hookrightarrow C(M)$, namely the construction of smooth Fréchet subalgebras of Banach algebras by using unbounded derivations. In particular, we obtain a smooth Fréchet subalgebra $\mathcal{S}_2(\Gamma)$ inside the reduced group C*-algebra $\mathrm{C}^*_{\mathrm{r}}\Gamma$ of a finitely generated group Γ by using the commutator derivation $a \mapsto [a, |\cdot|]$, where $|\cdot|$ denotes the word-length.

In Chapter 3, we look at the property of Rapid Decay for a finitely generated group Γ. Many groups are known to satisfy this property; we mention here the hyperbolic groups. Our emphasis is less on which groups satisfy property RD, and more on what property RD brings. For instance, property RD makes the smooth Fréchet subalgebra $\mathcal{S}_2(\Gamma)$ of $\mathrm{C}^*_{\mathrm{r}}\Gamma$ explicit: it consists of the rapidly decaying ℓ^2 functions on Γ. The material of this chapter, mostly well-known, is brought together in a novel exposition (compare [85, Ch.8]). A new direction is the idea of quantifying property RD. It is unclear at this point how interesting this idea will turn out, but we nevertheless find that Proposition III.6.2 is a lovely result one can prove in this direction.

Chapter 4, the longest of the thesis, is based on the preprint [62]. It is an introduction to stable ranks - updating Rieffel's 20 years old foundational paper [71] - spiced with some new results. Let us point them out. In Section IV.3, Theorem IV.3.2 is actually taken from [61], but Proposition IV.3.4 is the first non-trivial computation of the general stable rank that I know of. In Section IV.6 a number of new estimates are interspersed between known estimates. Section IV.8 is a new direction altogether, coming from the fact that we extended the horizon from Banach algebras to good Fréchet algebras. In Section IV.9, the general stable rank half of Proposition IV.9.1 is new. We come upon a more significant result in Section IV.11, where we solve Swan's problem for two of the four stable ranks under consideration (the part of Theorem IV.11.6 concerning the connected stable rank was first proved in [61]).

Stable ranks - particularly the connected stable rank - are relevant for stabilization of homotopy groups of the general linear groups over Banach algebras. This is the topic of Chapter 5. Section V.2 simplifies the results from [61] which were formulated in terms of certain higher connected stable ranks; in particular, we do away with higher connected stable ranks altogether. Section V.3, concerned with certain exact computations of stable homotopy levels, is completely new.

Chapters 6 and 7 are lifted from [61], essentially without any change. The motivation is to extend the scope of spectrality to situations where the spectral information is only known over a dense subalgebra. We investigate such "relatively spectral" morphisms; to be specific, a morphism $\phi : A \to B$ is *relatively spectral* if $\mathrm{sp}_B(\phi(x)) = \mathrm{sp}_A(x)$ for all x in some dense subalgebra X of A. We do not know examples of relatively spectral morphisms that are not spectral. Most likely, examples exist and they are not obvious. But, once again, the point of considering relatively spectral morphisms is to get by with less spectral information. For instance, when one compares two completions of a group algebra $\mathbb{C}\Gamma$, it suffices to consider the spectral behavior of finitely-supported elements. We prove a relative version of the Density Theorem regarding isomorphism in K-theory (Theorem VI.4.2) and we extend the invariance of the connected stable rank to the relatively spectral context (Theorem VI.4.4). Then we take on spectral K-functors, and we prove the Relative Density Theorem for spectral K-functors K_Ω, indexed by open subsets $\Omega \subseteq \mathbb{C}$ containing the origin

(Theorem VII.5.1). For suitable Ω, one recovers the usual K_0 and K_1 functors. The basic motivation for introducing spectral K-functors is the need for an alternate picture of K_0, in which the subset of idempotents is replaced by an open subset.

We collect some background information and terminology in the last chapter. A detailed introduction to Fréchet algebras can be found there.

CHAPTER I

PREQUEL: THE GELFAND TRANSFORM

The spectral aspect first appeared in the context of Gelfand transforms, in the early forties. Gelfand's conceptual proof of a difficult and seminal theorem of Wiener, saying that the algebra $A(T)$ of continuous functions with absolutely convergent Fourier series is spectral in $C(T)$, was the first sign of the potency of Gelfand's theory. (Here T denotes the 1-torus $\{z \in \mathbb{C} : |z| = 1\}$.)

As an introduction to the directions taken in thesis, we review some classical results on the Gelfand transform. This brief discussion is based on Taylor's papers [83], [84]; we use [43] as a reference on commutative Banach algebras.

Let A be a unital, commutative Banach algebra. Let X_A be the subset of the dual A^* consisting of the unital, multiplicative linear functionals on A. Under the induced weak*-topology (i.e., that of pointwise convergence), X_A is a compact, Hausdorff space. The space X_A is called the *maximal ideal space*, due to the fact that we have a bijective correspondence, given by $\phi \mapsto \ker\phi$, between functionals in X_A and maximal proper ideals of A.

For $a \in A$, let $\hat{a} \in C(X_A)$ denote the evaluation at a. The map $A \to C(X_A)$, given by $a \mapsto \hat{a}$, is called the *Gelfand transform* of A. Here are three examples:

· if A is a commutative C^*-algebra, then the Gelfand transform is an isomorphism;

· if $A = \ell^1(\mathbb{Z}^d)$, then X_A is homeomorphic to the d-torus T^d so, after suitable identifications, we view the inclusion $\ell^1(\mathbb{Z}^d) \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\mathbb{Z}^d$ as being the Gelfand transform;

· if A is the disk algebra $A(D)$, namely the closed subalgebra of $C(D)$ consisting of those continuous functions on the disk which are holomorphic in the interior, then X_A is homeomorphic to D; with this identification, the Gelfand transform is the inclusion $A(D) \hookrightarrow C(D)$. (Here D is the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.)

In all examples above, the Gelfand transform is injective, but this is not true in general. The Gelfand transform is injective if and only if A is semisimple.

Theorem I.0.1. *The Gelfand transform $A \to C(X_A)$ is a unital, continuous morphism with the property that a is invertible in A if and only if $\hat{a}$ is invertible in $C(X_A)$.*

Strong relations between the structural properties of A and those of $C(X_A)$ - which are, in fact, topological properties of X_A - can be established across the Gelfand transform. The first result of this kind is Shilov's Idempotent Theorem (1953), saying that every idempotent in $C(X_A)$ - that is, the characteristic function of a closed and open subset of X_A - is the image under the Gelfand transform of an idempotent in A. In hindsight, this gives a strong hint that a K_0-isomorphism might be lurking in the background.

Arens (1963) proved the following:

Theorem I.0.2. *The Gelfand transform $A \to C(X_A)$ induces, for each n, an isomorphism*

$$\pi_0\big(\mathrm{GL}_n(A)\big) \to \pi_0\big(\mathrm{GL}_n(C(X_A))\big).$$

Here the K-theoretic implication is obvious: the Gelfand transform $A \to C(X_A)$ induces an isomorphism $K_1(A) \to K_1(C(X_A))$. This was pointed out by Novodvorskii (1967) and, independently, by Eidlin (1967). In fact, Novodvorskii also handled the K_0-isomorphism. Thus:

Theorem I.0.3. *The Gelfand transform $A \to C(X_A)$ induces an isomorphism $K_*(A) \to K_*(C(X_A))$.*

In order to state the next result, also due to Novodvorskii, we recall the following definition: the *joint spectrum* of an n-tuple $(a_1, \dots, a_n) \in A^n$ is the subset of $\mathbb{C}^n$ given by

$$\mathrm{sp}(a_1, \dots, a_n) = \{(\lambda_1, \dots, \lambda_n) : A(a_1 - \lambda_1) + \cdots + A(a_n - \lambda_n) \neq A\}.$$

Theorem I.0.4. *Let Ω be a homogeneous open subset of $\mathbb{C}^n$. Put*

$$A_\Omega = \{(a_1, \dots, a_n) \in A^n : \ \mathrm{sp}(a_1, \dots, a_n) \subseteq \Omega\}.$$

Then the Gelfand transform $A \to C(X_A)$ induces an isomorphism $\pi_0(A_\Omega) \to \pi_0(C(X_A)_\Omega)$.

The open subset $\Omega = \mathbb{C}^n \setminus \{0\}$ is homogeneous, since it admits a transitive action of the complex Lie group $\mathrm{GL}_n(\mathbb{C})$. With an eye towards our discussion of stable ranks, we re-denote the set A_Ω by

$$\mathrm{Lg}_n(A) = \{(a_1, \dots, a_n) \in A^n : Aa_1 + \cdots + Aa_n = A\}.$$

In this case, Novodvorskii's theorem reads as follows:

Theorem I.0.5. *The Gelfand transform $A \to C(X_A)$ induces an isomorphism*

$$\pi_0\big(\mathrm{Lg}_n(A)\big) \to \pi_0\big(\mathrm{Lg}_n(C(X_A))\big).$$

Notice that the last column of a matrix in $\mathrm{GL}_n(A)$ is an n-tuple in $\mathrm{Lg}_n(A)$. The question arises whether the converse holds, namely: given an n-tuple in $\mathrm{Lg}_n(A)$, is there an matrix in $\mathrm{GL}_n(A)$ having the n-tuple as its last column?

Lin (1973) proved the following:

Theorem I.0.6. *An n-tuple $(a_1, \dots, a_n) \in \mathrm{Lg}_n(A)$ can be filled out to a matrix in $\mathrm{GL}_n(A)$ if and only if the n-tuple $(\hat{a}_1, \dots, \hat{a}_n) \in \mathrm{Lg}_n(C(X_A))$ can be filled out to a matrix in $\mathrm{GL}_n(C(X_A))$.*

In what follows we shall be interested almost exclusively in spectral morphisms that are also dense; this is generally not the case with the Gelfand transform. However, the facts stated in this section bear some resemblance with directions taken in this thesis. Theorem I.0.3 is a commutative cousin of the Density Theorem in K-theory. Theorems I.0.5 and I.0.6 have to do with preservation of the connected and the general stable ranks. Theorem I.0.4 is, in retrospect, somewhat related to the idea of spectral K-functors.

CHAPTER II

SMOOTH SUBALGEBRAS

II.1 Generalities

In general, if A is a subalgebra of B then $\mathrm{sp}_B(a) \subseteq \mathrm{sp}_A(a)$ for each $a \in A$. We are interested in those special inclusions that are spectrum-preserving:

Definition II.1.1. An inclusion of algebras $A \hookrightarrow B$ is *spectral* if $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$ for each $a \in A$; equivalently, if every element of A that is invertible in B is actually invertible in A, i.e., $A \cap B^\times = A^\times$.

Throughout, inclusions are assumed to be unital. Furthermore, the algebras A and B will be topological algebras (usually A is Fréchet and B is Banach), in which case the inclusion is also assumed to be continuous.

Remark II.1.2. This property can be found under several names in the literature. Bourbaki (*Théories Spectrales*) says that A is "full" in B. Naimark (*Normed Algebras*) refers to a spectral inclusion as a "Wiener pair". In other sources, A is said to be "inverse-closed". Closer to our terminology is Schweitzer's [76] "spectral invariance" property.

The following result has been proved more than once [82, Lemma 2.1],[11, Prop. A.2.2], [76, Thm 2.1]. We give a proof due to Swan.

Theorem II.1.3. *Let A be a dense subalgebra of a Banach algebra B. If A is spectral in B, then $\mathrm{M}_n(A)$ is spectral in $\mathrm{M}_n(B)$.*

Proof. To show that a dense subalgebra A is spectral in B, it suffices to find U open in B with $A \cap U \subseteq A^\times$. Indeed, let $a \in A \cap B^\times$; by density, there is $a' \in a^{-1}U \cap A$, so $aa' \in U \cap A \subseteq A^\times$. Thus a has a right inverse in A, hence $a \in A^\times$.

We apply this observation to $\mathrm{M}_n(A) \hookrightarrow \mathrm{M}_n(B)$. Consider the neighborhood V of $1_n \in \mathrm{M}_n(B)$ given by

$$\begin{pmatrix} B^\times & U_0 & \dots & U_0 \\ U_0 & B^\times & \dots & U_0 \\ \dots & \dots & \dots & \dots \\ U_0 & U_0 & \dots & B^\times \end{pmatrix}$$

where U_0 is a suitable neighborhood of 0 in B. Let $(a_{ij}) \in \mathrm{M}_n(A) \cap V$. In particular, the diagonal has entries in $B^\times$, hence in $A^\times$ by spectrality. Row and column operations allow to clear first row and column, except for a_{11}. For suitable U_0, we may assume the diagonal of the resulting matrix to still have entries in $B^\times$, in fact in $A^\times$. We continue this process until we end up with a diagonal matrix with entries in $A^\times$. This means that the starting matrix (a_{ij}) is in $\mathrm{M}_n(A)^\times$. $\square$

Simplicity is inherited by dense and spectral subalgebras:

Proposition II.1.4. *Let A be a dense and spectral subalgebra of a Banach algebra B. If B is simple then A is simple.*

Proof. Let M be a maximal 2-sided ideal in A. Then $M \cap A^\times = \varnothing$, so $M \cap B^\times = \varnothing$ by spectrality. As $B^\times$ is open, the B-closure $\overline{M}$ also satisfies $\overline{M_A} \cap B^\times = \varnothing$. But $\overline{M}$ is a 2-sided ideal in $\overline{A} = B$, by continuity of operations. Hence $\overline{M}$ must be the 0 ideal of B. We conclude that $M = 0$. $\square$

Remark II.1.5. Schweitzer [77] provides an example in which A is a dense Banach subalgebra of a Banach algebra B, both A and B are simple, yet A is not spectral in B.

We adopt the following

Definition II.1.6. A dense and spectral subalgebra A of a topological algebra B is said to be a *smooth subalgebra*.

Let M be a compact manifold. Then $C^k(M) \hookrightarrow C(M)$ is a smooth inclusion for each $1 \le k \le \infty$.

Inspired by this commutative example, a smooth subalgebra A_∞ of a Banach algebra A is thought of as carrying the differential information, whereas A carries the topological information. Let us point out that some authors require a smooth subalgebra to be a Fréchet subalgebra under a Fréchet topology finer than the topology of A. We consider this situation, of smooth subalgebras being endowed with a Fréchet algebra structure, in the remainder of this chapter.

Proposition II.1.7. *Let $A \hookrightarrow B$ be a dense inclusion of Banach algebras. If $r_A(a) = r_B(a)$ for all $a \in A$ then A is spectral in B.*

Proof. Let $a \in A$ be invertible in B. Let $a' \in A$ with $\|1 - aa'\|_B < 1$. Then $r_A(1 - aa') \le r_B(1 - aa') < 1$ and so aa' is invertible in A. It follows that a is invertible in A. $\square$

Next we consider the connection between the "algebraic fullness" of a subalgebra as described by the spectral property and the "analytic fullness" of a subalgebra expressed by closure under holomorphic calculus.

Definition II.1.8. Let B be a Banach algebra. A subalgebra A is *holomorphically closed* in B if, for each $a \in A$, we have $f(a) \in A$ whenever f is holomorphic in the neighborhood of $\mathrm{sp}_B(a)$.

The following appears, for instance, in [76, Lemma 1.2]:

Proposition II.1.9. *Let A be a subalgebra of B.*

a) If A is holomorphically closed in B, then A is spectral in B.

b) Let A be a Fréchet subalgebra. If A is spectral in B, then A is holomorphically closed in B.

Proof. a) Let $a \in A \cap B^\times$. Since $0 \notin \mathrm{sp}_B(a)$, $z \mapsto z^{-1}$ is holomorphic near $\mathrm{sp}_B(a)$ and so $a^{-1} \in A$.

b) In the formula

$$f \mapsto f(a) = \frac{1}{2\pi i} \oint f(\lambda)(\lambda - a)^{-1} d\lambda$$

the integrand is A-valued and A-continuous (the inversion $a \mapsto a^{-1}$ is continuous in A). Thus $f(a)$, a limit of Riemann sums, must be in A by the completeness of A. $\square$

From the K-theoretic standpoint, the interest in spectral morphisms is motivated by the following useful criterion for K-theoretic isomorphism is known as the Density Theorem. Initial versions are due to Karoubi [44, p.109] and Swan [82, Sec.2.2 & 3.1]; see also Connes [15, Appendix 3]. The Density Theorem as stated below is taken from Bost [11, Thm.A.2.1].

Theorem II.1.10. *Let $\phi : A \to B$ be a dense and spectral morphism between good Fréchet algebras. Then ϕ induces an isomorphism $K_*(A) \simeq K_*(B)$.*

Recall, morphisms are assumed to be unital and continuous. A morphism $\phi : A \to B$ is *dense* if ϕ has dense image, and *spectral* if $\mathrm{sp}_B(\phi(a)) = \mathrm{sp}_A(a)$ for all $a \in A$.

II.2 Smooth Fréchet subalgebras coming from closed derivations

In this section, we show how a closed derivation on a Banach algebra gives rise to a Fréchet subalgebra. Furthermore, the Fréchet subalgebra associated to a closed derivation turns out to be a spectral and dense subalgebra in many cases.

We start with the linear aspect. Let X be a Banach space and $\xi : \mathrm{dom}(\xi) \subseteq X \to X$ an unbounded linear map, where $\mathrm{dom}(\xi)$ is a linear subspace of X. Recall, the operator ξ is said to be *densely defined* if $\mathrm{dom}(\xi)$ is dense in X, and *closed* if $x_n \to x$ and $\xi(x_n) \to y$ for $(x_n) \subseteq \mathrm{dom}(\xi)$ imply $x \in \mathrm{dom}(\xi)$ and $\xi(x) = y$.

To ξ we associate a Fréchet subspace on which ξ is continuous. For $k \geq 0$, the fact that ξ is closed gives that $\mathrm{dom}(\xi^k)$ is a Banach space under

$$\|x\|_k := \|x\| + \|\xi(x)\| + \cdots + \|\xi^k(x)\|$$

and $\xi : \mathrm{dom}(\xi^{k+1}) \to \mathrm{dom}(\xi^k)$ is continuous of norm at most 1. Therefore

$$\mathrm{dom}(\xi^\infty) = \bigcap_{k \geq 0} \mathrm{dom}(\xi^k)$$

is a Fréchet subspace of X under the seminorms $\{\|\cdot\|_k\}_{k\geq 0}$, and $\xi : \mathrm{dom}(\xi^\infty) \to \mathrm{dom}(\xi^\infty)$ is continuous.

The corresponding construction for algebras works best for derivations. Let B be a Banach algebra and $\delta : B \to B$ a closed unbounded derivation, i.e., a linear map satisfying

$$\delta(ab) = \delta(a)b + a\delta(b)$$

defined on a subalgebra $\mathrm{dom}(\delta)$. For $k \geq 0$, we equip $\mathrm{dom}(\delta^k)$ with the following norm:

$$\|a\|_k := \|a\| + \frac{\|\delta(a)\|}{1!} + \cdots + \frac{\|\delta^k(a)\|}{k!}$$

The iterated Leibniz formula:

$$\delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(a)\delta^{n-i}(b)$$

gives that $\mathrm{dom}(\delta^k)$ is closed under multiplication. Moreover, $\|\cdot\|_k$ is submultiplicative:

$$\begin{aligned} \|ab\|_k &= \sum_{i=0}^{k} \frac{\|\delta^i(ab)\|}{i!} \le \sum_{i=0}^{k}\sum_{j=0}^{i} \frac{\|\delta^j(a)\|}{j!}\frac{\|\delta^{i-j}(b)\|}{(i-j)!} \\ &\le \left(\sum_{i=0}^{k} \frac{\|\delta^i(a)\|}{i!}\right)\left(\sum_{j=0}^{k} \frac{\|\delta^j(b)\|}{j!}\right) = \|a\|_k\|b\|_k \end{aligned}$$

We conclude that each $\mathrm{dom}(\delta^k)$ is a Banach algebra. Consequently, $\mathrm{dom}(\delta^\infty)$ is a Fréchet subalgebra of B, and $\delta : \mathrm{dom}(\delta^\infty) \to \mathrm{dom}(\delta^\infty)$ is continuous.

In practice, it is useful to restrict the descending chain of Banach algebras

$$B = \mathrm{dom}(\delta^0) \supseteq \mathrm{dom}(\delta^1) \supseteq \mathrm{dom}(\delta^2) \supseteq \dots$$

to closed subalgebras.

The following theorem is due to R. Ji [38, Thm.1.2]:

Theorem II.2.1. *Let A be a closed subalgebra of B such that $A \cap \mathrm{dom}(\delta^\infty)$ is dense in A. Then $A \cap \mathrm{dom}(\delta^\infty)$ is a Fréchet subalgebra of A, and $A \cap \mathrm{dom}(\delta^n)$ is dense and spectral in A for every $1 \le n \le \infty$.*

Proof. It is clear that $A \cap \mathrm{dom}(\delta^\infty)$ is a Fréchet subalgebra of A. We argue spectrality in the case $n = \infty$, the finite case being done similarly.

For $a \in \mathrm{dom}(\delta)$, the formula

$$\delta(a^n) = \sum_{i=0}^{n-1} a^i \delta(a) a^{n-1-i}$$

shows that $\|\delta(a^n)\| \le n\|a\|^{n-1}\|\delta(a)\|$.

Let $a \in A \cap \mathrm{dom}(\delta^\infty)$ be invertible in A. Assume first that $\|1-a\| < 1$. We can then write $a^{-1} = \sum_{n\ge 0}(1-a)^n$ in A. We have

$$\|\delta\big((1-a)^n\big)\| \le n\|1-a\|^{n-1}\|\delta(1-a)\|$$

so $\sum_{n\ge 1}\delta\big((1-a)^n\big)$ is (absolutely) convergent in A. The fact that δ is closed implies $a^{-1} \in \mathrm{dom}(\delta^1)$ and $\delta(a^{-1}) = \sum_{n\ge 1}\delta\big((1-a)^n\big)$. This is the first step of an obvious induction, which ends up with the conclusion that $a^{-1} \in A \cap \mathrm{dom}(\delta^\infty)$.

In general, the density of $A \cap \mathrm{dom}(\delta^\infty)$ gives some $a' \in A \cap \mathrm{dom}(\delta^\infty)$ with

$$\|1 - aa'\| \le \|a\|\|a^{-1} - a'\| < 1.$$

We already know that $(aa')^{-1} \in A \cap \mathrm{dom}(\delta^\infty)$, so $a^{-1} \in A \cap \mathrm{dom}(\delta^\infty)$ as well. □

A closed, densely defined operator on X determines a closed derivation on $\mathcal{B}(X)$ and, by restriction, a closed derivation on every closed subalgebra of $\mathcal{B}(X)$. This is more naturally formulated in terms of strongly-closed subalgebras; the norm-closed case is an immediate consequence.

Lemma II.2.2. *Let $\xi : X \to X$ be a closed, densely defined linear map. Then $\delta_\xi : \mathcal{B}(X) \to \mathcal{B}(X)$ given by $\delta_\xi(T) = T\xi - \xi T$ is a closed derivation with domain*

$$\mathrm{dom}(\delta_\xi) = \{T \in \mathcal{B}(X) : T\big(\mathrm{dom}(\xi)\big) \subseteq \mathrm{dom}(\xi),\ \delta_\xi(T) \in \mathcal{B}(X)\}.$$

Proof. First, note that δ_ξ is a derivation and $\mathrm{dom}(\delta_\xi)$ is a subalgebra.

Let $(T_n) \subseteq \mathrm{dom}(\delta_\xi)$ and $T, S \in \mathcal{B}(X)$ such that $T_n \to T$ and $\delta_\xi(T_n) \to S$ strongly. Let $x \in \mathrm{dom}(\xi)$. Then $T_n(x) \to T(x)$ and $T_n\xi(x) - \xi T_n(x) \to S(x)$. Combining the latter with $T_n\xi(x) \to T\xi(x)$, we obtain $\xi T_n(x) \to T\xi(x) - S(x)$. As ξ is closed, we get $T(x) \in \mathrm{dom}(\xi)$ and $\xi T(x) = T\xi(x) - S(x)$, i.e., $\delta_\xi(T)(x) = S(x)$. By the density of $\mathrm{dom}(\xi)$, we obtain $\delta_\xi(T) = S$.

We have thus checked that $T \in \mathrm{dom}(\delta_\xi)$ and $\delta_\xi(T) = S$, which proves that δ_ξ is closed. □

Let Γ be a finitely generated group. Define $L : \ell^2\Gamma \to \ell^2\Gamma$ to be the pointwise multiplication by $1 + |\cdot|$, where $|\cdot|$ is a fixed word-length function. That is, $L(\sum a_g\delta_g) = \sum a_g(1+|g|)\delta_g$. Then L is a closed and densely defined, with domain

$$H^1\Gamma := \{\sum a_g\delta_g \in \ell^2\Gamma :\ \sum a_g(1+|g|)\delta_g \in \ell^2\Gamma\}.$$

More generally, for $k \geq 0$ put $H^k\Gamma := \{\sum a_g\delta_g \in \ell^2\Gamma :\ \sum a_g(1+|g|)^k\delta_g \in \ell^2\Gamma\}$, and $H^\infty\Gamma = \cap_{k\geq 0} H^k\Gamma$.

To L we associate a closed derivation $\delta : \mathcal{B}(\ell^2\Gamma) \to \mathcal{B}(\ell^2\Gamma)$ given by $\delta(T) = TL - LT$, with domain

$$\mathrm{dom}(\delta) = \{T \in \mathcal{B}(\ell^2\Gamma) : T(H^1\Gamma) \subseteq H^1\Gamma,\ \delta(T) \in \mathcal{B}(\ell^2\Gamma)\}.$$

Whenever it makes sense, the action of δ^k can be computed as follows:

$$\delta^k(\sum a_g g)(\sum b_h\delta_h) = \sum_x \Big(\sum_{gh=x} a_g b_h \big(|gh| - |h|\big)^k\Big)\delta_x$$

Lemma II.2.3. $\mathbb{C}\Gamma \subseteq \mathrm{dom}(\delta^k) \subseteq H^k\Gamma$ *for every* $0 \leq k \leq \infty$.

Proof. It suffices to consider k finite. In order to prove $\mathbb{C}\Gamma \subseteq \mathrm{dom}(\delta^k)$, we need to show that each $\delta^k(g)$ leaves $H^1\Gamma$ invariant and that $\delta^k(g)$ is (rather, it extends to) a bounded operator on $\ell^2\Gamma$. We have:

$$\delta^k(g)(\sum b_h\delta_h) = \sum b_h\big(|gh| - |h|\big)^k\delta_{gh}$$

Since $\big||gh| - |h|\big| \leq |g|$, we get $\|\delta^k(g)\| \leq |g|^k$. In fact, equality can be obtained by acting on $\{\delta_h\}_{h\in\Gamma}$. This proves $\delta^k(g) \in \mathcal{B}(\ell^2\Gamma)$. Similar estimates show that $\delta^k(g)$ leaves $H^1\Gamma$ invariant. As for the inclusion $\mathrm{dom}(\delta^k) \subseteq H^k\Gamma$, we have:

$$\delta^k(\sum a_g g)(\delta_1) = \sum a_g|g|^k\delta_g$$

Since $\sum a_g|g|^i\delta_g \in \ell^2\Gamma$ for all $0 \leq i \leq k$, we have $\sum a_g(1+|g|)^k\delta_g \in \ell^2\Gamma$, i.e., $\sum a_g\delta_g \in H^k\Gamma$. □

Putting

$$S_2(\Gamma) := \mathrm{C}^*_{\mathrm{r}}\Gamma \cap \mathrm{dom}(\delta^\infty)$$

we obtain:

Corollary II.2.4. *$S_2(\Gamma)$ is a dense and spectral Fréchet subalgebra of $\mathrm{C}^*_\mathrm{r}\Gamma$, as well as a dense subspace of $H^\infty\Gamma$.*

Remark II.2.5. The construction of $S_2(\Gamma)$, due to Connes and Moscovici [17], predates Ji's theorem II.2.1.

Remark II.2.6. It is easily seen that the spaces $H^k\Gamma$ ($0 \leq k \leq \infty$) of decaying functions on Γ do not depend on the choice of word-length on Γ. However, I do not know whether $S_2(\Gamma)$ is independent of the choice of word-length.

If we perform the ℓ^1 version of this construction, namely we take

- $L : \ell^1\Gamma \to \ell^1\Gamma$ to be the pointwise multiplication by $1 + |\cdot|$, i.e., $L(\sum a_g \delta_g) = \sum a_g (1+|g|)\delta_g$;
- $\ell^1_k\Gamma := \{\sum a_g\delta_g \in \ell^1\Gamma : \sum a_g(1+|g|)^k\delta_g \in \ell^1\Gamma\}$, and $\ell^1_\infty\Gamma = \cap_{k\geq 0}\,\ell^1_k\Gamma$

then we obtain $\mathrm{dom}(\delta^k) = \ell^1_k\Gamma$, with Banach norm

$$\Big\|\sum a_g g\Big\|_k = \sum |a_g|\big(1 + \frac{|g|}{1!} + \cdots + \frac{|g|^k}{k!}\big).$$

This norm is visibly equivalent to the standard norm on $\ell^1_k\Gamma$, namely $\|\sum a_g g\|_{1,k} = \sum |a_g|(1+|g|)^k$. Instead of restricting to $\mathrm{C}^*_\mathrm{r}\Gamma$, as we did in the ℓ^2 case, we now restrict to the group ℓ^1-algebra $\ell^1\Gamma$. We conclude that the ℓ^1 version of $S_2(\Gamma)$ is simply $S_1(\Gamma) = \ell^1_\infty\Gamma$.

Returning to the ℓ^2 context, the following question then appears: when is $H^\infty\Gamma$ included in $\mathrm{C}^*_\mathrm{r}\Gamma$? This leads us into the next chapter.

CHAPTER III

THE PROPERTY OF RAPID DECAY

III.1 Motivation: Sobolev spaces

Consider the d-torus T^d, parameterized as $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n})$ for $\theta = (\theta_1, \ldots, \theta_d) \in [0, 2\pi)^d$. Put $D_j = -i\frac{\partial}{\partial\theta_j}$, and $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. The algebra of k-times differentiable functions $C^k(T^d)$ consists of those $f \in C(T^d)$ for which $D^\alpha f \in C(T^d)$ whenever $\alpha \in \mathbb{N}^d$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq k$. The algebra of smooth functions $C^\infty(T^d)$ is the intersection of $C^k(T^d)$ over all $k \geq 0$.

For $s > 0$, the s-th Sobolev space $H^s(T^d)$ is the Banach space obtained by completing the algebra of trigonometric polynomials on T^d under the norm

$$\|f\|_{(s)} = \Big(\sum_{m\in\mathbb{Z}^d} |\widehat{f}(m)|^2(1+|m|)^{2s}\Big)^{1/2}.$$

In fact, $H^s(T^d)$ is a Hilbert space under the inner product

$$\langle f, g\rangle_{(s)} = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)\overline{\widehat{g}(m)}(1+|m|)^{2s}.$$

Here $|m|$ denotes the length of $m \in \mathbb{Z}^d$, i.e., $|m| = |m_1| + \ldots + |m_d|$. As usual, $\widehat{f}$ is the Fourier transform of f:

$$\widehat{f}(m) = (2\pi)^{-d} \int_{[0,2\pi]^d} f(e^{i\theta}) e^{-i\theta\cdot m} \mathrm{d}\theta$$

Then the following hold:

- $C^k(T^d) \subseteq H^k(T^d)$
- $H^{s+k}(T^d) \subseteq C^k(T^d)$ for $s > d/2$ (*Sobolev's embedding theorem*)
- $H^s(T^d)$ is a Banach algebra for $s > d/2$ (*Sobolev's multiplication theorem*)
- $H^s(T^d)$ is invariant under multiplication by $C^\infty(T^d)$
- for $s > t$, the inclusion $H^s(T^d) \subseteq H^t(T^d)$ is compact (*Rellich's compactness lemma*)

The algebra of rapidly decaying functions $H^\infty(T^d)$ is the intersection of $H^s(T^d)$ over all $s > 0$. By the first two facts, we have $C^\infty(T^d) = H^\infty(T^d)$.

The definition of the Sobolev spaces compels us to pass to the dual and to view the Sobolev spaces as being defined on $\mathbb{Z}^d$. From this perspective, the s-th Sobolev space $H^s(\mathbb{Z}^d)$ is the Banach space obtained by completing the group algebra of $\mathbb{Z}^d$ under the weighted ℓ^2-norm

$$\big\|\sum a_m m\big\|_{2,s} = \big(\sum |a_m|^2(1+|m|)^{2s}\big)^{1/2}.$$

We think of the H^s spaces on the $\mathbb{Z}^d$-side as being analogues of the C^k spaces on the T^d-side. More to the point, $H^\infty(\mathbb{Z}^d)$ corresponds to $C^\infty(T^d)$. The above properties for the Sobolev spaces over T^d turn into the following properties for the Sobolev spaces over $\mathbb{Z}^d$:

· $H^s(\mathbb{Z}^d) \subseteq \mathrm{C}^*_{\mathrm{r}}(\mathbb{Z}^d)$ for $s > d/2$
· $H^s(\mathbb{Z}^d)$ is a Banach algebra for $s > d/2$
· $H^s(\mathbb{Z}^d)$ is invariant under multiplication by $H^\infty(\mathbb{Z}^d)$
· for $s > t$, the inclusion $H^s(\mathbb{Z}^d) \subseteq H^t(\mathbb{Z}^d)$ is compact

Property RD extends this Sobolev context to other groups.

III.2 Haagerup's inequality for free groups

Much knowledge about reduced group C*-algebras emanates from a number of seminal results on the reduced group C*-algebra of free groups; simplicity of $\mathrm{C}^*_{\mathrm{r}}F_n$ (due to Powers), or the fact that $\mathrm{C}^*_{\mathrm{r}}F_n$ has no non-trivial idempotents (due to Pimsner and Voiculescu) immediately come to mind.

Another seminal work on $\mathrm{C}^*_{\mathrm{r}}F_n$ is due to Haagerup. Haagerup's aim in [31] was proving that, although $\mathrm{C}^*_{\mathrm{r}}F_n$ is not nuclear, it nevertheless satisfies a certain "metric approximation property". In doing so, Haagerup established two facts about free groups: a fact which would become the so-called Haagerup property, or a-T-menability, and a fact which was later formalized into the property of Rapid Decay. We prove the latter fact in what follows.

We work in a free group F_n with $n \geq 2$. By S_k we denote the sphere of radius k and by $a \mapsto a|_{S_k}$ the restriction $\sum a_g g \mapsto \sum_{g \in S_k} a_g g$.

Lemma III.2.1. *If* $\operatorname{supp}(a) \subseteq S_k$ *and* $\operatorname{supp}(b) \subseteq S_l$ *then* $\|(ab)|_{S_j}\|_2 \leq \|a\|_2 \|b\|_2$. *Moreover,* $(ab)|_{S_j} = 0$ *unless* $|k-l| \leq j \leq k+l$ *and* $k+l-j$ *is even.*

The condition on k, l, j from the lemma means that they are the sidelengths of a triangle in the Cayley graph. In particular, the condition on k, l, j is permutation-invariant.

Proof. Let $j = k+l-2m$ with $0 \leq m \leq \min\{k, l\}$. Write each g of length k as a product $g = g_1 g_2$ with $|g_1| = k-m, |g_2| = m$. Similarly, write each h of length l as a product $h = h_2 h_1$ with $|h_2| = m, |h_1| = l-m$. Note that $(gh)_j = \langle g_2, h_2^{-1} \rangle (g_1 h_1)|_{S_j}$. Putting a and b in the form

$$a = \sum a_{g_1 g_2} g_1 g_2, \qquad b = \sum b_{h_2 h_1} h_2 h_1$$

we get

$$(ab)|_{S_j} = \sum_{g_1, h_1} \Big(\sum_{g_2, h_2} a_{g_1 g_2} b_{h_2 h_1} \langle g_2, h_2^{-1} \rangle \Big) (g_1 h_1)|_{S_j}.$$

Then

$$\|(ab)|_{S_j}\|_2^2 \leq \sum_{g_1, h_1} \Big| \sum_{g_2, h_2} a_{g_1 g_2} b_{h_2 h_1} \langle g_2, h_2^{-1} \rangle \Big|^2 = \sum_{g_1, h_1} \Big| \langle \sum_{g_2} a_{g_1 g_2} g_2, \big(\sum_{h_2} b_{h_2 h_1} h_2\big)^* \rangle \Big|^2$$

and so

$$\|(ab)|_{S_j}\|_2^2 \leq \sum_{g_1, h_1} \Big\| \sum_{g_2} a_{g_1 g_2} g_2 \Big\|_2^2 \Big\| \sum_{h_2} b_{h_2 h_1} h_2 \Big\|_2^2 = \|a\|_2^2 \|b\|_2^2.$$

We conclude that $\|(ab)|_{S_j}\|_2 \leq \|a\|_2 \|b\|_2$. $\square$

Theorem III.2.2 (Haagerup). *If* $\mathrm{supp}(a) \subseteq S_k$ *then* $\|a\| \leq (k+1)\|a\|_2$.

Proof. We need to show $\|ab\|_2 \leq (k+1)\|a\|_2\|b\|_2$. First we estimate ab on spheres:

$$\|(ab)|_{S_j}\|_2 = \Big\| \sum_{l\geq 0} (ab|_{S_l})|_{S_j} \Big\|_2 \leq \sum_{l\geq 0} \|(ab|_{S_l})|_{S_j}\|_2$$

We have $(ab|_{S_l})|_{S_j} = 0$ unless $|k-l| \leq j \leq k+l$ and $k+l-j$ is even, which can be written as $l = k+j-2m$ with $0 \leq m \leq \min\{k, j\}$. Using the previous lemma, we continue:

$$\|(ab)|_{S_j}\|_2 \leq \|a\|_2 \sum_{m=0}^{\min\{k,j\}} \|b|_{S_{k+j-2m}}\|_2 \leq \sqrt{k+1}\|a\|_2 \sqrt{\sum_{m=0}^{\min\{k,j\}} \|b|_{S_{k+j-2m}}\|_2^2}$$

We get:

$$\begin{aligned}
\|ab\|_2^2 &= \sum_{j\geq 0} \|(ab)|_{S_j}\|_2^2 \\
&\leq (k+1)\|a\|_2^2 \sum_{j\geq 0} \sum_{m=0}^{\min\{k,j\}} \|b|_{S_{k+j-2m}}\|_2^2 \\
&= (k+1)\|a\|_2^2 \sum_{m=0}^{k} \sum_{j\geq m} \|b|_{S_{k+j-2m}}\|_2^2 \\
&\leq (k+1)\|a\|_2^2 \sum_{m=0}^{k} \|b\|_2^2 = (k+1)^2\|a\|_2^2\|b\|_2^2
\end{aligned}$$

That is, $\|ab\|_2 \leq (k+1)\|a\|_2\|b\|_2$ as desired. □

Cohen [14] has computed

$$\|\chi_{S_k}\| = (k+1)(2n-1)^{k/2} - (k-1)(2n-1)^{k/2-1}$$

where χ_{S_k} is the characteristic function of the sphere of radius $k \geq 1$ in F_n; this extends Kesten's well-known formula $\|\chi_{S_1}\| = 2\sqrt{2n-1}$. Writing

$$\|\chi_{S_k}\| = C_n\Big(k + \frac{n}{n-1}\Big)\|\chi_{S_k}\|_2, \qquad C_n = \frac{2n-2}{\sqrt{2n(2n-1)}}$$

we see that Haagerup's inequality provides the correct degree for k.

III.3 Property RD

Following Haagerup's seminal result, Jolissaint explicitly defined property RD in [40] (see also the K-theoretic companion [39]).

Definition III.3.1. Let Γ be a finitely generated group equipped with a word-length $|\cdot|$. We say that Γ has *property RD* if one of the following equivalent conditions holds:

(RD) there are $C, s > 0$ such that, for all $a = \sum a_g g \in \mathbb{C}\Gamma$, we have

$$\|\sum a_g g\| \leq C\Big(\sum |a_g|^2 (1+|g|)^{2s}\Big)^{1/2}$$

(RD$_\bullet$) there are $C, s > 0$ so that $\|a\| \leq C(1+n)^s \|a\|_2$ for all $a \in \mathbb{C}\Gamma$ supported on the n-ball

(RD$_\circ$) there are $C, s > 0$ so that $\|a\| \leq C(1+n)^s \|a\|_2$ for all $a \in \mathbb{C}\Gamma$ supported on the n-sphere

This definition does not depend on the choice of the word-length.

More generally, property RD can be defined relative to a length function. Recall, a length function on Γ is a map $l : \Gamma \to [0, \infty)$ that satisfies: i) $l(1) = 0$, ii) $l(g^{-1}) = l(g)$ for all $g \in \Gamma$, iii) $l(gh) \leq l(g) + l(h)$ for all $g, h \in \Gamma$. The advantage of this generalized approach to RD is that it allows groups that are not necessarily finitely generated. However, for finitely generated groups, having property RD with respect to some length function is equivalent to having property RD with respect to word-length. We confine ourselves to the finite generation situation in what follows.

Free groups have property RD: Haagerup's inequality amounts to (RD$_\circ$) with $s = 1$. One of Jolissaint's results from [40] is that groups of polynomial growth have property RD. In fact:

Theorem III.3.2. *Let Γ be amenable. Then Γ has property RD iff Γ has polynomial growth.*

Proof. We denote by $\rho_\bullet$ the growth function of the balls in Γ. We claim that Γ satisfies (RD$_\bullet$) of order $s > 0$ iff $\rho_\bullet(n) \prec n^{2s}$.

Let Γ satisfy (RD$_\bullet$) of order s. For χ_{B_n}, the characteristic function of the ball of radius n, it says that $\|\chi_{B_n}\| \leq C(1+n)^s \sqrt{\rho_\bullet(n)}$. On the other hand, we have $\big|\sum a_g\big| \leq \|\sum a_g g\|$ for all $\sum a_g g \in \mathbb{C}\Gamma$ by the amenability assumption. For χ_{B_n}, we obtain $\rho_\bullet(n) \leq \|\chi_{B_n}\|$. It follows that $\rho_\bullet(n) \prec (1+n)^{2s}$ by combining the two inequalities.

Conversely, assume Γ has polynomial growth, with $\rho_\bullet(n) \prec n^{2s}$. For $a = \sum a_g g \in \mathbb{C}\Gamma$ supported on the n-ball, we have

$$\|a\| \leq \sum |a_g| \leq \sqrt{\rho_\bullet(n)}\sqrt{\sum |a_g|^2} \leq C(1+n)^s \|a\|_2$$

meaning that Γ satisfies (RD$_\bullet$) of order s.

If $\rho_\circ$ denotes the growth function of the spheres in Γ, then a similar argument shows that Γ satisfies (RD$_\circ$) of order $s > 0$ iff $\rho_\circ(n) \prec n^{2s}$. $\square$

Many more groups enjoy property RD. Here is a sampler:

· hyperbolic groups ([22])

· groups that are relatively hyperbolic to subgroups having property RD ([26])

· groups acting freely on finite dimensional CAT(0) cube complexes ([12])

· cocompact lattices in $\mathrm{SL}_3(F)$ for F a local field ([70]) and $F = \mathbb{R}, \mathbb{C}$ ([46])

· mapping class groups ([8])

Jolissaint [40] showed that property RD is hereditary (a subgroup of an RD-group is also RD) and virtual (a group commensurable to an RD-group is also RD), and that property RD is preserved by amalgamating over finite subgroups.

On the other hand, a group containing an amenable group of superpolynomial growth (for example, $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$ contains a solvable group of exponential growth) does not have property RD. This is the only known obstruction to RD; it would be useful to find others.

A recent collection of questions on property RD can be found in [1]. We single out the following:

Problem. Is property RD invariant under quasi-isometries?

Informally, property RD for a group gives a concrete way to understand its reduced C*-algebra; the subsequent sections are meant to illustrate this. Unfortunately, establishing property RD for a given group is usually quite difficult. Leaving aside a detailed study of which groups have property RD, we focus in what follows on the benefits of property RD. The hope is that a weaker, and more flexible, version of property RD will emerge, a version which still delivers the goods of RD.

III.4 Decaying ℓ^2-functions

One of the most important aspects of property RD is that it leads to a subalgebra $H^\infty\Gamma$ of rapidly decaying function inside $\mathrm{C}^*_\mathrm{r}\Gamma$, hence the very name of Rapid Decay given to the property. This insight of Connes is explained in [39], [40].

Let $H^s\Gamma$ be the completion of $\mathbb{C}\Gamma$ under the weighted ℓ^2-norm

$$\big\|\sum a_g g\big\|_{2,s} = \sqrt{\sum |a_g|^2(1+|g|)^{2s}}.$$

Put $L = 1+|\cdot|$. Then we can also write $\|a\|_{2,s} = \|L^s a\|_2$, where multiplication by (powers of) L is pointwise.

Thinking of $H^s\Gamma$ as a noncommutative analogue of the Sobolev spaces, we shall see that property RD guarantees the facts we mentioned in Section III.1 for $H^s(\mathbb{Z}^d)$. We start by taking care of the Rellich Compactness Lemma, for which no assumption on Γ is needed: for $s>t$, we have a continuous inclusion $H^s\Gamma \subseteq H^t\Gamma$ which is the diagonal matrix $\mathrm{diag}(L^{t-s}(g))_{g\in\Gamma}$ with respect to the standard bases. Since L is proper, we have that $H^s\Gamma \subseteq H^t\Gamma$ is compact. Note that, if Γ has polynomial growth, then $H^s\Gamma \subseteq H^t\Gamma$ is in fact p-summable for sufficiently large p.

The role of the critical value $d/2$ we encountered there is played here by the "RD-dimension" of Γ, defined as $\mathrm{rd}(\Gamma) = \inf\{s : \Gamma \text{ satisfies (RD) of order } s\}$.

The following theorem is taken from [46]:

Theorem III.4.1 (Lafforgue)**.** *Assume Γ has property RD. If $s > \mathrm{rd}(\Gamma)$ then $H^s\Gamma$ is a dense and spectral Banach $*$-subalgebra of* $\mathrm{C}^*_\mathrm{r}\Gamma$.

Proof. The (RD) inequality says that $H^s\Gamma$ is continuously embedded in $\mathrm{C}^*_\mathrm{r}\Gamma$. Clearly, is dense in $\mathrm{C}^*_\mathrm{r}\Gamma$. Also, $H^s\Gamma$ is $*$-closed and $\|\cdot\|_{2,s}$ is a $*$-norm. We have to show that $H^s\Gamma$ is a Banach algebra, and that $H^s\Gamma$ is spectral in $\mathrm{C}^*_\mathrm{r}\Gamma$. We make two preparatory observations.

1) For $g_1,\dots,g_n\in\Gamma$ we have

$$L^s(g_1\dots g_n) \leq n^s\big(L^s(g_1)+\dots+L^s(g_n)\big)$$

where $L = 1 + |\cdot|$ as defined above. Indeed, $(a_1 + \cdots + a_n)^s \leq n^s(a_1^s + \cdots + a_n^s)$ whenever a_i are non-negative reals: if $s \geq 1$ then $(a_1 + \cdots + a_n)^s \leq n^{s-1}(a_1^s + \cdots + a_n^s)$ by the convexity of $f(x) = x^s$ on $[0, \infty)$, and if $0 < s < 1$, then we simply use $(a_1 + \cdots + a_n)^s \leq a_1^s + \cdots + a_n^s$.

Putting $a_i = L(g_i)$, and using $L(g_1 \ldots g_n) \leq L(g_1) + \cdots + L(g_n)$, we obtain the desired inequality.

2) We have the "twin" (RD) inequalities

$$\|ab\|_2 \leq C\|a\|_{2,s}\|b\|_2 \quad (a \in H^s\Gamma, b \in \ell^2\Gamma),$$

$$\|ab\|_2 \leq C\|a\|_2\|b\|_{2,s} \quad (a \in \ell^2\Gamma, b \in H^s\Gamma).$$

Indeed, the first is essentially the (RD) inequality. The second is obtained from the first:

$$\|ab\|_2 = \|(ab)^*\|_2 = \|b^*a^*\|_2 \leq C\|b^*\|_{2,s}\|a^*\|_2 = C\|a\|_2\|b\|_{2,s}$$

Let $|a| = \sum |a_g| g$ be the pointwise absolute value of $a = \sum a_g g$. For $a, b \in H^s\Gamma$ we have:

$$\begin{aligned}
\|ab\|_{2,s}^2 &= \sum_x \Big| \sum_{gh=x} a_g b_h \Big|^2 L^{2s}(x) \leq \sum_x \Big(\sum_{gh=x} |a_g||b_h| L^s(gh) \Big)^2 \\
&\leq 2^{2s} \sum_x \Big(\sum_{gh=x} |a_g||b_h| L^s(g) + \sum_{gh=x} |a_g||b_h| L^s(h) \Big)^2 \\
&\leq 2^{2s+1} \sum_x \Big(\sum_{gh=x} |a_g||b_h| L^s(g) \Big)^2 + 2^{2s+1} \sum_x \Big(\sum_{gh=x} |a_g||b_h| L^s(h) \Big)^2 \\
&= 2^{2s+1} \big(\big\| |L^s a||b| \big\|_2^2 + \big\| |a||L^s b| \big\|_2^2 \big)
\end{aligned}$$

The "twin" RD inequalities give

$$\big\| |L^s a||b| \big\|_2 \leq C \big\| |L^s a| \big\|_2 \big\| |b| \big\|_{2,s} = C\|a\|_{2,s}\|b\|_{2,s}$$

$$\big\| |a||L^s b| \big\|_2 \leq C \big\| |a| \big\|_{2,s} \big\| |L^s b| \big\|_2 = C\|a\|_{2,s}\|b\|_{2,s}$$

hence $\|ab\|_{2,s}^2 \leq 2^{2s+2}C^2\|a\|_{2,s}^2\|b\|_{2,s}^2$, that is, $\|ab\|_{2,s} \leq 2^{s+1}C\|a\|_{2,s}\|b\|_{2,s}$. We conclude that $\|\cdot\|_{2,s}$ is an algebra norm.

Next, we show that $H^s\Gamma$ is spectral in $\mathrm{C}^*_\mathrm{r}\Gamma$. We have:

$$\begin{aligned}
\|a^n\|_{2,s}^2 &= \sum_x \Big| \sum_{g_1 \ldots g_n = x} a_{g_1} \ldots a_{g_n} \Big|^2 L^{2s}(x) \\
&\leq \sum_x \Big(\sum_{g_1 \ldots g_n = x} |a_{g_1}| \ldots |a_{g_n}| L^s(g_1 \ldots g_n) \Big)^2
\end{aligned}$$

Using the inequality established in 1), we continue:

$$\begin{aligned}\|a^n\|_{2,s}^2 &\leq n^{2s}\sum_x\Big(\sum_{i=1}^n\sum_{g_1\ldots g_n=x}|a_{g_1}|\ldots|a_{g_n}|L^s(g_i)\Big)^2\\ &\leq n^{2s+1}\sum_x\sum_{i=1}^n\Big(\sum_{g_1\ldots g_n=x}|a_{g_1}|\ldots|a_{g_n}|L^s(g_i)\Big)^2\\ &= n^{2s+1}\sum_{i=1}^n\big\||a|\ldots|L^s a|\ldots|a|\big\|_2^2\end{aligned}$$

Pick $s > s' > d(\Gamma)$. A repeated application of the "twin" (RD) inequalities of order s' gives

$$\big\||a|\ldots|L^s a|\ldots|a|\big\|_2 \leq C^{n-1}\big\||a|\big\|_{2,s'}^{n-1}\big\||L^s a|\big\|_2 = C^{n-1}\|a\|_{2,s'}^{n-1}\|a\|_{2,s}$$

hence

$$\|a^n\|_{2,s} \leq C^{n-1}n^{s+1}\|a\|_{2,s'}^{n-1}\|a\|_{2,s}$$

which implies

$$r_{H^s\Gamma}(a) \leq C\|a\|_{2,s'}.$$

On the other hand, for $a \in H^s\Gamma$ we have

$$\|a\|_{2,s'} \leq \|a\|_{2,s}^{\frac{s'}{s}}\|a\|_2^{1-\frac{s'}{s}}$$

by the Hölder inequality. From

$$r_{H^s\Gamma}(a) \leq C\|a\|_{2,s}^{\frac{s'}{s}}\|a\|_2^{1-\frac{s'}{s}} \leq C\|a\|_{2,s}^{\frac{s'}{s}}\|a\|^{1-\frac{s'}{s}}$$

we obtain, by using a^n and then taking the n-th root, the inequality

$$r_{H^s\Gamma}(a) \leq r_{H^s\Gamma}(a)^{\frac{s'}{s}}\, r_{C^*_r\Gamma}(a)^{1-\frac{s'}{s}}.$$

and so $r_{H^s\Gamma}(a) \leq r_{C^*_r\Gamma}(a)$. We conclude that $H^s\Gamma$ is spectral in $C^*_r\Gamma$. □

If Γ has property RD, then

$$H^\infty\Gamma = \cap_{s>0} H^s\Gamma$$

is a dense, spectral Fréchet subalgebra of $C^*_r\Gamma$, thought of as the subalgebra of smooth functions on an imaginary "noncommutative space" $\widehat{\Gamma}$. We now see this fact - due to Connes - as a corollary of Theorem III.4.1. It should be mentioned, however, that it predates Lafforgue's theorem. Tying in with the discussion from Section II.2, if Γ satisfies (RD) of order s, then the estimate

$$\big\|\delta^k(\textstyle\sum a_g g)\big\| \leq \big\|\sum|a_g||g|^k g\big\| \leq C\big\|\sum a_g(1+|g|)^{k+s}g\big\|_2$$

shows that $H^{k+s}\Gamma \subseteq \mathrm{dom}(\delta^k)\ \big(\subseteq H^k\Gamma\big)$; therefore $\mathrm{dom}(\delta^\infty) = H^\infty\Gamma$. We conclude that $S_2(\Gamma)$, which is defined for all finitely generated groups, coincides with $H^\infty\Gamma$ whenever Γ has property RD.

We point out that, if $H^\infty\Gamma$ is contained in $\mathrm{C}^*_\mathrm{r}\Gamma$ as a subspace, then Γ has property RD. Indeed, both $H^\infty\Gamma$ and $\mathrm{C}^*_\mathrm{r}\Gamma$ appear as Fréchet subspaces of $\ell^2\Gamma$ so the inclusion $H^\infty\Gamma \hookrightarrow \mathrm{C}^*_\mathrm{r}\Gamma$ is necessarily continuous. As the family of norms $\{\|\cdot\|_{2,s}\}$ on $H^\infty\Gamma$ is nested, it follows that for some $C, s > 0$ we have $\|a\| \leq C\|a\|_{2,s}$ for all $a \in H^\infty\Gamma$. In particular, this holds on $\mathbb{C}\Gamma$ and so we get RD.

To complete our Sobolev analogy, we are left with showing the following

Proposition III.4.2. *Assume Γ has property RD. Then, for all $t \geq 0$, $H^t\Gamma$ is invariant under multiplication by $H^\infty\Gamma$.*

Proof. Assume Γ satisfies (RD) of order s. It is clear that $H^S\Gamma$ is invariant under multiplication by $H^\infty\Gamma$ whenever $S > s$, since $H^\infty\Gamma$ is a subalgebra of $H^S\Gamma$. However, the required statement is more general. Let $t \geq 0$. In the proof of Theorem III.4.1 we showed the following estimate, independent of property RD:

$$\|ab\|_{2,t}^2 \leq 2^{2t+1}\left(\big\||L^t a||b|\big\|_2^2 + \big\||a||L^t b|\big\|_2^2\right)$$

If $a \in H^t\Gamma$ and $b \in H^\infty\Gamma$, then using twice one of the "twin" (RD) inequalities of order s

$$\big\||L^t a||b|\big\|_2 \leq C\||L^t a|\|_2\||b|\|_{2,s} = C\|a\|_{2,t}\|b\|_{2,s} < \infty$$

$$\big\||a||L^t b|\big\|_2 \leq C\big\||a|\big\|_2\big\||L^t b|\big\|_{2,s} = C\|a\|_2\|b\|_{2,t+s} < \infty$$

we conclude that $ab \in H^t\Gamma$. Thus $H^t\Gamma$ is invariant under right-multiplication by $H^\infty\Gamma$; invariance under left-multiplication follows by using the involution. □

III.4.1 Decaying ℓ^1-functions (a comparison)

Under property RD, decaying ℓ^2-functions form a dense and spectral $*$-subalgebra of the reduced group C*-algebra. The ℓ^1 analogue is (too) generic: for any group, decaying ℓ^1-functions form a dense and spectral $*$-subalgebra of the group ℓ^1-algebra.

Let $\ell^1_s\Gamma$ denote the completion of $\mathbb{C}\Gamma$ under the weighted ℓ^1-norm

$$\big\|\sum a_g g\big\|_{1,s} := \sum |a_g|(1+|g|)^s = \|L^s a\|_1.$$

Proposition III.4.3. *$\ell^1_s\Gamma$ is a dense and spectral Banach $*$-subalgebra of $\ell^1\Gamma$.*

Proof. Let $a, b \in \ell^1_s\Gamma$. Using $L(gh) \leq L(g)L(h)$, we have:

$$\|ab\|_{1,s} = \sum_x \Big|\sum_{gh=x} a_g b_h\Big| L^s(x) \leq \sum_x \Big(\sum_{gh=x} |a_g||b_h|L^s(g)L^s(h)\Big) = \|a\|_{1,s}\|b\|_{1,s}$$

This shows that $\ell^1_s\Gamma$ is a Banach subalgebra of $\ell^1\Gamma$; it is clearly dense and $*$-closed. We show $\ell^1_s\Gamma \hookrightarrow \ell^1\Gamma$ is spectral. The inequality $L^s(g_1\dots g_n)\le n^s\big(L^s(g_1)+\dots+L^s(g_n)\big)$ is used again:

$$\begin{aligned}
\|a^n\|_{1,s} &= \sum_x\Big|\sum_{g_1\dots g_n=x} a_{g_1}\dots a_{g_n}\Big|L^s(x)\\
&\le \sum_x\Big(\sum_{g_1\dots g_n=x}|a_{g_1}|\dots|a_{g_n}|L^s(g_1\dots g_n)\Big)\\
&\le n^s\sum_x\Big(\sum_{i=1}^n\sum_{g_1\dots g_n=x}|a_{g_1}|\dots|a_{g_n}|L^s(g_i)\Big)=n^{s+1}\|a\|_1^{n-1}\|a\|_{1,s}
\end{aligned}$$

Taking the n-th root we get $r_{\ell^1_s\Gamma}(a)\le\|a\|_1$. We conclude that $\ell^1_s\Gamma\hookrightarrow\ell^1\Gamma$ is spectral. $\square$

It follows that

$$\ell^1_\infty\Gamma=\cap_{s>0}\,\ell^1_s\Gamma$$

is a dense, spectral Fréchet subalgebra of ℓ^1G.

Schematically, we have the following diagram of dense inclusions:

$$\begin{array}{ccccc}
H^\infty\Gamma & \longrightarrow & H^s\Gamma & \longrightarrow & C^*_r\Gamma\\
\uparrow & & \uparrow & & \uparrow\\
\ell^1_\infty\Gamma & \longrightarrow & \ell^1_s\Gamma & \longrightarrow & \ell^1\Gamma
\end{array}$$

For an arbitrary group, the diagram should be interpreted as a diagram of Banach and Fréchet *spaces*. For a group enjoying property RD, it becomes a diagram of Banach and Fréchet *algebras*, and the inclusions along the rows are spectral.

The spaces $H^\infty\Gamma$ and $\ell^1_\infty\Gamma$ coincide precisely when Γ has polynomial growth. Compare [40, Thm. 3.1.7].

Lemma III.4.4. *The following are equivalent:*

a) Γ has polynomial growth

b) $H^s\Gamma\subseteq\ell^1\Gamma$ for some $s>0$

c) $H^\infty\Gamma\subseteq\ell^1\Gamma$

d) $H^\infty\Gamma=\ell^1_\infty\Gamma$

Proof. $a)\Rightarrow b)$: if Γ has polynomial growth then $\sum(1+|g|)^{-2s}$ converges for s large enough; then Cauchy-Schwarz yields $\sum|a_g|\le\sqrt{\sum|a_g|^2(1+|g|)^{2s}}\sqrt{\sum(1+|g|)^{-2s}}$, that is $\|a\|_1\le C\|a\|_{2,s}$ for all $a\in H^s\Gamma$.

$b)\Rightarrow a)$: the inclusion $H^s\Gamma\subseteq\ell^1\Gamma$ is necessarily continuous, so there are $C,s>0$ such that $\|a\|_1\le C\|a\|_{2,s}$ for all $a\in H^s\Gamma$; in particular, for the characteristic function of the n-ball we obtain $|B_n|\le C(1+n)^s\sqrt{|B_n|}$ and so $|B_n|$ is dominated by n^{2s}.

$b)\Rightarrow d)$: if $H^s\Gamma\subseteq\ell^1\Gamma$ then $H^{s+r}\Gamma\subseteq\ell^1_r\Gamma$ for all $r>0$, so $H^\infty\Gamma\subseteq\ell^1_\infty\Gamma$; the other inclusion, $H^\infty\Gamma\supseteq\ell^1_\infty\Gamma$, is generic.

$d)\Rightarrow c)$: clear.

$c)\Rightarrow b)$: the inclusion $H^\infty\Gamma\subseteq\ell^1\Gamma$ is necessarily continuous, so there are $C,s>0$ such that $\|a\|_1\le C\|a\|_{2,s}$ for all $a\in\mathbb{C}\Gamma$; hence $H^s\Gamma\subseteq\ell^1\Gamma$. $\square$

III.5.1 The approach of Connes and Moscovici to the Novikov conjecture

Connes and Moscovici ([17]) were the first to use the Rapid Decay property for K-theoretic purposes: in order to define a certain pairing between K-theory and cohomology, one needs to pass from the reduced C^*-algebra of a group to a smaller algebra having the same K-theory.

Let M be a closed, oriented manifold and Γ a group. For every group cohomology class $[c] \in H^\bullet(\Gamma;\mathbb{Q})$, one attaches to every map $f : M \to B\Gamma$ a higher signature $\sigma_{[c]}(M, f)$. The Novikov conjecture is that all these higher signatures only depend on the homotopy type of M. For fixed M and arbitrary Γ, the relevant case is $\Gamma = \pi_1 M$ as it implies all others. However, the usual point of view is to fix Γ.

In [17] (see also [58] for an overview), Connes and Moscovici prove the following:

Theorem III.5.1. *If Γ has polynomial cohomology and property RD, then Γ satisfies the Novikov conjecture.*

Polynomial cohomology of Γ means that every cohomology class of even degree can be represented by a cocycle of polynomial growth. These two conditions of polynomial control, polynomial cohomology and property RD, come about in the following way.

Let $\mathcal{R}\Gamma$ be the group ring with coefficients in the infinite matrices (a_{ij}) that are rapidly decaying in the sense that $(i^\alpha j^\beta a_{ij})$ is bounded for all $\alpha, \beta \geq 1$. As a special case of their Index Theorem, Connes and Moscovici recover the higher signatures from a pairing between $H^{\text{even}}(\Gamma;\mathbb{C})$ and $K_0(\mathcal{R}\Gamma)$:

$$\sigma_{[c]}(M, f) = C\langle \tau_c, \operatorname{Ind}_{f,\Gamma} D_{\text{sign}} \rangle, \qquad [c] \in H^{2k}(\Gamma;\mathbb{C})$$

Here C is a constant depending only on the dimension of M and the cohomological degree, τ_c is a cyclic cocycle on $\mathcal{R}\Gamma$ associated to the group cocycle c, and $\operatorname{Ind}_{f,\Gamma} D_{\text{sign}}$ is a $K_0(\mathcal{R}\Gamma)$-valued index for the signature operator D_{sign} on $\widetilde{M}$.

The problem is now the homotopy invariance of $\operatorname{Ind}_\Gamma$. Let $\mathcal{K}$ denote, as usual, the compacts on a separable, infinite dimensional Hilbert space. The inclusion

$$\mathcal{R}\Gamma = \mathbb{C}\Gamma \otimes \mathcal{R} \hookrightarrow C^*_r\Gamma \otimes \mathcal{K}$$

induces a map $K_0(\mathcal{R}\Gamma) \to K_0(C^*_r\Gamma \otimes \mathcal{K}) = K_0(C^*_r\Gamma)$. The image of $\operatorname{Ind}_\Gamma$ through this map is a higher index, previously introduced by Mishchenko and Kasparov, which is known to be homotopy invariant. Thus one has to extend to a pairing between $H^{\text{even}}(\Gamma;\mathbb{C})$ and $K_0(C^*_r\Gamma)$. The result [16, p.242] is that, in the presence of property RD, one can extend the pairing with a cyclic cocycle of polynomial growth from $K_0(\mathcal{R}\Gamma)$ to $K_0(C^*_r\Gamma \otimes \mathcal{K})$.

On one hand, hyperbolic groups have property RD. On the other hand, they satisfy polynomial cohomology in the strongest possible way, namely, every cohomology class can be represented by a bounded cocycle; this has been stated by Gromov and proved by Mineyev in [56]. The conclusion is the following:

Corollary III.5.2. *Hyperbolic groups satisfy the Novikov conjecture.*

Nowadays much stronger results on the Novikov conjecture are known, mainly coming from a coarse geometric approach. See [91] for an overview.

In [53] it is proved that combable groups have polynomial cohomology. This raises the following

Problem. Do combable groups have property RD?

Kato [45] has showed, by different methods, that combable groups satisfy the Novikov conjecture. Another reason that lends credibility to the above problem is that hyperbolic groups, mapping class groups, and groups acting properly discontinuously on CAT(0) cube complexes are combable.

III.5.2 Lafforgue's work on the Baum-Connes conjecture

Another use of the property of Rapid Decay for K-theoretic purposes features in the remarkable work of Lafforgue [47].

The Baum-Connes conjecture is the statement that a certain assembly map

$$\mu_{\mathrm{r}} : K_*^{\mathrm{top}}(\Gamma) \to K_*(\mathrm{C}_{\mathrm{r}}^*\Gamma) \qquad (* = 0,1)$$

is an isomorphism. Here $K_*^{\mathrm{top}}(\Gamma)$ is a "topological" K-theory for Γ, whereas $K_*(\mathrm{C}_{\mathrm{r}}^*\Gamma)$ is an "analytical" K-theory for Γ. Higson and Kasparov [34] (see also the expositions [42], [33]) proved the Baum-Connes conjecture for groups with the Haagerup property. The reduced assembly map μ_{r} is the composition of a full assembly map $\mu : K_*^{\mathrm{top}}(\Gamma) \to K_*(\mathrm{C}^*\Gamma)$ with the map $K_*(\mathrm{C}^*\Gamma) \to K_*(\mathrm{C}_{\mathrm{r}}^*\Gamma)$ induced by the natural morphism $\mathrm{C}^*\Gamma \to \mathrm{C}_{\mathrm{r}}^*\Gamma$; Higson and Kasparov prove that the latter two maps are isomorphisms.

The method of Higson and Kasparov does not work for groups with property (T), for the map $K_0(\mathrm{C}^*\Gamma) \to K_0(\mathrm{C}_{\mathrm{r}}^*\Gamma)$ is no longer injective. Lafforgue managed to break the property (T) barrier in [47]. Sporadic results on the Baum-Connes conjecture in the (T) case were known before Lafforgue's work, but Lafforgue handled a large class of groups, including all hyperbolic groups.

Lafforgue introduces a version of KK-theory for Banach algebras in which many features of Kasparov's original KK-theory for C*-algebras have an analogue. However, $\mathrm{KK}^{\mathrm{ban}}$ does not have a product. The $\mathrm{KK}^{\mathrm{ban}}$ approach allows Lafforgue to prove that, for a certain class $\mathcal{C}'$ of groups, a variant of the assembly map

$$\mu_A : \mathrm{K}_*^{\mathrm{top}}(\Gamma) \to \mathrm{K}_*(A\Gamma) \qquad (* = 0,1)$$

is an isomorphism for any unconditional completion $A\Gamma$ of $\mathbb{C}\Gamma$ [47, Théorème 0.0.2]. An *unconditional completion* $A\Gamma$ is Banach algebra obtained by completing $\mathbb{C}\Gamma$ under a norm $\|f\|_A$ that only depends on $g \mapsto |f(g)|$; equivalently, a norm $\|\cdot\|_A$ for which $\|f\|_A \le \|f'\|_A$ whenever $f, f' \in \mathbb{C}\Gamma$ satisfy $|f| \le |f'|$ pointwise (to see that the first definition implies the second, note that the restriction of the norm $\|\cdot\|_A : [0,\infty)^{(\Gamma)} \to [0,\infty)$ is convex hence non-decreasing).

The class $\mathcal{C}'$ includes the following classes of groups:

· groups with the Haagerup property

· groups acting geometrically on a CAT(0) space

· hyperbolic groups ([57])

To obtain the actual Baum-Connes conjecture, one needs an unconditional completion $A\Gamma$ having the same K-theory as $C^*_r\Gamma$. For instance, an unconditional completion $A\Gamma$ that is a spectral $*$-subalgebra of $C^*_r\Gamma$. This is where property RD comes in, providing the Banach algebra $H^s\Gamma$ as the desired unconditional completion.

Summarizing, the groups in the class $\mathcal{C}'$ which have property RD - hyperbolic groups being an obvious highlight - satisfy the Baum-Connes conjecture. In particular, this approach handles many property (T) groups, groups which were inaccessible to the Higson-Kasparov method. There are, however, property (T) groups - notably $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$ - for which Lafforgue's approach is ineffective.

III.5.3 Stable ranks of some reduced group C*-algebras

The previous two applications of property RD were K-theoretic. Let us now see an application in a different direction.

All we need out of property RD is the following

Lemma III.5.3. *Assume Γ has property RD. Then, for all $a \in \mathbb{C}\Gamma$ we have*

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|_2^{1/n}.$$

Proof. Let $R(a) = \max\{|g| : g \in \mathrm{supp}(a)\}$ be the radius of the smallest ball supporting $a \in \mathbb{C}\Gamma$. If Γ has RD then $\|a\| \leq C\big(1+R(a)\big)^s\|a\|_2$ for some $C, s > 0$. Then:

$$\|a^n\|_2 \leq \|a^n\| \leq C\big(1+R(a^n)\big)^s\|a^n\|_2 \leq C\big(1+nR(a)\big)^s\|a^n\|_2$$

Taking the n-th root and letting $n \to \infty$ yields the desired equality of limits. □

In [27], a group Γ is said to have the "ℓ^2 spectral radius" property if it satisfies the conclusion of the lemma. Perhaps [23] is the first paper where this property receives attention. It is easy to show that groups of subexponential growth also enjoy the ℓ^2 spectral radius property

The next theorem, is a special case of a result due to Dykema, Haagerup and Rørdam [28]; we follow [27] and [73].

Theorem III.5.4. $C^*_r F_n$ *has stable rank* 1.

Stable ranks are discussed extensively in the next chapter. Here, we take the following as a definition: a unital C*-algebra A has stable rank 1 if the invertible group $A^\times$ is dense in A.

Proof. Let $a \in \mathbb{C}F_n$. There is $\gamma \in F_n$ such that the support of γa generates a free semigroup; take, for instance, $\gamma = x^N y^N$ where x, y are generators and N is large enough. Then $\|(\gamma a)^n\|_2 = \|\gamma a\|_2^n$. Using the ℓ^2 spectral radius property, we obtain:

$$r(\gamma a) = \lim_{n\to\infty} \|(\gamma a)^n\|_2^{1/n} = \|\gamma a\|_2 = \|a\|_2$$

In general, $r(a) \geq d(a, A^\times)$. Indeed, for every $\varepsilon > 0$ we have $r(a)+\varepsilon \notin \mathrm{sp}(a)$, i.e., $a - (r(a)+\varepsilon) \in A^\times$. Then $d(a, A^\times) \leq \|a - \big(a-(r(a)+\varepsilon)\big)\| = r(a)+\varepsilon$.

Therefore $\|a\|_2 = r(\gamma a) \geq d(\gamma a, A^{\times}) = d(a, A^{\times})$ for all $a \in \mathbb{C}F_n$. It follows that $\|a\|_2 \geq d(a, A^{\times})$ for all $a \in \mathrm{C}^*_{\mathrm{r}} F_n$.

Assume that the stable rank of $\mathrm{C}^*_{\mathrm{r}} F_n$ is greater than 1. A result of Rørdam (see [73]) says that there is $a \in \mathrm{C}^*_{\mathrm{r}} F_n$ with $\|a\| = d(a, A^{\times}) = 1$. We have

$$1 = \|a\| = d(a, A^{\times}) \leq \|a\|_2 \leq \|a\|$$

hence $\|a\| = 1 = \|a\|_2$. Then both $1 - aa^*$ and $1 - a^*a$ are positive (by the first equality), and have vanishing trace (by the second equality). We must have $1 - aa^* = 1 - a^*a = 0$, in particular a is invertible. This contradicts the fact that $d(a, A^{\times}) = 1$. □

The proof rests on the combination of the ℓ^2 spectral radius property (which allows us to pass from the operator norm to the 2-norm) with the "free semigroup" property that every finite subset has a translate that generates a free semigroup (which allows the computation of 2-norms). Hyperbolic groups have property RD, hence they satisfy the ℓ^2 spectral radius property. It turns out that torsion-free, non-elementary hyperbolic groups have the free semigroup property. Therefore:

Theorem III.5.5 ([27])**.** *If Γ is a torsion-free, non-elementary hyperbolic group, then $\mathrm{C}^*_{\mathrm{r}}\Gamma$ has stable rank* 1.

III.6 Quantifying property RD

For any finitely generated group Γ that satisfies property RD, we defined the "RD-dimension" of Γ as follows:

$$\mathrm{rd}(\Gamma) = \inf\{s : \ \Gamma \text{ satisfies (RD) of order } s\}$$

Similarly, we put:

$$\mathrm{rd}_\bullet(\Gamma) = \inf\{s : \ \Gamma \text{ satisfies (RD}_\bullet\text{) of order } s\}$$

$$\mathrm{rd}_\circ(\Gamma) = \inf\{s : \ \Gamma \text{ satisfies (RD}_\circ\text{) of order } s\}$$

Note that these definitions do not depend on the choice of word-length (this is stronger than saying that property RD does not depend of the choice of word-length.)

Lemma III.6.1. $\mathrm{rd}(\Gamma) = \mathrm{rd}_\bullet(\Gamma)$ *and* $\mathrm{rd}_\circ(\Gamma) \leq \mathrm{rd}_\bullet(\Gamma) \leq \mathrm{rd}_\circ(\Gamma) + \frac{1}{2}$.

Proof. That $\mathrm{rd}(\Gamma) \geq \mathrm{rd}_\bullet(\Gamma) \geq \mathrm{rd}_\circ(\Gamma)$ is trivial. The inequality $\mathrm{rd}(\Gamma) \leq \mathrm{rd}_\bullet(\Gamma)$, implicit in [46], works as follows. Let $s > \mathrm{rd}_\bullet(\Gamma)$; we show Γ satisfies (RD) of order s. Pick $s > s' > \mathrm{rd}_\bullet(\Gamma)$, so Γ satisfies (RD$_\bullet$) of order s'. Let A_n be the annulus $\{g : 2^n \leq |g| < 2^{n+1}\}$. Then:

$$\begin{aligned}
\|\sum a_g g\| &\leq |a_0| + \sum_{n\geq 0} \Big\| \sum_{g\in A_n} a_g g \Big\| \leq |a_0| + C\sum_{n\geq 0} 2^{(n+1)s'} \Big\| \sum_{g\in A_n} a_g g \Big\|_2 \\
&\leq |a_0| + C\sum_{n\geq 0} 2^{(n+1)s'-ns} 2^{ns} \Big(\sum_{g\in A_n} |a_g|^2 (1+|g|)^{2s} \Big)^{1/2} \\
&\leq C' \big(1+\sum_{n\geq 0} 4^{n(s'-s)}\big)^{1/2} \Big(\sum |a_g|^2 (1+|g|)^{2s} \Big)^{1/2}
\end{aligned}$$

Thus Γ satisfies (RD) of order s, as desired. We now prove the remaining inequality, namely $\mathrm{rd}_\bullet(\Gamma) \leq \mathrm{rd}_\circ(\Gamma) + \frac{1}{2}$. Assume that Γ satisfies ($\mathrm{RD}_\circ$) of order s; we show that Γ satisfies ($\mathrm{RD}_\bullet$) of order $s+\frac{1}{2}$. Let $\sum a_g g \in \mathbb{C}\Gamma$ be supported on the n-ball. We have:

$$\begin{aligned}
\|\sum a_g g\| &\leq \sum_{i\leq n} \Big\| \sum_{|g|=i} a_g g \Big\| \leq C\sum_{i\leq n} (1+i)^s \Big\| \sum_{|g|=i} a_g g \Big\|_2 \\
&\leq C\Big(\sum_{i\leq n} (1+i)^{2s} \Big)^{1/2} \|\sum a_g g\|_2 \leq C'(1+n)^{s+\frac{1}{2}} \|\sum a_g g\|_2
\end{aligned}$$

Thus ($\mathrm{RD}_\bullet$) of order $s+\frac{1}{2}$ holds. □

The proof of Theorem III.3.2 shows that, for Γ a group of polynomial growth, we have

$$\deg \rho_\bullet = 2\mathrm{rd}_\bullet(\Gamma), \quad \deg \rho_\circ = 2\mathrm{rd}_\circ(\Gamma)$$

where $\deg \rho_\bullet = \inf\{s : \rho_\bullet(n) \prec n^s\}$ and $\deg \rho_\circ = \inf\{s : \rho_\circ(n) \prec n^s\}$ denote the growth degree of balls, respectively of spheres. Note that $\deg \rho_\bullet$ is a non-negative integer: polynomial growth groups are precisely the virtually nilpotent groups (Gromov), while for nilpotent groups we have $\deg \rho_\bullet = \sum k \,\mathrm{rank}(\Gamma_k/\Gamma_{k+1})$ where $\Gamma_{k+1} = [\Gamma_k, \Gamma]$ is the lower central series of Γ (Bass). On the other hand, surprisingly little is known about $\deg \rho_\circ$ and how it relates to $\deg \rho_\bullet$. It is easy to see that $\deg \rho_\bullet - 1 \leq \deg \rho_\circ \leq \deg \rho_\bullet$, which translates precisely in the inequality $d_\circ(\Gamma) \leq d_\bullet(\Gamma) \leq d_\circ(\Gamma) + \frac{1}{2}$. Examples like $\mathbb{Z}^n$ and the interpretation of $\rho_\circ$ as the discrete derivative of $\rho_\bullet$ suggest $\deg \rho_\circ = \deg \rho_\bullet - 1$ should hold.

For free groups, we know that $\mathrm{rd}_\circ(F_n) = 1$; this follows by combining Haagerup's inequality and Cohen's computation (see Section III.2). Thus $1 \leq \mathrm{rd}_\bullet(F_n) \leq 3/2$. It would be interesting to compute the actual value of $\mathrm{rd}_\bullet(F_n)$. This is relevant to the following speculation:

What is the dimension of the (imaginary) dual of F_n?

One argues by analogy with the free abelian groups $\mathbb{Z}^n$. If we understand dimension from the perspective of the stable rank, then the noncommutative space $\widehat{F_n}$ should have dimension 1. Indeed, the fact that $\mathrm{tsr}\, C(\widehat{F_n}) = 1$ means that $\widehat{F_n}$ has dimension 0 or 1, and the fact that $\mathrm{csr}\, C(\widehat{F_n}) = 2$ rules out the possibility that the dimension is 0.

On the other hand, if we understand dimension from the RD viewpoint, then the noncommutative space $\widehat{F_n}$ should have dimension 2 or 3. We have just seen that, in the case of $\mathbb{Z}^n$, the dimension is recovered by taking twice the "RD-dimension" $\mathrm{rd}(\mathbb{Z}^n)$. Now $2 \leq 2\mathrm{rd}_\bullet(F_n) \leq 3$ so, assuming integrality, the dimension of $\widehat{F_n}$ is either 2 or 3.

If Γ is finite, then $\mathrm{rd}_\circ(\Gamma) = \mathrm{rd}_\bullet(\Gamma) = \mathrm{rd}_\circ(\Gamma) = 0$. On the other hand, for infinite groups we have:

Proposition III.6.2. *Let Γ be an infinite, finitely generated group. Then* $\mathrm{rd}(\Gamma) \geq \frac{1}{2}$.

Note that $\mathrm{rd}(\mathbb{Z}) = \frac{1}{2}$, so the lower bound is best possible.

The proof of the above proposition, although elementary, is intricate. Already the fact that an infinite group cannot satisfy (RD) of order 0 is non-trivial, despite its obviousness. Rajagopalan formulated in the early 60's the following "L^p-conjecture": if G is a locally compact group, and $L^p(G)$ is closed under convolution for some $p \in (1,\infty)$, then G is compact. The case when G is discrete and $p = 2$ (and this is exactly what having (RD) of order 0 means) was solved by Rajagopalan himself in [69] by using structural results on so-called H^*-algebras. Our $H^s\Gamma$ is both a Hilbert space and, for suitable s, a Banach $*$-algebra; however, it is not a H^*-algebra, so Rajagopalan's arguments do not apply. Instead, we find inspiration in the complete solution to the L^p-conjecture, given by Saeki [75] almost 30 years after its formulation.

Proof. We argue by contradiction. Assume (RD) of order $\varepsilon \in [0, \frac{1}{2})$ holds, hence ($\mathrm{RD}_\bullet$) of order ε holds as well with the same constant C. Fix a set of generators for Γ.

Claim 1: There is an integer $r > 0$ such that $|B_{(n+1)r}| > 4|B_{nr}|$ for all n.

If Γ has subexponential growth, then Γ is amenable, and the proof of Theorem III.3.2 says that Γ has in fact sublinear growth; but this is impossible for an infinite group. Thus Γ is of exponential growth, so let $c_2 \geq c_1 > 0$ be such that $c_1 e^n \leq |B_n| \leq c_2 e^n$ for all n. Let $r > 0$ be an integer with $e^r \geq 4c_2/c_1$. Then:

$$\frac{|B_{(n+1)r}|}{|B_{nr}|} \geq \frac{c_1 e^{(n+1)r}}{c_2 e^{nr}} = \frac{c_1}{c_2} e^r > 4$$

From now on, let r be as in Claim 1. For $k \geq 1$, denote by ϕ_k the characteristic function of B_{rk}, and let $b_k := |B_{rk}| = \|\phi_k\|_2^2$. Unless otherwise stated, all infinite sums start from 1 in what follows.

Claim 2: Let $a = \sum a_k \phi_k$, where $a_k \geq 0$ and $a_{k+1} \leq a_k/2$ for all k. Then $\|a\|_2 < \infty$ iff $\sum a_k^2 b_k < \infty$.

On one hand, we have $\|a\|_2^2 \geq \sum \|a_k \phi_k\|_2^2 = \sum a_k^2 b_k$. On the other hand, we show $\|a\|_2^2 \leq 4 \sum a_k^2 b_k$. For all k,l we have $\langle \phi_k, \phi_l \rangle = \|\phi_{\min\{k,l\}}\|_2^2 = b_{\min\{k,l\}}$. Hence:

$$\|a\|_2^2 = \sum_{k,l} a_k a_l \langle \phi_k, \phi_l \rangle = \sum_{k,l} a_k a_l b_{\min\{k,l\}} \leq 2 \sum_{k \leq l} a_k a_l b_k = 2 \sum_k \big(\sum_{l \geq k} a_l \big) a_k b_k$$

Since $a_{k+1} \leq a_k/2$ for all k, we have $\sum_{l \geq k} a_l \leq 2a_k$, thus $\|a\|_2^2 \leq 4 \sum a_k^2 b_k$ as desired.

Now we make our choice: let $a = \sum a_k \phi_k$ with

$$a_k = \frac{1}{\sqrt{k^\alpha b_k}}, \quad \alpha := \frac{3}{2} + \varepsilon.$$

Let $L := 1 + |\cdot|$, where $|\cdot|$ is the length function on Γ. Note that the multiplication by (powers of) L is pointwise.

Claim 3: $\|L^\varepsilon a\|_2 < \infty$.

Inequalities of functions being understood in the pointwise sense, we have:

$$L^\varepsilon a = \sum a_k(L^\varepsilon \phi_k) \leq \sum a_k(1+rk)^\varepsilon \phi_k \leq \sum a_k(2rk)^\varepsilon \phi_k = (2r)^\varepsilon \sum a_k k^\varepsilon \phi_k$$

The plan is to apply Claim 2 to the sum on right hand side. First, we check:

$$\frac{a_{k+1}(k+1)^\varepsilon}{a_k k^\varepsilon} = \sqrt{\frac{k^{\alpha-2\varepsilon} b_k}{(k+1)^{\alpha-2\varepsilon} b_{k+1}}} < \sqrt{\frac{b_k}{b_{k+1}}} < \frac{1}{2}$$

We have used the fact that $\alpha - 2\varepsilon > 0$, as well as the estimate from Claim 1. Actually, $\alpha - 2\varepsilon > 1$ so

$$\sum (a_k k^\varepsilon)^2 b_k = \sum \frac{1}{k^{\alpha-2\varepsilon}}$$

is convergent. By Claim 2, $\|\sum a_k k^\varepsilon \phi_k\|_2 < \infty$ hence $\|L^\varepsilon a\|_2 < \infty$ as well.

By (RD) of order ε, we have $\|aa'\|_2 \leq C\|L^\varepsilon a\|_2 \|a'\|_2$; for $a' := L^\varepsilon a$, claim 3 implies that $\|a(L^\varepsilon a)\|_2 < \infty$. The contradiction will arise from the following:

Claim 4: $\|a(L^\varepsilon a)\|_2 = \infty$.

As a preparation, we first prove the following estimate:

$$\sum_{g\in B_{rn}} (1+|g|)^\varepsilon = \sum_{0\leq i\leq rn} (1+i)^\varepsilon |S_i| \geq \frac{1}{2}(rn)^\varepsilon b_n \tag{III.1}$$

We use the following well-known inequality, a consequence of the rearrangement inequality: if $x_1 \leq x_2 \leq \cdots \leq x_N$ and $y_1 \leq y_2 \leq \cdots \leq y_N$, then

$$\sum_{1\leq i\leq N} x_i y_i \geq \frac{1}{N}\Big(\sum_{1\leq i\leq N} x_i\Big)\Big(\sum_{1\leq i\leq N} y_i\Big).$$

We cannot apply directly the above inequality to the two sequences $\{(1+i)^\varepsilon\}_{0\leq i\leq rn}$ and $\{|S_i|\}_{0\leq i\leq rn}$ because we are not guaranteed to have the latter sequence non-decreasing. Instead, we shall use the r-thick annuli $A_i = \{g : r(i-1) < |g| \leq ri\}$. When $i = 1$, it is convenient to set $A_1 = \{g : |g| \leq r\}$. The estimate of Claim 1 yields that the sequence $\{|A_i|\}_{1\leq i\leq n}$ increases; in fact $|A_{i+1}| > 3|A_i|$. Then:

$$\sum_{0\leq i\leq rn} (1+i)^\varepsilon |S_i| \geq \sum_{1\leq i\leq n} \big(1+r(i-1)\big)^\varepsilon |A_i| \geq \frac{1}{n}\Big(\sum_{1\leq i\leq n} \big(1+r(i-1)\big)^\varepsilon\Big)\Big(\sum_{1\leq i\leq n} |A_i|\Big)$$

As $0 \leq \varepsilon < 1$, we have $x^\varepsilon + y^\varepsilon \geq (x+y)^\varepsilon$ whenever $x, y \geq 0$. Hence:

$$\begin{aligned}
\sum_{1\leq i\leq n} \big(1+r(i-1)\big)^\varepsilon &= \frac{1}{2}\sum_{1\leq i\leq n} \Big(\big(1+r(i-1)\big)^\varepsilon + \big(1+r(n-i+1)\big)^\varepsilon\Big) \\
&\geq \frac{1}{2}\sum_{1\leq i\leq n} (2+rn)^\varepsilon \geq \frac{1}{2} n (rn)^\varepsilon
\end{aligned}$$

Using $\sum_{1\leq i\leq n} |A_i| = b_n$, we conclude that $\sum_{0\leq i\leq rn} (1+i)^\varepsilon |S_i| \geq \frac{1}{2}(rn)^\varepsilon b_n$, so (III.1) is proved.

Next, we show the following estimate:

$$\phi_{n+k}(L^{\varepsilon}\phi_n) \geq \frac{1}{2}(rn)^{\varepsilon} b_n \phi_k \qquad (n,k \geq 1) \tag{III.2}$$

Recall that the inequality is understood pointwise, L^{ε} multiplies pointwise, and $\phi_{n+k}(L^{\varepsilon}\phi_n)$ is the convolution product. Pick $h \in \mathrm{supp}(\phi_k) = B_{rk}$. For each $g \in \mathrm{supp}(\phi_n) = B_{rn}$, we have $hg^{-1} \in \mathrm{supp}(\phi_{n+k}) = B_{rn+rk}$, so h appears with coefficient at least

$$\sum_{g \in B_{rn}} (1+|g|)^{\varepsilon}$$

in the convolution $\phi_{n+k}(L^{\varepsilon}\phi_n)$. Using (III.1), we obtain (III.2).

Now (III.2) yields:

$$\begin{aligned} a(L^{\varepsilon}a) &= \sum_{m,n} a_m a_n \phi_m(L^{\varepsilon}\phi_n) \geq \sum_{m>n} a_m a_n \phi_m(L^{\varepsilon}\phi_n) \\ &= \sum_k \sum_n a_{n+k} a_n \phi_{n+k}(L^{\varepsilon}\phi_n) \geq \frac{1}{2}\sum_k \sum_n a_{n+k} a_n (rn)^{\varepsilon} b_n \phi_k \end{aligned}$$

Let $q \geq 1$. For $n \leq qk$, we have

$$a_n \geq \frac{1}{\sqrt{(qk)^{\alpha} b_n}}, \quad a_{n+k} \geq \frac{1}{\sqrt{((q+1)k)^{\alpha} b_{n+k}}} \geq \frac{1}{\sqrt{(2qk)^{\alpha} b_{n+k}}}$$

so, using the generic inequality $b_{n+k} \leq b_n b_k$, we get

$$a_{n+k} a_n b_n \geq \frac{1}{2^{\alpha/2}(qk)^{\alpha}} \sqrt{\frac{b_n}{b_{n+k}}} \geq \frac{1}{2^{\alpha/2}(qk)^{\alpha}\sqrt{b_k}}$$

On the other hand, for $n > (q-1)k$ we have $rn \geq \frac{1}{2}rqk$ as soon as $q \geq 2$. So, for $q \geq 2$ and $(q-1)k < n \leq qk$ we get

$$a_{n+k} a_n b_n (rn)^{\varepsilon} \geq \frac{2^{-\varepsilon} r^{\varepsilon} (qk)^{\varepsilon}}{2^{\alpha/2}(qk)^{\alpha}\sqrt{b_k}} = \frac{K}{(qk)^{3/2}\sqrt{b_k}}$$

by absorbing the constants into a new constant K; recall, $\alpha = \frac{3}{2} + \varepsilon$. We now sum over n:

$$\begin{aligned} \sum_n a_{n+k} a_n (rn)^{\varepsilon} b_n &\geq \sum_{q \geq 2} \left(\sum_{(q-1)k < n \leq qk} a_{n+k} a_n (rn)^{\varepsilon} b_n \right) \\ &\geq \sum_{q \geq 2} \left(\sum_{(q-1)k < n \leq qk} \frac{K}{(qk)^{3/2}\sqrt{b_k}} \right) \\ &= K \sum_{q \geq 2} \frac{k}{(qk)^{3/2}\sqrt{b_k}} = K' \frac{1}{\sqrt{k b_k}} \end{aligned}$$

Here $K' = K\sum_{q\geq 2} q^{-3/2}$. We now return to our estimate of $a(L^{\varepsilon}a)$:

$$a(L^{\varepsilon}a) \quad \geq \quad \frac{1}{2}\sum_k \sum_n a_{n+k}a_n(rn)^{\varepsilon}b_n\phi_k \geq \frac{K'}{2}\sum_k \frac{1}{\sqrt{kb_k}}\phi_k$$

We apply the conclusion of Claim 2 to the sum on right hand side. Using the estimate from Claim 1, we check:

$$\frac{1}{\sqrt{(k+1)b_{k+1}}} < \frac{1}{\sqrt{kb_{k+1}}} < \frac{1}{2\sqrt{kb_k}}$$

This time

$$\sum_k \left(\frac{1}{\sqrt{kb_k}}\right)^2 b_k = \sum_k \frac{1}{k} = \infty$$

so $\|a(L^{\varepsilon}a)\|_2 = \infty$. Claim 4 is proved.

This ends the proof. $\square$

Proposition III.6.2 prompts the following two questions.

Question 1. Is it true that $\mathrm{rd}(\Gamma) = \frac{1}{2}$ precisely when Γ is virtually infinite cyclic?

The answer is, very likely, affirmative. We know this is the case in the polynomial growth regime, so one needs to show $\mathrm{rd}(\Gamma) > \frac{1}{2}$ whenever Γ has exponential growth.

The next question looks much harder, and a negative answer seems more likely.

Question 2. Is it true that $2\mathrm{rd}(\Gamma)$ is always an integer, if finite?

CHAPTER IV

STABLE RANKS

IV.1 Introduction

The theory of C*-algebras can be thought of as "noncommutative" topology. This viewpoint is rooted in Gelfand's foundational theorem, saying that a commutative C*-algebra is C*-isomorphic to $C_0(X)$ for some locally compact, Hausdorff space X.

Extending topological invariants to the C*-context is, consequently, a key aspect in the study of C*-algebras. The topological K-theory of Atiyah and Hirzebruch has been vastly generalized in this way to operator K-theory for C*-algebras. In fact, operator K-theory is most naturally defined for Banach algebras.

We are interested in notions of dimensions for C*-algebras and, more generally, Banach algebras, which extend the notion of (covering) dimension for topological spaces. Recall, we put $\dim X = d$ if d is the least integer with the property that every open cover of X can be refined in such a way that no point of X lies in more than $d+1$ sets; if no such d exists, we put $\dim X = \infty$.

Let us sketch the properties we would expect from such a "rank" for Banach algebras. First, we would like to recover the usual notion of dimension:

D. $\operatorname{rk} C(X) = \dim X$ for a compact space X

Throughout, spaces are assumed to be Hausdorff. Inspired by properties of dimension for topological spaces, we also expect the "rank" to fulfill the following:

Q. if B is a quotient of A, then $\operatorname{rk} A \geq \operatorname{rk} B$

I. if I is an ideal in A, then $\operatorname{rk} I \leq \operatorname{rk} A$

E. if $0 \to I \to A \to B \to 0$ is an extension, then $\operatorname{rk} A \leq \max\{\operatorname{rk} I, \operatorname{rk} B\}$

L. $\operatorname{rk}\left(\varinjlim A_i\right) \leq \underline{\lim}\ \operatorname{rk} A_i$

S. $\operatorname{rk}(A \oplus B) \leq \max\{\operatorname{rk} A, \operatorname{rk} B\}$

T. $\operatorname{rk}(A \otimes B) \leq \operatorname{rk} A + \operatorname{rk} B$

In what concerns amplifications, we might desire something more precise than a bound as provided by **T**. Namely:

M. for each $n \geq 1$, $\operatorname{rk} \mathrm{M}_n(A)$ can be computed from $\operatorname{rk} A$

By now, at least a dozen such ranks have been proposed in the literature. Here, we are interested in the following: the *Bass stable rank* (bsr), the *topological stable rank* (tsr), the *connected stable rank* (csr), and the *general stable rank* (gsr). The original notion of stable rank is the Bass stable rank, introduced by Bass in [7]. The other three stable ranks under consideration are due to Rieffel [71].

We find it useful to study these four stable ranks together. On one hand, they share many general properties. On the other hand, aspects that are particular to each stable rank are best displayed in contrast with the other stable ranks. The connected and the topological stable rank are topological in nature, whereas the Bass and the general stable rank are purely algebraic. However, similarities emerge between the Bass stable rank and the topological stable rank on one hand, and the connected stable rank and the general stable rank on the other hand.

So far, the Bass stable rank and the topological stable rank have been most thoroughly investigated due to their interpretation as noncommutative dimensions. Although harder to compute in general, the connected stable rank and its higher relatives are the better suited stable ranks for K-theoretic purposes. Of all four stable ranks, the general stable rank is the most difficult to compute.

The stable ranks we are considering are related by the following estimate:

$$\mathrm{gsr} \leq \mathrm{csr} \leq \mathrm{bsr} + 1 \leq \mathrm{tsr} + 1$$

For C*-algebras we have $\mathrm{bsr} = \mathrm{tsr}$, but in general the inequalities can be strict.

The expected properties that we listed above are only satisfied to a certain extent; nevertheless, they serve as a guiding light. The dimension property **D** is satisfactory for the Bass and the topological stable rank; it is more complicated for the connected stable rank, whereas for the general stable rank very little is known. Some estimates pertaining to properties **Q**, **E**, **I** are known, but one would expect more. Properties **L** and **S** are easily satisfied. Behavior under tensorization - property **T** - is far from being understood. Finally, for property **M** the currently known estimates are rather satisfactory.

Among the other "noncommutative dimensions" that have received attention in the literature, we mention the *real rank* introduced by Brown and Pedersen, the *analytic rank* of Murphy, H. Lin's *topological tracial rank*, and the *decomposition rank* of Kirchberg and Winter. These ranks are C*-algebraic, and they are typically used for classification purposes. It should be noted that they fulfill the dimension requirement **D** exactly.

Notations. In what follows, a notation like $(a_i) \in A^n$ means that the index i runs from 1 to n, so (a_i) stands for the vector $(a_1, \ldots, a_n)$. For an algebra A, the group of invertible elements of A is denoted by $A^\times$. The floor function $\lfloor \cdot \rfloor$ is given by $\lfloor x \rfloor$ = greatest integer that is not greater than x, and the ceiling function $\lceil \cdot \rceil$ is given by $\lceil x \rceil$ = smallest integer that is not smaller than x. The unit interval $[0,1]$ is denoted by I.

IV.2 Definitions of stable ranks

Let A be a unital topological algebra. Consider, for each $n \geq 1$, the collection of n-tuples that generate A as a left ideal:

$$\mathrm{Lg}_n(A) = \{(a_1, \ldots, a_n) : Aa_1 + \cdots + Aa_n = A\} \subseteq A^n$$

Elements of $\mathrm{Lg}_n(A)$ are also called unimodular vectors. Note that $\mathrm{Lg}_n(A)$ carries a left action of $\mathrm{GL}_n(A)$, namely left-multiplying the transpose of a unimodular vector by an invertible matrix.

The stable ranks under consideration capture the stabilization of certain properties in the sequence

$\left(\mathrm{Lg}_n(A)\right)_{n\geq 1}$:

Definition IV.2.1. Let A be a topological algebra. Then:

$\cdot$ the *Bass stable rank* of A, denoted $\mathrm{bsr}\, A$, is the least n for which $\mathrm{Lg}_{n+1}(A)$ is reducible to $\mathrm{Lg}_n(A)$ in the following sense: if $(a_i) \in \mathrm{Lg}_{n+1}(A)$ then $(a_i + x_i a_{n+1}) \in \mathrm{Lg}_n(A)$ for some $(x_i) \in A^n$;

$\cdot$ the *topological stable rank* of A, denoted $\mathrm{tsr}\, A$, is the least n for which $\mathrm{Lg}_n(A)$ is dense in A^n;

$\cdot$ the *connected stable rank* of A, denoted $\mathrm{csr}\, A$, is the least n for which $\mathrm{Lg}_m(A)$ is connected for all $m \geq n$;

$\cdot$ the *general stable rank* of A, denoted $\mathrm{gsr}\, A$, is the least n such that $\mathrm{GL}_m(A)$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m \geq n$.

Notation. In what follows, the generic sr denotes any one of bsr, tsr, csr, gsr.

Naturally, we put $\mathrm{sr}\, A = \infty$ whenever there is no integer n satisfying the required stable rank condition. Also, if A is non-unital, then we put $\mathrm{sr}\, A := \mathrm{sr}\, A^+$, where A^+ is the unitization of A.

Remark IV.2.2. If A is a Banach algebra then it is easy to see that $\mathrm{Lg}_n(A)$ is open in A^n; in fact, this holds whenever A is a good topological algebra (Lemma VIII.7.5). In particular, $\mathrm{Lg}_n(A)$ is connected if and only if $\mathrm{Lg}_n(A)$ is path-connected. Indeed, a topological algebra is locally path-connected, since each point has a local basis consisting of balanced sets, so path-connectivity and connectivity are equivalent for open sets.

Since the openness of $\mathrm{Lg}_n(A)$ is the only topological feature we need in much of what follows, we formulate the results for good topological algebras whenever we can.

Remark IV.2.3. For a good topological algebra A, we let $A_0^\times$ denote the connected component of the identity in $A^\times$. Correspondingly, we let $\mathrm{GL}_n(A)_0$ denote the connected component of the identity in $\mathrm{GL}_n(A)$. We note the following

Fact. The action of $\mathrm{GL}_n(A)_0$ on $\mathrm{Lg}_n(A)$ has open orbits.

Indeed, fix $(a_i) \in \mathrm{Lg}_n(A)$. Pick some $(b_i) \in A^n$ such that $\sum b_i a_i = 1$. Let $U \subseteq A$ be a neighborhood of 0 such that $1_n + (u_i b_j)_{1\leq i,j\leq n} \in \mathrm{GL}_n(A)_0$ whenever $u_i \in U$. We claim that (a_i') is in the $\mathrm{GL}_n(A)_0$-orbit of (a_i) whenever $a_i' \in a_i + U$. To verify the claim, put $a_i' = a_i + u_i$ with $u_i \in U$. Then

$$\begin{pmatrix} u_1 b_1 & u_1 b_2 & \dots & u_1 b_n \\ u_2 b_1 & u_2 b_2 & \dots & u_2 b_n \\ & \vdots & & \\ u_n b_1 & u_n b_2 & \dots & u_n b_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

so $1_n + (u_i b_j)_{1\leq i,j\leq n} \in \mathrm{GL}_n(A)_0$ takes (a_i) to $(a_i + u_i) = (a_i')$. The fact is proved.

A consequence is that $\mathrm{Lg}_n(A)$ is connected if and only if $\mathrm{GL}_n(A)_0$ acts transitively on $\mathrm{Lg}_n(A)$. Therefore, we can reformulate the definition of the connected stable rank as follows:

$\cdot$ $\mathrm{csr}\, A$ is the least n for which $\mathrm{GL}_m(A)_0$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m \geq n$

Remark IV.2.4. Strictly speaking, we have defined the *left* Bass, topological, and connected stable rank. The definition of the right counterpart for each stable rank is based on the right generating n-tuples

$$\mathrm{Rg}_n(A) = \{(a_1,\ldots,a_n) :\ a_1A+\cdots+a_nA = A\}.$$

Let us consider the symmetry of the stable ranks. Note that, in the case of topological $*$-algebras, there is no difference between left and right stable ranks.

The Bass stable rank is left-right symmetric (Vaserstein [87]; see also [52, Prop.11.3.4]), and so are the connected stable rank (at least for Banach algebras, as shown by Corach - Larotonda [19]) and the general stable rank ([52, Lem.11.1.13]). On the other hand, the topological stable rank is not left-right symmetric. Examples of Banach algebras for which ltsr $\neq$ rtsr were given recently in [25]. The context is that of *nest algebras*, which can be thought of as algebras of upper triangular infinite matrices. A *nest* $\mathcal{N}$ on a Hilbert space H is a chain of closed subspaces of H satisfying the following two conditions:

· $0 \in \mathcal{N}$ and $H \in \mathcal{N}$

· $\bigcap N_i \in \mathcal{N}$ and $\overline{\bigcup N_i} \in \mathcal{N}$ whenever $\{N_i\}_{i\in I}$ is a family of subspaces in $\mathcal{N}$

For a nest $\mathcal{N}$, the weakly-closed algebra of operators that leave each subspace in $\mathcal{N}$ invariant

$$T(\mathcal{N}) = \{T \in \mathcal{B}(H) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}$$

is called the nest algebra determined by $\mathcal{N}$. It turns out that at least one of $\mathrm{ltsr}\big(T(\mathcal{N})\big)$ or $\mathrm{rtsr}\big(T(\mathcal{N})\big)$ is infinite, and in many cases both are infinite in fact. However, there are nest algebras where a one-sided topological stable rank is 2. Let $(d_k)_{k\geq 1}$ be an increasing sequence of natural numbers and let $(e_n)_{n\geq 1}$ be an orthonormal basis for the separable Hilbert space H. Consider the nest $\mathcal{N}$ consisting of the d_k-dimensional spaces $N_k = \langle e_i :\ 1 \leq i \leq d_k\rangle$, together with $N_0 = 0$ and $N_\infty = H$. The main result of [25] is that, if $(d_k)_{k\geq 1}$ grows exponentially, then the corresponding nest algebra has ltsr $= \infty$ and rtsr $= 2$. To obtain a nest algebra with rtsr $= \infty$ and ltsr $= 2$, one need only dualize: if $\mathcal{N}$ is a nest, then $\mathcal{N}^\perp = \{N^\perp :\ N \in \mathcal{N}\}$ is a nest whose corresponding algebra $T(\mathcal{N}^\perp)$ equals $T(\mathcal{N})^*$.

Remark IV.2.5. In the definition above, $\mathrm{sr}\,A$ is defined as the least n such that a certain condition sr_n involving $\mathrm{Lg}_n(A)$ holds. Note that sr_k implies sr_{k+1}. This is clear for tsr, as $A \times \mathrm{Lg}_k(A) \subseteq \mathrm{Lg}_{k+1}(A)$. For csr and gsr, it holds by design. As for bsr, it is shown as follows. Observe first that the converse of the implication in bsr_k is always true, i.e., if $(a_i + x_i a_{k+1}) \in \mathrm{Lg}_k(A)$ for some $(x_i) \in A^k$, then $(a_i) \in \mathrm{Lg}_{k+1}(A)$. Let now $(a_i) \in \mathrm{Lg}_{k+2}(A)$. Then $(a_1,\ldots,a_k, c_{k+1}a_{k+1} + c_{k+2}a_{k+2}) \in \mathrm{Lg}_{k+1}(A)$ for some $c_{k+1}, c_{k+2} \in A$. By bsr_k, we have $\big(a_i + z_i(c_{k+1}a_{k+1} + c_{k+2}a_{k+2})\big) \in \mathrm{Lg}_k(A)$ for some $(z_i) \in A^k$. We get $(a_i + z'_i a_{k+2} + z''_i a_{k+1}) \in \mathrm{Lg}_k(A)$ after renaming the coefficients, and the previous observation yields

$$(a_1 + z'_1 a_{k+2},\ldots,a_k + z'_k a_{k+2}, a_{k+1}) \in \mathrm{Lg}_{k+1}(A)$$

which proves bsr_{k+1}.

The very first computation of stable ranks is, of course, the following

$\mathrm{sr}\,\mathbb{C} = 1$.

IV.3 Stable ranks for $C(X)$

For a compact space X, the stable ranks of $C(X)$ can be computed - or at least estimated - in terms of the dimension of X. As in manifold theory, we use the notation X^d to indicate that X is d-dimensional.

The following is due to Vaserstein [87, Thm.7] for the Bass stable rank, to Rieffel [71, Prop.1.7] for the topological stable rank, and to Nistor [63, Cor.2.5] for the connected stable rank.

Theorem IV.3.1. *Let X^d be a compact space. Then:*

$$\mathrm{bsr}\, C(X^d) = \mathrm{tsr}\, C(X^d) = \lfloor d/2 \rfloor + 1$$

$$\mathrm{gsr}\, C(X^d) \leq \mathrm{csr}\, C(X^d) \leq \lceil d/2 \rceil + 1$$

In general, the dimensional upper bound for the connected stable rank is not attained; this is due to the fact that the connected stable rank is homotopy invariant (cf. proof of Theorem IV.3.2 below; see also Section IV.9). Thus, if we want the dimensional upper bound for the connected stable rank to be achieved, then we need an assumption which guarantees that a space is not homotopy equivalent to a space of lower dimension (a dimension 1 drop from an even dimension to an odd dimension may be allowed, however).

The following criterion was proved in [61, Prop.6.7]. Roughly, the hypothesis is that the top cohomology group in $H^{\mathrm{odd} \leq d}(X^d)$ is non-vanishing:

Theorem IV.3.2. *Let X^d be a compact space. Then*

$$\mathrm{csr}\, C(X^d) = \lceil d/2 \rceil + 1$$

provided $\check{H}^d(X^d) \neq 0$ for odd d, or $\check{H}^{d-1}(X^d) \neq 0$ for even d.

In what follows, the convention is that cohomology groups are taken with integer coefficients.

Proof. For simplicity, we assume that X^d is a d-dimensional finite CW-complex and we work with the usual cohomology. For a general compact space, cohomology is understood in the Čech sense and results from [35, p. 149-150] are the used in a similar way.

We identify $\mathrm{Lg}_m C(X^d)$ with the space of continuous maps $\mathrm{Map}(X^d, \mathbb{C}^m \setminus \{0\})$. Consequently, we have a based identification $\pi_0\big(\mathrm{Lg}_m C(X^d)\big) \simeq [X^d, S^{2m-1}]$. Hence, $\mathrm{csr}\, C(X^d)$ is the least n such that $[X^d, S^{2m-1}] = 0$ for all $m \geq n$. On one hand, this shows that $\mathrm{csr}\, C(X^d)$ only depends on the homotopy type of X^d. On the other hand, we directly recover the dimensional upper bound $\mathrm{csr}\, C(X^d) \leq \lceil d/2 \rceil + 1$. Indeed, if $m \geq \lceil d/2 \rceil + 1$ then $2m - 1 \geq d + 1$, so $[X^d, S^{2m-1}] = 0$.

We have $\mathrm{csr}\, C(X^d) = \lceil d/2 \rceil + 1$ if and only if $[X^d, S^{2m-1}] \neq 0$ for $m = \lceil d/2 \rceil$. That is, for odd d we need to check $[X^d, S^d] \neq 0$, and for even d we need to check $[X^d, S^{d-1}] \neq 0$. This is achieved by the following

Fact. Let X^d be a d-dimensional finite CW-complex. Then:

a) $H^d(X^d) \neq 0$ if and only if $[X^d, S^d] \neq 0$;

b) $H^{d-1}(X^d) \neq 0$ implies $[X^d, S^{d-1}] \neq 0$.

To prove the fact, we recall that $H^d(X^d)$ can be identified with $[X^d, K(\mathbb{Z},d)]$. Realize the $K(\mathbb{Z},d)$ CW-complex by starting with S^d as its d-skeleton and then adding cells of dimension $\geq d+2$ that kill the higher homotopy.

For a) we claim that the natural map $[X^d, S^d] \to [X^d, K(\mathbb{Z},d)]$ is bijective. Surjectivity is clearly a consequence of the Cellular Approximation Theorem. But so is injectivity: a homotopy in $K(\mathbb{Z},d)$ between $f_0, f_1 : X^d \to S^d$ can be assumed cellular, so it maps to the $(d+1)$-skeleton of $K(\mathbb{Z},d)$ which is S^d. That is, the maps $f_0, f_1 : X^d \to S^d$ are actually homotopic over S^d.

For b), note that the natural map $[X^d, S^{d-1}] \to [X^d, K(\mathbb{Z},d-1)]$ is surjective as well. Again, this follows from the Cellular Approximation Theorem, together with the fact that the d-skeleton of $K(\mathbb{Z},d-1)$ is S^{d-1}. □

Remark IV.3.3. The proof actually shows that, for odd d, the criterion is best possible: namely, we have $\operatorname{csr} C(X^d) = \lceil d/2 \rceil + 1$ if and only if $H^d(X^d) \neq 0$.

Theorem IV.3.2 applies to tori of arbitrary dimension, orientable and non-orientable surfaces with the exception of the 2-sphere, closed odd-dimensional manifolds, etc. On the other hand, Theorem IV.3.2 does not apply directly to (even-dimensional) spheres. We treat them in the following:

For S^d, the d-sphere, we have:

$$\operatorname{csr} C(S^d) = \begin{cases} \lceil d/2 \rceil + 1 & \text{if } d \neq 2 \\ 1 & \text{if } d = 2 \end{cases}$$

We need to revisit the proof of Theorem IV.3.2 in the case d is even. Then $\operatorname{csr} C(S^d) = \lceil d/2 \rceil + 1$ if and only if $[S^d, S^{d-1}] \neq 0$. This holds when $d \neq 2$: it is well-known that $\pi_d(S^{d-1}) = \mathbb{Z}/_2$ for $d \geq 4$. For $d = 2$, the fact that $\pi_2(S^1) = 0$ gives $\operatorname{csr} C(S^2) = 1$.

If the computation of $\operatorname{csr} C(X)$ was problematic, the computation of $\operatorname{gsr} C(X)$ is significantly more difficult. No general facts are known, so let us state, somewhat loosely, the following:

Problem. Give conditions on a compact space X which allow for an explicit computation of $\operatorname{gsr} C(X)$.

The following result seems to be the first non-trivial computation of general stable rank:

Proposition IV.3.4. *The general stable rank of $C(S^d)$ is given as follows:*

$$\operatorname{gsr} C(S^d) = \begin{cases} \lceil d/2 \rceil + 1 & \text{if } d > 4 \text{ and } d \notin 4\mathbb{Z} \\ \lceil d/2 \rceil & \text{if } d > 4 \text{ and } d \in 4\mathbb{Z} \\ 1 & \text{if } d \leq 4 \end{cases}$$

Proof. We need two facts:

- Let X be a compact space. According to [71, Prop.10.5], $\operatorname{gsr} C(X)$ is the least positive integer n with the following property: for all $m \geq n$, if W is a left $C(X)$-module with $W \oplus C(X) \simeq C(X)^m$ then $W \simeq C(X)^{m-1}$. Via the Serre-Swan dictionary, we can translate this module-theoretic characterization of the general stable rank into a geometric one about complex vector bundles. Namely, $\operatorname{gsr} C(X)$ is the least positive integer n with the following property: for all $m \geq n$, if E is an $(m-1)$-dimensional vector bundle over X which is trivialized by adding a line bundle over X, then E is trivial.

- Let X be, again, a compact space. Then there is a bijective correspondence

$$\big[\mathrm{Vect}_n(SX)\big] \longleftrightarrow \big[X, \mathrm{GL}_n(\mathbb{C})\big]$$

between the isomorphism classes of n-dimensional complex vector bundles over the suspension SX and the homotopy classes of continuous maps $X \to \mathrm{GL}_n(\mathbb{C})$; see [44, p.36]. This correspondence is implemented by clutching. View SX as the union of two cones over X, denoted X_+ and X_-. On X_+, respectively X_-, take the trivial vector bundle $X_+ \times \mathbb{C}^n$, respectively $X_- \times \mathbb{C}^n$. We glue these trivial bundles along $X = X_+ \cap X_-$ by a continuous map $f : X \to \mathrm{GL}_n(\mathbb{C})$; specifically, the two copies of $\mathbb{C}^n$ above each $x \in X$ get identified by the linear isomorphism $f(x)$. We thus have an n-dimensional vector bundle E_f over SX for each continuous map $f : X \to \mathrm{GL}_n(\mathbb{C})$. Note that each n-dimensional vector bundle over SX arises (up to isomorphism) in this way. Indeed, as X_+ and X_- are contractible, every vector bundle over X restricts to trivial vector bundles over X_+ and X_-. Thus all that matters is the way these two trivial vector bundles fit together over X. Furthermore, the direct sum of clutching-type bundles behaves as expected [36, p.136]:

$$E_f \oplus E_g \simeq E_{\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)}$$

Therefore, if E_f is the n-dimensional bundle given by $f : X \to \mathrm{GL}_n(\mathbb{C})$ then E_f is trivial iff f vanishes in $\big[X, \mathrm{GL}_n(\mathbb{C})\big]$, and E_f is stably trivial iff f vanishes in $\big[X, \mathrm{GL}_m(\mathbb{C})\big]$ for some $m \geq n$. To put it differently, there is a bijective correspondence between non-zero elements in the kernel of $\big[X, \mathrm{GL}_n(\mathbb{C})\big] \to \big[X, \mathrm{GL}_{n+1}(\mathbb{C})\big]$, and n-dimensional vector bundles which are not trivial, but they become trivial after adding a line bundle.

Combining these two facts, we conclude that $\mathrm{gsr}\, C(SX)$ is the least positive integer n with the property that $\big[X, \mathrm{GL}_{m-1}(\mathbb{C})\big] \to \big[X, \mathrm{GL}_m(\mathbb{C})\big]$ is injective for all $m \geq n$.

We now let $X = S^*$ to be a sphere, and we recall that the unitary group $\mathrm{U}(n)$ is a retract of $\mathrm{GL}_n(\mathbb{C})$. Then $\mathrm{gsr}\, C(S^{*+1})$ is the least positive integer n for which $\pi_* \mathrm{U}(m-1) \to \pi_* \mathrm{U}(m)$ is injective for all $m \geq n$. We are thus led to investigating the injectivity of the connecting maps in the (π_*) sequence of the unitary groups.

The long exact homotopy sequence associated to the fibration $\mathrm{U}(n) \to \mathrm{U}(n+1) \to S^{2n+1}$ yields that $\pi_* \mathrm{U}(n) \to \pi_* \mathrm{U}(n+1)$ is bijective for $n > */2$. When $* \leq 3$, one easily checks that all maps in the (π_*) sequence are injective. Therefore $\mathrm{gsr}\, C(S^d) = 1$ for $d \leq 4$. (This fact is actually true in general; see item 5) in Remark IV.10.3.)

Assume $* \geq 4$. In order to see what happens right before the stable range $n > */2$, we use some computations of homotopy groups of unitary groups as tabulated in [51, p.254]. We split the analysis according to the parity of $*$:

· (even $*$) Put $* = 2k$ with $k \geq 2$. The homotopy groups $\big\{\pi_{2k} \mathrm{U}(n)\big\}_{n \geq 1}$ stabilize starting from $\pi_{2k} \mathrm{U}(k+1)$, and $\pi_{2k} \mathrm{U}(k+1) \simeq \pi_{2k} \mathrm{U}(\infty) \simeq 0$ by Bott periodicity. The last unstable group is $\pi_{2k} \mathrm{U}(k) \simeq \mathbb{Z}_{k!}$, so the map $\pi_{2k}\mathrm{U}(k) \to \pi_{2k}\mathrm{U}(k+1)$ is not injective. Thus $\mathrm{gsr}\, C(S^{2k+1}) = k+2$ for $k \geq 2$.

$\cdot$ (odd $*$) Put $* = 2k+1$ with $k \geq 2$. The homotopy groups $\{\pi_{2k+1}\mathrm{U}(n)\}_{n\geq 1}$ stabilize starting from $\pi_{2k+1}\mathrm{U}(k+1)$, and $\pi_{2k+1}\mathrm{U}(k+1) \simeq \pi_{2k+1}\mathrm{U}(\infty) \simeq \mathbb{Z}$ by Bott periodicity. The last unstable group $\pi_{2k+1}\mathrm{U}(k)$ is given as follows:

$$\pi_{2k+1}\mathrm{U}(k) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

If k is even, then $\pi_{2k+1}\mathrm{U}(k) \to \pi_{2k+1}\mathrm{U}(k+1)$ is not injective. Thus $\mathrm{gsr}\, C(S^{2k+2}) = k+2$ for even $k \geq 2$. For k odd, we must look at the map $\pi_{2k+1}\mathrm{U}(k-1) \to \pi_{2k+1}\mathrm{U}(k)$ in order to see the failure of injectivity. Indeed, $\pi_{2k+1}\mathrm{U}(k-1)$ has a cyclic group of order $\gcd(k-1, 8)$ (which is not 1, since k is odd) as a direct summand. Thus $\mathrm{gsr}\, C(S^{2k+2}) = k+1$ for odd $k \geq 2$. $\square$

Remark IV.3.5. A related, but incomplete, discussion along these lines is carried out by Sheu [78, p. 369-370]; in particular, the previous proposition confirms his conjectural Remark [78, p.370]. Let us note that the homotopical information used in the proof of Proposition IV.3.4 quickly leads to an exact computation of the constant C_G that appears in the main theorem (5.4) of [78].

The next step would be a computation of $\mathrm{gsr}\,(T^d)$, where T^d stand for the d-dimensional torus. To get a sense of why this is much more challenging than the computation of $\mathrm{gsr}\,(S^d)$, one need only glance at the proof of Packer and Rieffel [65] that $\mathrm{gsr}\,(T^5) > 1$.

IV.4 Matrix algebras

Theorem IV.4.1. *Let A be a good topological algebra. Then:*

$$\mathrm{tsr}\,\mathrm{M}_n(A) = \left\lceil \frac{\mathrm{tsr}\,A - 1}{n} \right\rceil + 1 \qquad \mathrm{csr}\,\mathrm{M}_n(A) \leq \left\lceil \frac{\mathrm{csr}\,A - 1}{n} \right\rceil + 1$$

$$\mathrm{bsr}\,\mathrm{M}_n(A) = \left\lceil \frac{\mathrm{bsr}\,A - 1}{n} \right\rceil + 1 \qquad \mathrm{gsr}\,\mathrm{M}_n(A) \leq \left\lceil \frac{\mathrm{gsr}\,A - 1}{n} \right\rceil + 1$$

This was proved by Vaserstein [87, Thm.3] for the Bass stable rank, by Rieffel [71, Thm.6.1] for the topological stable rank, and by Nistor [63, Prop.2.10] and Rieffel [72, Thm.4.7] for the connected stable rank. Corollary 11.5.13 of [52] proves the general stable rank estimate (note the dimension shift: gsr as defined in [52] equals $\mathrm{gsr} - 1$ as defined here).

We record the following important consequence:

Corollary IV.4.2. *If* $\mathrm{sr}\,A = 1$ *then* $\mathrm{sr}\,\mathrm{M}_n(A) = 1$.

IV.5 Relations between stable ranks

Theorem IV.5.1. *Over good topological algebras, we have:*

$$\mathrm{gsr} - 1 \leq \mathrm{csr} - 1 \leq \mathrm{bsr} \leq \mathrm{tsr}$$

This is due, quite predictably, to Rieffel [71]; he works with Banach algebras, but the proof works in fact for good topological algebras. The last three inequalities are implicit in Corach - Larotonda [18]. We include the proof, as an exercise in definitions.

Proof. Let A be a good topological algebra. Without loss of generality, we may assume A is unital.

The inequality $\operatorname{gsr} A \leq \operatorname{csr} A$ clearly follows from the equivalent definition for the connected stable rank given in Remark IV.2.3.

We prove that $\operatorname{csr} A \leq \operatorname{bsr} A + 1$. Letting $m \geq \operatorname{bsr} A$, we need to show that $\mathrm{Lg}_{m+1}(A)$ is path-connected. Let $(a_1, \ldots, a_{m+1}) \in \mathrm{Lg}_{m+1}(A)$, so $(a_1 + x_1 a_{m+1}, \ldots, a_m + x_{m+1} a_{m+1}) \in \mathrm{Lg}_m(A)$ for some $x_1, \ldots, x_m \in A$. The path

$$t \mapsto (a_1 + t x_1 a_{m+1}, \ldots, a_m + t x_{m+1} a_{m+1}, (1-t) a_{m+1})$$

connects $(a_1, \ldots, a_{m+1})$ to $(a_1 + x_1 a_{m+1}, \ldots, a_m + x_{m+1} a_{m+1}, 0)$, and lies in $\mathrm{Lg}_{m+1}(A)$. Indeed, we have $\sum c_i (a_i + t x_i a_{m+1}) + \left(\sum c_i x_i\right)(1-t) a_{m+1} = 1$ whenever $\sum c_i (a_i + x_i a_{m+1}) = 1$.

Next one connects any element of $\mathrm{Lg}_{m+1}(A)$ of the form $(a'_1, \ldots, a'_m, 0)$ to $(0, \ldots, 0, 1)$ via

$$t \mapsto (t a'_1, \ldots, t a'_m, 1-t).$$

This path lies in $\mathrm{Lg}_{m+1}(A)$ as $\sum c'_i (t a'_i) + (1-t) = 1$ whenever $\sum c'_i a'_i = 1$. We have thus connected an arbitrary $(a_1, \ldots, a_{m+1})$ to $(0, \ldots, 0, 1)$ in $\mathrm{Lg}_{m+1}(A)$.

Alternatively, we can prove $\operatorname{csr} A \leq \operatorname{bsr} A + 1$ by showing that $\mathrm{GL}^0_{m+1}(A)$ acts transitively on $\mathrm{Lg}_{m+1}(A)$ for $m \geq \operatorname{bsr} A$. It suffices to show that each $(a_1, \ldots, a_{m+1}) \in \mathrm{Lg}_{m+1}(A)$ is taken by a product of elementary matrices to the last basis vector $(0, \ldots, 0, 1) \in \mathrm{Lg}_{m+1}(A)$. Since $m \geq \operatorname{bsr} A$, we have $(a_i + x_i a_{m+1}) \in \mathrm{Lg}_m(A)$ for some $(x_i) \in A^m$. We first have:

$$\begin{pmatrix} 1 & & & x_1 \\ & \ddots & & \vdots \\ & & 1 & x_m \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_m \\ a_{m+1} \end{pmatrix} = \begin{pmatrix} a_1 + x_1 a_{m+1} \\ \vdots \\ a_m + x_m a_{m+1} \\ a_{m+1} \end{pmatrix}$$

Let $(c_i) \in A^m$ such that $\sum c_i (a_i + x_i a_{m+1}) = 1$ and put $c'_i = (1 - a_{m+1}) c_i$. Then:

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ c'_1 & \ldots & c'_m & 1 \end{pmatrix} \begin{pmatrix} a_1 + x_1 a_{m+1} \\ \vdots \\ a_m + x_m a_{m+1} \\ a_{m+1} \end{pmatrix} = \begin{pmatrix} a_1 + x_1 a_{m+1} \\ \vdots \\ a_m + x_m a_{m+1} \\ 1 \end{pmatrix}$$

Now one uses the 1 in the last entry to turn all the other entries of the right-hand side column to 0, thereby reaching $(0, \ldots, 0, 1)$.

We prove that $\operatorname{bsr} A \leq \operatorname{tsr} A$. We actually show that the (left) topological stable rank is no less than the right Bass stable rank, and then we invoke the left-right symmetry of the Bass stable rank. Put $\operatorname{tsr} A = n$. Let

$(a_i) \in \mathrm{Rg}_{n+1}(A)$, so $\sum a_i d_i = 1$ for some $(d_i) \in A^{n+1}$. In A^n, the open set $\{(x_1,\dots,x_n) : a_1x_1 + \cdots + a_nx_n + a_{n+1}d_{n+1} \in A^\times\}$ is nonempty, for it contains $(d_1,\dots,d_n)$, so it meets the dense subset $\mathrm{Lg}_n(A)$. Let $(x_1,\dots,x_n)$ be a common point and pick $(c_i) \in A^n$ with $\sum c_i x_i = 1$. As $\sum(a_i + a_{n+1}d_{n+1}c_i)x_i = \sum a_i x_i + a_{n+1}d_{n+1} \in A^\times$, we get $(a_i + a_{n+1}d_{n+1}c_i) \in \mathrm{Rg}_n(A)$. We conclude that the right Bass stable rank is at most n. $\square$

Herman and Vaserstein [32] have shown that, in the C*-case, there is no distinction between the Bass and the topological stable rank:

Theorem IV.5.2. *If A is a* C**-algebra then* $\mathrm{bsr}\,A = \mathrm{tsr}\,A$.

For Banach algebras, this is not true in general:

[[41], [21]] Let $A(D)$ be the disk algebra, that is, the closed subalgebra of $C(D)$ consisting of the functions that are holomorphic in the interior of D. Then $\mathrm{bsr}\,A(D) = 1$ and $\mathrm{tsr}\,A(D) = 2$.

We now provide examples of C*-algebras having all stable ranks infinite. To that end, it suffices to show that the general stable rank is infinite. The following lemma is essentially [29, Prop.1.4] (compare [71, Prop.6.5]):

Lemma IV.5.3. *Let A be a unital* C**-algebra containing $n \geq 2$ isometries $(s_i)_{1\leq i\leq n}$ with $\sum_1^n s_i s_i^* = 1$. Then* $\mathrm{gsr}\,A = \infty$.

Proof. Assume $\mathrm{gsr}\,A$ is finite and put $N = \max\{n, \mathrm{gsr}\,A\}$. In A, one can manufacture N isometries $(s_i)_{1\leq i\leq N}$ with $\sum_1^N s_i s_i^* = 1$; note that the isometries s_i are automatically orthogonal, i.e., $s_i^* s_j = 0$ for $i \neq j$. Since $(s_i^*)_{1\leq i\leq N} \in \mathrm{Lg}_N(A)$, there is $\alpha \in \mathrm{GL}_N(A)$ such that

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1N} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2N} \\ \vdots & \vdots & & \vdots \\ \alpha_{N1} & \alpha_{N2} & \dots & \alpha_{NN} \end{pmatrix} \begin{pmatrix} s_1^* \\ s_2^* \\ \vdots \\ s_N^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In particular, $\sum_1^N \alpha_{2i} s_i^* = 1$. Right multiplying by s_j leads to $\alpha_{2j} = 0$ for all j, contradicting the fact that α is invertible. $\square$

The following C*-algebras have all their stable ranks (Bass, topological, connected, or general) infinite:

· $\mathcal{B}$, the C*-algebra of bounded operators on a separable, infinite dimensional Hilbert space;

· the Calkin algebra $\mathcal{Q}$ (the quotient of $\mathcal{B}$ by the ideal $\mathcal{K}$ of compact operators);

· the Cuntz algebra $\mathcal{O}_n$ (for $2 \leq n < \infty$, $\mathcal{O}_n$ is the C*-algebra generated by n isometries $(s_i)_{1\leq i\leq n}$ subject to $\sum_1^n s_i s_i^* = 1$; this definition is independent of the choice of isometries).

Remark IV.5.4. For the Cuntz algebra $\mathcal{O}_\infty$ (the C*-algebra generated by isometries $(s_i)_{1\leq i<\infty}$ subject to $\sum_1^\infty s_i s_i^* = 1$, where - again - the definition is independent of the choice of isometries), we have $\mathrm{tsr}\,\mathcal{O}_\infty = \infty$ and $\mathrm{csr}\,\mathcal{O}_\infty = \mathrm{gsr}\,\mathcal{O}_\infty = 2$.

The fact that $\mathrm{tsr}\,\mathcal{O}_\infty = \infty$ is an application of Proposition 6.5 of [71], which states the following variant of Lemma IV.5.3: if a unital C*-algebra A contains two orthogonal isometries, then $\mathrm{tsr}\,A = \infty$.

That csr $O_\infty = 2$ is due to Xue [90]. In fact, Xue shows the following: a purely infinite, simple unital C*-algebra has connected stable rank 2 or ∞ according to whether the order of $[1]$ in the K_0 group is infinite or finite. Here $K_0(O_\infty) \simeq \mathbb{Z}$ with generator $[1]$, whereas for finite $n \geq 2$ we have $K_0(O_n) \simeq \mathbb{Z}/_{n-1}$ with generator $[1]$.

As O_∞ is infinite, csr $O_\infty = 2$ implies that gsr $O_\infty = 2$.

IV.6 Dense morphisms and quotients

If $\phi : A \to B$ is a morphism then, abusing notation, we let $\phi : A^n \to B^n$ be the obvious map. Clearly $\phi(\mathrm{Lg}_n(A)) \subseteq \mathrm{Lg}_n(B)$. A morphism $\phi : A \to B$ is *dense* if $\phi(A)$ is dense in B. Although they mean the same thing in the C*-context, dense morphisms are more natural than onto morphisms for Banach and Fréchet algebras.

Proposition IV.6.1. *Let $\phi : A \to B$ be a dense morphism. Then:*

$$\operatorname{tsr} B \leq \operatorname{tsr} A \qquad\qquad \operatorname{csr} B \leq \max\{\operatorname{csr} A, \operatorname{tsr} A\}$$
$$\operatorname{gsr} B \leq \max\{\operatorname{gsr} A, \operatorname{tsr} A\}$$

The tsr estimate is folklore; the other two are new.

Proof. If $\mathrm{Lg}_n(A)$ is dense in A^n, then $\phi(\mathrm{Lg}_n(A))$ is dense in $\phi(A^n)$ which in turn is dense in B^n. As $\phi(\mathrm{Lg}_n(A)) \subseteq \mathrm{Lg}_n(B)$, it follows that $\mathrm{Lg}_n(B)$ is dense in B^n. This proves that $\operatorname{tsr} B \leq \operatorname{tsr} A$.

Let $m \geq \max\{\operatorname{csr} A, \operatorname{tsr} A\}$. As $m \geq \operatorname{tsr} A$, we have $\phi(\mathrm{Lg}_m(A))$ dense in $\mathrm{Lg}_m(B)$ as above. Since $m \geq \operatorname{csr} A$, $\mathrm{Lg}_m(A)$ is connected and so $\phi(\mathrm{Lg}_m(A))$ is connected. It follows that $\mathrm{Lg}_m(B)$ is connected, since it contains a dense connected subset. We conclude that $\operatorname{csr} B \leq \max\{\operatorname{csr} A, \operatorname{tsr} A\}$.

Alternatively, we can show that the $\mathrm{GL}_m(B)_0$-action on $\mathrm{Lg}_m(B)$ is transitive. Let $(b_i) \in \mathrm{Lg}_m(B)$, and pick $(a_i) \in \mathrm{Lg}_m(A)$ such that $\phi\big((a_i)\big)$ is in the $\mathrm{GL}_m(B)_0$-orbit of (b_i). Since $m \geq \operatorname{csr} A$, there is $\alpha \in \mathrm{GL}_m(A)_0$ taking $(0,\dots,0,1) \in A^m$ to (a_i). Then $\phi(\alpha) \in \mathrm{GL}_m(B)_0$ takes $(0,\dots,0,1) \in B^m$ to $\phi\big((a_i)\big)$. Therefore $(b_i) \in \mathrm{Lg}_m(B)$ is in the $\mathrm{GL}_m(B)_0$-orbit of $(0,\dots,0,1) \in B^m$. We conclude that $\operatorname{csr} B \leq \max\{\operatorname{csr} A, \operatorname{tsr} A\}$.

Although slightly longer, the second argument has the advantage of being easily adaptable so as to yield the general stable rank estimate. To spell it out, let $m \geq \max\{\operatorname{gsr} A, \operatorname{tsr} A\}$. As $m \geq \operatorname{tsr} A$, we have $\phi(\mathrm{Lg}_m(A))$ dense in $\mathrm{Lg}_m(B)$. As before, let $(b_i) \in \mathrm{Lg}_m(B)$, and pick $(a_i) \in \mathrm{Lg}_m(A)$ such that $\phi\big((a_i)\big)$ is in the $\mathrm{GL}_m(B)_0$-orbit of (b_i). Since $m \geq \operatorname{gsr} A$, there is $\alpha \in \mathrm{GL}_m(A)$ taking the last basis vector $(0,\dots,0,1) \in A^m$ to (a_i). Then $\phi(\alpha) \in \mathrm{GL}_m(B)$ takes $(0,\dots,0,1) \in B^m$ to $\phi\big((a_i)\big)$. Therefore $(b_i) \in \mathrm{Lg}_m(B)$ is in the $\mathrm{GL}_m(B)$-orbit of $(0,\dots,0,1) \in B^m$. Therefore $\operatorname{gsr} B \leq \max\{\operatorname{gsr} A, \operatorname{tsr} A\}$. $\square$

If morphisms are onto rather than dense, one can say a bit more. The next result is due to Vaserstein [87, Thm.7] for bsr, to Rieffel [71, Thm.4.3] for tsr, and to Elhage-Hassan [29, Thm.1.1] for csr; the gsr estimate is new. Actually, the statement concerning the topological stable rank is the one from Proposition IV.6.1; we include it for the sake of completeness.

Proposition IV.6.2. *Let $\phi : A \to B$ be an onto morphism. Then:*

$$\operatorname{tsr} B \leq \operatorname{tsr} A \qquad \operatorname{csr} B \leq \max\{\operatorname{csr} A, \operatorname{bsr} A\}$$
$$\operatorname{bsr} B \leq \operatorname{bsr} A \qquad \operatorname{gsr} B \leq \max\{\operatorname{gsr} A, \operatorname{bsr} A\}$$

Proof. We use the following fact ([7, Lem.4.1]): if $\phi : A \to B$ is onto, then $\phi : \mathrm{Lg}_n(A) \to \mathrm{Lg}_n(B)$ is onto for $n \geq \operatorname{bsr} A$.

Indeed, let $(b_i) \in \mathrm{Lg}_n(B)$ and let $(a_i) \in A^n$ with $\phi(a_i) = b_i$. If $\sum b_i' b_i = 1$ and we set $\phi(a_i') = b_i'$, where $(a_i') \in A^n$, then $\phi(\sum a_i' a_i) = 1$. In other words, $\sum a_i' a_i = 1 + k$ for some $k \in \ker\phi$. Since $(a_1, \ldots, a_n, k) \in \mathrm{Lg}_{n+1}(A)$ and $n \geq \operatorname{bsr} A$, we get $(a_i + c_i k) \in \mathrm{Lg}_n(A)$ for some $(c_i) \in A^n$. Now ϕ maps $(a_i + c_i k)$ to (b_i).

We look at the stable rank estimates one by one.

· [bsr] Put $n = \operatorname{bsr} A$. Let $(b_i) \in \mathrm{Lg}_{n+1}(B)$, so $\phi(a_i) = b_i$ for some $(a_i) \in \mathrm{Lg}_{n+1}(A)$. Reduce to $(a_i + c_i a_{n+1}) \in \mathrm{Lg}_n(A)$, which maps to $(b_i + \phi(c_i) b_{n+1}) \in \mathrm{Lg}_n(B)$ via ϕ. Thus $n \geq \operatorname{bsr} B$.

· [csr] Let $m \geq \max\{\operatorname{csr} A, \operatorname{bsr} A\}$. Since $\mathrm{Lg}_m(A)$ connected and $\mathrm{Lg}_m(A)$ maps onto $\mathrm{Lg}_m(B)$, we have that $\mathrm{Lg}_m(B)$ is connected. Thus $\operatorname{csr} B \leq \max\{\operatorname{csr} A, \operatorname{bsr} A\}$.

· [gsr] Let $m \geq \max\{\operatorname{gsr} A, \operatorname{bsr} A\}$. We show that $\mathrm{GL}_m(B)$ acts transitively on $\mathrm{Lg}_m(B)$. Let $(b_i), (b_i') \in \mathrm{Lg}_m(B)$, and let $(a_i), (a_i') \in \mathrm{Lg}_m(A)$ be lifts through ϕ. There is $\alpha \in \mathrm{GL}_m(A)$ such that $\alpha \cdot (a_i)^T = (a_i')^T$, which leads to $\phi(\alpha) \cdot (a_i)^T = (a_i')^T$ with $\phi(\alpha) \in \mathrm{GL}_m(B)$. We conclude that $\operatorname{gsr} B \leq \max\{\operatorname{gsr} A, \operatorname{bsr} A\}$. □

Remark IV.6.3. If $\phi : A \to B$ is onto, it does not follow that $\operatorname{csr} B \leq \operatorname{csr} A$, or that $\operatorname{gsr} B \leq \operatorname{gsr} A$. Consider, for example, the epimorphism $C(I^{d+1}) \twoheadrightarrow C(S^d)$ induced by the inclusion of $S^d = \partial I^{d+1}$ into I^{d+1}. Here $\operatorname{csr} C(I^{d+1}) = 1$, as I^{d+1} is contractible, hence $\operatorname{gsr} C(I^{d+1}) = 1$ as well. On the other hand, both $\operatorname{csr} C(S^d)$ and $\operatorname{gsr} C(S^d)$ are on the order of d.

Comparing Prop. IV.6.1 and Prop. IV.6.2, the following question arises: if $\phi : A \to B$ is a dense morphism, is $\operatorname{bsr} A \geq \operatorname{bsr} B$?

IV.7 Direct products. Direct limits

Proposition IV.7.1. $\mathrm{sr}\,(A \oplus B) = \max\{\mathrm{sr}\,(A), \mathrm{sr}\,(B)\}$.

Proof. We have $\mathrm{Lg}_n(A \oplus B) = \mathrm{Lg}_n(A) \oplus \mathrm{Lg}_n(B)$ under the identification of $(A \oplus B)^n$ with $A^n \oplus B^n$. The rest is an exercise in definitions. □

If A is a finite dimensional C*-algebra, then $\operatorname{sr} A = 1$.

Indeed, a finite dimensional C*-algebra is C*-isomorphic to a direct sum of matrix algebras over $\mathbb{C}$, and each such matrix algebra has sr $= 1$.

Proposition IV.7.2. $\mathrm{sr}\,(\varinjlim A_i) \leq \liminf\ \mathrm{sr}\,(A_i)$.

[AF-algebras] In the C* category, an inductive limit of finite dimensional algebras is called an *approximately finite dimensional algebra*, abbreviated to *AF-algebra*. A distinguished example of an AF-algebra is $\mathcal{K}$, the C*-algebra of compact operators on a separable, infinite dimensional Hilbert space.

If A is an AF-algebra then $\operatorname{sr} A = 1$.

IV.8 Inverse limits

In this section, we compute compute stable ranks for Fréchet algebras from their Arens - Michael realization as inverse limits of Banach algebras with dense connecting morphisms. For terminology and basic facts, we refer to Section VIII.8.

Let A_∞ be a Fréchet algebra arising as the inverse limit of an inverse system

$$\dots \longrightarrow A_{k+1} \xrightarrow{\theta_k} A_k \longrightarrow \dots \longrightarrow A_2 \xrightarrow{\theta_1} A_1 \xrightarrow{\theta_0} A_0$$

where A_k are Banach algebras, and the connecting morphisms θ_k have dense image.

Proposition IV.8.1. *If A_∞ is a good Fréchet algebra, then* $\operatorname{tsr} A_\infty = \sup_k(\operatorname{tsr} A_k)$.

Proof. Since the connecting morphisms $\theta_k : A_{k+1} \to A_k$ and the coordinate morphisms $\pi_k : A_\infty \to A_k$ are dense, Proposition IV.6.1 yields the following estimates:

$$\operatorname{tsr} A_0 \leq \operatorname{tsr} A_1 \leq \cdots \leq \operatorname{tsr} A_\infty$$

If $\sup_k(\operatorname{tsr} A_k) = \infty$, there is nothing to prove. Let us assume $\sup_k(\operatorname{tsr} A_k) = n$. Then $\operatorname{tsr} A_k = n$ for large enough k, so, up to discarding a finite initial segment from the inverse system $(A_k)_{k\geq 0}$, we only have to prove the following particular case:

Claim: if $\operatorname{tsr} A_k = n$ for all k, then $\operatorname{tsr} A_\infty = n$.

We need to prove that $\operatorname{tsr} A_\infty \leq n$, i.e., that $\mathrm{Lg}_n(A_\infty)$ is dense in A_∞^n. By [2, Thm. 4.2], we have $\mathrm{Lg}_n(A_\infty) = \bigcap \pi_k^{-1}(\mathrm{Lg}_n(A_k))$. Putting

$$W_k = \pi_k^{-1}(\mathrm{Lg}_n(A_k))$$

we have that each W_k is open in A_∞^n. We claim that each W_k is dense in A_∞^n. Then the Baire Category Theorem will tell us that $\mathrm{Lg}_n(A_\infty)$ is dense in A_∞^n. Instead of checking that each W_k is dense in A_∞^n, we check that $\pi_j(W_k)$ is dense in A_j^n for all j,k (see Corollary VIII.8.4).

For $j = k$, we have $\pi_k(W_k) = \pi_k(A_\infty^n) \cap \mathrm{Lg}_n(A_k)$ dense in A_k^n because $\pi_k(A_\infty^n)$ is dense and $\mathrm{Lg}_n(A_k)$ is open and dense.

Let $j > k$. We claim that $W_j \subseteq W_k$. Indeed, if $w_j \in W_j$ then $\pi_j(w_j) \in \mathrm{Lg}_n(A_j)$; now $\pi_k(w_j) = \theta_k \dots \theta_{j-1}\pi_j(w_j) \subseteq \mathrm{Lg}_n(A_k)$, so $w_j \in W_k$ as desired. Thus $\pi_j(W_j) \subseteq \pi_j(W_k)$, so $\pi_j(W_k)$ is dense in $\mathrm{Lg}_n(A_j)$ because we already know that $\pi_j(W_j)$ is dense in A_j^n.

Let $j < k$. As $\pi_k(W_k)$ is dense in A_k^n, we get $\theta_j \dots \theta_{k-1}\pi_k(W_k) = \pi_j(W_k)$ dense in A_j^n. $\square$

The previous proposition seems to be the first result about stable ranks of inverse limits. It is very likely that corresponding statements can be proved for the other stable ranks.

IV.9 Homotopy invariance of the connected and the general stable ranks

Let A, B be Banach algebras. Two morphisms $\phi_0, \phi_1 : A \to B$ are *homotopic* if they are the endpoints of a path of morphisms $\{\phi_t\}_{0\leq t\leq 1} : A \to B$, where the continuity of $t \mapsto \phi_t$ is in the pointwise sense, namely

$t \mapsto \phi_t(a)$ is continuous for each $a \in A$. If there are morphisms $\alpha : A \to B$ and $\beta : B \to A$ with $\beta\alpha$ homotopic to id_A and $\alpha\beta$ homotopic to id_B, then A and B are said to be *homotopy equivalent*. This notion generalizes the usual homotopy equivalence: two compact spaces X and Y are homotopy equivalent (as spaces) if and only if $C(X)$ and $C(Y)$ are homotopy equivalent (as algebras).

It is clear that the topological and the Bass stable ranks are not homotopy invariant. On the other hand, the connected and the general stable ranks are homotopy invariant. For the connected stable rank, the following result is due to Nistor [63]; for the general stable rank, it is new.

Proposition IV.9.1. *Let A and B be Banach algebras. If A and B are homotopy equivalent then* $\mathrm{csr}\, A = \mathrm{csr}\, B$ *and* $\mathrm{gsr}\, A = \mathrm{gsr}\, B$.

Proof. Assume that there exist morphisms $\alpha : A \to B$ and $\beta : B \to A$ with $\beta\alpha$ homotopic to id_A. If $(a_i) \in \mathrm{Lg}_m(A)$, then $(\beta\alpha)\big((a_i)\big)$ is in the same component as (a_i); in other words, in the same $\mathrm{GL}_m(A)_0$-orbit.

We show that $\mathrm{csr}\, A \leq \mathrm{csr}\, B$. Let $m \geq \mathrm{csr}\, B$, and pick $(a_i), (a'_i) \in \mathrm{Lg}_m(A)$. Then $\alpha\big((a_i)\big), \alpha\big((a'_i)\big)$ are in $\mathrm{Lg}_m(B)$, so there is $S_B \in \mathrm{GL}_m(B)_0$ taking $\alpha\big((a_i)\big)$ to $\alpha\big((a'_i)\big)$. Sending this through β, we get $\beta(S_B) \in \mathrm{GL}_m(A)_0$ taking $(\beta\alpha)\big((a_i)\big)$ to $(\beta\alpha)\big((a'_i)\big)$. We conclude that (a_i) and (a'_i) are in the same $\mathrm{GL}_m(A)_0$-orbit.

We show that $\mathrm{gsr}\, A \leq \mathrm{gsr}\, B$. Let $m \geq \mathrm{gsr}\, B$, and pick $(a_i), (a'_i) \in \mathrm{Lg}_m(A)$. The argument runs just like the one for csr, except that we get some $T_B \in \mathrm{GL}_m(B)$, rather than some $S_B \in \mathrm{GL}_m(B)_0$, taking $\alpha\big((a_i)\big)$ to $\alpha\big((a'_i)\big)$. The conclusion is that (a_i) and (a'_i) are in the same $\mathrm{GL}_m(A)$-orbit. $\square$

Corollary IV.9.2. *Let X and Y be compact spaces. If X and Y are homotopy equivalent, then* $\mathrm{csr}\, C(X) = \mathrm{csr}\, C(Y)$ *and* $\mathrm{gsr}\, C(X) = \mathrm{gsr}\, C(Y)$.

For instance, if the compact space X is contractible then $\mathrm{csr}\, C(X) = \mathrm{gsr}\, C(X) = 1$, whereas $\mathrm{tsr}\, C(X) = \mathrm{bsr}\, C(X)$ can be made arbitrarily large.

IV.10 The case of stable rank one

In this section, we look closely at the case when then stable ranks take on their smallest possible value, namely 1. This discussion is related to a notion of finiteness, which we now recall. A unital algebra A is *finite* if left-invertible implies invertible in A, equivalently, right-invertible implies invertible in A; we say that A is *stably finite* if each matrix algebra $\mathrm{M}_n(A)$ is finite.

Lemma IV.10.1. *Over good topological algebras, we have the following implications:*

$$\begin{array}{ccccc} \mathrm{tsr} = 1 & \Longrightarrow & \mathrm{bsr} = 1 & \Longrightarrow & \mathrm{gsr} = 1 \\ & & & & \Uparrow \\ & & & & \mathrm{csr} = 1 \end{array}$$

Proof. Let A be a good topological algebra; we may assume A is unital. Two of the implications, that $\mathrm{tsr} = 1$ implies $\mathrm{bsr} = 1$, and that $\mathrm{csr} = 1$ implies $\mathrm{gsr} = 1$, follow from Theorem IV.5.1. For the remaining implication, assume $\mathrm{bsr}\, A = 1$, so $\mathrm{gsr}\, A \leq 2$. In order to conclude that $\mathrm{gsr}\, A = 1$, we are left with arguing that $\mathrm{GL}_1(A) = A^\times$ acts transitively on $\mathrm{Lg}_1(A)$, i.e., that A is finite. Let $a \in A$ be left invertible; we show that a is invertible. Say $ba=1$ with $b\in A$. Then $ab+(1-ab)=1$ says that $(b,1-ab)\in \mathrm{Lg}_2(A)$. As $\mathrm{bsr}\,A=1$, we get $b+c(1-ab)\in \mathrm{Lg}_1(A)$ for some $c\in A$. But $\big(b+c(1-ab)\big)a=1$, so $a\big(b+c(1-ab)\big)=1$. This shows that a is right-invertible as well, therefore invertible. □

We have (compare [71, Prop.3.1] and [29, Prop.1.15]):

Proposition IV.10.2. *Let A be a unital, good topological algebra. If* $\mathrm{sr}\,A=1$ *then A is stably finite.*

Proof. It suffices to handle the case $\mathrm{gsr}\,A=1$. Then $\mathrm{GL}_1(A)=A^\times$ acts transitively on $\mathrm{Lg}_1(A)$, in other words A is finite. Since we actually have $\mathrm{gsr}\,\mathrm{M}_n(A)=1$ for each n, we conclude that A is stably finite. □

Remark IV.10.3. 1) In general, the implications in Lemma IV.10.1 cannot be reversed, and there is no general relationship between having tsr $=1$ or bsr $=1$, and having csr $=1$.

2) Having $\mathrm{tsr}\,A=1$ means that $A^\times$ is dense in A ([71, Prop.3.1]).

3) Having $\mathrm{csr}\,A=1$ implies that $K_1(A)=0$ ([29, Prop.1.15]). Indeed, if $\mathrm{csr}\,A=1$ then $\mathrm{Lg}_1(A)=A^\times$ is connected. Since $\mathrm{csr}\,\mathrm{M}_n(A)=1$ for each n, we obtain that $\mathrm{GL}_n(A)$ is connected for each n. Therefore $K_1(A)=0$.

In particular, if $\mathrm{tsr}\,A=1$ and $K_1(A)\neq 0$, then $\mathrm{csr}\,A=2$.

4) Having $\mathrm{gsr}\,A=1$ means that every unimodular row over A is the first row of an invertible matrix over A. This property appears under different names in the literature: in [49, p.27] it is termed the "Hermite" property, and in [24, Def.2.2] Davidson and Ji call it "complete finiteness" (we note, however, that both [49] and [24] work on the right). As showed in [49], $\mathrm{gsr}\,A=1$ if and only if every finitely generated, stably free left A-module is in fact free.

5) If A is a commutative Banach algebra and $\mathrm{csr}\,A\leq 3$, then $\mathrm{gsr}\,A=1$. In particular, for $d\leq 4$ we have $\mathrm{gsr}\,C(X^d)=1$.

Indeed, from $\mathrm{csr}\,A\leq 3$ we deduce that $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$ for $n\geq 3$. Also $\mathrm{GL}_2(A)$ acts transitively on $\mathrm{Lg}_2(A)$, because every 2-unimodular vector is the last column of a 2×2-matrix when A is commutative. Finally, $\mathrm{GL}_1(A)=A^\times$ acts transitively on $\mathrm{Lg}_1(A)=A^\times$. Summarizing, $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$ for $n\geq 1$, so $\mathrm{gsr}\,A=1$.

[Irrational rotation algebras] Let $\theta\in(0,1)$ be irrational. The irrational rotation algebra A_θ is the C*-algebra generated by unitaries u,v subject to $vu=\exp(2\pi i\theta)uv$; this definition is independent of the choice of unitaries. It is known that A_θ is simple, and $K_0(A_\theta)=K_1(A_\theta)=\mathbb{Z}^2$.

Putnam [68] has shown that $\mathrm{tsr}\,A_\theta=1$. Thus $\mathrm{csr}\,A_\theta=2$ by using the non-vanishing of K_1.

We have seen that $\mathrm{tsr}\,\mathrm{C}^*_\mathrm{r}(F_n)=1$ (Theorem III.5.4). It follows that $\mathrm{csr}\,\mathrm{C}^*_\mathrm{r}(F_n)=2$, for $K_1(\mathrm{C}^*_\mathrm{r}(F_n))=\mathbb{Z}^n$ as shown by Pimsner - Voiculescu.

More generally, $\mathrm{tsr}\,\mathrm{C}^*_\mathrm{r}(\Gamma)=1$ for any torsion-free, non-elementary hyperbolic group Γ.

IV.11 Swan's problem

Swan [82, p.206] implicitly raised the following

Problem. If $\phi: A\to B$ is a dense and spectral morphism, are the stable ranks of A and B equal?

Swan's problem should be viewed as a stable rank analogue of the Density Theorem in K-theory. In [82], Swan was working with the Bass stable rank and the projective stable rank; however, the above problem has since been considered for many other stable ranks (see, e.g., [5]).

The following lemma is useful in this context:

Lemma IV.11.1. *Let $\phi : A \to B$ be a spectral, dense morphism between good topological algebras. Let $(a_i) \in A^n$. Then $(a_i) \in \mathrm{Lg}_n(A)$ if and only if $(\phi(a_i)) \in \mathrm{Lg}_n(B)$.*

Furthermore, $\phi\big(\mathrm{Lg}_n(A)\big)$ is dense in $\mathrm{Lg}_n(B)$.

Proof. Let $(\phi(a_i)) \in \mathrm{Lg}_n(B)$. Thus $\sum_i b_i\phi(a_i) \in B^\times$ for some $(b_i) \in B^n$. As $\phi(A)$ is dense in B and $B^\times$ is open, we may assume $b_i = \phi(a_i')$ with $a_i' \in A$. Then $\phi\big(\sum a_i' a_i\big) \in B^\times$ which means, by spectrality, that $\sum a_i' a_i \in A^\times$, i.e., $(a_i) \in \mathrm{Lg}_n(A)$. The other implication is trivial.

For the second part, note that $\phi(A^n)$ is dense in B^n so $\phi(A^n) \cap \mathrm{Lg}_n(B)$ - which equals $\phi\big(\mathrm{Lg}_n(A)\big)$, according to the first part - is dense in $\mathrm{Lg}_n(B)$. □

Swan observed [82, Thm.2.2 c)] the following:

Proposition IV.11.2. *Let $\phi : A \to B$ be a spectral, dense morphism between good topological algebras. Then $\mathrm{bsr}\, B \geq \mathrm{bsr}\, A$.*

Proof. Let $n = \mathrm{bsr}\, B$. If $(a_i) \in \mathrm{Lg}_{n+1}(A)$ then $(\phi(a_i)) \in \mathrm{Lg}_{n+1}(B)$, so $(\phi(a_i) + z_i\phi(a_{n+1})) \in \mathrm{Lg}_n(B)$ for some $(z_i) \in B^n$. As ϕ is dense and $\mathrm{Lg}_n(B)$ is open, we may assume that $z_i = \phi(x_i)$ for some $x_i \in A$. Thus $(\phi(a_i + x_i a_{n+1})) \in \mathrm{Lg}_n(B)$, and Lemma IV.11.1 yields $(a_i + x_i a_{n+1}) \in \mathrm{Lg}_n(A)$. We conclude that $n \geq \mathrm{bsr}\, A$. □

Note that a positive answer to item 1) of Remark IV.6.3 would solve Swan's problem for the Bass stable rank. Note also that, if $A \to B$ is a spectral, dense morphism between Banach algebras, then

$$\mathrm{tsr}\, A \geq \mathrm{tsr}\, B \geq \mathrm{bsr}\, B \geq \mathrm{bsr}\, A$$

since the mere density of the morphism yields $\mathrm{tsr}\, A \geq \mathrm{tsr}\, B$ (Prop. IV.6.1), and $\mathrm{tsr}\, B \geq \mathrm{bsr}\, B$ holds generically. Thus, generalizations of the Herman-Vaserstein theorem (IV.5.2) to, say, smooth subalgebras of C^*-algebras would solve Swan's problem for both the topological and the Bass stable rank whenever A is such a subalgebra.

Swan's theorem for the Bass stable rank has been proved in certain cases. Vaserstein's foundational paper [87] already contains a result in this direction. In our language, Theorem 7 in [87] reads as follows:

Theorem IV.11.3. *Let X be a compact space. If A is a spectral, dense subalgebra of $C(X)$, then $\mathrm{bsr}\,(A) = \mathrm{bsr}\, C(X)$.*

Perhaps more notable is the following result of Badea [5, Thm.1.1, Cor.4.10], which solves Swan's problem for the Bass stable rank for a natural class of dense and spectral inclusions:

Theorem IV.11.4. *Let A be a dense and spectral Fréchet $*$-subalgebra of a C^*-algebra B. If A is closed under C^∞-functional calculus for self-adjoint elements, then $\mathrm{bsr}\, A = \mathrm{bsr}\, B$.*

In particular, this applies to smooth subalgebras coming from derivations (cf. Theorem II.2.1). Let δ be a closed derivation on the C*-algebra B, and let $B(\delta^n) := B \cap \mathrm{dom}(\delta^n)$ for $1 \leq n \leq \infty$. If $B(\delta^n)$ is dense in B, then $B(\delta^n)$ is a dense and spectral Fréchet $*$-subalgebra of B which is closed under C^∞-functional calculus for self-adjoint elements; hence $\mathrm{bsr}\, B(\delta^n) = \mathrm{bsr}\, B$. For instance, $\mathrm{bsr}\, H^\infty\Gamma = \mathrm{bsr}\, \mathrm{C}^*_{\mathrm{r}}\Gamma$ for an RD group Γ.

Remark IV.11.5. Badea's results on the Swan problem for the topological stable rank [5, Thm.4.13, Cor. 4.14] have unnatural hypotheses. Badea's earlier paper [4] suffers from the same flaw, although not explicitly.

We consider now the connected stable rank and the general stable rank. For these, one can give a positive answer to Swan's problem in full generality:

Theorem IV.11.6. *Let $\phi : A \to B$ be a spectral, dense morphism between good topological algebras. Then:*
a) $\mathrm{gsr}\, A = \mathrm{gsr}\, B$*;*
b) $\mathrm{csr}\, A = \mathrm{csr}\, B$*, provided B is Fréchet.*

Proof. a) First, we show that $\mathrm{gsr}\, A \geq \mathrm{gsr}\, B$. The gsr part of the proof of Proposition IV.6.1 applies here almost verbatim. The point is to have $\phi\big(\mathrm{Lg}_m(A)\big)$ dense in $\mathrm{Lg}_m(B)$; in the proof of Proposition IV.6.1 this was guaranteed as soon as $m \geq \mathrm{tsr}\, A$, whereas here it holds for all m according to Lemma IV.11.1.

Second, we show that $\mathrm{gsr}\, B \geq \mathrm{gsr}\, A$. Let $m \geq \mathrm{gsr}\, B$. We claim that each unimodular m-vector over A appears as the last column of a matrix in $\mathrm{GL}_m(A)$; this means that $\mathrm{GL}_m(A)$ acts transitively on $\mathrm{Lg}_m(A)$, which then leads to $\mathrm{gsr}\, A \leq \mathrm{gsr}\, B$. Let $(a_i) \in \mathrm{Lg}_m(A)$. Then $\big(\phi(a_i)\big) \in \mathrm{Lg}_m(B)$, so - by the transitivity of the action of $\mathrm{GL}_m(B)$ on $\mathrm{Lg}_m(B)$ - there is a matrix $\beta \in \mathrm{GL}_m(B)$ having $\big(\phi(a_i)\big)$ as its last column. As A is dense in B, we can approximate the entries of β, except for the last column, so as to get a matrix β' which has all its entries in $\phi(A)$, has $\big(\phi(a_i)\big)$ as its last column, and β' still is invertible in $\mathrm{M}_m(B)$. Put $\beta' = \phi(\alpha)$, where $\alpha \in \mathrm{M}_m(A)$ has (a_i) as its last column. The inclusion $\mathrm{M}_m(A) \hookrightarrow \mathrm{M}_m(B)$ being spectral, we get $\alpha \in \mathrm{GL}_m(A)$. The claim is proved.

b) First, we show $\mathrm{csr}\, B \geq \mathrm{csr}\, A$. Let $m \geq \mathrm{csr}\, A$. Then $\mathrm{Lg}_m(A)$ is connected, so $\phi\big(\mathrm{Lg}_m(A)\big)$ is connected as well. But $\phi\big(\mathrm{Lg}_m(A)\big)$ is dense in $\mathrm{Lg}_m(B)$, so it follows that $\mathrm{Lg}_m(B)$ is connected. This argument is akin to the one used in the proof of Proposition IV.6.1.

Next, we show $\mathrm{csr}\, A \geq \mathrm{csr}\, B$. Let $m \geq \mathrm{csr}\, B$, and we claim $\mathrm{Lg}_m(A)$ is connected; it will then follow that $\mathrm{csr}\, A \geq \mathrm{csr}\, B$. Let $(a_i), (a'_i) \in \mathrm{Lg}_m(A)$. Since $\phi\big((a_i)\big), \phi\big((a'_i)\big) \in \mathrm{Lg}_m(B)$, and $\mathrm{Lg}_m(B)$ is path-connected, there is a path $p : [0,1] \to \mathrm{Lg}_m(B)$ such that $p(0) = \phi\big((a_i)\big)$ and $p(1) = \phi\big((a'_i)\big)$. We shall obtain a broken-line path connecting (a_i) to (a'_i) and lying entirely in $\mathrm{Lg}_m(A)$. For each $t \in [0,1]$, let V_t be an open, *convex* neighborhood of $p(t)$ contained in $\mathrm{Lg}_m(B)$. (Here we use the fact that B is Fréchet; for a general topological algebra, we only have star-convex local bases.) Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that $\{V_{t_j}\}_{0 \leq j \leq k}$ is an open cover of $p[0,1]$. Connectivity of $p[0,1]$ tells us that we can extract a sub-index set $0 = s_0 < s_1 < \cdots < s_l = 1$ such that $V_{s_{j-1}}$ meets V_{s_j} for $1 \leq j \leq l$. By the density of $\phi(A^m)$ in B^m, we can pick $(x_i^j) \in \mathrm{Lg}_m(A)$ for $0 \leq j \leq l+1$ such that $(x_i^0) = (a_i)$, $(x_i^{l+1}) = (a'_i)$, and $\phi\big((x_i^j)\big) \in V_{s_{j-1}} \cap V_{s_j}$ for $1 \leq j \leq l$. Let q_A be the broken-line from $(x_i^0) = (a_i)$ to $(x_i^{l+1}) = (a'_i)$ with successive vertices (x_i^j), and let q_B be the image of q_A through ϕ. Then q_B is a broken-line from $\phi\big((a_i)\big)$ to $\phi\big((a'_i)\big)$ with successive vertices $\phi\big((x_i^j)\big)$. Each line segment from $\phi\big((x_i^{j-1})\big)$ to $\phi\big((x_i^{j})\big)$ lies in the convex set $V_{s_{j-1}}$, which in turn lies in $\mathrm{Lg}_m(B)$. Thus q_B is contained in $\mathrm{Lg}_m(B)$. By Lemma IV.11.1, we conclude that q_A lies entirely in $\mathrm{Lg}_m(A)$. $\square$

Remark IV.11.7. For the connected stable rank part, a proof similar to the one in part a) can be given if one uses the definition involving the transitivity of the $\mathrm{GL}_m(\cdot)_0$-action on $\mathrm{Lg}_m(\cdot)$. However, one still needs the assumption that B is Fréchet at the point of replacing a $\beta \in \mathrm{GL}_m(B)_0$ by $\phi(\alpha)$ with $\alpha \in \mathrm{GL}_m(A)_0$. The approach in terms of the connectivity of $\mathrm{Lg}_m(\cdot)$ is more convenient for a subsequent generalization (Theorem VI.4.4).

Remark IV.11.8. The connected stable rank result from Theorem IV.11.6 significantly generalizes [5, Thm.4.15] by removing the commutativity assumption on the algebras.

The inclusion $\ell^1(\mathbb{Z}^d) \hookrightarrow \mathrm{C}^*_\mathrm{r}(\mathbb{Z}^d)$ is dense and spectral. We have

$$\operatorname{bsr} \ell^1(\mathbb{Z}^d) = \operatorname{bsr} \mathrm{C}^*_\mathrm{r}(\mathbb{Z}^d) \quad \big(= \operatorname{bsr} C(T^d) = \lfloor d/2 \rfloor + 1\big)$$

by Theorem IV.11.3. Note that the result on the connected stable rank follows from Theorem I.0.5 as well. Also,

$$\operatorname{csr} \ell^1(\mathbb{Z}^d) = \operatorname{csr} \mathrm{C}^*_\mathrm{r}(\mathbb{Z}^d) \quad \big(= \operatorname{csr} C(T^d) = \lceil d/2 \rceil + 1\big)$$
$$\operatorname{gsr} \ell^1(\mathbb{Z}^d) = \operatorname{gsr} \mathrm{C}^*_\mathrm{r}(\mathbb{Z}^d) \quad \big(= \operatorname{gsr} C(T^d) = ?????\big)$$

by Theorem IV.11.6. The topological stable rank of $\ell^1(\mathbb{Z}^d)$ is known ([66], [55]), and we have once again

$$\operatorname{tsr} \ell^1(\mathbb{Z}^d) = \operatorname{tsr} \mathrm{C}^*_\mathrm{r}(\mathbb{Z}^d) \quad \big(= \operatorname{tsr} C(T^d) = \lfloor d/2 \rfloor + 1\big).$$

IV.12 Extensions

Given a short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

we want to relate the stable ranks $\operatorname{sr} J$, $\operatorname{sr} A$, $\operatorname{sr} B$. Initially, we might hope that $\operatorname{sr} A = \max\{\operatorname{sr} J, \operatorname{sr} B\}$ but this turns to be false in general for each stable rank under consideration (for csr and gsr, see item 2 of Remark IV.6.3; for bsr and tsr, see Example IV.12 below). We are, however, guided by the above equality in what follows.

We have already bounded the stable ranks of B interms of the stable ranks of A in Proposition IV.6.2. The goal is to bound the stable ranks of J in terms of those of A, and the stable ranks of A in terms of those for J and B.

The setting for the results of this section is that of Banach algebras. In some of the results below, we need the closed ideal J to have a *bounded approximate identity*. Recall, a bounded approximate identity for J is a net (j_α) of elements of J having uniformly bounded norm, and such that $j_\alpha j \to j$ and $j j_\alpha \to j$ for all

$j \in J$. In the C*-setting, this is automatic: every closed ideal in a C*-algebras has a bounded approximate identity.

As far as stable ranks are concerned, there is no loss of generality if we assume that A is unital (else, just "unitize" A and B in the exact sequence). Let J^+ be the unital Banach subalgebra of A obtained by adjoining the unit of A to J. Note that the (closed) inclusion $J^+ \hookrightarrow A$ is spectral: if $(\lambda + j)a = 1$ for some a in A, then $\lambda \neq 0$ and $a = \lambda^{-1} - \lambda^{-1} ja \in J^+$.

First, we prove a lemma that will help us recognize unimodular vectors over J^+:

Lemma IV.12.1. *Assume J has an approximate identity. Then* $\mathrm{Lg}_n(J^+) = \mathrm{Lg}_n(A) \cap (J^+)^n$.

Proof. The inclusion "$\subseteq$" is obvious; we show the other inclusion. Let $(\lambda_i + j_i) \in (J^+)^n$, where $\lambda_i \in \mathbb{C}$ and $j_i \in J$, and assume $(\lambda_i + j_i) \in \mathrm{Lg}_n(A)$. Let $(a_i) \in A^n$ such that $\sum a_i(\lambda_i + j_i) = 1$. In particular, there is i_0 such that $\lambda_{i_0} \neq 0$ (else $\sum a_i(\lambda_i + j_i)$ lies in J).

Let $(j_\alpha) \subseteq J$ be an approximate identity for J. We look for $(a_i') \in (J^+)^n$ such that $\sum a_i'(\lambda_i + j_i)$ is close enough to 1 as to make it invertible in A. Since $\sum a_i'(\lambda_i + j_i) \in J^+$, it is actually invertible in J^+, allowing us to conclude that $(\lambda_i + j_i) \in \mathrm{Lg}_n(J^+)$ as desired.

Put

$$a_{i_0}' := a_{i_0} + \sum_{i \neq i_0} \frac{\lambda_i}{\lambda_{i_0}} a_i (1 - j_\alpha), \qquad a_i' := a_i j_\alpha \quad (i \neq i_0)$$

with α still to be chosen. Observe that $(a_i') \in (J^+)^n$. Indeed, for $i \neq i_0$ this is obvious, and we only have to check that $a_{i_0}' \in J^+$. From $\sum a_i(\lambda_i + j_i) = 1$ we deduce that $\sum \lambda_i a_i \in 1 + J$, so we obtain

$$a_{i_0} + \sum_{i \neq i_0} \frac{\lambda_i}{\lambda_{i_0}} a_i (1 - j_\alpha) \in \Big(a_{i_0} + \sum_{i \neq i_0} \frac{\lambda_i}{\lambda_{i_0}} a_i\Big) + J = \frac{1}{\lambda_{i_0}} \big(\sum \lambda_i a_i\big) + J \subseteq J^+.$$

On the other hand,

$$\begin{aligned}\sum a_i'(\lambda_i + j_i) &= \Big(a_{i_0} + \sum_{i \neq i_0} \frac{\lambda_i}{\lambda_{i_0}} a_i (1 - j_\alpha)\Big)(\lambda_{i_0} + j_{i_0}) + \sum_{i \neq i_0} (a_i j_\alpha)(\lambda_i + j_i) \\ &= \big(\sum \lambda_i a_i\big)\Big(1 + \frac{j_{i_0}}{\lambda_{i_0}}\Big) + \sum_{i \neq i_0} a_i j_\alpha \Big(j_i - \frac{\lambda_i}{\lambda_{i_0}} j_{i_0}\Big)\end{aligned}$$

which converges to

$$\big(\sum \lambda_i a_i\big)\Big(1 + \frac{j_{i_0}}{\lambda_{i_0}}\Big) + \sum_{i \neq i_0} a_i \Big(j_i - \frac{\lambda_i}{\lambda_{i_0}} j_{i_0}\Big) = a_{i_0}(\lambda_{i_0} + j_{i_0}) + \sum_{i \neq i_0} a_i(\lambda_i + j_i) = 1.$$

Thus, we pick α such that $\sum a_i'(\lambda_i + j_i)$ is invertible in A. This ends the proof. □

As expected, for the dimensional stable ranks (bsr and tsr) we can estimate the stable rank of J in terms of the stable rank of A. In the next proposition, the Bass stable rank result is due to Vaserstein [87, Thm.4], and it is a purely ring-theoretic fact. The topological stable rank result is due to Rieffel [71, Thm.4.4].

Proposition IV.12.2. *Let J be a closed ideal in A. Then* $\mathrm{bsr}\, J \leq \mathrm{bsr}\, A$. *If J has a bounded approximate identity, then* $\mathrm{tsr}\, J \leq \mathrm{tsr}\, A$.

Not surprisingly, for the homotopic stable ranks (csr and gsr) such a result is not true, that is, neither $\operatorname{csr} J \leq \operatorname{csr} A$ nor $\operatorname{gsr} J \leq \operatorname{gsr} A$ hold in general. Consider again the situation of Remark IV.6.3, namely the epimorphism $C(I^{d+1}) \twoheadrightarrow C(S^d)$ induced by the inclusion of $S^d = \partial I^{d+1}$ into I^{d+1}. We have the following exact sequence:

$$0 \longrightarrow C_0(I^{d+1} \setminus \partial I^{d+1}) \longrightarrow C(I^{d+1}) \longrightarrow C(S^d) \longrightarrow 0$$

As $C_0(I^{d+1} \setminus \partial I^{d+1})^+ = C(I^{d+1}/\partial I^{d+1}) = C(S^{d+1})$, we have that both $\operatorname{csr} C_0(I^{d+1} \setminus \partial I^{d+1})$ and $\operatorname{gsr} C_0(I^{d+1} \setminus \partial I^{d+1})$ are on the order of d, whereas $\operatorname{csr} C(I^{d+1}) = \operatorname{gsr} C(I^{d+1}) = 1$ since I^{d+1} is contractible.

Next, we estimate the stable rank of A in terms of the stable rank of J and the stable rank of B. The bsr estimate is due to Vaserstein [87, Thm.4], the tsr estimate to Rieffel [71, Thm.4.11], and the csr estimate is due to Nagy [60, Lem.2] and independently to Sheu [78, Thm.3.9]. We observe that a gsr estimate can be established in the same way.

Proposition IV.12.3. *Let $0 \to J \to A \to B \to 0$ be an exact sequence of Banach algebras. Then:*

$$\operatorname{tsr} A \leq \max\{\operatorname{tsr} J, \operatorname{tsr} B, \operatorname{csr} B\} \qquad \operatorname{csr} A \leq \max\{\operatorname{csr} J, \operatorname{csr} B\}$$

$$\operatorname{bsr} A \leq \max\{\operatorname{bsr} J, \operatorname{bsr} B + 1\} \qquad \operatorname{gsr} A \leq \max\{\operatorname{gsr} J, \operatorname{csr} B\}$$

For the csr *and* gsr *estimates, it is assumed that J has an approximate identity.*

Proof. We justify the csr and the gsr estimates. Let $\pi : A \to B$ denote the quotient map; as usual, we abuse notation and we denote by π all other maps naturally induced by π.

Let $m \geq \max\{\operatorname{csr} J, \operatorname{csr} B\}$. Let $(a_i) \in \operatorname{Lg}_m(A)$, so $\pi(a_i) \in \operatorname{Lg}_m(B)$. As $m \geq \operatorname{csr} B$, there is $\beta \in \mathrm{GL}_m(B)_0$ such that $\beta(\pi(a_1), \dots, \pi(a_m))^T = (1, 0, \dots, 0)^T \in B^m$. Since $\pi : \mathrm{GL}_m(A)_0 \to \mathrm{GL}_m(B)_0$ is onto ([83]), there is $\alpha \in \mathrm{GL}_m(A)_0$ with $\pi(\alpha) = \beta$ and so $\alpha(a_1, \dots, a_m)^T = (j_1 + 1, j_2, \dots, j_m)^T$ for some $(j_i) \in J^m$. It follows that $(j_1 + 1, j_2, \dots, j_m) \in \operatorname{Lg}_m(A) \cap (J^+)^m = \operatorname{Lg}_m(J^+)$. As $m \geq \operatorname{csr} J$, there is $\mu \in \mathrm{GL}_m(J^+)_0$ such that $\mu(j_1 + 1, j_2, \dots, j_m)^T = (1, 0, \dots, 0)^T$. Thus $\mathrm{GL}_m(A)_0$ acts transitively on $\operatorname{Lg}_m(A)$.

For the estimate $\operatorname{gsr} A \leq \max\{\operatorname{gsr} J, \operatorname{csr} B\}$, the steps are the same up to the appearance of μ. In this case, μ is in $\mathrm{GL}_m(J^+)$, and the conclusion is that $\mathrm{GL}_m(A)$ acts transitively on $\operatorname{Lg}_m(A)$. $\square$

The following C*-algebraic example will show that $\operatorname{tsr} A \leq \max\{\operatorname{tsr} J, \operatorname{tsr} B\}$ does not hold in general. Since the Bass and the topological stable ranks coincide for C*-algebras, the same example shows that $\operatorname{bsr} A \leq \max\{\operatorname{bsr} J, \operatorname{bsr} B\}$ does not hold in general.

The Toeplitz algebra $\mathcal{T}$ is the C*-algebra generated by a non-unitary isometry. The Toeplitz algebra can be realized as an extension as follows:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

Therefore:

$$\operatorname{tsr} \mathcal{T} \leq \max\{\operatorname{tsr} \mathcal{K}, \operatorname{tsr} C(S^1), \operatorname{csr} C(S^1)\}$$

$$\operatorname{csr} \mathcal{T} \leq \max\{\operatorname{csr} \mathcal{K}, \operatorname{csr} C(S^1)\}$$

Since $\operatorname{csr} \mathcal{K} = \operatorname{tsr} \mathcal{K} = \operatorname{tsr} C(S^1) = 1$ and $\operatorname{csr} C(S^1) = 2$, we get $\operatorname{tsr} \mathcal{T} \leq 2$ and $\operatorname{gsr} \mathcal{T} \leq \operatorname{csr} \mathcal{T} \leq 2$. As $\mathcal{T}$ is infinite, we conclude that $\operatorname{sr} \mathcal{T} = 2$.

IV.13 Problems

1. We know the stable ranks of $C^*_r F_n$. What are the stable ranks of other group algebras associated to the free groups? In the case of the full C*-algebra, Rieffel [71, Thm.6.7] showed that $\operatorname{tsr} C^*(F_n) = \infty$.

Problem. Compute $\operatorname{csr} C^*(F_n)$ and $\operatorname{gsr} C^*(F_n)$.

Problem. Compute the stable ranks of $\ell^1(F_n)$.

It is tempting to conjecture that, in general, the stable ranks of $\ell^1\Gamma$ equal the corresponding stable ranks of $C^*_r\Gamma$.

2. For a Banach algebra A and a compact space X, let $A(X)$ denote the Banach algebra of continuous maps from X to A. Rieffel [71, Cor.7.2] proved that $\operatorname{tsr} A(I) \leq \operatorname{tsr} A + 1$ for any unital C*-algebra A, and asked the following [71, Ques.1.8]:

Problem. Is it true that

$$(R) \qquad \operatorname{tsr} A(I^2) \leq \operatorname{tsr} A + 1$$

whenever A is a unital Banach algebra?

The motivation comes from the commutative case: it is easily checked that the answer is positive whenever A is the commutative C*-algebra of continuous functions on a compact space.

We point out that Rieffel's problem is equivalent, at least in the C*-setting, to the following statement: $\operatorname{tsr}(A \otimes C(X^d)) \leq \operatorname{tsr} A + \lceil d/2 \rceil$ whenever X^d is d-dimensional compact space and A is a unital C*-algebra. This, in turn, is slightly stronger than axiom **T** for tensor products by $C(X)$, which puts forth the estimate $\operatorname{tsr}(A \otimes C(X)) \leq \operatorname{tsr} A + \operatorname{tsr} C(X)$ whenever X is compact and A is a C*-algebra.

The non-trivial direction of the equivalence is shown in [59] (p. 990, comments after Cor.1.12). It is proved there that $\operatorname{tsr}(A \otimes C(X^d)) = \operatorname{tsr} A(I^d)$ whenever X is a finite CW-complex of dimension d, and that $\operatorname{tsr}(A \otimes C(X^d)) \leq \operatorname{tsr} A(I^d)$ whenever X is a compact space of dimension d. On the other hand, the estimate $\operatorname{tsr} A(I^2) \leq \operatorname{tsr} A + 1$ leads to $\operatorname{tsr} A(I^d) \leq \operatorname{tsr} A + \lceil d/2 \rceil$.

Sudo claims in [80] a proof for (R); however, Proposition 1 of [80] cannot hold. For if it is true, then it implies $(\dagger)$ $\operatorname{tsr} A(I) = \max\{\operatorname{csr} A, \operatorname{tsr} A\}$ for all C*-algebras A, by using Nistor's description for $\operatorname{tsr} A(I)$ as the absolute connected stable rank of A. Putting $A = C(I^k)$ in $(\dagger)$ leads to $\operatorname{tsr} C(I^{k+1}) = \operatorname{tsr} C(I^k)$ for all $k \geq 1$, since $\operatorname{csr} C(I^k) = 1$ by the homotopy invariance of the connected stable rank. This is a contradiction. The reader might enjoy finding elementary counterexamples to [80, Prop.1].

As we shall see in the next chapter, the estimate (R) has consequences in the problem of estimating stabilization of homotopy groups attached to the GL_n-sequence.

CHAPTER V

THE HOMOTOPY GROUPS OF THE GENERAL LINEAR GROUPS

In this chapter we are interested in stabilization theorems in topological K-theory and, more generally, in stabilization phenomena for the homotopy groups of the general linear group over Banach algebras. The stable ranks - the connected one, especially - play the key role in this discussion.

V.1 Motivation: Stabilization for the algebraic K_1-group

Bass [7] devised his "stable range" condition as a way to control the sequence

$$\cdots \to \mathrm{GL}_n(R)/\mathrm{E}_n(R) \to \mathrm{GL}_{n+1}(R)/\mathrm{E}_{n+1}(R) \to \dots$$

which yields the algebraic K_1-group $K_1^{\mathrm{alg}}(R)$ in the limit. For a unital ring R, the elementary group $\mathrm{E}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by the elementary matrices $\{1_n + e_{ij}\}_{1\leq i\neq j\leq n}$. The quotient $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$ is not necessarily a group, i.e., $\mathrm{E}_n(R)$ may not normal in $\mathrm{GL}_n(R)$. The map $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$, given by $* \mapsto \left(\begin{smallmatrix} * & 0 \\ 0 & 1 \end{smallmatrix}\right)$, sends $\mathrm{E}_n(R)$ to $\mathrm{E}_{n+1}(R)$, so it induces a well-defined map $\mathrm{GL}_n(R)/\mathrm{E}_n(R) \to \mathrm{GL}_{n+1}(R)/\mathrm{E}_{n+1}(R)$.

The following theorem was conjectured by Bass in [7]. Bass proved the surjectivity part, whereas the injectivity part, significantly more difficult, is due to Vaserstein [86].

Theorem V.1.1. *Let R be a unital ring. The map*

$$\mathrm{GL}_n(R)/\mathrm{E}_n(R) \to \mathrm{GL}_{n+1}(R)/\mathrm{E}_{n+1}(R)$$

is surjective for $n \geq \mathrm{bsr}\, R$, and injective for $n \geq \mathrm{bsr}\, R+1$. In particular, $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$ is a group isomorphic to $K_1^{\mathrm{alg}}(R)$ for $n \geq \mathrm{bsr}\, R+1$.

Similar stabilization phenomena occur in higher algebraic K-theory (cf. Suslin [81]). For a sequence of non-stable K_i-groups

$$K_{i,1}(R) \to K_{i,2}(R) \to \cdots \to K_{i,n}(R) \to K_{i,n+1}(R) \to \dots$$

we expect stabilization whenever n is larger than the Bass stable rank. The issue here is how to define the non-stable K_i-groups. Two possibilities are $K^V_{i,n}(R)$ in the sense of Volodin, and $K^Q_{i,n}(R)$ in the sense of Quillen (see [81] for the definitions); there is a canonical map $K^V_{i,n}(R) \to K^Q_{i,n}(R)$ for $n \geq 2i+1$ ([81, Thm.8.1]). The stabilization results for the two choices of non-stable K_i-groups read as follows ([81, Thm.4.1, Thm.8.2]):

· the map $K^V_{i,n}(R) \to K^V_{i,n+1}(R)$ is surjective for $n \geq \mathrm{bsr}\, R+i-1$, and bijective for $n \geq \mathrm{bsr}\, R+i$

· the map $K^Q_{i,n}(R) \to K^Q_{i,n+1}(R)$ is surjective for $n \geq \max\{i+1, \mathrm{bsr}\, R\}+i-1$, and bijective for $n \geq \max\{i+1, \mathrm{bsr}\, R\}+i$

V.2 Stabilization of homotopy groups

Let A be a unital, good topological algebra. The sequence

$$\{1\} = \mathrm{GL}_0(A) \hookrightarrow A^\times = \mathrm{GL}_1(A) \hookrightarrow \mathrm{GL}_2(A) \hookrightarrow \dots$$

induces, for each $k \geq 0$, a sequence of (identity-based) homotopy groups:

$$(\pi_k) \qquad \pi_k(\mathrm{GL}_0(A)) \to \pi_k(\mathrm{GL}_1(A)) \to \pi_k(\mathrm{GL}_2(A)) \to \dots$$

In order to formulate our results, we introduce the following notations:

· $\mathrm{surj}_k A$ is the least $n \geq 1$ such that $\pi_k(\mathrm{GL}_{m-1}(A)) \to \pi_k(\mathrm{GL}_m(A))$ is surjective for all $m \geq n$

· $\mathrm{inj}_k A$ is the least $n \geq 1$ such that $\pi_k(\mathrm{GL}_{m-1}(A)) \to \pi_k(\mathrm{GL}_m(A))$ is injective for all $m \geq n$

· $\mathrm{bij}_k A$ is the least $n \geq 1$ such that $\pi_k(\mathrm{GL}_{m-1}(A)) \to \pi_k(\mathrm{GL}_m(A))$ is bijective for all $m \geq n$

Of course, $\mathrm{bij}_k A = \max\{\mathrm{surj}_k A, \mathrm{inj}_k A\}$. With these notations, we state

Proposition V.2.1. *For $k \geq 0$, we have:*

a) $\mathrm{surj}_k A \leq \mathrm{csr} A(S^k)$ *and* $\mathrm{inj}_k A \leq \mathrm{csr} A(S^{k+1})$

b) $\max\{\mathrm{surj}_{k+1} A, \mathrm{inj}_k A, \mathrm{csr} A\} = \mathrm{csr} A(S^{k+1})$

Before we prove the proposition, we record a lemma (first stated by Elhage Hassan [29]) which says, in particular, that $\mathrm{csr} A(S^k) \geq \mathrm{csr} A$ for all $k \geq 0$.

Lemma V.2.2. *For every compact space X, we have* $\mathrm{csr} A(X) \geq \mathrm{csr} A$.

Proof. Identifying $\mathrm{Lg}_m(A(X))$ with the continuous maps from X to $\mathrm{Lg}_m(A)$, we see that the evaluation at an arbitrary $x \in X$ induces an onto map $\mathrm{Lg}_m(A(X)) \twoheadrightarrow \mathrm{Lg}_m(A)$. Hence $\mathrm{Lg}_m(A)$ is connected whenever $\mathrm{Lg}_m(A(X))$ is connected. □

Proof of Proposition V.2.1. Let $m \geq \mathrm{csr} A$. According to [20, Cor.1.6] one has a long exact homotopy sequence:

$$\cdots \to \pi_{k+1}(\mathrm{Lg}_m(A)) \to \pi_k(\mathrm{GL}_{m-1}(A)) \to \pi_k(\mathrm{GL}_m(A)) \to \pi_k(\mathrm{Lg}_m(A)) \to \cdots$$
$$\cdots \to \pi_1(\mathrm{Lg}_m(A)) \to \pi_0(\mathrm{GL}_{m-1}(A)) \to \pi_0(\mathrm{GL}_m(A)) \to 0$$

Also, for $m \geq \mathrm{csr} A$ we can simply identify

$$\pi_k\big(\mathrm{Lg}_m(A)\big) = \pi_0\big((\mathrm{Lg}_m(A))(S^k)\big) = \pi_0\big(\mathrm{Lg}_m(A(S^k))\big)$$

without having to keep track of basepoints.

a) If $m \geq \mathrm{csr} A(S^k) (\geq \mathrm{csr} A)$, then $\mathrm{Lg}_m(A(S^k))$ is connected, so $\pi_k\big(\mathrm{Lg}_m(A)\big)$ vanishes. Therefore $\pi_k(\mathrm{GL}_{m-1}(A)) \to \pi_k(\mathrm{GL}_m(A))$ is surjective. We conclude that $\mathrm{surj}_k A \leq \mathrm{csr} A(S^k)$.

If $m \geq \mathrm{csr} A(S^{k+1}) (\geq \mathrm{csr} A)$, then $\mathrm{Lg}_m(A(S^{k+1}))$ is connected, so $\pi_{k+1}\big(\mathrm{Lg}_m(A)\big)$ vanishes. Therefore $\pi_k(\mathrm{GL}_{m-1}(A)) \to \pi_k(\mathrm{GL}_m(A))$ is injective. We conclude that $\mathrm{inj}_k A \leq \mathrm{csr} A(S^{k+1})$.

b) By part a), we have $\operatorname{surj}_{k+1} A \leq \operatorname{csr} A(S^{k+1})$ and $\operatorname{inj}_k A \leq \operatorname{csr} A(S^{k+1})$. Since $\operatorname{csr} A \leq \operatorname{csr} A(S^{k+1})$, we get $\max\{\operatorname{surj}_{k+1} A, \operatorname{inj}_k A, \operatorname{csr} A\} \leq \operatorname{csr} A(S^{k+1})$.

To prove the reversed inequality, we let $m \geq \max\{\operatorname{surj}_{k+1} A, \operatorname{inj}_k A, \operatorname{csr} A\}$. In the following piece of the long exact homotopy sequence

$$\pi_{k+1}(\mathrm{GL}_{m-1}(A)) \to \pi_{k+1}(\mathrm{GL}_m(A)) \to \pi_{k+1}(\mathrm{Lg}_m(A)) \to \pi_k(\mathrm{GL}_{m-1}(A)) \to \pi_k(\mathrm{GL}_m(A))$$

the first map is surjective and the last map is injective. A simple analysis of exactness (kernel equals image) shows that $\pi_{k+1}(\mathrm{Lg}_m(A))$ must vanish, hence $\mathrm{Lg}_m(A(S^{k+1}))$ is connected. We conclude that $\operatorname{csr} A(S^{k+1}) \leq \max\{\operatorname{surj}_{k+1} A, \operatorname{inj}_k A, \operatorname{csr} A\}$. $\square$

Corollary V.2.3. *For $k \geq 0$, we have:*

a) $\operatorname{bij}_k A \leq \max\{\operatorname{csr} A(S^k), \operatorname{csr} A(S^{k+1})\}$

b) $\operatorname{csr} A(S^{k+1}) \leq \max\{\operatorname{bij}_k A, \operatorname{bij}_{k+1} A, \operatorname{csr} A\}$

Dimensional upper bounds for surj_k, inj_k, or bij_k are obtained by using the following

Lemma V.2.4. *For $k \geq 0$ we have $\operatorname{csr} A(S^k) \leq \operatorname{bsr} A(I^{k+1})$. In particular, $\operatorname{csr} A(S^k) \leq \operatorname{tsr} A + k + 1$. If Rieffel's estimate (R) holds, then $\operatorname{csr} A(S^k) \leq \operatorname{tsr} A + \lfloor k/2 \rfloor + 1$.*

Proof. First, we handle the case $k = 0$: we show that $\operatorname{csr} A \leq \operatorname{bsr} A(I)$. We use again the following ([7, Lem.4.1]): if $A \to B$ is onto and $m \geq \operatorname{bsr} A$ then $\mathrm{Lg}_m(A) \to \mathrm{Lg}_m(B)$ is onto. Now consider the quotient map $A(I) \twoheadrightarrow A \oplus A$, obtained by evaluating at endpoints. Let $m \geq \operatorname{bsr} A(I)$; then the induced map $\big(\mathrm{Lg}_m(A)\big)(I) \to \mathrm{Lg}_m(A) \oplus \mathrm{Lg}_m(A)$ is onto. This means precisely that any two points in $\mathrm{Lg}_m(A)$ can be connected by a path in $\mathrm{Lg}_m(A)$.

For the general case, start from the fact that $A(S^k)$ is a quotient of $A(I^{k+1})$. Hence

$$\operatorname{csr} A(S^k) \leq \max\{\operatorname{csr} A(I^{k+1}), \operatorname{bsr} A(I^{k+1})\}$$

by Proposition IV.6.2. Note that $\operatorname{csr} A(I^{k+1}) = \operatorname{csr} A$ by homotopy invariance. Since $A(I)$ is a quotient of $A(I^{k+1})$, we have $\operatorname{bsr} A(I^{k+1}) \geq \operatorname{bsr} A(I) \geq \operatorname{csr} A$. We conclude that $\operatorname{csr} A(S^k) \leq \operatorname{bsr} A(I^{k+1})$.

For the estimates in terms of the topological stable rank, we first recall that $\operatorname{bsr} \leq \operatorname{tsr}$. From $\operatorname{tsr} A(I) \leq \operatorname{tsr} A + 1$ one obtains $\operatorname{tsr} A(I^{k+1}) \leq \operatorname{tsr} A + k + 1$, hence $\operatorname{csr} A(S^k) \leq \operatorname{tsr} A + k + 1$. If (R) holds, then $\operatorname{tsr} A(I^{k+1}) \leq \operatorname{tsr} A + \lfloor k/2 \rfloor + 1$, hence $\operatorname{csr} A(S^k) \leq \operatorname{tsr} A + \lfloor k/2 \rfloor + 1$. $\square$

The conjectured upper bound $\operatorname{csr} A(S^k) \leq \operatorname{tsr} A + \lfloor k/2 \rfloor + 1$ is known to hold when $k = 1$: namely, Corollary 8.6 of [71] says that $\operatorname{csr} A(S^1) \leq \operatorname{tsr} A + 1$ whenever A is a unital Banach algebra. (In [71], this is stated for unital C*-algebras but the proof works for unital Banach algebras as well.)

We thus get a further corollary (recovering Theorem 10.10 of [71] and Theorem 2.10 of [72]):

Corollary V.2.5. *Let A be a unital Banach algebra. Then the natural map*

$$\pi_0(\mathrm{GL}_m(A)) \to \pi_0(\mathrm{GL}_{m+1}(A))$$

is surjective for $m \geq \operatorname{csr} A - 1$*, and bijective for* $m \geq \operatorname{csr} A(S^1) - 1$*; in particular, it is bijective for* $m \geq \operatorname{tsr} A$.

Remark V.2.6. i) From the surjectivity part we recover the fact that $K_1(A) = 0$ when $\operatorname{csr} A = 1$.

ii) Assume $\operatorname{tsr} A = 1$. We have already observed that $K_1(A) \neq 0$ implies $\operatorname{csr} A = 2$ (Remark IV.10.3). Conversely, $\operatorname{csr} A = 2$ implies $K_1(A) \neq 0$. Indeed, assume $K_1(A) = 0$. Then $\operatorname{Lg}_1(A) = A^\times$ (since $\operatorname{tsr} A = 1$ implies A is finite) and $\pi_0(A^\times) \simeq K_1(A) = 0$, meaning that $\operatorname{Lg}_1(A)$ is connected. Together with $\operatorname{csr} A \leq 2$ (again, due to $\operatorname{tsr} A = 1$), we obtain $\operatorname{csr} A = 1$, a contradiction.

One should compare part a) of Proposition V.2.1, in conjunction with Lemma V.2.4, with Theorem V.1.1, Theorem 6.3 in [20], Theorems 10.12 in [71], and Proposition 6 of [88]. On the other hand, Theorem 4.1 of [6] (see also Theorem 2.9 of [72]) improves our injectivity upper bound $\operatorname{inj}_k A \leq \operatorname{csr} A(S^{k+1})$ to $\operatorname{inj}_k A \leq \operatorname{gsr} A(S^{k+1})$ [1]. This may help in certain instances (for example in low dimensions, by using item 5) in Remark IV.10.3), but otherwise the general stable rank is much harder to compute than the connected stable rank.

Summarizing, our upper bounds for stabilization in homotopy essentially recover the known results on this topic. But the lower bounds for homotopic stabilization are new. We shall exploit them in the next section.

Remark V.2.7. For Banach algebras, the homotopy stabilization has a K-theoretic interpretation. The K_1 group and the K_0 group are the limit groups of the direct sequences (π_0) and (π_1):

$$K_1(A) = \varinjlim \pi_0(\mathrm{GL}_n(A)), \qquad K_0(A) \simeq \varinjlim \pi_1(\mathrm{GL}_n(A))$$

We stress the following aspect: stabilization for (π_0) is indeed stabilization for K_1, whereas stabilization for (π_1) is construed as stabilization for K_0.

Using the "homotopic" bounds provided by the connected stable rank, we have:

· $K_1(A) \simeq \pi_0(\mathrm{GL}_n(A))$ for $n \geq \operatorname{csr} A(S^1) - 1$;

· $K_1(A) \simeq \pi_0(\mathrm{GL}_n(A))$ and $K_0(A) \simeq \pi_1(\mathrm{GL}_n(A))$ for $n \geq \max\{\operatorname{csr} A(S^1), \operatorname{csr} A(S^2)\} - 1$.

In terms of the "dimensional" bounds, we have:

· $K_1(A) \simeq \pi_0(\mathrm{GL}_n(A))$ for $n \geq \operatorname{tsr} A$;

· $K_1(A) \simeq \pi_0(\mathrm{GL}_n(A))$ and $K_0(A) \simeq \pi_1(\mathrm{GL}_n(A))$ for $n \geq \operatorname{tsr} A + 2$.

It is likely that one can use in fact $\operatorname{tsr} A + 1$, instead of $\operatorname{tsr} A + 2$, in the last line.

Remark V.2.8. In connection with the homotopy stabilization phenomena discussed in this section, we mention the following remarkable family of examples constructed by Villadsen [88]:

> For each $N \geq 1$, there is a unital, simple C*-algebra A_N having $\operatorname{tsr} A_N = N$ with the following property: for all $k \geq 0$, the natural map $\pi_k\big(\mathrm{GL}_{m-1}(A_N)\big) \to \pi_k\big(\mathrm{GL}_m(A_N)\big)$ is bijective when $m \geq N + 1$, but not surjective when $m = N$.

[1] This bound, together with the proof of Proposition IV.3.4, suggest that $\operatorname{inj}_k A$ equals the general stable rank of the k-th suspension of A.

The coexistence of simplicity with high topological stable rank is already interesting. It was an open problem for a long time whether there are simple C*-algebras having a given topological stable rank $N \geq 2$. (Note that simple, finite C*-algebras tend to have stable rank 1, whereas simple, infinite C*-algebras always have infinite topological stable rank). The first examples were provided by Villadsen in 1999, and the above examples should be viewed as a significant refinement of his original construction.

We end this section by emphasizing the following ideas:

· the stable rank that is best suited for controlling stabilization in the homotopy sequences (π_*) is the connected stable rank;

· estimates in terms of connected stable ranks admit, in turn, upper bounds by the dimensional stable ranks (the topological stable rank and the Bass stable rank) which are typically easier to compute;

· Rieffel's conjectured inequality (R) plays a crucial role in providing good dimensional estimates for the connected stable ranks, hence in estimating homotopy stabilization.

V.3 Two examples concerning stabilization in K-theory

Let A be a Banach algebra. The direct limit of the (π_k) homotopy sequence is, by definition, the K-group $K_{k+1}(A)$; by Bott periodicity, $K_{k+1}(A)$ is isomorphic to $K_0(A)$ or $K_1(A)$ according to whether k is odd or even.

According to our notations, the natural map

$$\pi_k(\mathrm{GL}_n(A)) \to K_{k+1}(A)$$

becomes an isomorphism starting from $n = \mathrm{bij}_k A - 1$ precisely. We have upper bounds for $\mathrm{bij}_k A - 1$; primarily in terms of the connected stable rank, and secondarily in terms of the topological stable rank. But what is $\mathrm{bij}_k A - 1$ exactly? Unfortunately, we are unable to provide exact computations for, say, the class of commutative C*-algebras. The non-triviality of such an exact computation is apparent even in the trivial case $A = \mathbb{C}$ (this is related to, but easier than the proof of Proposition IV.3.4).

What we can compute, however, is the exact level of stabilization for *pairs* of consecutive homotopy sequences. Namely, we consider the problem of finding the least n for which the natural maps induce isomorphisms

$$\begin{cases} \pi_k\big(\mathrm{GL}_n(A)\big) \simeq K_{k+1}(A) \\ \pi_{k+1}\big(\mathrm{GL}_n(A)\big) \simeq K_{k+2}(A) \end{cases}$$

simultaneously in the homotopy sequences (π_k) and (π_{k+1}).

Note that each such pairing recovers the K-theory of A in the limit. If we think of stabilization in a single homotopy sequence as stabilization in one of the two K-groups, then the joint stabilization we want to study is truly stabilization for the K-theory (i.e. for both K-groups) of A.

For the results in this section, the criterion provided by Theorem IV.3.2 is crucial. We recall: if d is odd and $\check{H}^d(X^d) \neq 0$, then $\mathrm{csr}\, C(X^d) = 1 + \lceil d/2 \rceil$. Here X^d denotes a d-dimensional compact space.

V.3.1 K-theoretic stabilization for $C(X)$

We consider the problem of joint stabilization, as described above, for $A = C(X^d)$. First of all, part a) of Proposition V.2.1 becomes the following upper bound:

Proposition V.3.1. *Let $A = C(X^d)$. For $k \geq 0$, the natural map $\pi_k(\mathrm{GL}_n(A)) \to K_{k+1}(A)$ is an isomorphism starting from $n = 1 + \lfloor (d+k)/2 \rfloor$.*

Proof. We have $\mathrm{bij}_k\, C(X^d) \leq \max\{\mathrm{csr}\, C(X^d \times S^k), \mathrm{csr}\, C(X^d \times S^{k+1})\}$, and

$$\mathrm{csr}\, C(X^d \times S^k) \leq 1 + \lceil (d+k)/2 \rceil, \quad \mathrm{csr}\, C(X^d \times S^{k+1}) \leq 1 + \lceil (d+k+1)/2 \rceil.$$

Thus $\mathrm{bij}_k\, C(X^d) - 1 \leq \lceil (d+k+1)/2 \rceil = 1 + \lfloor (d+k)/2 \rfloor$. □

Proposition V.3.2. *Let $A = C(X^d)$, where the compact space X^d satisfies $\check{H}^d(X^d) \neq 0$. Let $k \geq 0$ have the same parity as d. Then $n = 1 + \frac{1}{2}(d+k)$ is the least integer for which the natural maps induce isomorphisms*

$$\begin{cases} \pi_k\big(\mathrm{GL}_n(A)\big) \simeq K_{k+1}(A) \\ \pi_{k+1}\big(\mathrm{GL}_n(A)\big) \simeq K_{k+2}(A) \end{cases}$$

simultaneously.

Proof. The least integer for which we have the required simultaneous isomorphisms is

$$C_k := \max\{\mathrm{bij}_k A, \mathrm{bij}_{k+1} A\} - 1.$$

We know $C_k \leq 1 + \lfloor (d+k+1)/2 \rfloor = 1 + \frac{1}{2}(d+k)$ from the previous proposition. On the other hand, Corollary V.2.3 says that $\max\{C_k, \mathrm{csr}\, C(X^d) - 1\} \geq \mathrm{csr}\, C(X^d \times S^{k+1}) - 1$. By the Kunneth formula, we have $\check{H}^{d+k+1}(X^d \times S^{k+1}) \simeq \check{H}^d(X^d) \neq 0$. Therefore

$$\mathrm{csr}\, C(X^d \times S^{k+1}) - 1 = \lceil (d+k+1)/2 \rceil = 1 + \frac{1}{2}(d+k).$$

As $1 + \frac{1}{2}(d+k) > \lceil d/2 \rceil \geq \mathrm{csr}\, C(X^d) - 1$ it follows that $C_k = 1 + \frac{1}{2}(d+k)$. □

V.3.2 K-theoretic stabilization for tensor products of C*-extensions of $\mathcal{K}$ by $C(X)$

In Section V.3.1, we computed K-theoretic stabilization over commutative C* algebras. In this section, we do the same for some noncommutative examples:

Theorem V.3.3. *For $1 \leq i \leq n$, let A_i be an extension of $\mathcal{K}$ by $C(X_i)$ with X_i a compact metrizable space. Put $A := A_1 \otimes \cdots \otimes A_n$, and $X := X_1 \times \cdots \times X_n$. Let $d = \dim X$, and let $k \geq 0$.*

a) The natural map $\pi_k(\mathrm{GL}_n(A)) \to K_{k+1}(A)$ is an isomorphism for $n \geq 1 + \lfloor (d+k)/2 \rfloor$.

b) Assume $d \neq 0$, $\check{H}^d(X) \neq 0$, and k has the parity of d. Then $n = 1 + \frac{1}{2}(d+k)$ is the least integer for which the natural maps induce isomorphisms

$$\begin{cases} \pi_k\big(\mathrm{GL}_n(A)\big) \simeq K_{k+1}(A) \\ \pi_{k+1}\big(\mathrm{GL}_n(A)\big) \simeq K_{k+2}(A) \end{cases}$$

simultaneously.

For instance, the Toeplitz algebra $\mathcal{T}$ fits in an exact sequence $0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0$. So $\mathcal{T}^{\otimes n}$, the n-fold tensor power of the Toeplitz algebra, would be an example of the type of C*-algebras under consideration.

Theorem V.3.3 is completely analogous to the result of the previous section, but the proof is more involved. The key point is Proposition V.3.7, which computes $\operatorname{csr} A(S^{k+1})$ for k having the same parity as d. We remind the reader that the behavior of stable ranks under tensor products is poorly understood, so we are very far from being able to derive Theorem V.3.3 from the result of the previous section.

We need to work with the connected stable rank, as well as with the topological stable rank. However, the reader should keep in mind the following slogan: if each A_i is an extension of $\mathcal{K}$ by the symbol space $C(X_i)$, then the stable ranks of $A_1 \otimes \cdots \otimes A_n$ equal the corresponding stable ranks of the "symbol space" $C(X_1) \otimes \cdots \otimes C(X_n) \simeq C(X_1 \times \cdots \times X_n)$; furthermore, this remains true after tensoring both A and $C(X)$ by some $C(Z)$. Although not always true, this slogan is valid enough to capture the spirit of the results.

In preparation for the first lemma, we collect some estimates we will use along the way:

· (extensions) Let $0 \to J \to A \to B \to 0$ be an exact sequence of C*-algebras. Then:

$$\max\{\operatorname{tsr} J, \operatorname{tsr} B\} \leq \operatorname{tsr} A \leq \max\{\operatorname{tsr} J, \operatorname{tsr} B, \operatorname{csr} B\}$$
$$\operatorname{csr} A \leq \max\{\operatorname{csr} J, \operatorname{csr} B\}, \quad \operatorname{csr} B \leq \max\{\operatorname{tsr} A, \operatorname{csr} A\}$$

· (tensoring by $\mathcal{K}$) Let A be a unital C*-algebra. Then:

$$\operatorname{tsr}(A \otimes \mathcal{K}) \leq \operatorname{tsr} A, \quad \operatorname{csr}(A \otimes \mathcal{K}) \leq \operatorname{csr} A$$

Rieffel [71, Thm.6.4] showed that $\operatorname{tsr}(A \otimes \mathcal{K}) \leq 2$, and that $\operatorname{tsr} A = 1$ implies $\operatorname{tsr}(A \otimes \mathcal{K}) = 1$. A similar situation occurs for the connected stable rank (Nistor [63, Cor.2.5&2.12], Sheu [78, Thm.3.10]).

· (tensoring by $C(X)$) Let A be a unital C*-algebra, and X a compact space. Then:

$$\operatorname{tsr}(A \otimes C(X)) \geq \operatorname{tsr} A, \quad \operatorname{csr}(A \otimes C(X)) \geq \operatorname{csr} A$$

The first estimate is due to the fact that the topological stable rank does not increase by passing to quotients. The second estimate is proved in Lemma V.2.2.

In what follows, it is convenient to use the notation $\operatorname{msr} A := \max\{\operatorname{tsr} A, \operatorname{csr} A\}$ ([60]).

Lemma V.3.4. *Let $0 \to \mathcal{K} \to A \to C(Y) \to 0$ be an exact* C*-sequence with A unital and Y compact. Then, for each nuclear unital* C*-algebra D, we have:*

$$\begin{aligned} \operatorname{msr}(D \otimes A) &= \operatorname{msr}(D \otimes C(Y)) \\ \operatorname{tsr}(D \otimes A) &\geq \operatorname{tsr}(D \otimes C(Y)) \\ \operatorname{csr}(D \otimes A) &\leq \operatorname{csr}(D \otimes C(Y)) \end{aligned}$$

Proof. We use the stable rank estimates for the exact sequence

$$0 \to D \otimes \mathcal{K} \to D \otimes A \to D \otimes C(Y) \to 0.$$

For the topological stable rank, we know

$$\max\left\{\operatorname{tsr}(D \otimes \mathcal{K}), \operatorname{tsr}(D \otimes C(Y))\right\} \leq \operatorname{tsr}(D \otimes A)$$
$$\operatorname{tsr}(D \otimes A) \leq \max\left\{\operatorname{tsr}(D \otimes \mathcal{K}), \operatorname{tsr}(D \otimes C(Y)), \operatorname{csr}(D \otimes C(Y))\right\}$$

which, using $\operatorname{tsr}(D \otimes \mathcal{K}) \leq \operatorname{tsr} D \leq \operatorname{tsr}(D \otimes C(Y))$, simplify to

$$\operatorname{tsr}(D \otimes C(Y)) \leq \operatorname{tsr}(D \otimes A) \tag{V.1}$$
$$\operatorname{tsr}(D \otimes A) \leq \max\left\{\operatorname{tsr}(D \otimes C(Y)), \operatorname{csr}(D \otimes C(Y))\right\}. \tag{V.2}$$

Similarly, for the connected stable rank we have:

$$\operatorname{csr}(D \otimes A) \leq \max\left\{\operatorname{csr}(D \otimes \mathcal{K}), \operatorname{csr}(D \otimes C(Y))\right\}$$
$$\operatorname{csr}(D \otimes C(Y)) \leq \max\left\{\operatorname{tsr}(D \otimes A), \operatorname{csr}(D \otimes A)\right\}$$

Since $\operatorname{csr}(D \otimes \mathcal{K}) \leq \operatorname{csr} D \leq \operatorname{csr} D \otimes C(Y)$, we get:

$$\operatorname{csr}(D \otimes A) \leq \operatorname{csr}(D \otimes C(Y)) \tag{V.3}$$
$$\operatorname{csr}(D \otimes C(Y)) \leq \max\left\{\operatorname{tsr}(D \otimes A), \operatorname{csr}(D \otimes A)\right\} \tag{V.4}$$

Taken together, the numbered relations are equivalent to the required relations. □

Proposition V.3.5. *Let A and X be as in Theorem V.3.3. Then, for every compact space Z we have:*

$$\begin{aligned} \operatorname{msr}(A \otimes C(Z)) &= \operatorname{msr} C(X \times Z) \\ \operatorname{tsr}(A \otimes C(Z)) &\geq \operatorname{tsr} C(X \times Z) \\ \operatorname{csr}(A \otimes C(Z)) &\leq \operatorname{csr} C(X \times Z) \end{aligned}$$

Proof. We argue by induction on n. The base case $n = 1$ is obtained by setting $D = C(Z)$ in Lemma V.3.4:

$$\begin{aligned}
\operatorname{msr}(A_1 \otimes C(Z)) &= \operatorname{msr} C(X_1 \times Z) \\
\operatorname{tsr}(A_1 \otimes C(Z)) &\geq \operatorname{tsr} C(X_1 \times Z) \\
\operatorname{csr}(A_1 \otimes C(Z)) &\leq \operatorname{csr} C(X_1 \times Z)
\end{aligned}$$

Next, we handle the induction step. Assume the conclusion of the proposition is valid for $n = k$; we show it holds for $n = k+1$. We need to check the following:

$$\begin{aligned}
\operatorname{msr}(\otimes_{i=1}^{k+1} A_i \otimes C(Z)) &= \operatorname{msr} C(\times_{i=1}^{k+1} X_i \times Z) \\
\operatorname{tsr}(\otimes_{i=1}^{k+1} A_i \otimes C(Z)) &\geq \operatorname{tsr} C(\times_{i=1}^{k+1} X_i \times Z) \\
\operatorname{csr}(\otimes_{i=1}^{k+1} A_i \otimes C(Z)) &\leq \operatorname{csr} C(\times_{i=1}^{k+1} X_i \times Z)
\end{aligned}$$

Setting $D = \otimes_{i=1}^{k} A_i \otimes C(Z)$ in Lemma V.3.4, and then invoking the induction step, we have:

$$\begin{aligned}
\operatorname{msr}(\otimes_{i=1}^{k} A_i \otimes A_{k+1} \otimes C(Z)) &= \operatorname{msr}(\otimes_{i=1}^{k} A_i \otimes C(X_{k+1} \times Z)) = \operatorname{msr} C(\times_{i=1}^{k} X_i \times X_{k+1} \times Z) \\
\operatorname{tsr}(\otimes_{i=1}^{k} A_i \otimes A_{k+1} \otimes C(Z)) &\geq \operatorname{tsr}(\otimes_{i=1}^{k} A_i \otimes C(X_{k+1} \times Z)) \geq \operatorname{tsr} C(\times_{i=1}^{k} X_i \times X_{k+1} \times Z) \\
\operatorname{csr}(\otimes_{i=1}^{k} A_i \otimes A_{k+1} \otimes C(Z)) &\leq \operatorname{csr}(\otimes_{i=1}^{k} A_i \otimes C(X_{k+1} \times Z)) \leq \operatorname{csr} C(\times_{i=1}^{k} X_i \times X_{k+1} \times Z)
\end{aligned}$$

The induction step is proved. □

We state, at this point, the following result of Nistor [63, Thm.4.4]:

Theorem V.3.6. *Let A and X be as in Theorem V.3.3, and let Z be a compact metrizable space. Assume* $\dim(X \times Z) \neq 1$.

Then $\operatorname{tsr}(A \otimes C(Z)) = \operatorname{tsr} C(X \times Z) = 1 + \lfloor \dim(X \times Z)/2 \rfloor$.

For instance, we obtain that $\operatorname{tsr} \mathcal{T}^{\otimes n} = 1 + \lfloor n/2 \rfloor$ for $n \geq 2$. However, $\operatorname{tsr} \mathcal{T} = 2$ so the previous formula cannot hold for $n = 1$; in particular, the assumption $\dim X \neq 1$ cannot be removed in the above theorem.

We mention that Nistor states the above result for those compact spaces X_i which can be realized as an inverse limit of finite CW-complexes of dimension $\dim X_i$ (see assumptions before Lemma 3.7 in [63]). He then points out that Theorem V.3.6 holds whenever X_i is a compact manifold. However, a result of Freudenthal from the 1920's says that every compact metrizable space can be realized as such an inverse limit.

Our Proposition V.3.5 strengthens Proposition 3.4 of [63]; the additional information we have is the equality $\operatorname{msr}(A \otimes C(Z)) = \operatorname{msr} C(X \times Z)$. We use it by exploiting the fact that the connected stable rank is larger than the topological stable rank in odd dimensions:

Proposition V.3.7. *Let A and X be as in Theorem V.3.3, and let Z be a compact metrizable space. Assume* $\dim(X \times Z)$ *is odd,* $\dim(X \times Z) \neq 1$, *and* $\check{H}^d(X \times Z) \neq 0$.

Then $\operatorname{csr}(A \otimes C(Z)) = \operatorname{csr} C(X \times Z) = 1 + \lceil \dim(X \times Z)/2 \rceil$.

Proof. The assumptions on $X \times Z$ imply that $\operatorname{csr} C(X \times Z) = 1 + \lceil \dim(X \times Z)/2 \rceil$, which is greater than $\operatorname{tsr} C(X \times Z) = 1 + \lfloor \dim(X \times Z)/2 \rfloor$ since $\dim(X \times Z)$ is odd. Together with Theorem V.3.6, this gives

$$\operatorname{msr} C(X \times Z) = \operatorname{csr} C(X \times Z) > \operatorname{tsr} C(X \times Z) = \operatorname{tsr}(A \otimes C(Z)).$$

Now $\operatorname{msr}(A \otimes C(Z)) = \operatorname{msr} C(X \times Z)$ forces $\operatorname{csr}(A \otimes C(Z)) = \operatorname{csr} C(X \times Z)$. $\square$

Finally, we give:

Proof of Theorem V.3.3. a) By Proposition V.3.5, we have

$$\operatorname{csr}\left(A \otimes C(S^{k+1})\right) \leq \operatorname{csr} C(X \times S^{k+1}) \leq 1 + \lceil (d+k+1)/2 \rceil = 2 + \lfloor (d+k)/2 \rfloor$$

It follows that $\operatorname{bij}_k A - 1 \leq 1 + \lfloor (d+k)/2 \rfloor$.

b) Now let $k \geq 0$ have the same parity as $d = \dim X$. Recall that the least integer for which we have the required simultaneous isomorphisms is

$$C_k := \max\{\operatorname{bij}_k A, \operatorname{bij}_{k+1} A\} - 1.$$

From part a) we know $C_k \leq 1 + \lfloor (d+k+1)/2 \rfloor = 1 + \frac{1}{2}(d+k)$. Since $\dim(X \times S^{k+1}) = d+k+1$ is odd and different from 1, and $\check{H}^{d+k+1}(X \times S^{k+1}) \simeq \check{H}^d(X) \neq 0$, we can apply Proposition V.3.7 to get

$$\operatorname{csr}\left(A \otimes C(S^{k+1})\right) = 1 + \lceil (d+k+1)/2 \rceil = 2 + \frac{1}{2}(d+k).$$

From Corollary V.2.3 we get:

$$\max\{C_k, \operatorname{csr} A - 1\} \geq \operatorname{csr} C(A \otimes C(S^{k+1})) - 1 = 1 + \frac{1}{2}(d+k)$$

But $1 + \frac{1}{2}(d+k) > \lceil d/2 \rceil \geq \operatorname{csr} C(X) - 1 \geq \operatorname{csr} A - 1$; for the last estimate, we applied Proposition V.3.5 in the case when Z is a singleton. It follows that $C_k = 1 + \frac{1}{2}(d+k)$. $\square$

Remark V.3.8. Each A_i is in the "bootstrap class" ([9, Def.22.3.4]), so the K-theory of A can be computed in terms of the K-theory of the A_i's by using Schochet's Künneth formula for tensor products ([9, Thm.23.1.3]). In particular, if the A_i's have torsion-free K-theory, then the K-theory of A is simply

$$K_*(A) \simeq \otimes_{i=1}^n K_*(A_i).$$

Remark V.3.9. For $n \geq 1$, let $\mathcal{T}_n$ be the Toeplitz C*-algebra on the odd-dimensional sphere S^{2n-1}; these higher analogues of the usual Toeplitz algebra $\mathcal{T}$ were introduced by Coburn [13]. Then $\mathcal{T}_n$ is a C*-extension of $\mathcal{K}$ by $C(S^{2n-1})$:

$$0 \to \mathcal{K} \to \mathcal{T}_n \to C(S^{2n-1}) \to 0$$

When $n > 1$, we have that $\operatorname{csr} \mathcal{T}_n = n+1$ (by Proposition V.3.7) and $\operatorname{tsr} \mathcal{T}_n = n$ (by Theorem V.3.6). For $n = 1$, we already know that $\operatorname{csr} \mathcal{T} = \operatorname{tsr} \mathcal{T} = 2$ (Example IV.12).

CHAPTER VI

RELATIVELY SPECTRAL MORPHISMS

VI.1 Relatively spectral morphisms

In this section, we discuss relatively spectral morphisms. We emphasize the comparison between relatively spectral morphisms and spectral morphisms.

Recall that a morphism $\phi : A \to B$ is *spectral* if $\mathrm{sp}_B(\phi(a)) = \mathrm{sp}_A(a)$ for all $a \in A$; equivalently, for $a \in A$ we have $a \in A^\times \Leftrightarrow \phi(a) \in B^\times$. We are concerned with the following relative notion:

Definition VI.1.1. A morphism $\phi : A \to B$ is *spectral relative to a subalgebra $X \subseteq A$* if $\mathrm{sp}_B(\phi(x)) = \mathrm{sp}_A(x)$ for all $x \in X$; equivalently, for $x \in X$ we have $x \in A^\times \Leftrightarrow \phi(x) \in B^\times$. A morphism $\phi : A \to B$ is *relatively spectral* if ϕ is spectral relative to some dense subalgebra of A.

If $\phi : A \to B$ and $\psi : B \to C$ are morphisms, then $\psi\phi$ is spectral relative to X if and only if ϕ is spectral relative to X and ψ is spectral relative to $\phi(X)$. This shows, in particular, that the passage from surjective to dense morphisms naturally entails a passage from spectral to relatively spectral morphisms. It also follows that a morphism $\phi : A \to B$ is relatively spectral if and only if both the dense morphism $\phi : A \to \overline{\phi(A)}$ and the inclusion $\overline{\phi(A)} \hookrightarrow B$ are relatively spectral (where $\overline{\phi(A)}$ is the closure of $\phi(A)$ in B). In other words, relative spectrality has two aspects: the dense morphism case and the closed-subalgebra case. We are interested in the dense morphism case in this paper.

In practice, the following criterion for relative spectrality is useful:

Proposition VI.1.2. *Let $\phi : A \to B$ be a dense morphism between good Fréchet algebras. Let $X \subseteq A$ be a dense subalgebra. The following are equivalent:*

i) $\mathrm{sp}_B(\phi(x)) = \mathrm{sp}_A(x)$ *for all* $x \in X$

ii) $r_B(\phi(x)) = r_A(x)$ *for all* $x \in X$.

Proof. For ii) $\Rightarrow$ i), let $x \in X$ with $\phi(x) \in B^\times$. Let $(x_n) \subseteq X$ with $\phi(x_n) \to \phi(x)^{-1}$. We have $r_B(1 - \phi(x_n)\phi(x)) \to 0$, that is, $r_A(1 - x_n x) \to 0$. In particular, $x_n x \in A^\times$ for large n so x is left-invertible. A similar argument shows that x is right-invertible. Thus $x \in A^\times$. □

Typically, the domain of a dense, relatively spectral morphism cannot be a C*-algebra:

Lemma VI.1.3. *Let $\phi : A \to B$ be a $*$-morphism, where A is a* C**-algebra and B is a Banach $*$-algebra. If ϕ is dense and spectral relative to a dense $*$-subalgebra, then ϕ is an isomorphism.*

Proof. Note first that ϕ is onto, as $\phi(A)$ is both dense and closed. Let X be a dense $*$-subalgebra of A relative to which ϕ is spectral. For $x \in X$ we have:

$$\|x\|_A^2 = \|xx^*\|_A = r_A(xx^*) = r_B(\phi(xx^*)) \leq \|\phi(xx^*)\|_B \leq \|\phi(x)\|_B^2$$

It follows that $\|a\|_A \leq \|\phi(a)\|_B \leq C\|a\|_A$ for all $a \in A$, where $C > 0$. Thus ϕ is an algebraic isomorphism, and ϕ can be made into an isometric isomorphism by re-norming B. □

Let us point out two disadvantages in working with relatively spectral morphisms.

First, if $\phi : A \to B$ is a spectral morphism between Fréchet algebras and B is good, then A is good as well. If ϕ is only relatively spectral, then the knowledge that A is good has to come from elsewhere; that is why our examples involve Banach algebras only.

Second, spectrality is well-behaved under amplifications: if $\phi : A \to B$ is a dense and spectral morphism then each $\mathrm{M}_n(\phi) : \mathrm{M}_n(A) \to \mathrm{M}_n(B)$ is a dense and spectral morphism ([82, Lem.2.1],[11, Prop.A.2.2], [76, Thm.2.1]). We have been unable to prove a similar result for relatively spectral morphisms.

Problem. Let $\phi : A \to B$ be a dense and relatively spectral morphism. Is $\mathrm{M}_n(\phi) : \mathrm{M}_n(A) \to \mathrm{M}_n(B)$ a relatively spectral morphism for each $n \geq 1$?

We thus have to introduce a stronger property that describes relative spectrality at all matrix levels:

Definition VI.1.4. A morphism $\phi : A \to B$ is *completely spectral relative to a subalgebra $X \subseteq A$* if each $\mathrm{M}_n(\phi) : \mathrm{M}_n(A) \to \mathrm{M}_n(B)$ is spectral relative to $\mathrm{M}_n(X)$. A morphism $\phi : A \to B$ is *completely relatively spectral* if each $\mathrm{M}_n(\phi) : \mathrm{M}_n(A) \to \mathrm{M}_n(B)$ is relatively spectral.

In concrete situations, one has a "subexponential norm control" which guarantees relative spectrality at all matrix levels; see Section VI.5.

Let Γ be a finitely-generated amenable group, and consider the dense inclusion $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$.

If Γ has polynomial growth then $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ is spectral (Ludwig [50]). On the other hand, if Γ contains a free subsemigroup on two generators then $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ is not spectral (Jenkins [37]). In between these two results, say for groups of intermediate growth, it is unknown whether $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ is spectral or not.

Turning to relative spectrality, it is easy to see that $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ is spectral relative to $\mathbb{C}\Gamma$ if Γ has subexponential growth. Indeed, we show that $r_{\ell^1\Gamma}(a) = r_{\mathrm{C}^*_{\mathrm{r}}\Gamma}(a)$ for $a \in \mathbb{C}\Gamma$. We have $r_{\mathrm{C}^*_{\mathrm{r}}\Gamma}(a) \leq r_{\ell^1\Gamma}(a)$ from $\|a\| \leq \|a\|_1$, so it suffices to prove that $r_{\ell^1\Gamma}(a) \leq r_{\mathrm{C}^*_{\mathrm{r}}\Gamma}(a)$. If $a \in \mathbb{C}\Gamma$ is supported in a ball of radius R, then a^n is supported in a ball of radius nR. We then have

$$\|a^n\|_1 \leq \sqrt{\mathrm{vol}\, B(nR)}\|a^n\|_2 \leq \sqrt{\mathrm{vol}\, B(nR)}\|a^n\|.$$

Taking the n-th root and letting $n \to \infty$, we obtain $r_{\ell^1\Gamma}(a) \leq r_{\mathrm{C}^*_{\mathrm{r}}\Gamma}(a)$.

This example serves as a preview for Example VI.5, where we show that $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ is in fact completely spectral relative to $\mathbb{C}\Gamma$, and for Section VI.6, where we investigate the groups Γ for which $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ is spectral relative to $\mathbb{C}\Gamma$.

In general, it is hard to decide whether a relatively spectral morphism is spectral or not. For instance, we do not know any examples of relatively spectral morphisms that are not spectral. Under spectral continuity assumptions, however, it is easy to show that relatively spectral morphisms are in fact spectral. We discuss this point in the next subsection.

VI.2 Spectral continuity

A relatively spectral morphism $\phi : A \to B$ is described by a spectral condition over a dense subalgebra of A. In the presence of spectral continuity, this spectral condition can be then extended to the whole of A, i.e., ϕ

is a spectral morphism. It might seem, at first sight, that both A and B need to have spectral continuity for this to work, but in fact spectral continuity for A suffices.

Spectral continuity can be interpreted in three different ways. The strongest form is to view the spectrum as a map from a good Fréchet algebra to the non-empty compact subsets of the complex plane, and to require continuity of the spectrum with respect to the Hausdorff distance. Recall, the Hausdorff distance between two non-empty compact subsets of the complex plane is given by $d_H(C,C') = \inf\{\varepsilon > 0 :\ C \subseteq C'_\varepsilon, C' \subseteq C_\varepsilon\}$, where C_ε denotes the open ε-neighborhood of C. Letting $\mathcal{S}$ denote the class of good Fréchet algebras with continuous spectrum in this sense, we have:

Proposition VI.2.1. *Let $\phi : A \to B$ be a relatively spectral morphism between good Fréchet algebras. If $A \in \mathcal{S}$ then ϕ is spectral.*

Proof. Let $a \in A$; we need to show that $\mathrm{sp}_A(a) \subseteq \mathrm{sp}_B(\phi(a))$. Let $\varepsilon > 0$ and pick $x_n \to a$ such that $\mathrm{sp}_A(x_n) = \mathrm{sp}_B(\phi(x_n))$. For $n \gg 1$ we have $\mathrm{sp}_B(\phi(x_n)) \subseteq \mathrm{sp}_B(\phi(a))_\varepsilon$ by Lemma VIII.7.3. On the other hand, the continuity of the spectrum in A gives $\mathrm{sp}_A(a) \subseteq \mathrm{sp}_A(x_n)_\varepsilon$ for $n \gg 1$. Combining these two facts, we get $\mathrm{sp}_A(a) \subseteq \mathrm{sp}_A(x_n)_\varepsilon = \mathrm{sp}_B(\phi(x_n))_\varepsilon \subseteq \mathrm{sp}_B(\phi(a))_{2\varepsilon}$ for $n \gg 1$. As ε is arbitrary, we are done. □

The usefulness of Proposition VI.2.1 is limited by the knowledge about the class $\mathcal{S}$. There are surprisingly few results on $\mathcal{S}$; we did not find in the literature any results complementing the ones contained in Aupetit's survey [3]. The following list summarizes the results mentioned by Aupetit:

a) if A is a commutative Banach algebra then A is in $\mathcal{S}$

b) if A is a commutative Banach algebra and B is a Banach algebra in $\mathcal{S}$, then the projective tensor product $A \otimes B$ is in $\mathcal{S}$ ([3, Thm.5, p.139])

c) if A is a commutative Banach algebra then $\mathrm{M}_n(A) \in \mathcal{S}$ ([3, Cor.1, p.139])

d) if A is a Banach algebra in $\mathcal{S}$, then every closed subalgebra of A is in $\mathcal{S}$ ([3, Thm.3, p.138])

A result of Kakutani says that $B(H)$, the algebra of bounded operators on an infinite-dimensional Hilbert space, is not in $\mathcal{S}$ ([3, p.34]).

The second form of spectral continuity, weaker than the one above, is adapted to $*$-algebras. We now require continuity of the spectrum (with respect to the Hausdorff distance) on self-adjoint elements only. Denoting by $\mathcal{S}_{\mathrm{sa}}$ the class of good Fréchet $*$-algebras whose spectrum is continuous on the self-adjoint elements, we have:

Proposition VI.2.2. *Let $\phi : A \to B$ be a relatively spectral $*$-morphism between good Fréchet $*$-algebras. If $A \in \mathcal{S}_{\mathrm{sa}}$ then ϕ is spectral.*

Proof. Arguing as in the proof of Proposition VI.2.1, we see that $\mathrm{sp}_A(a^*a) = \mathrm{sp}_B(\phi(a^*a))$ for all $a \in A$. Let $a \in A$ with $\phi(a) \in B^\times$. Then $\phi(a^*a), \phi(aa^*) \in B^\times$, and the previous equality of spectra gives $a^*a, aa^* \in A^\times$. Therefore $a \in A^\times$. □

According to [3, Cor.4, p.143], symmetric Banach $*$-algebras are in $\mathcal{S}_{\mathrm{sa}}$. Recall that a Banach $*$-algebra is *symmetric* if every self-adjoint element has real spectrum. We obtain:

Corollary VI.2.3. *Let $\phi : A \to B$ be a relatively spectral $*$-morphism between Banach $*$-algebras. If A is symmetric then ϕ is spectral.*

The third meaning that one can give to spectral continuity is continuity of the spectral radius. Let $\mathcal{R}$ denote the class of good Fréchet algebras with continuous spectral radius. Continuity of the spectrum (with respect to the Hausdorff distance) implies continuity of the spectral radius, that is, $\mathcal{S} \subseteq \mathcal{R}$; the inclusion is strict ([3, p.38]). Kakutani's result, mentioned above, actually says that $B(H)$ is not in $\mathcal{R}$.

Proposition VI.2.4. *Let $\phi : A \to B$ be a dense and relatively spectral morphism between good Fréchet algebras. If $A \in \mathcal{R}$ then ϕ is spectral.*

Proof. The proof is very similar to that of Proposition VI.2.1. Let $a \in A$; we need to show that $r_A(a) \leq r_B(\phi(a))$. For then Proposition VI.1.2 allows us to conclude that ϕ is spectral. Let $\varepsilon > 0$ and pick $x_n \to a$ such that $r_A(x_n) = r_B(\phi(x_n))$. For $n \gg 1$ we have $r_B(\phi(x_n)) \leq r_B(\phi(a)) + \varepsilon$, i.e., $r_A(x_n) \leq r_B(\phi(a)) + \varepsilon$. Letting $n \to \infty$ and using the continuity of the spectral radius in A, we get $r_A(a) \leq r_B(\phi(a)) + \varepsilon$. As ε is arbitrary, we are done. $\square$

Finally, note that if B satisfies one of the above forms of spectral continuity and $\phi : A \to B$ is spectral, then A satisfies that spectral continuity as well.

VI.3 On finiteness

Recall, an algebra A is *finite* if the left-invertibles of A are actually invertible, equivalently, if the right-invertibles of A are actually invertible. An algebra A is *stably finite* if each matrix algebra $\mathrm{M}_n(A)$ is finite. Stable-finiteness is relevant in K-theory, for it guarantees the non-vanishing of K_0.

Proposition VI.3.1. *Let $\phi : A \to B$ be a dense morphism between good topological algebras.*

a) Assume ϕ is relatively spectral. Then A is finite if and only if B is finite.

b) Assume ϕ is completely relatively spectral. Then A is stably finite if and only if B is stably finite.

Proof. We prove a). Part b) is an obvious corollary.

Letting $L(A)$ denote the left-invertibles of A, note that the density of $A^\times$ in $L(A)$ suffices for finiteness. Indeed, if $a \in L(A)$ then $u_n \to a$ for some invertibles u_n. There is $a' \in A$ with $a'a = 1$, so $a'u_n \to 1$. Hence $a'u_n \in A^\times$ for large n, thus $a' \in A^\times$ and we conclude $a \in A^\times$.

Let X be a dense subalgebra of A relative to which ϕ is spectral.

Assume B is finite. Let $a \in L(A)$ and let $x_n \to a$ where $x_n \in X$. Then $\phi(x_n) \to \phi(a) \in L(B) = B^\times$ so, by relative spectrality, x_n is eventually invertible. Thus $A^\times$ is dense in $L(A)$.

Assume A is finite. We claim that $\phi(L(A))$ is dense in $L(B)$; as $\phi(L(A)) = \phi(A^\times) \subseteq B^\times \subseteq L(B)$, it will follow that $B^\times$ is dense in $L(B)$. Let $b \in L(B)$ with $b'b = 1$. Pick sequences (x_n), (x_n') in X so that $\phi(x_n) \to b$, $\phi(x_n') \to b'$. Since $\phi(x_n'x_n) \to 1$, relative spectrality gives that x_n is left-invertible for large n. This proves that $\phi(L(A))$ is dense in $L(B)$. $\square$

VI.4 The Relative Density Theorem

We adapt Proposition A.2.6 of [11] as follows:

Lemma VI.4.1. *Let A, B be Fréchet spaces and $\phi : A \to B$ be a continuous linear map with dense image. Let $U \subseteq A$, $V \subseteq B$ be open with $\phi(U) \subseteq V$. If X is a dense subspace of A such that $U \cap X = \phi^{-1}(V) \cap X$ then ϕ induces a weak homotopy equivalence between U and V.*

Proof. First, we show that ϕ induces a bijection $\pi_0(U) \to \pi_0(V)$. We make repeated use of the local convexity. For surjectivity, pick $v \in V$. As V is open and $\phi(X)$ is dense, there is $v_X \in V \cap \phi(X)$ such that v and v_X can be connected by a segment in V. If $u_X \in X$ is a pre-image of v_X, then $u_X \in U$. For injectivity, let $u, u' \in U$ such that $\phi(u), \phi(u')$ are connected in V. As U is open and X is dense, there are $u_X, u'_X \in U \cap X$ such that u and u_X, respectively u' and u'_X, can be connected by a segment in U. That is, we may assume that we start with $u, u' \in U \cap X$. As V is open and $\phi(X)$ is dense, if $\phi(u)$ and $\phi(u')$ can be connected by a path in V then they can be connected by a piecewise-linear path p_B lying entirely in $V \cap \phi(X)$. Take pre-images in X for the vertices of p_B and extend to a piecewise-linear path p_A connecting u to u'. Since the path p_A lies in X and is mapped inside V, it follows that p_A lies in U. We conclude that u, u' are connected in U.

Next, let $k \geq 1$. We show that ϕ induces a bijection $\pi_k(U, a) \to \pi_k(V, \phi(a))$ for each $a \in U$. Up to translating U by $-a$ and V by $-\phi(a)$, we may assume that $a = 0$. Fix a basepoint $\bullet$ on S^k. The dense linear map $\phi : A \to B$ induces a dense linear map $\phi_k : A(S^k) \to B(S^k)$, which restricts to a dense linear map $\phi_k^\bullet : A(S^k)^\bullet \to B(S^k)^\bullet$. The $\bullet$-decoration stands for restricting to the based maps sending $\bullet$ to 0. With $U(S^k)^\bullet$, $V(S^k)^\bullet$, $X(S^k)^\bullet$ playing the roles of U, V, X, we get by the first part that $\phi_k^\bullet$ induces a bijection $\pi_0(U(S^k)^\bullet) \to \pi_0(V(S^k)^\bullet)$. That means precisely that ϕ induces a bijection $\pi_k(U, 0) \to \pi_k(V, 0)$.

Let us give some details on the previous paragraph. That $A(S^k)^\bullet$, respectively $B(S^k)^\bullet$, is a Fréchet space follows by viewing it as a closed subspace of the Fréchet space $A(S^k)$, respectively $B(S^k)$. The map $\phi_k : A(S^k) \to B(S^k)$ is given by $\phi_k(f) = \phi \circ f$, and it is obviously linear and continuous. Consequently, the restriction $\phi_k^\bullet : A(S^k)^\bullet \to B(S^k)^\bullet$ is linear and continuous. The fact that $X(S^k)^\bullet$ is dense in $A(S^k)^\bullet$ is showed using a partition of unity argument. This also proves the density of ϕ_k and $\phi_k^\bullet$. Next, $U(S^k)$ is open in $A(S^k)$ (Remark VIII.8.12) so $U(S^k)^\bullet$ is open in $A(S^k)^\bullet$; similarly, $V(S^k)^\bullet$ is open in $B(S^k)^\bullet$. Finally, $\phi_k^\bullet\big(U(S^k)^\bullet\big) \subseteq V(S^k)^\bullet$ and

$$U(S^k)^\bullet \cap X(S^k)^\bullet = (\phi_k^\bullet)^{-1}\big(V(S^k)^\bullet\big) \cap X(S^k)^\bullet$$

are immediate to check: they boil down to $\phi(U) \subseteq V$ and $U \cap X = \phi^{-1}(V) \cap X$ respectively. $\square$

It is not hard to imagine that, modulo notational complications, the idea used to treat π_0 has a higher-dimensional analogue. The underlying phenomenon is that homotopical considerations for open subsets of Fréchet spaces can be carried out in a piecewise-affine fashion over a dense subspace. Bost's elegant approach of upgrading π_0 knowledge to higher homotopy groups is, however, more economical.

As we shall see, the above Lemma immediately yields the Relative Density Theorem in spectral K-theory (Theorem VII.5.1) for good Fréchet algebras. First, a quick proof for the Relative Density Theorem in the Banach algebra case:

Theorem VI.4.2. *Let $\phi : A \to B$ be a dense and completely relatively spectral morphism between Banach algebras. Then ϕ induces an isomorphism $K_*(A) \simeq K_*(B)$.*

Proof. Let $\phi : A \to B$ be a dense morphism that is spectral relative to a dense subalgebra $X \subseteq A$, that is, $A^\times \cap X = \phi^{-1}(B^\times) \cap X$. Lemma VI.4.1 says that ϕ induces a bijection $\pi_*(A^\times) \to \pi_*(B^\times)$ for $* = 0, 1$. Thus, if $\phi : A \to B$ is a dense and completely relatively spectral morphism then ϕ induces a bijection $\pi_*(\mathrm{GL}_n(A)) \to \pi_*(\mathrm{GL}_n(B))$. It follows that ϕ induces a bijection

$$\varinjlim \pi_*(\mathrm{GL}_n(A)) \to \varinjlim \pi_*(\mathrm{GL}_n(B)).$$

That is, ϕ induces an isomorphism $K_{*+1}(A) \simeq K_{*+1}(B)$. As the Bott isomorphism $K_0(A) \simeq K_2(A)$ is natural, we conclude that ϕ induces an isomorphism $K_*(A) \simeq K_*(B)$. $\square$

Remark VI.4.3. The surjectivity part in the Relative Density Theorem was known to Lafforgue ([48, Lem.3.1.1] and comments after [47, Cor.0.0.3]).

Another application of Lemma VI.4.1 is the following strengthening of Theorem IV.11.6:

Theorem VI.4.4. *Let $\phi : A \to B$ be a dense and relatively spectral morphism, where A and B are good Fréchet algebras. Then $\mathrm{csr}\, A = \mathrm{csr}\, B$.*

Proof. We have $\phi(\mathrm{Lg}_m(A)) \subseteq \mathrm{Lg}_m(B)$. We claim that $\mathrm{Lg}_m(A) \cap X^m = \phi^{-1}(\mathrm{Lg}_m(B)) \cap X^m$, where X is a dense subalgebra of A relative to which ϕ is spectral. Let $(x_i) \in X^m$ with $(\phi(x_i)) \in \mathrm{Lg}_m(B)$, so $\sum c_i \phi(x_i) = 1$ for some $(c_i) \in B^m$. Density of ϕ allows to approximate each c_i by some $\phi(x_i')$, where $x_i' \in X$, so as to get $\sum \phi(x_i')\phi(x_i) = \phi\big(\sum x_i' x_i\big) \in B^\times$. Now $\sum x_i' x_i \in X$, so relative spectrality gives $\sum x_i' x_i \in A^\times$, i.e., $(x_i) \in \mathrm{Lg}_m(A)$ as desired.

Lemma VI.4.1 applied to the dense morphism $\phi : A^m \to B^m$ gives that ϕ induces a weak homotopy equivalence between $\mathrm{Lg}_m(A)$ and $\mathrm{Lg}_m(B)$. In particular, $\mathrm{Lg}_m(A)$ is connected if and only if $\mathrm{Lg}_m(B)$ is connected. $\square$

VI.5 Subexponential norm control

Let A and B be Banach algebra completions of an algebra X. Say that A and B are *spectrally equivalent over X* if $\mathrm{sp}_A(x) = \mathrm{sp}_B(x)$ for all $x \in X$, and *completely spectrally equivalent over X* if $\mathrm{M}_n(A)$ and $\mathrm{M}_n(B)$ are spectrally equivalent over $\mathrm{M}_n(X)$ for each $n \geq 1$.

Proposition VI.5.1. *The connected stable ranks and the property of being finite are invariant under spectral equivalence. Furthermore, K-theory and the property of being stably finite are invariant under complete spectral equivalence.*

Proof. Let A, B be Banach algebra completions of X. Let $\overline{X}$ be the Banach algebra obtained by completing X under the norm $\|x\| := \|x\|_A + \|x\|_B$. Then $r_{\overline{X}}(x) = \max\{r_A(x), r_B(x)\}$ for all $x \in X$. Now, if A and B are spectrally equivalent over X then $r_{\overline{X}}(x) = r_A(x) = r_B(x)$ for all $x \in X$. We obtain dense and relatively spectral morphisms $\overline{X} \to A$, $\overline{X} \to B$ by extending the inclusions $X \hookrightarrow A$, $X \hookrightarrow B$. Thus $\operatorname{csr} A = \operatorname{csr} \overline{X} = \operatorname{csr} B$, and A is finite if and only if $\overline{X}$ is finite if and only if B is finite.

If A and B are completely spectrally equivalent over X then we obtain morphisms $\overline{X} \to A$, $\overline{X} \to B$ that are completely spectral relative to X. Hence $K_*(A) \simeq K_*(\overline{X}) \simeq K_*(B)$, and A is stably finite if and only if $\overline{X}$ is stably finite if and only if B is stably finite. $\square$

Two algebras that are spectrally equivalent need not be connected by a morphism. Consider, however, the following $*$-context: A is a Banach $*$-algebra, B is a C^*-algebra, and X is a dense $*$-subalgebra of A and B. The argument used in the proof of Lemma VI.1.3 shows the following: if A and B are spectrally equivalent over X, then there is a dense and relatively spectral morphism $A \to B$ extending the identity on X.

In practice, complete spectral equivalence arises from subexponential norm control. We formulate this as follows.

A *gauge function* on a group Γ is a map $S \mapsto \omega(S) \in [1, \infty)$ on the finite nonempty subsets of Γ which is non-decreasing, i.e., $S \subseteq S'$ implies $\omega(S) \leq \omega(S')$. A gauge function ω is *subexponential* if $\omega(S^n)^{1/n} \to 1$ for each finite nonempty subset $S \subseteq \Gamma$.

Proposition VI.5.2. *Let $A\Gamma$ and $B\Gamma$ be Banach algebra completions of a group algebra $\mathbb{C}\Gamma$. Assume that there are subexponential gauge functions ω, ω' such that*

$$\|a\|_B \leq \omega(\operatorname{supp} a)\|a\|_A, \quad \|a\|_A \leq \omega'(\operatorname{supp} a)\|a\|_B$$

for all $a \in \mathbb{C}\Gamma$. Then $A\Gamma$ and $B\Gamma$ are completely spectrally equivalent over $\mathbb{C}\Gamma$.

Proof. Let $k \geq 1$. For all $(a_{ij}) \in \mathrm{M}_k(\mathbb{C}\Gamma)$ we have analogous subexponential estimates in the matrix algebras $\mathrm{M}_k(A\Gamma)$ and $\mathrm{M}_k(B\Gamma)$:

$$\|(a_{ij})\|_B \leq \omega\big(\operatorname{supp}\,(a_{ij})\big)\|(a_{ij})\|_A, \quad \|(a_{ij})\|_A \leq \omega'\big(\operatorname{supp}\,(a_{ij})\big)\|(a_{ij})\|_B$$

As $\omega\big(\operatorname{supp}\,(a_{ij})^n\big) \leq \omega\big((\operatorname{supp}\,(a_{ij}))^n\big)$, we get $r_A\big((a_{ij})\big) = r_B\big((a_{ij})\big)$ for all $(a_{ij}) \in \mathrm{M}_k(\mathbb{C}\Gamma)$. $\square$

Let Γ be a group of subexponential growth. We claim that $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ is completely spectral relative to $\mathbb{C}\Gamma$.

Indeed, let ω be the subexponential gauge function on Γ given by $\omega(S) = \sqrt{\operatorname{vol} B(S)}$, where $B(S)$ denotes the ball centered at the identity that circumscribes S. For $a \in \mathbb{C}\Gamma$ we have $\|a\| \leq \|a\|_1$ and $\|a\|_1 \leq \omega(\operatorname{supp} a)\|a\|_2 \leq \omega(\operatorname{supp} a)\|a\|$, i.e., we are in the conditions of Proposition VI.5.2. We infer that $K_*(\ell^1\Gamma) \simeq K_*(\mathrm{C}^*_{\mathrm{r}}\Gamma)$.

Remark VI.5.3. Consider, at this point, the following conjecture:

> For any discrete countable group Γ, the inclusion $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_{\mathrm{r}}\Gamma$ induces an isomorphism $K_*(\ell^1\Gamma) \simeq K_*(\mathrm{C}^*_{\mathrm{r}}\Gamma)$.

Let us refer to this statement as the BBC conjecture, for it connects the Bost conjecture with the Baum-Connes conjecture. We find the BBC conjecture to be a natural question on its own, outside of the Bost-Baum-Connes context. One knows, by combining results of Higson and Kasparov on the Baum-Connes side with results of Lafforgue on the Bost side, that the BBC conjecture holds for groups with the Haagerup property. On the other hand, the spectral approach allowed us to verify the BBC conjecture for groups of subexponential growth. The severe limitations of the spectral approach are made evident by this comparison.

Let Γ be a finitely generated group with property RD. Recall, this means that there are constants $C, d > 0$ such that $\|a\| \leq C\|a\|_{2,d}$ for all $a \in \mathbb{C}\Gamma$, where the weighted ℓ^2-norm $\|\cdot\|_{2,d}$ is given by

$$\big\|\sum a_g g\big\|_{2,d} = \sqrt{\sum |a_g|^2 (1+|g|)^{2d}}.$$

For $s > d$, the weighted ℓ^2-space

$$H^s\Gamma = \big\{\sum a_g g : \ \big\|\sum a_g g\big\|_{2,s} < \infty\big\}$$

is a Banach subalgebra of $\mathrm{C}^*_\mathrm{r}\Gamma$. The goal being the isomorphism $K_*(H^s\Gamma) \simeq K_*(\mathrm{C}^*_\mathrm{r}\Gamma)$, we showed in Theorem III.4.1 that $H^s\Gamma$ is a spectral subalgebra of $\mathrm{C}^*_\mathrm{r}\Gamma$.

The relative perspective for establishing the desired K-theoretic isomorphism is much easier. Let ω_s be the subexponential gauge function on Γ given by $\omega_s(S) = \big(1+R(S)\big)^s$, where $R(S)$ is the radius of the ball centered at the identity that circumscribes S. For $s > d$, we have $\|a\| \leq C\|a\|_{2,s}$ and $\|a\|_{2,s} \leq \omega_s(\text{supp } a)\|a\|_2 \leq \omega_s(\text{supp } a)\|a\|$ for all $a \in \mathbb{C}\Gamma$. From Proposition VI.5.2 it follows that $K_*(H^s\Gamma) \simeq K_*(\mathrm{C}^*_\mathrm{r}\Gamma)$.

VI.6 Σ_1-groups

We consider groups Γ for which the inclusion $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_\mathrm{r}\Gamma$ is spectral relative to $\mathbb{C}\Gamma$. Let us refer to such groups Γ as Σ_1-*groups*. That is, Γ is a Σ_1-group if $r_{\ell^1\Gamma}(a) \leq \|a\|$ for all $a \in \mathbb{C}\Gamma$. The Σ_1 condition is the ℓ^1 analogue of the ℓ^2-spectral radius property discussed in Section III.5.3. The major difference is that the "ℓ^2-spectral radius property" is satisfied by many non-amenable groups (e.g. by hyperbolic groups) whereas Σ_1-groups are necessarily amenable:

Proposition VI.6.1. *We have:*

a) Σ_1 is closed under taking subgroups;

b) Σ_1 is closed under taking directed unions;

c) Σ_1-groups are amenable;

d) Σ_1-groups do not contain FS_2, the free semigroup on two generators;

e) groups of subexponential growth are Σ_1-groups.

Proof. a) An embedding $\Lambda \hookrightarrow \Gamma$ induces isometric embeddings $\ell^1\Lambda \hookrightarrow \ell^1\Gamma$, $\mathrm{C}^*_\mathrm{r}\Lambda \hookrightarrow \mathrm{C}^*_\mathrm{r}\Gamma$.

b) Let Γ be the directed union of the Σ_1-groups $(\Gamma_i)_{i\in I}$. If $a \in \mathbb{C}\Gamma$, then $a \in \mathbb{C}\Gamma_i$ for some i and we know that $r_{\ell^1\Gamma_i}(a) = r_{\mathrm{C}^*_\mathrm{r}\Gamma_i}(a)$ can also be read as $r_{\ell^1\Gamma}(a) = r_{\mathrm{C}^*_\mathrm{r}\Gamma}(a)$.

c) Let Γ be a Σ_1-group. The inclusion $\ell^1\Gamma \hookrightarrow \mathrm{C}^*_\mathrm{r}\Gamma$ factors as $\ell^1\Gamma \hookrightarrow \mathrm{C}^*\Gamma \twoheadrightarrow \mathrm{C}^*_\mathrm{r}\Gamma$. It follows that $\mathrm{C}^*\Gamma \twoheadrightarrow \mathrm{C}^*_\mathrm{r}\Gamma$ is spectral relative to $\mathbb{C}\Gamma$, so necessarily $\mathrm{C}^*\Gamma \twoheadrightarrow \mathrm{C}^*_\mathrm{r}\Gamma$ is an isomorphism, i.e., Γ is amenable.

Here is another argument. By a), it suffices to show that Γ is amenable in the case Γ is finitely generated, say by a finite symmetric set S. For $a \in \mathbb{R}_+\Gamma$ we have $\|a^n\|_1 = \|a\|_1^n$, so $r_{\ell^1\Gamma}(a) = \|a\|_1$. We obtain $r_{\mathrm{C}^*_\mathrm{r}\Gamma}(\chi_S) = r_{\ell^1\Gamma}(\chi_S) = \|\chi_S\|_1 = \#S$. By Kesten's criterion, Γ is amenable.

d) Let Γ be a Σ_1-group and assume, on the contrary, that $x, y \in \Gamma$ generate a free subsemigroup. If the support of $a \in \mathbb{C}\Gamma$ generates a free subsemigroup then $\|a^n\|_1 = \|a\|_1^n$, hence $r_{\ell^1\Gamma}(a) = \|a\|_1$. Together with the Σ_1 condition, this gives $\|a\| = \|a\|_1$ for those $a \in \mathbb{C}\Gamma$ whose support generates a free subsemigroup. Consider now $a = 1 + ix + ix^{-1} \in \mathbb{C}\Gamma$. On one hand, the support of $(yx)a$ generates a free subsemigroup, so $\|a\| = \|(yx)a\| = \|(yx)a\|_1 = \|a\|_1 = 3$. On the other hand, we have $\|a\|^2 = \|aa^*\| = \|3 + x^2 + x^{-2}\| \leq 5$, a contradiction.

e) This was proved in Example VI.5. □

Thus Σ_1-groups include the subexponential groups and are included among amenable groups without free subsemigroups. Note that there are examples, first constructed by Grigorchuk, of amenable groups of exponential growth that do not contain a copy of FS_2. A well-known result of Rosenblatt says that such examples cannot occur among solvable groups: that is, a solvable group either has subexponential growth, or it contains FS_2.

Remark VI.6.2. If one considers the groups Γ for which the inclusion $\ell^1\Gamma \hookrightarrow \mathrm{C}^*\Gamma$ is spectral relative to $\mathbb{C}\Gamma$, then the analogues of parts a), b), d), e) of Proposition VI.6.1 still hold.

After Lafforgue, we say that a Banach algebra completion $A\Gamma$ of $\mathbb{C}\Gamma$ is an *unconditional completion* if the norm $\|\cdot\|_A$ of $A\Gamma$ satisfies $\||a|\|_A = \|a\|_A$ for all $a \in \mathbb{C}\Gamma$, where $|a|$ denotes the pointwise absolute value. The simplest example of unconditional completion is $\ell^1\Gamma$. For groups with property RD, $H^s\Gamma$ ($s \gg 1$) is an unconditional completion. In light of [47], one is interested in finding unconditional completions $A\Gamma$ having the same K-theory as $\mathrm{C}^*_\mathrm{r}\Gamma$. One way to achieve $K_*(A\Gamma) \simeq K_*(\mathrm{C}^*_\mathrm{r}\Gamma)$ would be to have $A\Gamma$ and $\mathrm{C}^*_\mathrm{r}\Gamma$ completely spectrally equivalent over $\mathbb{C}\Gamma$. For amenable groups, however, one cannot do better than Σ_1-groups:

Proposition VI.6.3. *Let Γ be amenable. If there is an unconditional completion $A\Gamma$ with the property that $A\Gamma$ and $\mathrm{C}^*_\mathrm{r}\Gamma$ are spectrally equivalent over $\mathbb{C}\Gamma$, then in fact $\ell^1\Gamma$ and $\mathrm{C}^*_\mathrm{r}\Gamma$ are spectral over $\mathbb{C}\Gamma$, i.e., Γ is a Σ_1-group.*

Proof. We first show that $r_{\ell^1\Gamma}(a) = r_{\mathrm{C}^*_\mathrm{r}\Gamma}(a)$ whenever $a \in \mathbb{C}\Gamma$ is self-adjoint. The spectral equivalence of $A\Gamma$ and $\mathrm{C}^*_\mathrm{r}\Gamma$ yields $\|a\| = r_{\mathrm{C}^*_\mathrm{r}\Gamma}(a) = r_{A\Gamma}(a) \leq \|a\|_A$ for all self-adjoint $a \in \mathbb{C}\Gamma$. Since Γ is amenable, we get $\|a\|_1 = \||a|\| \leq \||a|\|_A = \|a\|_A$ for all self-adjoint $a \in \mathbb{C}\Gamma$, so $\|a\| \leq \|a\|_1 \leq \|a\|_A$ for all self-adjoint $a \in \mathbb{C}\Gamma$. It follows that $r_{\ell^1\Gamma}(a) = r_{\mathrm{C}^*_\mathrm{r}\Gamma}(a)$ for all self-adjoint $a \in \mathbb{C}\Gamma$.

Next, one needs to adapt the proof of Proposition VI.1.2 to the following $*$-version:

> Let $\phi : A \to B$ be a dense $*$-morphism between Banach $*$-algebras, and let $X \subseteq A$ be a dense $*$-subalgebra. If $r_B(\phi(x)) = r_A(x)$ for all self-adjoint $x \in X$, then $\phi : A \to B$ is spectral relative to X.

Indeed, let $x \in X$ with $\phi(x)$ invertible in B; we show x invertible in A. Pick $(x_n) \subseteq X$ such that $\phi(x_n) \to \phi(x)^{-1}$. Then $\phi((xx_n)(xx_n)^*) \to 1$, so

$$r_A(1-(xx_n)(xx_n)^*) = r_B\big(\phi(1-(xx_n)(xx_n)^*)\big) \to 0.$$

It follows that $(xx_n)(xx_n)^*$ is invertible in A for large n. Similarly, $(x_nx)^*(x_nx)$ is invertible in A for large n. We conclude that x is invertible in A. $\square$

The above proof becomes even simpler if one makes the (natural) assumption that the unconditional completion $A\Gamma$ is a Banach $*$-algebra. One then argues, as in the proof of Lemma VI.1.3, that $\|a\| \leq \|a\|_A$ for all $a \in \mathbb{C}\Gamma$.

CHAPTER VII

SPECTRAL K-FUNCTORS

By a *K-functor* we simply mean a functor from good Fréchet algebras to abelian groups. The notion of K-scheme we introduce below provides a general framework for constructing K-functors. A K-functor is usually required to be stable, homotopy-invariant, half-exact and continuous. These properties do not concern us here. We point out, however, that the functors induced by K-schemes are stable and homotopy-invariant by construction.

Roughly speaking, a K-scheme is a selection of elements in each good Fréchet algebra in such a way that morphisms and amplifications preserve the selection. An open set $\Omega \subseteq \mathbb{C}$ containing the origin selects, in every good Fréchet algebra, those elements whose spectrum is contained in Ω. We can thus associate to each Ω a K-functor K_Ω; these K-functors are called spectral K-functors. For suitable choices of Ω, one recovers the K_0 and K_1 functors. We investigate how K_Ω depends on Ω and we show, for instance, that conformally equivalent domains yield naturally equivalent spectral K-functors. Finally, we prove the Relative Density Theorem for spectral K-functors.

VII.1 K-schemes and induced K-functors

A *K-scheme* S associates to each good Fréchet algebra A a subset $A_S \subseteq A$, such that the following axioms are satisfied:

(S_1) $0 \in A_S$

(S_2) if $a \in \mathrm{M}_p(A)_S$ and $b \in \mathrm{M}_q(A)_S$ then $\mathrm{diag}(a,b) \in \mathrm{M}_{p+q}(A)_S$

(S_3) if $\phi : A \to B$ is a morphism then $\phi(A_S) \subseteq B_S$

Two trivial examples are the *zero K-scheme* $A \mapsto \{0_A\}$ and the *full K-scheme* $A \mapsto A$; more interesting examples will appear shortly. Observe, at this point, the following consequence of (S_3): the set of S-elements in a good Fréchet algebra is invariant under conjugation.

A K-scheme S gives rise to a K-functor K_S. We first construct a functor V_S with values in abelian monoids, then we obtain a functor with values in abelian groups via the Grothendieck functor.

Let A be a good Fréchet algebra. The embeddings $\mathrm{M}_n(A) \hookrightarrow \mathrm{M}_{n+1}(A)$, given by $a \mapsto \mathrm{diag}(a,0)$, restrict to embeddings $\mathrm{M}_n(A)_S \hookrightarrow \mathrm{M}_{n+1}(A)_S$. Put

$$V_S(A) = \Big(\bigsqcup_{n \geq 1} \mathrm{M}_n(A)_S \Big) / \sim$$

where the equivalence relation $\sim$ is that of eventual homotopy, that is, $a \in \mathrm{M}_p(A)_S$ and $b \in \mathrm{M}_q(A)_S$ are equivalent if $\mathrm{diag}(a, 0_{n-p})$ and $\mathrm{diag}(b, 0_{n-q})$ are path-homotopic in $\mathrm{M}_n(A)_S$ for some $n \geq p,q$.

In other words:

$$V_S(A) = \varinjlim \pi_0\big(\mathrm{M}_n(A)_S\big)$$

Lemma VII.1.1. *$V_S(A)$ is an abelian monoid under $[a]+[b] := [\mathrm{diag}(a,b)]$.*

Proof. For $a, b \in \mathrm{M}_n(A)$, conjugating by $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, where $0 \leq t \leq \frac{\pi}{2}$, gives a a path-homotopy in $\mathrm{M}_{2n}(A)$ between $\mathrm{diag}(a,b)$ and $\mathrm{diag}(b,a)$. Next we show that for $a \in \mathrm{M}_p(A)$ and $b \in \mathrm{M}_q(A)$ we have a path-homotopy in $\mathrm{M}_{4p+4q}(A)$ between $\mathrm{diag}(a,b,0_{3p+3q})$ and $\mathrm{diag}(b,a,0_{3p+3q})$. By (S_2), we can do the following shuffling of blocks:

$$\begin{aligned}
\mathrm{diag}(a,b,0_{3p+3q}) \quad &\sim_h \quad \mathrm{diag}(0_{2p+2q},a,b,0_{p+q}) = \mathrm{diag}(0_{2p+q},0_q,a,b,0_p,0_q) \\
&\sim_h \quad \mathrm{diag}(0_{2p+q},b,0_p,0_q,a,0_q) \\
&\sim_h \quad \mathrm{diag}(0_{2p+q},b,a,0_q,0_p,0_q) = \mathrm{diag}(0_{2p+q},b,a,0_p,0_{2q}) \\
&\sim_h \quad \mathrm{diag}(b,a,0_p,0_{2p+q},0_{2q}) = \mathrm{diag}(b,a,0_{3p+3q})
\end{aligned}$$

It is easy to check that addition is well-defined, associative, commutative, and has $[0]$ as identity. $\square$

Let $\phi : A \to B$ be a morphism. We then have a monoid morphism $V_S(\phi) : V_S(A) \to V_S(B)$ given by $V_S(\phi)([a]) = [\phi(a)]$. Here we use ϕ to denote each of the amplified morphisms $\mathrm{M}_n(A) \to \mathrm{M}_n(B)$.

So far, we have that V_S is a monoid-valued functor on good Fréchet algebras. Let K_S be obtained by applying the Grothendieck functor to V_S. We conclude:

Proposition VII.1.2. *K_S is a K-functor.*

The K-functor associated to a K-scheme is modeled after the K_0 functor, which arises in this way from the *idempotent K-scheme* $A \mapsto \mathrm{Idem}(A)$. One could introduce a multiplicative version of K-scheme, by replacing the axiom $0 \in A_S$ with the axiom $1 \in A_S$, and suitably define a corresponding K-functor so that one recovers the K_1 functor from the (multiplicative) K-scheme $A \mapsto A^\times$. However, the difference between the "additive" axiom $0 \in A_S$ and the "multiplicative" axiom $1 \in A_S$ is a unit shift, which makes such a "multiplicative" K-scheme essentially redundant. Therefore, up to the natural equivalence induced by the unit shift, we may think of the K_1 functor as arising form the *shifted invertible K-scheme* $A \mapsto A^\times - 1$.

A *morphism of K-schemes* $f : S \to S'$ associates to each good Fréchet algebra A a continuous map $f : A_S \to A_{S'}$ such that the following axioms are satisfied:

(MS_1) $f(0) = 0$

(MS_2) if $a \in \mathrm{M}_p(A)_S$ and $b \in \mathrm{M}_q(A)_S$ then $\mathrm{diag}(f(a), f(b)) = f(\mathrm{diag}(a,b))$ in $\mathrm{M}_{p+q}(A)_{S'}$

(MS_3) if $\phi : A \to B$ is a morphism then the following diagram commutes:

$$\begin{array}{ccc}
A_S & \xrightarrow{\phi} & B_S \\
{\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\
A_{S'} & \xrightarrow{\phi} & B_{S'}
\end{array}$$

Proposition VII.1.3. *A morphism of K-schemes $f : S \to S'$ induces a natural transformation of K-functors $K_f : K_S \to K_{S'}$.*

Proof. If suffices to show that $f : S \to S'$ induces a natural transformation $V_f : V_S \to V_{S'}$. First, we show that for any good Fréchet algebra A there is a monoid morphism $f_A : V_S(A) \to V_{S'}(A)$. Second, we show that for any morphism $\phi : A \to B$ the following diagram commutes:

$$\begin{array}{ccc} V_S(A) & \xrightarrow{V_S(\phi)} & V_S(B) \\ {\scriptstyle f_A}\downarrow & & {\scriptstyle f_B}\downarrow \\ V_{S'}(A) & \xrightarrow{V_{S'}(\phi)} & V_{S'}(B) \end{array}$$

Let A be a good Fréchet algebra. Define $f_A : V_S(A) \to V_{S'}(A)$ by $f_A([a]) = [f(a)]$. Note that f_A is well-defined: if $a \in M_p(A)_S$ and $b \in \mathrm{M}_q(A)_S$ are eventually homotopic, i.e., $\mathrm{diag}(a, 0_{n-p})$ and $\mathrm{diag}(b, 0_{n-q})$ are path-homotopic in some $\mathrm{M}_n(A)_S$, then

$$f(\mathrm{diag}(a, 0_{n-p})) = \mathrm{diag}(f(a), 0_{n-p})$$

and

$$f(\mathrm{diag}(b, 0_{n-q})) = \mathrm{diag}(f(b), 0_{n-q})$$

are path-homotopic in $\mathrm{M}_n(A)_{S'}$, i.e., $f(a) \in M_p(A)_{S'}$ and $f(b) \in \mathrm{M}_q(A)_{S'}$ are eventually homotopic. Clearly f_A is a monoid morphism: we have $f_A([0]) = [f(0)] = [0]$ by (MS_1), while by (MS_2) we have

$$\begin{aligned} f_A([a]+[b]) = f_A[\mathrm{diag}(a,b)] &= [f(\mathrm{diag}(a,b))] \\ &= [\mathrm{diag}(f(a), f(b))] = [f(a)] + [f(b)] = f_A([a]) + f_A([b]). \end{aligned}$$

The commutativity of the diagram follows from (MS_3). □

Note that, if $f : S \to S'$ and $g : S' \to S''$ are morphisms of K-schemes then $gf : S \to S''$ is a morphism of K-schemes and $K_{gf} = K_g K_f$.

The morphisms of K-schemes $f, g : S \to S'$ are *homotopic* if $f, g : A_S \to A_{S'}$ are homotopic for any good Fréchet algebra A. If that is the case, then f and g induce the same natural transformation $K_f = K_g : K_S \to K_{S'}$. The K-schemes S, S' are *homotopy equivalent* if there are morphisms of K-schemes $f : S \to S'$ and $g : S' \to S$ such that gf is homotopic to id_S and fg is homotopic to $\mathrm{id}_{S'}$. A K-scheme S is *contractible* if S is homotopy equivalent to the zero K-scheme $A \mapsto \{0_A\}$.

Proposition VII.1.4. *If the K-schemes S and S' are homotopy equivalent, then the K-functors K_S and $K_{S'}$ are naturally equivalent. In particular, if the K-scheme S is contractible then K_S is the zero functor.*

VII.2 Spectral K-functors: defining K_Ω

Let $0 \in \Omega \subseteq \mathbb{C}$ be an open set. The *spectral K-scheme* S_Ω associates to each good Fréchet algebra A the subset $A_\Omega = \{a : \mathrm{sp}(a) \subseteq \Omega\}$; note that A_Ω is open (Lemma VIII.7.4).

The axioms for a K-scheme are easy to check: (S_1) is due to the fact that we imposed $0 \in \Omega$, (S_2) follows from

$$\mathrm{sp}_{M_{p+q}(A)}\big(\mathrm{diag}(a,b)\big) = \mathrm{sp}_{M_p(A)}(a) \cup \mathrm{sp}_{M_q(A)}(b)$$

while (S_3) follows from the fact that morphisms are non-increasing on spectra.

The full K-scheme and the shifted invertible K-scheme are spectral K-schemes; they are obtained by taking $\Omega = \mathbb{C}$, respectively $\Omega = \mathbb{C} \setminus \{-1\}$.

On the other hand, the zero K-scheme and the idempotent K-scheme are not spectral K-schemes.

The K-functor associated to the spectral K-scheme S_Ω is denoted by K_Ω and is referred to as a spectral K-functor.

We compute $K_\Omega(\mathbb{C})$.

Assume, for simplicity, that Ω has finitely many connected components $\Omega_0, \Omega_1, \ldots, \Omega_k$, where Ω_0 is the component of 0. For $1 \le i \le k$, let $\#_i(a)$ denote the number of eigenvalues, counted with multiplicity, of $a \in \mathrm{M}_n(\mathbb{C})$ that lie in Ω_i. Then the eigenvalue-counting map $\# : V_\Omega(\mathbb{C}) \to \mathbb{N}^k$ given by $\#([a]) = (\#_1(a), \ldots, \#_k(a))$ is well-defined. Visibly, $\#$ is a surjective morphism of monoids. We show $\#$ is injective. For $1 \le i \le k$, pick a basepoint $\lambda_i \in \Omega_i$ and think of

$$\lambda(a) = \mathrm{diag}\big(\underbrace{\lambda_1, \ldots, \lambda_1}_{\#_1(a)}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{\#_k(a)}\big)$$

as a normal form for $a \in \mathrm{M}_n(\mathbb{C})$. It suffices to show that a is eventually homotopic to $\lambda(a)$. Similar matrices are eventually homotopic; this is true for any K-scheme in fact. Thus a is eventually homotopic to its Jordan normal form. Up to a spectrum-preserving homotopy, one can assume that the Jordan normal form is in fact diagonal. Finally, a homotopy sends all eigenvalues in each Ω_i to the chosen basepoint λ_i, and all eigenvalues in Ω_0 to 0. That is, we can reach a diagonal matrix $\mathrm{diag}(\lambda(a), 0, \ldots, 0)$.

Therefore $V_\Omega(\mathbb{C})$ is isomorphic to $\mathbb{N}^k$ and, consequently, $K_\Omega(\mathbb{C})$ is isomorphic to $\mathbb{Z}^k$.

VII.3 Spectral K-functors: dependence on Ω

An open set $\Omega \subseteq \mathbb{C}$ containing 0 is thought of as a *based* open set. Correspondingly, a holomorphic map $f : \Omega \to \Omega'$ sending 0 to 0 is called a *based* holomorphic map.

Proposition VII.3.1. *A based holomorphic map $f : \Omega \to \Omega'$ induces a morphism of K-schemes $f_* : S_\Omega \to S_{\Omega'}$, hence a natural transformation of K-functors $K_{f_*} : K_\Omega \to K_{\Omega'}$. In particular, a based conformal equivalence between Ω and Ω' induces a natural equivalence between K_Ω and $K_{\Omega'}$.*

Proof. Let A be a good Fréchet algebra. We have a map $f_* : A_\Omega \to A_{\Omega'}$ given by $f_*(a) := f(a) = O_a(f)$, where $O_a : O(\Omega) \to A$ is the holomorphic calculus of $a \in A_\Omega$. We claim that f_* is continuous. Indeed, let $a_n \to a$ in A_Ω. Pick a (topologically tame) cycle γ containing $\mathrm{sp}(a)$ in its interior. Since the set of elements whose spectrum is contained in the interior of γ is open, we may assume without loss of generality that all the a_n's have their spectrum contained in the interior of γ. Then

$$f_*(a_n) - f_*(a) = \frac{1}{2\pi i} \oint_\gamma f(\lambda)\big((\lambda - a_n)^{-1} - (\lambda - a)^{-1}\big) d\lambda$$

and so $f_*(a_n) \to f_*(a)$ since the integrand converges to 0. We show that f_* is a morphism of K-schemes.

Axiom (MS_1) is obvious.

For (MS_2) we use the uniqueness of holomorphic calculus. Let $\mathcal{O}_a : \mathcal{O}(\Omega) \to \mathrm{M}_p(A)$ be the holomorphic calculus for a, and $\mathcal{O}_b : \mathcal{O}(\Omega) \to \mathrm{M}_q(A)$ be the holomorphic calculus for b. Then $\mathrm{diag}(\mathcal{O}_a, \mathcal{O}_b) : \mathcal{O}(\Omega) \to \mathrm{M}_{p+q}(A)$ given by $g \mapsto \mathrm{diag}(\mathcal{O}_a(g), \mathcal{O}_b(g))$ is a holomorphic calculus for $\mathrm{diag}(a,b)$; thus $\mathcal{O}_{\mathrm{diag}(a,b)} = \mathrm{diag}(\mathcal{O}_a, \mathcal{O}_b)$, in particular $f_*(\mathrm{diag}(a,b)) = \mathrm{diag}(f_*(a), f_*(b))$.

For (MS_3), we use again the uniqueness of holomorphic calculus. If $\mathcal{O}_a : \mathcal{O}(\Omega) \to A$ is the holomorphic calculus of $a \in A_\Omega$ then $\phi \circ \mathcal{O}_a : \mathcal{O}(\Omega) \to B$ is a holomorphic calculus for $\phi(a)$ hence $\phi \circ \mathcal{O}_a = \mathcal{O}_{\phi(a)}$, in particular $\phi(f_*(a)) = f_*(\phi(a))$ for all $a \in A_\Omega$.

For the second part, if suffices to check that $(gf)_* = g_* f_*$ for based holomorphic maps $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$. That is, we need $(gf)(a) = g(f(a))$ for all $a \in A_\Omega$. This follows once again by the uniqueness of holomorphic calculus: as both $h \mapsto (hf)(a)$ and $h \mapsto h\big(f(a)\big)$ are holomorphic calculi $\mathcal{O}(\Omega') \to A$ for $f(a)$, we get $(hf)(a) = h\big(f(a)\big)$ for all $h \in \mathcal{O}(\Omega')$, in particular for g. □

Corollary VII.3.2. *If Ω is connected and simply connected then K_Ω is the zero functor.*

Proof. By the conformal invariance of K_Ω and the Riemann Mapping Theorem, we may assume that Ω is either the entire complex plane, or the open unit disk. To show that K_Ω is the zero functor, it suffices to get $\pi_0(A_\Omega) = 0$ for every good Fréchet algebra A. Indeed, each $a \in A_\Omega$ can be connected to 0_A by the path $t \mapsto ta$, path which lies in A_Ω since $t\Omega \subseteq \Omega$ for $0 \leq t \leq 1$.

In a more sophisticated formulation, S_Ω is contractible. □

Proposition VII.3.1 shows that the functor K_Ω depends only on the conformal type of Ω. But more is true, in fact: K_Ω depends only on the holomorphic homotopy type of Ω. Roughly speaking, the notion of holomorphic homotopy type is obtained from the usual notion of homotopy type by requiring the maps to be holomorphic rather than continuous.

The based holomorphic maps $f, g : \Omega \to \Omega'$ are *holomorphically homotopic* if there is a family of based holomorphic maps $\{h_t\}_{0 \leq t \leq 1} : \Omega \to \Omega'$ such that $h_0 = f$, $h_1 = g$ and $t \mapsto h_t \in \mathcal{O}(\Omega)$ is continuous. If that is the case, then the induced morphisms of K-schemes $f_*, g_* : S_\Omega \to S_{\Omega'}$ are homotopic. If there are based holomorphic maps $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega$ such that fg is holomorphically homotopic to $\mathrm{id}_{\Omega'}$ and gf is holomorphically homotopic to id_Ω, then Ω and Ω' are said to be *holomorphic-homotopy equivalent*. If that is the case, then the K-schemes S_Ω and $S_{\Omega'}$ are homotopy equivalent in the sense described in the previous subsection. Therefore:

Proposition VII.3.3. *If Ω and Ω' are holomorphic-homotopy equivalent, then K_Ω and $K_{\Omega'}$ are naturally equivalent.*

VII.4 Spectral K-functors: recovering K_0 and K_1

The K_1 functor can be obtained from the shifted invertible K-scheme $A \mapsto A^\times - 1$. That is, K_1 is naturally equivalent to K_Ω for $\Omega = \mathbb{C} \setminus \{-1\}$. The K_0 functor can be obtained from the idempotent K-scheme $A \mapsto \mathrm{Idem}(A)$. However, the idempotent scheme is not a spectral K-scheme. We now realize the K_0 functor as a spectral K-functor.

Proposition VII.4.1. *$K_0 = K_\Omega$ for $\Omega = \mathbb{C} \setminus \{\mathrm{Re} = \frac{1}{2}\}$.*

Proof. Let χ denote the function defined on Ω as $\chi = 0$ on $\{\mathrm{Re} < \frac{1}{2}\}$ and $\chi = 1$ on $\{\mathrm{Re} > \frac{1}{2}\}$. Consider the holomorphic functions $\{h_t\}_{0 \le t \le 1} : \Omega \to \Omega$ defined as $h_t = (1-t)\mathrm{id} + t\chi$. We claim that $\{(h_t)_*\}_{0 \le t \le 1}$ is a strong deformation of S_Ω to the idempotent K-scheme. It will then follow that $K_\Omega = K_0$.

Let A be a good Fréchet algebra. We need to show that $\{(h_t)_*\}_{0 \le t \le 1}$ is a strong deformation from A_Ω to the idempotents of A. Each $(h_t)_* : A_\Omega \to A_\Omega$ is continuous. The map $t \mapsto h_t$ is $O(\Omega)$-continuous, so $t \mapsto (h_t)_*$ is A-continuous. At $t = 0$, $(h_0)_* = \mathrm{id}_{A_\Omega}$. At $t = 1$, $(h_1)_* = \chi_*$ takes idempotent values since $\chi^2 = \chi$. Finally, each $(h_t)_*$ acts identically on idempotents. Indeed, it suffices to show that χ_* acts identically on idempotents. Letting e be an idempotent, we have

$$\chi(e) = \frac{1}{2\pi i} \oint_{\gamma_1} (\lambda - e)^{-1} d\lambda = \frac{1}{2\pi i} \oint_{\gamma_1} \Big(\frac{1-e}{\lambda} + \frac{e}{\lambda - 1} \Big) d\lambda = e$$

where γ_1 is a curve around 1. $\square$

Remark VII.4.2. The fact that K_0 can be described in terms of elements with spectrum contained in $\mathbb{C} \setminus \{\mathrm{Re} = \frac{1}{2}\}$ is discussed in [74, pp. 193-196]; the above proof is essentially the one given there. It is this alternate picture for K_0 that inspired us in defining the K_Ω groups. Note that K_Ω is naturally equivalent to K_0 whenever Ω is the disjoint union of two open subsets of $\mathbb{C}$ which are connected and simply connected.

VII.5 Spectral K-functors: the Relative Density Theorem

Finally, we prove the Relative Density Theorem for spectral K-functors. In particular, we obtain the Relative Density Theorem for the usual K-theory. In the case of K_0, this proof is more elementary than the proof of Theorem VI.4.2, which used the Bott periodicity.

Theorem VII.5.1. *If $\phi : A \to B$ is a dense and completely relatively spectral morphism between good Fréchet algebras, then ϕ induces an isomorphism $K_\Omega(A) \simeq K_\Omega(B)$.*

Proof. Let $\phi : A \to B$ be a dense morphism that is spectral relative to a dense subalgebra $X \subseteq A$. Obviously $A_\Omega \cap X = \phi^{-1}(B_\Omega) \cap X$, so ϕ induces a bijection $\pi_0(A_\Omega) \to \pi_0(B_\Omega)$ by Lemma VI.4.1. Thus, if $\phi : A \to B$ is a dense and completely relatively spectral morphism then ϕ induces a bijection $\pi_0(\mathrm{M}_n(A)_\Omega) \to \pi_0(\mathrm{M}_n(B)_\Omega)$ for each $n \ge 1$. It follows that ϕ induces a bijection

$$\varinjlim \pi_0(\mathrm{M}_n(A)_\Omega) \to \varinjlim \pi_0(\mathrm{M}_n(B)_\Omega).$$

In other words, $V_\Omega(\phi) : V_\Omega(A) \to V_\Omega(B)$ is a bijection. We conclude that ϕ induces an isomorphism $K_\Omega(A) \to K_\Omega(B)$. $\square$

CHAPTER VIII

BACK MATTER

VIII.1 Spectrum

Let A be a unital Banach algebra, and let $a \in A$. The *spectrum* of a is $\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{\times}\}$. The spectrum is a non-empty compact set. The *spectral radius* of a is $r(a) = \max\{|\lambda| : \lambda \in \mathrm{sp}(a)\}$. The spectral radius formula is the following equality:

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

In particular $r(a) \leq \|a\|$. Note that $r(1-a) < 1$ implies $a \in A^{\times}$.

VIII.2 Holomorphic functional calculus

Let A be a unital Banach algebra, and fix $a \in A$ and $\Omega \subseteq \mathbb{C}$ an open neighborhood of $\mathrm{sp}(a)$. Let $O(\Omega)$ be the unital algebra of functions that are holomorphic in Ω, endowed with the topology of uniform convergence on compacts. There is a morphism $O(\Omega) \to A$ given as follows:

$$f \mapsto f(a) = \frac{1}{2\pi i} \oint f(\lambda)(\lambda - a)^{-1} d\lambda$$

The integral is taken around a finite collection of paths in Ω where f, such that the winding number is 1 for each point of $\mathrm{sp}(a)$.

The morphism $O(\Omega) \to A$ is unital and continuous; furthermore, we have the spectral mapping property $\mathrm{sp} f(a) = f(\mathrm{sp}(a))$ for each $f \in O(\Omega)$.

VIII.3 Finiteness

A unital complex algebra A is *finite* if every left-invertible element of A is invertible in A, equivalently, if every right-invertible element of A is invertible in A. A unital complex algebra A is *stably finite* if $\mathrm{M}_n(A)$ is finite for all $n \geq 1$.

This definition makes sense for rings. Using the terminology “finite” in that case would be ill-advised; instead, one calls a ring with the above property “Dedekind finite”.

The main source of stably finite C^*-algebras is given by tracial C^*-algebras. Recall, a *trace* on a unital $*$-algebra A is a functional $\mathrm{tr} : A \to \mathbb{C}$ satisfying $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$, and $\mathrm{tr}(a^*a) \geq 0$ for all $a \in A$. A trace tr is *faithful* if $\mathrm{tr}(a^*a) = 0$ implies $a = 0$, and *normalized* if $\mathrm{tr}(1) = 1$. A *tracial $*$-algebra* is a $*$-algebra endowed with a faithful, normalized trace. If A is a tracial $*$-algebra, then $\mathrm{M}_n(A)$ is a tracial $*$-algebra as well: set $\mathrm{tr}_n\big((a_{ij})\big) = \frac{1}{n}\sum a_{ii}$.

Lemma VIII.3.1. *Let A be a tracial C^*-algebra. If $e \in A$ is an idempotent then $0 \leq \mathrm{tr}(e) \leq 1$, with $\mathrm{tr}(e) = 0$ iff $e = 0$ and $\mathrm{tr}(e) = 1$ iff $e = 1$.*

Proof. We may assume e is a projection ($z = \big((2e^*-1)(2e-1)+1\big)^{1/2}$ is invertible and zez^{-1} is easily checked to be a projection with $\mathrm{tr}(zez^{-1}) = \mathrm{tr}(e)$). Then $\mathrm{tr}(e) = \mathrm{tr}(e^*e) \geq 0$, with equality iff $e = 0$. Applying this to the projection $1-e$, we get $\mathrm{tr}(e) \leq 1$, with equality iff $e = 1$. □

Proposition VIII.3.2. *Let A be a tracial* C*-*algebra. Then A is stably finite.*

Proof. Let $a, b \in A$, and assume that $ab = 1$. Then ba is an idempotent, and $\mathrm{tr}(ba) = 1$. By the previous lemma, we must have $ba = 1$. This shows that A is finite. Applying this to matrix algebras over A, we conclude that A is stably finite. □

If A is a tracial C*-algebra, then the trace tr induces a monomorphism $\mathrm{tr} : K_0(A) \to \mathbb{R}$ sending the class of 1_A to 1; in particular, the class of 1_A has infinite order in $K_0(A)$. The latter fact is actually true for stably finite C*-algebras. This is certainly well-known, but we could not find a reference in the literature so we prove it in the next

Lemma VIII.3.3. *Let R be a unital ring. If R is stably finite in the above sense, then the class of 1_R has infinite order in $K_0(R)$.*

Proof. Assume $k \cdot [1] = [0]$ in $K_0(R)$ for some positive integer k. This means that there is an idempotent e in a matrix ring $\mathrm{M}_n(R)$ such that $[e] + k \cdot [1] = [e] + [0]$ in $V_0(R)$. Let N be a multiple of k such that $N \geq n$. After adding 0's down the diagonal, we may view e as an idempotent in $\mathrm{M}_N(R)$. We have $[e] + N \cdot [1_R] = [e] + [0_R]$, i.e., $[e] + [1_{\mathrm{M}_N(R)}] = [e] + [0_{\mathrm{M}_N(R)}]$. Thus, up to replacing R by $\mathrm{M}_N(R)$, we may assume that we have $[e] + [1_R] = [e] + [0_R]$ in $V_0(R)$ for some idempotent e in R. Now, this means that in some matrix ring $\mathrm{M}_q(R)$ we have

$$U \begin{pmatrix} e & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} e & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} U$$

for some invertible matrix $U = (u_{ij})$. Performing the products, we get:

$$\begin{pmatrix} u_{11}e & u_{12} & 0 & \dots & 0 \\ u_{21}e & u_{22} & 0 & \dots & 0 \\ & \dots & & \dots & \\ u_{q1}e & u_{q2} & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} eu_{11} & eu_{12} & eu_{13} & \dots & eu_{1q} \\ 0 & 0 & 0 & \dots & 0 \\ & \dots & & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Comparison in the first column gives $u_{21}e = \cdots = u_{q1}e = 0$. Comparison in the second column yields

$u_{22} = \cdots = u_{q2} = 0$ and $u_{12} = eu_{12}$. Now let (x_{ij}) be the inverse of (u_{ij}). From

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1q} \\ x_{21} & x_{22} & \dots & x_{2q} \\ \dots & \dots & \dots & \dots \\ x_{q1} & x_{q2} & \dots & x_{qq} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1q} \\ u_{21} & 0 & \dots & u_{2q} \\ \dots & \dots & \dots & \dots \\ u_{q1} & 0 & \dots & u_{qq} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

we get $x_{21}u_{12} = 1$. By finiteness, we get u_{12} invertible in R. Then $e = 1$, hence $u_{21} = \cdots = u_{q1} = 0$. Also, $x_{11}u_{12} = 0$ so $x_{11} = 0$. We now have

$$\begin{pmatrix} 0 & x_{12} & \dots & x_{1q} \\ x_{21} & x_{22} & \dots & x_{2q} \\ \dots & \dots & \dots & \dots \\ x_{q1} & x_{q2} & \dots & x_{qq} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1q} \\ 0 & 0 & \dots & u_{2q} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & u_{qq} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

which gives a contradiction in the $(1,1)$-position. □

Let us mention the following counterpoint to the above lemma: for the Cuntz algebra O_∞, the class of the identity has infinite order in K_0; however, O_∞ is infinite (in fact, "very" infinite in a certain technical sense).

VIII.4 Group algebras

Let Γ be a (discrete, countable) group. The starting object is the *group algebra* $\mathbb{C}\Gamma$. As a vector space, $\mathbb{C}\Gamma$ is the $\mathbb{C}$-span of Γ. The multiplication on $\mathbb{C}\Gamma$ is by convolution:

$$\left(\sum a_g g\right)\left(\sum b_h h\right) = \sum \left(\sum_{gh=x} a_g b_h\right) x$$

Note here that inversion in Γ turns $\mathbb{C}\Gamma$ into a $*$-algebra by defining $(\sum a_g g)^* = \sum \overline{a}_g g^{-1}$.

By an analytic group algebra associated to Γ we mean a completion of the group algebra $\mathbb{C}\Gamma$; the $*$-operation usually extends, so analytic group algebras are $*$-algebras. Such an analytic group algebra can be a Fréchet algebra, a Banach algebra, a C*-algebra, or a von Neumann algebra. Most analytic group algebras are defined as algebras of operators; others appear as sequence spaces with summability conditions (though they can also be viewed as algebras of operators).

The broad question of a topic that could be called *Analytic Group Theory* is to understand how properties of Γ are reflected in properties of analytic group algebras associated to Γ. Analytic Group Theory can thus be viewed as a bridge between Group Theory and Noncommutative Geometry. Typically, it is the Noncommutative Geometry end that is the busiest, for the passage from a group to an analytic group algebra is a tremendous increase in complexity. This is partly due to the fact that we are trading a relatively small object, the group, for a much larger object. But the key source of mystery is the fact that most analytic group algebras are defined implicitly. It is only rarely the case, e.g. within the commutative world, that analytic group algebras can be made explicit.

There are many, many analytic group algebras one can associate to a group Γ. Here, we recall three of the most fundamental ones: the Banach algebra $\ell^1\Gamma$, and the C*-algebras $\mathrm{C}^*_{\mathrm{r}}\Gamma$ and $\mathrm{C}^*\Gamma$. Another fundamental group algebra is the von Neumann algebra $L\Gamma$; in the coarse setting, an important group C*-algebra is the uniform (Roe) algebra $\mathrm{C}^*_{\mathrm{u}}\Gamma$.

The group ℓ^1-algebra $\ell^1\Gamma$ is the Banach algebra completion of $\mathbb{C}\Gamma$ under the norm $\|\sum a_g g\|_1 = \sum |a_g|$. Concretely, $\ell^1\Gamma = \{\sum a_g g : \sum |a_g| < \infty\}$; this algebra is the prominent example of a "sequential" group algebra.

The reduced C*-algebra $\mathrm{C}^*_{\mathrm{r}}\Gamma$ is the completion of $\mathbb{C}\Gamma$ with respect to the left regular representation. Recall, the left regular representation $\lambda : \Gamma \to \mathcal{B}(\ell^2\Gamma)$ is the unitary representation given by $\lambda(g)(\sum a_h \delta_h) = \sum a_h \delta_{gh}$. The representation λ extends by linearity to a faithful $*$-representation $\lambda : \mathbb{C}\Gamma \to \mathcal{B}(\ell^2\Gamma)$, and $\mathrm{C}^*_{\mathrm{r}}\Gamma$ is defined as the operator-norm closure of $\mathbb{C}\Gamma$ inside $\mathcal{B}(\ell^2\Gamma)$. An alternate notation for the reduced C*-algebra of Γ is $\mathrm{C}^*_{\lambda}\Gamma$.

The full (or maximal, or universal) C*-algebra $\mathrm{C}^*\Gamma$ is the completion of $\mathbb{C}\Gamma$ under the norm $\|a\|_* = \sup \|\pi(a)\|$ where π runs over the equivalence classes of unitary (cyclic) representations of Γ. More set-theoretically comforting is to define $\mathrm{C}^*\Gamma$ as the enveloping C*-algebra of $\ell^1\Gamma$. That is, $\mathrm{C}^*\Gamma$ is the completion of $\ell^1\Gamma$ under the norm $\|a\|_* = \|\pi_{\mathrm{univ}}(a)\|$, where the universal representation $\pi_{\mathrm{univ}} = \oplus \pi_f$ of $\ell^1\Gamma$ collects the representations π_f induced by the states f of $\ell^1\Gamma$.

For $a \in \mathbb{C}\Gamma$ we have $\|a\| \leq \|a\|_* \leq \|a\|_1$. It follows that the identity on $\mathbb{C}\Gamma$ extends to $*$-morphisms in such a way that the following diagram commutes:

$$\begin{tikzcd} \mathrm{C}^*\Gamma \arrow[r] & \mathrm{C}^*_{\mathrm{r}}\Gamma \\ \ell^1\Gamma \arrow[u] \arrow[ur] & \end{tikzcd}$$

The morphism $\ell^1\Gamma \to \mathrm{C}^*_{\mathrm{r}}\Gamma$ is injective since $\ell^1\Gamma$ is faithfully represented on $\ell^2\Gamma$. Hence $\ell^1\Gamma \to \mathrm{C}^*\Gamma$ is injective as well. The morphism $\mathrm{C}^*\Gamma \to \mathrm{C}^*_{\mathrm{r}}\Gamma$ is surjective; the reason is that a $*$-morphism between C*-algebras has closed image. We record the following important

Fact. $\mathrm{C}^*\Gamma \to \mathrm{C}^*_{\mathrm{r}}\Gamma$ is an isomorphism iff Γ is amenable.

Outside the amenable world, the two group C*-algebras may differ drastically.

· [*functoriality*] The full C*-algebra is functorial: a group morphism $\Gamma \to \Gamma'$ induces a C*-morphism $\mathrm{C}^*\Gamma \to \mathrm{C}^*\Gamma'$. The reduced C*-algebra is, however, not functorial. The simplicity of $\mathrm{C}^*_{\mathrm{r}}(F_n)$ (due to Powers) is the most dramatic illustration of this fact: although a free group has plenty of non-trivial quotients, its reduced C*-algebra has none!

· [*traces*] There is a standard trace $\mathrm{tr}(a) = \langle a(\delta_e), \delta_e\rangle$ on $\mathrm{C}^*_{\mathrm{r}}\Gamma$, making $\mathrm{C}^*_{\mathrm{r}}\Gamma$ a tracial C*-algebra. On the other hand, $\mathrm{C}^*\Gamma$ may not be tracial, e.g., for $\Gamma = \mathrm{SL}_{n\geq 3}(\mathbb{Z})$ (due to Bekka).

· [*idempotents*] For Γ torsion-free, the Kaplansky-Kadison conjecture claims that $\mathrm{C}^*_{\mathrm{r}}\Gamma$ has no idempotents other than 0 and 1. By now, this has been verified for many classes of groups, e.g. hyperbolic groups.

On the other hand, the full C*-algebra C$^*\Gamma$ may contain non-trivial idempotents: this is the case whenever Γ has Kazhdan's property (T). Connecting these two statements is the fact that many hyperbolic groups have Kazhdan's property (T).

VIII.5 Nuclearity

In general, there are many C*-completions for the algebraic tensor product $A \odot B$ of the C*-algebras A and B. Every C*-norm on $A \odot B$ turns out to be a cross-norm, i.e., $\|a \otimes b\| = \|a\|\|b\|$ for $a \in A$, $b \in B$. Among all C*-norms on $A \odot B$, there is a least one ($\|\cdot\|_{\min}$) and a greatest one ($\|\cdot\|_{\max}$). The minimal (or spatial) tensor norm $\|\cdot\|_{\min}$ comes from the representation $A \odot B \hookrightarrow \mathcal{B}(H_A \otimes H_B)$, independent of the chosen representations $A \hookrightarrow \mathcal{B}(H_A)$, $B \hookrightarrow \mathcal{B}(H_B)$. The completion of $A \odot B$ under $\|\cdot\|_{\min}$ is denoted $A \otimes_{\min} B$. The maximal tensor norm $\|\cdot\|_{\max}$ is given by

$$\|\sum a_i \otimes b_i\| := \sup \|\sum \pi_A(a_i)\pi_B(b_i)\|$$

where the supremum being taken over all pairs of commuting representations $\pi_A : A \to \mathcal{B}(H)$, $\pi_B : B \to \mathcal{B}(H)$. The completion of $A \odot B$ under $\|\cdot\|_{\max}$ is denoted $A \otimes_{\max} B$. See [89, Appendix T] for details.

A C*-algebra A is *nuclear* if $A \otimes_{\min} B \simeq A \otimes_{\max} B$ for each C*-algebra B; in other words, for each C*-algebra B, the algebraic tensor product $A \odot B$ admits a unique C*-norm. Basic examples of nuclear C*-algebras are the finite-dimensional C*-algebras (e.g. $\mathrm{M}_n(\mathbb{C})$) and the commutative C*-algebras (i.e. $C_0(X)$ for X a locally compact Hausdorff space). However, $\mathcal{B}$ is not nuclear.

We have the following important

Fact. The class of nuclear C*-algebras is closed under taking ideals, quotients, extensions, inductive limits, tensor products, and crossed products by amenable groups.

Nuclearity is not inherited by subalgebras in general. However, subalgebras of nuclear C*-algebras are exact. A C*-algebra A is said to be *exact* if min-tensoring by A preserves short exact sequences, i.e., for each short exact sequence of C*-algebras $0 \to J \to B \to Q \to 0$, the sequence

$$0 \to J \otimes_{\min} A \to B \otimes_{\min} A \to Q \otimes_{\min} A \to 0$$

is also exact. In other words, exactness is C*-flatness with respect to the minimal tensor product. It should be noted that the maximal tensor product, on the other hand, is functorial and preserves short exact sequences.

VIII.6 Hyperbolic groups

A geodesic metric space X is *hyperbolic* if there is $\delta \geq 0$ such that every geodesic triangle in X is δ-thin, i.e., each side of the triangle is contained in the δ-neighborhood of the union of the other two sides. It is a fundamental fact that hyperbolicity is invariant under quasi-isometries.

Hyperbolicity is a *coarse* notion of negative curvature. A *metric* notion of negative curvature is expressed by the CAT(κ) condition, where $\kappa < 0$, which says that geodesic triangles are thinner than comparison geodesic triangles in the standard hyperbolic plane of curvature κ. Clearly, a CAT(κ) space is hyperbolic.

A finitely generated group Γ is *(Gromov) hyperbolic* if its Cayley graph is hyperbolic; this definition is independent of the choice of generating set.

Virtually cyclic groups, i.e., finite groups and virtually-$\mathbb{Z}$ groups, are called *elementary*; they are hyperbolic groups and should be considered as low-dimensional accidents. More interesting examples of hyperbolic groups are provided by

· finitely generated free groups (all geodesic triangles in a tree are 0-thin);

· fundamental groups of compact Riemannian manifolds of negative sectional curvature (if M is such a manifold, then $\pi_1(M)$ is quasi-isometric to the universal cover $\widetilde{M}$; but $\widetilde{M}$ is hyperbolic since the negative sectional curvature assumption makes $\widetilde{M}$ a CAT(κ) space for some $\kappa < 0$);

· $C'(1/6)$ small-cancellation groups (to explain what the $C'(1/6)$-condition means, consider a finite presentation $\langle A \mid R\rangle$ whose relators are cyclically reduced, let $R^{\pm} = R \cup R^{-1}$ and think of the elements of $R^{\pm}$ as circular words; the $C'(1/6)$-condition for $\langle A \mid R\rangle$ requires that a common arc of two distinct circular words from $R^{\pm}$ has length less than a sixth of each word's length).

A subgroup of a hyperbolic group is either an elementary group, or it contains the free group on two generators. In particular, hyperbolic groups are amenable precisely when they are elementary.

VIII.7 Good topological algebras

A (complex) topological algebra is a complex algebra endowed with a topological vector space structure such that multiplication is separately continuous. The following class of topological algebras is of interest in this work:

Definition VIII.7.1. A topological algebra A is a *good topological algebra* if i) the group of invertibles $A^{\times}$ is open, and ii) the inversion $a \mapsto a^{-1}$ is continuous on $A^{\times}$.

This terminology is taken from [11]. Elsewhere in older literature, they are also called Waelbroeck algebras.

The main condition in the above definition is i). Some of what follows requires i) only, but we include ii) for several reasons. First, we do not know examples of topological algebras which satisfy i) but fail ii). Second, ii) is automatic for the topological algebras we are particularly interested in, namely Fréchet algebras. Third, the algebras that satisfy i) only have an uninspiring name: *topological Q-algebras*.

The next result is due to Swan [82, Corollary 1.2]:

Lemma VIII.7.2. *If A is a good topological algebra then* $\mathrm{M}_n(A)$ *is a good topological algebra.*

Let us substantiate the terminology of "good" by the following facts. The first fact implies, in particular, that the spectral radius is well-defined in a good topological algebra.

Lemma VIII.7.3. *If A is a good topological algebra, then* $\mathrm{sp}(a)$ *is compact for all $a \in A$.*

Proof. Let $a \in A$. We show $\mathrm{sp}(a)$ is closed: if $\lambda_n \in \mathrm{sp}(a)$ and $\lambda_n \to \lambda$, then $(\lambda_n - a)_n$ is a sequence of non-invertibles converging to $\lambda - a$, hence $\lambda - a$ is non-invertible i.e. $\lambda \in \mathrm{sp}(a)$. We show $\mathrm{sp}(a)$ is bounded: assuming, on the contrary, that there is $\lambda_n \in \mathrm{sp}(a)$ with $\lambda_n \to \infty$, we get that $(1 - \lambda_n^{-1}a)_n$ is a sequence of non-invertibles converging to 1, a contradiction. $\square$

Lemma VIII.7.4. *Let A be a good topological algebra. Then $\{a : \mathrm{sp}(a) \subseteq U\}$ is open in A for any open set $U \subseteq \mathbb{C}$.*

Proof. Putting $\Lambda = \mathbb{C} \setminus U$, we need $\{a : \mathrm{sp}(a) \text{ meets } \Lambda\}$ closed in A. So let $a_n \to a$ with $\mathrm{sp}(a_n)$ meeting Λ at, say, λ_n. If $(\lambda_n)_n$ is unbounded, by passing to a subsequence we may assume $\lambda_n \to \infty$, so $(1 - \lambda_n^{-1} a_n)_n$ is a sequence of non-invertibles converging to 1, a contradiction. Thus $(\lambda_n)_n$ is bounded and by passing to a subsequence we may assume that $\lambda_n \to \lambda \in \Lambda$. Then $(\lambda_n - a_n)_n$ is a sequence of non-invertibles converging to $\lambda - a$, thus $\lambda - a$ is non-invertible i.e. $\lambda \in \mathrm{sp}(a)$. We conclude that $\mathrm{sp}(a)$ meets Λ. □

Lemma VIII.7.5. *Let A be a good topological algebra, and $n \geq 1$. Then $\mathrm{Lg}_n(A)$ is open in A^n.*

Proof. Let $(a_i) \in \mathrm{Lg}_n(A)$, so $\sum b_i a_i = 1$ for some $(b_i) \in A^n$. There is a neighborhood U of 0 such that $\sum b_i U \subseteq A^\times - 1$. Then, for $a_i' \in a_i + U$, we have

$$\sum b_i a_i' \in \sum b_i a_i + \sum b_i U = 1 + \sum b_i U \subseteq 1 + (A^\times - 1) = A^\times$$

hence $(a_i') \in \mathrm{Lg}_n(A)$. □

VIII.8 Fréchet spaces and Fréchet algebras

In this section, we provide the definitions and the salient facts of the theory of Fréchet spaces and Fréchet algebras. Our exposition is quite detailed and essentially self-contained.

We shall deal with inverse systems and inverse limits, so let us recall these notions first. Categorically speaking, a countable *inverse system* in a category $\mathcal{C}$ is given by a sequence of objects $(X_k)_{k\geq 0}$ together with connecting morphisms $\xi_k : X_{k+1} \to X_k$. Diagrammatically:

$$\ldots \longrightarrow X_{k+1} \xrightarrow{\xi_k} X_k \longrightarrow \ldots \longrightarrow X_2 \xrightarrow{\xi_1} X_1 \xrightarrow{\xi_0} X_0$$

An object X equipped with morphisms $\pi_k : X \to X_k$ which are compatible with the connecting morphisms, in the sense that $\xi_k \circ \pi_{k+1} = \pi_k$, is an *inverse limit* for the inverse system $(X_k)_{k\geq 0}$ if it has the following universal property: for every object X' equipped with morphisms $\pi_k' : X' \to X_k$ satisfying $\xi_k \circ \pi_{k+1}' = \pi_k'$, there is a unique morphism $\phi : X' \to X$ such that $\pi_k' = \pi_k \circ \phi$. If it exists, the inverse limit is unique up to isomorphism and is denoted $\varprojlim X_k$. If the category $\mathcal{C}$ has products, then inverse limits exist in $\mathcal{C}$: the subobject X_∞ of $\prod X_k$ consisting of the *coherent sequences*

$$X_\infty = \{(x_k)_{k\geq 0} : \ \xi_k(x_{k+1}) = x_k\}$$

equipped with the (restrictions of the) natural projections $\pi_k : X_\infty \subseteq \prod X_k \to X_k$, is an inverse limit for the inverse system $(X_k)_{k\geq 0}$. We call X_∞ the *realization* of $\varprojlim X_k$. Note that the inverse limit is unchanged if one passes to a subsequence. More precisely, if $(X_{k_j})_{j\geq 0}$ is the inverse system whose connecting maps are obtained by composing the connecting maps of the inverse system $(X_k)_{k\geq 0}$, then $\varprojlim X_{k_j} \simeq \varprojlim X_k$. This can be shown abstractly, or concretely by noticing that a coherent sequence is uniquely determined by its entries over a subsequence $(k_j)_{j\geq 0}$.

Definition VIII.8.1. A topological vector space X is a *Fréchet space* if the topology of X is given by a family of seminorms $\{\|\cdot\|_k\}_{k\geq 0}$ which is

· separating: $\|x\|_k = 0$ for all $k \geq 0$ implies $x = 0$;

· ordered: $\|\cdot\|_k \leq \|\cdot\|_{k+1}$ for all $k \geq 0$;

· complete: if (x_n) is Cauchy under each $\|\cdot\|_k$, then there is $x \in X$ such that (x_n) converges to x under each $\|\cdot\|_k$.

We start by looking at Fréchet spaces which are inverse limits of Banach spaces; further in this section we shall see that every Fréchet space can be realized in this way. Consider an inverse system of Banach spaces

$$\ldots \longrightarrow X_{k+1} \xrightarrow{\xi_k} X_k \longrightarrow \ldots \longrightarrow X_2 \xrightarrow{\xi_1} X_1 \xrightarrow{\xi_0} X_0$$

where each X_k is equipped with a Banach norm $\|\cdot\|_k$. We may assume that the connecting maps $\xi_k : X_{k+1} \to X_k$ are non-expanding; clearly every inverse system of Banach spaces can be normalized as such by inductively rescaling the norms.

Endow $X_\infty = \left\{(x_k)_{k\geq 0} : \ \xi_k(x_{k+1}) = x_k\right\} \subseteq \prod X_k$ with the following family of seminorms:

$$\|(x_k)_{k\geq 0}\|_{(k)} := \|x_k\|_k$$

Using the (restrictions of the) natural projections $\pi_k : X_\infty \subseteq \prod X_k \to X_k$, we can write these seminorms as $\|x_\infty\|_{(k)} := \|\pi_k(x_\infty)\|_k$. It is easy to check that X_∞ is closed in $\prod X_k$, and that X_∞ is a Fréchet space under $\{\|\cdot\|_{(k)}\}_{k\geq 0}$.

In the category of Fréchet spaces (!), X_∞ is the inverse limit of the system of Banach spaces $(X_k, \|\cdot\|_k)$.

The next lemma will be useful later on:

Lemma VIII.8.2. *Let $Z \subseteq X_\infty$. Then Z is dense in X_∞ iff $\pi_k(Z)$ is dense in $\pi_k(X_\infty)$ for all k.*

Proof. The forward direction is trivial. For the converse, we show that Z meets every (non-empty) basic open set

$$X_\infty \cap \pi_{k_1}^{-1}(U_{k_1}) \cap \cdots \cap \pi_{k_j}^{-1}(U_{k_j})$$

where each U_{k_j} is an open subset of X_{k_j}. Let $N = \max\{k_1, \ldots, k_j\} + 1$ and pull back each U_{k_j} in X_N using the connecting maps, i.e., define the following open sets:

$$V_N^j = (\xi_{k_j} \circ \cdots \circ \xi_{N-1})^{-1}(U_{k_j}) \subseteq X_N$$

Then

$$X_\infty \cap \pi_N^{-1}(V_N^j) = X_\infty \cap (\xi_{k_j} \circ \cdots \circ \xi_{N-1} \circ \pi_N)^{-1}(U_{k_j}) = X_\infty \cap \pi_{k_j}^{-1}(U_{k_j})$$

hence

$$\begin{aligned} X_\infty \cap \pi_{k_1}^{-1}(U_{k_1}) \cap \cdots \cap \pi_{k_j}^{-1}(U_{k_j}) &= X_\infty \cap \pi_N^{-1}(V_N^1) \cap \cdots \cap \pi_N^{-1}(V_N^j) \\ &= X_\infty \cap \pi_N^{-1}(V_N^1 \cap \cdots \cap V_N^j) \end{aligned}$$

which meets Z by the density of $\pi_N(Z)$ in $\pi_N(X_\infty)$. □

In general, it is possible that the inverse limit X_∞ ends up being an uneventful 0-space. An obvious way to counteract this possibility is to demand the connecting maps to be surjective (in which case the projection maps would be surjective as well), but that would not fit well with many natural situations (see Example VIII.8 below). Instead, we consider an "almost" surjectivity condition on the connecting maps ξ_k which will make the projections maps π_k "almost" surjective as well.

The next statement is known as the abstract Mittag-Leffler theorem and it is due to Bourbaki.

Theorem VIII.8.3. *Assume that each connecting map $\xi_k : X_{k+1} \to X_k$ has dense image. Then each projection $\pi_k : X_\infty \to X_k$ has dense image.*

Proof. It suffices to show that $\pi_0(X_\infty)$ is dense in X_0, for one has $\pi_0(\varprojlim X_{k\geq i}) = \pi_i(\varprojlim X_{k\geq 0})$. Pick $x_0 \in X_0$ and $\varepsilon > 0$. For each k, pick $x_{k+1} \in X_{k+1}$ such that $\|x_k - \xi_k(x_{k+1})\|_k < \varepsilon/2^k$.

Fix k. For $n \geq k$ let $z_k^{(n)} = (\xi_k \circ \cdots \circ \xi_{n-1})(x_n) \in X_k$, where $z_k^{(k)} = x_k$. As each connecting map is non-expanding, we have:

$$\|z_k^{(n)} - z_k^{(n+1)}\|_k \leq \|z_n^{(n)} - z_n^{(n+1)}\|_n = \|x_n - \xi_n(x_{n+1})\|_n < \varepsilon/2^n$$

Thus $\big(z_k^{(n)}\big)_{n\geq k}$ is Cauchy; let $z_k^{(n)} \to y_k$ in X_k. In particular, for $k = 0$ we get that $z_0^{(0)} = x_0$ is within 2ε of y_0. We are therefore done once we show $(y_k)_{k\geq 0} \in X_\infty$. But $z_{k+1}^{(n)} \to y_{k+1}$ implies $z_k^{(n)} = \xi_k\big(z_{k+1}^{(n)}\big) \to \xi_k(y_{k+1})$, hence $\xi_k(y_{k+1}) = y_k$. □

The abstract Mittag-Leffler theorem holds, more generally, in the context of complete metric spaces. Interpreting the density of the projections as a non-vanishing result for the inverse limit, Bourbaki obtains the classical Mittag-Leffler theorem in complex analysis concerning the existence of meromorphic functions with prescribed poles. Density as a means for existence is the motivation that underlies the Baire category theorem, as well; in fact, one can easily derive the latter from the abstract Mittag-Leffler theorem; see [30].

We derive two corollaries. The first is an improvement on Lemma VIII.8.2:

Corollary VIII.8.4. *Assume that each connecting map $\xi_k : X_{k+1} \to X_k$ has dense image, and let $Z \subseteq X_\infty$. Then Z is dense in X_∞ iff $\pi_k(Z)$ is dense in X_k for all k.*

The second corollary says that an inverse system of Banach spaces with dense connecting maps is not normable, unless it is "stationary" from a certain point on:

Corollary VIII.8.5. *Assume that each connecting map $\xi_k : X_{k+1} \to X_k$ has dense image. Then X_∞ is normable iff the connecting maps are eventually bijective.*

Proof. Assume ν is a Banach norm on X_∞ which induces the Fréchet topology we already have on X_∞. Then $\nu \leq C\|\cdot\|_{(N)}$ for some N, and $\|\cdot\|_{(k)} \leq C_k\nu$ for all k; hence $\|\cdot\|_{(k)} \leq C'_k\|\cdot\|_{(N)}$ for all k. When $k \geq N$, we obtain $\|\cdot\|_{(k+1)} \leq C'_{k+1}\|\cdot\|_{(N)} \leq C'_{k+1}\|\cdot\|_{(k)}$. Thus:

$$c_k\|\cdot\|_{(k+1)} \leq \|\cdot\|_{(k)} \leq \|\cdot\|_{(k+1)} \quad (k \geq N)$$

This leads to

$$c_k\|\cdot\|_{k+1} \leq \|\xi_k(\cdot)\|_k \leq \|\cdot\|_{k+1} \quad (k \geq N)$$

on $\pi_{k+1}(X_\infty)$. Since $\pi_{k+1}(X_\infty)$ is dense in X_{k+1}, it follows that the previous double inequality actually holds on X_{k+1}. That makes ξ_k bijective for $k \geq N$.

Conversely, if ξ_k bijective for $k \geq N$ then the Open Mapping Theorem guarantees that a double inequality as above holds for $k \geq N$. We quickly obtain the equivalence of the seminorms $\|\cdot\|_{(k)}$ for $k \geq N$, and X_∞ is then normed by $\|\cdot\|_{(N)}$. $\square$

Next, we show that every Fréchet space can be realized as an inverse limit of Banach spaces. This is the Arens - Michael theorem.

Let X be a Fréchet space under the family of seminorms $\{\|\cdot\|_k\}_{k\geq 0}$. Let X_k denote the $\|\cdot\|_k$-completion of X modulo the degenerate subspace of $\|\cdot\|_k$. The ordering on the seminorms implies that, before completions, we had an inverse system of normed spaces with surjective and non-expanding connecting maps:

$$\cdots \overset{\theta_2}{\twoheadrightarrow} X/_{\{\|\cdot\|_2=0\}} \overset{\theta_1}{\twoheadrightarrow} X/_{\{\|\cdot\|_1=0\}} \overset{\theta_0}{\twoheadrightarrow} X/_{\{\|\cdot\|_0=0\}}$$

After completions, we have an inverse system of Banach spaces with dense and non-expanding maps:

$$\cdots \xrightarrow{\overline{\theta}_2} X_2 \xrightarrow{\overline{\theta}_1} X_1 \xrightarrow{\overline{\theta}_0} X_0$$

The inverse limit of this inverse system is a Fréchet space denoted X_∞.

Proposition VIII.8.6. *X_∞ is isometrically isomorphic to X.*

Proof. There is a natural map $p_k : X \to X_k$, and $\|x\|_k = \|p_k(x)\|_{X_k}$. Let $p = \prod p_k : X \to \prod X_k$. The fact that $\{\|\cdot\|_k\}_{k\geq 0}$ is separating translates into p being injective. As $\overline{\theta}_k(p_{k+1}(x)) = p_k(x)$, we have $p(X) \subseteq X_\infty$. Furthermore, for each k, $\|x\|_k = \|p_k(x)\|_{X_k} = \|p(x)\|_{(k)}$ meaning that $p : X \to X_\infty$ is isometric as a map between Fréchet spaces. We are left with showing the surjectivity of p. By the completeness of X, it suffices to show that $p(X)$ dense in X_∞. For that, it suffices to show that $\pi_k(p(X))$ is dense in X_k for all k. But that is obvious: $\pi_k(p(X)) = p_k(X)$, which is dense in X_k. $\square$

The inverse limit of a descending chain of Banach spaces

$$\cdots \subseteq X_2 \subseteq X_1 \subseteq X_0$$

can be identified with $\bigcap_{k\geq 0} X_k$ via the diagonal embedding. The Fréchet structure on $\bigcap_{k\geq 0} X_k$ is simply the restriction of the family $\{\|\cdot\|_k\}_{k\geq 0}$ to $\bigcap_{k\geq 0} X_k$.

Assume each X_{k+1} is dense in X_k. Then Theorem VIII.8.3 says that X_∞ is dense in each X_k. Furthermore, if the descending sequence (X_k) does not stabilize, then X_∞ is not normable by Corollary VIII.8.5.

We end our discussion by mentioning that Fréchet spaces enjoy the three fundamental principles of functional analysis: the Open Mapping Theorem, the Closed Graph Theorem, and the Uniform Boundedness

Principle. As an application of the Closed Graph Theorem for Fréchet spaces, we state the following useful fact:

Lemma VIII.8.7. *Let X_∞, Y_∞ be Fréchet subspaces of Banach spaces X, Y such that the inclusions $X_\infty \hookrightarrow X$ and $Y_\infty \hookrightarrow Y$ are continuous. Then a restriction $\phi| : X_\infty \to Y_\infty$ of a continuous linear map $\phi : X \to Y$ is continuous.*

In particular, if X_∞, X'_∞ are Fréchet subspaces of the Banach space X such that the inclusions $X_\infty \hookrightarrow X$ and $X'_\infty \hookrightarrow X$ are continuous, and if $X_\infty \subseteq X'_\infty$, then the inclusion $X_\infty \hookrightarrow X'_\infty$ is continuous.

We now move to Fréchet algebras. Recall, algebras and their morphisms are assumed to be unital; however, in the next definition we do not require the seminorms to be unital.

Definition VIII.8.8. A *Fréchet algebra A* is an algebra endowed with a family of seminorms $\{\|\cdot\|_k\}_{k\geq 0}$ such that:

· A is a Fréchet space under the seminorms $\{\|\cdot\|_k\}_{k\geq 0}$

· for each k the seminorm $\|\cdot\|_k$ is submultiplicative, i.e., $\|ab\|_k \leq \|a\|_k\|b\|_k$ for all $a, b \in A$

If A is a Fréchet algebra under the seminorms $\{\|\cdot\|_k\}_{k\geq 0}$ then $\mathrm{M}_n(A)$ is a Fréchet algebra under the seminorms given by $\|(a_{ij})\|_k = \sum_{i,j}\|a_{ij}\|_k$; this will be the standard Fréchet structure on matrix algebras in what follows.

Remark VIII.8.9. It is also common to call a Fréchet algebra a topological algebra that is a Fréchet space as a topological vector space. A Fréchet algebra with submultiplicative seminorms is then called (following Michael [54]) a *locally multiplicatively-convex* or, simply, an *m-convex* Fréchet algebra.

The terminology "σ-Banach algebra" for the type of algebra described in Definition VIII.8.8 would be more suggestive, and it would avoid the confusion around what is meant by a Fréchet algebra. Furthermore, it would be consistent with the notion of σ-C*-algebras which is already in use. Alas, terminologies are what they are.

Mutatis mutandis, our discussion of Fréchet spaces extends to the context of Fréchet algebras. Consider an inverse system of Banach algebras

$$\ldots \longrightarrow A_{k+1} \xrightarrow{\alpha_k} A_k \longrightarrow \ldots \longrightarrow A_2 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_0} A_0$$

where each A_k is equipped with a Banach algebra norm $\|\cdot\|_k$. We may assume the connecting morphisms $\alpha_k : A_{k+1} \to A_k$ to be non-expanding. The inverse limit $\varprojlim A_k$ in the category of Fréchet algebras is realized by the (unital) Fréchet algebra

$$A_\infty = \big\{(a_k)_{k\geq 0} :\ \alpha_k(a_{k+1}) = a_k\big\} \subseteq \prod A_k$$

equipped with the family of seminorms described by $\|a\|_{(k)} := \|\pi_k(a)\|_k$.

The inverse limit of an inverse system of C*-algebras with surjective (equivalently, dense) connecting morphisms is termed a σ-C*-algebra.

Conversely, every Fréchet algebra as an inverse limit of Banach algebras in a canonical way. Let A be a Fréchet algebra under the family of seminorms $\{\|\cdot\|_k\}_{k\geq 0}$. Let A_k be the (unital) Banach algebra obtained by completing A modulo the degeneracy ideal of $\|\cdot\|_k$. We obtain an inverse system of Banach algebras with non-expanding, dense morphisms:

$$\cdots \xrightarrow{\bar\theta_2} A_2 \xrightarrow{\bar\theta_1} A_1 \xrightarrow{\bar\theta_0} A_0$$

Let A_∞ denote the Fréchet algebra arising as the inverse limit of the inverse system thus obtained. Then A_∞ is isometrically isomorphic to A.

Proposition VIII.8.10. *Let A_∞ denote the inverse limit $\varprojlim A_k$. Then:*

a) $a \in A^\times$ iff $\pi_k(a) \in A_k^\times$ for all k;

b) $\mathrm{sp}_{A_\infty}(a) = \bigcup \mathrm{sp}_{A_k}(\pi_k(a))$ for $a \in A_\infty$. In particular, $\mathrm{sp}(a)$ is nonempty for all $a \in A_\infty$;

$r_{A_\infty}(a) = \sup_k r_{A_k}(\pi_k(a))$ for $a \in A_\infty$. In particular, if $r_{A_\infty}(1-a) < 1$ then $a \in A_\infty^\times$.

Proof. a) The forward direction is clear; we show the converse. For each k let $b_k \in A_k$ be the inverse of $\pi_k(a)$. As b_{k+1} is the inverse of $\pi_{k+1}(a)$, we get $\alpha_k(b_{k+1})$ is the inverse of $\alpha_k(\pi_{k+1}(a)) = \pi_k(a)$. Thus $\alpha_k(b_{k+1}) = b_k$, and we conclude that $b = (b_k)_{k\geq 0}$ is an inverse of a in A_∞.

b) is a spectral reformulation of a), and c) follows easily. □

Let $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ be a descending chain of Banach algebras with continuous inclusions. Then $\cap_{k\geq 0} A_k$ is a Fréchet subalgebra of A.

If each A_{k+1} is dense in A_k (equivalently, if each A_k is dense in A) then $\cap_{k\geq 0} A_k$ is dense in A.

If each A_{k+1} is spectral in A_k (equivalently, if each A_k is spectral in A) then $\cap_{k\geq 0} A_k$ is spectral in A.

Inversion is a continuous operation on $A_\infty^\times$; this is justified by mimicking the Banach algebra proof. However, $A_\infty^\times$ may not be open, i.e., A_∞ may not be a good topological algebra.

Let X be a σ-compact and locally compact Hausdorff space, for instance a proper metric space. We can exhaust X by a non-decreasing sequence of compact subsets $(X_k)_{k\geq 0}$ with $X_k \subseteq \mathrm{int}\, X_{k+1}$; the advantage of such an exhaustion is the fact that f is continuous on X iff the restriction of f to each X_k is continuous. Then $C(X)$ has a natural Fréchet algebra structure, given by the seminorms $\|f\|_k = \sup_{x\in X_k} |f(x)|$. Its canonical inverse system of Banach algebras is

$$\ldots \twoheadrightarrow C(X_2) \twoheadrightarrow C(X_1) \twoheadrightarrow C(X_0).$$

The invertible group of $C(X)$, consisting of the non-vanishing functions, is not open. Indeed, let f_k be a continuous extension of the characteristic function of X_k. Then (f_k) is a sequence of non-invertibles converging to 1.

The Fréchet algebra $C(X)$ is a (commutative) σ-C*-algebra.

Inspired by the above example, one can formulate a general situation where a Fréchet algebra fails to have an open group of invertibles.

Proposition VIII.8.11. *Let $\cdots \overset{\alpha_2}{\twoheadrightarrow} A_2 \overset{\alpha_1}{\twoheadrightarrow} A_1 \overset{\alpha_0}{\twoheadrightarrow} A_0$ be an inverse system of semisimple Banach algebras and let A_∞ be its inverse limit. If $A_\infty^\times$ is open, then the connecting morphisms α_k are eventually isomorphisms.*

In particular, if a σ-C^**-algebra has an open group of invertibles, then it is actually a* C^**-algebra.*

Proof. If A is semisimple, and $\alpha : A \twoheadrightarrow B$ is not injective, then there is a non-invertible $a \in A$ such that $\alpha(a) = 1$. Indeed, the non-zero ideal $\ker\alpha$ cannot be contained in $1 + A^\times$, by the semisimplicity of A. Hence, the identity of B can be lifted to a non-invertible in A.

Arguing by contradiction, say that there is a subsequence (α_{k_j}) of non-injectives. For each j, let $a^{(j)} \in A_\infty$ be obtained as follows: lift $1 \in A_0$ to $1 \in A_1$ and so on to $1 \in A_{k_j}$, then to a non-invertible in A_{k_j+1}, after which lift arbitrarily. Then $a^{(j)}$ is non-invertible, and $a^{(j)} \to 1$ in A_∞. $\square$

Remark VIII.8.12. If X, Y are topological spaces then $X(Y)$ denotes the continuous maps from Y to X. Let X be a compact Hausdorff space and let A be a Fréchet algebra under the seminorms $\{\|\cdot\|_k\}_{k\geq 0}$. Then $A(X)$ is a Fréchet algebra under the seminorms $\|f\|_k := \sup_{x\in X}\|f(x)\|_k$. As inversion is continuous in $A^\times$, we have $A(X)^\times = A^\times(X)$. Since $V(X)$ is open in $A(X)$ whenever V is an open subset of A, we obtain in particular that $A(X)$ is good whenever A is good.

K-theory for Fréchet algebras is rather technical. In [67], Phillips defines and studies a "representable" K-theory which coincides with the usual K-theory in the case of good Fréchet algebras ([67, Thm.7.7]). Phillips is partly motivated by the context of σ-C^*-algebras - which, we have seen above, are not good in the non-trivial case. A systematic use is made of the fact that a Fréchet algebra is the inverse limit of an inverse system of Banach algebras with dense connecting morphisms; however, the details of lifting Banach algebraic results to the Fréchet algebraic level of generality are painful.

As a particular case of [67, Thm.6.5], we have the following Milnor $\varprojlim^1$-sequence:

Theorem VIII.8.13. *Let* $A_\infty = \varprojlim A_k$ *be a good Fréchet algebra, arising as the inverse limit of an inverse system of Banach algebras with dense connecting maps:*

$$\ldots \longrightarrow A_{k+1} \xrightarrow{\alpha_k} A_k \longrightarrow \ldots \longrightarrow A_2 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_0} A_0$$

Then there is a natural short exact sequence

$$0 \longrightarrow \varprojlim{}^1 K_{1-*}(A_k) \longrightarrow K_*(A_\infty) \longrightarrow \varprojlim K_*(A_k) \longrightarrow 0$$

Remark VIII.8.14. Let us describe the functor $\varprojlim^1$. Consider an inverse system of abelian groups

$$\ldots \longrightarrow G_{k+1} \xrightarrow{\phi_k} G_k \longrightarrow \ldots \longrightarrow G_2 \xrightarrow{\phi_1} G_1 \xrightarrow{\phi_0} G_0$$

and let $\Delta : \prod G_k \to \prod G_k$ be given by

$$\Delta\big((g_k)_{k\geq 0}\big) = \big(g_k - \phi_k(g_{k+1})\big)_{k\geq 0}.$$

Then $\varprojlim G_k$ is (isomorphic to) the kernel of Δ, and we define $\varprojlim^1 G_k$ to be the cokernel of Δ. We therefore have an exact sequence

$$0 \longrightarrow \varprojlim{}^1 G_k \longrightarrow \prod G_k \xrightarrow{\Delta} \prod G_k \longrightarrow \varprojlim G_k \longrightarrow 0.$$

If the connecting maps ϕ_k are surjective, then $\varprojlim^1 G_k = 0$.

Holomorphic functional calculus for Fréchet algebras can be obtained from the holomorphic functional calculus for Banach algebras; this is done via the Arens - Michael realization of Fréchet algebras as inverse limits of Banach algebras ([67, Lem.1.3]).

For good Fréchet algebras, we have a direct extension of the holomorphic functional calculus for Banach algebras:

Proposition VIII.8.15. *Let A be a good Fréchet algebra, $a \in A$ and $\Omega \subseteq \mathbb{C}$ an open neighborhood of* $\mathrm{sp}(a)$. *Then there is a unique (unital and continuous) morphism $O(\Omega) \to A$ sending* id_Ω *to a, given by*

$$O_a(f) = f(a) := \frac{1}{2\pi i} \oint f(\lambda)(\lambda - a)^{-1} d\lambda$$

where the integral is taken around a cycle (finite union of closed paths) in Ω containing $\mathrm{sp}(a)$ *in its interior. Furthermore, we have* $\mathrm{sp} f(a) = f(\mathrm{sp}(a))$ *for each $f \in O(\Omega)$.*

The unique morphism indicated by the previous proposition is referred to as the holomorphic calculus for a. Here $O(\Omega)$, the unital algebra of functions that are holomorphic in Ω, is endowed with the topology of uniform convergence on compacts.

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