# The Novikov Conjecture, the Group of Volume Preserving Diffeomorphisms and Non-Positively Curved Hilbert Manifolds

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*To my parents, Zhihao Wu and Chunhui Li*

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# CHAPTER I

## INTRODUCTION

Since the advent of topology, many numerical topological invariants have been found to be computable with drastically different methods. For example, the winding number of a curve in a plane around the origin can be computed by various counting algorithms, or by evaluating, after equating the plane with the set of complex numbers, the integral $\frac{1}{2\pi i}\int df/f$; the Euler characteristic of a surface can also be computed by counting on the combinatoric structure, counting of singularities of a vector field, and integrating the Gaussian curvature over the entire surface. These examples hint at the deep interplay between topology and analysis.

Since the 19th century, a series of theorems have been discovered to demonstrate such an interplay, such as the Riemann-Roch theorem, the Chern-Gauss-Bonnet theorem and the Hirzebruch signature theorem, cumulating in the 1960's to the celebrated Atiyah-Singer index theorem, which generalizes all previous theorems by stating that, for an elliptic pseudo-differential operator $D$ on a compact manifold $M$, its Fredholm index — a numerical invariant associated to dimensions of solution spaces of certain differential equations — is indeed topological, and can be computed in terms of the coefficients of $D$.

In the first proof they published ([2, 3]), Atiyah and Singer made extensive use of the newly-developed topological $K$-theory, a kind of cohomological theory that has proved to be the right tool to study various index problems. Later on, Atiyah([1]) and Kasparov([14]) discovered that the $K$-homology $K_0(M)$, the dual theory to the topological $K$-theory, classifies all elliptic pseudo-differential operators on $M$ up to homotopy, and assigning index constitutes a natural map $\mu : K_0(M) \to \mathbb{Z}$. Further attempts at generalizing the Atiyah-Singer index theorem led Alain Connes to develop the theory of noncommutative geometry.

One of the most prolific ideas in modern mathematics is that one can study the structure of a space by investigating all functions on the space compatible with the structure. Its incarnation in the area of topology is the commutative Gelfand-Naimark theorem: every *commutative* C*-algebra (i.e. a commutative closed subalgebra of all bounded linear operators from a Hilbert space to itself that is closed under taking adjoints) can be represented as $C_0(X)$, the algebra of all complex-valued continuous functions on a *locally compact, Hausdorff* space $X$ that vanish at infinity, and consequently, studying locally compact Hausdorff topological spaces is equivalent to studying commutative C*-algebras.

By analogy, a noncommutative C*-algebra can be considered the algebra of all continuous "functions" on a "noncommutative topological space". This is not merely an artificial generalization, as it allows one to study various pathological (e.g. non-Hausdorff) spaces that refuse conventional treatments; for instance, the quotient space of the unit circle by an irrational rotation can be studied via a noncommutative C*-algebra, namely the irrational rotation algebra([7]). This new perspective also incorporates harmonic analysis of any locally compact non-abelian group $G$ by examining the "noncommutative space" underlying the *reduced group C*-algebra* $C_r^*(G)$, which in the case when $G$ is discrete, is defined to be the norm closure of the group algebra $\mathbb{C}[G]$, viewed as an algebra of operators over the Hilbert space $L^2(G)$ defined using the Haar measure.

One may also study algebraic topology on noncommutative spaces. In particular, the topological $K$-theory extends immediately to the operator $K$-theory of C*-algebras, and they agree in the case of commutative spaces. Operator $K$-theory is a powerful tool for distinguishing C*-algebras. For instance, it is the main ingredient in the so-called dimension group which Elliott employed to give a complete classification of approximately finite dimensional (AF) C*-algebras ([9]). And by the work of Pimsner, Voiculescu and Cuntz ([20, 8]), one can use $K$-theory to answer (positively) the question whether the reduced free group C*-algebras $C_r^*(F_n)$ are non-isomorphic for different $n$'s, a feat yet to be accomplished for the free group *von Neumann algebras* in the study of noncommutative measure theory.

The ideas of noncommutative geometry relate to the index theorem in the following way: Suppose we are given a family of pseudo-differential operators parametrized by a compact, Hausdorff space $X$, then we should acquire an index in $K^*(X)$, the $K$-theory of $X$ (in the classical setting, $X = \{\text{point}\}$ and thus $K^0(X) \cong \mathbb{Z}$, giving us the usual Fredholm index). Now instead suppose the pseudo-differential operator carries some intrinsic symmetry: the underlying manifold is acted upon properly and co-compactly by a group $G$ and the operator is equivariant with regard to the action. Then we can think of the operator as a family of operators parametrized by the (usually noncommutative) space given by the reduced group C*-algebra $C_r^*(G)$, and obtain an index map $\mu : K_*^G(M) \to K_*(C_r^*(G))$ from the *equivariant* $K$-homology of $M$ to the operator $K$-theory of $C_r^*(G)$. This is usually called the higher index as it provides more refined information about the operator.

This simple generalization has profound implications. These can be seen in two directions: On the one hand, when $G$ is non-abelian, $K_*(C_r^*(G))$ is often very hard to calculate, as it concerns the representation theory of $G$, while $K_*^G(M)$ is readily computable via, for instance, Mayer-Vietoris sequences, and thus the index map $\mu$ supplies one with a tool to probe the former. Of course this method recovers the entire $K_*(C_r^*(G))$ at once only if $\mu$ is *surjective*. On the other hand, several analytic properties can a priori only be read off from the index in $K_*(C_r^*(G))$; for example, a spin manifold carries a Riemannian structure with everywhere positive scalar curvature only if the index of the Dirac operator $D$ is 0 (the identity) in $K_*(C_r^*(G))$, and therefore comparing it with the non-trivial topological invariant $[D] \in K_*^G(M)$, one concludes that the *injectivity* of the index map would creat a topological obstruction for $M$ to carry *any* Riemannian structure with positive scalar curvature.

Of course, one can hope for $\mu$ to be bijective only if the topology of $M$ carries no other data than those coming from $G$. A natural choice would be taking $M$ to be the universal proper $G$-space $\underline{E}G$, i.e. a space that classifies all proper actions of G. $\underline{E}G$ is only defined up to homotopy, and may well not be a manifold, but there is no technical difficulty in defining its $G$-equivariant $K$-homology; however, one needs a remedy for the possibility that $\underline{E}G$ may not be co-compact with respect to the $G$-action, and thus one replaces $K_*^G(\underline{E}G)$ with $K_*^G(\underline{E}G)$, its equivariant $K$-homology with $G$-co-compact supports([14]). In 1982, Baum and Connes ([4]) put out (essentially) the conjecture that the index map $\mu : K_*^G(\underline{E}G) \to K_*(C_r^*(G))$ is an isomorphism.

The statement is for each individual group, and the index map $\mu$ is also known as the *assembly map*. Morally, it says the analytic information of $G$ given by the right-hand side of the arrow is entirely captured by its geometric information given by the left-hand side. This insightful conjecture, if true, would take quite a number of famous conjectures as corollaries. For instance, for a discrete group $\Gamma$, the injectivity of $\mu$ would imply the Novikov conjecture in surgery theory, while its surjectivity would imply the Kadison-Kaplansky conjecture on idempotents in $C_r^*(G)$.

Until this day, many classes of groups have been proved to satisfy the conjecture. These include a-T-menable groups ([12]), Gromov hyperbolic groups ([18, 16]), one-relator groups ([5]), fundamental groups of Haken 3-manifolds ([19, 21]), etc. The simplest groups for which the conjecture is not known are the integral special linear groups $SL(n, \mathbb{Z})$ for $n > 2$. The study of this conjecture has been and continues to be a major driving force in noncommutative geometry, and inspired a great deal of novel ideas and tools.

A natural question along this line is: Can one prove the Novikov conjecture for a discrete group $\Gamma$ that acts isometrically and properly on a simply-connected, non-positively-curved *Hilbert* manifold $M$ (the infinite-dimensional counterpart of a Riemannian manifold)? Here the properness of an action can only be defined metrically: For any *bounded* subset $X$ of $M$, all but finitely many elements in $\Gamma$ moves $X$ to a subset disjoint from $X$.

The flexibility of infinite-dimensional manifolds allows one to include many examples, such as the finite-dimensional cases, the Hilbert space, and the following, which is more or less our main motivating example:

**Example I.1.** *when a group $\Gamma$ acts as volume-form-preserving diffeomorphisms on a compact orientable $n$-dimensional manifold $N$ with a given volume form $\omega$ ([10]). Then the space of all Riemannian metrics on $N$ having $\omega$ as their volume form carries a natural Rieman-*

*nian inner product structure, and taking completion with regard to it gives rise to a Hilbert manifold consisting of the so-called $L^2$-Riemannian metrics on $N$, which can be seen as the space of $L^2$-sections on an $SL(n,R)/SO(n)$-fiber bundle over $N$. Now as $SL(n,\mathbb{R})/SO(n)$ is a simply-connected, non-positively-curved Riemannian manifold, the Hilbert manifold we obtain inherits these properties, regardless of the geometry of $N$, and $\Gamma$ acts on this Hilbert manifold by permuting these Riemannian metrics. The study of the Novikov conjecture for diffeomorphism group was pioneered by Connes [6].*

The main result in this paper is the following:

**Theorem I.2.** *Let $\Gamma$ be a discrete torsion-free group that admits an isometric, metrically proper action on a complete, simply-connected non-positively curved Hilbert manifold $M$ that has a completely geodesic finite-dimensional submanifolds whose union is dense in $M$. Assume that the action is null-homotopic. Then $\Gamma$ satisfies the rational analytic Novikov conjecture, i.e. the rational assembly map*

$$\mu : K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C_r^*(\Gamma)) \otimes \mathbb{Q}$$

*is injective.*

The work of Higson and Kasparov [12] on a-T-menable groups shows that if the underlying Hilbert manifold is flat, i.e. simply a Hilbert space, then the group actually satisfies the stronger Baum-Connes conjecture. Yet the non-flat case is all the more difficult, as even in the finite-dimensional case, which already includes examples like $SL(n,\mathbb{Z})$, it is not known how to prove the surjectivity part of the Baum-Connes conjecture. Moreover, certain constructions used in their paper, innovative as they were, are specific to the structure of Hilbert spaces, and are not known to be generalizable to encompass the case of non-positively curved Hilbert manifolds. Therefore, in order the prove the Novikov conjecture for groups acting isometrically and metrically properly on non-positively curved Hilbert manifolds, it is necessary for us to develop new techniques that apply to the more general structure of these Hilbert manifolds.

CHAPTER II

# PRELIMINARIES

## II.1 $C^*$-algebras

Given a complex Hilbert space $\mathcal{H}$, we form $\mathfrak{B}(\mathcal{H})$, the set of all bounded linear operators from $\mathcal{H}$ to itself. This is an algebra: a vector space (over $\mathbb{C}$) together with a multiplication $(T_1, T_2) \mapsto T_1T_2$ given by composition of operators. It is easy to check that the multiplication is associative and bilinear as a binary operation. There is one more crucial operation, the "adjoint" operation, also called the "star" operation, taking $T \in \mathfrak{B}(\mathcal{H})$ to its adjoint $T^*$, the unique operator that satisfies

$$< T\xi, \eta >=< \xi, T^*\eta >, \ \forall \xi, \eta \in \mathcal{H}$$

This unary operation is said to be a conjugate linear involution:

$$(T_1 + T_2)^* = T_1^* + T_2^* \tag{II.1}$$

$$(\lambda T)^* = \overline{\lambda} T^* \tag{II.2}$$

$$T^{**} = T \tag{II.3}$$

$$(T_1T_2)^* = T_2^*T_1^* \tag{II.4}$$

On the other hand, it is a Banach space under the operator norm

$$\|T\| := \sup_{\xi \in \mathcal{H}, \ \|\xi\|=1} \|T\xi\|$$

The topology it determines is called the norm topology, with regard to which all those operations are continuous. Besides the basic axioms of a Banach space, a version of the triangle inequality for multiplication holds and can be easily checked:

$$\|T_1T_2\| \leq \|T_1\|\|T_2\| \tag{II.5}$$

The relation between the norm and the adjoint operation, however, is more subtle. One can easily see that the star operation is an isometry, that is

$$\|T^*\| = \|T\|$$

But there is a more essential equality, called the $C^*$-identity:

$$\|T^*T\| = \|T\|^2 \tag{II.6}$$

This can be proved by noticing that the left-hand side is equal to

$$\sup_{\|\xi\|=\|\eta\|=1} < T^*T\xi, \eta > = \sup_{\|\xi\|=\|\eta\|=1} < T\xi, T\eta >$$

which is equal to the right-hand side.

**Definition II.1.1.** A *concrete $C^*$-algebra* $\mathcal{A}$ is a norm-closed $*$-subalgebra of $\mathfrak{B}(\mathcal{H})$, that is, a linear subspace closed under the norm topology, and also closed under multiplication and the star operation.

**Example II.1.2.** $\mathfrak{B}(\mathcal{H})$ itself is an example of a concrete $C^*$-algebra. Its subset consisting of all the finite-rank (i.e. finite-dimensional image) operators is clearly a $*$-subalgebra; thus by the norm-continuity of the three basic operations, taking norm-closure of this subset would give us a concrete $C^*$-algebra, denoted by $\mathfrak{K}(\mathcal{H})$, whose elements are called the compact operators. This is also a closed ideal of $\mathfrak{B}(\mathcal{H})$: $\forall T \in \mathfrak{B}(\mathcal{H})$ and $K \in \mathfrak{K}(\mathcal{H})$, we have $KT \in \mathfrak{K}(\mathcal{H})$ and $TK \in \mathfrak{K}(\mathcal{H})$. In the case where $\mathcal{H}$ is finite-dimensional, both $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{K}(\mathcal{H})$ are equal to $\mathcal{M}_n(\mathbb{C})$, where $n$ is the dimension of $\mathcal{H}$.

Notice that the three basic operations restrict to a concrete $C^*$-algebra $\mathcal{A}$, and the above equalities and inequalities, being merely quantifier-free formulas over elements of $\mathfrak{B}(\mathcal{H})$, are still satisfied in $\mathcal{A}$. This observation leads to the following:

**Definition II.1.3.** An *abstract $C^*$-algebra* is a complex normed algebra which is complete under the norm topology, has another unary operation called adjoint, and satisfies the axioms (II.1) - (II.6) above. $\mathcal{A}$ is called *unital* if it has a multiplicative identity $I$.

**Remark II.1.4.** If we drop condition (II.6), we obtain the more general notion of a Banach $*$-algebra, and if we further drop the adjoint operation, we obtain the notion of a Banach algebra.

**Remark II.1.5.** It is worthwile to notice that conditions (II.3), (II.5) and (II.6) are sufficient to force the adjoint operation to be an isometry. Indeed,

$$\|T\|^2 = \|T^*T\| \leq \|T^*\|\|T\|$$

implies $\|T^*\| \geq \|T\|$ and the fact that adjoint is an involution gives the opposite inequality.

**Remark II.1.6.** Every non-unital $C^*$-algebra can be embedded into a unital one.

Although one loses the underlying Hilbert space, some common notions in $\mathfrak{B}(\mathcal{H})$ are carried over to this abstract setting: an operator $T \in \mathcal{A}$ is called

1. self-adjoint if $T^* = T$,
2. normal if $T^*T = TT^*$,
3. an idempotent if $T^2 = T$,
4. a projection if $T = T^* = T^2$,
5. invertible if $\exists T'$ such that $T'T = TT' = I$,
6. unitary if $T^*T = TT^* = I$,
7. isometry if $T^*T = I$,
8. partial isometry if $TT^*T = T$.

(5)-(7) only make sense when $\mathcal{A}$ is unital.

The flexibility of the abstract definition leads to a natural definition of morphisms between abstract $C^*$-algebras.

**Definition II.1.7.**

1. A $*$-*homomorphism* $\Phi : \mathcal{A} \to \mathcal{B}$ is a continuous linear map that preserves multiplication and adjoint.
2. The *kernel* of $\Phi$ is the ideal $\mathrm{Ker}(\Phi) := \{T \in \mathcal{A} \colon \Phi(T) = 0\}$.
3. $\Phi$ is an *isometric $*$-isomorphism* if it is bijective and preserves norm.
4. A *representation* of $\mathcal{A}$ is a $*$-homomorphism$\rho : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

In a short moment we will see that we can drop the continuity and isometry requirement in the definitions of $*$-homomorphisms and $*$-isomorphisms. But first, we need to introduce an important example.

**Example II.1.8.** Let $X$ be a compact Hausdorff space. The collection $C(X)$ of all continuous complex-valued functions on $X$ is an abstract unital $C^*$-algebra with norm being the supremum norm $\|f\| := \sup_{x \in X} \|f(x)\|$, the multiplication and adjoint being pointwise multiplication and conjugation.

**Example II.1.9.** More generally, for a locally compact[1] Hausdorff topological space $X$, the collection $C_b(X)$ of all bounded continuous complex-valued functions on $X$ is an abstract unital $C^*$-algebra with the same definition of norm, multiplication and adjoint. It has a non-closed $*$-subalgebra and ideal $C_c(X)$ consisting of all continuous complex-valued functions with compact support, i.e. zero outside a compact subset of $X$. Its closure $C_0(X)$ is a non-unital abstract $C^*$-algebra, also a closed ideal of $C_b(X)$. (In the compact case these algebras are all equal to $C(X)$)

**Example II.1.10.** If $X$ is equipped with a metric $d$, $C_b(X)$ has another $C^*$-subalgebra $C_u(X)$ made up of all bounded uniformly continuous complex-valued functions on $X$. It is unital as it contains $I \in C_b(X)$.

Unlike the algebras $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{K}(\mathcal{H})$, th $C^*$-algebras in these examples are *commutative*, that is

$$T_1T_2 = T_2T_1$$

for all $T_1$ and $T_2$ in these algebras. A far-reaching theorem by Gelfand shows all commutative $C^*$-algebras can be obtained in this way.

**Theorem II.1.11.** *For every unital commutative $C^*$-algebra $\mathcal{A}$, there is an isomorphism $\Gamma : \mathcal{A} \to C(\mathrm{Spec}(\mathcal{A}))$, where $\mathrm{Spec}(\mathcal{A})$ is the space of nontrivial multiplicative bounded linear functionals from $\mathcal{A}$ to $\mathbb{C}$, and $\Gamma$ takes $a$ to $(\hat{a} : \mathrm{Spec}(\mathcal{A}) \to \mathbb{C}, \Phi \mapsto \Phi(a)) \in C(\mathrm{Spec}(\mathcal{A}))$.*

One consequence of this theorem is that, since the norm in $C_0(X)$ is determined by its algebraic structure via

$$\|f\| = \sup\{\|\lambda\| : f - \lambda \text{ is not invertible}\}$$

thus so is the norm in any commutative $C^*$-algebra, and moreover, the norm in any $C^*$-algebra, because $\|T\|^2 = \|T^*T\|$, the normal operator $T^*T$ generates a commutative algebra, and invertibility is inherited by $C^*$-subalgebras. Using this, one can also show that every $*$-homomorphism is non-expanding and thus continuous, and every $*$-isomorphism is isometric.

Another natrual question to ask is whether every abstract $C^*$-algebra can be realized as a concrete one, or more precisely, whether there exists a concrete $C^*$-algebra that is $*$-isomorphic to the given abstract $C^*$-algebra.

**Example II.1.12.** The commutative $C^*$-algebras in example II.1.8, II.1.9 and II.1.10 can all be realized as concrete $C^*$-algebras. Indeed, if we take a regular Borel measure $\mu$ on $X$

[1] The function spaces we are going to define may not be very interesting if $X$ is not locally compact.

and form the Hilbert space $L^2(X,\mu)$, then each bounded continuous function $f \in C_b(X)$ determines an "multiplication" operator $M_f \in \mathfrak{B}(\mathcal{H})$ such that $M_f(\xi) = f \cdot \xi$. It is evident that the map $f \to M_f$ is an injective $*$-homomorphism, that is, an embedding of $C^*$-algebras (in this case also called a *faithful* representation). Therefore $C_b(X)$, $C_0(X)$ and $C_u(X)$ are isomorphic to some concrete $C^*$-algebras in $\mathfrak{B}(\mathcal{H})$.

By theorem II.1.11, every commutative $C^*$-algebra is concrete. More generally, Gelfand and Neumark proved that the abstract and concrete definitions are in fact equivalent:

**Theorem II.1.13.** *Every abstract $C^*$-algebra is $*$-isomorphic to a concrete $C^*$-algebra.*

## II.2 Clifford algebras and grading

Let $\mathcal{H}$ be a (possibly finite dimensional) *real* Hilbert space, and $\mathcal{H}_\mathbb{C}$ be its complexification. The $k$-th exterior power of $\mathcal{H}_\mathbb{C}$, denoted $\Lambda^k\mathcal{H}_\mathbb{C}$, is defined to be the quotient of $\bigotimes^k \mathcal{H}_\mathbb{C}$ by equating $\xi_1 \otimes \cdots \otimes \xi_k$ with $\mathrm{sgn}(\sigma)\ \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k)}$ for any $k$-permutation $\sigma$, with the equivalence class denoted as $\xi_1 \wedge \cdots \wedge \xi_k$. We also equate the 0-th exterior power with $\mathbb{C}$. The *antisymmetric Fock space* of $\mathcal{H}_\mathbb{C}$ is then defined to be the Hilbert space

$$\Lambda^*\mathcal{H}_\mathbb{C} := \bigoplus_{k=0}^{\infty} \Lambda^k\mathcal{H}_\mathbb{C}.$$

Notice that this is finite dimensional when $\mathcal{H}$ is, since $\Lambda^k\mathcal{H}_\mathbb{C}$ vanishes when $k > \dim(\mathcal{H})$. $\Lambda^*\mathcal{H}_\mathbb{C}$ is generated by the pure tensors $\xi_1 \wedge \cdots \wedge \xi_k$ with arbitrary $k$, with $k$ being called the *degree* of this pure tensor.

For each *real vector* $\nu \in \mathcal{H}$, we may define its *creation operator* $C(\nu) : \Lambda^*\mathcal{H}_\mathbb{C} \to \Lambda^*\mathcal{H}_\mathbb{C}$ such that

$$C(\nu)(\xi_1 \wedge \cdots \wedge \xi_k) := \nu \wedge \xi_1 \wedge \cdots \wedge \xi_k$$

It raises the grading of a pure operator by 1, and in particular, maps $1 \in \Lambda^0\mathcal{H}_\mathbb{C}$ to $\xi$. Also $\|C(\nu)\| \leq \|\nu\|$, and its adjoint is the *annihilation operator* $C^*(\nu)$, whose effect on a pure tensor can be easily checked:

$$C^*(\nu)(\xi_1 \wedge \cdots \wedge \xi_k) := \sum_{i=1}^{k} (-1)^{k-1} < \nu, \xi_i > \ \xi_1 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k$$

Again, checking on pure tensors, one finds the following relations between these two operators

$$C(\nu)C(\nu') + C(\nu')C(\nu) = 0$$

$$C^*(\nu)C^*(\nu') + C^*(\nu')C^*(\nu) = 0$$

$$C(\nu)C^*(\nu') + C^*(\nu')C(\nu) = C^*(\nu)C(\nu') + C(\nu')C^*(\nu) = 2(\nu, \nu')$$

for any $\nu, \nu' \in \mathcal{H}$. Therefore, if we define an adjoint operator

$$\hat{\nu} := C(\nu) + C^*(\nu)$$

for each $\nu \in \mathcal{H}$, then we have the following relation

$$\hat{\nu}\hat{\nu'} + \hat{\nu'}\hat{\nu} = 2 < \nu, \nu' > \tag{II.7}$$

for any $\nu, \nu' \in \mathcal{H}$. In particular,

$$\hat{\nu}^2 = \|\nu\|^2 \tag{II.8}$$

a scalar multiplication.

**Definition II.2.1.** The *(complex) Clifford algebra* $\mathfrak{Cl}_{\mathbb{C}}(\mathcal{H})$ of $\mathcal{H}$ is the subalgebra of $\mathfrak{B}(\Lambda^*\mathcal{H}_{\mathbb{C}})$ generated by $\{\hat{\nu} : \ \nu \in \mathcal{H}\}$.

Given an orthonormal basis $\{\nu_\alpha : \ \alpha \in A\}$ of $\mathcal{H}$, then (II.7) implies that any product of $\hat{\nu_\alpha}$'s can be simplified to $\pm\hat{\nu_{\alpha_1}}\hat{\nu_{\alpha_2}}\cdots\hat{\nu_{\alpha_k}}$, unique up to permutation, and thus all linear combinations of such simplified products form an algebra, which by definition is dense in $\mathfrak{Cl}_{\mathbb{C}}(\mathcal{H})$.

An important feature that makes Clifford algebras so useful in the study of geometry is that any isometry $V : \mathcal{H} \to \mathcal{H}'$ induces a $*$-homomorphism $\hat{V} : \mathfrak{Cl}_{\mathbb{C}}(\mathcal{H}) \to \mathfrak{Cl}_{\mathbb{C}}(\mathcal{H}')$ such that $\hat{V}(\hat{\nu}) = \widehat{V\nu}$. In fact, the construction of Clifford algebras and such $*$-homomorphisms between them gives a functor from the category of Hilbert spaces and isometries to that of $C^*$-algebras. In particular, a unitary operator $U : \mathcal{H} \to \mathcal{H}$ induces a $*$-automorphism of $\mathfrak{Cl}_{\mathbb{C}}(\mathcal{H})$.

The $*$-automorphism of $\mathfrak{Cl}_{\mathbb{C}}(\mathcal{H})$ induced by the inversion $\alpha : \mathcal{H} \to \mathcal{H}, \nu \mapsto -\nu$ deserves special interest, as it suggests a more general structure.

**Definition II.2.2.** Let $\mathcal{A}$ be a $C^*$-algebra. A *grading* on $\mathcal{A}$ is given by an involutive $*$-automorphism of $\mathcal{A}$.

Alternatively, since an involution always split $\mathcal{A}$ into two eigenspaces, corresponding respectively to the eigenvalues +1 and -1, we may say a grading on $\mathcal{A}$ is a splitting $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ such that

$$\mathcal{A}_0 \cdot \mathcal{A}_0 \subset \mathcal{A}_0, \ \mathcal{A}_0 \cdot \mathcal{A}_1 \subset \mathcal{A}_1, \ \mathcal{A}_1 \cdot \mathcal{A}_1 \subset \mathcal{A}_0$$

where $\mathcal{A}_i$ corresponds to the eigenvalue $(-1)^i$, and the multiplicative relations come from the fact that the splitting from an involutive $*$-automorphism has to be compatible with multiplication in $\mathcal{A}$.

$\mathcal{A}_0$ is called the even degree part and $\mathcal{A}_1$ the odd degree part. In the case of the Clifford algebras where the involution comes from $\alpha : \nu \mapsto -\nu$, the even (respectively, odd) degree part is generated by linear combinations of simplified products $\pm\hat{\nu_{\alpha_1}}\hat{\nu_{\alpha_2}}\cdots\hat{\nu_{\alpha_k}}$ with $k$ even (respectively, odd). In particular, the generators $\hat{\nu}$ are all of odd degree.

On graded $C^*$-algebras, we can carry out the following important constructions: For pure-degree elements $a, a' \in \mathcal{A}$, the graded commutator is defined as

$$[a, a'] := aa' - (-1)^{\deg(a)\deg(a')}a'a$$

For example, the Clifford relation (II.7) can be simplified as

$$[\hat{\nu}, \hat{\nu'}] = 2 < \nu, \nu' > . \tag{II.9}$$

## II.3 Group actions and universal spaces

We would like to study actions of discrete groups on topological spaces, as the interplay between the two reveals structures of the group and often produces very interesting phenomena. Let $\Gamma$ be a discrete group and $\alpha : \Gamma \times X \to X$ be its isometric action on $X$.

**Definition II.3.1.** A group action $\Gamma \curvearrowright X$ is called *effective* if no element of $\Gamma$ acts trivially (i.e. $\gamma \cdot x = x$, $\forall x \in X$), and *free* if it does not fix any point, or equivalently, the stabilizer group of each point is trivial.

The idea is to disallow degeneracy and force the action to be 'spread out'. Another related notion expressing a similar idea, but with topology of $X$ involved and behavior of finite groups ignored, is the following:

**Definition II.3.2.** Let $X$ be a locally compact topological space with an action by a discrete group $\Gamma$. Then the action is called (topologically) proper if for every compact subset $Y \subset X$, $\{\gamma \in \Gamma \mid \gamma \cdot Y \cap Y \neq \varnothing\}$ is finite.

Later we will discuss an analogous notion for (possibly non-locally compact) metric spaces. Intuitively, for a proper action, group elements that are 'far away' from the identity need to move points in $X$ by a great distance. In particular, it is easy to see that the stabilizer group for each $x \in X$ must be finite, although this alone is not sufficient to ensure metric properness, as the following example shows.

**Example II.3.3.** An irrational rotation is a transformation of $S^1 = \{e^{2\pi it} \mid t \in \mathbb{R}\}$ given by multiplication by $e^{2\pi is}$ with $s$ irrational. It is easy to see that the action is free, i.e. all stabilizer groups are trivial. But it is not proper since each orbit is dense in $S^1$.

Proper actions have some nice properties that eliminates pathologies. For instance, the quotient space $M/\Gamma$ is Hausdorff.

Of great importance in the study of group actions are a kind of topological objects that classifies a class of actions up to homotopy.

**Definition II.3.4.** Let $\Gamma$ be a group, and let $\mathcal{C}$ be a category of topological spaces with $\Gamma$-actions and homotopy classes of equivariant maps between them. Then an object $X$ in $\mathcal{C}$ is universal if for any object $Y$ in $\mathcal{C}$, there is a unique morphism from $Y$ to $X$; in other words, there are always equivariant maps $Y \to X$ and any two such maps are equivariant homotopic.

The universal object for the category of all free actions is denoted $E\Gamma$, and for the category of all proper actions, $\underline{E}\Gamma$.

Note that these objects are only determined up to equivariant homotopy. Thus there are many different possible models for them. Here we introduce one way of constructing such models.

Recall that for a topological space $X$, the cone $CX$ over $X$ is the quotient of $X \times [0, 1]$ obtained by regarding all points in $X \times 0$ as one point. We write a typical point in $CX$ as $tx$. The *infinite join* $X * X * \cdots$ is the set of sequence $(t_1x_1, t_2x_2, \cdots)$ such that $t_k = 0$ for almost all $k$ and $\sum t_k = 1$. It is given the weakest topology such that all the projection maps

$$(t_1x_1, t_2x_2, \cdots) \mapsto t_i$$

$$(t_1x_1, t_2x_2, \cdots) \mapsto x_i \text{ for } t_i \neq 0$$

are continuous. If $X$ carries a $\Gamma$-action, then $\Gamma$ may act on $X * X * \cdots$ diagonally by

$$\gamma \cdot (t_1x_1, t_2x_2, \cdots) = (t_1(\gamma \cdot x_1), t_2(\gamma \cdot x_2), \cdots)$$

When $X$ is discrete, this space may also be given a CW-structure such that one $k$-cell is attached to each $k$-tuple of elements in $X$.

We may take

$$E\Gamma = \Gamma * \Gamma * \cdots$$

$$\underline{E}\Gamma = X * X * \cdots$$

where $X$ is the disjoint union of all homogeneous spaces $\Gamma/G$ for $G$ a finite subgroup of $\Gamma$. In both cases, $\Gamma$ acts diagonally on all copies.

## II.4 The crossed product construction

The crossed product construction for $C^*$-algebras is a useful tool for studying unitary group actions on $C^*$-algebras. Let us start by making precise the notion of a group action on a $C^*$-algebra.

**Definition II.4.1.** Let $\Gamma$ be a discrete group. A *$C^*$-dynamical system* $(\mathcal{A}, \Gamma, \alpha)$ consists of a $C^*$-algebra $\mathcal{A}$ together with a homomorphism $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{A})$. We will denote by $\alpha_\gamma$ the automorphism $\alpha(\gamma)$ for $\gamma \in \Gamma$ and say the group $\Gamma$ *acts*[2] on $\mathcal{A}$ by $\alpha$.
Given a $C^*$-dynamical system, a *covariant representation* is a pair $(\rho, U)$ where $\rho$ is a $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and $\gamma \mapsto U_\gamma$ is a unitary representation of $\Gamma$ on $\mathcal{H}$ such that

$$U_\gamma \rho(a) U_\gamma^* = \rho(\alpha_\gamma(a)) \; \forall a \in \mathcal{A}, \gamma \in \Gamma.$$

**Example II.4.2.** Let $X$ be a locally compact Hausdorff topological space and let $\Gamma \curvearrowright X$ by homeomorphisms. Then this action induces a $C^*$-dynamical system $(C_0(X), \Gamma, \alpha)$ where

$$\alpha_\gamma(f)(x) := f(\gamma^{-1} \cdot x), \;\; \forall f \in C_0(X),\; \gamma \in \Gamma \text{ and } x \in X.$$

There is also a covariant representation $(m, U)$ on $L^2(X)$ by:

$$m_f(\xi)(x) := f(x)\xi(x), \;\; \forall f \in C_0(X),\; \xi \in L^2(X) \text{ and } x \in X;$$

$$U_\gamma(\xi)(x) := \xi(\gamma^{-1} \cdot x), \;\; \forall \gamma \in \Gamma,\; \xi \in L^2(X) \text{ and } x \in X.$$

**Definition II.4.3.** Let $(\mathcal{A}, \Gamma, \alpha)$ be a $C^*$-dynamical system. The *algebraic crossed product* algebra $\mathcal{A} \rtimes_{\alpha, alg} \Gamma$ consists of all finite sums $f = \sum_{\gamma \in \Gamma} a_\gamma \gamma$ where $a_\gamma \in \mathcal{A}$. Multiplication is determined by the formal rules $\gamma a \gamma^{-1} = \alpha_\gamma(a)$ and $\gamma^* = \gamma^{-1}$. We shall often write $\mathcal{A} \rtimes_{alg} \Gamma$ for $\mathcal{A} \rtimes_{\alpha, alg} \Gamma$ when the group action is understood.

Thus if $g = \sum_{\lambda \in \Gamma} \mathcal{B}_\lambda \lambda$ is another finite sum in $\mathcal{A} \rtimes_{alg} \Gamma$, then

$$fg = \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Gamma} (a_\gamma \gamma \cdot \mathcal{B}_\lambda \lambda) = \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Gamma} a_\gamma (\gamma \mathcal{B}_\lambda \gamma^{-1}) \gamma \lambda$$

[2] Here we are using left action, but the right action case is completely analogous: simply replace $\Gamma$ with $\Gamma^{\,\mathrm{op}}$, the group with reversed multiplication.

$$= \sum_{\gamma\in\Gamma}\sum_{\lambda\in\Gamma} a_\gamma \alpha_\gamma(\mathcal{B}_\lambda)\gamma\lambda \xlongequal{\beta:=\gamma\lambda} \sum_{\beta\in\Gamma}\left(\sum_{\gamma\in\Gamma} a_\gamma \alpha_\gamma(\mathcal{B}_{\gamma^{-1}\lambda})\right)\beta.$$

Also since

$$(a\gamma)^* = \gamma^* a^* = \gamma^{-1} a^* \gamma\gamma^{-1} = \alpha_\gamma^{-1}(a^*)\gamma^{-1},$$

then

$$f^* = \sum_{\gamma\in\Gamma} \alpha_\gamma(a^*_{\gamma^{-1}})\gamma.$$

The algebra $\mathcal{A} \rtimes_{alg} \Gamma$ encodes the $C^*$-dynamical system$(\mathcal{A}, \Gamma, \alpha)$. Furthermore, it is immediate that a covariant representation $(\rho, U)$ corresponds to a $*$-representation $\tilde{\rho}$ of $\mathcal{A} \rtimes_{alg} \Gamma$, and vice versa. More explicitly, $\tilde{\rho}$ is defined by

$$\tilde{\rho}(f) = \sum_{\gamma\in\Gamma} \rho(a_\gamma)U_\gamma.$$

There is a kind of covariant representations of particular interest to us. Let $\rho$ be a $*$-representation of $\mathcal{A}$ on $\mathcal{H}$. Then we form the "twisted tensor product" of this representation with the left regular representation of $\Gamma$, which is a covariant representation $(\tilde{\rho}, L)$ of $(\mathcal{A}, \Gamma, \alpha)$ on $L\Gamma \otimes \mathcal{H}$ such that:

$$\tilde{\rho}(a) \cdot (\delta_\gamma \otimes \xi) := \delta_\gamma \otimes (\rho(\alpha_\gamma^{-1}(a)) \cdot \xi)$$

$$L_\lambda \cdot (\delta_\gamma \otimes \xi) := \delta_{\lambda\gamma} \otimes \xi$$

where $a \in \mathcal{A},\ \xi \in \mathcal{H},\ \gamma, \lambda \in \Gamma$, and $\delta_\gamma$ is the characteristic function of the singleton $\{\gamma\}$. We also denote the corresponding $*$-representation of $\mathcal{A} \rtimes_{alg} \Gamma$ on $L\Gamma \otimes \mathcal{H}$ by $\tilde{\rho}$, and by a slight abuse of language, call it the *left regular representation of* $\mathcal{A} \rtimes_{alg} \Gamma$ *with regard to* $\rho$. Hence

$$\begin{aligned}
\tilde{\rho}(\sum_{\lambda\in\Gamma} a_\lambda \lambda) \cdot (\delta_\gamma \otimes \xi) &= \sum_{\lambda\in\Gamma} \tilde{\rho}(a_\lambda) \cdot (\delta_{\lambda\gamma} \otimes \xi) \\
&= \sum_{\lambda\in\Gamma} \delta_{\lambda\gamma} \otimes (\rho(\alpha_{\lambda\gamma}^{-1}(a_\lambda)) \cdot \xi) \\
&= \sum_{\lambda\in\Gamma} \delta_{\lambda\gamma} \otimes \left(\rho\left((\lambda\gamma)^{-1} a_\lambda (\lambda\gamma)\right) \cdot \xi\right).
\end{aligned}$$

This may be compared with the rules of multiplication in $\mathcal{A} \rtimes_{alg} \Gamma$. It is also evident that if $\rho : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ is faithful, then so is $\tilde{\rho} : \mathcal{A} \rtimes_{alg} \Gamma \to \mathfrak{B}(L\Gamma \otimes \mathcal{H})$.

Now $\tilde{\rho}$ induces a $C^*$-norm

$$\|f\|_r = \|\tilde{\rho}(f)\|$$

on $\mathcal{A} \rtimes_{alg} \Gamma$. Yet as seen in the construction of the group $C^*$-algebras, another standard practice is taking the maximal $C^*$-norm

$$\|f\| = \sup_\rho \|\phi(f)\|$$

where $\rho$ runs over all $*$-representations of $\mathcal{A} \rtimes_{alg} \Gamma$. It is well defined because there is at least one $*$-representation (the left regular representation) and the supremum is bounded above by $\sum_{\gamma\in\Gamma} \|a_\gamma\|$. This maximal norm is induced by the universal representation of $\mathcal{A} \rtimes_{alg} \Gamma$.

**Definition II.4.4.** Let $\rho : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ be a faithful representation of $\mathcal{A}$. Then the *reduced crossed product* $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ is the norm closure of $\tilde{\rho}(\mathcal{A} \rtimes_{\alpha,alg} \Gamma)$ in $\mathfrak{B}(L\Gamma \otimes \mathcal{H})$, and the *(unreduced) crossed product*, denoted by $\mathcal{A} \rtimes_\alpha \Gamma$, is the norm closure of the image of the universal representation of $\mathcal{A} \rtimes_{\alpha,alg} \Gamma$.

Equivalently, we may say $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ and $\mathcal{A} \rtimes_\alpha \Gamma$ are completions of $\mathcal{A} \rtimes_{\alpha,alg} \Gamma$ with respect to the norms $\|\cdot\|_r$ and $\|\cdot\|$, respectively. We shall often drop the homomorphism $\alpha$ in the notation, if it causes no confusion.

**Example II.4.5.** If we take $\mathcal{A} = \mathbb{C}$ and $\alpha$ to be the trivial homomorphism, then $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ and $\mathcal{A} \rtimes_\alpha \Gamma$ are nothing but the reduced and full group $C^*$-algebras, respectively.

## II.5 $KK$-theory

Kasparov's equivariant $KK$-theory (cf. [14, 15]) associates to a locally compact and $\sigma$-compact group $\Gamma$ and two separable $\Gamma$-$C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ (meaning that $\Gamma$ acts on them) the abelian group $KK^\Gamma(A, B)$. The group $KK^\Gamma(A, B)$ contains, among other things, elements $[\varphi]$ induced from equivariant $*$-homomorphisms $\varphi\colon A \to B$. It is contravariant in $A$ and covariant in $B$, both with respect to equivariant $*$-homomorphisms. It is equivariant-homotopy-invariant, stably invariant, preserves equivariant split exact sequences, and satisfies *Bott periodicity*:

$$KK^\Gamma(A, B) \cong KK^\Gamma(\Sigma^2 A, B) \cong KK^\Gamma(\Sigma A, \Sigma B) \cong KK^\Gamma(A, \Sigma^2 B)$$

where $\Sigma^i A$ stands for $C_0(\mathbb{R}^i, A)$ with $i \in \mathbb{N}$ and $\Gamma$ acting trivially on $\mathbb{R}$. These properties ensure that a short exact sequence $0 \to J \to E \to A$ of $C^*$-algebras and $*$-homomorphisms induces *six-term exact sequences* in both variables. When one of the two variables is $\mathbb{C}$, equivariant $KK$-theory recovers equivariant $K$-theory: $KK^\Gamma(\mathbb{C}, B) \cong K_0^\Gamma(B)$ and equivariant $K$-homology: $KK^\Gamma(A, \mathbb{C}) \cong K^0_\Gamma(B)$.

The most powerful feature of equivariant $KK$-theory is the *Kasparov product*, which assigns a group homomorphism

$$KK^{\Gamma}(A, B) \otimes_{\mathbb{Z}} KK^{\Gamma}(B, C) \to KK^{\Gamma}(A, C)$$

for any three separable $\Gamma$-$C^*$-algebras $A$, $B$, and $C$. The Kasparov product of two elements $x \in KK^{\Gamma}(A, B)$ and $y \in KK^{\Gamma}(B, C)$ is often denoted by $x \otimes_B y$. The Kasparov product is associative and taking Kasparov product with an element $[\varphi]$ induced from an equivariant $*$-homomorphisms $\varphi$ coincides with the functorial properties of equivariant $KK$-theory. In categorical language, this can be summarized as saying that $KK^{\Gamma}(-, -)$ constitutes a bifunctor from the category of separable $\Gamma$-$C^*$-algebras and $*$-homomorphism to the additive category $KK^{\Gamma}$ of separable $\Gamma$-$C^*$-algebras and equivariant $KK$-group elements, with the Kasparov product as its composition of morphisms. When the acting group $\Gamma$ is the trivial group, we simply write $KK(A, B)$ for $KK^{\Gamma}(A, B)$ and drop the word "equivariant" everywhere. There is a forgetful functor from $KK^{\Gamma}$ to $KK$.

When $\Gamma$ is a countable discrete group and its action on $B$ is trivial, it is immediate from the definition that there is a natural isomorphism $KK^{\Gamma}(A, B) \cong KK(A \rtimes \Gamma, B)$ where $A \rtimes \Gamma$ is the maximal crossed product. In particular, if $A = C_0(X)$ for a locally compact second countable space $X$ and $\Gamma$ acts freely and properly on $X$, then since $C_0(X) \rtimes \Gamma$ is stably isomorphic to $C_0(X/\Gamma)$, we have a natural isomorphism $KK^{\Gamma}(C_0(X), B) \cong KK(C_0(X/\Gamma), B)$.

Also, when $\Gamma$ is a countable discrete group and $A = C_0(\Gamma, D)$ with an action of $\Gamma$ by translation on the domain $\Gamma$, there is a natural isomorphism $KK^{\Gamma}(C_0(\Gamma, D), B) \xrightarrow{\cong} KK(D, B)$ given by first applying the forgetful functor and then composing with the embedding $D \cong C(\{1_{\Gamma}\}, D) \hookrightarrow C_0(\Gamma, D)$.

In this paper, we will focus on the case when the first variable $A$ in $KK^{\Gamma}(A, B)$ is commutative and view the theory as a homological theory on the spectrum of $A$. In fact, we will need a variant of it that may be thought of as homology with $\Gamma$-compact support. Recall that a subset of a topological space $X$, on which $\Gamma$ acts, is called $\Gamma$-*compact* if it is contained in $\{g \cdot x \colon g \in \Gamma,\ x \in K\}$ for some compact subset $K$ in $X$.

**Definition II.5.1.** Given a countable discrete group $\Gamma$, a Hausdorff space $X$ with a $\Gamma$-action, a $\Gamma$-$C^*$-algebra $B$, and $i \in \mathbb{N}$, we write $KK_i^{\Gamma}(X, B)$ for the inductive limit of the equivariant $KK$-groups $KK^{\Gamma}(C_0(Z), C_0(\mathbb{R}^i, A))$, where $Z$ ranges over $\Gamma$-invariant and $\Gamma$-compact subsets of $X$ and $A$ ranges over $\Gamma$-invariant separable $C^*$-subalgebras of $B$, both directed by inclusion.

We write $K_i^{\Gamma}(X)$ for $KK_i^{\Gamma}(X, \mathbb{C})$ and call it the $\Gamma$-*equivariant $K$-homology of $X$ with $\Gamma$-compact supports.*

It is clear from Bott periodicity that there is a natural isomorphism $KK_i^{\Gamma}(X, B) \cong$

$KK^\Gamma_{i+2}(X,B)$. Thus we can view the index $i$ as an element of $\mathbb{Z}/2\mathbb{Z}$. Also note that this construction is covariant both in $X$ with respect to continuous maps and in $B$ with respect to equivariant $*$-homomorphisms. Partially generalizing the functoriality in the second variable, the Kasparov product gives us a natural product $KK^\Gamma_i(X,B)\otimes_{\mathbb{Z}} KK^\Gamma(B,C) \to KK^\Gamma_i(X,C)$ for any separable $\Gamma$-$C^*$-algebras $B$ and $C$ (the separability condition can be dropped by extending the definition of $KK^\Gamma(B,C)$ through taking limits).

In the non-equivariant case, we may think of $KK_i(-,B)$ as an extraordinary homology theory in the sense of Eilenberg-Steenrod. In particular, there is a Chern character which is an isomorphism

$$\mathrm{ch}\colon KK_i(X,B)\otimes\mathbb{Q} \xrightarrow{\cong} \bigoplus_{j\in\mathbb{Z}/2\mathbb{Z}} K_j(X)\otimes_{\mathbb{Z}} K_{i-j}(B)\otimes\mathbb{Q}\ .$$

The *reduced Baum-Connes assembly map* for a countable discrete group $\Gamma$ and a $\Gamma$-$C^*$-algebra $B$ is a group homomorphism

$$\mu\colon KK^\Gamma_i(\underline{E}\Gamma, B) \to K_i(B\rtimes_{\mathrm{r}}\Gamma)\ .$$

It is natural in $B$ with respect to $\Gamma$-equivariant $*$-homomorphisms or more generally with respect to taking Kasparov products, in the sense that any element $\delta\in KK^\Gamma(B,C)$ induces a commuting diagram

$$\begin{tikzcd} KK^\Gamma_i(\underline{E}\Gamma, B) \arrow[r, "\mu"] \arrow[d, "\delta"] & K_i(B\rtimes_{\mathrm{r}}\Gamma) \arrow[d, "\delta\rtimes_{\mathrm{r}}\Gamma"] \\ KK^\Gamma_i(\underline{E}\Gamma, C) \arrow[r, "\mu"] & K_i(C\rtimes_{\mathrm{r}}\Gamma) \end{tikzcd} \tag{II.10}$$

for an induced group homomorphism $\delta\rtimes_{\mathrm{r}}\Gamma$.

The case when $B=\mathbb{C}$ is of special interest. The *rational strong Novikov conjecture* asserts that the composition

$$K^\Gamma_i(E\Gamma)\to K^\Gamma_i(\underline{E}\Gamma)\xrightarrow{\mu} K_i(C^*_{\mathrm{r}}\Gamma)$$

is injective after tensoring each term by $\mathbb{Q}$. It implies the classical Novikov conjecture.

On the other hand, it has proven extremely useful to have the flexibility of a general $\Gamma$-algebra $B$ in the picture, largely due to the following key observation, which is based on a five lemma argument.

**Theorem II.5.2** (cf. [11, Theorem 13.1])**.** *For any countable discrete group $\Gamma$, and a $\Gamma$-$C^*$-algebra $B$, if $B$ is a proper $\Gamma$-$X$-$C^*$-algebra for some locally compact Hausdorff space $X$,* *then the reduced Baum-Connes assembly map*

$$\mu\colon KK_i^{\Gamma}(\underline{E}\Gamma, B) \to K_i(B \rtimes_{\mathrm{r}} \Gamma)\ .$$

*is a bijection.* □

## II.6 Non-positively curved Hilbert manifolds

We give a summary of the basic theory of Hilbert manifolds, following [17]. A Hilbert manifold is an infinite-dimensional analogue of Riemannian manifold, with the neighborhood of a point in the manifold modelled after a Hilbert space instead of an Euclidean space. Fortunately, most of the core constructions in classical Riemannian geometry carries over to the infinite-dimensional case, albeit with some impasses and caveats here and there.

**Definition II.6.1.** A *topological manifold modelled after a Hilbert space* $\mathcal{H}$ is a separable, metrizable space which is locally homeomorphic to $\mathcal{H}$.

**Definition II.6.2.** A *differentiable atlas* on a topological manifold $M$ modelled after a Hilbert space $\mathcal{H}$ is a family of charts $(\phi_\alpha, M_\alpha)_{\alpha\in A}$ such that $\{M_\alpha\}_{\alpha\in A}$ is an open covering of $M$, and for any $\alpha \in A$, $\phi_\alpha\colon M_\alpha \to U_\alpha$ is a homeomorphism onto an open subset $U_\alpha$ of $\mathcal{H}$ and for any two charts indexed by $\alpha$ and $\beta$ in $A$, the transition map $\phi_\alpha \circ \phi_\beta^{-1} :: \phi_\beta(M_\alpha \cap M_\beta) \to \phi_\alpha(M_\alpha \cap M_\beta)$ is a diffeomorphism, i.e., it is a homeomorphism such that both itself and its inverse are differentiable for any number of times.

Two differentiable atlases are said to be equivalent if their union is also a differentiable atlas. And finally, a *differentiable Hilbert manifold* is a topological Hilbert manifold together with a chosen equivalence class of differentiable atlases.

**Definition II.6.3.** Let $B$ be a Banach space and $M$ a differentiable manifold modelled after a Hilbert space $\mathcal{H}$. Then a *differentiable $B$-vector bundle over $M$* is given by $(P, M, \pi)$, where $P$ is a topological space and $\pi$ a surjective continuous map such that $\pi^{-1}(x)$ is a Banach space isomorphic to $B$ and there is an *atlas of bundle charts*, that is, a family of charts $(\phi_\alpha, M_\alpha)_{\alpha\in A}$ such that $\{M_\alpha\}_{\alpha\in A}$ is an open covering of $M$, and for any $\alpha \in A$, $\phi_\alpha : \pi^{-1}(M_\alpha) \to U_\alpha \times B$ is a homeomorphism, where $U_\alpha$ is an open subset of $\mathcal{H}$, and for any two charts indexed by $\alpha$ and $\beta$ in $A$, the transition map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\pi^{-1}(M_\alpha \cap M_\beta)) \to \phi_\alpha(\pi^{-1}(M_\alpha \cap M_\beta))$ is a diffeomorphism.

The definition of differentiable maps, tangent bundles, derivatives, vector fields are completely analogous to the finite dimensional case. But the definition of covariant derivatives requires some additional structure on the Hilbert manifold.

**Definition II.6.4.** Let $M$ be a differentiable Hilbert manifold. Assume that for every chart $(\phi_\alpha, M_\alpha)$ of an atlas of $M$ there is given a differentiable map

$$\Gamma_\alpha : \phi_\alpha \mapsto \Gamma(\phi_\alpha) \in \mathrm{L}(\mathcal{H}, \mathcal{H}, \mathcal{H}^*; \mathbb{R}) = \mathrm{L}(\mathcal{H}, \mathcal{H}; \mathcal{H})$$

called a Christoffel symbol, with the following property: If $(\phi_\beta, M_\beta)$ is a second chart, then $\Gamma_\alpha | \phi_\alpha(M_\alpha \cap M_\beta)$ and $\Gamma_\beta | \phi_\beta(M_\alpha \cap M_\beta)$ are related by

$$D(\phi_\beta \circ \phi_\alpha^{-1})\Gamma_\alpha(\phi_\alpha) = D^2(\phi_\beta \circ \phi_\alpha^{-1}) + \Gamma_\beta(\phi_\beta) \circ (D(\phi_\beta \circ \phi_\alpha^{-1}) \times D(\phi_\beta \circ \phi_\alpha^{-1}))$$

Given a family Christoffel symbols on $M$ we define a covariant derivation

$$(X, Y) \mapsto \bigtriangledown_X Y$$

by taking for the principal part $\bigtriangledown_X Y(\phi_\alpha)$ of $\bigtriangledown_X Y$ with respect to a chart $(\phi_\alpha, M_\alpha)$ the expression

$$\bigtriangledown_X Y(\phi_\alpha) = DY(\phi_\alpha) \cdot X(\phi_\alpha) + \Gamma(\phi_\alpha)(X(\phi_\alpha), Y(\alpha))$$

When $M$ is endowed with an atlas for which we have Christoffel symbols we also say $M$ possesses a covariant derivation.

**Definition II.6.5.** Let $\bigtriangledown$ be a given covariant derivation on $M$. The curvature tensor is defined by the following formula on a local chart $(\phi_\alpha, M_\alpha)$:

$$\begin{aligned}(X, Y, Z) \mapsto\ & R(\phi_\alpha)(X, Y)Z \\ &= D\Gamma(\phi_\alpha)\ X(Y, Z) - D\Gamma(\phi_\alpha)\ Y(X, Z) \\ &\quad + \Gamma(\phi_\alpha)(X, \Gamma(\phi_a)(X, Z)) - \Gamma(\phi_\alpha)(Y, \Gamma(\phi_\alpha)(X, Y))\end{aligned}$$

Once we have a covariant derivative, geodesics and exponential maps are again defined in the same manner as in the finite dimensional case. To define Hilbert-Riemannian metrics, we consider $\mathrm{Sym}_2\mathcal{H}$, the Banach space of symmetric bilinear maps from $\mathcal{H} \times \mathcal{H}$ into $\mathbb{R}$. Such a bilinear map $g$ is called a positive definite quadratic form on $\mathcal{H}$ if there exists $\epsilon > 0$ such that

$$g(X, X) \geq \epsilon < X, X >$$

where $< \bullet, \bullet >$ is the inner product on the Hilbert space $\mathcal{H}$.

**Definition II.6.6.** Let $M$ be a differentiable Hilbert manifold. A Hilbert-Riemannian metric on $M$ is a section of the bundle $\mathrm{Sym}TM$ with positive definite quadratic forms as values. A Hilbert-Riemannian manifold is $M$ together with the metric $g$.

Hilbert-Riemannian metrics always exist for a given Hilbert manifold, thanks to the existence of partitions of unity subordinate to any given atlas. And any such metric uniquely determines a covariant derivation $\bigtriangledown$, called the Levi-Civita derivation, with the defining relation

$$\begin{aligned} & 2 < \bigtriangledown_X Y, Z > \\ = & X < Y, Z > + Y < Z, X > - Z < X, Y > \\ & + < Z, [X, Y] > + < Y, [Z, X] > - < X, [Y, Z] > \end{aligned}$$

for all vector fields $X, Y, Z$.

**Definition II.6.7.** Let $(M, g)$ be a Hilbert-Riemannian manifold, and let $(X, Y, Z) \mapsto R(X, Y)Z$ be the curvature tensor of the Levi-Civita derivation. Then for any $x \in M$ and any two linearly independent tangent vector $\overrightarrow{v}, \overrightarrow{v}' \in T_x M$, the sectional curvature for the 2-dimensional linear subspace $V$ spanned by $\overrightarrow{v}$ and $\overrightarrow{v}'$ is defined to be

$$K(V) := \frac{< R(X, Y)Y, X >}{< X, X >< Y, Y > - < X, Y >^2}$$

Intuitively, the sectional curvature measures how large the infinitesmal area spanned by $\overrightarrow{v}$ and $\overrightarrow{v}'$ in $M$ is compared to such an area in a Hilbert space. When the sectional curvatures are all non-positive, the manifold locally resembles a hyperbolic space in that all the geodesic triangles are thin, and if in addition, the manifold is complete and simply connected, then such a property holds on the global scale. In particular, any two points is connected by a unique geodesic. This is the idea behind the Hadamard theorem, whose proof can be found in [17].

**Proposition II.6.8.** *(Hadamard theorem) Let $M$ be a complete, simply-connected (Hilbert) manifold with non-positive sectional curvatures. Then for any $x \in M$, the exponential map*

$$\exp_x : T_x M \to M$$

*is a diffeomorphism and metric-expanding.* □

# CHAPTER III

## A NONCOMMUTATIVE $C^*$-ALGEBRA ASSOCIATED TO A NON-POSITIVELY CURVED HILBERT MANIFOLD

The main strategy of the proof is inspired by the Dirac-dual-Dirac approach, which has been very successful in the study of the strong Novikov conjecture and the Baum-Connes conjecture, in particular, in the proof of the strong Novikov conjecture for groups acting isometrically and properly on a non-positively curved Riemannian manifold. This approach can be outlined as follows: Assuming for simplicity that $\Gamma$ acts on $M$ freely, one makes use of two deep results in different areas:

1. Cartan-Hadamard theorem, which states that a complete, connected, simply-connected, and non-positively curved Riemannian manifold is diffeomorphic to $\mathbb{R}^n$, and

2. Bott periodicity, which states $\mathbb{R}^n$ has the same $K$-theory as a point (up to a dimension shift).

Thus one can make a connection between the $K$-theory of the highly noncommutative space given by $C_r^*(\Gamma)$ and the $K$-theory of the *commutative* space $M/\Gamma$, the latter readily checked to be isomorphic, via the assembly map *with coefficient algebra* $C_0(M)$, to the $KK$-group $KK_*^\Gamma(\underline{E}\Gamma, C_0(M))$, which, by a strong form of Bott periodicity, is isomorphic to $RK_*^\Gamma(\underline{E}\Gamma)$ (again up to a dimension shift), hence we end up with the following commutative diagram:

$$\begin{tikzcd} K_*^\Gamma(\underline{E}\Gamma) \arrow[r, "\mu"] \arrow[d, "\cong"] & K_*(C_r^*(\Gamma)) \arrow[d] \\ KK_*^\Gamma(\underline{E}\Gamma, C_0(M)) \arrow[r, "\mu", "\cong"'] & K_*(C_r^*(\Gamma, C_0(M))) \end{tikzcd}$$

and it is immediate that the map on the top is injective.

A major difficulty with adapting this proof by Kasparov is that now the manifold $M$, being infinite-dimensional, is not locally compact, and thus falls out of the scope of the Gelfand-Naimark theorem, and has $C_0(M) = 0$, rendering the traditional approach useless. However, using ideas from noncommutative geometry, we can develop a novel construction of a C*-algebra $\mathcal{A}(M)$, which encodes the geometric data of $M$ and can largely replace the role $C_0(M)$ plays in the finite-dimensional case. In the case when $M$ is flat, our construction yields the same algebra as the one appearing in Higson and Kasparov [12], but unlike their construction, ours gives a more intrinsic description without evoking the affine structure of a Hilbert space.

Now we would like to generalize the construction to a non-positively curved Hilbert manifold, still denoted by $M$. We are going to construct an appropriate $C^*$-algebra $\mathcal{A}(M)$ which should play an analogous role as the one in the finite-dimensional case in terms of $K$-theoretic calculations.

### III.1 Sections of the Clifford bundle

One may hope the construction to be completely analogous to the finite dimensional case. However, the complexity of the infinite dimensionality displays itself when we try to define $\mathcal{A}(M)$. If one naively attempt to define the algebra exactly as before — the closure of the algebra consisting of all continuous sections of the Clifford bundle $\mathfrak{Cl}_{\mathbb{C}}M$ that vanish outside of a compact set, one could see that, because $M$ is not locally compact anywhere — there is no non-empty open set with compact closure — any continuous section vanishing outside of a compact set must be globally zero, and hence the entire algebra becomes trivial. This demands us to relax our condition when picking the suitable sections to form an algebra; yet not so much that the algebra fails to be $\Gamma$-proper or the $K$-theory of the algebra collapses.

Recall that to any Riemannian vector bundle $E$ over $X$ one may associate the (complex) Clifford bundle $\mathfrak{Cl}_{\mathbb{C}}E$ over $X$ and the graded $*$-algebra of continuous sections $C(X, \mathfrak{Cl}_{\mathbb{C}}E)$, where the multiplication and the $*$-operation are defined pointwise as

$$(\sigma \cdot \sigma')(x) := \sigma(x) \cdot \sigma'(x) \in \mathfrak{Cl}_{\mathbb{C}}E_x \ , \ (\sigma^*)(x) := (\sigma(x))^* \in \mathfrak{Cl}_{\mathbb{C}}E_x$$

for any $x \in X$ and $\sigma, \sigma' \in C(X, \mathfrak{Cl}_{\mathbb{C}}E)$, and any $\sigma$ is even (respectively, odd) if it is even (respectively, odd) at any point. One also gets a graded subalgebra $C_b(X, \mathfrak{Cl}_{\mathbb{C}}E)$ which is a $C^*$-algebra when one restricts to bounded continuous sections, where the norm is given by the sup norm:

$$\|\sigma\| := \sup\{\|\sigma(x)\|_{\mathfrak{Cl}_{\mathbb{C}}E_x} : x \in X\} \ ,$$

where at each point $x$, the norm $\| - \|_{\mathfrak{Cl}_{\mathbb{C}}E_x}$ is the canonical $C^*$-norm on the Clifford algebra $\mathfrak{Cl}_{\mathbb{C}}E_x$.

We are interested in certain subalgebras of $C_b\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right)$.

**Definition III.1.1.** Define $C_{00,\mathrm{met}}\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right)$ to be the graded $*$-subalgebra of $C_b\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right)$ consisting of sections that vanish outside of a bounded set in $M$, i.e.,

$$\{\sigma \in C_b\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right) : \ \exists K \subset M \text{ bounded s.t. } \sigma(x) = 0, \ \forall x \in M \setminus K\} \ .$$

Let $C_{0,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ be the closure of $C_{00,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ under the norm topology, which is then a graded $C^*$-subalgebra of $C_b(M, \mathfrak{Cl}_{\mathbb{C}}TM)$.

**Remark III.1.2.** One may also directly characterize $C_{0,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ as containing those sections such that for any $\varepsilon > 0$, there exists a bounded set $K \subset M$ such that $\|\sigma(x)\|_{\mathfrak{Cl}_{\mathbb{C}}T_xM} < \varepsilon$ for any $x \in M \setminus K$.

**Remark III.1.3.** Notice that there is a canonical central embedding

$$C_b(M) \hookrightarrow \mathcal{Z}(C_b(M, \mathfrak{Cl}_{\mathbb{C}}TM))$$

given by the scalar sections over $M$. It is obviously equivariant with regard to $\alpha\colon \Gamma \curvearrowright M$ and maps the subalgebra

$$C_{0,\mathrm{met}}(M) := \overline{\{f \in C_b(M) |\ \exists K \subset M \text{ bounded s.t. } f(x) = 0,\ \forall x \in M \setminus K\}}$$

into the center of $C_{0,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$. Thus we will consider $C_b(M)$ as a sub-$C^*$-algebra of $C_b(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ via this embedding, when there is no danger of confusion.

**Example III.1.4.** Let $M = V$ be a (real) Hilbert space. Then under the canonical identification of $TV$ with the trivial bundle $V \times V$, we have (graded) canonical isomorphisms $C_b(V, \mathfrak{Cl}_{\mathbb{C}}TV) \cong C_b(V, \mathfrak{Cl}_{\mathbb{C}}V)$ and $C_{0,\mathrm{met}}(V, \mathfrak{Cl}_{\mathbb{C}}TV) \cong C_{0,\mathrm{met}}(V, \mathfrak{Cl}_{\mathbb{C}}V)$, where $C_{0,\mathrm{met}}(V, \mathfrak{Cl}_{\mathbb{C}}V)$ is generated by boundedly supported functions in $C_b(V, \mathfrak{Cl}_{\mathbb{C}}V)$.

**Example III.1.5.** When $M$ is finite-dimensional, it is locally compact and bounded sets are precisely precompact subsets of $M$. Thus $C_{0,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ is equal to $C_0(M, \mathfrak{Cl}_{\mathbb{C}}TM)$, the $C^*$-subalgebra of $C_b(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ consisting of all sections such that for any $\varepsilon > 0$, there exists a compact set $K \subset M$ such that $\|\sigma(x)\|_{\mathfrak{Cl}_{\mathbb{C}}T_xM} < \varepsilon$ for any $x \in M \setminus K$.

Now given an isometry $\phi$ of $M$, we may construct a $*$-automorphism $\phi_*$ of $C(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ defined by

$$(\phi_*)(\sigma)(x) := \mathfrak{Cl}_{\mathbb{C}}(D_x\phi)^{-1}\ \left(\sigma(\phi^{-1}(x))\right) \tag{III.1}$$

for any $x \in M$ and $\sigma \in C(M, \mathfrak{Cl}_{\mathbb{C}}TM)$. The automorphism clearly preserves the $*$-subalgebras $C_b(M, \mathfrak{Cl}_{\mathbb{C}}TM)$, $C_{00,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ and $C_{0,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$. The assignment $\phi \mapsto \phi_*$ give rise to group homomorphisms from the isometry group $\mathrm{Isom}(M)$ to the groups of $*$-automorphisms of $C(M, \mathfrak{Cl}_{\mathbb{C}}TM)$, etc. We denote the resulting fixed-point subalgebras with regard to $\phi_*$ as $C(M, \mathfrak{Cl}_{\mathbb{C}}TM)^{\phi}$, etc.

**Example III.1.6.** Let $M = \mathbb{R}$ together with the $\mathbb{Z}_2$-action given by the reflection $R$ around the origin. Then we have graded isomorphisms

$$\begin{aligned} C_{0,\mathrm{met}}\left(\mathbb{R}, \mathfrak{Cl}_{\mathbb{C}} T\mathbb{R}\right)^R &\cong C_0(\mathbb{R}, \mathfrak{Cl}_{\mathbb{C}}\mathbb{R})^R \\ &\cong \{\sigma \in C_0([0,\infty), \mathfrak{Cl}_{\mathbb{C}}(\mathbb{R}))\colon \mathfrak{Cl}_{\mathbb{C}}(R) f(0) = f(0) \in \mathfrak{Cl}_{\mathbb{C}}(\mathbb{R})\} \\ &\cong \{\sigma \in C_0([0,\infty), \mathfrak{Cl}_{\mathbb{C}}(\mathbb{R}))\colon f(0) \in \mathbb{C} \cdot 1_{\mathfrak{Cl}_{\mathbb{C}}(\mathbb{R})}\} \\ &\cong \{\sigma \in C_0([0,\infty), \mathbb{C} \oplus \mathbb{C})\colon f(0) \in \mathbb{C} \cdot (1,1)\} \\ &\cong C_0(\mathbb{R}) \end{aligned}$$

where the identification $\mathfrak{Cl}_{\mathbb{C}}(\mathbb{R}) \cong \mathbb{C} \oplus \mathbb{C}$ is induced by mapping $v \in \mathbb{R} \subset \mathfrak{Cl}_{\mathbb{C}}(\mathbb{R})$ to $(v, -v)$, with the grading on $\mathbb{C} \oplus \mathbb{C}$ given by swapping the two coordinates, and the last isomorphism above is given by

$$f \in C_0(\mathbb{R}) \mapsto \ (\ x \to (f(x), f(-x))\ ) \in \{\sigma \in C_0([0,\infty), \mathbb{C} \oplus \mathbb{C})\colon f(0) \in \mathbb{C} \cdot 1\},$$

with the grading on $C_0(\mathbb{R})$ induced by the reflection $R$ around the origin. Thus the grade-zero part consists of all even functions, and the grade-one part consists of all odd functions. This algebra will play an important role in our construction, and will be denoted as $\mathcal{S}$.

Returning to the study of the Novikov conjecture, we have at hand a simply-connected, complete Hilbert manifold $M$ with non-positive sectional curvatures and an isometric action by $\Gamma$. As in Example III.1.5, if $M$ is $2n$-dimensional for $n < \infty$, then $C_{0,\mathrm{met}}\left(M, \mathfrak{Cl}_{\mathbb{C}} TM\right)$ is the same as $C_0\left(M, \mathfrak{Cl}_{\mathbb{C}} TM\right)$, the algebra generated by sections of compact supports, and thus is isomorphic to $C_0(M, \mathcal{M}_{2^n}(\mathbb{C}))$, which is suitable to serve as the coefficient algebra for the Dirac-dual-Dirac method. On the other hand, unlike $C_0\left(M, \mathfrak{Cl}_{\mathbb{C}} TM\right)$, the larger algebra $C_{0,\mathrm{met}}\left(M, \mathfrak{Cl}_{\mathbb{C}} TM\right)$ remains at least non-trivial when we pass to the infinite dimensional case. Thus $C_{0,\mathrm{met}}\left(M, \mathfrak{Cl}_{\mathbb{C}} TM\right)$ appears to be a good candidate for the coefficient algebra in order to apply the Dirac-dual-Dirac method.

However, $C_{0,\mathrm{met}}\left(M, \mathfrak{Cl}_{\mathbb{C}} TM\right)$ turns out to be too large for our purpose. Instead, it acts as a playground in which the coefficient algebra we are going to use, $\mathcal{A}(M)$, will be assembled.

## III.2 Bott homomorphisms

Roughly speaking, $\mathcal{A}(M)$ is defined as the smallest algebra generated by the images of the so-called Bott homomorphisms, $*$-homomorphisms that induce a suitable version of the Bott map on $K$-theory. Let us first explain how these homomorphisms are constructed for a Hilbert manifold.

**Definition III.2.1.** Let $M$ be a nonpositively curved Hilbert manifold with a proper isometric action $\alpha$ of $\Gamma$. For any point $x_0 \in M$, we define the *Clifford multiplier* $C_{x_0}$ to be the (unbounded) continuous section of the Clifford bundle $\mathfrak{Cl}_{\mathbb{C}}TM$ that maps $x \in M$ to $-\exp_x^{-1}(x_0) \in T_xM \subset \mathfrak{Cl}_{\mathbb{C}}T_xM$.

**Example III.2.2.** When $M = V$ is a (real) Hilbert *space*, $C_{x_0}(x) = x - x_0$, after we identify $M$ with its own tangent spaces $T_xM$.

**Proposition III.2.3.** *For any point $x_0 \in M$, let $\beta_{x_0}$ be the map that sends $f \in \mathcal{S}$ to the section*

$$\beta_{x_0}(f) : x \mapsto f(C_{x_0}(x)),\ \forall x \in M,$$

*of the bundle $\mathfrak{Cl}_{\mathbb{C}}TM$. Then $\beta_{x_0}$ is a graded $*$-homomorphism from $\mathcal{S}$ to $C_{0,\mathrm{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$. Here the left-hand side is given by the functional calculus applied to the self-adjoint element $C_{x_0}(x) \in \mathfrak{Cl}_{\mathbb{C}}T_xM$ and the function $f \in \mathcal{S} = C_0(\mathbb{R})$.*

*Proof.* For any $f_1, f_2, f_3 \in \mathcal{S}$, $\lambda \in \mathbb{C}$ and $x \in M$, we have

$$\begin{aligned}
&\left(\beta_{x_0}(f_1) \cdot \beta_{x_0}(f_2) + \lambda \cdot \beta_{x_0}(f_2)^*\right)(x) \\
=\ & \beta_{x_0}(f_1)(x) \cdot \beta_{x_0}(f_2)(x) + \lambda \cdot \beta_{x_0}(f_3)^*(x) \\
=\ & f_1(C_{x_0}(x)) \cdot f_2(C_{x_0}(x)) + \lambda \cdot (f_3(C_{x_0}(x)))^* \\
=\ & (f_1 f_2 + \lambda \overline{f_3})(C_{x_0}(x)) \\
=\ & \beta_{x_0}(f_1 f_2 + \lambda \overline{f_3})(x)
\end{aligned}$$

and

$$\|\beta_{x_0}(f_1)(x)\| = \|f_1(C_{x_0}(x))\| \leq \|f_1\|.$$

These, combined with the continuity of functional calculus, show that $\beta_{x_0}$ is a $*$-homomorphism from $\mathcal{S}$ to $C_b(M, \mathfrak{Cl}_{\mathbb{C}}TM)$.

Note that for any $x \in M$, $C_{x_0}(x) \in \mathfrak{Cl}_{\mathbb{C}}T_xM$ is an odd element, while

$$C_{x_0}^2(x) = d(x_0, x)^2$$

is a nonnegative scalar, and in particular, an even element. Now when $f \in \mathcal{S}$ is even, we may write $f(t) = g(t^2)$ for some $g \in C_0([0, \infty))$ and thus

$$\beta_{x_0}(f)(x) = g(C_{x_0}^2(x))$$

is even for all $x \in M$. And when $f$ is odd, we write $f(t) = tg(t^2)$ for some $g \in C_0([0, \infty))$

and gets

$$\beta_{x_0}(f)(x) = C_{x_0}(x) \cdot g(C_{x_0}^2(x))$$

is odd for all $x \in M$. Therefore $\beta_{x_0}$ preserves grading.

Lastly, to show that the image of $\beta_{x_0}$ is contained in $C_{0,\text{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$, we observe that the compactly supported functions on $\mathbb{R}$ form a dense subalgebra of $\mathcal{S}$, and claim that $\beta_{x_0}$ maps these functions to boundedly supported sections of $\mathfrak{Cl}_{\mathbb{C}}TM$. Indeed, if $\operatorname{supp} f$ is contained in the closed $R$-ball around $0 \in \mathbb{R}$, then for any $x \in M$ outside the closed $R$-ball around $x_0$, we have $\|C_{x_0}(x)\| = d(x, x_0) > R$, and thus $\beta_{x_0}(f)(x) = f(C_{x_0}(x)) = 0$. It follows that boundedly supported sections of $\mathfrak{Cl}_{\mathbb{C}}TM$ form a dense subset of the image of $\beta_{x_0}$, and thus $\beta_{x_0}(\mathcal{S}) \subset C_{0,\text{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$. □

**Definition III.2.4.** For any point $x_0 \in M$, the graded $*$-homomorphism

$$\beta_{x_0} : \mathcal{S} \to C_{0,\text{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$$

is called the *Bott homomorphism* centered at $x_0$.

We discuss some important features of $\beta_{x_0}$, the first of which we have already hinted at in the proof of the last proposition. Let us denote the even part of $\mathcal{S}$ as $\mathcal{S}_{\text{ev}}$, which is comprised of all even functions.

**Proposition III.2.5.** *For any $x_0 \in M$, $\beta_{x_0}$ maps $\mathcal{S}_{ev}$ into the subalgebra $C_b(M)$ in the center of $C_b(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ (see Remark III.1.3).*

*Proof.* For any $x \in M$ and $f \in \mathcal{S}_{\text{ev}}$, we can write $f(t) = g(t^2)$ for some $g \in C_0([0, \infty))$ and thus

$$\beta_{x_0}(f)(x) = g\left(C_{x_0}^2(x)\right) = g\left(d(x_0, x)^2\right)$$

is a scalar. □

Next we study the relation between Bott homomorphisms and isometries on $M$.

**Lemma III.2.6.** *For any $\phi \in \operatorname{Isom}(M)$ and any $x_0 \in M$, the following holds:*

$$\phi_* \circ \beta_{x_0} = \beta_{\phi(x_0)}$$

*where $\phi_*$ is the induced $*$-automorphism of $C_{0,\text{met}}(M, \mathfrak{Cl}_{\mathbb{C}}TM)$ (c.f.* (III.1)*).*

*Proof.* For any $f \in \mathcal{S}$ and $x \in M$, we have

$$\begin{aligned}
\phi_* \left(\beta_{x_0}(f)\right)(x) &= \mathfrak{Cl}_{\mathbb{C}} \left(D_{\phi^{-1}(x)}\phi\right) \left(\beta_{x_0}(f)(\phi^{-1}(x))\right) \\
&= \mathfrak{Cl}_{\mathbb{C}} \left(D_{\phi^{-1}(x)}\phi\right) \left(f\left(C_{x_0}(\phi^{-1}(x))\right)\right) \\
&= f\left(\mathfrak{Cl}_{\mathbb{C}} \left(D_{\phi^{-1}(x)}\phi\right) \left(C_{x_0}(\phi^{-1}(x))\right)\right) \\
&= f\left(\left(D_{\phi^{-1}(x)}\phi\right) \left(-\exp^{-1}_{\phi^{-1}(x)}(x_0)\right)\right) \\
&= f\left(-\exp^{-1}_{x}(\phi(x_0))\right) \\
&= f\left(C_{\phi(x_0)}(x)\right) \\
&= \beta_{\phi(x_0)}(f)(x).
\end{aligned}$$

Here, we used that functional calculus commutes with automorphisms of $C^*$-algebras, and that being an isometry, $\phi : M \to M$ carries the geodesic line segment $[\phi^{-1}(x), x_0]$ to $[x, \phi(x_0)]$. □

**Remark III.2.7.** In fact, when we view $C_{x_0}$ as an element of $C\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right)$ and use the automorphism $\phi_*$ of $C\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right)$ induced by $\phi$, we have

$$\phi_*\left(C_{x_0}\right) = C_{\phi(x_0)}$$

for any $x_0 \in M$.

**Corollary III.2.8.** *If $x_0$ is fixed by an isometry $\phi$ of $M$, then*

$$\beta_{x_0}(\mathcal{S}) \subset C_{0,\mathrm{met}}\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right)^{\phi} \ ,$$

*the fixed-point subalgebra with regard to $\phi_*$.*

*Proof.* For any $f \in \mathcal{S}$, we have

$$\left(\phi_*\right)\left(\beta_{x_0}(f)\right) = \beta_{\phi(x_0)}(f) = \beta_{x_0}(f)$$

and thus $\beta_{x_0}(f) \in C_{0,\mathrm{met}}\left(M, \mathfrak{Cl}_{\mathbb{C}}TM\right)^{\phi}$. □

Lastly we show that although the definition of a Bott homomorphism depends on the choice of the base point, this dependence is a rather mild one.

**Lemma III.2.9.** *For any $x_0, x_1 \in M$, we have $C_{x_0} - C_{x_1}$ is a bounded function and*

$$\|C_{x_0} - C_{x_1}\| \leq d(x_0, x_1)$$

*Proof.* Direct computation yields

$$\|C_{x_0}(x) - C_{x_1}(x)\| = \| - \exp_x^{-1}(x_0) + \exp_x^{-1}(x_1)\| \leq d(x_0, x_1)$$

where the inequality comes from the fact that the exponential maps for non-positively curved manifolds are metric-expanding, by Hadamard's theorem. □

For any $f \in \mathcal{S}$, let us define its *$r$-oscillation* by

$$\Omega_r f := \sup \{|f(t) - f(t')| : \ t, t' \in \mathbb{R}, |t - t'| \leq r\} . \tag{III.2}$$

Then we have the following estimate.

**Proposition III.2.10.** *For any $x_0, x_1 \in M$ and any $f \in \mathcal{S}$, let $r := d(x_0, x_1)$ Then*

$$\|\beta_{x_0}(f) - \beta_{x_1}(f)\| \leq 2\ \Omega_r f + r\|f\|.$$

*Proof.* Since $C_{x_0}^2$ is a non-negative scalar function, we have the non-negative scalar function $|C_{x_0}| := \sqrt{C_{x_0}^2}$ that satisfies

$$|C_{x_0}|(x) = \|C_{x_0}(x)\|$$

for any $x \in M$ and thus

$$\begin{aligned} |(|C_{x_0}| - |C_{x_1}|)(x)| &= \ |\|C_{x_0}(x)\| - \|C_{x_1}(x)\|| \\ &\leq \ \|C_{x_0}(x) - C_{x_1}(x)\| \\ &= \ \|C_{x_0} - C_{x_1}\| \leq d(x_0, x_1) = r. \end{aligned}$$

Now if $f \in \mathcal{S}$ is an even function, then it is of the form $f(t) = g(t^2)$ and since

$$\beta_{x_0}(f)(x) = g(C_{x_0}^2(x)) = f(|C_{x_0}|(x)),$$

we have

$$\|(\beta_{x_0}(f) - \beta_{x_1}(f))(x)\| = |f(|C_{x_0}|(x)) - f(|C_{x_1}|(x))| \leq \Omega_r f.$$

Taking supreme yields $\|\beta_{x_0}(f) - \beta_{x_1}(f)\| \leq \Omega_r f$.

If $f \in \mathcal{S}$ is of the form $f(s) = sg(s)$ for an even function $g \in \mathcal{S}$, then

$$\begin{aligned}
&\|(\beta_{x_0}(f) - \beta_{x_1}(f))(x)\| \\
= &\| \ g\left(|C_{x_0}|(x)\right) \cdot C_{x_0}(x) - g\left(|C_{x_1}|(x)\right) \cdot C_{x_1}(x) \ \| \\
\leq &| \ g\left(|C_{x_0}|(x)\right) \cdot |C_{x_0}|(x) - g\left(|C_{x_1}|(x)\right) \cdot |C_{x_1}|(x) \ | \\
&+ \max\{g\left(|C_{x_0}|(x)\right), g\left(|C_{x_1}|(x)\right)\} \cdot \|C_{x_0}(x) - C_{x_1}(x)\| \\
\leq &\Omega_r f + r \cdot \|f\|.
\end{aligned}$$

Here the crucial observation is that $C_{x_0}(x)$ and $C_{x_1}(x)$ are vectors in $T_xM$ and we can use planar geometry to get an estimate. Taking supreme yields the desired bound.

Since for every function on $\mathbb{R}$, its canonical decomposition into a sum of an odd function and an even function produces summand functions with no larger oscillations or norms, the general estimate follows. □

**Corollary III.2.11.** *If a sequence $\{x_n \in M\}_{n \in \mathbb{N}}$ converges to $x_0 \in M$, then for any $f \in \mathcal{S}$, we have*

$$\|\beta_{x_0}(f) - \beta_{x_n}(f)\| \xrightarrow{n \to \infty} 0.$$

*Proof.* This follows from the previous observation and the fact that any $f \in C_0(\mathbb{R})$ is uniformly continuous, and thus $\lim_{r \to 0} \Omega_r(f) = 0$. □

## III.3 The coefficient algebra $\mathcal{A}(M)$

Now we are ready to define the desired coefficient algebra $\mathcal{A}(M)$. We fix a nonpositively curved Hilbert manifold $M$. Consider the manifold $\mathbb{R} \times M$ equipped with the product Hilbert-Riemannian metric. Thus $\mathbb{R} \times M$ is also a nonpositively curved Hilbert manifold. Note that $\mathrm{Isom}(M)$ embeds into $\mathrm{Isom}(\mathbb{R} \times M)$ by acting on the second coordinate. This action commutes with the reflection map around 0 on the first factor $\mathbb{R}$, i.e., the involutive isometry $R$ on $\mathbb{R} \times M$ given by

$$R(t, x) := (-t, x) \tag{III.3}$$

for any $t \in \mathbb{R}$ and $x \in M$. Viewing $R$ as an isometric action by $\mathbb{Z}_2$, we have thus obtained an isometric action of $\mathbb{R} \times M$ by $\mathbb{Z}_2 \times \mathrm{Isom}(M)$. For any $x_0 \in M$, since $(0, x_0)$ is fixed by $R$, by Corollary III.2.8, the image of the Bott homomorphism

$$\beta_{(0,x_0)} : \mathcal{S} \to C_{0,\mathrm{met}}\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)$$

is contained in the fixed-point algebra $C_{0,\mathrm{met}}\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$.

**Definition III.3.1.** Let $M$ be a nonpositively curved Hilbert manifold. The algebra $\mathcal{A}(M)$ is the $C^*$-subalgebra of $C_{0,\mathrm{met}}\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$ generated by

$$\{\beta_{(0,x_0)}(f) \colon x_0 \in M,\ f \in \mathcal{S}\ \}.$$

We also define $\mathcal{A}_{\mathrm{ev}}(M)$ to be the $C^*$-subalgebra of $\mathcal{A}(M)$ generated by

$$\{\beta_{(0,x_0)}(f) \colon x_0 \in M,\ f \in \mathcal{S}_{\mathrm{ev}}\ \}.$$

We discuss some basic properties of these algebras.

**Proposition III.3.2.** *$\mathcal{A}(M)$ is separable.*

*Proof.* By the separability of $M$ and $\mathcal{S}$, there are countable dense subsets $X$ and $F$ of $M$ and $\mathcal{S}$, respectively. It follows then by Lemma III.2.11 that $\{\beta_{(0,x_0)}(f) \mid x_0 \in X,\ f \in F\}$ is a dense subset of $\mathcal{A}(M)$. □

**Proposition III.3.3.** *For any isometry $\phi$ of $M$, the induced $*$-automorphism $\phi_*$ of $C_{0,\mathrm{met}}(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times$ given in* (III.1) *preserves $\mathcal{A}(M)$ and $\mathcal{A}_{\mathrm{ev}}(M)$.*

*Proof.* This follows directly from Lemma III.2.6 and the fact that $\phi_*$ commutes with $R_*$. □

Next we investigate the center of $\mathcal{A}(M)$. We first observe that by Proposition III.2.5, $\mathcal{A}_{\mathrm{ev}}(M)$ is contained in $C_b(\mathbb{R} \times M) \cap \mathcal{A}(M)$, and is thus in the center of $\mathcal{A}(M)$. In fact, we have the following proposition.

**Proposition III.3.4.** *$\mathcal{A}(M)$ is an $\mathrm{Isom}(M)$-$\mathcal{A}_{\mathrm{ev}}(M)$-$C^*$-algebra.*

*Proof.* Since we already know by Proposition III.3.3 that $\mathcal{A}_{\mathrm{ev}}(M)$ is an $\mathrm{Isom}(M)$-invariant sub-$C^*$-algebra of the center of $\mathcal{A}(M)$, it suffices to show that $\mathcal{A}_{\mathrm{ev}}(M) \cdot \mathcal{A}(M)$ is dense in $\mathcal{A}(M)$. Indeed, this follows from the definition of $\mathcal{A}(M)$ and the fact that every $f \in \mathcal{S}$ can be written as a product $f = f_1 f_2$ where $f_1 \in \mathcal{S}_{\mathrm{ev}}$ and $f_2 \in \mathcal{S}$. □

To determine the spectrum of $\mathcal{A}_{\mathrm{ev}}(M)$, we notice that each pair $(t, x) \in \mathbb{R} \times M$ induces a character $\mathsf{ev}_{(t,x)} : C_b(\mathbb{R} \times M) \to \mathbb{C}$ by evaluation at $(t, x)$, and this in turn induces a character of $\mathcal{A}_{\mathrm{ev}}(M)$ by restriction, as it is a non-zero map: for example, for the non-negative even function $f(s) = (1 + |t| - |s|)_+$, we have $\mathsf{ev}_{(t,x)}(\beta_{(0,x)}(f)) = 1$. In the following proposition, we equip $\mathbb{R}^{\geq 0} \times M \subset \mathbb{R} \times M$ with the product topology.

**Proposition III.3.5.** *The map $\mathsf{ev} : \mathbb{R}^{\geq 0} \times M \to \widehat{\mathcal{A}_{\mathrm{ev}}(M)}$, $(t, x) \mapsto \mathsf{ev}_{(t,x)}$ is an $\mathrm{Isom}(M)$-equivariant continuous bijection, and restricts to a homeomorphism between $\{0\} \times M$ $(\cong M)$*

*and its image. Moreover, the pull-back of the topology of $\widehat{\mathcal{A}_{\mathrm{ev}}(M)}$ onto $\mathbb{R}^{\geq 0} \times M$ is the weakest topology that makes the functions*

$$\rho_{x_0} \colon \mathbb{R}^{\geq 0} \times M \to \mathbb{R}^{\geq 0}\ ,\ \ (t,x) \mapsto |t|^2 + d(x,x_0)^2$$

*continuous, for all $x_0 \in M$. This topology has a subbase consisting of "open half-shells[1]"*

$$S_{x_0,(a,b)} := \mathsf{ev}\left(\{(t,x) \in \mathbb{R}^{\geq 0} \times M \colon a < |t|^2 + d(x,x_0)^2 < b\}\right)$$

*for all $x_0 \in M$ and $a, b \in \mathbb{R}$ such that $b > a$.*

*Proof.* The equivariance and continuity can be seen by realizing $\mathsf{ev}$ as a composition of equivariant continuous maps

$$\mathbb{R}^{\geq 0} \times M \hookrightarrow \mathbb{R} \times M \hookrightarrow \beta(\mathbb{R} \times M) \cong C_b(\mathbb{R} \times M)\widehat{\ } \to (\mathcal{A}_{\mathrm{ev}}(M)^+)\widehat{\ } = \widehat{\mathcal{A}_{\mathrm{ev}}(M)}^+,$$

where $\beta(\mathbb{R} \times M)$ is the Stone-Čech compactification of $\mathbb{R} \times M$, $\mathcal{A}_{\mathrm{ev}}(M)^+$ is the unitalization of $\mathcal{A}_{\mathrm{ev}}(M)$ while $\widehat{\mathcal{A}_{\mathrm{ev}}(M)}^+$ is the one-point compactification of $\widehat{\mathcal{A}_{\mathrm{ev}}(M)}$, with the point at infinity being the character mapping the unit to 1 and $\mathcal{A}_{\mathrm{ev}}(M)$ to 0 (and hence outside of the image of the composition).

The injectivity follows from the fact that the family of functions

$$\{\beta_{(0,x)}(f) \colon x \in M, f \in \mathcal{S}_{\mathrm{ev}}\}$$

separates points: for any different $(t_1, x_1), (t_2, x_2) \in \mathbb{R}^{\geq 0} \times M$, we can always find some $x_0 \in M$ such that $t_1^2 + d(x_1, x_0)^2 \neq t_2^2 + d(x_2, x_0)^2$, which then guarantees that, for any $a > 0$ strictly between these two values, the function $\beta_{(0,x_0)}(f) \in \mathcal{A}_{\mathrm{ev}}(M)$ for $f : s \mapsto (\sqrt{a} - |s|)_+$ will take value 0 at one of the two points and attain a nonzero value at the other.

Next we prove the statement about the pull-back topology on $\mathbb{R}^{\geq 0} \times M$, which we denote by $\mathcal{T}$. By the definition of $\mathcal{A}_{\mathrm{ev}}(M)$, we know that $\mathcal{T}$ is the weakest topology that makes the functions $(t,x) \mapsto \mathsf{ev}_{(t,x)}(\beta(f))$ continuous for every $x_0 \in M$ and every even function $f \in \mathcal{S}$. Since any even function $f$ in $\mathcal{S}$ can be written as $f(t) = g(t^2)$ for some $g \in C_0(\mathbb{R}^{\geq 0})$ and vice versa, we see that $\mathcal{T}$ is the weakest topology that makes the functions

$$(t,x) \mapsto \mathsf{ev}_{(t,x)}(\beta(f)) = g(C^2_{(0,x_0)}(t,x)) = g(|t|^2 + d(x,x_0)^2) = (g \circ \rho_{x_0})(t,x)$$

continuous for every $x_0 \in M$ and every function $g \in C_0(\mathbb{R}^{\geq 0})$. Since the standard (Euclidean)

[1] These include open balls, which occur when $a < 0$ in the definition.

topology on $\mathbb{R}^{\geq 0}$ is the weakest that makes any function $g \in C_0(\mathbb{R}^{\geq 0})$ continuous, we see that $\mathcal{T}$ is the weakest topology that makes $\rho_{x_0}$ continuous for all $x_0 \in M$. The claim about the subbase follows from the well-known fact that the standard topology on $\mathbb{R}^{\geq 0}$ has a base consisting of (relatively) open intervals.

In order to show the surjectivity of $\mathsf{ev}$, we know from the local compactness of $\widehat{\mathcal{A}_{\mathrm{ev}}(M)}$ that it suffices to show $\mathsf{ev}$ has dense image and is a proper map when we equip $\mathbb{R}^{\geq 0} \times M$ with the topology $\mathcal{T}$. The density of its image follows from the fact that any element in $C_b(\mathbb{R} \times M)^R$, and thus any in $\mathcal{A}_{\mathrm{ev}}(M)$, that vanishes at each point in $\mathbb{R}^{\geq 0} \times M \subset \mathbb{R} \times M$ must be zero. On the other hand, we notice that for any $x_0 \in M$ and any $R > 0$, the set

$$\{(t,x) \in \mathbb{R}^{\geq 0} \times M \colon t^2 + d(x,x_0)^2 \leq R^2\}$$

is compact with respect to $\mathcal{T}$, because its image under $\rho_{x_1}$ for any $x_1 \in M$ is compact. Combining this with the statement about the subbase of $\mathcal{T}$ that we have proved above, we see that $\mathbb{R}^{\geq 0} \times M$ is locally compact with regard to $\mathcal{T}$. Thus the properness of $\mathsf{ev}$ follows from that for any $a \in \mathcal{A}_{\mathrm{ev}}(M)$, the function $(t,x) \mapsto \mathsf{ev}_{(t,x)}(a)$ on $\mathbb{R}^{\geq 0} \times M$ vanishes at infinity (with regard to $\mathcal{T}$): indeed, since $a \in C_{0,\mathrm{met}}(\mathbb{R} \times M)$, for any $\varepsilon > 0$, we can find some $x_0 \in M$ and $R > 0$ such that $|\mathsf{ev}_{(t,x)}(a)| < \varepsilon$ for any $(t,x)$ outside the compact set

$$\{(t,x) \in \mathbb{R}^{\geq 0} \times M \colon t^2 + d(x,x_0)^2 \leq R^2\} \,.$$

Therefore $\mathsf{ev}$ is also surjective.

Lastly, to show the restriction of $\mathsf{ev}$ to $\{0\} \times M$ is a homeomorphism onto its image, it suffices to show that, for any $x_0 \in M$ and $R > 0$, the open ball

$$\{0\} \times \{x \in M \colon d(x,x_0) < R\}$$

is still open under the subspace topology on $\{0\} \times M$ induced from $\mathcal{T}$, because these open balls form a basis of $\{0\} \times M$. This is indeed true, as this open ball is precisely the intersection of $\{0\} \times M$ and the $\mathcal{T}$-open set

$$S_{x_0,(-1,R^2)} := \mathsf{ev}\left(\{(t,x) \in \mathbb{R}^{\geq 0} \times M \colon |t|^2 + d(x,x_0)^2 < R^2\}\right) \,.$$

Therefore we have proved all the claims. □

**Corollary III.3.6.** *The map* $\mathsf{ev}$ *is a homeomorphism if and only if* $M$ *is finite-dimensional.*

*Proof.* If $M$ is infinite-dimensional, then the domain of $\mathsf{ev}$ is not locally compact but the range is; thus they are not homeomorphic. On the other hand, if $M$ is finite-dimensional, then since the functions $\rho_{x_0}$ defined in Proposition III.3.5 separate points on $\mathbb{R}^{\geq 0} \times M$, the pull-back of the topology of $\widehat{\mathcal{A}_{\mathrm{ev}}(M)}$ onto $\mathbb{R}^{\geq 0} \times M$ agrees with the original, locally compact topology of $\mathbb{R}^{\geq 0} \times M$ by the Stone-Weierstrass theorem, which means $\mathsf{ev}$ is a homeomorphism. $\square$

As a consequence of Proposition III.3.5, the fibers of $\mathcal{A}(M)$ (as an $\mathcal{A}_{\mathrm{ev}}(M)$-$C^*$-algebra) can be indexed by $\mathbb{R}^{\geq 0} \times M$. Thus for each $(t, x) \in \mathbb{R}^{\geq 0} \times M$, we write $\mathcal{A}(M)_{(t,x)}$ for the fiber associated to the character $\mathsf{ev}_{(t,x)} \in \widehat{\mathcal{A}_{\mathrm{ev}}(M)}$. It is sometimes helpful to think of $\mathcal{A}(M) \subset C_{0,\mathrm{met}}\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$ as consisting of sections over $\mathbb{R}^{\geq 0} \times M$, viewed as a manifold with boundary, in the same spirit as in Example III.1.6.

**Proposition III.3.7.** *For any $x \in M$ and $t > 0$, there are isomorphisms*

$$\mathcal{A}(M)_{(0,x)} \cong \mathfrak{Cl}_{\mathbb{C}}(T_x M) \text{ and } \mathcal{A}(M)_{(t,x)} \cong \mathfrak{Cl}_{\mathbb{C}}(\mathbb{R} \oplus T_x M) .$$

*Proof.* The algebra $C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$ is a $C_b(\mathbb{R} \times M)^R$-$C^*$-algebra, and is thus fibered over $\left(C_b(\mathbb{R} \times M)^R\right)^\wedge \cong \left(C_b(\mathbb{R}^{\geq 0} \times M)\right)^\wedge \cong \beta(\mathbb{R}^{\geq 0} \times M)$, the latter containing $\widehat{\mathcal{A}_{\mathrm{ev}}(M)}$ as a subspace. Thus for any $(t,x) \in \mathbb{R}^{\geq 0} \times M \subset \beta(\mathbb{R}^{\geq 0} \times M)$, the fiber $\left(C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R\right)_{(t,x)}$ is isomorphic to

$$C(\{(-t,x),(t,x)\}, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M))^R \cong \begin{cases} \mathfrak{Cl}_{\mathbb{C}} T_{(t,x)}(\mathbb{R} \times M)\,, & t > 0 \\ \mathfrak{Cl}_{\mathbb{C}} T_{(0,x)}(\mathbb{R} \times M)^R\,, & t = 0 \end{cases}$$

where in the case where $t = 0$, $\mathfrak{Cl}_{\mathbb{C}} T_{(0,x)}(\mathbb{R} \times M)^R$ is the fixed-point subalgebra of $\mathfrak{Cl}_{\mathbb{C}} T_{(0,x)}(\mathbb{R} \times M)$ by the involution $R_*$ induced by reversing the $\mathbb{R}$ direction, and thus we have $\mathfrak{Cl}_{\mathbb{C}} T_{(0,x)}(\mathbb{R} \times M)^R \cong \mathfrak{Cl}_{\mathbb{C}} T_x(M)$. It remains to show that the inclusion map $\mathcal{A}(M) \hookrightarrow C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$ induce surjections on each such fiber. This is indeed true: for any $(t,x) \in \mathbb{R}^{\geq 0} \times M$, if $t > 0$, then the fiber $\left(C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R\right)_{(t,x)}$, which is isomorphic to $\mathfrak{Cl}_{\mathbb{C}} T_{(t,x)}(\mathbb{R} \times M)$, is generated by the vectors in $T_{(t,x)}(\mathbb{R} \times M)$ (viewed as elements in the Clifford algebra), and it suffices to take all $(s,v) \in T_{(t,x)}(\mathbb{R} \times M)$ with $s = t$ as they span $T_{(t,x)}(\mathbb{R} \times M)$, but this vector $(t,v)$ is precisely the value of $C_{(0,\exp_x(-v))}$ at $(t,x)$; the case when $t = 0$ is similar except that we need all $(0,v) \in T_{(0,x)}(\mathbb{R} \times M)^R$, which is again provided by $C_{(0,\exp_x(-v))}$. $\square$

**Corollary III.3.8.** *When $M$ is infinite-dimensional, then each fiber of $\mathcal{A}(M)$ is isomorphic to the CAR algebra $\mathcal{M}_{2^\infty}$ and $\mathcal{A}_{\mathrm{ev}}(M)$ is equal to the center of $\mathcal{A}(M)$.*

*Proof.* The first claim is a direct consequence of the fact that the complex Clifford algebra of an infinite-dimensional separable Hilbert space is isomorphic to the CAR algebra, and the second claim follows from that the CAR algebra is simple, and in particular, centerless. $\square$

**Corollary III.3.9.** *$\mathcal{A}(M)$ is a nuclear $C^*$-algebra.*

*Proof.* This follows directly from the nuclearity of all the fibers. $\square$

**Proposition III.3.10.** *When $M$ is finite-dimensional, we have*

$$\mathcal{A}(M) = C_0\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$$

*as subalgebras of $C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$. More explicitly, if $M$ has dimension $2n$, then*

$$\mathcal{A}(M) \cong C_0(\mathbb{R} \times M, \mathcal{M}_{2^n})$$

*and if $M$ has dimension $2n+1$, then*

$$\mathcal{A}(M) \cong \{f \in C_0(\mathbb{R}^{\geq 0} \times M, \mathcal{M}_{2^{n+1}}) \colon f(\{0\} \times M) \subset \mathcal{M}_{2^n} \oplus \mathcal{M}_{2^n}\ \}\ ,$$

*where $\mathcal{M}_{2^n} \oplus \mathcal{M}_{2^n}$ embeds into $\mathcal{M}_{2^{n+1}}$ diagonally.*

*Proof.* By Corollary III.3, we have $\mathcal{A}_{\mathrm{ev}}(M) = C_0(\mathbb{R} \times M)^R$ as subalgebras of $C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$. Now both $\mathcal{A}(M)$ and $C_0\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$ are fibered over $\mathbb{R}^{\geq 0} \times M$. By Proposition III.3.7, the algebras $\mathcal{A}(M)$ and $C_0(\mathbb{R} \times M, \mathcal{M}_{2^n})$ agree on each fiber. Thus the first claim follows.

For the second claim, we first see by the contractibility of $(\mathbb{R} \times M)/R$ that the bundle $T(\mathbb{R} \times M)$ can be $R$-equivariantly trivilized to $(\mathbb{R} \times M) \times (\mathbb{R} \oplus \mathbb{R}^d)$, where $d$ is the dimension of $M$ and $R$ reverses both of the $\mathbb{R}$-factors of $(\mathbb{R} \times M) \times (\mathbb{R} \oplus \mathbb{R}^d)$ at the same time. It follows that $C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)$ can be $R$-equivariantly trivialized to $C_b(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} \mathbb{R} \hat{\otimes} \mathfrak{Cl}_{\mathbb{C}} \mathbb{R}^d)$. Thus $\mathcal{A}(M)$ is isomorphic to

$$\begin{aligned} & C_0(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} \mathbb{R} \hat{\otimes} \mathfrak{Cl}_{\mathbb{C}} \mathbb{R}^d)^R \\ \cong\ & \{f \in C_0(\mathbb{R}^{\geq 0} \times M, \mathfrak{Cl}_{\mathbb{C}} \mathbb{R} \hat{\otimes} \mathfrak{Cl}_{\mathbb{C}} \mathbb{R}^d) \colon f(\{0\} \times M) \subset 1 \hat{\otimes} \mathfrak{Cl}_{\mathbb{C}} \mathbb{R}^d\ \}\ . \end{aligned}$$

Hence when $d = 2n+1$, the result follows from the isomorphisms $\mathbb{R}^{2n+2}) \cong \mathcal{M}_{2^{n+1}}$ and $\mathfrak{Cl}_{\mathbb{C}} \mathbb{R}^{2n+1} \cong \mathcal{M}_{2^n} \oplus \mathcal{M}_{2^n}$. When $d = 2n$, we have, similarly,

$$\mathcal{A}(M) \cong \{f \in C_0(\mathbb{R}^{\geq 0} \times M, \mathcal{M}_{2^n} \oplus \mathcal{M}_{2^n}) \colon f(\{0\} \times M) \subset \mathcal{M}_{2^n}\ \}\ ,$$

where $\mathcal{M}_{2^n}$ embeds into $\mathcal{M}_{2^n} \oplus \mathcal{M}_{2^n}$ diagonally. The latter algebra is clearly isomorphic to $C_0(\mathbb{R} \times M, \mathcal{M}_{2^n})$. $\square$

One of the key properties of $\mathcal{A}(M)$ is that any isometric, metrically proper action of $M$

by a discrete group $\Gamma$ induces by means of Proposition III.3.3 a proper action on $\mathcal{A}(M)$. Here $\mathcal{A}_{\mathrm{ev}}(M)$ will play an important role, as we will show the action of $\Gamma$ on the spectrum of $\mathcal{A}_{\mathrm{ev}}(M)$ is proper.

**Lemma III.3.11.** *Let $X$ be a locally compact Hausdorff space, $\Gamma$ a discrete group, and $\alpha\colon \Gamma \curvearrowright X$ a continuous action. Let $\alpha_*$ be the induced action on $C_0(X)$. Then this action is (topologically) proper if and only if for any $f \in C_0(X)$,*

$$\lim_{\gamma\to\infty} \|\big((\alpha_*)_\gamma(f)\big) \cdot f\| \to 0 ,$$

*i.e., for any $\epsilon > 0$, there is a finite subset $F \subset \Gamma$ such that for any $\gamma \in \Gamma \setminus F$,*

$$\|\big((\alpha_*)_\gamma(f)\big) \cdot f\| < \epsilon.$$

*Proof.* If the action $\Gamma \curvearrowright X$ is proper, then for any $f \in C_c(X)$, there is a finite subset $F \subset \Gamma$ such that for any $\gamma \in \Gamma \setminus F$, $\|(\alpha_*)_\gamma(f) \cdot f\| = 0$. The statement for a general $f \in C_0(X)$ follows by approximation.

On the other hand, if every element of $C_0(X)$ satisfies the condition in the statement, then for any compact subset $K \subset X$, picking a positive function $f \in C_0(X)$ such that $f(x) \geq 1$ for $x \in K$, we can find, according to the condition, a finite $F \subset \Gamma$ such that for any $\gamma \notin F$, $\|(\alpha_*)_\gamma(f) \cdot f\| < \frac{1}{2}$, which implies that $\alpha_\gamma(K) \cap K = \varnothing$. $\square$

**Proposition III.3.12.** *Let $\Gamma$ be a discrete group and $\alpha\colon \Gamma \to \mathrm{Isom}(M)$ an isometric, metrically proper action on $M$. Then the induced action on $\mathcal{A}(M)$ (also denoted by $\alpha$) given in Proposition III.3.3 makes $\mathcal{A}(M)$ into a proper $\Gamma$-$\mathcal{A}_{\mathrm{ev}}(M)$-$C^*$-algebra.*

*Proof.* Observe that by definition, each $a \in C_{00,\mathrm{met}}\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)$ is supported in a bounded subset of $M$. Thus because of the metric properness of the action $\Gamma \curvearrowright M$, all but finitely many elements $\gamma$ of $\Gamma$ satisfy $(\alpha_\gamma(a)) \cdot a = 0$. Since $C_{0,\mathrm{met}}\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)$ is the closure of the previous algebra, each element $b$ in it satisfies

$$\lim_{\gamma\to\infty} \|(\alpha_\gamma(b)) \cdot b\| = 0 .$$

The same is thus true for the $\Gamma$-invariant subalgebra $\mathcal{A}_{\mathrm{ev}}(M)$, which ensures the action of $\Gamma$ on the spectrum of $\mathcal{A}_{\mathrm{ev}}(M)$ is (topologically) proper by Lemma III.3.11, i.e. $\mathcal{A}_{\mathrm{ev}}(M)$ is a commutative proper $\Gamma$-$C^*$-algebra. It follows that $\mathcal{A}(M)$ is a proper $\Gamma$-$\mathcal{A}_{\mathrm{ev}}(M)$-$C^*$-algebra. $\square$

CHAPTER IV

# THE NOVIKOV CONJECTURE FOR GROUPS ACTING ON NON-POSITIVELY CURVED MANIFOLDS

In the last section we prove our main result, which will make use of the $C^*$-algebra we have constructed.

## IV.1 The $K$-theory of $\mathcal{A}(M)$

The algebra $\mathcal{A}(M)$ is constructed to imitate the setting for the Bott periodicity, an isomorphism on $K$-theory induced by the Bott homomorphism in the finite dimensional case. However, when $M$ is infinite dimensional, proving the full periodicity (injectivity *and* surjectivity of the Bott map) appears difficult. In particular, it is not clear how to carry out the construction of the *Dirac element*, which is the inverse of the Bott map in $KK$-theory and plays a crucial role in [12].

Here we use an approximation technique to show the Bott homomorphism induces an injective map on $K$-theory.

**Definition IV.1.1.** Let $N \subset M$ be a subset. We define $\mathcal{A}(M, N)$ (respectively $\mathcal{A}_{\mathrm{ev}}(M, N)$) to be the $*$-subalgebra of $\mathcal{A}(M)$ generated by

$$\{\beta_{(0,x_0)}(f)\colon x_0 \in N \subset M,\ f \in \mathcal{S} \text{ (respectively, } \mathcal{S}_{\mathrm{ev}})\}.$$

We list some immediate consequences of the definition.

**Lemma IV.1.2.** *Let $N_1, N_2, \ldots$ be subsets of $M$.*

1. $\mathcal{A}(M, M) = \mathcal{A}(M)$.
2. *If $N_1 \subset N_2$ then $\mathcal{A}(M, N_1) \subset \mathcal{A}(M, N_2)$.*
3. *If $\overline{N_1}$ is the closure of $N_1$, then $\mathcal{A}(M, \overline{N_1}) \subset \mathcal{A}(M, N_1)$.*
4. *If $N_1 \subset N_2 \subset \ldots$, then $\mathcal{A}(M, \overline{\bigcup_{k=1}^{\infty} N_k})$ is the direct limit of the sequence $\mathcal{A}(M, N_1) \subset \mathcal{A}(M, N_2) \subset \ldots$ of subalgebras.*

*Proof.* The first and second claims are immediate from the definition. The third claim is a consequence of Corollary III.2.11. The last claim follows from the second and the third. $\square$

The construction of $\mathcal{A}(M,N)$ is particularly interesting when $N$ is a *complete geodesic submanifold* of $M$, that is, for any two points $x_0, x_1$, the (unique) geodesic in $M$ that runs through these two points lies entirely in $N$. In this case, $N$ is again a complete Hilbert manifold with non-positive sectional curvatures. We let $T_{\mathbb{R}\times N}(\mathbb{R}\times M)$ be the pull-back of the tangent bundle $T(\mathbb{R}\times M)$ by the inclusion map $\mathbb{R}\times N \hookrightarrow \mathbb{R}\times M$, and $\mathfrak{Cl}_{\mathbb{C}} T_{\mathbb{R}\times N}(\mathbb{R}\times M)$ its complex Clifford bundle. Consider the $C^*$-algebra $C_b\left(\mathbb{R}\times N, \mathfrak{Cl}_{\mathbb{C}} T_{\mathbb{R}\times N}(\mathbb{R}\times M)\right)$ consisting of all bounded continuous sections of $\mathfrak{Cl}_{\mathbb{C}} T_{\mathbb{R}\times N}(\mathbb{R}\times M)$. Thus the inclusion map $\mathbb{R}\times N \hookrightarrow \mathbb{R}\times M$ defines by means of restriction a $*$-homomorphism

$$\pi_{M,N} \colon C_b\left(\mathbb{R}\times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R}\times M)\right) \to C_b\left(\mathbb{R}\times N, \mathfrak{Cl}_{\mathbb{C}} T_{\mathbb{R}\times N}(\mathbb{R}\times M)\right)$$

while the inclusion of the tangent bundle $T(\mathbb{R}\times N)$ into $T_{\mathbb{R}\times N}(\mathbb{R}\times M)$ induces an embedding

$$\iota_{M,N} \colon C_b\left(\mathbb{R}\times N, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R}\times N)\right) \hookrightarrow C_b\left(\mathbb{R}\times N, \mathfrak{Cl}_{\mathbb{C}} T_{\mathbb{R}\times N}(\mathbb{R}\times M)\right) \ .$$

**Lemma IV.1.3.** *Let $N$ be a complete geodesic submanifold $M$. Then for any $x_0 \in N$, we have*

$$\pi_{M,N} \circ \beta^M_{(0,x_0)} = \iota_{M,N} \circ \beta^N_{(0,x_0)} \ ,$$

*where $\beta^M_{(0,x_0)}$ is the Bott homomorphism into $\mathcal{A}(M,N) \subset \mathcal{A}(M)$ and $\beta^N_{(0,x_0)}$ is the Bott homomorphism into $\mathcal{A}(N)$. In particular,*

$$\pi_{M,N}(\mathcal{A}(M,N)) = \iota_{M,N}(\mathcal{A}(N)) \ .$$

*Proof.* We check that for any $(t,x) \in \mathbb{R}\times N$ and any $f \in \mathcal{S}$, we have, inside $\mathfrak{Cl}_{\mathbb{C}} T_{(t,x)}(\mathbb{R}\times M)$,

$$\pi_{M,N} \circ \beta^M_{(0,x_0)}(f)(t,x) = \pi_{M,N}\left(f(C^M_{(0,x_0)}(t,x))\right) = f\left(C^M_{(0,x_0)}(t,x)\right)$$

and

$$\iota_{M,N} \circ \beta^N_{(0,x_0)}(f)(t,x) = \iota_{M,N}\left(f(C^N_{(0,x_0)}(t,x))\right) = f\left(C^N_{(0,x_0)}(t,x)\right) \ .$$

They give the same element because the geodesic in $M$ connecting $(t_0, x_0, v_0)$ to $(t,x)$ coincides with the geodesic in the submanifold $N$ connecting the same two points. $\square$

For the next lemma, we observe that when $M$ is a simply connected non-positively curved Hilbert manifold, so is $M \times M$.

**Lemma IV.1.4.** *If $M$ is a finite-dimensional simply connected non-positively curved Hilbert manifold, then for any $x_0 \in M$, the Bott homomorphism $\beta_{(0,x_0)} \colon \mathcal{S} \to \mathcal{A}(M)$ induces an* *isomorphism on $K$-theory.*

*Proof.* By Proposition III.3.10, we have

$$\mathcal{A}(M) = C_0\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$$

as subalgebras of $C_b\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R$. Let $d$ be the dimension of $M$. Note that the exponential map $\exp_{x_0} \colon \mathbb{R}^d \to M$, which is a diffeomorphism by the Hadamard theorem, induces an isomorphism

$$C_0\left(\mathbb{R} \times M, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times M)\right)^R \cong C_0\left(\mathbb{R} \times \mathbb{R}^d, \mathfrak{Cl}_{\mathbb{C}} T(\mathbb{R} \times \mathbb{R}^d)\right)^R = \mathcal{A}(\mathbb{R}^d)$$

that intertwines the Bott homomorphisms $\beta^M_{(0,x_0)}$ and $\beta^{\mathbb{R}^d}_{(0,0)}$. [13] $\square$

**Lemma IV.1.5.** *For any two points $x_0, x_1 \in M$, the Bott homomorphisms*

$$\beta_{(0,x_0)}, \beta_{(0,x_1)} \colon \mathcal{S} \to \mathcal{A}(M)$$

*are homotopic to each other.*

*Proof.* Let $(x_s)_{s\in[0,1]}$ be a path in $M$ connecting $x_0$ and $x_1$ (e.g., the geodesic between the two points). Then by Corollary III.2.11, the family $\left(\beta_{(0,x_s)}\right)_{s\in[0,1]}$ constitutes a homotopy between $\beta_{(0,x_0)}$ and $\beta_{(0,x_1)}$. $\square$

Recall that $M$ is said to have a filtration of finite-dimensional complete geodesic submanifolds if there are finite-dimensional complete geodesic submanifolds $M_1 \subset M_2 \subset \ldots$ such that $M = \bigcup_{k=1}^{\infty} M_k$.

**Proposition IV.1.6.** *Suppose that $M$ has a filtration of finite-dimensional complete geodesic submanifolds. Then for any $x_0 \in M$, the Bott homomorphism*

$$\beta_{(0,x_0)} \colon \mathcal{S} \to \mathcal{A}(M)$$

*induces an injection on $K$-theory.*

*Proof.* Let $M_1 \subset M_2 \subset \ldots$ be a sequence of finite-dimensional complete geodesic submanifolds such that $M = \bigcup_{k=1}^{\infty} M_k$. By Lemma IV.1.5, the Bott homomorphism associated to any two base points agree on $K$-theory. Hence we may assume without loss of generality that

$x_0 \in TM_1$. By Lemma IV.1.2, we see that $\mathcal{A}(M)$ is the direct limit of the increasing sequence of subalgebras $\mathcal{A}(M, M_1) \subset \mathcal{A}(M, M_2) \subset \ldots$. Since the image of $\beta_{(0,x_0)}$ is contained in $\mathcal{A}(M, M_k)$ for any $k \in \mathbb{Z}^{>0}$, by the continuity of the $K$-theory functor with regard to direct limits, it suffices to show that

$$\beta_{(0,x_0)} \colon \mathcal{S} \to \mathcal{A}(M, M_k)$$

induces an injection on $K$-theory for every $k \in \mathbb{Z}^{>0}$. To this end, we fix an arbitrary $k \in \mathbb{Z}^{>0}$ and consider the composition

$$\pi_{M,M_k} \circ \beta_{(0,x_0)} \colon \mathcal{S} \to C_b\left(\mathbb{R} \times M_k, \mathfrak{Cl}_{\mathbb{C}} T_{\mathbb{R}\times M_k}(\mathbb{R} \times M)\right)$$

as given in Lemma IV.1.3, where we know that its image falls in $\iota_{M,M_k}(\mathcal{A}(M_k))$ and that it is equal to $\iota_{M,M_k} \circ \beta^{M_k}_{(0,x_0)}$ where $\iota_{M,M_k}$ is the embedding of $\mathcal{A}(M_k)$ into $C_b\left(\mathbb{R} \times M_k, \mathfrak{Cl}_{\mathbb{C}} T_{\mathbb{R}\times M_k}(\mathbb{R} \times M)\right)$ and $\beta^{M_k}_{(0,x_0)}$ is the Bott homomorphism into $\mathcal{A}(M_k)$ based at $(0, x_0)$. By Lemma IV.1.4, we know that the latter induces an isomorphism on $K$-theory. Hence the composition

$$\pi_{M,M_k} \circ \beta_{(0,x_0)} \colon \mathcal{S} \to \iota_{M,N}(\mathcal{A}(N))$$

also induces an isomorphism on $K$-theory. It follows that $\beta_{(0,x_0)} \colon \mathcal{S} \to \mathcal{A}(M, M_k)$ induces an injection on $K$-theory, which suffices as we argued above. $\square$

## IV.2 The Bott element

Using the Bott homomorphism $\beta_{(0,x_0)}$ we would like to construct a $KK$-element

$$[\beta] \in KK^{\Gamma}(\mathcal{S}, \mathcal{A}(M))$$

Notice that $\beta_{(0,x_0)}$ already defines an element in $KK(\mathcal{S}, \mathcal{A}(M))$. Moreover, if $G < \Gamma$ fixes $x_0$, then $\beta_{(0,x_0)}$ is $G$-equivariant, and thus defines an element in $KK^G(\mathcal{S}, \mathcal{A}(M))$. Therefore, in order to define an element in $KK^{\Gamma}(\mathcal{S}, \mathcal{A}(M))$, one must get around the issue of the non-equivariance of $\beta_{(0,x_0)}$ with respect to the $\Gamma$-action.

Fortunately, the non-positively curved manifold $M$, like a Hilbert space, possess a kind of dilation property such that for any function on $M$, we may always make it as flat as we want, thus asymptotically it commutes with the action of $\Gamma$. To make the idea precise, we are going to deform each Bott homomorphism to make an asymptotic morphism.

A key idea in this section is to use a deformation technique to simplify the calculation of equivariant $KK$-groups involving the $C^*$-algebra of a complete simply connected nonpositively curved Hilbert manifold. To formalize this deformation technique, we introduce the following $C^*$-algebra.

**Definition IV.2.1.** Let

$$\sigma\colon \mathbb{R}^+ \curvearrowright \mathcal{S}$$

be the rescaling action given by

$$(s\cdot f)(t) = f(s^{-1}t)$$

for any $s \in \mathbb{R}^+$, $f \in \mathcal{S}$, and $t \in \mathbb{R}$. This action preserves the grading.

**Proposition IV.2.2.** *For any $f \in \mathcal{S}$, $x_0 \in M$ and $\varphi \in \mathrm{Isom}(M)$, we have*

$$\lim_{s\to\infty} \left\| \beta_{(0,x_0)}(\sigma_s(f)) - \gamma \cdot \beta_{(0,x_0)}(\sigma_s(f)) \right\| = 0 \ .$$

*Therefore the pointwise continuous family $\left\{\beta_{(0,x_0)} \circ \sigma_s\right\}_{s\in[1,\infty)}$ constitutes an $\mathrm{Isom}(M)$-equivariant asymptotic morphism from $\mathcal{S}$ to $\mathcal{A}(M)$, where the action on $\mathcal{S}$ is trivial.*

*Proof.* This follows from Lemma III.2.6, Proposition III.2.10, and the computation in the proof of III.2.11. □

**Corollary IV.2.3.** *The family of $*$-homomorphisms $\left\{\beta_{(0,x_0)} \circ \sigma_s\right\}_{s\in[1,\infty)}$ gives rise to an element*

$$[\beta] \in KK_1^\Gamma(\mathbb{C}, \mathcal{A}(M))$$

*which recovers $[\beta_{(0,x_0)}] \in KK_1(\mathbb{C}, \mathcal{A}(M))$ under the forgetful map*

$$KK_1^\Gamma(\mathbb{C}, \mathcal{A}(M)) \to KK_1(\mathbb{C}, \mathcal{A}(M)).$$

*Proof.* This follows from Proposition IV.2.2 and [12, 7.4]. □

We would like to be able to calculate the group $KK^\Gamma_{*+1}(E\Gamma, \mathcal{A}(M))$ and compare it with $K_*^\Gamma(E\Gamma)$ using the homomorphism

$$K_*^\Gamma(E\Gamma) \to KK^\Gamma_{*+1}(E\Gamma, \mathcal{A}(M))$$

given by taking the Kasparov product with $[\beta]$. In the case $\beta_{(0,x_0)}$ induces an isomorphism $K_*(\mathbb{C}) \to K_{*+1}(\mathcal{A}(M))$, a standard cutting-and-pasting argument using the Mayer-Vietoris exact sequence and five lemma implies that the above homomorphism is bijective. However, in Section IV.1, we are only able to show $\beta_{(0,x_0)}$ induces an injection $K_*(\mathbb{C}) \to K_{*+1}(\mathcal{A}(M))$.

Thus our calculation will have to be different. The key idea is to deform the action of $\Gamma$ on $K_{*+1}(\mathcal{A}(M))$ to the trivial action.

To this end, let us discuss how the $K$-theory of $\mathcal{A}(M)$ behaves under the action by $\mathrm{Isom}(M)$ given in Proposition III.3.3. We are particularly interested in those isometries of $M$ whose induced automorphism on $\mathcal{A}(M)$ is trivial on $K$-theory.

Let us endow $\mathrm{Isom}(M)$ with the *topology of pointwise convergence*, which is generated by a local basis around the identity consisting of the sets

$$U(x_1,\ldots,x_n;V_1,\ldots,V_n) := \{\varphi \in \mathrm{Isom}(M)\colon \varphi(x_k) \in V_k, \text{ for } k = 1,\ldots,n\}$$

where $n$ is any natural number, $x,\ldots,x_n \in M$ and each $V_k$ is an open neighborhood of $x_k$. Equivalently, the topology is characterized by the requirement that a net $\{\varphi_i\}_{i\in I}$ converges to the identity if and only if $\lim_{i\in I}\varphi_i(x) = x$ for any $x \in M$.

Similarly, the group of $*$-automorphisms of $\mathcal{A}(M)$ is also given the *topology of pointwise (norm) convergence*, so that a net $\{\varphi_i\}_{i\in I}$ converges to the identity if and only if $\lim_{i\in I}\varphi_i(a) = a$ in norm for any $a \in \mathcal{A}(M)$. Note that it suffices to check the latter condition for any $a$ in a generating set of $\mathcal{A}(M)$, e.g., for all $a$ of the form $\beta_{(0,x_0)}(f)$ for $(0,x_0) \in \mathbb{R}\times M$ and $f \in \mathcal{S}$.

**Lemma IV.2.4.** *When both* $\mathrm{Isom}(M)$ *and* $\mathrm{Aut}(\mathcal{A}(M))$ *carry the topology of pointwise convergence, the canonical action* $\mathrm{Isom}(M) \to \mathrm{Aut}(\mathcal{A}(M))$ *given in Proposition III.3.3 is continuous.*

*Proof.* It suffices to show that for any net $\{\varphi_i\}_{i\in I}$ in $\mathrm{Isom}(M)$ that converges to the identity, the induced net $\{(\varphi_i)_*\}_{i\in I}$ in $\mathrm{Aut}(\mathcal{A}(M))$ also converges to the identity. Since $\mathcal{A}(M)$ is generated by $\beta_{(0,x_0)}(f)$ for $(0,x_0) \in \mathbb{R}\times M$ and $f \in \mathcal{S}$, it suffices to check

$$\lim_{i\in I}(\varphi_i)_*(\beta_{(0,x_0)}(f)) = \beta_{(0,x_0)}(f)$$

for any $(0,x_0) \in \mathbb{R}\times M$ and $f \in \mathcal{S}$. By Lemma III.2.6, the left-hand side is equal to $\lim_{i\in I}\beta_{(0,\varphi_i(x_0))}(f)$, which by Lemma III.2.11 is equal to the right-hand side, as, by assumption, $\lim_{i\in I}\varphi_i(x_0) = x_0$. $\square$

**Proposition IV.2.5.** *Let* $\alpha\colon \Gamma \to \mathrm{Isom}(M)$ *be an isometric action that is null-homotopic in the sense that there is a family* $\{\alpha_t\colon \Gamma \to \mathrm{Isom}(M)\}_{t\in[0,1]}$ *of isometric actions which is pointwise continuous with* $\alpha_1 = \alpha$ *and* $\alpha_0$ *being the trivial action. Then there is an isomorphism*

$$KK_*^\Gamma(E\Gamma,\mathcal{A}(M)) \cong KK_*(B\Gamma,\mathcal{A}(M))$$

*that makes the following diagram commute:*

$$\begin{tikzcd} K_*^\Gamma(E\Gamma) \arrow[d, "{[\beta]}"] \arrow[r, phantom, "\cong"] & K_*(B\Gamma) \arrow[d, "{[\beta]}"] \\ KK_*^\Gamma(E\Gamma, \mathcal{A}(M)) \arrow[r, phantom, "\cong"] & KK_*(B\Gamma, \mathcal{A}(M)) \end{tikzcd}$$

*Proof.* By Lemma IV.2.4, the continuous family $\{\alpha_t \colon \Gamma \to \mathrm{Isom}(M)\}_{t\in[0,1]}$ of actions induces a continuous family $\{\alpha_t \colon \Gamma \to \mathrm{Aut}(\mathcal{A}(M))\}_{t\in[0,1]}$, and thus an action $\alpha_{[0,1]} \colon \Gamma \to \mathrm{Aut}(C([0,1], \mathcal{A}(M)))$ such that for any $\gamma \in \Gamma$, $f \in C([0,1], \mathcal{A}(M))$ and $t \in [0,1]$, we have $(\alpha_{[0,1]})_\gamma(f)(t) = (\alpha_t)_\gamma(f(t))$. For $t = 0, 1$, consider the evaluation map $\mathrm{ev}_t \colon C([0,1], \mathcal{A}(M)) \to \mathcal{A}(M)$, which clearly intertwines the actions $\alpha_{[0,1]}$ and $\alpha_t$. It is also clear that each $\mathrm{ev}_t$ is a homotopy equivalence, without considering the group actions. Therefore a standard argument using the five lemma gives us isomorphisms

$$KK_*^\Gamma(E\Gamma, \mathcal{A}(M)) \cong KK_*^\Gamma(E\Gamma, C([0,1], \mathcal{A}(M))) \cong KK_*^\Gamma(E\Gamma, \mathcal{A}(M))$$

where the actions on $\mathcal{A}(M)$ on the two sides are $\alpha_0$ and $\alpha_1$, respectively. Since $\alpha_0$ is the identity action, we have the isomorphism $KK_*^\Gamma(E\Gamma, \mathcal{A}(M)) \cong KK_*(B\Gamma, \mathcal{A}(M))$. Also notice that there is a Bott homomorphism

$$\beta_{(0,x_0)}^{[0,1]} \colon \mathcal{S} \to C([0,1], \mathcal{A}(M))$$

given by composing the usual Bott homomorphism with the inclusion of $\mathcal{A}(M)$ into $C([0,1], \mathcal{A}(M))$ as constant functions. Since the evaluation maps clearly intertwines the Bott homomorphisms, we obtain the desired commuting diagram. $\square$

**Proposition IV.2.6.** *Let $\alpha \colon \Gamma \to \mathrm{Isom}(M)$ be an isometric action that is null-homotopic. Then the $K$-theory homomorphism given by taking Kasparov product with $[\beta]$*

$$\otimes_{\mathbb{C}}[\beta] : K_*^\Gamma(E\Gamma) \otimes \mathbb{Q} \to KK_{*+1}^\Gamma(E\Gamma, \mathcal{A}(M)) \otimes \mathbb{Q}$$

*is injective.*

*Proof.* By the previous proposition, it suffices to show the homomorphism

$$\otimes_{\mathbb{C}}[\beta] : K_*(B\Gamma) \otimes \mathbb{Q} \to KK_{*+1}(B\Gamma, \mathcal{A}(M)) \otimes \mathbb{Q}$$

is injective. But this follows from the naturality of the Chern character and Proposition IV.1.6. $\square$

## IV.3 The main result

The following is our main result regarding the Novikov conjecture.

**Theorem IV.3.1.** *Let $\Gamma$ be a discrete torsion-free group that admits an isometric, metrically proper action on a complete, simply-connected non-positively curved Hilbert manifold $M$ that has a completely geodesic finite-dimensional submanifolds whose union is dense in $M$. Assume that the action is null-homotopic. Then $\Gamma$ satisfies the rational analytic Novikov conjecture, i.e. the rational assembly map*

$$\mu : K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C_r^*(\Gamma)) \otimes \mathbb{Q}$$

*is injective.*

*Proof.* As indicated before, the proof of the main theorem follows the dual Dirac method. Let us look at the following commutative diagram:

$$\begin{tikzcd} K_*(B\Gamma) \otimes \mathbb{Q} \arrow[r, "\mu"] \arrow[d, "\otimes_{\mathbb{C}}[\beta]"] & K_*(C_r^*(\Gamma)) \otimes \mathbb{Q} \arrow[d, "C_r^*(\Gamma, \otimes_{\mathbb{C}}[\beta])"] \\ KK^{\Gamma}_{*+1}(E\Gamma, \mathcal{A}(M)) \otimes \mathbb{Q} \arrow[r, "\mu"] & K_*(C_r^*(\Gamma, \mathcal{A}(M))) \otimes \mathbb{Q} \end{tikzcd} \tag{IV.1}$$

Since the algebra $\mathcal{A}(M)$ is $\Gamma$-proper, the horizontal map in the bottom is bijective by [11]. On the other hand, in the last subsection we proved that the vertical map on the left is also injective. By the commutativity of the diagram, the composition of the horizontal map on the top and the vertical map on the right is also injective, which forces the horizontal map on the top, i.e. the assembly map for $\mathcal{S}$, to be injective. □

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