The word and conjugacy problems in lacunary hyperbolic groups

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In loving memory of my grandfather, Alexander Petrovich Darbinyan

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Chapter 0

Introduction

Traditionally, computability questions of *word and conjugacy problems* in groups, along with the groups isomorphism problem, are considered as some of the most important properties and questions in combinatorial and geometric group theory. For a given finitely generated group $G = \langle X \rangle$, $|X| < \infty$, the word problem is an algorithmic problem of deciding whether any arbitrarily given word $W \in X^*$ represents the trivial element in G or not. Here and later, whenever a set, say X, is a set of group generators, by X^* we denote the set of all words in the alphabet $X \cup X^{-1}$. Otherwise, if X is merely a (finite) set, then X^* means the set of all finite words composed by letters from X.

The conjugacy problem considers on input an arbitrary pair $(U, V) \in X^* \times X^*$ and decides whether U is conjugate to V in G or not. If for the word problem in G such a decision algorithm exists, then it is said that the word problem (briefly, WP) is decidable in G. Analogously, if there is an decision algorithm for the conjugacy problem (briefly, CP) in G, then it is said that the conjugacy problem is decidable in G.

Observe that, since the triviality of an element of G is equivalent to the fact that it is conjugate to the trivial element of G, decidability of the conjugacy problem in G implies decidability of the word problem. Another obvious observation is that the decidability of WP and CP do not depend on the choice of the finite generating set.

Word and conjugacy problems in groups first were introduced by Max Dehn in 1911. A bit later, in 1912, Max Dehn described algorithms for word and conjugacy problems for surface groups (i.e. fundamental groups of two dimensional manifolds) for surfaces of genus $g \geqslant 2$. The algorithm described by him for the word problem is one of the most important word problem solving algorithms. It is one of the most important word problem solving algorithms not only because of its simplicity and good time complexity behavior or, say, because of its historical importance, but also because, based on generalizations of underlying properties of surface groups, this algorithm was generalized to a much broader class of groups, called hyperbolic groups (or, word hyperbolic groups). The notion of hyperbolic groups was first introduced by Gromov in his seminal paper [28]. In fact, it is well-known that hyperbolic groups are essentially the finitely presented groups on which one can extend Dehn's original algorithm for the word problem in surface groups. See, for example, [28, 41].

To describe Dehn's algorithm, let us consider any finitely presented group G with its finite presentation

$$G = \langle X \mid \mathcal{R} \rangle. \tag{0.1}$$

Then the presentation (0.1) is said to be *Dehn's presentation* if the following property holds: $\mathcal{R}$ is a finite symmetric set of words (i.e. it is closed under operations of taking cyclic shifts and inverses of words); for any freely cyclically reduced word $W \in X^*$, if $W =_G 1$ (i.e. W represents the trivial element in G), then there exists a word $R = R_1R_2 \in \mathcal{R}$ such that $\|R_1\| > \|R_2\|$, and a cyclic shift W' of W such that $W' = W_1R_1W_2$. (Throughout this text, by the symbol $\|\cdot\|$ I denote lengths of words in a given alphabet. Another notation which I use in this work extensively is the following: For $G = \langle X \rangle$ suppose $U, V \in X^*$, then $U =_G V$ means that the words U and V represent the same element from G.)

Note that if (0.1) is a Dehn's presentation, then to check whether or not a cyclically reduced word $W \in X^*$ is trivial in G, one can simply consider all cyclic shifts of W and all relator words from $\mathcal{R}$ in order to find the above mentioned cyclic shift W' and relator word $R = R_1R_2$. Then the key observation is that $W =_G 1$ if and only if $W_1R_2^{-1}W_2 =_G 1$. But $\|W_1R_2^{-1}W_2\| < \|W\|$. Thus the word problem for W is reduced to the word problem for a strictly shorter word $W_1R_2^{-1}W_2$. Next, in order to check whether or not $W_1R_2^{-1}W_2 =_G 1$, in a similar way as for W, we can try to reduce this question to the word problem for a shorter word. If at some point this shortening procedure cannot be applied anymore, then it means that either we obtained an empty word, hence we conclude $W =_G 1$ or, otherwise, we conclude $W \neq_G 1$. Also it is clear that this procedure of shortening can be applied only finitely many times (bounded from above by $\|W\|$), hence the process will eventually halt, giving us the wanted answer about triviality of W in G. Since this procedure is based on the original algorithm of Dehn, following the established tradition, we call it *Dehn's algorithm.*

Note that there exist finitely presented groups with undecidable word problem. In fact, the first examples of finitely presented groups with algorithmically undecidable word problem were given by Novikov in 1955, see [48] and independently by Boone in 1958, see [9]. These results of Novikov and Boone are considered as one of the most important and classical results in the algorithmic theory of groups. Another famous example is a construction by Kharlampovich (see [37]), where the first example of finitely presented solvable group with undecidable word problem was constructed, answering a long standing open problem by Adian.

Speaking about word and conjugacy problems in finitely generated groups, there are several key aspects one might consider. Below we mention some of them.

(a). Whether or not the WP (resp. CP) is decidable?

(b). If it is undecidable, what is the Turing degree of undecidability of the WP (resp. CP)?

(c). If it is decidable, what computational complexity classes does it belong to?

Note that for a given group, the answer to (a) reveals not only computational properties of the group, but also its algebraic properties. This follows, for example, from a classical theorem of Boone and Higman, [14, 40], which says that a finitely generated group G has decidable word problem if and only if G can be embedded in a simple subgroup of a finitely presented group. Moreover, after the works of Gromov [29], Sapir, Birget, Rips [65], Birget, Olshanskii, Rips, Sapir [8], Olshanskii [56], Grigorchuk, Ivanov [26], Bridson [11] and others, it becomes apparent that the answer to (c) may reveal information not only about the computational properties of the group, but also about its topological and geometric properties. Therefore, in the light of modern developments in the theory of groups, investigation of these questions is important from the perspective of computational, algebraic, topological and geometric points of view. Note that since for any two finite sets of generators X and Y of a given group, the words in X^* can be in linear time translated into corresponding words in Y^*, the answer to the above formulated questions (a), (b) and (c) is independent of finite sets of group generators.

We would like to mention that even the question of existence of a lacunary hyperbolic group with decidable word problem and undecidable conjugacy problem was still open. This question was asked by Olshanskii, Osin and Sapir as Problem 7.5 in [57]. A positive answer to this question follows from Theorems 5 and 6 of the current paper.

In this thesis we systematically study all the above mentioned aspects of word and conjugacy problems in the class of so called lacunary hyperbolic groups, with a special emphasize on the ones obtained via small cancellation techniques.

The formal definition of the class of lacunary hyperbolic groups (more briefly, LHG) was first introduced by Olshanskii, Osin and Sapir in [57]. Intuitively, lacunary hyperbolic groups can be thought of as the finitely generated but not necessarily finitely presented versions of word hyperbolic groups. In the next sections we will recall the mathematically rigorous definitions of both hyperbolic and lacunary hyperbolic groups. But for this introductory part let us just add to the already mentioned that all lacunary hyperbolic groups are inductive limits of hyperbolic groups as it is established in [57] and recalled in Lemma 16 of the current work.

Speaking about inductive limits of hyperbolic groups, here we would like to mention that many such groups were constructed by using various generalized small cancellation techniques and many of them possess various exotic group theoretical properties. For example, this way Olshanskii constructed Burnside groups of large exponents [51] and Tarski Monsters [49, 50]. See also Ivanov's proof of the Burnside's problem for even exponents [35]. For a more complete exposition of these constructions see also [53].

Following an already established tradition, we call the groups which possess exotic properties and are obtained as inductive limits of hyperbolic groups via small cancellation techniques, *monster groups*.

For the monster groups appearing, for example, in [53], in the currently existing literature there are no known time complexity effective algorithms for the basic decision problems such as the word and conjugacy problems. The methods developed in this work help us to construct monster groups with effective word and conjugacy problems. See Theorems 3 and 4.

In this thesis, besides the developed frameworks and tools, we prove sever theorems which answer several natural questions about the nature of the word and conjugacy problems. Some of these questions previously were formulated by other authors and were known to be open. In the next chapter we describe our main results in more details.

Chapter 1

Main results

The main objective of this manuscript is twofold.

First, based on the small cancellation theory of Olshanskii (see [55]), we describe general constructions of lacunary hyperbolic groups under which the word and conjugacy problems can be effectively reduced to much simpler problems.

Even more, we develop a general framework in Sections 2.5–5.1 which provides with necessary tools to understand the rich nature of the word and conjugacy problems in the class of LHG. In fact, this framework will allow us to shed light on the rich nature of word and conjugacy problems in LHG from several perspectives. More specifically:

1. From the perspective of computability, e.g. in Theorem 10 we formulate a necessary and sufficient condition for decidability of WP. Also we develop necessary tools to construct lacunary hyperbolic groups with decidable word problem and undecidable conjugacy problem;

2. From the perspective of computational complexity theory; and

3. From the perspective of interconnection of WP and CP in the class of LHG, both in terms of computability and computational complexity.

Second, we use the developed framework to formulate the main theorems of this paper, that is Theorems 3, 4, 5 and 6. The first two theorems, in particular, show that versions of some of the most prominent groups of the class of LHG can be constructed in such a way that they will have fast WP and CP. The third theorem shows in particular that WP and CP are 'almost' completely independent one of another in the class of lacunary hyperbolic groups, not only in terms of computability, but also in terms of computational complexity.

Below we describe the content of the paper in more details.

Let us define the group $\bar{G} = \langle X \rangle$, $|X| < \infty$, as the inductive limit of the chain of group epimorphisms

$$G_1 \overset{\alpha_1}{\twoheadrightarrow} G_2 \overset{\alpha_2}{\twoheadrightarrow} \dots, \tag{1.1}$$

where $\alpha_i : G_i \twoheadrightarrow G_{i+1}$ is the induced epimorphism from the identity map $id : X \to X$ for all $i \in \mathbb{N}$ and $G_i = \langle X \mid \bar{\mathcal{R}}_i \rangle$ is a finitely presented hyperbolic group.

Even though the original definition of lacunary hyperbolic groups involves the concept of asymptotic cones, there exist equivalent and more algebraic definitions. In this work we employ the following definition of lacunary hyperbolicity (see Lemma 16 and Remark 5).

Definition 1.1. A finitely genrated group $\bar{G} = \langle X \rangle$ is lacunary hyperbolic if and only if $\bar{G}$ is the inductive limit of a chain of group epimorphisms of type (1.1) such that the hyperbolicity constant of G_i (relative to X) is little o of the radius of α_i, where *radius* is defined as follows: For $G = \langle X \rangle$ and $\alpha : G \to G'$, the radius of α is the maximal radius of a ball in the Cayley graph $\Gamma(G, X)$ centered at 1_G such that all elements from that ball map to non-trivial elements in G' except for 1_G.

The sequence of radiuses of (1.1) is the sequence of radiuses of epimorphisms α_i.

Definition 1.2. We say that (1.1) along with the group presentations $G_i = \langle X \mid \bar{\mathcal{R}}_i \rangle$ is a *graded recursive presentation* of $\bar{G}$ by hyperbolic groups if the map $i \mapsto \bar{\mathcal{R}}_i$ is computable, i.e. the set $\{(i, \bar{\mathcal{R}}_i) \mid i \in \mathbb{N}\}$ is recursive.

In Section 2.4 we prove the following theorems.

Theorem 1 (Theorem 10)**.** *Let $\bar{G}$ be an inductive limit of hyperbolic groups connected by epimorphisms. Then $\bar{G}$ has decidable word problem if and only if it has a graded recursive presentation by hyperbolic groups and increasing sequence of radiuses over that presentation (i.e. the radiuses of the epimorphisms are increasing).*

Theorem 2 (Corollary 8)**.** *A lacunary hyperbolic group has a decidable word problem if and only if either G is a hyperbolic group or G is the direct limit of a sequence of δ_i-hyperbolic groups*

$G_i = \langle X_i \rangle$ (X_i is finite) and epimorphisms

$$G_1 \overset{\alpha_1}{\twoheadrightarrow} G_2 \overset{\alpha_2}{\twoheadrightarrow} \dots, \tag{1.2}$$

where $\alpha_i(X_i) = X_{i+1}$, such that

1. *for all i, G_i is a δ_i-hyperbolic group, where $(\delta_i)_{i=1}^{\infty}$ is an increasing sequence of positive integers,*

2. *the sequence $(r_i)_{i=1}^{\infty}$ is strictly increasing, where r_i is the radius for the epimorphism $\alpha_i : G_i \to G_{i+1}$,*

3. *the sequence $(\frac{\delta_i}{r_i})_{i=1}^{\infty}$ is strictly decreasing and converges to 0,*

4. *the groups G_i have presentation $G_i = \langle X \mid \mathcal{R}_i \rangle$ such that the map $i \mapsto \mathcal{R}_i$ is computable (i.e. the presentation $G = \langle X \mid \cup \mathcal{R}_i \rangle$ is a graded recursive presentation by hyperbolic groups).*

Remark 1. Note that the conditions (1)-(3) in fact give a general characterization of arbitrary non-hyperbolic lacunary hyperbolic groups.

One of the main object of investigation in this paper is the following type of chains of hyperbolic groups satisfying some special conditions.

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \dots. \tag{1.3}$$

If we denote $\alpha_i = \gamma_{i+1} \circ \beta_i$, then we always assume that α_i is surjective for $i = 1, 2, \dots$. All the groups in this chain are assumed to be hyperbolic. Let $G_0 = \langle X \mid \mathcal{R}_0 \rangle$ be given with its initial finite presentation, and let for all $i \in \mathbb{N}$,

$$H_i = G_{i-1} * F(Y_i) / \ll \mathcal{S}_i \gg, \tag{1.4}$$

where $|Y_i| < \infty$, $Y_i \cap \beta_{i-1}(G_{i-1}) = \varnothing$, $\mathcal{S}_i$ is a finite (symmetric) set of words from $(X \cup Y_i)^*$ and

$F(Y_i)$ is the free group with basis Y_i. Also

$$G_i = H_i / \ll \mathcal{R}_i \gg, \tag{1.5}$$

where $\mathcal{R}_i$ is a finite symmetric set of words from $(X \cup Y_i)^*$ satisfying certain small cancellation conditions.

The main group of our interest is the group $\bar{G} = \langle X \rangle$, $|X| < \infty$, defined as the inductive limit

$$\bar{G} = \lim_{i} (G_i, \alpha_i).$$

In Section 4 we introduce the concepts of G- and H-conjugacies with respect to (1.3) for the group $\bar{G}$ defined as follows: For $u, v \in X^*$ we say that u is H-conjugate to v if there exists $i \in \mathbb{N}$ such that u is conjugate to v in H_i but nevertheless u is not conjugate to v in G_{i-1}. G-conjugacy is defined analogously, namely, u is G-conjugate to v in $\bar{G}$ if either u is conjugate to v in G_0 or there exists $i \in \mathbb{N}$ such that u is conjugate to v in G_i but u is not conjugate to v in H_i. Clearly, u is conjugate to v in $\bar{G}$ if and only if either u is H-conjugate to v or G-conjugate to v in $\bar{G}$. In the same section we introduce a special small cancellation condition
$C'\big(\mathcal{TM}, (g_i)_{i=1}^{\infty}, (\rho_i)_{i=1}^{\infty}\big)$ which assures that the word and G-conjugacy problems for $\bar{G}$ can be solved in polynomial time provided that the words $\mathcal{R}_i$, $i = 1, 2, \ldots$, are polynomial time computable. Here we would like to highlight that this condition does not tell us about the effectiveness of the H-conjugacy problem. In fact, as the proof of Theorem 5 reveals, H-conjugacy problem in general can have an arbitrary behavior not depending on the behavior of, say, G-conjugacy problem.

In Subsections 2.5.2 and 2.5.3 we describe constructions of words which can be highly effectively constructed, have appropriate small-cancellation properties, and they will serve in Sections 5.2, 5.3 and 5.4 as the main ingredient for defining the words $\mathcal{R}_i$, $i = 1, 2, \ldots$ for corresponding constructions. It is worthwhile to mention here that Sections 5.2, 5.3 and 5.4 provide the proofs of the main applications of the general framework, that is the proofs of Theorems 3, 4 and 5, and all the proofs are constructive and based on a general scheme described in Section 5.1. On its own turn, the general scheme from Section 5.1 is based on the already mentioned general framework developed mostly in Sections 2.5, 2.7, 3 and 4.

Concerning the groups H_i, $i = 1, 2, \ldots$, in the main applications in Sections 5.2, 5.3, 5.4, we consider two main situations: First, when $H_i = G_{i-1}$ and $\beta_{i-1} = id$ and second, when H_i-s are obtained as HNN-extensions of G_{i-1}.

Definition 1.3. Let $f : \mathbb{N} \to \mathbb{N}$ be a positive integer valued function, and let $\mathcal{D}$ be any decision problem. Then we say that $\mathcal{D}$ can be solved in almost $f(n)$ time, if for any $\varepsilon > 0$ the problem $\mathcal{D}$ belongs to $DTime\big(n^{\varepsilon} f(n)\big)$, or in other words, it belongs to $\bigcap_{k=1}^{\infty} DTime\big(n^{1/k} f(n)\big)$. If $f(n) = n$, $n \in \mathbb{N}$, then we say that $\mathcal{D}$ is decidable in almost linear time (similarly we define almost quadratic time, etc).

1.1 Main theorems

The next theorem shows that every non-elementary, torsion-free hyperbolic group has a non-trivial verbally complete quotient with almost-linear time word problem and polynomial time conjugacy problem.

Recall that the group $G' = \langle X \rangle$ is *verbally complete* if for any element $g \in G'$ and for any non-trivial element w from a countably generated free group $F = F(y_1, y_2, \ldots)$, the equation $w = g$ has a solution in G', where the letters of w are regarded as the variables of the equation. In other words, there exists a homomorphism $h : F \to G'$ such that $h : w \mapsto g$.

Theorem 3. *Let G be an arbitrary torsion-free, non-elementary hyperbolic group. Then there exists a lacunary hyperbolic infinite torsion-free quotient $\check{G}$ of G such that the following is true about $\check{G}$.*

(i). *$\check{G}$ is a verbally complete group,*

(ii). *The word problem in $\check{G}$ is decidable in almost quadratic time and the conjugacy problem in $\check{G}$ is decidable in polynomial time.*

Note that part (i) of Theorem 3 appears in the work of Mikhajlovskii and Olshanskii, [44]. Also, since verbally complete groups are divisible groups, Mikhajlovskii and Olshanskii's work can be regarded as a generalization of a result of Guba from 1987, [30], which answered a long standing open question about the existence of finitely generated non-trivial divisible groups. To achieve the result of Theorem 3, we elaborate the original construction of Mikhajlovskii and Olshanskii and combine it with the machinery developed in this paper.

One interesting corollary from Theorem 3 is that for the group $\check{G}$, there exists an algorithm such that for all inputs $w \in F(y_1, y_2, \ldots)\backslash\{1\}$ and $\check{g} \in \check{G}$, the algorithm finds a solution for the equation $w = \check{g}$ in $\check{G}$. Indeed, to solve the equation $w = \check{g}$ in $\check{G}$, one can just check for all possible values of variables $y_1, y_2, \ldots$, whether $w = \check{g}$ in $\check{G}$ or not. Since the word problem in $\check{G}$ is decidable and $\check{G}$ is verbally complete, this procedure will eventually halt.

Theorem 4. *Let G be an arbitrary torsion-free, non-elementary hyperbolic group. Then there exists a non-cyclic torsion-free lacunary hyperbolic quotient $\hat{G}$ of G such that the following is true about $\hat{G}$.*

(i). Every proper subgroup of $\hat{G}$ is an infinite cyclic group,

(ii). The word problem in $\hat{G}$ is decidable in almost quadratic time and the conjugacy problem in $\hat{G}$ is decidable in polynomial time.

Note that the first example of an infinite non-cyclic group with the property of the part (i) appears in [49] and the exact statement of Theorem 4 without the part (ii) appears in [55]. Construction of $\hat{G}$ can be regarded as a more elaborated version of the corresponding result from [55] combined with the machinery developed in this paper.

Let us also mention that from the method by which the groups $\check{G}$ and $\hat{G}$ are constructed it follows that for every torsion-free, non-elementary hyperbolic G, there are continuum many pairwise non-isomorphic quotients of G satisfying the statements (i) of Theorems 3 and 4, respectively. However, the cardinality of groups satisfying all the conditions of Theorems 3 and 4, respectively, is $\aleph_0$. (In fact, the cardinality of finitely generated groups with decidable word problem is $\aleph_0$.)

Definition 1.4 (Strong (many-one) reduction)**.** Let $\mathcal{L}_1 \subseteq \mathcal{A}_1^*$ and $\mathcal{L}_2 \subseteq \mathcal{A}_2^*$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite alphabets. Then $\mathcal{L}_1$ is strongly (many-one) reducible to $\mathcal{L}_2$ if there exists a computable function $\phi : \mathcal{A}_1^* \to \mathcal{A}_2^*$ and a constant $C > 0$ such that for any $x \in \mathcal{A}_1^*$ we have $\|\phi(x)\|_{\mathcal{A}_2} \leqslant C\|x\|_{\mathcal{A}_1}$ and $\phi(\mathcal{L}_1) = \mathcal{L}_2$, $\phi(\mathcal{A}_1^* \backslash \mathcal{L}_1) \subseteq \mathcal{A}_2^* \backslash \mathcal{L}_2$. Moreover, if for some $g : \mathbb{N} \to \mathbb{N}$ and for all $x \in \mathcal{L}_1$ the value of $\phi(x)$ can be computed in time $\mathcal{O}(g(\|x\|_{\mathcal{A}_1}))$, then we say that $\mathcal{L}_1$ is strongly reducible to $\mathcal{L}_2$ in time $g(n)$.

Theorem 5. *Let $\mathcal{A}$ be any finite alphabet, and let $\mathcal{L} \subseteq \mathcal{A}^*$ be any recursively enumerable subset (i.e., r.e. language) of $\mathcal{A}^*$. Then there exists a lacunary hyperbolic group $G_{\mathcal{L}}$ such that the following is true about $G_{\mathcal{L}}$.*

(I). The word problem in $G_{\mathcal{L}}$ is decidable in almost linear time.

(II.i). The conjugacy problem in $G_{\mathcal{L}}$ can be strongly reduced to the decidability problem in $\mathcal{L}$ in almost linear time;

(II.ii). The decidability problem in $\mathcal{L}$ can be strongly reduced to the conjugacy problem in $G_{\mathcal{L}}$ in linear time;

In particular, if the membership problem for $\mathcal{L}$ belongs to $DTime(f(n))$, then the conjugacy problem in $G_{\mathcal{L}}$ is decidable in almost $f(n)$ time, and if the conjugacy problem in $G_{\mathcal{L}}$ belongs to $DTime(g(n))$, then the membership problem in $\mathcal{L}$ also belongs to $DTime(g(n))$.

(II.iii). For every fixed $g_0 \in G_{\mathcal{L}}$, the problem of deciding if an arbitrary $g \in G_{\mathcal{L}}$ is conjugate to g_0 is decidable in almost linear time.

The individual conjugacy problem in regard to a fixed element $g_0 \in G$, shortly $ICP(g_0)$, for any input element $g \in G$ asks whether or not g is conjugate to g_0 in G. Note that $ICP(1)$ coincides with the word problem in G. The statement (II.iii) of Theorem 5 says that for every $g_0 \in G$, $ICP(g_0)$ belongs to $\bigcap_{k=1}^{\infty} DTime\big(n^{1+\frac{1}{k}}\big)$.

Note that, in particular, Theorem 5 implies that there exist finitely generated groups with almost linear time individual conjugacy problems and (uniform) conjugacy problem which belongs to one of the following time complexity classes:

- NP-complete, co-NP-complete, PP-complete, PSpace-complete, etc; or
- belongs to $DTime(f(n)) \backslash DTime(g(n))$ where the time constructible functions f and g are such that $DTime(f(n)) \backslash DTime(g(n)) \neq \emptyset$ and $f(n) > n^{1+\varepsilon}$ for some $\varepsilon > 0$; or
- the conjugacy problem is undecidable and has any given recursively enumerable Turing degree of undecidability.

In particular, Theorem 5 extends the main results of Miasnikov and Schupp from [43].

In [17], Cannonito classified finitely generated groups with decidable word problem based on the complexity of the word problem. As a measure of complexity the author considered *Grzegorczyk hierarchy*. (For the details of the results and definition of Grzegorczyk hierarchy and its link to word problem, we refer to [31] and [17].)

In the same paper [17], the author mentions a question posed by Boone (see page 391, [17]) which was formulated as follows:

> *A very interesting problem suggested by W. W. Boone is the following: Do there exist any f.g. groups with conjugacy problem $\mathcal{E}_*^{\alpha}$-decidable, and word problem $\mathcal{E}^{\beta}$-decidable such that $\beta < \alpha$?*

This question was also touched in [39].

Parts (I) and (II) of Theorem 5 imply the following stronger statement.

Corollary 1. *For every $\alpha \geqslant 3$, there exists a finitely generated (lacunary hyperbolic) group $\tilde{G}$ with $\mathcal{E}^3$-decidable word problem and $\mathcal{E}_*^{\alpha}$-decidable conjugacy problem.*

Remark 2. We would like to note that Corollary 1 follows also from the main results of [43].

In [57], the authors, Olshanskii, Osin and Sapir, asked about the existence of a lacunary hyperbolic group with decidable word problem but undecidable conjugacy problem. See Problem 7.5 in [57]. Another immediate corollary from Theorem 5, parts (I) and $(II.ii)$, answers this question in positive.

Corollary 2. *There exists a lacunary hyperbolic group with decidable word problem but undecidable conjugacy problem.*

Proof. Indeed, take any recursively enumerable but not recursive set $\mathcal{L}$. Then, according to Theorem 5, the group $G_{\mathcal{L}}$ has decidable word problem but undecidable conjugacy problem. $\square$

Theorem 5 provides a reasonably complete classification of the conjugacy problem in finitely generated groups in terms of time computational complexity for groups with decidable word problem and for recursively presented groups with undecidable conjugacy problem - in terms of recursively enumerable Turing degrees. It is worth mentioning that similar classifications were obtained for the word problem, for example, by the following authors: By Cannonito [17] in terms of Grzegorczyk hierarchy; by Valiev and Trakhtenbrot [69, 68] in terms of space complexity, by Stillwell [67] in terms of time complexity. However, in spirit, probably the closest result to parts $(II.i)$ and $(II.ii)$ of Theorem 5 is the following result of Birget-Olshanskii-Rips-Sapir from [8] stated for the word problem in finitely presented groups and mentioned as "an important corollary" (see Corollary 1.1, [8]).

> There exists a finitely presented group with NP-complete word problem. Moreover, for every language $L \subseteq \mathcal{A}^*$ from some finite alphabet $\mathcal{A}$, there exists a finitely presented group G such that the nondeterministic time complexity of G is polynomially equivalent to the nondeterministic time complexity of L.

The first examples of groups with decidable word problem and undecidable conjugacy problem of arbitrary r.e. Turing degree for finitely generated groups were constructed by Miller [45], and for finitely presented groups by Collins [20]. It was shown in [10] that in Miller's group from [45] even though the conjugacy problem is undecidable, the individual conjugacy problems $ICP(g)$ are solvable in polynomial time for all g from an exponentially generic subset of G. This and other observations led Miasnikov and Schupp to formulate the following question in [43].

Question. Are there recursively presented groups G with solvable word problem such that if the individual conjugacy problems are decidable on a computably enumerable subset $Y \subseteq G$ then Y is negligible, or indeed exponentially negligible?

We answer this question in positive by showing the following much stronger existence result.

Theorem 6. *There exist lacunary hyperbolic groups $\tilde{G} = \langle X \rangle$ with word problem decidable in almost linear time and such that for $g \in \tilde{G}$ the individual conjugacy problem $ICP(g)$ is decidable if and only if $g = 1$.*

Remark 3. In fact, the group $\tilde{G}$, which is constructed in Section 5.5, is non-amenable, hence, by Grigorchuk's co-growth criterium of amenability, we have $\{w \in X^* \mid w =_{\tilde{G}} 1\}$ is exponentially negligible.

In self-contained Chapter 6, we answer a well-known question of Collins, asked in early 1970's, about the embeddability of torsion-free groups with decidable word problem into groups with decidable conjugacy problem. Our main theorem of that chapter is the following.

Theorem 7 (Theorem 17)**.** *There exists a finitely presented torsion-free group $\mathcal{G}$ with decidable word problem such that $\mathcal{G}$ cannot be embedded into a group with decidable conjugacy problem.*

Chapter 2

Preliminaries

Let $(\mathcal{X}, d)$ be a geodesic metric space. Given a geodesic triangle ABC in $\mathcal{X}$ with vertices A, B and C, for any $\delta > 0$, ABC is called δ-*slim* if each side of the triangle ABC is contained in the δ-neighborhood of the union of other two sides of ABC.

For a given constant $\delta > 0$, $\mathcal{X}$ is called δ-*hyperbolic* space, if all the geodesic triangles in $\mathcal{X}$ are δ-slim. Throughout this text, when we consider a triangle with vertices A, B and C, we denote by AB, BC, CA the sides of the triangle joining the corresponding vertices. The same convention we use also for other polygons.

Let $G = \langle X \rangle$ be a finitely generated group with a finite generating set X. Note that the Cayley graph $\Gamma(G, X)$ possesses a natural geodesic metric, d_G, called word metric. That is for any $g, h \in G$, $d_G(g, h)$ is the length of a smallest word from X^* representing the word $g^{-1}h \in G$. Moreover, since each edge of a Cayley graph is isometric to the unit line, the metric d_G can be extended to a geodesic metric on any pair of points from $\Gamma(G, X)$. By $|g|_G$ (or just by $|g|_X$ or $|g|$, depending on the context and convenience) we denote the distance $d_G(1, g)$. In the current work, whenever it does not lead to ambiguities, instead of using the notation d_G we will simply write d. Depending on the convenience derived from the context, we will use sometimes instead of d_G, d_X or simply d, if it does not lead to ambiguities.

Note that, at the first glance, it would be more appropriate to use notations d_X and $|\cdot|_X$ instead of d_G and $|\cdot|_G$. However, this notation we use by purpose, because in many applications in this paper, we interchangeably consider metrics on different Cayley graphs of groups with presentations $\langle X \mid \mathcal{R}_1 \rangle$ and $\langle X \mid \mathcal{R}_2 \rangle$, where $\mathcal{R}_1 \neq \mathcal{R}_1$.

The group $G = \langle X \rangle$ is called δ-hyperbolic, if its Cayley graph $\Gamma(G, X)$ is δ-hyperbolic. In general, we say G is hyperbolic if the Cayley graph $\Gamma(G, X)$ is δ-hyperbolic for some $\delta \geqslant 0$. It is a well-know fact that the property of hyperbolicity does not depend on the choice of finite generating sets (see [28]). However, the hyperbolicity constant δ may depend on the choice of the generating set. In this thesis, whenever we say that some group or space is δ-hyperbolic, by default we assume that δ is a positive integer.

The following are well-known algorithmic properties of hyperbolic groups.

1. The class of hyperbolic groups is exactly the class of finitely presented groups with Dehn presentation. See [28] and also [41].

2. It was established by Epstein and Holt in [24] that given a hyperbolic group G with finite Dehn presentation, there exists an algorithm solving the conjugacy problem in G in linear time.

3. It was established by Papasoglu in [62] (see also [61] for background) that there exists a partial algorithm which detects hyperbolicity of finitely presented hyperbolic groups. In other words, the set of finite presentations of hyperbolic groups is recursively enumerable. See also [22].

4. There exists an algorithm which computes a slimness constant δ for any finite presentation of a hyperbolic group. See, for example, [22].

5. There exists an algorithm which for any input of finite presentation of a hyperbolic group computes its Dehn presentation. It follows from [62] and [22].

Now consider a path p in $(\mathcal{X}, d)$ with a natural parametrization by length. The path p is called (λ, c)-*quasi-geodesic* for some $\lambda \geqslant 1$ and $c \geqslant 0$, if for any points $p(s)$ and $p(t)$ on p, we have

$$|s-t| \leqslant \lambda d(p(s), p(t)) + c.$$

Hereafter, whenever it is not stated otherwise, we assume that the quasi-geodesity constants λ and c are integers. We denote the origin of p with respect to this parametrization (i.e. the point $p(0)$) by p_- and the terminal point by p_+.

We say that a word $W \in X^*$ is a *geodesic word* (in $\Gamma(G, X)$), if the paths in $\Gamma(G, X)$ with label W are geodesics, and we say that a word $W \in X^*$ is *cyclically geodesic* if any cyclic shift of W is a geodesic word in $\Gamma(G, X)$. Analogously, for $\lambda \geqslant 1, c \geqslant 0$, we say W is (λ, c)*-quasi-geodesic* (in $\Gamma(G, X)$) if the corresponding paths in $\Gamma(G, X)$ are (λ, c)-quasi-geodesic. The length of the word W we denote by $\|W\|$ and by $|W|$ we denote the length of the shortest word representing the same element as W in G. Clearly, W is a geodesic word if and only if $\|W\| = |W|$.

For any $W' \in X^*$ the notation $W' \sim_{conj} W$ *in* G means that W' represents an element in G conjugate to the element represented by W in G.

We say that $V \in X^*$ is a *cyclically minimal* representative of W if $V \sim_{conj} W$ in G and V has the smallest length among all such words. For V satisfying this assumption, we also define $|W|_c = \|V\|$. If $\|W\| = |W|_c$, then we say that W is *cyclically minimal*. Clearly, if W is cyclically minimal, then it is cyclically geodesic.

Now suppose that p is a path in $\Gamma(G, X)$. Then, as we said, we will denote its initial and terminal points by p_- and p_+, respectively. If A, B are some points on p, then by $[A, B]$ we denote the subpath q of p between A and B such that $q_- = A$ and $q_+ = B$. Also we denote the length of p by $\|p\|$ and, context based, we denote the length of q by $\|q\|$ or by $\big\|[A, B]\big\|$. Since all the edges in Cayley graphs are labeled by the letters of $X \cup X^{-1}$, any path p in $\Gamma(G, X)$ in fact is a labeled path. We denote the label of p by $lab(p)$.

Lemma 1 (Theorem III.1.7, [12])**.** *Let p be a (λ, c)-quasi-geodesic path in the Cayley graph $\Gamma(G, X)$, where $\lambda \geqslant 1, c \geqslant 0$ and $G = \langle X \rangle$ is a hyperbolic group. Then there exists an effectively calculable constant $R_{\lambda,c} \in \mathbb{N}$ depending on λ, c and G, such that the Hausdorff distance between p and any geodesic path joining p_- to p_+ is bounded by $R_{\lambda,c}$.*

In this text, whenever we use the notation $R_{\lambda,c}$, we refer to the constant from Lemma 1.

Corollary 3. *Let p and q be (λ_1, c_1)- and (λ_2, c_2)-quasi-geodesic paths in $\Gamma(G, X)$ respectively. Also let $d(p_-, q_-) \leqslant L$, $d(p_+, q_+) \leqslant L$ for some constant L, then the Hausdorff distance between*

p and q is bounded from above by $L + R_{\lambda_1,c_1} + R_{\lambda_2,c_2} + 2\delta$, where δ is a hyperbolicity constant of $\Gamma(G,X)$. Moreover, if we join p_- to q_- and p_+ to q_+ by some geodesics, then we get a quadrangle such that the distance from any point on p (or q) to the union of the other three sides is bounded from above by $R_{\lambda_1,c_1} + R_{\lambda_2,c_2} + 2\delta$. In case p and q are geodesics, this distance is bounded from above by 2δ.

Proof. It follows from Lemma 1 that it would be enough to prove the statement for the case when p and q are geodesic paths and correspondingly $R_{\lambda_1,c_1} = R_{\lambda_2,c_2} = 0$.

Now assume that p and q are geodesics. Let p_-, q_- and p_+, q_+ be joined by some geodesics f_1 and f_2, respectively. Also let e be a geodesic path joining q_- to p_+.

By the definition of hyperbolicity constant, for any point $o_1 \in q$, there exists $o_2 \in e \cup f_2$ such that $d(o_1, o_2) \leqslant \delta$. Now, if $o_2 \in f_2$, then since $\|f_2\| \leqslant L$, the statement of the corollary follows for o_1 immediately. Otherwise, if $o_2 \in e$, the statement follows for o_1 immediately from the observation that $dist(o_2, f_1 \cup p) \leqslant \delta$ and $\|f_1\| \leqslant L$. If o_1 belongs to one of the other three sides, then we can deal with that case analogously. □

Corollary 4. *Let p and q be (λ_1, c_1)- and (λ_2, c_2)-quasi-geodesic paths in $\Gamma(G,X)$ respectively, and let $d(p_-, q_-) \leqslant L$, $d(p_+, q_+) \leqslant L$ for some constants $\lambda_1 \geqslant 1, c_1 \geqslant 0$, $\lambda_2 \geqslant 1, c_2 \geqslant 0$, $L \geqslant 0$. Then for any point $o \in p$ such that $d(o, p_-), d(o, p_+) \geqslant L + R_{\lambda_1,c_1} + 2\delta$, we have $dist(o, q) \leqslant R_{\lambda_1,c_1} + R_{\lambda_2,c_2} + 2\delta$, where δ is the hyperbolicity constant of $\Gamma(G,X)$.*

Proof. Let p_-, q_- and p_+, q_+ be joined by some geodesics f_1 and f_2, respectively. Also let p', q' be geodesic paths joining p_- to p_+ and q_- to q_+, respectively.

By Lemma 1 there exists $o_1 \in p'$ such that $d(o, o_1) \leqslant R_{\lambda_1,c_1}$. Now, by Corollary 3, $dist(o_1, f_1 \cup f_2 \cup q') \leqslant 2\delta$.

On the other hand, if $dist(o_1, f_1) \leqslant 2\delta$, then $dist(o, f_1) \leqslant d(o, o_1) + dist(o_1, f_1) \leqslant R_{\lambda_1,c_1} + 2\delta$. Hence, by the triangle inequality, this would imply $d(o, p_-) \leqslant L + R_{\lambda,c} + 2\delta$, which is a contradiction. This contradiction implies that $dist(o_1, f_1) > 2\delta$. Similarly, we get that $dist(o_1, f_2) > 2\delta$. Therefore, $dist(o_1, q') \leqslant 2\delta$, and hence $dist(o_1, f_1 \cup f_2 \cup q') \leqslant 2\delta$ implies that $dist(o_1, q') \leqslant 2\delta$.

Therefore, since $d(o, o_1) \leqslant R_{\lambda_1,c_1}$ and the Hausdorff distance between q' and q is bounded from above by R_{λ_2,c_2}, we get that $dist(o, q) \leqslant R_{\lambda_1,c_1} + R_{\lambda_2,c_2} + 2\delta$. □

Given a path p and $k \geqslant 0$, $\lambda \geqslant 1, c \geqslant 0$, we say that p is *k-local (λ, c)-quasi-geodesic*, if each subpath of p, of length at most k, is (λ, c)-quasi-geodesic. In case $\lambda = 1$, $c = 0$, we say that p is *k-local geodesic*.

Lemma 2 (Theorem III.H.1.13, [12]). *Let $\mathcal{X}$ be a δ-hyperbolic geodesic space and p be a k-local geodesic, where $k > 8\delta$. Then for every geodesic segment q joining p_- to p_+ we have:*
(1) p is contained in the 2δ-neighborhood of q;
(2) q is contained in the 3δ-neighborhood of p;
(3) p is a (λ, c)-quasi-geodesic, where $\lambda = (k + 4\delta)/(k - 4\delta)$ and $c = 2\delta$.

The next lemma is a generalization of the previous one. It can be found in [32].

Lemma 3 (See Theorem 25 in [32]). *Let $\mathcal{X}$ be a δ-hyperbolic space. Then there exists an effectively computable constant $\mathcal{K} = \mathcal{K}(\delta, \lambda, c) \in \mathbb{N}$ such that for any $k \geqslant \mathcal{K}$, if p is a k-local (λ, c)-quasi-geodesic path in $\mathcal{X}$, then p is $(\mathcal{K}, \mathcal{K})$-quasi-geodesic.*

For any metric space $(\mathcal{X}, d)$ and for any $x, y, z \in \mathcal{X}$ the *Gromov product* of y and z at x, denoted $(y \cdot z)_x$, is defined by

$$(y \cdot z)_x = \frac{1}{2}\big(d(x, y) + d(x, z) - d(y, z)\big).$$

Lemma 4 (see Lemma 5, [36]). *Let $G = \langle X \rangle$ be a δ-hyperbolic group. Let $\alpha \geqslant 14\delta$, $\alpha_1 \geqslant 12(\alpha + \delta)$, and a geodesic n-gon $A_1 A_2 \ldots A_n$ with $n \geqslant 3$ satisfies the following conditions: $d(A_{i-1}, A_i) > \alpha_1$ for $i = 2, ..., n$ and $(A_{i-2} \cdot A_i)_{A_{i-1}} \leqslant \alpha$ for $i = 3, ..., n$. Then the polygonal line $p = A_1 A_2 \cup \ldots \cup A_{n-1} A_n$ is contained in the closed 2α-neighborhood of the side $A_n A_1$ and the side $A_n A_1$ is contained in the closed 14δ-neighborhood of p. In addition, $d(A_1, A_n) > 6(n-1)(\alpha + \delta)$.*

Lemma 5 (see Lemma 1.17, [55], also Lemma 8, [36]). *Let g be an element of infinite order in a hyperbolic group G and an equality $x g^k x^{-1} = g^l$ holds in G, where $x \in G$, $l \neq 0$. Then $k = \pm l$.*

Lemma 6. *Let $G = \langle X \rangle$ be a δ-hyperbolic group, and let $W, V, T \in X^*$ be such that V is freely cyclically reduced non-empty word and*

$$W =_G T^{-1}VT.$$

Suppose that for some $k \in \mathbb{N}$ and $\lambda \geqslant 1$, $c \geqslant 0$, V^k is a (λ, c)-quasi-geodesic word. Then W^k is a $\big(\lambda\|W\|, (2\lambda\|T\| + c + 2)\|W\|\big)$-quasi-geodesic word.

Proof. First of all, note that for all $l \geqslant 0$, we have

$$l\|V\| = \|V^l\| \leqslant \lambda|V^l| + c,$$

hence

$$l \leqslant \frac{\lambda|V^l| + c}{\|V\|} \leqslant \lambda|V^l| + c. \tag{2.1}$$

Now note that every subword of W^k is of the form $W_1W^lW_2$, where $l \geqslant 0$ and W_1, W_2 are (possibly empty) suffix and prefix of W respectively.

Now for $W_1W^lW_2$ we have

$$\begin{aligned}
\|W_1W^lW_2\| &\leqslant \|W_1\| + \|W^l\| + \|W_2\| = \|W_1\| + \|W_2\| + l\|W\| \\
\text{by (2.1),} \quad &\leqslant \|W_1\| + \|W_2\| + (\lambda|V^l| + c)\|W\| \\
&= \|W_1\| + \|W_2\| + (\lambda|TW^lT^{-1}| + c)\|W\| \\
&\leqslant 2\|W\| + (\lambda|W^l| + 2\lambda\|T\| + c)\|W\| \\
&= \lambda\|W\||W^l| + 2\|W\| + 2\lambda\|T\|\|W\| + c\|W\| \\
&= \lambda\|W\||W^l| + (2\lambda\|T\| + c + 2)\|W\|.
\end{aligned}$$

Now, since $W_1W^lW_2$ was chosen to be an arbitrary subword of W^k, we conclude that W^k is a $\big(\lambda\|W\|, (2\lambda\|T\| + c + 2)\|W\|\big)$-quasi-geodesic word. □

Lemma 7. *Let $G = \langle X \rangle$ be a δ-hyperbolic group, and let $V \in X^*$ be a cyclically minimal word such that $\|V\| \geqslant \alpha$, where $\alpha = 12 \cdot 15\delta = 180\delta$. Then for each $k \in \mathbb{Z}$, V^k is a $(4, 2520\delta)$-quasi-*

geodesic word.

Proof. Without loss of generality let us assume that $k \in \mathbb{N}$. We want to show that V^k is $(4, 2520\delta)$-quasi-geodesic for any cyclically minimal word $V \in X^*$ such that $\|V\| \geqslant \alpha$.

For that reason, let us decompose V as

$$V = V_1 V_2 \dots V_s,$$

where $s = \left\lfloor \frac{\|V\|}{\alpha} \right\rfloor$ and $\alpha \leqslant \|V_i\| < 2\alpha$ for $i = 1, \dots, s$. Then, since V is cyclically minimal and the word $V_s V_1$ along with the words $V_1 V_2, \dots, V_{s-1} V_s$ are subwords of (a cyclic shift of) V, we get

$$|V_1| + |V_s| - |V_s V_1| = \|V_1\| + \|V_s\| - \|V_s V_1\| = 0$$

and

$$|V_i| + |V_{i+1}| - |V_i V_{i+1}| = \|V_i\| + \|V_{i+1}\| - \|V_i V_{i+1}\| = 0, \text{ for } i = 1, \dots, s-1.$$

The last equations suggest that we can apply Lemma 4 on subwords of V^k to conclude that for any subword V' of V^k such that $\|V'\| \geqslant 3\alpha$, which is indeed of the form

$$V' = U_1 V_{i_1} \dots V_{i_t} U_2,$$

where U_1 and U_2 are suffix and prefix of words from $\{V_1, \dots, V_s\}$, we have

$$|V'| \geqslant |V_{i_1} \dots V_{i_t}| - \|U_1\| - \|U_2\| > 6(t-1)15\delta - 2\alpha = 90\delta t - 450\delta. \tag{2.2}$$

(Lemma 4 was used to obtain $|V_{i_1} \dots V_{i_t}| > 6(t-1)15\delta$).

On the other hand

$$\|V'\| < (t+2)2\alpha = (t+2)360\delta = (360\delta t - 1800\delta) + 1800\delta + 720\delta$$

$$\text{by (2.2)}, \ \leqslant 4|V'| + 2520\delta.$$

Therefore, since V' is an arbitrary subword of V^k of length $\geqslant 3\alpha$, we conclude that V^k is a

$$(4, 2520\delta)\text{-quasi-geodesic word.} \tag{2.3}$$

□

Lemma 8. *Let $G = \langle X \rangle$ be a δ-hyperbolic group, where X is symmetric (i.e. $X = X^{-1}$), and let $W \in X^*$ be a geodesic word representing an element of G of infinite order. Then for every $k \in \mathbb{Z}$, the word W^k is (λ_W, c_W)-quasi-geodesic in the Cayley graph $\Gamma(G, X)$, where λ_W and c_W are given by the formulas*

$$\lambda_W = 4|X|^\alpha \|W\|, \tag{2.4}$$

and

$$c_W = 5|X|^{2\alpha}\|W\|^2 \tag{2.5}$$

where $\alpha = 180\delta$. Moreover, if W is cyclically minimal, then W^k is $(4\alpha|X|^\alpha, 5\alpha^2|X|^{2\alpha})$-quasi-geodesic.

Proof. First, let us show that there exists an integer $1 \leqslant m \leqslant |X|^\alpha$ such that $|W^m|_c > \alpha$ (recall that we assume $X = X^{-1}$). Indeed, assume that there is no such m. Then, by the pigeonhole principle, there exist $1 \leqslant m_1 < m_2 \leqslant |X|^\alpha$ and $V \in X^*$, $T_1, T_2 \in X^*$, such that $\|V\| \leqslant \alpha$, $\|V\| = |W^m|_c$ and

$$W^{m_1} =_G T_1^{-1} V T_1,\ W^{m_2} =_G T_2^{-1} V T_2.$$

But this means that W^{m_1} and W^{m_2} are conjugate in G, which on its own turn, by Lemma 5, implies that $m_1 = \pm m_2$. A contradiction.

Thus there exists $1 \leqslant m \leqslant |X|^\alpha$ such that

$$W^m =_G T^{-1} V T, \tag{2.6}$$

where $T, V \in X^*$, $\|V\| = |W^m|_c$ and

$$\|V\| > \alpha. \tag{2.7}$$

Note that the equation $\|V\| = |W^m|_c$ implies that V is cyclically geodesic. Also, without loss of generality assume that T has the smallest length among all the words T satisfying the equation (2.6) for some V with $\|V\| = |W^m|_c$.

Let us assume that $W^m =_G U$ for some geodesic word $U \in X^*$. Let us consider a geodesic quadrangle $ABCD$ in $\Gamma(G, X)$ such that $lab(AB) = lab(DC) = T$, $lab(AD) = V$ and $lab(BC) = W^m$, i.e. the boundary of $ABCD$ corresponds to the equation $W^m =_G T^{-1}VT$.

The first observation is that $\|T^{-1}\| = \|T\| = dist(B, AD)$ $(= dist(C, AD))$. Indeed, if there exists a point $O \in AD$ such that $d(B, O) < \|T\|$, then there exists a path joining B to O, whose label is a word Q such that $\|Q\| < \|T\|$. Now, if we denote $lab(AO) = V_1$, $lab(O, D) = V_2$, we get $W^m =_G U =_G Q(V_2V_1)Q^{-1}$. See Figure 2.1. But because of the minimality assumption on $\|T\|$, the inequality $\|Q\| < \|T\|$ leads to a contradiction. Thus the first observation is proved.

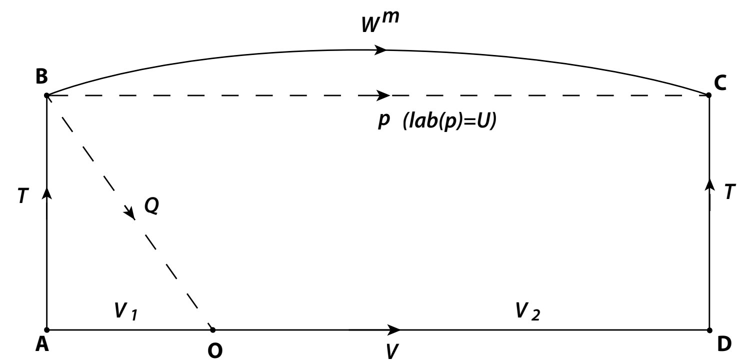

Figure 2.1

The second observation is that for any point $O_1 \in AD$ such that $d(A, O_1), d(O_1, D) > 4\delta$ (note that such a point exists, because $\|V\| > \alpha$), we have $dist(O_1, p) \leqslant 2\delta$, where p is the path joining B to C with the label U. To show this, first notice that $dist(O_1, AB \cup p \cup CD) \leqslant 2\delta$ (see Corollary 3). Also, because of the minimality assumption on $\|T\|$, we get $d(B, O_1) \geqslant d(B, A)$. Now suppose that there is a point $O_2 \in AB$ such that $d(O_1, O_2) \leqslant 2\delta$. Then, since $d(B, O_1) \geqslant d(B, A)$, we get

$$d(B, A) = d(B, O_2) + d(O_2, A) \leqslant d(B, O_1) \leqslant d(B, O_2) + d(O_2, O_1).$$

Therefore, $d(O_2, A) \leqslant d(O_2, O_1) \leqslant 2\delta$ and as a consequence, by the triangle inequality, we get $d(A, O_1) \leqslant d(A, O_2) + d(O_2, O_1) \leqslant 4\delta$. But since $d(A, O_1) > 4\delta$, we obtain a contradiction.

The last contradiction implies that $dist(O_1, AB) > 2\delta$. The same way we get $dist(O_1, CD) > 2\delta$. Therefore, the inequality $dist(O_1, AB \cup p \cup CD) \leqslant 2\delta$ implies that $dist(O_1, p) \leqslant 2\delta$, and consequently, since the length of p is bounded from above by $\|W^m\|$, we get that $d(O_1, B) \leqslant \|W^m\| + 2\delta$. Therefore, from the minimality assumption on $\|T\|$, we get

$$\|T\| \leqslant |X|^{\alpha}\|W\| + 2\delta. \tag{2.8}$$

Now, since $W^m = T^{-1}VT$ and $|V| > \alpha$, it follows immediately from Lemmas 6, 7 and the inequality (2.8) that for all $k \in \mathbb{Z}$, W^{km} is a $\big(4\|W^m\|, (2|X|^{\alpha}\|W\| + 8\delta + 2520\delta + 2)\|W^m\|\big)$-quasi-geodesic word. Also, taken into the account the fact that W^k is a subword of W^{km} and the inequalities $m \leqslant |X|^{\alpha}$ and $2520\delta + 2 \leqslant |X|^{\alpha}\|W\|$, we conclude that W^k is a

$$(4|X|^{\alpha}\|W\|, 5|X|^{2\alpha}\|W\|^2)\text{-quasi-geodesic.} \tag{2.9}$$

Finally, since for cyclically minimal words V satisfying $\|V\| > \alpha$, we showed that V^k is $(4, 2520\delta)$-quasi-geodesic, by taking $\|W\| = \alpha$ in (2.9), we get that for every cyclically minimal $V \in X^*$, regardless their lengths, V^k is $(4\alpha|X|^{\alpha}, 5\alpha^2|X|^{2\alpha})$-quasi-geodesic. □

2.1 Isoperimetric functions of hyperbolic groups

Let G be a group with a finite presentation $G = \langle X \mid r_1, \ldots, r_k\rangle$. A function $f : \mathbb{N} \to \mathbb{N}$ is called an *isoperimetric function* for G (w.r.t. the given presentation), if for every reduced word $W \in X^*$ such that $W =_G 1$, W can be presented as

$$W = \prod_{i=1}^{n} u_i r_{j_i}^{\pm 1} u_i^{-1}$$

where $n \leqslant f(\|W\|)$. The minimal among the isoperimetric functions is traditionally called *the Dehn function* of the given presentation. If n is the minimal number for which such a decomposition exists, then n is called *the area* of W and denoted $n = Area(W)$. Another, equivalent definition of the isoperimetric function is the following: let p be a closed path in $\Gamma(G, X)$, then p can be tessellated by at most $f(\|p\|)$ labeled discs whose labels belong to $\{r_1^{\pm 1}, \ldots, r_k^{\pm 1}\}$.

It is a well known fact that a group is hyperbolic if and only if it has a finite presentation with linear (equivalently, subquadratic) isoperimetric function. See for example [12, 54]. Moreover, if $G = \langle X \rangle$ is δ-hyperbolic and $\mathcal{F} = \{U \in X^* \mid \|U\| \leqslant 16\delta + 1,\ U =_G 1\}$, then G can be given by the following presentation

$$G = \langle X \mid \mathcal{F} \rangle, \tag{2.10}$$

and for this presentation, for all reduced words $W \in \ll \mathcal{F} \gg$, we have $Area(W) \leqslant n$. Let us call this presentation the (X, δ)-*full presentation* of G with respect to X and δ. If from the context it is clear what are X and δ, then we will just call it *the full presentation* of G.

An important observatoin about full-presentations follows from Lemma 2. Namely, the full presentations (2.10) is in fact Dehn presentations. It follows from Lemma 2 and from the observation that in the Cayley graph $\Gamma(G, X)$ the only $(8\delta + 1)$-local geodesic loop is the loop with length 0, i.e. a point. For more details see [12] or Proposition 1.

For a given presentation $G = \langle X \mid \mathcal{R} \rangle$ of a hyperbolic group, let $f(n) \leqslant An$ for some constant $A > 0$. Then we call A an *isoperimetry coefficient* (w.r.t. $G = \langle X \mid \mathcal{R} \rangle$).

Proposition 1. *(1). For any Dehn presentation $G = \langle X \mid \mathcal{R} \rangle$ the isoperimetry coefficient is equal to* 1.

(2). If G is δ-hyperbolic, then the full presentation $G = \langle X \mid \mathcal{F} \rangle$ is a Dehn presentation.

Proof. (1). Let $G = \langle X \mid \mathcal{R} \rangle$ be a Dehn presentation and let p be a loop in $\Gamma(G, X)$. Then, since $G = \langle X \mid \mathcal{R} \rangle$ is a Dehn presentation, p contains a subpath q such that for another path q' we have $\|q\| > \|q'\|$ and $lab(q^{-1}q') \in \mathcal{R}$. Then $q^{-1}q'$ can be filled with one cell from $\mathcal{R}$. Based on this observation, it is clear that there is a van Kampen diagram over $G = \langle X \mid \mathcal{R} \rangle$ with boundary p and number of cells not exceeding p. Hence the first part of the proposition is proved.

(2). Indeed, let p be a closed path in $\Gamma(G, X)$ with its ends on 1. Then, by Lemma 2, there exists a closed $8\delta + 1$-local geodesic path q with its ends on 1 such that pq can be tesselated by at most $\|p\|$ cells with labels from $\mathcal{F}$.

On the other hand, again by Lemma 2, q is $(3, 2\delta)$-quasi-geodesic. Now, since q is a closed 8δ-local geodesic, we get that either q has 0 length, or $\|q\| \geqslant 8\delta$. But since q is $(3, 2\delta)$-quasi-geodesic, the last inequality cannot happen. Hence q has length 0. This means that the loop p can be tessellated by at most $\|p\|$ cells with labels from $\mathcal{F}$. Thus the proposition is proved. $\square$

It is well-known that a finitely presentable group is hyperbolic if and only if with respect to any finite presentation the Dehn function of the group is linear. See, for example, [28, 54, 5]. The next lemma tells that if with respect to some finite presentation $\langle X \mid r_1, r_2, \ldots, r_l\rangle$ of a hyperbolic group G, an isoperimetric coefficient A is given, then one can effectively find $\delta > 0$ such that G will be δ-hyperbolic with respect to the generating set X.

Lemma 9 (See [41], [5])**.** *Suppose G is a hyperbolic group given with a finite presentation $G = \langle X \mid r_1, r_2, \ldots, r_l\rangle$. Also suppose that $f : \mathbb{N} \to \mathbb{N}$ is an isoperimetric function with respect to this presentation such that $f(n) \leqslant An$ for some positive integer A. Then G is $\tilde{f}(A, M)$-hyperbolic with respect to the generating set X, where $M = \max\{\|r_1\|, \ldots, \|r_l\|\}$ and $\tilde{f} : \mathbb{N}^2 \to \mathbb{N}$ is a computable function independent of G.*

2.2 Elementary subgroups of hyperbolic groups

A group is called *elementary* if it has a cyclic subgroup of finite index. It is a well know fact that in a hyperbolic group each element g of infinite order is contained in a unique maximal elementary subgroup, usually denoted by $E(g)$, see for example [55].

By the lemmas 1.16 and 1.17 of [55], for a hyperbolic group G and for any $g \in G$ of infinite order, the following holds:

$$E(g) = \{x \in G \mid xg^n x^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{N}\}$$

and

$$E(g) = \{x \in G \mid xg^k x^{-1} = g^l \text{ for some } k, l \in \mathbb{Z}\backslash\{0\} \}.$$

Also we need the following definitions:

$$E^{-}(g) = \{x \in G \mid xg^n x^{-1} = g^{-n} \text{ for some } n \in \mathbb{N}\},$$

$$E^{+}(g) = \{x \in G \mid xg^n x^{-1} = g^{n} \text{ for some } n \in \mathbb{N}\}.$$

Note that the equivalence of the two descriptions of $E(g)$ given above, follows from Lemma 5.

Since, as it is well-known, in every torsion-free hyperbolic group G each elementary subgroup is cyclic, it follows that for all $g \in G\backslash\{1\}$, the subgroup $E(g)$ is of the form $\langle g_0 \rangle$, where g is a power of g_0 and $E(g_0) = \langle g_0 \rangle$.

For any $U \in X^*$, we denote by $E(U)$ the group $E(g)$, where $g \in G$ and $U =_G g$. Similarly, we define $E^{\pm}(U)$. For $V \in X^*$, we say that $V \in E(U)$, if for some $h \in G$, $V =_G h$ and $h \in E(U)$.

Definition 2.1 (The root elements)**.** If $G = \langle X \rangle$ is a torsion-free hyperbolic group, then for a word $U \in X^*$ we say that U represents a root element in G, if $U =_G g_0$ and $E(g_0) = \langle g_0 \rangle$. Correspondingly, if $E(g_0) = \langle g_0 \rangle$, then g_0 is called root element.

If for some $g \in G$, $E(g) = \langle g_0 \rangle$, then g_0 is called a root of g. (Note that each element $g \in G\backslash\{1\}$ has two different roots, g_0 and g_0^{-1}.)

Lemma 10 (See Lemma 2.1 in [55])**.** *Let $G = \langle X \rangle$ be a δ-hyperbolic group, X be symmetric, and let $U, V \in X^*$ be geodesic words with respect to $\Gamma(G, X)$. Let $\lambda \geqslant 1$ and $c \geqslant 0$ be constants such that U^k and V^k are (λ, c)-quasi-geodesic words w.r.t. $\Gamma(G, X)$ for all $k \in \mathbb{Z}$. (According to Lemma 8, such (λ, c) always exist.) Let $T_1, T_2 \in X^*$ be arbitrary elements in G. Denote $L = \max\{\|T_1\|, \|T_2\|\}$. Then, there exists a computable function $f : \mathbb{N}^5 \to \mathbb{N}$ independent of G such that for any integer m satisfying the inequality*

$$m \geqslant f(|X|, \delta, \lambda, c, \|V\|),$$

either

$$L > \frac{\|U\|}{12\lambda} m,$$

or the equation

$$T_1U^mT_2 =_G V^n$$

implies that $T_1UT_1^{-1}, T_2^{-1}UT_2 \in E(V)$. *Moreover, if* $U =_G V$, *then* $T_1, T_2 \in E(U)(= E(V))$. *More precisely,* $T_1, T_2 \in E^+(U)$ *for* $n > 0$ *and* $T_1, T_2 \in E^-(U)$ *for* $n \leqslant 0$.

For the purpose of completeness we present a proof of Lemma 10 in Appendix.
Also, for the reason of convenience, for the constants mentioned in Lemma 10 we introduce the following notations

$$\upsilon = \upsilon(U) = \frac{\|U\|}{12\lambda}. \tag{2.11}$$

and, assuming that the values of $|X|, \delta, \lambda, c, \|V\|$ are already known, we denote

$$\mathcal{M} = \mathcal{M}(U, V) = f(|X|, \delta, \lambda, c, \|V\|). \tag{2.12}$$

Lemma 11 (See Theorem 2 and Theorem 3 in [41])**.** *Let* $G = \langle X \rangle$ *be a torsion-free* δ*-hyperbolic group given with its* (X, δ)*-full-presentation. Then there exists an algorithm such that for any input* $U \in X^*$ *it finds a word* $V \in X^*$ *such that* $E(U) = \langle V \rangle$*, i.e. there exists an algorithm computing roots of the elements of* G.

Corollary 5. *There exists an algorithm which for any input hyperbolic group* $G = \langle X \mid \mathcal{R} \rangle$ *given by a finite presentation and for any input word* $U \in X^*$ *finds* $V \in X^*$ *such that* V *represents a root element of* U *in* G.

Proof. The set of finite group presentations for hyperbolic groups is recursively enumerable (Papasoglu [62]) and there is an algorithm which finds a thinness constant δ for any input finitely presented hyperbolic group $G = \langle X \mid \mathcal{R} \rangle$ (see, for example, [22]) and moreover, with respect to this constant one can find the (X, δ)-full-presentation of G. Combination of these observations with Lemma 11 implies Corollary 5. $\square$

Let $G = \langle X \mid \mathcal{R} \rangle$ is a finitely generated group and $A, B \leqslant G$ are some isomorphic subgroups of G, and $\phi : A \to B$ is a group isomorphism between A and B. Then the *HNN-extension* of G with respect to $\phi : A \to B$ is defined as $H_\phi^G = \langle X \cup \{t\} \mid \mathcal{R}, t^{-1}at = \phi(a)\ \forall a \in A \rangle$. Note that in this text, since mostly from the context it will be clear what is ϕ , for the HNN-extension H_ϕ^G we will use the notation $H_\phi^G = H = \langle G, t \mid t^{-1}At = B \rangle$.

We are mostly interested in the case when $A = \langle a \rangle$, $B = \langle b \rangle$ are infinite cyclic groups. For this case by the notation $H = \langle G, t \mid t^{-1}at = b \rangle$ we denote the HNN-extension H_ϕ^G, where $\phi : A \to B$ is induced by the map $\phi : a \mapsto b$.

Let us consider the product

$$u = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots t^{\epsilon_n} g_n, \tag{2.13}$$

where for $0 \leqslant i \leqslant n$, $g_i \in G$ and for $1 \leqslant j \leqslant n$, $\epsilon_j \in \{\pm 1\}$. We say that this decomposition corresponds to the sequence $(g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \dots, t^{\epsilon_n}, g_n)$ and we say that a decomposition is a cyclic shift of (2.13) if it corresponds to a cyclic shift of the sequence $(t^{\epsilon_1}, g_1, t^{\epsilon_2}, \dots, t^{\epsilon_n}, g_n g_0)$.

Also the decomposition from (2.13) is said to be *t-reduced* if for $0 \leqslant i \leqslant n$, we have $g_i \in G$ and no subproduct of the form $t^{-1}at$, $a \in A$ or of the form tbt^{-1}, $b \in B$, appears in (2.13). And it is said to be *cyclically t-reduced* if all cyclic shifts of the product (2.13) are *t-reduced*.

The word

$$w = u_0 t^{\epsilon_1} u_1 t^{\epsilon_2} \dots t^{\epsilon_n} u_n \in (X \cup \{t\})^*$$

is called *reduced word* with respect to the HNN-extension H, if $u_i \in X^*$ for $0 \leqslant i \leqslant n$ and the corresponding sequence $(u_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, u_n)$ is t-reduced. Analogously, w is said to be *cyclically reduced* with respect to the HNN-extension H if all cyclic shifts of $(t^{\epsilon_1}, u_1 \dots, u_{n-1}, t^{\epsilon_n}, u_n u_0)$ are t-reduced. Also we define θ as

$$\theta(w) = n$$

and for $h \in H$, define

$$\theta(h) = \min\{\theta(w) \mid w \in (X \cup \{t\})^*, w =_H h\}.$$

An element $u \in H_\phi^G$ is said to be *cyclically t-reduced* if its t-reduced decomposition is in fact cyclically t-reduced. Again, this is a well-defined definition. Also every element $u \in H_\phi^G$ is conjugate to a cyclically t-reduced element u' which we call *t-cyclic-reduction* of u. See [40].

The next lemma is a very well-known and in literature sometimes is called Britton's Lemma.

Lemma 12 (Britton's Lemma)**.** *Let $w \in (X \cup \{t\})^*$ be a reduced word with respect to the HNN-extension $H = \langle G, t \mid t^{-1}At = B\rangle$ and $\theta(w) > 0$, then $w \neq_H 1$.*

Then next lemma is a well-known fact as well and in literature is usually called Collins' Lemma. See, for example, [47, 25]

Lemma 13 (Collins' Lemma)**.** *Let*

$$u = u_0 t^{\alpha_0} u_1 t^{\alpha_1} \dots u_n t^{\alpha_n}$$

and

$$v = v_0 t^{\beta_0} v_1 t^{\beta_1} \dots v_m t^{\beta_m}$$

be cyclically reduced words with respect to the HNN-extension H such that $u_i, v_j \in X^*$ and $\alpha_i, \beta_j \in \mathbb{Z}$. If u and v are conjugate in $H_\phi^G = \langle G, t \mid t^{-1}At = B\rangle$, then one of the following holds:

- *u, v are words in X^* which are conjugate in G;*
- *There is a finite chain of words in G*

$$u = w_0, w_1', w_1, w_2', w_2, ..., w_k', w_k, w_{k+1}' = v$$

such that $w_i = \phi^{\pm 1}(w_i')$, as group elements, w_i', w_i represent elements from $A \cup B$, and for each $i = 0, \dots, k$, w_i is conjugate to w_{i+1}' in G;

- $\theta(u), \theta(v) > 0$ *and* $p = q$, *and* u *is conjugate in* H *to some cyclic shift of* v *by an element from* $A \cup B$.

Corollary 6. *Let* $H = \langle X \cup \{t\} \mid \mathcal{R}, t^{-1}at = \phi(a)\ \forall a \in A\rangle$ *be an HNN-extension of* $G = \langle X \mid \mathcal{R}\rangle$*, and suppose* $g_1 \in G$ *is not a proper power of any element in* G*. Then the image of* g_1 *in* H *(which we again denote by* g_1*) is a proper power in* H *if and only if there exists* $k \geqslant 2$ *and* $g_2 \in G$ *such that* $g_1 \sim_{conj} g_2^k$ *in* H*.*

Proof. Let for some $u \in X^*$, $u =_G g_1$, and $w \in \big(X \cup \{t\}\big)^*$ such that $u =_H w^k$ for some $k \geqslant 2$. Now let $w' \in \big(X \cup \{t\}\big)^*$ be a t-cyclic reduction of w. Then for some $T \in \big(X \cup \{t\}\big)^*$, we have $w = Tw'T^{-1}$. Also note that for any $k \geqslant 2$, $(w')^k$ is also t-cyclically reduced. Therefore, by Lemma 13, it must be that $w' \in X^*$, namely w' represents an element in H which is an image of an element from G.

The inverse statement of the corollary is obvious. $\square$

Lemma 14. *Let* $H = \langle G \cup \{t\} \mid t^{-1}at = b\rangle$ *be an HNN-extension of* $G = \langle X\rangle$ *where* $a, b \in G$ *are elements of infinite order which are not proper powers. Then, for any* $g_0 \in G$*, if* g_0 *is not a proper power in G, then its image in* H *is also not a proper power.*

Proof. Assume that for some $u \in X^*$, $u =_G g_0$ and also assume that there exists a word $w \in \big(X \cup \{t\}\big)^*$ such that $u =_H w^k$ for some $k \geqslant 2$. Then, by Corollary 6, there exists a word $T \in \big(X \cup \{t\}\big)^*$ and a word $w' \in X^*$ such that $u =_H T^{-1}(w')^kT$. If T does not contain $t^{\pm 1}$, then clearly we get a contradiction to the fact that u is not a proper power in G. Therefore, it must be that $\theta(T) \geqslant 1$, i.e. its t-reduced decompositions contains $t^{\pm 1}$. Assume that w' and T are chosen such that $\|T\|$ is minimal for all possible such triples (u, w', T).

Since $Tu^{-1}T^{-1}(w')^k =_H 1$, the word $Tu^{-1}T^{-1}(w')^k$ must contain a subword of the form $t^{\epsilon}vt^{-\epsilon}$, where for some $l \in \mathbb{Z}$, $v =_G a^l$ if $\epsilon = -1$ or $v =_G b^l$ if $\epsilon = 1$. Moreover, v is of the form $v_1u^{-1}v_1^{-1}$, where $v_1 \in X^*$ is a suffix of T. But this contradicts the minimality assumption of T.

$\square$

Lemma 15. *Let* $H = \langle G \cup \{t\} \mid t^{-1}u_0t = v_0\rangle$ *be an HNN-extension of* $G = \langle X\rangle$*, where* $u_0, v_0 \in X^*$*. Suppose that* $u, v \in X^*$ *such that* $u \sim_{conj} v$ *in* H*. Then, either* $u \sim_{conj} v$ *in* G *or* u *and* v *commensurate with at least one of* u_0 *and* v_0 *in* G*.*

Proof. It follows immediately from Theorem 2 in [47]. □

The following theorem can be found in [44] (it can be also regarded as a corollary from the combination theorem of Bestvina and Feighn, [13]).

Theorem 8. *Let G be a hyperbolic group with isomorphic infinite elementary subgroups A and B, and let ϕ be an isomorphism from A to B. Then the HNN-extension $H = \langle G, t \mid t^{-1}at = \phi(a), a \in A\rangle$ of G with associated subgroups A and B is hyperbolic if and only if the following two conditions hold:*

1. *either A or B is a maximal elementary subgroup of G;*

2. *for all $g \in G$ the subgroup $gAg^{-1} \cap B$ is finite.*

Remark 4. In this work we need Theorem 8 in case when G is a torsion-free hyperbolic group. Note that in case G is a torsion free hyperbolic group, the subgroups A and B, being maximal elementary subgroups, are cyclic. Therefore, in this case, the second condition in the statement of Theorem 8 can be replaced with this: for all $g \in G$, the subgroup $gAg^{-1} \cap B$ is trivial.

2.4 Lacunary hyperbolic groups

Let $G = \langle X \rangle$, $|X| < \infty$. Let $\bar{d} = (d_i)_{i=1}^{\infty}$ be an unbounded sequence of positive constants, called scaling constants, and let $\bar{x} = (x_i)_{i=1}^{\infty}$ be any fixed sequence of points from $\Gamma(G, X)$, called observation points. Then the ultralimit of the sequence of spaces with basepoints $(\Gamma(G, X), d/d_i, x_i)$ with respect to some non-principal ultrafilter ω over $\mathbb{N}$ is called the asymptotic cone of $G = \langle X \rangle$ with respect to $\bar{d}$ and ω, where d is the word metric over $\Gamma(G, X)$. It is denoted by $Con^{\omega}(G, \bar{d})$. The term *asymptotic cone* was first introduced by Gromov in [27]. Since, in this paper, we do not work with asymptotic cones, for more detailed definitions we refer to [27, 57].

As it was discovered by Gromov (see, for example, [29, 28]) many basic algebraic properties of groups can be translated into geometric or topological ones via studying asymptotic cones of Cayley graphs of groups. For example, hyperbolicity of a group is equivalent to the fact that all the asymptotic cones of the group are $\mathbb{R}$-trees. Moreover, as it is shown by Kapovich and Kleiner (see [57]), if for a finitely presented group at least one of the asymptotic cones is an $\mathbb{R}$-tree, then the group is hyperbolic. However, if the group is not finitely presentable, then this statement is not true anymore. In fact, *lacunary hyperbolic groups* are defined to be the groups which have at least one asymptotic cone that is an $\mathbb{R}$-tree, see [57].

Definition 2.2 (Lacunary hyperbolic groups)**.** A finitely presented group G is lacunary hyperbolic if for some unbounded sequence $\bar{d} = (d_i)_{i=1}^{\infty}$ of scaling constants, $Con^{\omega}(G, \bar{d})$ is an $\mathcal{R}$-tree.

Let $\alpha : G \to G'$ be a homomorphism, $G = \langle X \rangle$. As it is mentioned in the introductory Section 1, the *radius* of α is the maximal radius of a ball in the Cayley graph $\Gamma(G, X)$ centered at 1_G such that all elements from that ball map to non-trivial elements in G' except for 1_G.

The next lemma is essentially Theorem 1.1 from [57].

Lemma 16 (Theorem 1.1, [57])**.** *A finitely generated group G is lacunary hyperbolic if and only if G is the direct limit of a sequence of δ_i-hyperbolic groups $G_i = \langle X_i \rangle$ (X_i is finite) and epimorphisms*

$$G_1 \overset{\alpha_1}{\twoheadrightarrow} G_2 \overset{\alpha_2}{\twoheadrightarrow} \dots, \tag{2.14}$$

where $\alpha_i(X_i) = X_{i+1}$, and the hyperbolicity constant δ_i of G_i (relative to X_i) is little o of the radius of α_i.

Remark 5. Note that in Part (3) of Lemma 16, for almost all indices i, $|X_i| = |X_{i+1}|$, therefore we can identify X_i with X_{i+1} by $x = \alpha_i(x)$ for $x \in X_i$ and regard α_i as the identity map from X_i to X_{i+1}.

Corollary 7. *A finitely generated group G is lacunary hyperbolic if and only if either G is a hyperbolic group or G is the direct limit of a sequence of δ_i'-hyperbolic groups $H_i = \langle X_i \rangle$ (X_i is finite) and epimorphisms*

$$H_1 \overset{\beta_1}{\twoheadrightarrow} H_2 \overset{\beta_2}{\twoheadrightarrow} \dots, \tag{2.15}$$

where $\beta_i(X_i) = X_{i+1}$, such that

1. *for all i, H_i is a δ_i'-hyperbolic group, where $(\delta_i')_{i=1}^{\infty}$ is an increasing sequence of positive integers,*

2. *the sequence $(r_i)_{i=1}^{\infty}$ is strictly increasing, where r_i is the radius for the epimorphism β_i : $H_i \to H_{i+1}$,*

3. *the sequence* $(\frac{\delta_i'}{r_i})_{i=1}^{\infty}$ *is strictly decreasing and converges to* 0.

Proof. First of all, notice that for any fixed δ and X the number of δ-hyperbolic groups generated by X is finite. Therefore, there exists an increasing subsequence $(i_j)_{j=1}^{\infty}$ such that for $j < k$, $\delta_{i_j} < \delta_{i_k}$, where we borrow some of the notations from Lemma 16. Now define $H_j' = G_{j_i}$ and $\beta_j' = \alpha_{i_j} \circ \ldots \circ \alpha_{i_{j+1}}$. Then, the sequence $H_1' \overset{\beta_1'}{\twoheadrightarrow} H_2' \overset{\beta_2'}{\twoheadrightarrow} \ldots$ satisfies the conditions of Lemma 16, because the radius of a composition of epimorphims is equal to the minimal of the radiuses of the epimorphisms. Thus we can assume that the condition of (1) takes place. To obtain the properties of parts (2) and (3), one can simply use the same 'infiltration' trick. Here, one needs just to notice that the set of radiuses of the epimorphisms is an unbounded set of bounded integers if G is not hyperbolic (this easily follows from the fact that hyperbolic groups are Hopfian).

□

2.4.1 The word problem in lacunary hyperbolic groups

Let $\bar{G} = \langle X \rangle$ be a finitely presented group given as an inductive limit of the chain of epimorphims

$$G_1 \overset{\alpha_1}{\twoheadrightarrow} G_2 \overset{\alpha_2}{\twoheadrightarrow} \ldots \tag{2.16}$$

where $\alpha_i : G_i \twoheadrightarrow G_{i+1}$ is the induced epimorphism from the identity map $id : X \to X$, and for $i \in \mathbb{N}$, $G_i = \langle X \mid \mathcal{R}_i \rangle$ is finitely presented.

Recall that $\bar{G}$ has a *graded recursive presentation* with respect to (2.16) if the function $i \mapsto \mathcal{R}_i$ is computable. In general, if $\bar{G}$ has a graded recursive presentation with respect to some sequence of type (2.16) then we say that $\bar{G}$ has a graded recursive presentation. If, in addition, all the groups G_i, $i \in \mathbb{N}$, are hyperbolic, then we say that the presentation is a *graded recursive presentation by hyperbolic groups*.

Proposition 2. *If the limit group* $\bar{G}$ *is lacunary hyperbolic and* G_i *is hyperbolic for all* $i \in \mathbb{N}$, *then either* $\bar{G}$ *is finitely presented, hence hyperbolic, or*

$$\limsup_{i\to\infty} r_i = \infty \textit{ and } \limsup_{i\to\infty} \delta_i = \infty,$$

where r_i is the radius of $\alpha : G_i \twoheadrightarrow G_{i+1}$ and δ_i is a hyperbolicity constant for G_i.

Proof. Indeed, if $\bar{G}$ is finitely presented, then starting from some $i \in \mathbb{N}$, for all $j > i$, the normal closure of $\mathcal{R}_{j+1}$ in G_i coincides with the normal closure of $\mathcal{R}_{i+1}$ in G_i. Therefore, $\bar{G}$ coincides with G_{i+1}, hence is hyperbolic.

Now let us assume that $\bar{G}$ is lacunary hyperbolic, but it is not hyperbolic. Then, since $\bar{G}$ is not finitely presented for each $N > 0$ there is $n \in \mathbb{N}$ and $R \in \mathcal{R}_n$ such that there is no $U \in X^*$ such that $\|U\| < N$ and $R =_{G_{n-1}} U$. Therefore, $\limsup_{i\to\infty} r_i = \infty$ and by the statement (3) of Lemma 16, also $\limsup_{i\to\infty} \delta_i = \infty$. □

In particular, from Lemma 2 it follows that for infinitely presented lacunary hyperbolic groups all supradius functions are unbounded.

For the proof of the next theorem we need the following definition from [6], which is a slight generalization of the standard notion of the Dehn's presentation.

Definition 2.3 (See Definition 1 in [6])**.** For $\frac{1}{2} \leqslant \alpha < 1$, the group $G = \langle X \mid \mathcal{R} \rangle$ given with a finite presentation, where $\mathcal{R}$ is symmetric, is said to be α-Dehn presented, if for any freely cyclically reduced word $W \in X^*$ representing the trivial element of G, for some cyclic shift W' of W, W' contains a subword u, such that u is a prefix of some word $R \in \mathcal{R}$ and $\|u\| > \alpha\|R\|$.

It is a well-known fact that hyperbolic groups admit α-Dehn presentations for all $\frac{1}{2} \leqslant \alpha < 1$. See, for example, [28, 5, 6].

It was shown by Arzhantseva in [6] that the property that a finite presentation of a group is an α-Dehn presentation for some $\frac{3}{4} \leqslant \alpha < 1$ can be detected algorithmically as it is stated below.

Theorem 9 (See [6])**.** *There exists an algorithm determining whether or not a finite presentation of a group is an α-Dehn presentation for some $\frac{3}{4} \leqslant \alpha < 1$.*

Note that if $G = \langle X \mid \mathcal{R} \rangle = \langle X \mid \mathcal{R}' \rangle$ and $\mathcal{R} \subseteq \mathcal{R}'$, then the presentation $G = \langle X \mid \mathcal{R} \rangle$ is a α-Dehn presentation implies that the presentation $\langle X \mid \mathcal{R}' \rangle$ is a α-Dehn presentation too. Also, as we already mentioned in preliminaries, if $G = \langle X \mid \mathcal{R} \rangle$ is a finite presentation for a hyperbolic group, then there is an algorithm which constructs a Dehn presentation for G.

Theorem 10. *Let $\bar{G}$ be an inductive limit of hyperbolic groups connected by epimorphisms. Then $\bar{G}$ has decidable word problem if and only if it is either hyperbolic or it has a graded recursive presentation by hyperbolic groups with strictly increasing corresponding radiuses.*

Proof. First, let us show that if $\bar{G}$ has a decidable word problem, then $\bar{G}$ possesses the mentioned properties.

Indeed, let $\bar{G} = \langle X \rangle$, $|X| < \infty$. For $n \in \mathbb{N}$, let us define $\mathcal{S}_n = \{W \in X^* \mid W =_{\bar{G}} 1, \|W\| \leqslant n\}$. Since the word problem in $\bar{G}$ is decidable, we get that the sets $\mathcal{S}_1, \mathcal{S}_2, \ldots$ are algorithmically constructible - one just needs to check for each $W \in X^*$, $\|W\| \leqslant n$, if $W =_{\bar{G}} 1$ or not, in order to construct $\mathcal{S}_n$.

We are going to define a sequence $\mathcal{S}'_{n_1} \subsetneq \mathcal{S}'_{n_2} \subsetneq \ldots$ such that each $\mathcal{S}'_{n_i}$ is the minimal set containing the set $\mathcal{S}_{n_i}$ and the group $G_i := \langle X \mid \mathcal{S}'_{n_i} \rangle$ is hyperbolic. In particular, we get $n_1 < n_2 < \ldots$.

Suppose that the set $\mathcal{S}'_{n_i} \supseteq \mathcal{S}_{n_i}$ is already constructed. Let j be the minimal index such that $\mathcal{S}'_{n_i} \subsetneq \mathcal{S}_j$. Then, define $n_{i+1} = j$ and $\mathcal{S}'_{n_{i+1}} \supseteq \mathcal{S}_{n_{i+1}}$ as the minimal set such that the presentation $\langle X \mid \mathcal{S}'_{n_{i+1}} \rangle$ is an α-Dehn presentation for $\frac{3}{4} \leqslant \alpha < 1$. The existence of such $\mathcal{S}'_{n_{i+1}}$ follows from the basic properties of hyperbolicity and the assumption that $\bar{G}$ is an inductive limit of hyperbolic groups. The set $\mathcal{S}'_{n_{i+1}}$ can be found algorithmically because of Theorem 9.

Now, clearly, $\bar{G}$ is a direct limit of $G_1 \overset{\alpha_1}{\twoheadrightarrow} G_2 \ldots$, where $\alpha_i : G_i \to G_{i+1}$ is the induced homomorphism of the identity map $id : X \to X$. On the other hand, since $\mathcal{S}_{n_i} \subseteq \mathcal{S}'_{n_i} \subsetneq \mathcal{S}_{n_{i+1}} \subseteq \mathcal{S}'_{n_{i+1}}$, we get that the radius r_i of α_i satisfies $n_i < r_i < n_{i+1}$. In particular, $r_1 < r_2 < \ldots$.

Thus the first part of the theorem is proved.

Now assume that $\bar{G}$ is the inductive limit of

$$G_1 \overset{\alpha_1}{\twoheadrightarrow} G_2 \overset{\alpha_2}{\twoheadrightarrow} \ldots, \tag{2.17}$$

where for $i \in \mathbb{N}$, the groups $G_i = \langle X \mid \mathcal{R}_i \rangle$ are hyperbolic groups, the corresponding sequence on radiuses in increasing, and $\bar{G} = \langle X \mid \cup_{i=1}^{\infty} \mathcal{R}_i \rangle$ be a graded recursive presentation. Then, clearly, for any $W \in X^*$, $W =_{\bar{G}} 1$ if and only if $W =_{G_n} 1$. Therefore, since the groups $G_1, G_2, \ldots$ have decidable word problem, we get that $\bar{G}$ had decidable word problem as well.

Thus the theorem is proved. □

Remark 6. Note that the proof of Theorem 10 does not give any idea about complexity of the word problem in lacunary hyperbolic groups. Hence we need to obtain more detailed structure of presentations of classes of lacunary hyperbolic groups in order to describe efficient word problem solving algorithms on them. Description of subclasses of LHG with effective word (and conjugacy) problems is one of the primary goals in the next sections.

Corollary 8. *A lacunary hyperbolic group has a decidable word problem if and only if either G is a hyperbolic group or G is the direct limit of a sequence of δ_i-hyperbolic groups $G_i = \langle X_i \rangle$ (X_i is finite) and epimorphisms*

$$G_1 \overset{\alpha_1}{\twoheadrightarrow} G_2 \overset{\alpha_2}{\twoheadrightarrow} \dots, \tag{2.18}$$

where $\alpha_i(X_i) = X_{i+1}$, such that

1. *for all i, G_i is a δ_i-hyperbolic group, where $(\delta_i)_{i=1}^{\infty}$ is an increasing sequence of positive integers,*
2. *the sequence $(r_i)_{i=1}^{\infty}$ is strictly increasing, where r_i is the radius for the epimorphism $\alpha_i : G_i \to G_{i+1}$,*
3. *the sequence $(\frac{\delta_i}{r_i})_{i=1}^{\infty}$ is strictly decreasing and converges to 0,*
4. *the groups G_i have presentation $G_i = \langle X \mid \mathcal{R}_i \rangle$ such that the map $i \mapsto \mathcal{R}_i$ is computable (i.e. the presentation $G = \langle X \mid \cup \mathcal{R}_i \rangle$ is a graded recursive presentation by hyperbolic groups).*

Proof. The proof is similar to the proof of Corollary 7 when combined with Theorem 10. □

2.5 Small cancellation conditions

In this section we are going to recall some small cancellation concepts for hyperbolic groups introduced in [55] and then describe a class of special words which possess small cancellation conditions.

Let $G = \langle X \rangle$ be a finitely generated group, and let $\mathcal{R}$ be a symmetric set of words from X^*. A subword U of a word $R \in \mathcal{R}$ is called an *ϵ-piece* for $\epsilon \geqslant 0$ if there exists a word $R' \in \mathcal{R}$ such that

1. $R \equiv UV$, $R' \equiv U'V'$ for some $V, U', V' \in X^*$;

2. $U' =_G YUZ$ for some $Y, Z \in X^*$ where $\|Y\|, \|Z\| \leqslant \epsilon$;

3. $YRY^{-1} \neq_G R'$.

It is said that the system $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$-*condition* for some $\lambda \geqslant 1$, $c \geqslant 0$,$\epsilon \geqslant 0$, $\mu > 0$, $\rho > 0$, if

(1.1) $\|R\| \geqslant \rho$ for any $R \in \mathcal{R}$;

(1.2) any word $R \in \mathcal{R}$ is (λ, c)-quasi-geodesic;

(1.3) for any ϵ-piece of any word $R \in \mathcal{R}$, the inequalities $\|U\|, \|U'\| < \mu\|R\|$ hold.

Now suppose that for a word $R \in \mathcal{R}$ we have

(2.1) $R = UVU'V'$ for some $U, V, U', V' \in X^*$;

(2.2) $U' = YU^{\pm 1}Z$ in the group G for some words $Y, Z \in X^*$ where $\|Y\|, \|Z\| \leqslant \epsilon$;

then the word U is called an ϵ'-*piece* of the word R. If $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$-condition and, in addition, for all $R \in \mathcal{R}$, the above described decomposition of R implies $\|U\|, \|U'\| < \mu\|R\|$ then, like in [55], we say that $\mathcal{R}$ satisfies the $C'(\lambda, c, \epsilon, \mu, \rho)$-*condition* .

2.5.1 Auxiliary parameters, lowest parameter principle (LPP) and the main conventions

In the context of the definition of the small cancellation condition $C(\lambda, c, \epsilon, \mu, \rho)$ the parameters $\delta, \lambda, c, \epsilon, \mu, \rho$ were introduced. In this paper, whenever we mention the small-cancellation condition $C(\lambda, c, \epsilon, \mu, \rho)$, we assume that the parameters $\delta, \lambda, c, \epsilon, \mu, \rho$ satisfy some relations. More specifically, ϵ depends on λ and c; μ depends on λ, c and ϵ; and ρ depends on λ, c, ϵ and μ (see, for instance, Lemma 19 for an example where the condition $C(\lambda, c, \epsilon, \mu, \rho)$ is involved).

Based on a similar concept introduced in [53] (see §15 in [53]), we introduce the notation $\succ$ between parameters defined as follows: if $\alpha_1, \alpha_2, \ldots$ are some parameters, then $\alpha_1 \succ \alpha_2 \succ \ldots$ means that the value of α_i is being chosen after the parameters $\alpha_1, \ldots \alpha_{i-1}$ were chosen. In other words, the parameters $\alpha_1, \ldots \alpha_{i-1}$ are independent of α_i, but α_i depends on the values of $\alpha_1, \ldots, \alpha_{i-1}$. If α and β are some parameters such that $\alpha \succ \beta$ then we say that α is a *higher* parameter (correspondingly, β is a *lower* parameter), alternatively, we say that α has higher priority with respect to β and β has lower priority with respect to α.

Convention 1. Throughout this text we will deal with statements involving parameters $\lambda, c, \epsilon, \mu, \rho$ and their indexed versions $\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i$ for $i \in \mathbb{N}$. For all these parameters we assume that $\lambda \succ c \succ \epsilon \succ \mu \succ \rho$. Analogously, $\lambda_i \succ c_i \succ \epsilon_i \succ \mu_i \succ \rho_i$. Also we assume that parameters with lower indexes are higher with respect to $\succ$.

We also will deal with parameters δ_i, δ'_i. For them we assume $\rho_i \succ \delta_i$ and $\delta'_1 \succ \lambda_1$, $\rho_{i-1} \succ \delta'_i \succ \lambda_i$ for $i = 2, 3, \ldots$.

Convention 2. Throughout this text, for parameters $\delta, \lambda, c, \epsilon, \mu^{-1}, \rho$ and their indexed versions $\lambda_i, c_i, \epsilon_i, \mu_i^{-1}, \rho_i$ when we say that some parameter, say α, is *large enough* we mean that there is a finite number of parameters of higher priority, say $\beta_1, \ldots, \beta_k$, and a computable function $f_{\alpha,\beta_1,\ldots,\beta_k} : \mathbb{N}^k \to \mathbb{N}$ such that α can be chosen to have any value greater than $f(\beta_1, \ldots, \beta_k)$. For example, if for ρ_i "large enough" means $\rho_i > \lambda_i \mu_i$, then we think of i to be an arbitrary index from $\mathbb{N}$.

Definition 2.4 (The standard parameters)**.** The parameters $\delta, \lambda, c, \epsilon, \mu, \rho$ and the indexed parameters $\delta_i, \delta'_i, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i$, which are intensively used in this paper, we call the standard parameters.

Definition 2.5 (Sparse enough standard parameters)**.** We will say that the sequence of standard parameters $(\delta'_i, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i, \delta_i)_{i=1}^{\infty}$ is *sparse enough* if for each parameter α_{i_0}, where i_0 is the index of the parameter, we assume that

$$\alpha_{i_0}^{\iota} > f_{i_0,i_1,\ldots,i_k}(\alpha_{i_1}, \ldots, \alpha_{i_k}), \tag{2.19}$$

where $\alpha_{i_1}, \ldots, \alpha_{i_k}$ are parameters of higher priority with indices $i_1, \ldots, i_k$, $\iota = -1$ if $\alpha_{i_0} \in \{\mu_1, \mu_2, \ldots\}$ and $\iota = 1$ otherwise, and $f_{i_0,i_1,\ldots,i_k}$ is a computable function such that $f_{i_0,i_1,\ldots,i_k} = f_{i_0+t,i_1+t,\ldots,i_k+t}$ for all $t \geqslant 0$, and the map $i_0 \mapsto f_{i_0,i_1,\ldots,i_k}$ is computable as well.

Convention 3 (Lowest parameter principle (LPP))**.** In order many results of the current paper to hold (for example, Theorems 14, 5, etc.), we require from the standard parameters to be sparse enough. Therefore, whenever we mention some relation of the form (2.19) involving the standard parameters, for example, $\epsilon > \lambda\delta + c$ or $\epsilon_i > \mu_i\rho_i + c$ (the last one is equivalent to $\mu_i^{-1} > (\epsilon_i - c)^{-1}$), then we say that this relation holds by lower parameter principle – simply, by LPP.

2.5.2 Words with small cancellation conditions

Hereafter, if it is not stated otherwise, we assume that $G = \langle X \rangle$ is a non-trivial, non-elementary, torsion free δ-hyperbolic group for some $\delta > 0$.

Let us consider a set $\mathcal{R}$ consisting of words of the form

$$R_i = z_i U^{m_{i,1}} V U^{m_{i,2}} V U^{m_{i,3}} \dots V U^{m_{i,j_i}}, \ \ i = 1, 2, \dots, k \tag{2.20}$$

and their cyclic shifts, where $k \in \mathbb{N}$, $U, V, z_1, \dots, z_k \in X^*$ are geodesic words, $U, V \neq_G 1$, and $m_{i,t} \in \mathbb{N}$ for $1 \leqslant i \leqslant k$, $1 \leqslant t \leqslant j_i$. Denote $Z = \{z_1, \dots, z_k\}$, $L = \max\{\|U\|, \|V\|, \|z_1\|, \dots, \|z_k\|\}$.

Let $\tilde{\lambda}, \tilde{c} \in \mathbb{N}$ be such that U^n is $(\tilde{\lambda}, \tilde{c})$-quasi-geodesic in $\Gamma(G, X)$ for all $n \in \mathbb{Z}$. Note that the existence of $\tilde{\lambda}$ and $\tilde{c}$ follows from (8). Moreover, given the δ-hyperbolic group $G = \langle X \rangle$ and the word U, one can find such a pair $(\tilde{\lambda}, \tilde{c})$ algorithmically.

Now let $\underline{m} = \min\{m_{i,t} \mid 1 \leqslant i \leqslant k \text{ and } 1 \leqslant t \leqslant j_i\}$, $\overline{m}_i = \max\{m_{i,t} \mid 1 \leqslant t \leqslant j_i\}$ for $1 \leqslant i \leqslant k$. Then the following holds.

Lemma 17 (Compare with Lemma 2.3 in [55])**.** *For the set of words $\mathcal{R}$ suppose that $V \notin E(U)$, $z_i \notin E(U)$ for $1 \leqslant i \leqslant k$. Then there exist constants $\lambda = c = \tilde{K} \in \mathbb{N}$, computably depending on G, U, V and Z, such that the words of the system* (2.20) *are (λ, c)-quasi-geodesic in $\Gamma(G, X)$, provided that $\underline{m} \geqslant \tilde{K}$.*

Proof. We will show that λ, c and $\tilde{K}$ can be effectively computed by the following formulas

$$\lambda = c = \tilde{K} = \mathcal{K}\big(24\tilde{\lambda}, (2\bar{\mathcal{M}} + 2)L\big) \tag{2.21}$$

where $\mathcal{K}(\ )$ is defined as in Lemma 3, $L = \max\{\|U\|, \|V\|, \|z_1\|, \dots, \|z_k\|\}$ and

$$\bar{\mathcal{M}} = \max\big\{24\tilde{\lambda} + \tilde{c},\ \mathcal{M}(U, V_1) \mid V_1 \in \big\{V^{\pm 1}, z_1^{\pm 1}, \dots, z_k^{\pm 1}\big\}\big\}$$

where $\mathcal{M}()$ is given by the formula (2.12).

First, we will show that all the paths in $\Gamma(G, X)$ with labels of the form

$$W_1 U^{a_1} V_1^b U^{a_2} W_2, \tag{2.22}$$

where W_1 and W_2 are subwords of some words from $\{U^{\pm 1}, V^{\pm 1}, z_1^{\pm 1}, \ldots, z_k^{\pm 1}\}$, $V_1 \in \{V^{\pm 1}, z_1^{\pm 1}, \ldots, z_k^{\pm 1}\}$ and $b \in \{0, \pm 1\}$, are $\Big(24\tilde{\lambda}, (2\bar{\mathcal{M}} + 2)L\Big)$-quasi-geodesic.

For that let us fix an arbitrary such path q, with $lab(q) = W_1 U^{a_1} V_1^b U^{a_2} W_2$. Note that since all the subwords of $lab(q)$ are also of the form (2.22), to show that q is $\Big(24\tilde{\lambda}, (2\bar{\mathcal{M}} + 2)L\Big)$-quasi-geodesic, it is enough to show that

$$\|W_1 U^{a_1} V_1^b U^{a_2} W_2\| \leqslant 24\lambda |W_1 U^{a_1} V_1^b U^{a_2} W_2| + (2\mathcal{M} + 2)L.$$

To this end we will separately consider three cases:

1. when $b \neq 0$ and $\max\{a_1, a_2\} < \bar{\mathcal{M}}$;

2. when $b \neq 0$ and $\max\{a_1, a_2\} \geqslant \bar{\mathcal{M}}$; and

3. when $b = 0$.

Case 1. If $b \neq 0$ and $\max\{a_1, a_2\} < \bar{\mathcal{M}}$, then

$$\begin{aligned} \|W_1 U^{a_1} V_1^b U^{a_2} W_2\| \leqslant & \|W_1\| + a_1\|U\| + \|V_1\| + a_2\|U\| + \|W_2\| \\ \leqslant & \|W_1\| + \|V_1\| + \|W_2\| + (2\bar{\mathcal{M}} - 1)\|U\| \leqslant (2\bar{\mathcal{M}} + 2)L. \end{aligned}$$

Case 2. If $b \neq 0$ and $\max\{a_1, a_2\} \geqslant \bar{\mathcal{M}}$, then, by Lemma 10, either $|U^{a_1} V_1^b U^{a_2}| \geqslant \upsilon \max\{a_1, a_2\}$ or $V_1 \in E(U)$, where $\upsilon = \|U\|/12\tilde{\lambda}$.

Since, by our assumptions, $V_1 \notin E(U)$, we get that $|U^{a_1} V_1^b U^{a_2}| \geqslant \upsilon \max\{a_1, a_2\}$. Therefore,

$$\begin{aligned} |W_1 U^{a_1} V_1^b U^{a_2} W_2| \geqslant & |U^{a_1} V_1^b U^{a_2}| - |W_1| - |W_2| \\ \geqslant & \upsilon \max\{a_1, a_2\} - 2L. \end{aligned} \tag{2.23}$$

On the other hand,

$$\begin{aligned}\|W_1U^{a_1}V_1^bU^{a_2}W_2\| &\leqslant \|W_1\| + a_1\|U\| + \|V_1\| + a_2\|U\| + \|W_2\| \\ &\leqslant 2\max\{a_1, a_2\}\|U\| + 3L \\ \text{by (2.23)}, \ &\leqslant 2\Big(\frac{|W_1U^{a_1}V_1^bU^{a_2}W_2| + 2L}{\upsilon}\Big)\|U\| + 3L \\ &\leqslant 24\tilde{\lambda}|W_1U^{a_1}V_1^bU^{a_2}W_2| + (48\tilde{\lambda}+1)L \\ &\leqslant 24\tilde{\lambda}|W_1U^{a_1}V_1^bU^{a_2}W_2| + (2\bar{\mathcal{M}}+2)L.\end{aligned}$$

Case 3. If $b = 0$, then since $U^{a_1+a_2}$ is a $(\tilde{\lambda}, \tilde{c})$-quasi-geodesic word, we get

$$\begin{aligned}\|W_1U^{a_1}V_1^bU^{a_2}W_2\| = \|W_1U^{a_1+a_2}W_2\| &\leqslant \|U^{a_1+a_2}\| + \|W_1\| + \|W_2\| \\ &\leqslant \lambda|U^{a_1+a_2}| + c + 2L < \lambda(|W_1U^{a_1+a_2}W_2| + 2L) + c + 2L \\ &< 24\lambda|W_1U^{a_1+a_2}W_2| + (2\bar{\mathcal{M}}+2)L.\end{aligned}$$

Formula (2.12) implies that $2(\tilde{\lambda}+1)L + \tilde{c} < (2\bar{\mathcal{M}}+2)L$, hence the last inequality is true.

Now, let p be a path in $\Gamma(G, X)$ whose label corresponds to a word from $\mathcal{R}$. Since $\tilde{K} \leqslant m$, all the subpaths of p of the lengths bounded from above by $\tilde{K}$ are of the form (2.22). Therefore, p is $\tilde{K}$-local $\Big(24\tilde{\lambda}, (2\bar{\mathcal{M}}+2)L\Big)$-quasi-geodesic. Therefore, taken into account the formula for $\tilde{K}$ from (2.21) and the inequality $m \geqslant \tilde{K}$, by Lemma 3, p is $(\tilde{K}, \tilde{K})$-quasi-geodesic.

□

Assume that in the system (2.20), for all $1 \leqslant i, i' \leqslant k$ and $1 \leqslant t \leqslant j_i$, $1 \leqslant t' \leqslant j_{i'}$, $m_{i,t} \neq m_{i',t'}$ if $(i,t) \neq (i',t')$.

Recall that in Lemma 17 we required

$$V \notin E(U) \text{ and } z_i \notin E(U) \text{ for } 1 \leqslant i \leqslant k. \tag{2.24}$$

Let us introduce the following notations: For a given $\epsilon > 0$, $\epsilon_0 = \epsilon + 2L$, $\epsilon_i = \epsilon_0 + i\big(2R_{\lambda,c} + 182\delta + \frac{L}{2}\big)$ for $1 \leqslant i \leqslant 5$, where $R_{\lambda,c}$ is defined as in Lemma 1 and, as before,

$L = \max\{\|U\|, \|V\|, \|z_1\|, \ldots, \|z_k\|\}$. Let $\tilde{K}$ be defined by the formula (2.21) and $\lambda = c = \tilde{K}$. Now, with respect to the given constants $\epsilon \geqslant 0$, $\mu > 0$, $\rho > 0$ assume that

$$\|R\| \geqslant \rho, \text{ for all } R \in \mathcal{R}, \tag{2.25}$$

$$\underline{\mathrm{m}} \geqslant \tilde{K}, \tag{2.26}$$

hence, by Lemma 17, the words from $\mathcal{R}$ are (λ, c)-quasi-geodesics in $\Gamma(G, X)$. Next, we require the following.

$$\mu\|R_i\| \geqslant 6L(\overline{m}_i + 1) \tag{2.27}$$

and

$$\underline{\mathrm{m}} \geqslant \frac{2\epsilon_5}{\upsilon} \tag{2.28}$$

where $\upsilon = \upsilon(U)$ is defined by formula (2.11).

Lemma 18. *Using the setting of the previous lemma and assuming that the above described conditions take place, let us consider the system of words $\mathcal{R}$ given by* (2.20)*. Let λ, c be defined by the formulas* (2.21)*. Then, if for the given constants $\epsilon \geqslant 0$, $\mu > 0$, $\rho > 0$, the conditions* (2.25), (2.26), (2.27) *and* (2.28) *are satisfied, then the system $\mathcal{R}$ satisfies the $C'(\lambda, c, \epsilon, \mu, \rho)$-condition.*

Moreover, if two words $R_1, R_2 \in \mathcal{R}$ are not equal up to cyclic shifts, then there are no subwords U_1 and U_2 of R_1 and R_2, respectively, such that $\|U_1\| \geqslant \mu\|R_1\|$ and for some $T_1, T_2 \in X^*$, $\|T_1\|, \|T_2\| \leqslant \epsilon$ and

$$T_1^{-1} U_1 T_2 =_G U_2.$$

Proof. First of all, let us assume that the constants $\epsilon \geqslant 0$, $\mu > 0$, $\rho > 0$ are already given.

Assume by contradiction that there exist two different words from $\mathcal{R}$, W_1 and W_2, which have common ϵ-pieces. Suppose that W_1 and W_2 are cyclic shifts of the words

$$z_i U^{m_{i,1}} V U^{m_{i,2}} V U^{m_{i,3}} \dots V U^{m_{i,j_i}}$$

and

$$z_{i'} U^{m_{i',1}} V U^{m_{i',2}} V U^{m_{i',3}} \dots V U^{m_{i',j_{i'}}}$$

or of their inverses, for some $1 \leqslant i, i' \leqslant k$.

Existence of a common ϵ-piece for the words W_1 and W_2 implies that there is a rectangle $ABCD$ in $\Gamma(G, X)$ such that the labels of AD and BC are prefixes of W_1 and W_2 with length at least $\mu\|W_1\|$ and $\mu\|W_2\|$, respectively, and AB, CD are geodesics with length at most ϵ.

Let us call vertices on AD and BC *phase vertices* if they are either origin or endpoint of a subpath with label $U^{\pm 1}$, $V^{\pm 1}$, $z_i^{\pm 1}$ or $z_{i'}^{\pm 1}$.

Note that after making AB and CD longer by at most $2y$, we can ensure that A, B, C and D are phase vertices. Hereafter, let us assume that the length of AB and CD are bounded by $\epsilon + 2L = \epsilon_0$ and the vertices A, B, C and D are phase vertices.

We will call a subpath of AD or BC *special* if it is labeled by V, z_i or $z_{i'}$. If a special segment on AD or BC is between other special segments then we call this special segment *inner*, otherwise, we call it *boundary special segment*. Note that for any point $O \in AD$ (or $O \in BC$), there is a phase vertex $O' \in AD$ (or, correspondingly, $O' \in BC$), such that $\|[O, O']\| \leqslant L/2$.

Before proceeding further, let us state and prove the following auxiliary claims.

Claim 1. For the rectangle $ABCD$ let us consider any inner special segment P_1P_2 on one of the sides AD or BC. For concreteness let us assume that P_1P_2 belongs to AD. Then for any phase vertex $Q_1 \in BC$, if $d(P_1, Q_1) \leqslant \epsilon_3$, then either $lab(P_1Q_1) \in E(g)$ in G or $lab(P_2Q_1) \in E(g)$ in G, where by $lab(P_1Q_1)$ and $lab(P_2Q_1)$ we mean the labels of any paths joining P_1 to Q_1 and P_2 to Q_1, respectively.

Proof. Let P_3P_4 and P_5P_6 be the closest to P_1P_2 special segments on AD such that P_1P_2 is between P_3P_4 and P_5P_6 (their existence follows from the assumption that P_1P_2 is an inner special segment). Let $Q_1 \in BC$ be a fixed phase vertex such that $d(P_1, Q_1) \leqslant \epsilon_3$, and let Q_4 be the closest to P_4 phase vertex on BQ_1. See Figure 2.2.

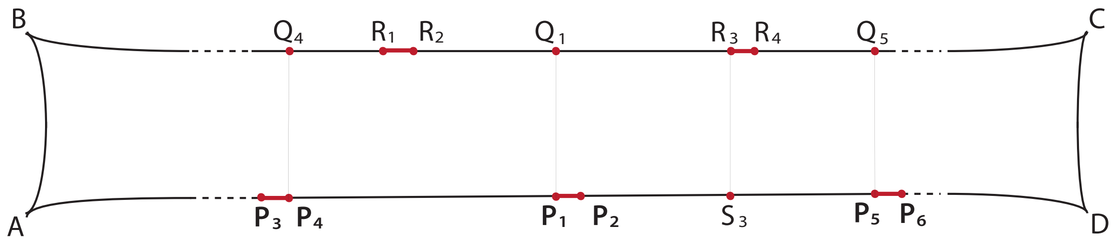

Figure 2.2

Since $d(P_1, Q_1), d(A, B) \leqslant \epsilon_3$, by Corollary 3, we get $dist(P_4, BQ_1) \leqslant \epsilon_3 + 2R_{\lambda,c} + 2\delta$. Therefore, $d(P_4, Q_4) \leqslant \epsilon_3 + 2R_{\lambda,c} + 2\delta + \frac{L}{2} = \epsilon_4$.

There are two possibilities which we are going to discuss separately: either $lab(Q_4Q_1)$ is a power of U or Q_4Q_1 contains a special segment.

In case $lab(Q_4Q_1)$ is a power of U, since $lab(P_4P_1) = U^{\xi_1}$ for $\xi_1 \geqslant \underline{\mathrm{m}} \geqslant \mathcal{M}$ and $d(P_4, Q_4), d(P_1, Q_1) \leqslant \epsilon_4 \leqslant \upsilon\underline{\mathrm{m}}$ (the last inequality follows from (2.28)), by Lemma 10, the equality

$$lab(P_4Q_4)lab(Q_4Q_1)lab(Q_1P_1)lab(P_1P_4) =_G 1$$

implies that $lab(P_1Q_1) \in E(U)$. Thus we are done with this case.

Now let us consider the case when Q_4Q_1 contains a special segment. Let Q_5 be a phase vertex on Q_1C closest to P_5. By Corollary 3, we again get $d(P_5, Q_5) \leqslant \epsilon_4$. Again, if $lab(Q_1Q_5)$ is a power of U, then, similarly to the previous case, by Lemma 10, $lab(P_2Q_1) \in E(g)$ in G. Thus we are left only with the case when both Q_4Q_1 and Q_1Q_5 contain special segments. Let us consider this case in more details.

Let R_1R_2 and R_3R_4 be the closest to Q_1 special segments on Q_4Q_1 and Q_1Q_5, respectively. See Figure 2.2. Since $lab(R_2R_3)$ has a form U^{ξ_2}, where $|\xi_2| \geqslant \underline{\mathrm{m}}$, at least one of $lab(R_2Q_1)$ and $lab(Q_1R_3)$ is of the form U^{ξ_3}, where $|\xi_3| \geqslant \underline{\mathrm{m}}/2 \geqslant \mathcal{M}$. Without loss of generality, assume that $lab(Q_1R_3) = U^{\xi_3}$ for $|\xi_3| \geqslant \underline{\mathrm{m}}/2$. Then, let S_3 be a phase vertex on P_1P_5 closest to R_3. Then, by

Corollary 3, $d(R_3, Q_3) \leqslant \epsilon_5$. Therefore, since by (2.28), $\upsilon\underline{m} \geqslant 2\epsilon_5$, by Lemma 10, the equality

$$lab(P_2Q_1)lab(Q_1R_3)lab(R_3S_3)lab(S_3P_2) =_G 1$$

implies that $lab(Q_1P_2) \in E(U)$.

□

Claim 2. If A_1A_2, A_3A_4 and A_5A_6 are three consecutive inner special segments belonging either to AD or to BC, then A_3A_4 is a special segment on $AD \cap BC$.

Proof. Firstly, without loss of generality let us assume that A_1A_2, A_3A_4 and A_5A_6 belong to AD. Let B_3 be the closest to A_3 phase vertex on BC, B_2 be the closest to A_2 phase vertex on BB_3 and B_5 be the closest to A_5 phase vertex on B_3C. See Figure 2.3.

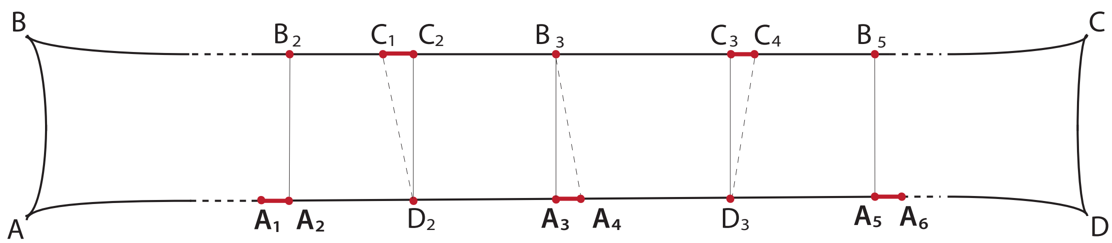

Figure 2.3

We will consider the case when B_3B_5 contains special segment(s) and the case when it does not contain any special segment separetely.

First let us consider the case when B_3B_5 contains special segment(s). Let C_1C_2 be the closest to B_3 special segment on B_2B_3 and C_3C_4 be the closest to B_3 special segment on B_3B_5. In case B_2B_3 does not contain any special segments, we take $C_2 = B_2$.

Correspondingly, let D_2 be the closest to C_2 phase vertex on A_2A_3 and D_3 be the closest to C_3 phase vertex on A_4A_5. Then, by Corollary 3, $d(A_3, B_3) \leqslant \epsilon_1$, $d(A_2, B_2), d(A_5, B_5) \leqslant \epsilon_2$, and hence $d(C_2, D_2), d(C_3, D_3) \leqslant \epsilon_3$. Therefore, by Claim 1, one word from each pair $(lab(C_1D_2)$, $lab(C_2D_2))$; $(lab(A_3B_3), lab(A_4, B_3))$ and $(lab(D_3, C_3), lab(D_3, C_4))$ belongs to $E(U)$.

Note that if $lab(D_3C_4) \in E(U)$, then it cannot be so that $lab(A_4B_3) \in E(U)$, because otherwise it would imply that $lab(C_3C_4) \in E(U)$ as well, which is not true by our assumptions (see the condition (2.24)). Therefore, in case $lab(D_3C_4) \in E(U)$, it must be that $lab(A_3B_3) \in E(U)$. But, since $lab(A_3C_3), lab(A_4C_4) \in E(U)$ in that case, by condition (2.24), it would mean that $d(A_3, C_3) = d(A_4, C_4) = 0$ or, in other words, A_3A_4 coincides with C_3C_4.

Now, if $lab(D_3C_3) \in E(U)$, then $lab(A_4B_3) \in E(U)$. Therefore, because of the condition (2.24), $lab(C_2D_2)$ cannot belong to $E(U)$. Finally, in case $C_2 = B_2$, by Claim 1, this would mean that $lab(A_1B_2) \in E(U)$, which is impossible because of the condition (2.24). Otherwise, again by Claim 1, $lab(C_1D_2) \in E(g)$ in G. Therefore, by the condition (2.24), since in this case $lab(A_3C_1) \in E(U)$ and $lab(A_4C_4) \in E(U)$, we would get $A_3 = C_1$ and $A_4 = C_2$.

Now let us turn to the case when B_3B_5 does not contain a special segment. In this case, by applying Lemma 10 to the boundary label of the rectangle $A_4B_3B_5A_5$ we get that $lab(A_4B_3)$ and $lab(A_5B_5)$ belong to $E(U)$. Then, by repeating previous arguments, we obtain that $lab(D_2C_1) \in E(U)$ and consequently $A_3 = C_1$ and $A_4 = C_2$. Thus Claim 2 is proved. □

Inequality (2.27) assures us that on AD one can find six consecutive special segments A_1A_2, A_3A_4, A_5A_6, A_7A_8, A_9A_{10} and $A_{11}A_{12}$. By Claim 2, A_5A_6, A_7A_8 belong to $AD \cap BC$. See Figure 2.4.

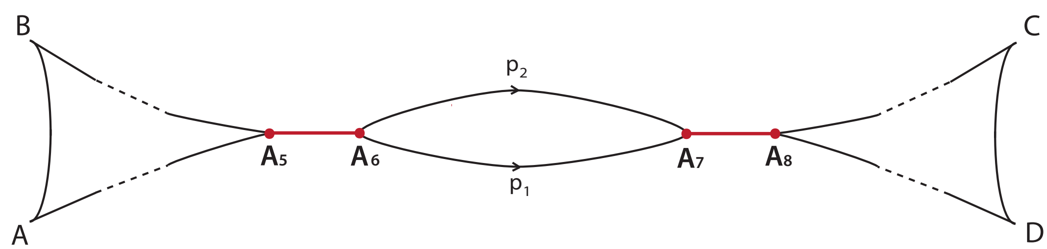

Figure 2.4

As it is shown in Figure 2.4, let us denote the subpaths of AD and BC restricted between A_6 and A_7 by p_1 and p_2, respectively. Since A_5A_6 and A_7A_8 are consecutive special segments, the label of p_1 is a power of U. Now, assuming that p_2 contains a special segment, just like it was done in the proof of Claim 2, we can show that that special segment must also belong to p_1, which is impossible since p_1 does not contain any special segments. Therefore, it must be that p_2 also does not contain any special segments. In other words, the label of p_2 is a power of U as well. This means that the label of the closed path $p_2p_1^{-1}$ is also a power of U. But since U represents an element $g \in G$ of infinite order, this can happen if and only if the label of $p_2 p_1^{-1}$ is the empty word, i.e., when p_2 coincides with p_1.

Now, since for all $1 \leqslant t \leqslant j_i$, $1 \leqslant t' \leqslant j_{i'}$, $m_{i,t} \neq m_{i',t'}$ if $(i,t) \neq (i',t')$, the last observation implies that, in fact, $i = i'$ and W_1 is a cyclic shift of W_2. Moreover, we get that either $lab(BA)$ is a suffix of $lab(AA_5)$ or $lab(AA_5)$ is a suffix of $lab(BA)$. This means that either $lab(AB)$ is equal to a prefix of W_1 in G or $lab(BA)$ is equal to a prefix of W_2 in G; but this is impossible, because it contradicts condition (3) in the definition of ϵ-pieces.

Now it follows from (2.25) and Lemma 17 that $\mathcal{R}$ satisfies the small cancellation condition $C'(\lambda, c, \epsilon, \mu, \rho)$.

□

2.5.3 A special subclass of small cancellation words

Using the already established setting of Subsection 2.5.2, let us define the positive integer $m_{1,1}$ as the smallest positive integer satisfying all the constraints put on it in Subsection 2.5.2.

Now let us assume that in the set of words $\mathcal{R}$ we have that $m_{1,1}$ is defined as above and for all $1 \leqslant i \leqslant k$, $m_{i,1} = 2^{i-1} m_{1,1}$, $j_i = m_{i,1} - 1$ and for all $1 \leqslant t \leqslant j_i$, $m_{i,t} = m_{i,1} + (t-1)$.

If all these equations are satisfied, then we denote the system of words $\mathcal{R}$ by

$$\mathcal{R} = \mathcal{R}\big(\boldsymbol{Z}, U, V, \delta, \lambda, c, \epsilon, \mu, \rho\big). \tag{2.29}$$

where $\boldsymbol{Z}$ is the ordered set $\{z_1 < z_2 < \ldots < z_k\}$.

Note that the set of words $\mathcal{R}$ defined this way satisfies all the conditions prescribed for Lemmas 17 and 18. A little bit less obvious among this conditions seems to be condition (2.27). Let us show that condition (2.27) holds as well.

Indeed, the length of each word R_i from $\mathcal{R}\big(\boldsymbol{Z}, U, V, \delta, \lambda, c, \epsilon, \mu, \rho\big)$ is not smaller than $m_{i,1} + (m_{i,1}+1) + \ldots + (2m_{i,1}-1) > m_{i,1}^2$ and $\overline{m}_i = 2m_{i,1} - 1$. Therefore, for each $1 \leqslant i \leqslant k$, $\mu \|R_i\| \geqslant \mu m_{i,1}^2$. Now we have $\mu\|R_i\| \geqslant \mu m_{i,1}^2 \geqslant 12 L m_{i,1} = 6L(\overline{m}_i + 1)$. Note that the last inequality follows from the property A2.

Thus, by Lemma 17 and Lemma 18, the set of words $\mathcal{R}\big(\boldsymbol{Z}, U, V, \delta, \lambda, c, \epsilon, \mu, \rho\big)$ satisfies the small cancellation condition $C'(\lambda, c, \epsilon, \mu, \rho)$.

Let $f : \mathbb{N} \to \mathbb{N}$ be a linear time computable function. Then for all $n \in \mathbb{N}$, define

$$ {}_n\mathcal{R} = \big\{ R \in \mathcal{R} \mid \|R\| \leqslant f(n) \big\}. $$

Assuming that $\boldsymbol{Z}, U, V, \delta, \lambda, c, \epsilon, \mu, \rho$ are already computed, from the structure of (2.29), it is not hard to see that the set ${}_n\mathcal{R}$ can be computed in time bounded from above by $Af(n)$, where $A > 0$ does not depend on the parameters of ${}_n\mathcal{R}$. Thus we get the following property.

Property 1. *${}_n\mathcal{R}$ can be computed in time bounded from above by $Af(n)$, where A is a positive constant not depending on $\boldsymbol{Z}, U, V, \delta, \lambda, c, \epsilon, \mu, \rho$.*

For the applications, let us introduce the following convention:

$$ \mathcal{R}\big(\varnothing, U, V, \delta, \lambda, c, \epsilon, \mu, \rho\big) = \varnothing. \tag{2.30} $$

2.5.4 Planar diagrams over hyperbolic groups and van Kampen's lemma

Let $H = \langle X \mid \mathcal{O} \rangle$, where $|X| < \infty$. A *map* is a finite, planar connected 2-complex. A *diagram* Δ over X is a map whose edges e are labeled by letters $lab(e) \in X^{\pm 1}$ such that $lab(e)^{-1} = lab(e^{-1})$. The label of a path $p = e_1 \dots e_n$ in Δ is, by definition, the word $lab(e_1) \dots lab(e_n)$. A diagram over X is called a *diagram over the group $H = \langle X \mid \mathcal{O} \rangle$* if the label of the boundary path of every cell of Δ is a cyclic shift of some relator from $\mathcal{O}$.

A van Kampen lemma states that a word $W \in X^*$ represents the identity of the group H if and only if there is a simply connected diagram Δ over $H = \langle X \mid \mathcal{O} \rangle$ such that the boundary label of Δ is W. Hence, for a given $W \in X^*$ we call such a Δ *van Kampen's diagram* with label W over $H = \langle X \mid \mathcal{O} \rangle$. In this paper we only use simply connected diagrams. Therefore, hereafter by diagrams we will mean simply connected diagrams.

Note that for any diagram Δ over $H = \langle X \mid \mathcal{O} \rangle$, Δ can be naturally projected into the Cayley graph $\Gamma(H, X)$ such that all the labels are preserved. Moreover, if we fix arbitrary vertex o_1 of Δ and arbitrary vertex o_2 of $\Gamma(H, X)$, then the projection which maps o_1 to o_2 is defined uniquely.

We denote this projection by $Proj_{o_1}^{o_2}(\Delta)$. Since in the applications of the current paper, we do not need to specify o_1 and o_2, we will simply use the notation $Proj(\Delta)$. This projection allows us to consider word metric on Δ, by simply considering the word metric on the projection of Δ.

Let G be a quotient of H. When considering group G we will partition the defining relators into two sets. The first set $\mathcal{O}$ will consist of all relators (not only defining) of H with a fixed generating set X. The second set, $\mathcal{R}$, will be some symmetrized set of additional relators. We shall write

$$G = \langle X \mid \mathcal{O} \cup \mathcal{R} \rangle = \langle H \mid \mathcal{R} \rangle. \tag{2.31}$$

Using the terminology of [55] we call the cells of a diagram with boundary labels from $\mathcal{O}$ (from $\mathcal{R}$) 0-cells ($\mathcal{R}$-cells). Diagram is called *reduced* if it contains minimal number of $\mathcal{R}$-cells among all diagrams with the same boundary label.

Now consider a simple closed path $w = p_1 q_1 p_2 q_2$ in a diagram Δ over G, such that q_1 and q_2 are subpaths of boundary cycles of $\mathcal{R}$-cells Π_1 and Π_2, and $\|p_1\|$, $\|p_2\| \leqslant \epsilon$ for a fixed constant ϵ. Assuming that the subdiagram Γ of Δ bounded by w has no hole and no $\mathcal{R}$-cell and $\Pi_1 \neq \Pi_2$, following Olshanskii, we call Γ a *ϵ-contiguity* (or simply, *contiguity*) subdiagram of Π_1 and Π_2. The same term will be used if $\Pi_1 = \Pi_2$ and Γ contains no holes.

In case q_2 instead of being a subpath of Π_2 is a subpath of of a connected path q on $\partial\Delta$, Γ is called *outer ϵ-contiguity subdiagram* (from Π_1 to $\partial\Delta$ or to q). The notation $\partial(\Pi_1, \Gamma, \Pi_2)$ (or $\partial(\Pi_1, \Gamma, q)) = p_1 q_1 p_2 q_2$ will define the partition of the contour w of Γ. The above subpaths q_1 and q_2 are called the contiguity arcs while p_1 and p_2 are called the side arcs of the contiguity subdiagram Γ.

Hereafter we will denote by $\partial\Pi$ the loop in $\Gamma(G, X)$ with the label equal to the label of Π. By $\|\Pi\|$ we denote the length of the boundary label of a cell Π. The ratio $\|q_1\|/\|\Pi_1\|$ for a contiguity subdiagram of a cell Π_1 to a cell Π_2 (or to a section q), is called the contiguity degree of Π_1 to Π_2 via Γ (or of Π_1 to q). It is denoted (Π_1, Γ, Π_2) (or (Π_1, Γ, q)). For a matter of convenience, instead of the notation (Π_1, Γ, q) we will simply use the notation $\partial\Gamma$ if it does not lead to ambiguities.

If for a contiguity subdiagram $\Gamma \in \mathcal{M}$, $p_1^{-1} q_1 p_2 q_2^{-1} = \partial\Gamma$, q_2 belongs to $\partial\Delta$, then q_2 is called *outer* contiguity arc, and correspondingly q_1 is called *inner* contiguity arc. Whenever it is not mentioned otherwise, hereafter we will denote the outer arc of Γ by $\hat{q}_\Gamma$ and the inner arc by $\check{q}_\Gamma$. Also let us denote $p_1 = p_\Gamma$ and $p_2 = p'_\Gamma$.

2.5.5 Quotients of hyperbolic groups by normal closures of words with small cancellation conditions

Hereafter, if Δ is a diagram over the quotient $G = H/\ll \mathcal{R} \gg$, then by saying that the boundary $\partial\Delta$ of Δ is a (λ, c)-quasi-geodesic t-gon, we mean that $\partial\Delta$ is partitioned into t connected pieces such that they are (λ, c)-quasi-geodesic in $\Gamma(H, X)$.

Lemma 19 (see Lemma 4.6 in [57] and Lemma 6.6 in [55])**.** *For appropriately chosen parameters based on the lowest parameter principle with respect to the order $\lambda \succ c \succ \epsilon \succ \mu \succ \rho$, if the presentation $G = H/ \ll \mathcal{R} \gg$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$-condition, then for any reduced disk diagram Δ over the presentation $G = H/ \ll \mathcal{R} \gg$ whose boundary is a (λ, c)-quasi-geodesic t-gon for $1 \leqslant t \leqslant 12$ and which contains an $\mathcal{R}$-cell, there exists an $\mathcal{R}$-cell Π in Δ and disjoint outer ϵ-contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_t$ of Π to different sides of the (λ, c)-q.g. t-gon $\partial\Delta$, such that*

$$\sum_{i=1}^{t} (\Pi, \Gamma_i, \hat{q}_{\Gamma_i}) > 1 - 23\mu. \tag{2.32}$$

Moreover, the quotient $G = H/ \ll \mathcal{R} \gg$ is $4L$-hyperbolic, where $L = \max\{\|R\| \mid R \in \mathcal{R}\}$.

Remark. Note that, in fact, in Lemma 19 some of the subdiagrams $\Gamma_1, \ldots, \Gamma_t$, say Γ_1, may not exist, in which case we would call Γ_1 *empty* contiguity subdiagram and take $(\Pi, \Gamma_1, \hat{q}_{\Gamma_1}) = 0$. The important thing is that, according to Lemma 19, some of $\Gamma_1, \ldots, \Gamma_t$ are not empty, so that the inequality (2.32) holds.

Lemma 20 (Lemma 7.2, [55])**.** *Let $H = \langle X \rangle$ by a non-elementary hyperbolic group. Let G be a group with a presentation (2.31) such that $\mathcal{R}$ satisfies the $C'(\lambda, c, \epsilon, \mu, \rho)$-condition for appropriately chosen parameters $\lambda \succ c \succ \epsilon \succ \mu \succ \rho$. Then G is non-cyclic, each $R \in \mathcal{R}$ represents an element of infinite order in H, and a word $W \in X^*$ has a finite order in G if and only if W is conjugate in G to an element having finite order in H.*

Remark 7. Note that if H is a non-elementary torsion-free hyperbolic group, then Lemma 20 implies that G is also a non-elementary torsion-free hyperbolic group.

Definition 2.6 (*Essential* cells and contiguity subdiagrams)**.** Let Π be an $\mathcal{R}$-cell in a reduced van Kampen diagram Δ with (λ, c)-quasi-geodesic t-gon boundary for $1 \leqslant t \leqslant 12$, and let Π be connected to the sides of the t-gon $\partial\Delta$ by disjoint outer ϵ-contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_t$ such that

$$\sum_{i=1}^{t} (\Pi, \Gamma_i, \hat{q}_{\Gamma_i}) > 1 - 23\mu. \tag{2.33}$$

Then we call Π *an essential* cell, and the contiguity subdiagrams $\Gamma_1, \ldots, \Gamma_t$ – *essential* contiguity subdiagrams.

2.5.6 Auxiliary definitions and lemmas

In this subsection we discuss some auxiliary lemmas and definitions for $G = H/ \ll \mathcal{R} \gg$, where $H = \langle X \rangle$ is hyperbolic and $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$-condition. Also, $0 \leqslant \eta \leqslant 1 \leqslant \epsilon_0$ are some constants.

Definition 2.7 ((ϵ_0, η)-arcs and (ϵ_0, η)-words)**.** $W_0 \in X^*$ is an (ϵ_0, η)-word (associated with a word $R \in \mathcal{R}$) with respect to the quotient $G = H/ \ll \mathcal{R} \gg$, if there exist words $T_1, T_2 \in X^*$, $\|T_1\|, \|T_2\| \leqslant \epsilon_0$ and a word $R \in \mathcal{R}$ such that $R = UV$, $\|U\| \geqslant \eta\|R\|$ and

$$W_0 =_H T_1^{-1} U T_2.$$

A subpath p' of a path p from $\Gamma(G, X)$ is called (ϵ_0, η)-arc (or (ϵ_0, η)-subpath) if its label is a (ϵ_0, η)-word.

Lemma 21. *Suppose that $W \in X^*$ contains a (ϵ_0, η)-subword associated with some word $R \in \mathcal{R}$. Then*

$$\|R\| \leqslant \frac{\lambda(\|W\| + 2\epsilon_0) + c)}{\eta}.$$

Proof. Follows from the definition of the (ϵ_0, η)-subwords and the fact that the word from $\mathcal{R}$ are (λ, c)-quasi-geodesics in $\Gamma(H, X)$. We just need to apply the triangle inequality. $\square$

Lemma 22. *Let us consider the quotient $G = H/ \ll \mathcal{R} \gg$, where $H = \langle X \rangle$ and $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$-condition. Then for any constants $\epsilon_0 \geqslant 0$ and $K > 0$, if μ is small enough and ρ is large enough, then there is no (λ, c)-quasi-geodesic path in $\Gamma(G, X)$ containing an $(\epsilon_0, 1 - K\lambda\mu)$-arc.*

Proof. All the metric notations which we use in this proof are in Cayley graph $\Gamma(G, X)$.

First of all, assume that $0 < \mu < \frac{1}{K\lambda}$ so that we have $0 < 1 - K\lambda\mu < 1$.

Now assume that there exists a (λ, c)-quasi-geodesic path p in $\Gamma(H, X)$ which contains an $(\epsilon_0, 1 - K\lambda\mu)$-arc p'. Then, by definition, there exist words $T_1, T_2 \in X^*$, $\|T_1\|, \|T_2\| \leqslant \epsilon_0$ and a word $R \in \mathcal{R}$, such that $R = UV$, $\|U\| \geqslant (1 - K\lambda\mu)\|R\|$ and

$$lab(p') =_H T_1^{-1} U T_2.$$

Then, combining the last equation with the triangle inequality and with the inequality $\|U\| \geqslant (1 - K\lambda\mu)\|R\|$, we get

$$|p'| \geqslant \frac{(1 - K\lambda\mu)\|R\| - c}{\lambda} - 2\epsilon_0. \tag{2.34}$$

On the other hand, by the triangle inequality, we have

$$|p'| \leqslant 2\epsilon_0 + \|V\| \leqslant 2\epsilon_0 + (1 - (1 - K\lambda\mu))\|R\| = 2\epsilon_0 + K\lambda\mu\|R\|. \tag{2.35}$$

Finally, note that, since $\|R\| \geqslant \rho$, if ρ is large enough, then the system of inequalities (2.34) and (2.35) is not consistent, which contradicts the existence of p' $\square$

Definition 2.8 (Truncated diagrams)**.** If a van Kampen diagram Δ over $G = H/ \ll \mathcal{R} \gg$ has a rectangular boundary $\partial\Delta = ABCD$ such that the following conditions hold

1. $[A, D]$ and $[B, C]$ are (λ, c)-quasi-geodesics in $\Gamma(G, X)$,
2. $[A, B]$ and $[D, C]$ are geodesic,
3. $d_G(A, B) = dist_G(A, [B, C])$, $d_G(D, C) = dist_G(D, [B, C])$,

then Δ is called a truncated diagram.

Lemma 23. *Suppose that Δ is a reduced diagram over $G = H/ \ll \mathcal{R} \gg$ such that $\partial\Delta = ABCD$, Δ is truncated and the following holds:*

$$d_G(A, D) \geqslant \lambda\big(L + \big\|[A, B]\big\| + \big\|[D, C]\big\| + 2\epsilon\big) + c, \tag{2.36}$$

where $L = \max\{\|R\| \mid R \in \mathcal{R}\}$.

Suppose that Δ contains an essential $\mathcal{R}$-cell Π connected to $[A, B]$, $[B, C]$, $[C, D]$ and $[D, A]$ by essential ϵ-contiguity subdiagrams Γ_1, Γ_2, Γ_3 and Γ_4, respectively. Then, if the standard parameters are large enough, we have

(i) *either Γ_1 or Γ_3 is empty;*

(ii) $(\Pi, \Gamma_1, [A, B]) + (\Pi, \Gamma_2, [B, C]) + (\Pi, \Gamma_3, [B, C]) \leqslant 1 - 26\mu$*; and*

(iii) $(\Pi, \Gamma_4, [A, D]) > \mu$.

Proof. First of all, if both Γ_1 and Γ_3 are not empty, then the distance between $[A, B]$ and $[D, C]$ is bounded by $2\epsilon + \|\Pi\| \leqslant 2\epsilon + L$. Therefore, since $[A, D]$ is (λ, c)-quasi-geodesic in $\Gamma(G, X)$, by the triangle inequality, we have $d_G(A, D) \leqslant \lambda\big(\big\|[A, B]\big\| + \big\|[D, C]\big\| + 2\epsilon + L\big) + c$, which contradicts the condition (2.36) in the statement of the lemma. Therefore, without loss of generality we can assume that Γ_3 is empty.

Now let us prove that

$$(\Pi, \Gamma_1, [A, B]) + (\Pi, \Gamma_2, [B, C]) < 1 - 26\mu.$$

For that let us denote $\kappa_1 = (\Pi, \Gamma_1, [A, B])$ and $\kappa_2 = (\Pi, \Gamma_2, [B, C])$, and by contradiction, assume that $\kappa_1 + \kappa_2 \geqslant 1 - 26\mu$. Then, since $d_G(A, B) = dist_G(A, [B, C])$, we get $d_G(A, B) \leqslant d_G(A, (\hat{q}_{\Gamma_2})_+)$, and consequently,

$$d_G\big((\hat{q}_{\Gamma_1})_-, B\big) \leqslant d_G\big((\hat{q}_{\Gamma_1})_-, (\hat{q}_{\Gamma_2})_+\big) \leqslant 2\epsilon + (1 - \kappa_1 - \kappa_2)\|\Pi\| < 2\epsilon + 26\mu\|\Pi\|. \tag{2.37}$$

See Figure 2.5. Since $\partial\Pi$ is (λ, c)-quasi-geodesic, we also have

$$\frac{\kappa_1\|\Pi\| - c}{\lambda} - 2\epsilon \leqslant \|\hat{q}_{\Gamma_1}\| \leqslant d_G((\hat{q}_{\Gamma_1})_-, B). \tag{2.38}$$

Combining (2.37) and (2.38), we get $\kappa_1\|\Pi\| \leqslant \lambda(4\epsilon + 26\mu\|\Pi\|) + c <^{\text{by LPP}} 27\lambda\mu\|\Pi\|$, and consequently, we get $\kappa_1 < 27\lambda\mu$. Therefore, $\kappa_2 > 1 - 26\mu - 27\lambda\mu > 1 - 53\lambda\mu$, or in other words, $\hat{q}_{\Gamma_2}$ is a $(\epsilon, 1 - 53\lambda\mu)$-arc, by Lemma 22, but for large enough ρ this is impossible, because $[B, C]$ is (λ, c)-quasi-geodesic.

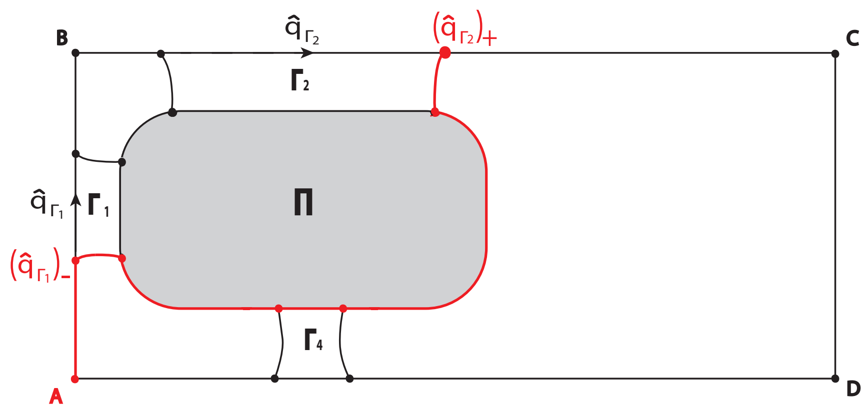

Figure 2.5

∎

Finally, since the system of contiguity diagrams Γ_i, $i = 1, 2, 3, 4$, is essential and Γ_3 is empty, we get that $(\Pi, \Gamma_4, [A, D]) \geqslant 1 - 23\mu - (\kappa_1 + \kappa_2) > 3\mu > \mu$.

Lemma 24. *Suppose that $\mathcal{R}$ satisfies $C'(\lambda, c, \epsilon, \mu, \rho)$-condition and ρ is large enough. Suppose that $R \in \mathcal{R}$ and U, V are disjoint subwords from R such that for some words $T_1, T_2 \in X^*$, $\|T_1\|, \|T_2\| \leqslant 2\epsilon$ and $T_1^{-1}UT_2 =_H V^{\pm 1}$.*

Then $\|U\|, \|V\| \leqslant 2\mu\|R\|$.

Proof. The statement follows from the definition of the small cancellation condition $C'(\lambda, c, \epsilon, \mu, \rho)$ (see properties (2.1) and (2.2) in the definition of $C'(\lambda, c, \epsilon, \mu, \rho)$) and Corollary 4. ∎

2.6 Van Kampen diagrams over HNN-extensions

Let $G = \langle X \mid \mathcal{R}\rangle$ be a group presentation. Let S be a subset of X. Then an *S-band* B is a sequence of cells Π_1,...,Π_n in a van Kampen diagram such that

- Each two consecutive cells in this sequence have a common edge labeled by a letter from S;

- Each cell Π_i, $i = 1, ..., n$, has exactly two S-edges (i.e. edges labeled by a letter from S) having opposite orientations.

We call the sides of an S-band with labels from S *ends* of the band, and the sides of the S-band which are separated by the ends *sides* of the S-band.

The concept of bands naturally occurs when one considers HNN-extensions of groups as follows.

Let $H = \langle X, t \mid t^{-1}At = B\rangle$, where $A, B \leqslant G = \langle X \mid R\rangle$ are isomorphic subgroups by some isomorphism $\phi : A \to B$. Then, from van Kampen's lemma it follows that for each $W \in (X \cup \{t\})^*$ such that $W =_H 1$, there exists a disc diagram (=van Kampen diagram) Δ over the presentation

$$H = \left\langle X \cup \{t\} \mid \mathcal{R} \cup \{t^{-1}at\phi(a)^{-1} \mid a \in A\}\right\rangle \tag{2.39}$$

such that $lab(\partial\Delta) = W$ and Δ is reduced, in the sense that it contains minimal number of t-bands among all van Kampen diagrams with boundary label W. A well-known fact is that either Δ does not contain t-bands (which implies that $W \in X^*$ and $W =_G 1$) or all the t-bands of Δ have their edges with label $t^{\pm 1}$ on the boundary $\partial\Delta$ of Δ. For more details see, for example, [47, 64].

Analogously, if H is obtained from G by multiple HNN-extensions with respect to isomorphic subgroups $\phi_1 : A_1 \to B_1, \ldots, \phi_n : A_n \to B_n$, namely,

$$H = \left\langle X \cup \{t_1, \ldots, t_n\} \mid \mathcal{R} \cup \{t_i^{-1}a_i t_i \phi_i(a_i)^{-1} \mid 1 \leqslant i \leqslant n, a_i \in A_i\}\right\rangle,$$

Then for each $1 \leqslant i \leqslant n$, either W does not contain letters from $\{t_i^{\pm 1}\}$ or in Δ all t_i-bands have their ends on $\partial\Delta$ and moreover, every edge of $\partial\Delta$ with a label from $\{t_i^{\pm 1}\}$ is connected with a t_i-band to another edge on $\partial\Delta$ with the same label.

2.7 Slender conjugacy diagrams and their geometry

2.7.1 Slender conjugacy diagrams over hyperbolic groups

Let $H = \langle X \mid \mathcal{O} \rangle$, $|\mathcal{O}| < \infty$, $|X| < \infty$, be a non-cyclic δ-hyperbolic group with respect to $\Gamma(G, X)$ for some $\delta \in \mathbb{N}$.

We call a disk diagram Δ with quadrilateral boundary $ABCD$, a (U, V)-*conjugacy diagram* over $\langle X \mid \mathcal{O} \rangle$ if $lab(AB) = lab(DC)$ and $lab(BC) = U$, $lab(AD) = V$.

We say that Δ is a *slender* (U, V)-conjugacy diagram over $\langle X \mid \mathcal{O} \rangle$, if AB has minimal length among all (U, V)-conjugacy diagrams over $\langle X \mid \mathcal{O} \rangle$. Also we say Δ is a *cyclically slender* (U, V)-conjugacy diagram over $\langle X \mid \mathcal{O} \rangle$ if it is a (U', V')-conjugacy diagram for some cyclic shifts U' and V' of U and V, respectively, and in addition,

$$\begin{aligned} &\|lab(AB)\| = \\ &\qquad \min\{\|lab(A'B')\| \mid \forall (U', V')\text{-conj. diagram } \Delta' \text{ with } \partial\Delta' = A'B'C'D', \\ &\qquad\qquad \text{where } U' \text{ and } V' \text{ are, respectively, cyclic shifts of } U \text{ and } V\}. \end{aligned}$$

For arbitrary points $O \in AB$ and $O' \in DC$, let us call them *mirroring* points if $lab(AO) = lab(DO')$.

Lemma 25. *If (U, V)-conjugacy diagram Δ has two different pairs of mirroring points (O_1, O_1') and (O_2, O_2') such that in $Proj(\Delta)$, O_1 is joined to O_1' by a path p_1 and O_2 is joined to O_2' by a path p_2 such that $lab(p_1) \equiv lab(p_2)$, then Δ is not slender.*

Proof. Indeed, if the statement of Lemma 25 holds, then we can remove the subdiagram in Δ bounded between O_1, O_2, O_2' and O_1' and obtain a new diagram Δ' with $\partial\Delta' = A'B'C'D'$, where $A'B'$ is shorter than AB. This procedure is depicted in Figure 2.6. Since the boundary label of the newly obtained diagram Δ' represents the trivial element of G, by van Kampen's lemma, there exists a disk diagram over $\langle X \mid \mathcal{O} \rangle$ with boundary of Δ'. Since $lab(A'B') = lab(D'C')$, then, in fact, the new disk diagram is a (U, V)-conjugacy diagram over $\langle X \mid \mathcal{O} \rangle$ as well. Finally, since the length of $A'B'$ is strictly shorter than the length of AB, by definition, Δ is not a slender diagram.

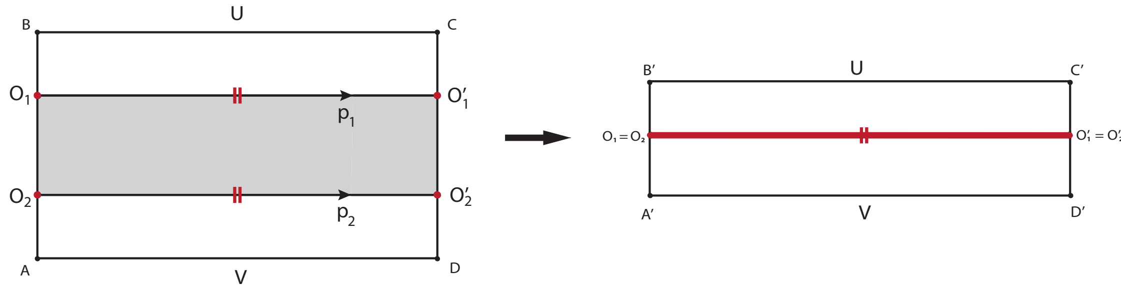

Figure 2.6: The left diagram is $Proj(\Delta)$, $lab(p_1) = lab(p_2)$. The right diagram is Δ', which is obtained after making a surgery on Δ to remove the colored subdiagram.

□

Based on Lemma 25 and Corollary 4, it is not hard to see that the following is true.

Lemma 26. *Let $U, V \in X^*$ be cyclically (λ, c)-geodesic words such that $U \sim_{conj} V$ in H. Suppose that Δ is a slender (U, V)-conjugcy diagram with the standard boundary $ABCD$. Then $\|AB\| = \|DC\| \leqslant \tau(|X|, \delta, \lambda, c)$, where $\tau : \mathbb{N}^4 \to \mathbb{N}$ is a computable function independent of H. In other words, there exist cyclic shifts $U', V' \in X^*$ of U and V, respectively, and a word $T \in X^*$ such that $\|T\| \leqslant \tau(|X|, \delta, \lambda, c)$ and $U' =_H T^{-1}V'T$.*

In fact, Lemma 26 is a slight variation of Lemma 10 in [41] and Proposition 3 in [4].

Definition 2.9 ((Cyclically) minimal conjugacy diagrams over HNN-extensions)**.** Let $H = \langle X \cup \{t\} \mid t^{-1}At = B\rangle$ be an HNN-extension of a group $G = \langle X \mid \mathcal{R}\rangle$. Suppose that $U, V \in (X \cup \{t\})^*$ are such that $U \sim_{conj} V$ in H. Let Δ be a (U, V)-conjugacy diagram over the presentation $H = \langle X \cup \{t\} \mid \mathcal{R} \cup \{t^{-1}at\phi(a)^{-1} \mid a \in A\}\rangle$ such that it contains minimal number of t-bands among all (U, V)-conjugacy diagrams. Then we call Δ minimal (U, V)-conjugacy diagram over the HNN-extension $H = \langle X \cup \{t\} \mid t^{-1}At = B\rangle$.

If Δ is a (U', V')-conjugacy diagram for some cyclic shifts U' and V' of U and V, respectively, and if Δ contains minimal number of t-bands among all such diagrams, then we say that Δ is a cyclically minimal (U, V)-conjugacy diagram over the HNN-extension $H = \langle X \cup \{t\} \mid t^{-1}At = B\rangle$.

Lemma 27 (About the conjugacy diagrams over HNN-extensions)**.** *(i). Let Δ be a cyclically minimal (U, V)-conjugacy diagram over the presentation* (2.39) *such that $\partial\Delta = ABCD$, $lab(AB) =$*

$lab(DC)$, and $lab(AD)$ and $lab(BC)$ are cyclic shifts of U and V for $U, V \in (X \cup \{t\})^*$, respectively. Then either Δ does not contain t-bands joining AB to DC or if there exists such a band with its ends $e_1 \in AB$ and $e_2 \in DC$, then $(e_1)_+$ and $(e_2)_+$ are mirroring pair of points.

(ii). Let Δ be a minimal (U, V)-conjugacy diagram over the presentation (2.39) *such that $\partial\Delta = ABCD$, $lab(AB) = lab(DC)$, and $U, V \in X^*$ (i.e. U and V do not contain letters from $\{t^{\pm 1}\}$). Then either Δ does not contain t-bands joining AB to DC or if there exists such a band with its ends $e_1 \in AB$ and $e_2 \in DC$, then $(e_1)_+$ and $(e_2)_+$ are a mirroring pair of points.*

Proof. The lemma is an easy consequence of Collins' Lemma. □

2.7.2 Cyclically slender conjugacy diagrams over quotient groups with small cancellation conditions

Definition 2.10 ((cyclically) $(\lambda, c, \epsilon, \eta)$-reduced words)**.** For $\epsilon > 0$, $0 < \eta \leqslant 1$, a cyclically reduced word $W \in X^*$ is called $(\lambda, c, \epsilon, \eta)$-reduced over the quotient $G = H/ \ll \mathcal{R} \gg$ if W is (λ, c)-quasi-geodesic in $\Gamma(H, X)$ and moreover, W does not contain a (ϵ, η)-subword. And it is called cyclically $(\lambda, c, \epsilon, \eta)$-reduced, if all cyclic shifts of W are $(\lambda, c, \epsilon, \eta)$-reduced.

For the next lemma, let $H = \langle X \rangle$ be a δ-hyperbolic group with respect to the generating set X, and let $G = H/ \ll \mathcal{R} \gg = \langle H \mid \mathcal{R} \rangle$, where $\mathcal{R}$ is a finite symmetric set of words satisfying the small cancellation condition $C'(\lambda, c, \epsilon, \mu, \rho)$ for appropriately chosen parameters $\lambda > c > \epsilon > \mu > \rho$.

Lemma 28. *Let $U, V \in X^*$ be cyclically $(\lambda, c, \epsilon, 1 - 121\lambda\mu)$-reduced words. Then for any reduced cyclically slender (U, V)-conjugacy diagram Δ with $\partial\Delta = ABCD$, assuming that Δ contains an $\mathcal{R}$-cell, we get that Δ contains an essential $\mathcal{R}$-cell Π which is connected to AB, BC, CD and DA by contiguity subdiagrams Γ_1, Γ_2, Γ_3 and Γ_4, respectively, and the following hold*

1. *Γ_2 and Γ_4 are non-empty;*

2. $(\Pi, \Gamma_2, BC) + (\Pi, \Gamma_4, DA) \geqslant 1 - 121\lambda\mu$; *and*

3. (Π, Γ_1, AB) *and* (Π, Γ_3, CD) *are either empty or smaller than* $49\lambda\mu$.

Proof. Proof of Lemma 28 is given in Appendix (see Subsection A.0.2). ☐

Definition 2.11. If Δ is a cyclically slender (U, V)-conjugacy diagram over the quotient $G = \langle H \mid \mathcal{R}\rangle$, then we say that Δ is a (U, V)-conjugacy H-diagram if G does not contain an $\mathcal{R}$-cell, otherwise, we say that Δ is a (U, V)-conjugacy G-diagram.

Convention 4. In the rest of the text for the quotient $G = H/ \ll \mathcal{R} \gg$ we assume that the parameters $\lambda, c, \epsilon, \mu, \rho$ are chosen so that $1 > 1 - 122\lambda\mu > 0$ and no (λ, c)-quasi-geodesic path in $\Gamma(G, X)$ contains an $(\epsilon, 1 - 122\lambda\mu)$-arc (with respect to $G = H/ \ll \mathcal{R} \gg$). Note that these assumptions can be made without loss of generality because of Lemma 22.

2.7.3 An application of Lemma 28

Lemma 28 together with Lemma 24 implies the following.

Lemma 29. *Let* $H = \langle X \rangle$ *be a torsion-free non-elementary hyperbolic group and* $G = H/ \ll \mathcal{R} \gg$ *satisfies the* $C'(\lambda, c, \epsilon, \mu, \rho)$*-condition for sparse enough parameters* $\lambda > c > \epsilon > \mu > \rho$.

(i) For any $U \in X^*$ *and* $k \in \mathbb{N}$ *such that* U *is a cyclically minimal word in* $\Gamma(G, X)$*,* U^k *does not contain a* $(\epsilon, 1 - 122\lambda\mu)$*-subword with respect to the quotient* $G = H/ \ll \mathcal{R} \gg$.

(ii) Suppose that $U, W \in X^*$ *are such that* $U =_G W^k$ *for some* $k \geqslant 2$ *and*

$$\|U\| < \frac{\mu\rho - c}{\lambda} - 2\epsilon. \tag{2.40}$$

Then $U =_H W^k$*. In particular, if* $U \notin E(W)$ *in G, then* $U \notin E(W)$ *in H.*

Proof. (i). Let U be as in the statement of the lemma. By contradiction, assume that U contains a $(\epsilon, 1 - 122\lambda\mu)$-subword V.

For sparse enough standard parameters λ, c, μ, ρ, by Lemma 8, U^k is a (λ, c)-quasi-geodesic word in $\Gamma(G, X)$. Then, since by the assumption U is cyclically minimal in $\Gamma\big(G, X\big)$, by Convention 4, we get that V is not a subword of a cyclic shift of U. This means that V is of the form

$$lab(\hat{q}_{\Gamma_2}) = (U')^n Q,$$

where U' is a cyclic shift of U, $n \geqslant 1$, and Q is a prefix of U'.

By the definition of $(\epsilon, 1 - 122\lambda\mu)$-subwords, in the Cayley graph $\Gamma(G, X)$, there exist paths p and q such that $lab(p)$ is a subword of a word R from $\mathcal{R}$, $lab(q) = U^k$ and $d(p_-, q_-), d(p_+, q_+) \leqslant \epsilon$.

Note that, by Corollary 3, the Hausdorff distance between p and q is bounded from above by $\epsilon + 2R_{\lambda,c} + 2\delta <^{\text{by LPP}} 2\epsilon$.

Now let us separately consider the cases when $n = 1$ and when $n > 1$.

Case 1:$(n = 1)$. In this case, let us partition $q = q_1 q_2 q_3$, where $lab(q_1) = lab(q_3) = Q$. Let us also partition $p = p_1 p_2 p_3$ such that $(p_1)_+$ and $(p_2)_+$ are the closest points on p_{Γ_2} correspondingly to $(q_1)_+$ and to $(q_2)_+$. Since $lab(q_1) = lab(q_3)$ and $d_{Haus}(p, q) < 2\epsilon$, from Lemma 24 it follows that $\|p_1\|, \|p_3\| \leqslant 2\mu\|R\| < \mu\lambda\|R\|$. Then, combining this with (2.42), we get that $\|p_1 p_2\| > 1 - 23\lambda\mu$. But, since $\|U'\| = \big\|p_1 p_2\big\|$ and U' is a geodesic word in $\Gamma\big(G, X\big)$, by Convention 4, we get a contradiction. Thus we are done with the case $n = 1$.

Case 2:$(n > 1)$. In this case, again we partition q into three parts $q = q_1 q_2 q_3$ such that $lab(q_1) = lab(q_3)$ and $lab(q_2)$ is a suffix of U'. Then, since $lab(q) = (U')^n Q$ and $n \geqslant 2$, we get that $\|q_1\| = \|q_3\| > \frac{1}{3}\|q_2\|$, hence $\|q\| < 3\|q_1\|$. Also just like we showed in case $n = 1$, by Lemma 24, in this case $\|q_1\|, \|q_3\| \leqslant 2\mu\|R\|$ as well. Therefore, $\|q\| < 6\mu\|R\|$. But if ρ and μ are chosen sparse enough, then the last inequality, combined with $\|R\| \geqslant \rho$, contradicts the assumption that $lab(q)$ contains a $(\epsilon, 1 - 122\lambda\mu)$-subword associated with R.

(ii). Suppose that U and W are as in the statement of the lemma and we have $U =_G W^k$ for some $k \geqslant 2$. Also, by contradiction, assume that $U \neq_H W^k$.

Let $U' \in X^*$ be a cyclically $(\lambda, c, \epsilon, 1 - 121\lambda\mu)$-reduced word such that $U' \sim_{conj} U$ in G (clearly such a word exists). Then there exists a word $W' \in X^*$ such that $U' =_G (W')^k$.

Now, let $W'' \in X^*$ be a cyclically minimal representative of W' with respect to G. This means that there exists $T \in X^*$ such that $W' =_G TW''T^{-1}$ and W'' has minimal length among all such words. In particular, this means that $U' =_G T(W'')^k T^{-1}$ and W'' is cyclically geodesic in $\Gamma\big(G, X\big)$. Note that, since G is a quotient of H, we get that W'' is also cyclically geodesic in $\Gamma\big(H, X\big)$. Therefore, if λ and c are large enough, then by Lemma 8, $(W'')^k$ is cyclically (λ, c)-quasi-geodesic in $\Gamma\big(H, X\big)$.

Since W'' is conjugate to W' in G and $U' =_G (W')^k$, there exists a $(U', (W'')^k)$-conjugacy diagram over G. Hence there exists a cyclically slender $(U', (W'')^k)$-conjugacy diagram over G. Let Δ be such a diagram. As before, let us denote $\partial\Delta = ABCD$, where $lab(BC), lab(AD)$ are cyclic shifts of $(W'')^k$ and U', respectively, and $lab(AB) = lab(DC)$ are geodesic words in $\Gamma\big(G, X\big)$.

Since U' is cyclically (λ, c)-quasi-geodesic in $\Gamma(H, X)$, by Lemma 19, Δ contains an essential $\mathcal{R}$-cell, Π. Let Γ_1, Γ_2, Γ_3 and Γ_4 be essential ϵ-contiguity subdiagrams connecting Π to AB, BC, CD and DA, respectively. Since we chose Δ to be cyclically slender, by Lemma 28, Γ_2 and Γ_4 are non-empty and

$$(\Pi, \Gamma_2, BC) + (\Pi, \Gamma_4, DA) \geqslant 1 - 121\lambda\mu. \tag{2.41}$$

Also, by (2.40), using triangle inequalities and the fact that $\partial\Pi$ is (λ, c)-quasi-geodesic, we get

$$\|\check{q}_{\Gamma_4}\| \leqslant \big(\|U\| + 2\epsilon\big)\lambda + c \leqslant^{\text{by (2.40)}} \mu\rho.$$

Therefore,

$$(\Pi, \Gamma_4, DA) < \frac{\mu\rho}{\rho} = \mu.$$

Combining this with (2.41), we get

$$(\Pi, \Gamma_2, BC) > (1 - 121\lambda\mu) - \mu > 1 - 122\lambda\mu. \tag{2.42}$$

Since $lab(BC) = (W'')^k$ and W'' is cyclically minimal in $\Gamma(G, X)$, by Part (i) of the current lemma, we get a contradiction.

□

Chapter 3

Algorithms

This section is dedicated to the description of the core algorithms needed for the further exposition.

Convention: In this section, for our purposes, it will be convenient to consider any word $W \in X^*$ as a labeled circle σ such that its label is equal to W when we read it in the clockwise direction starting from some point on it. We denote the length of σ by $\|\sigma\|$.

Throughout this subsection we are interested in words up to their cyclic shifts. Taken this into account, for a *labeled circle* σ, we will say that $lab(\sigma) = W$ if its label is equal to W if we read it in the clockwise direction starting from some point on it. For the circle σ we introduce the following quasi-metrics, $\overrightarrow{d}$ and $\overleftarrow{d}$: for any points $A, B \in \sigma$, $\overrightarrow{d}(A, B)$ is the length of the arc connecting A to B in the clockwise direction and $\overleftarrow{d}(A, B) = \|\sigma\| - \overrightarrow{d}(A, B)$. Also, for $\varepsilon > 0$, we say that B is in ε-neighborhood of A (denoted $B \in \mathcal{N}_\varepsilon(A)$) if either $\overrightarrow{d}(A, B) \leqslant \epsilon$ or $\overleftarrow{d}(A, B) \leqslant \epsilon$.

Oriented arc on σ which, in a clockwise direction, starts at A and ends at B we denote by $[A, B]$. The length of the label of $[A, B]$ we denote by $\big\|[A, B]\big\|$.

In the further exposition, we will regards $\mathcal{R}$-cells defined in Section 2.5, as labeled circles. Therefore, all the notations on circles are applicable for $\mathcal{R}$-cells.

Also, throughout this section $H = \langle X \mid \mathcal{F} \rangle$, $|X| < \infty$, is a δ-hyperbolic group with respect to the generating set X given with its (X, δ)-full-presentation.

3.1 A few auxiliary algorithms

Below we are going to describe a few auxiliary algorithms which will be used to construct effective algorithms for word and conjugacy problems in the limit groups over chains of type (1.3).

Algorithm `QuasiGeodesic-Smoothing`. Let us assume that σ is a labeled circle with a label from X^*, and let $A_1, \dots, A_K$ be points on σ such that the labels of the arcs $[A_1, A_2]$, $[A_2, A_3]$, $\dots [A_K, A_1]$ are $(8\delta + 1)$-local geodesics in $\Gamma(H, X)$. Then we call the points $A_1, \dots, A_K$ (λ, c)-*break-points* of σ.

Below we describe an algorithm which on input receives σ along with the break points $A_1, \dots, A_K$ and outputs another labeled circle σ' such that $lab(\sigma')$ is $(8\delta + 1)$-local geodesic in $\Gamma(H, X)$ and $lab(\sigma') \sim_{conj} lab(\sigma)$ in H.

First, suppose that $lab(\sigma) = W_0 =_H h_0$, and suppose that the break points $A_1, \dots, A_K$ are recorded in a list which may change after each step of the following described procedure.

<u>Step 1.</u> Choose a break point A on the circle σ and search for a pair of points $B, B' \in \mathcal{N}_{8\delta+1}$ such that $\vec{d}(B, B') = 8\delta + 1$ but $[B, B']$ is not a geodesic arc (i.e. $lab([B, B'])$ is not a geodesic word). If such a pair is found, then go to Step 1.1, otherwise, go to Step 1.2.

<u>Step 1.1.</u> If $[B, B']$ is not a geodesic arc, then replace the arc $[B, B']$ of σ with a shorter arc whose label represents the same element of H. As a result, we obtain a new labeled circle whose label represents an element of H conjugate to h_0. Also, add the points B, B' to the list of the break points and remove the break points which are not on the newly obtained circle from the current list of break points.

<u>Step 1.2.</u> If such points B, B' are not found, then remove A from the list of break points.

<u>Step 2.</u> Repeat the procedure of Step 1 for the next break point until there is no break point left in the list.

<u>Step 3.</u> If there is no break point left, then return the current circle.

Clearly the procedure described in Step 1 and Step 2 will eventually halt, since after each call of Step 1 either the newly obtained circle gets shorter (Step 1.1.) or the number of break point in the list decreases. Also, it is clear that the newly obtained labeled circle σ' is such that $lab(\sigma')$ is

$8\delta + 1$-local geodesic and $lab(\sigma) \sim_{conj} lab(\sigma')$ in H.

The following observation will be used for the main algorithm.

Observation 8.0. Suppose that the $A_1, \ldots, A_K \in \sigma$ are enumerated in the clockwise direction and $[A_1, A_K]$ is marked with some number, say, with 0. Suppose also that after replacing an arc of the current circle with a new arc on Step 1.1, we mark the edges of the new arc with the label 0. If σ' is the returned labeled circle of `QuasiGeodesic-Smoothing` algorithm, then, clearly, the edges of σ' which are marked with 0 compose a connected arc.

Now, suppose that $\|\sigma\| - \|\sigma'\| = d$, $d \geqslant 0$. Then, note that Step 1 was called during this procedure not more than d times, and since after Step 1.1 the number of break points in the list increases at most by 1, while after Step 1.2 it decreases by 1, we get that Step 1.2. was called during this procedure at most $K + d$ times. Therefore, Step 1 was called during this procedure at most $d + (K + d) = K + 2d$ times. Consequently, the total time required for this procedure is bounded from above by

$$f_S(\delta, |X|)(K + d), \tag{3.1}$$

where $f_S : \mathbb{N} \to \mathbb{N}$ is a computable function depending only on δ and $|X|$. After summarizing, we get to the following lemma.

Lemma 30. *For input σ, $A_1, \ldots, A_K \in \sigma$ if the output of the algorithm `QuasiGeodesic-Smoothing` is shorter than σ by d, then the time which `QuasiGeodesic-Smoothing` spent before halting is bounded from above by $f_S(\delta, |X|)(K + d)$, where $f_S : \mathbb{N} \to \mathbb{N}$ is a computable function depending only on δ and $|X|$.*

As we already mentioned the procedure of Steps 1 and 2 ends up with a cyclically $8\delta + 1$-local geodesic word. However, as it is apparent from Lemma 2, for large enough constants λ, c, $8\delta + 1$-local geodesiceness implies cyclically (λ, c)-quasi-geodesicness. Therefore, since our primary interest in this procedure is about obtaining cyclically (λ, c)-quasi-geodesic word conjugate to W_0 in H, we name this algorithm `QuasiGeodesic-Smoothing` (with respect to the input σ and $A_1, \ldots, A_K \in \sigma$).

Also, note that if we consider all the points on σ as break points, then the (λ, c)-`smoothing` algorithm becomes the well-known algorithm for finding $(8\delta + 1)$-local geodesic word conjugate to the given word (see, for example, [12]). In case all the points on σ are regarded as break points then we call this algorithm (λ_i, c_i)-`cyclic-reduction`.

Algorithm `ShortLex`. (Shapiro's Algorithm on ShortLex normal forms). As in [24], for a given element $g \in H$, we define the $ShortLex_H$ *normal form* of g with respect to X as lexicographically the least word $W' \in X^*$ such that $W' =_H g$. Analogously, for $W \in X^*$, we denote $W' = ShortLex_H(W)$ if W' is lexicographically the least word in X^* such that $W =_H W'$.

According to Shapiro's theorem described in [24], there is a linear-time algorithm which for any input $W \in X^*$ finds $W' = ShortLex_H(W)$. Moreover, as it follows from the proof of the theorem in [24], the time complexity of this procedure is bounded from above by

$$f_{\mathrm{SL}}(|X|, \delta)\|W\|, \tag{3.2}$$

where f_{SL} is a computable function independent of H and W. We name this algorithm simply `ShortLex`.

3.2 The main algorithm

($(\lambda, c, \epsilon, \eta)$-`cyclic-reduction`.)

As above, let $H = \langle X \rangle$ be a δ-hyperbolic group with (X, δ)-full presentation $H = \langle X \mid \mathcal{F} \rangle$. Let $G = H/ \ll \mathcal{R} \gg$, where $\mathcal{R}$ is a finite set of words satisfying the small cancellation condition $C'(\lambda, c, \epsilon, \mu, \rho)$ for large enough standard parameters $\lambda \succ c \succ \epsilon \succ \mu^{-1} \succ \rho$. Note that, as it follows from 19, the group G is a non-elementary hyperbolic group.

Let $0 < \eta < 1$ be a fixed rational constant such that

$$2\eta - 3/2 > 3\lambda(1 - \eta). \tag{3.3}$$

In this subsection, our goal is to describe an algorithm (see $(\lambda, c, \epsilon, \eta)$-`cyclic-reduction` algorithm below) which for an input word $W \in X^*$ (inputed as a labeled circle), outputs a word $W' \in X^*$ such that $W \sim_{conj} W'$ in G and W' is cyclically $(\lambda, c, \epsilon, \eta)$-reduced in $\Gamma(G, X)$. Moreover, if W and all its cyclic shifts do not contain (ϵ, η)-subwords, then $W \sim_{conj} W'$ in H. The algorithm will be universal in the sense that it does not depend on the choice of H and G. We also would like to note that the main technical difficulties for this algorithm are connected with making it as fast as possible.

Lemma 31. *Suppose that $W \in X^*$ is (λ, c)-quasi-geodesic word in H and W' is a geodesic word in H such that $W =_H W'$. If W contains an (ϵ, η)-subword, then W' contains a $(2\epsilon, \eta)$-subword.*

Observation 8.1. As it follows from Lemma 21, if a cyclic shift of a word $W \in X^*$ contains an (ϵ, η)-subword, then this subword must be associated with a word from $\mathcal{R}$ whose length is bounded from above by $\frac{\lambda(\|W\|+2\epsilon)+c}{\eta}$. This boservation leads us to the following definition.

Define

$$_n\mathcal{R} = \left\{ R \in \mathcal{R} \mid \|R\| \leqslant \frac{\lambda(\|W\| + 2\epsilon) + c}{\eta} \right\}$$

where $n = \|W\|$, and let $\mathcal{C}(_n\mathcal{R})$ be an upper bound of time required for constructing a set of representatives of $_n\mathcal{R}$ up to cyclic shift.

Let us denote

$$L_n = \max\{\|R\| \mid R \in {}_n\mathcal{R}\} \text{ and } l_n = \min\{\|R\| \mid R \in {}_n\mathcal{R}\} \tag{3.4}$$

and

$$\tilde{L}_n = \lceil \lambda(\eta L_n + 2\epsilon) + c \rceil$$

The following simple key observation is the main motivation for considering $\tilde{L}_n$.

Observation 8.2. Suppose that $A_1, \ldots, A_s \in \sigma$ are such that $\overrightarrow{d}(A_i, A_{i+1}) \leqslant \tilde{L}_n$ for $1 \leqslant i < s$ and $\overrightarrow{d}(A_s, A_1) \leqslant \tilde{L}_n$. Then, if σ contains an (ϵ, η)-arc, there exists a point $A_i \in \{A_1, \ldots, A_s\}$ such that the $\tilde{L}_n$-neighborhood of A_i contains an (ϵ, η)-arc.

Let $k_n = \#\{\ {}_n\mathcal{R}/ \sim_{cyclic\ shifts}\ \}$, where $\sim_{cyclic\ shifts}$ is the equivalence relation for cyclic shifts. Now let ${}_n\tilde{\mathcal{R}} \stackrel{def}{=} \{R_1, \ldots, R_{k_n}\} \subset {}_n\mathcal{R}$ be a set of representatives of the set ${}_n\mathcal{R}/ \sim_{cyclic\ shifts}$.

Next, let us partition the elements from ${}_n\tilde{\mathcal{R}}$ in the following way: partition the words $R_i \in_n \tilde{\mathcal{R}}$, $i = 1, \ldots, k_n$, in the form

$$R_i = U_i^1 U_i^2 \ldots U_i^{s_i},$$

where $\lfloor \frac{1}{1-\eta} \rfloor - 1 < s_i \leqslant \lceil \frac{1}{1-\eta} \rceil$, $\|U_i^j\| = \lfloor (1-\eta)\|R_i\| \rfloor$, for $j = 1, \ldots, s_i - 1$, and $\lfloor (1-\eta)\|R_i\| \rfloor \leqslant \|U_i^{s_i}\| < 2\lfloor (1-\eta)\|R_i\| \rfloor$.

Now let us define

$$\begin{gathered}{}_n\mathcal{R}' = \left\{\hat{U}_i^j \stackrel{\text{def}}{=} U_i^1 \ldots U_i^{j-2} U_i^{j+1} U_i^{j+2} \ldots U_i^{s_i} \mid (i,j), 1 \leqslant i \leqslant k_n,\ 1 < j < s_i\right\} \\ \bigcup \{\hat{U}_i^1 \stackrel{\text{def}}{=} U_i^2 U_i^3 \ldots U_i^{s_i - 1}\} \bigcup \{\hat{U}_i^{s_i} \stackrel{\text{def}}{=} U_i^1 U_i^2 \ldots U_i^{s_i - 2}\}\end{gathered}$$

and also, by using the convention $U_i^0 \stackrel{\text{def}}{=} U_i^{s_i}$, define

$${}_n\mathcal{R}'' = \{U_i^{j-1} U_i^j \mid (i,j), 1 \leqslant i \leqslant k_n,\ 1 \leqslant j \leqslant s_i\}. \tag{3.5}$$

Then we have

$$\begin{aligned} (2\eta - 1)\|R_i\| \leqslant \hat{U}_i^j \leqslant (3\eta - 1)\|R_i\|, \\ 2(1-\eta)\|R_i\| \leqslant \|U_i^{j-1} U_i^j\| \leqslant 3(1-\eta)\|R_i\|. \end{aligned} \tag{3.6}$$

One of the motivations for considering the sets ${}_n\mathcal{R}'$ and ${}_n\mathcal{R}''$ is revealed in the following proposition.

Proposition 3. *Let $W \in X^*$ be a word containing a (ϵ, η)-subword V. Suppose $n = \|W\|$. Then V contains a subword V' of the form*

$$V' =_H E_1^{-1} U' E_2,$$

where $U' \in {}_n\mathcal{R}'$ and $E_1, E_2 \in X^*$, $\|E_1\|, \|E_2\| \leqslant 2\epsilon$. Hence V' is a $(2\epsilon, 3\eta - 2)$-subword of W.

Proof. By definition and Observation 8.1, there exist $T_1, T_2 \in X^*$, $\|T_1\|, \|T_2\| \leqslant \epsilon$ and $R \in {}_n\mathcal{R}$ such that for some subword U of R, $\|U\| \geqslant \eta\|R\|$ and

$$V =_H T_1^{-1} U T_2.$$

Let $R = U\bar{U}$. Then, since $\|\bar{U}\| \leqslant (1-\eta)\|R\|$, there exists a word $U_i^{j-1}U_i^j$ such that $\bar{U}$ is a subword of $U_i^{j-1}U_i^j$. But this means that $\hat{U}_i^j$ is a subword of U. Therefore, by the inequality $\epsilon >^{\text{by LPP}} 2R_{\lambda,c} + 2\delta$ and by Corollary 3, we get that there exists a subword V' of V and $E_1, E_2 \in X^*$, $\|E_1\|, \|E_2\| \leqslant 2\epsilon$, such that $V' =_H E_1^{-1}U'E_2$. Finally, since we have $\|\hat{U}_i^j\| > \|R\| - 3\lfloor(1-\eta)\|R\|\rfloor \geqslant (3\eta - 2)\|R\|$, we get that V' is a $(2\epsilon, 3\eta - 2)$-subword of W.

□

Now let us define

$$\mathcal{E}_0({}_n\mathcal{R}') = \{ShortLex(T_1^{-1}\hat{U}_i^j T_2) \mid (i,j), 1 \leqslant i \leqslant k_n,\ 1 \leqslant j \leqslant s_i,$$
$$T_1, T_2 \in X^*, \|T_1\|, \|T_2\| \leqslant 3\epsilon\}.$$

Lemma 32. *The sum of the lengths of the words from $\mathcal{E}_0({}_n\mathcal{R}')$ is bounded from above by $f_{\mathcal{E}_0}(|X|, \epsilon, \eta)L_n k_n$, where $f_{\mathcal{E}_0} : \mathbb{N} \to \mathbb{N}$ is a computable function depending only on $|X|, \epsilon$ and η.*

Proof. Indeed, first of all note that

$$\#\mathcal{E}_0({}_n\mathcal{R}') \leqslant k_n \left\lceil \frac{1}{2(1-\eta)} \right\rceil |X|^{6\epsilon}.$$

Therefore,

$$\sum_{W \in \mathcal{E}_0({}_n\mathcal{R}')} \|W\| \leqslant (\eta' L_n + 6\epsilon)\#\mathcal{E}_0({}_n\mathcal{R}') \leqslant (\eta' L_n + 6\epsilon)k_n \left\lceil \frac{1}{2(1-\eta)} \right\rceil |X|^{6\epsilon}$$
$$< (\eta' + 6\epsilon)\left\lceil \frac{1}{2(1-\eta)} \right\rceil |X|^{6\epsilon} L_n k_n.$$

Now define $f_{\mathcal{E}_0}(|X|, \epsilon, \eta) = (\eta' + 6\epsilon)\left\lceil \frac{1}{2(1-\eta)} \right\rceil |X|^{6\epsilon}$. $\square$

The main motivation for considering the set $\mathcal{E}_0({}_n\mathcal{R}')$ is observed in the following proposition.

Proposition 4. *Suppose that $W \in X^*$ is a (λ, c)-quasi-geodesic word in $\Gamma(H, X)$ containing an (ϵ, η)-subword and $\|W\| \leqslant n$. Then the word $W' = ShortLex_H(W)$ contains a subword from $\mathcal{E}_0({}_n\mathcal{R}')$ which is also a $(3\epsilon, \eta')$-subword.*

Proof. First of all, note that, by Proposition 3, W contains a $(2\epsilon, \eta')$-subword, say, V.

Let us consider a bigon in $\Gamma(H, X)$ with boundary pq^{-1} such that $lab(p) = W$ and $lab(q) = W'$. Also, let p_1 be a subpath on p such that $lab(p_1) = V$. By Corollary 3, the Hausdorff distance between p and q is bounded from above by $2\delta + R_{\lambda,c}$. Therefore, since $\epsilon > 2\delta + R_{\lambda,c}$, we get that there is a subpath q_1 on q such that $d((p_1)_-, (q_1)_-), d((p_1)_+, (q_1)_+) < \epsilon$, which implies that $lab(q_1)$ is a $(3\epsilon, \eta')$-subword of W'.

Now, since all subword of a word in ShortLex form are also in ShortLex form, combining with Observation 8.1, we get that $lab(q_1) \in \mathcal{E}_0({}_n\mathcal{R}')$. $\square$

Now let us describe the $(\lambda, c, \epsilon, \eta)$-`cyclic-reduction` algorithm.

3.2.0.1 Description of $(\lambda, c, \epsilon, \eta)$-`cyclic-reduction`

Input/Output. As an input the algorithm receives a labeled circle σ with $lab(W) \in X^*$ and outputs a word W' such that $W' \sim_{conj} W$ in G and W' is cyclically $(\lambda, c, \epsilon, \eta)$-reduced. Let $\|W\| = n$.

Step 0. Compute W_0 such that W_0 is cyclically $8\delta + 1$-local geodesic (hence, W_0 is (λ, c)-quasi-geodesic in $\Gamma(H, X)$) and $W_0 \sim_{conj} W$ in H.

Let σ_0 be a labeled circle such that $lab(\sigma_0) = W_0$.

Step 1. If $\|\sigma_0\| \geqslant 2\tilde{L}_n$, then partition σ_0 by points $A_1, A_2, \ldots, A_s \in \sigma_0$ such that $\overrightarrow{d}(A_i, A_{i+1}) = \tilde{L}_n$ for $1 \leqslant i < s$ and $\overrightarrow{d}(A_s, A_1) \leqslant \tilde{L}_n$. Then $s = \lfloor \frac{\|\sigma_0\|}{\tilde{L}_m} \rfloor + 1$.

Otherwise, if $\|\sigma_0\| \leqslant 2\tilde{L}_n$ take $A_1 \in \sigma_0$ arbitrarily and define $A_2 \in \sigma_0$ as the opposite to A_1 point on σ_0 in the sense that $\overrightarrow{d}(A_1, A) = \overrightarrow{d}(A, A_1) \pm 1$.

Include the elements $A_1, A_2, \ldots, A_s$ in a list of special points which we simply call `List`.

Let us save the value of σ_0 in a special variable σ' which is by itself a labeled circle.

Now for all elements $A \in$ `List` do the procedure of Step 2 as follows.

Step 2. If $\|\sigma'\| < 2\tilde{L}_n$, then consider the points $B_1, B_2 \in \sigma'$ such that $B_1 = B_2$ and $\overrightarrow{d}(B_1, A) = \overrightarrow{d}(A, B_1) \pm 1$ (thus B_1 is the opposite vertex point of A on σ'). Otherwise, if $\|\sigma'\| \geqslant 2\tilde{L}_n$ choose $B_1, B_2 \in \sigma'$ such that $\overrightarrow{d}(B_1, A) = \overrightarrow{d}(A, B_2) = \tilde{L}_n$. Then go to Step 2.1 as follows.

Step 2.1. Compute $W_A \overset{\text{def}}{=} ShortLex(lab[B_1, B_2])$ and go to Step 2.2.

Step 2.2. Search for a subword from $\mathcal{E}_0({}_n\mathcal{R}')$ in W_A using Aho-Corasick's string search algorithm. (A formal description of Step 2.2 via pseudo-code is given in Algorithm 1).

If such a subword is not found, then conclude that $[B_1, B_2]$ does not contain a (ϵ, η)-subword and go to Step 2.2.1 as follows, otherwise go to Step 2.2.2.

Step 2.2.1. Remove A from `List`. Then, if `List` is not empty, choose another point from `List` and return to Step 2 with the chosen point as the input. Otherwise, return $lab(\sigma')$ and halt.

Step 2.2.2. Suppose that W_A contains a subword from $\mathcal{E}_0({}_n\mathcal{R}')$ of the form $ShortLex(T_1^{-1}\hat{U}_i^j T_2)$. Then,

1. In W_A replace the subword $ShortLex(T_1^{-1}\hat{U}_i^j T_2)$ with the word $T_1^{-1}U_i^{j-1}U_i^j T_2$. Denote

the new word by W'_A,

2. Compute $W''_A \stackrel{\text{def}}{=} ShortLex(W'_A)$ and replace the arc $[B_1, B_2]$ of σ' with a new arc $[B'_1, B'_2]$ such that $lab([B'_1, B'_2]) = W''_A$,

3. Change the value of σ' by prescribing to it the newly obtained labeled circle,

4. Add the points B'_1 and B'_2 to `List`,

5. If the point A was the i-th point which was checked in Step 2, then mark the arc $[B'_1, B'_2]$ of σ' with i,

6. Go to Step 2.2.3 as follows.

Step 2.2.3. Apply the `QuasiGeodesic-Smoothing` algorithm with inputs $lab(\sigma'), B'_1, B'_2$ and then mark all the newly obtained edges during the process of running `QuasiGeodesic-Smoothing`$(lab(\sigma'), B'_1, B'_2)$ with i. Save the newly obtained labeled circle again in the variable σ'.

Step 2.2.4. Suppose that the new labeled circle σ', obtained after Steps 2.2.2 and 2.2.3, has an arc marked with i which is bounded between some points $O, O' \in \sigma'$ (the fact that the edges marked with i form an arc follows from Observation 8.0). Then partition the arc $[O, O']$ with the points $O_1, \ldots, O_t \in [O, O']$ such that $O_1 = O$, $O_2 = O'$ and for $1 \leqslant i < t$, $\overrightarrow{d}(O_i, O_{i+1}) = \tilde{L}_n$ and $\overrightarrow{d}(O_{t-1}, O_t) \leqslant \tilde{L}_n$.

Add the points $O_1, \ldots, O_t$ to `List` and then choose another point from `List` and go to Step 2 with the chosen point as the input.

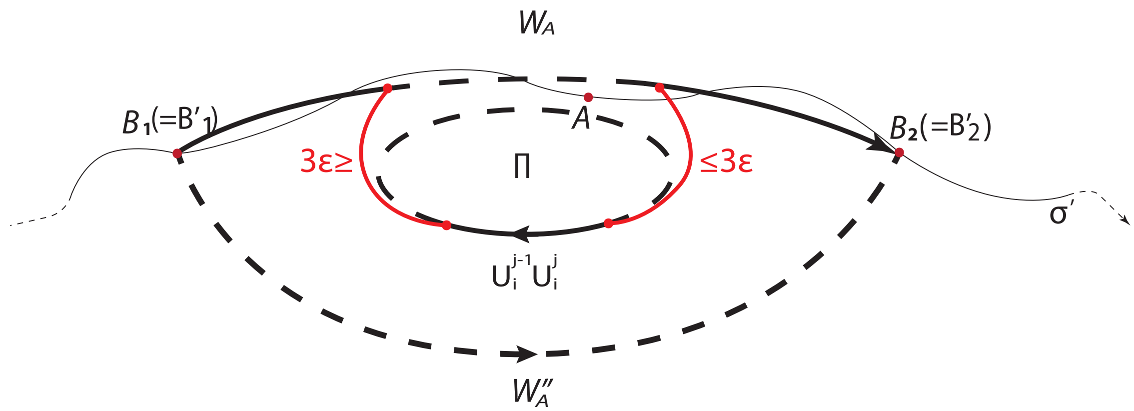

Figure 3.1: Graphical explanation of Steps 2-2.2.4 of $(\lambda, c, \epsilon, \eta)$-`cyclic-reduction` algorithm in case when the result of search in Step 2.2 is positive. In the figure W_A is the ShortLex for of the label of the arc $[B_1, B_2]$ which gets replaced with a shorter arc labeled with label W''_A as in Step 2.2.2.

Below we give a more formal description of Step 2.2 based on Aho-Corasick's famous algorithm (see [3] for the description of the algorithm) for multiple string search.

Algorithm 1 Searching for $(3\epsilon, \eta')$-arcs

Input: $W \in X^*$, $\mathcal{E}_0({}_n\mathcal{R}')$. $\{W$ *is given in its* $ShortLex_H$ *form.*$\}$
Output: An $(3\epsilon, \eta')$-subword of W if such a subword exists and 0 otherwise.
Apply Aho-Corasick's string searching algorithm to find all subwords of W from $\mathcal{E}_0({}_n\mathcal{R}')$ $\{$*For the description of Aho-Corasick's algorithmic see* [3].$\}$
if at least one such subword is found **then**
return one of the found subwords $\{$*By definition, this subword will be a* $(3\epsilon, \eta')$*-subword.* $\}$

elsereturn 0
=0

3.2.0.2 Time complexity of $(\lambda, c, \epsilon, \eta)$-**`cyclic-reduction`**

First, we will estimate the time that $(\lambda, c, \epsilon, \eta)$-`cyclic-reduction` algorithm spends on Step 2. To this end suppose $\sigma_1, \sigma_2, \ldots, \sigma_s$ are the circles which Step 2 outputs in cases when it changes the input circle (i.e. when on Step 2.2 algorithm finds a subword from $\mathcal{E}_0({}_n\mathcal{R}')$). According to Step 2.2.2 (5), this means that for each $1 \leqslant i \leqslant s$, σ_i contains an arc whose edges are marked by i and σ_i does not contain edges marked with $i+1$. Let us denote this arc by q_i and suppose that q_i

replaced an arc p_i of σ_{i-1}.

Note that on Step 2.2.2 the algorithm replaces an arc $[B_1, B_2]$ with an arc $[B_1', B_2']$. For sparse enough standard parameters $\lambda, c, \epsilon, \mu, \rho$ we have $\|[B_1', B_2']\| < \|[B_1, B_2]\|$. Therefore, after each run of Steps 2-2.2.2 either the number of elements in `List` is decreasing or the newly obtained circle is shorter, hence the algorithm eventually halts. On the other hand, it follows from Observation 8.2 and Proposition 4 that the output circle of $(\lambda, c, \epsilon, \eta)$-`cyclic-reduction` does not contain an (ϵ, η)-arc.

Lemma 33. $\frac{\|[B_1', B_2']\|}{\|[B_1, B_2]\|} \leqslant \Lambda_0 < 1$*, where* $\Lambda_0 = \Lambda_0(\lambda, c, \epsilon, \eta, \frac{L_n}{l_n})$ *is a constant depending only on* $\lambda, c, \epsilon, \eta$ *and* $\frac{L_n}{l_n}$.

Proof. Indeed, direct computations show

$$\begin{aligned}\|[B_1', B_2']\| &\leqslant \|[B_1, B_2]\| - |T_1^{-1}\hat{U}_i^j T_2| + \|T_1^{-1}U_i^{j-1}U_i^j T_2\| \\ &\leqslant^{\text{we use (3.6)}} 2\tilde{L}_n - \frac{(2\eta - 1)\|R_i\|}{\lambda} + 12\epsilon + 3(1-\eta)\|R_i\| \\ &\leqslant^{\text{by LPP}} 2\tilde{L}_n - \frac{(2\eta - 3/2 + 3\lambda(1-\eta))\|R_i\|}{\lambda} <^{\text{by (3.3)}} 2\tilde{L}_n.\end{aligned}$$

Therefore, we get that $\frac{\|[B_1', B_2']\|}{\|[B_1, B_2]\|}$ is of the forms described in the statement of the lemma. □

Lemma 34. *For each* $1 \leqslant i \leqslant s$, $\frac{\|q_i\|}{\|p_i\|} \leqslant \Lambda < 1$*, where* $\Lambda = \Lambda(\lambda, c, \epsilon, \eta, \frac{L_n}{l_n})$ *is a constant depending only on* $\lambda, c, \epsilon, \eta$ *and* $\frac{L_n}{l_n}$.

Proof. q_i is obtained from p_i after replacing arcs of p_i after applying Step 2.2.2. (5) and replacing arcs of lengths $8\delta + 1$ with shorter arcs after applying Step 2.2.3. Therefore, taken into account Lemma 33, Λ can be taken as $\Lambda = \min\{\Lambda_0, 8\delta/(8\delta+1)\}$, where Λ_0 is defined as in Lemma 33. □

Corollary 9. $\sum_{i=1}^{s} \|q_i\| \leqslant \frac{\Lambda}{1-\Lambda} n$.

Proof. Indeed, since by Lemma 34, q_i, $1 \leqslant i \leqslant s$ are obtained by replacing an arc p_i of σ_{i-1} of lengths at least $\|q_i\|/\Lambda$, we get that

$$\sum_{i=1}^{s} \|q_i\| \leqslant \Lambda n + \Lambda^2 n + \ldots = \frac{\Lambda}{1-\Lambda} n.$$

□

Lemma 35. *During the run of* $(\lambda, c, \epsilon, \eta)$`-cyclic-reduction` *algorithm, the total number of points added to* `List` *(See Steps 1, 2.2.2 (5) and 2.2.4) is bounded from above by*

$$\frac{2}{(1-\Lambda)\tilde{L}_n} n.$$

Hence Step 2 of $(\lambda, c, \epsilon, \eta)$`-cyclic-reduction` *algorithm is being called not more than* $\frac{2}{(1-\Lambda)\tilde{L}_n} n$ *times for input of lengths* n.

Proof. It follows directly from the description of $(\lambda, c, \epsilon, \eta)$`-cyclic-reduction` algorithm and Corollary 9. □

Corollary 10. *On Step 2.1 in summary —- spends time bounded from above by* $f_1(|X|, \lambda, c, \epsilon, \mu, \eta, k) n^{1+\upsilon}$*, where* $f_1 : \mathbb{N}^7 \to \mathbb{N}$ *is a computable function depending only on* δ *and* $|X|$*,* k *is the number of elements in* $\mathcal{R}$ *up to cyclic shifts, and* $\upsilon = 0$ *if* $k = 1$*, otherwise* $\upsilon = 1$.

Proof. It follows directly from Lemma 35 and from the time complexity properties of the `ShortLex` algorithm. See (3.2).

□

Corollary 11. *On Step 2.2 in summary the algorithm spends time bounded from above by* $f_2(|X|, \lambda, c, \epsilon, \mu, \eta, k) n^{1+\upsilon}$*, where* $f_2 : \mathbb{N}^7 \to \mathbb{N}$ *is a computable function depending only on* δ *and* $|X|$*,* k *is the number of elements in* $\mathcal{R}$ *up to cyclic shifts, and* $\upsilon = 0$ *if* $k = 1$*, otherwise* $\upsilon = 1$.

Proof. Indeed, it follows from Lemma 35 and from the time complexity properties of Aho-Corasick's string search algorithm. See [3]. □

Corollary 12. *On Steps 2.2.1-2.2.4 in summary the algorithm spends time bounded from above by* $f_3(|X|, \lambda, c, \epsilon, \mu, \eta, k) n^{1+\upsilon}$*, where* $f_3 : \mathbb{N}^7 \to \mathbb{N}$ *is a computable function depending only on* δ *and* $|X|$*,* k *is the number of elements in* $\mathcal{R}$ *up to cyclic shifts, and* $\upsilon = 0$ *if* $k = 1$*, otherwise* $\upsilon = 1$.

Proof. Indeed, in terms of time complexity, the hardest part among the Steps 2.2.1-2.2.4 is Step 2.2.3, and taken this into account, the claim of the corollary follows immediately from Lemma 30. □

Lemma 36. *Suppose that the above define set* $_n\mathcal{R}'$ *is already computed. Then there exists a computable function* $\Psi : \mathbb{Q}^7 \to \mathbb{N}$ *such that if the constants* $\lambda, c, \epsilon, \mu^{-1}, \rho$ *are fixed and large enough, then for any word* $W \in X^*$*,* $\|W\| = n$*, a cyclic* $(\lambda, c, \epsilon, \eta)$*-reduction of* W *can be computed in time bounded from above by*

$$\Psi(|X|, \lambda, c, \epsilon, \mu, \eta, k)n^{1+\upsilon} \tag{3.7}$$

where k *is the number of elements in* $\mathcal{R}$ *up to cyclic shifts, and* $\upsilon = 0$ *if* $k = 1$*, otherwise* $\upsilon = 1$.

Proof. Indeed, it directly follows from Corollaries 18, 11 and 17. □

Chapter 4

A subclass of lacunary hyperbolic groups with effectively decidable word problem and G-conjugacy problem

In this section we describe a subclass of lacunary hyperbolic groups for which there is a fast algorithm solving the word problem. As it will be shown in corresponding sections, the groups which are described in Theorems 3, 4 and 5 are constructed so that they belong to that subclass.

4.1 Small cancellation conditions in chains of hyperbolic groups

Let us consider the chain of hyperbolic groups (1.3), that is

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \dots, \tag{4.1}$$

where $\alpha_i = \gamma_{i+1} \circ \beta_i$ is surjective for $i = 1, 2, \ldots$. All the groups in this chain are hyperbolic.

Suppose that for all $i \geqslant 0$, G_i is δ_i hyperbolic and for all $j \geqslant 1$, H_i is δ'_j hyperbolic, where $\delta_i, \delta'_j \in \mathbb{N}$. Also suppose $G_0 = \langle X \mid \mathcal{R}_0 \rangle$ is given with its initial symmetric finite presentation and for all $i \in \mathbb{N}$

$$H_i = G_{i-1} * F(Y_i) / \ll \mathcal{S}_i \gg, \tag{4.2}$$

where $|Y_i| < \infty$, $Y_i \cap \beta_{i-1}(G_{i-1}) = \varnothing$ and $\mathcal{S}_i$ is a finite symmetric set of words from $(X \cup Y_i)^*$, and

$$G_i = H_i / \ll \mathcal{R}_i \gg, \tag{4.3}$$

where $\mathcal{R}_i$ is a finite symmetric set of words from $(X \cup Y_i)^*$ as well.

Let us denote the sequences $(\lambda_i)_{i=1}^{\infty}, (c_i)_{i=1}^{\infty}, (\epsilon_i)_{i=1}^{\infty}$, $(\mu_i)_{i=1}^{\infty}$, $(\rho_i)_{i=1}^{\infty}$ by $\boldsymbol{\lambda}$, $\boldsymbol{c}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\rho}$, respectively.

Definition 4.1 (Small cancellation conditions $C'\big(\boldsymbol{\lambda}, \boldsymbol{c}, \boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\rho}\big)$ and $C''\big(\boldsymbol{\lambda}, \boldsymbol{c}, \boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\rho}\big)$**).** We say that the chain (4.1) satisfies (alternatively, based on the context, we may say $\bar{G} = \lim_i (G_i, \alpha_i)$ satisfies) the

$$C'\big(\boldsymbol{\lambda}, \boldsymbol{c}, \boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\rho}\big)\text{-condition}$$

of small cancellation, if

(a) The set $\mathcal{R}_i$ satisfies the $C'(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$-condition with respect to $H_i = \langle X \bigcup \cup_{j=1}^{i} Y_j \rangle$;

(b) The following sequences are computable:

$$\begin{gathered} (Y_i)_{i=1}^{\infty}, (\mathcal{R}_i)_{i=0}^{\infty}, (\mathcal{S}_i)_{i=1}^{\infty} \text{ and} \\ (\delta_i)_{i=0}^{\infty}, (\delta'_i)_{i=1}^{\infty}, (\lambda_i)_{i=1}^{\infty}, (c_i)_{i=1}^{\infty}, (\epsilon_i)_{i=1}^{\infty}, (\mu_i)_{i=1}^{\infty}, (\rho_i)_{i=1}^{\infty}. \end{gathered} \tag{4.4}$$

That is there exists an algorithm which on input $i \geqslant 1$ returns $(Y_i, \mathcal{R}_i, \mathcal{S}_i, \delta_i, \delta'_i, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$.

If, in addition, the following condition is satisfied, then in the notations we replace $C'(\boldsymbol{\lambda}, \boldsymbol{c}, \boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\rho})$ with $C''(\boldsymbol{\lambda}, \boldsymbol{c}, \boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\rho})$.

(d) For each pair (i, j), $0 < i < j$, and each $R_i \in \mathcal{R}_i$, $R_j \in \mathcal{R}_j$, there are no subwords U_i and U_j of R_i and R_j, respectively, such that $\|U_i\| \geqslant \mu_i \|R_i\|$ and there exist $v_1, v_2 \in (X \cup Y_i)^*$, $\|v_1\|, \|v_2\| \leqslant \epsilon_i$, such that $v_1 U_i v_2 U_j =_{H_i} 1$.

Definition 4.2 (Rank of contiguity subdiagrams)**.** We say that a diagram over (4.1) is a ϵ_i-contiguity subdiagram of rank i if the diagram can be regarded as a ϵ_i-contiguity subdiagram over the quotient $G_i = H_i / \ll \mathcal{R}_i \gg$.

4.2 An auxiliary theorem

Now let $G = \langle H \mid \mathcal{R} \rangle$ be fixed and suppose $\mathcal{R}$ satisfies the $C'(\lambda, c, \epsilon, \mu, \rho)$ small cancellation condition. Let $\eta = 1 - 23\mu$. Then the following theorem holds.

Theorem 11. *Using the above described setting, suppose that $\lambda \succ c \succ \epsilon \succ \mu^{-1} \succ \rho$ are large enough. Then there exists a computable function $\Psi : \mathbb{Q}^6 \to \mathbb{Q}$ such that for any given $W \in X^*$, the checking $W =_G 1$ can be done in time bounded from above by*

$$a\Psi(|X|, \lambda, c, \epsilon, \mu, k)n^{1+\upsilon} \tag{4.5}$$

where $n = \|W\|$, k is the number of elements in $\mathcal{R}$ up to cyclic shifts, and $\varepsilon = 0$ when $k = 1$ and $\varepsilon = 1$ when $k > 1$.

Proof. First of all, notice that if ρ are large enough then the restrictions put on η and η' in the beginning of Subsection 3.2 are satisfied if η is defined as $\eta = 1 - 23\mu$.

For the given word $W \in X^*$, let σ be a labeled circle such that $lab(\sigma) = W$. Let σ' be the output of the $(\lambda, c, \epsilon, \eta)$-cyclic-reduction algorithm, and let $lab(\sigma') = W'$ for $W' \in X^*$. We claim that W' is empty if and only if $W =_G 1$.

First of all, since $W' \sim_{conj} W$ in G, the emptiness of W' would imply that $W =_G 1$. Now let us prove the opposite. Suppose that W' is not empty. Then $W' \neq_H 1$, because W' is $8\delta + 1$-local geodesic word in $\Gamma(H, X)$ and the only $8\delta + 1$-geodesic word in $\Gamma(H, X)$ which represents the trivial element of H is the empty word.

On the other hand, since W' is a (λ, c)-quasi-geodesic word in $\Gamma(H, X)$, by Lemma 19, the equation $W' =_G 1$ would imply that W' contains a $(\epsilon, 1 - 23\mu)$-subword. But since W' is an output of the $(\lambda, c, \epsilon, \eta)$-cyclic-reduction algorithm, this cannot happen. A contradiction. Therefore, it must be that W' is empty if and only if $W =_G 1$.

Now the complexity formula in the statement of the lemma directly follows from Lemma 36.

□

Remark 8. In the settings of the current subsection, for any word $W \in X^*$, $W =_G 1$ if and only if the $(\lambda, c, \epsilon, \eta)$-cyclic-reduction algorithm returns the empty word for input W, as it was shown in the proof of Theorem 11.

4.3 The definition of the subclass

Definition 4.3 (SP-relation)**.** Let $H = \langle X \rangle$, $|X| < \infty$, be a δ-hyperbolic group, where δ is a given positive integer. We say that the 5-tuple of positive number $(\lambda, c, \epsilon, \mu, \rho)$ satisfies the standard parameters relation, or briefly SP-relation with respect to (H, δ) if the following holds.

1. $\lambda, c, \epsilon, \mu, \rho$ with respect to (H, δ) satisfy all the restrictions and relations put on $\lambda, c, \epsilon, \mu, \rho$ for all the lemmas and theorems above (see Section 2.5),
2. $\lambda, c, \epsilon, 1/\mu, \rho \in \mathbb{N}$, and
3. if we define $\eta = 1 - 121\lambda\mu$ and $\eta' = 3\eta - 2$, then η and η' satisfy all the restrictions put on η and η' in the beginning of Subsection 3.2.

Let us return to the chain of hyperbolic groups given by (1.3) in the introduction. That is the chain of hyperbolic group homomorphisms:

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \dots, \tag{4.6}$$

where $\alpha_i = \gamma_{i+1} \circ \beta_i$ is surjective for $i = 1, 2, \dots$. Recall that in Subsection 4.1 we described additional settings for this chain. Namely, for all integers i, $i \geqslant 0$, G_i is δ_i hyperbolic and for all $j \geqslant 1$, H_i is δ'_j hyperbolic with respect to the generating set $X \cup \bar{Y}_i$, where $\bar{Y}_i = \cup_{j=1}^{i} Y_j$ and $\delta_i, \delta'_j \in \mathbb{N}$. Also we suppose $G_0 = \langle X \mid \mathcal{R}_0 \rangle$ is given with its initial symmetric finite presentation and for all $i \in \mathbb{N}$,

$$H_i = G_{i-1} * F(Y_i) / \ll \mathcal{S}_i \gg, \tag{4.7}$$

where $|Y_i| < \infty$, $Y_i \cap \beta_{i-1}(G_{i-1}) = \varnothing$ and $\mathcal{S}_i$ is a finite symmetric set of words from $(X \cup Y_i)^*$, and

$$G_i = H_i / \ll \mathcal{R}_i \gg, \tag{4.8}$$

where $\mathcal{R}_i$ is a finite symmetric set of words from $(X \cup Y_i)^*$. Then

$$\bar{G} \overset{\text{def}}{=} \lim_i (G_i, \alpha_i).$$

In addition, we suppose that the fixed sequences $(\lambda_i)_{i=1}^{\infty}$, $(c_i)_{i=1}^{\infty}$, $(\epsilon_i)_{i=1}^{\infty}$, $(1/\mu_i)_{i=1}^{\infty}$, $(\rho_i)_{i=1}^{\infty}$ of positive integers are such that the chain (4.6) satisfies the small cancellation condition $C'\big(\boldsymbol{\lambda}, \boldsymbol{c}, \boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\rho}\big)$. Even more, hereafter we will assume that for all $i \geqslant 1$, the 5-tuple $(\lambda_i, c_i, \epsilon_i, 1 - 121\lambda_i\mu_i)$ satisfies the SP-relation described in Definition 4.3.

Also suppose that the sequences

$$\begin{gathered} (Y_i)_{i=1}^{\infty}, (\mathcal{R}_i)_{i=0}^{\infty}, (\mathcal{S}_i)_{i=1}^{\infty} \text{ and} \\ (\delta_i)_{i=0}^{\infty}, (\delta'_i)_{i=1}^{\infty}, (\lambda_i)_{i=1}^{\infty}, (c_i)_{i=1}^{\infty}, (\epsilon_i)_{i=1}^{\infty}, (\mu_i)_{i=1}^{\infty}, (\rho_i)_{i=1}^{\infty} \end{gathered} \tag{4.9}$$

are computable sequences of integers, i.e. there exists an algorithm which on input $i \geqslant 1$ returns $(Y_i, \mathcal{R}_i, \mathcal{S}_i, \delta_i, \delta'_i, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$.

Let us assume that $\Psi : \mathbb{Q}^7 \rightarrow \mathbb{Q}$ is a fixed computable function satisfying the conditions defined in Theorem 11 (see expression (4.5)). Let us denote $\Psi_i = \Psi(|X \cup \bar{Y}_i|, \lambda_i, c_i, \epsilon_i, \mu_i, k_i)$, where k_i is the cardinality of $\mathcal{R}_i$ up to cyclic shifts of its elements.

Let $g_1, g_2, \ldots : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a sequence of increasing functions such that for all $i \in \mathbb{N}$, $g_i^{-1}|_{\mathbb{N}}$ is integer valued and computable and

$$g_i = o(g_j) \text{ whenever } i > j.$$

For all $i \in \mathbb{N}$, let $f_{\bar{\rho}} : \mathbb{N} \rightarrow \mathbb{N}$ be a fixed computable function such that for $\bar{\rho}_i = f_{\bar{\rho}}(i)$ the 5-tuple

$(\lambda_i, c_i, \epsilon_i, \mu_i, \bar{\rho}_i)$ satisfies the SP-relation, and in addition,

$$\bar{\rho}_i \geqslant \frac{\lambda_i(g_i^{-1}(\Psi_i) + 2\epsilon_i) + c_i}{1 - 23\mu_i}. \tag{4.10}$$

Clearly, since $i \mapsto \frac{\lambda_i(g_i^{-1}(\Psi_i)+2\epsilon_i)+c_i}{1-23\mu_i}$ is computable, such functions $f_{\bar{\rho}}$ do exist.

Hereafter, by lowest parameter principle, we will always assume that $\rho_i \geqslant \bar{\rho}_i$ for all $i \in \mathbb{N}$.

Let us define $\xi, \bar{\xi} : \mathbb{N} \to \mathbb{N}$ as follows

$$\bar{\xi}(i) = \frac{(1 - 23\mu_i)\bar{\rho}_i - c_i}{\lambda_i} - 2\epsilon_i \text{ and } \xi(i) = \frac{(1 - 23\mu_i)\rho_i - c_i}{\lambda_i} - 2\epsilon_i.$$

Note that, in this notations, (4.10) immediately implies

$$g_i\big(\bar{\xi}(i)\big) \geqslant \Psi_i \text{ for } \ i = 1, 2, \ldots. \tag{4.11}$$

Lemma 37. *Let $W \in X^*$ and $W =_{\bar{G}} 1$, but $W \neq_{G_0} 1$. Suppose $i \geqslant 1$ is such that $W =_{G_i} 1$, but $W \neq_{G_{i-1}} 1$ (i.e. i is the minimum index such that $W =_{G_i} 1$). Then*

$$\|W\| > \xi(i).$$

Proof. First, note that, since the map $\beta_i : G_{i-1} \to H_i$ is an embedding, the relation $W \neq_{G_{i-1}} 1$ implies that $W \neq_{H_i} 1$.

Now let $W' \in (X \cup Y_i)^*$ be the (λ_i, c_i)-cyclic-reduction of W over $\Gamma(H_i, X \cup Y_i)$. Then, since $W \sim_{conj} W'$ in H_i and $W \neq_{H_i} 1$, we get that $W' \neq_{H_i} 1$. Therefore, if Δ is a reduced disk diagram over G_i with the boundary label W' then, by Lemma 19, Δ contains an $\mathcal{R}_i$-cell Π connected to $\partial\Delta$ by a ϵ_i-contiguity subdiagram Γ such that $(\Pi, \Gamma, \partial\Delta) > 1 - 23\mu_i > \eta_i$. Therefore, by the triangle inequality, we have

$$\|W\| \geqslant \|W'\| \geqslant \|\hat{q}_\Gamma\| \geqslant \frac{\|\check{q}_\Gamma\| - c_i}{\lambda_i} - 2\epsilon_i > \frac{\eta_i\|\Pi\| - c_i}{\lambda_i} - 2\epsilon_i \geqslant \frac{\eta_i\rho_i - c_i}{\lambda_i} - 2\epsilon_i = \xi(i).$$

$\square$

Remark 9. Note that, since $\xi(i) \geqslant \bar{\xi}(i)$, in the setting of Lemma 37, Lemma 37 implies $\|W\| > \bar{\xi}(i)$.

Definition 4.4 (i-th level data)**.** For any fixed $i \in \mathbb{N}$, the below described list of data we call the i-th level data for the chain (4.6).

1. The slimness constant $\delta'_i \in \mathbb{N}$ of $\Gamma(H_i, X \cup \bar{Y}_i)$, where recall that $\bar{Y}_i = \cup_{j=1}^{i} Y_j$;
2. The $(X \cup \bar{Y}_i, \delta'_i)$-full-presentation $H_i = \langle X \cup \bar{Y}_i \mid \mathcal{F}_i \rangle$ of H_i;
3. The constants δ_{i-1}, δ'_i, λ_i, c_i, ϵ_i, μ_i, $\bar{\rho}_i$;
4. $\bar{\xi}(i)$.

Note that, since the sequences (4.9) are computable by our assumption, there exists an algorithm which computes the i-the level data, i.e. there exists a (deterministic) Turing machine $\mathcal{TM}$ which, for the input $i \in \mathbb{N}$, outputs the i-th level data. Indeed, the computability of the data from parts (1), (2) and (4) of Definition 4.4 is straightforward.

Let $\mathcal{TM}$ be a fixed deterministic Turing machine which, for all inputs $i \geqslant 1$, computes the i-th level data for the presentation (4.6) of $\bar{G}$. Suppose that $\phi : \mathbb{N} \to \mathbb{N}$ is a recursive function, such that for $i \in \mathbb{N}$, $\phi(i)$ is the number of steps $\mathcal{TM}$ makes after input i before it halts. Let $\Phi : \mathbb{N} \to \mathbb{N}$ be a function defined as $\Phi(i) = \sum_{j=1}^{i} \phi(i)$, for $i \in \mathbb{N}$.

Let $\mathcal{I} : \mathbb{N} \to \mathbb{N}$ be the integer valued function such that

$$\Phi(\mathcal{I}(n)) \leqslant n < \Phi(\mathcal{I}(n) + 1).$$

In other words, if we run $\mathcal{TM}$ consecutively for inputs $i = 1, 2, \ldots$, then after the n-th step $\mathcal{I}(n)$-th level data will be computed but $\mathcal{I}(n) + 1$-th level data will not.

Now for $i, n \in \mathbb{N}$, in analogy with the set (3.5), let us define

$${}_n\mathcal{R}_i = \left\{ R \mid R \in \mathcal{R}_i, \|R\| \leqslant \frac{\lambda_i(n + 2\epsilon_i) + c_i}{1 - 23\mu_i} \right\}.$$

The motivation behind the definition of ${}_n\mathcal{R}_i$ is that if a word $W \in X^*$, $\|W\| = n$, is not trivial in H_i but is trivial in G_i, then W must be trivial also in the group $H_i / \ll_n \mathcal{R}_i \gg$. This follows from Lemma 19.

Since, by our assumptions, the sequences (4.9) are computable, the sets ${}_n\mathcal{R}_i$ are computable too, i.e. there exists an algorithm which for input (n, i) returns ${}_n\mathcal{R}_i$. Let $\mathcal{C} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a (time-constructible) function such that, for some fixed Turing machine computing the words ${}_n\mathcal{R}_i$, $\mathcal{C}(n, i)$ is the time the machine spends after the input (n, i) before it halts. Define

$$\mathcal{C}_{{}_n\mathcal{R}_i} = \max\{\mathcal{C}(n, j) \mid 1 \leqslant j \leqslant i\}.$$

The main theorem of this section is the following.

Theorem 12. *If the standard parameters are sparse enough, the word problem in $\bar{G}$ can be solved in time*

$$\mathcal{O}\big(\mathcal{C}_{{}_n\mathcal{R}_{\mathcal{I}(n)}} + g_k(n) n^{1+\upsilon}\big),$$

where $k \in \mathbb{N}$ is any positive integer, n is the length of the input word from X^* and $\upsilon = 0$ if for all but finitely many $i \geqslant 1$, $\mathcal{R}_i$ contains one word up to cyclic shifts, otherwise, $\upsilon = 1$.

Proof. For a given word $W \in X^*$, first of all, without loss of generality assume that $\|W\| \geqslant \xi(1)$ and $W \neq_{G_0} 1$. Now, to check whether $W =_{\bar{G}} 1$ or not, we can apply the following procedure:

S-1. Run the Turing machine $\mathcal{TM}$ consecutively for inputs $k = 1, 2, \ldots$ and stop after exactly $\|W\|$ steps. Suppose that, as a result, the i_0-th level data is constructed, but the $(i_0 + 1)$-st level data is not constructed, i.e. $i_0 = \mathcal{I}(\|W\|)$;

S-2. Find the maximum integer index i_1 from the interval $[1, i_0]$ such that $\bar{\xi}(i_1) \leqslant \|W\|$;

S-3. Construct the set ${}_n\mathcal{R}_{i_1}$, where $n = \|W\|$;

S-4. Run the $(\lambda_{i_1}, c_{i_1}, \epsilon_{i_1}, 1 - 121\lambda_{i_1}\mu_{i_1})$-`cyclic-reduction` algorithm with input circle σ such that $lab(\sigma) = W$. Note that, in order to run this algorithm, we need the i_1-th level data and the set ${}_n\mathcal{R}_{i_1}$.

Note that if $W =_{\bar{G}} 1$, then there is minimum $j_0 \geqslant 1$ (recall that we assumed $W \neq_{G_0} 1$) such that

$W =_{G_{j_0}} 1$.

Claim. $j_0 \leqslant i_0$.

Proof of the claim. First of all, we have $\Phi(i_0) \leqslant \|W\| < \Phi(i_0 + 1)$. We have

$$\xi(j_0) \overset{\text{by def}}{=} \frac{(1 - 23\mu_{j_0})\rho_{j_0} - 2\lambda_{j_0}\epsilon_{j_0} - c_{j_0}}{\lambda_{j_0}} \geqslant^{byLPP} \Phi(j_0).$$

Therefore, since by Lemma 37 we have $\|W\| > \xi(j_0)$, we get $\|W\| > \Phi(j_0)$. On the other hand, since $\|W\| < \Phi(i_0 + 1)$ and Φ is an increasing function, from the last inequality we get $i_0 + 1 > j_0$. Therefore, $i_0 \geqslant j_0$. The claim is proved. □

Since, by the above claim, $j_0 < i_0$, and by Remark 9, $\bar{\eta}(j_0) < \|W\|$, in view of the way i_1 was defined, we get that $j_0 \leqslant i_1$. Therefore, the equality $W =_{G_{j_0}} 1$ implies $W =_{G_{i_1}} 1$. Thus $W =_{\bar{G}} 1$ if and only if $W =_{G_{i_1}} 1$, hence on step S-4 the $(\lambda_{i_1}, c_{i_1}, \epsilon_{i_1}, \eta_{i_1})$-cyclic-reduction algorithm returns empty word for input W (see Remark 8).

Now we are in a position to show that the time complexity estimations in the statement of the theorem, in fact, are true. For that reason, first, notice that on steps S-1 and S-3 the procedure spends $\|W\| + \mathcal{C}_{\mathcal{R}^n_m} = \mathcal{O}\big(\mathcal{C}_{\mathcal{R}^n_m} + n^{1+\upsilon} g_{j_0}(n)\big)$ time. Next, since $i_0 < \|W\|$ and since before the step S-2 the i-th level data already was constructed for $i = 1, 2, \ldots, i_0$, we get that on step S-2 the procedure spends $\mathcal{O}(\|W\|)$ time. Finally, on step S-4, by Theorem 11, the procedure spends $a\Psi\big(|X \cup Y_{i_1}|, \lambda_{i_1}, c_{i_1}, \epsilon_{i_1}, \mu_{i_1}, k_{i_1}\big) n^{1+\upsilon} = a\Psi_{i_1} n^{1+\upsilon}$ time, where a is a constant not depending on $\bar{G}$ and k_{i_1} is the number of elements in $\mathcal{R}_{i_1}$ up to cyclic shifts. Since $\|W\| > \bar{\xi}(i_1)$ and g_{i_1} is increasing, by (4.11) we get $a\Psi_{i_1} n^{1+\upsilon} = \mathcal{O}\big(g_{i_1}(n) n^{1+\upsilon}\big)$. Thus we confirmed the estimations in the statement of the lemma.

□

Remark 10. Note that in Theorem 12, we did not put any restrictions on the relators $\mathcal{S}_i$, $i = 1, 2, \ldots$, other then that they are recursively enumerable and make the groups $H_i = G_{i-1} * F(Y_i) / \ll \mathcal{S}_i \gg$ hyperbolic. However, when instead of the word problem we consider conjugacy problem in $\bar{G}$, an analogue to the statement of Theorem 12 no longer holds unless the sets $\mathcal{S}_i$ possess additional properties. In fact, the group $G_{\mathcal{L}}$, constructed in the proof of Theorem 5, has the structural properties of the group $\bar{G}$ from Theorem 12 but, nevertheless, the conjugacy problem is undecidable in it whenever the underlying set $\mathcal{L}$ is not recursive.

However, if we restrict ourselves from the conjugacy problem to the so called G-conjugacy problem, then the analogue of Theorem 12 holds as it is shown in Theorem 14.

Remark 11. Note that in the proof of Theorem 12 we, in particular, showed that the construction of the $\mathcal{I}(n)$-th level data and the implementation of the $(\lambda_{i_1}, c_{i_1}, \epsilon_{i_1}, \eta_{i_1})$-`cyclic-reduction` algorithm, whenever ${}_n\mathcal{R}_{i_1}$ is not empty, can be done in time $\mathcal{O}\big(\mathcal{C}_{{}_n\mathcal{R}_{\mathcal{I}(n)}} + g_k(n)n^{1+\upsilon}\big)$.

Theorem 13. *If the standard parameters are sparse enough, then the group $\bar{G}$ from Theorem 12 is lacunary hyperbolic.*

Proof. First of all, the group $\bar{G}$ is an inductive limit of groups G_i, $i \in \mathbb{N}$, all of which are hyperbolic. More precisely, $\bar{G}$ is the inductive limit of the following sequence

$$G_0 \overset{\alpha_0}{\twoheadrightarrow} G_1 \overset{\alpha_1}{\twoheadrightarrow} \ldots .$$

An immediate corollary of Lemma 37 is that, for all $i \in \mathbb{N}$, the radius of $\alpha_i : G_i \to G_{i+1}$, which we denote by r_i, satisfies the following inequality

$$\xi(i+1) \overset{\text{by def}}{=} \frac{(1-23\mu_{i+1})\rho_{i+1} - c_{i+1}}{\lambda_{i+1}} - 2\epsilon_{i+1} < r_i.$$

Combining the last inequality with the inequality $\xi(i+1) \leqslant \delta_i \Phi(i+1)$, we get $\delta_i\Phi(i+1) < r_i$. Therefore,

$$\lim_{i\to\infty} \frac{\delta_i}{r_i} \leqslant \lim_{i\to\infty} \frac{\delta_i}{\delta_i\Phi(i+1)} = \lim_{i\to\infty} \frac{1}{\Phi(i+1)} = 0,$$

hence, by Lemma 16, this means that $\bar{G}$ is lacunary hyperbolic. $\square$

4.4 G- and H- conjugacy problems in $\bar{G}$. Effectiveness of the G-conjugacy problem in $\bar{G}$

The main goal of this subsection is to define the G-conjugacy problem for sequences of type (4.6) (see Definition 4.6) and then show that the G-conjugacy problem is effectively solvable when the sequences $(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)_{i=1}^{\infty}$ are sparse enough (see Theorem 14).

Let $\bar{G}$ be the group defined in Subsection 4.3 which also carries all the properties described there.

Definition 4.5 (G- and H-conjugates)**.** Let $U, V \in X^*$. Then we say that U is G-conjugate to V in $\bar{G}$ if either $U \sim_{conj} V$ in G_0 or there exists $i \in \mathbb{N}$ such that $U \sim_{conj} V$ in G_i but $U \nsim_{conj} V$ in H_i.

Analogously, if there exists $i \in \mathbb{N}$ such that $U \sim_{conj} V$ in H_i, but $U \nsim_{conj} V$ in G_{i-1}, then we say that U is H-conjugate to V in $\bar{G}$.

Definition 4.6 (G- and H-conjugacy problems)**.** For the presentation (4.6) of $\bar{G}$ the G-conjugacy problem asks whether there is an algorithm which for any pair of input words $U, V \in X^*$, decides whether U is G-conjugate to V in $\bar{G}$ or not. H-conjugacy problem is defined analogously.

Let us define $\zeta : \mathbb{N} \to \mathbb{N}$ as

$$\zeta(i) = \frac{(1 - 121\lambda_i\mu_i)\rho_i - 2c_i}{\lambda_i} - 4\epsilon_i.$$

Lemma 38. *Suppose that the standard parameters are sparse enough, and $U, V \in X^*$ are such that U is G-conjugate to V in $\bar{G}$. Then there exists $i \in \mathbb{N}$ such that $\zeta(i) \leqslant \|U\| + \|V\|$, $i \leqslant \mathcal{I}(\|U\| + \|V\|)$ and $U \sim_{conj} V$ in G_i, but $U \nsim_{conj} V$ in H_i.*

Proof. If $U \sim_{conj} V$ in G_0 then the statement is obvious. Now, without loss of generality assume that $U \nsim_{conj} V$ in G_0. Then there exists a minimal $i \in \mathbb{N}$ such that $U \sim_{conj} V$ in G_i, but $U \nsim_{conj} V$ in H_i. Suppose that $U', V' \in X^*$ are the (λ_i, c_i)-cyclic-reductions of U and V, respectively.

First, let us show that $\zeta(i) \leqslant \|U\| + \|V\|$. For that purpose, let us separately consider two different cases. The first case is when at least one of U', V', say U', is not cyclically $(\lambda_i, c_i, \epsilon_i, 1 -$

$121\lambda_i\mu_i)$-reduced. The second case is when both U' and V' are cyclically $(\lambda_i, c_i, \epsilon_i, 1-121\lambda_i\mu_i)$-reduced.

For the first case, by definition, some cyclic shift U'' of U' contains a $(\epsilon_i, 1-121\lambda_i\mu_i)$-subword. Therefore, by definition and by the triangle inequality,

$$\|U\| + \|V\| \geqslant \|U''\| \geqslant \frac{(1-121\lambda_i\mu_i)\rho_i - c_i}{\lambda_i} - 2\epsilon_i > \zeta(i). \tag{4.12}$$

Now let us consider the second case, i.e. when both U' and V' are $(\lambda_i, c_i, \epsilon_i, 1-121\lambda_i\mu_i)$-reduced. In this case, there exists a reduced cyclically slender (U', V')-conjugacy diagram Δ over $G_i = H_i / \ll \mathcal{R}_i \gg$ which contains an $\mathcal{R}_i$-cell. Let $\partial\Delta = ABCD$ and $lab(BC) = U'', lab(AD) = V''$, where U'' and V'' are some cyclic shifts of U' and V', respectively. Then, by Lemma 28, there exists an essential $\mathcal{R}_i$-cell Π in Δ connected to AB, BC, CD and DA by Γ_1, Γ_2, Γ_3 and Γ_4, respectively, such that

1. Γ_2 and Γ_4 are non-empty;
2. $(\Pi, \Gamma_2, BC) + (\Pi, \Gamma_4, DA) \geqslant 1 - 121\lambda_i\mu_i$; and

Therefore,

$$\begin{aligned}
\|U\| + \|V\| &\geqslant \|U'\| + \|V'\| = \|U''\| + \|V''\| \geqslant \|\hat{q}_{\Gamma_2}\| + \|\hat{q}_{\Gamma_4}\| \\
&\geqslant \left(\frac{(\Pi, \Gamma_2, \hat{q}_{\Gamma_2})\|\Pi\| - c_i}{\lambda_i} - 2\epsilon_i\right) + \left(\frac{(\Pi, \Gamma_4, \hat{q}_{\Gamma_4})\|\Pi\| - c_i}{\lambda_i} - 2\epsilon_i\right) \\
&\geqslant \frac{(1-121\lambda_i\mu_i)\|\Pi\| - 2c_i}{\lambda_i} - 4\epsilon_i \geqslant \frac{(1-121\lambda_i\mu_i)\rho_i - 2c_i}{\lambda_i} - 4\epsilon_i = \zeta(i).
\end{aligned} \tag{4.13}$$

The conclusion from (4.12) and (4.13) is that if $i \in \mathbb{N}$, $U \sim_{conj} V$ in G_i, but $U \nsim_{conj} V$ in H_i, then

$$\|U\| + \|V\| \geqslant \zeta(i).$$

Now let us show that $i \leqslant \mathcal{I}(\|U\| + \|V\|)$.

From the definition of $\mathcal{I}(\|U\| + \|V\|)$ it follows that $\Phi(\mathcal{I}(\|U\| + \|V\|) + 1) > \|U\| + \|V\|$. Therefore, from the last two inequalities we get

$$\frac{\delta_{i-1}\Phi(\mathcal{I}(\|U\| + \|V\|) + 1) + 4\lambda_i\epsilon_i + 2c_i}{1 - 121\lambda_i\mu_i} > \rho_i \geqslant^{\text{by LPP}} \frac{\delta_{i-1}\Phi(i) + 4\lambda_i\epsilon_i + 2c_i}{1 - 121\lambda_i\mu_i},$$

which implies that $\mathcal{I}(\|U\| + \|V\|) \geqslant i$. Thus the lemma is proved.

$\square$

An obvious corollary from Lemma 38 is the following lemma.

Lemma 39. *If $U \nsim_{conj} V$ in $G_{\mathcal{I}(n)}$, but $U \sim_{conj} V$ in $\bar{G}$, then U is H-conjugate to V in $\bar{G}$.*

Theorem 14. *If the standard parameters are sparse enough and the function $f(n) \overset{def}{=} \mathcal{C}_{n\mathcal{R}_{\mathcal{I}(n)}}$ is bounded by a polynomial, then the G-conjugacy problem in $\bar{G}$ is solvable in polynomial time.*

Proof. For any given words $U, V \in X^*$, by definition, U being G-conjugate to V in $\bar{G}$ means that either $U \sim_{conj} V$ in G_0 or there exists $i \geqslant 1$ such that $U \sim_{conj} V$ in G_i but $U \nsim_{conj} V$ in H_i. If it is so, then, by Lemma 39, $i \leqslant \mathcal{I}(n)$, where $n = \|U\| + \|V\|$.

From what we said, it becomes apparent that in order to show that U is G-conjugate to V in $\bar{G}$ it is enough to check if $U \sim_{conj} V$ in G_0 and if it is not, then for each $1 \leqslant i \leqslant \mathcal{I}(n)$ check whether

- $U \nsim_{conj} V$ in H_i, and
- $U \sim_{conj} V$ in G_i.

Now without loss of generality let us assume that $U \nsim_{conj} V$ in G_0.

Let $U', V' \in (X \cup Y_i)^*$ be cyclically (λ_i, c_i)-quasi-quasi geodesic word obtained by applying the (λ_i, c_i)-`cyclic-reduction` algorithm on U and V, respectively. Then, since U' and V' are conjugate to U and V in H_i respectively, we get that $U \sim_{conj} V$ in H_i if and only if $U' \sim_{conj} V'$ in H_i.

To check whether $U' \sim_{conj} V'$ in H_i, by Lemma 26, it is enough to check for all 3-tuples (T, U'', V''), where $T, U'', V'' \in (X \cup Y_i)^*$, U'', V'' are some cyclic shifts of U, V and $\|T\| \leqslant$

$\tau(|X|, \delta_i', \lambda_i, c_i)$ (where τ is defined as in Lemma 26) the equality

$$T^{-1}U''T =_{H_i} V''. \tag{4.14}$$

Clearly, since for large enough standard parameters, the word problem in $\bar{G}$ is decidable in polynomial time, then for large enough values of ρ_i this checking can be done in polynomial time.

Now, assuming that $U \nsim_{conj} V$ in H_i is already verified, in order to check whether $U \sim_{conj} V$ in G_i, we can apply $(\lambda, c, \epsilon, \eta)$-`cyclic-reduction` algorithm for $\eta = 1 - 121\lambda_i\mu_i$ to find cyclic $(\lambda_i, c_i, \epsilon_i, 1 - 121\lambda_i\mu_i)$-reductions U' and V' of U and V, respectively, and then check whether $U' \sim_{conj} V'$ in H_i or in G_i. Without loss of generality assume that $U' \nsim_{conj} V'$ in H_i, then, by Lemma 28, there exist $T_1, T_2, W \in (X \cup Y_i)^*$ such that $\|T_1\|, \|T_2\| \leqslant 2\epsilon_i$, W is a subword of a word $R \in {}_n\mathcal{R}_i$ of length $\|W\| \leqslant \lambda_i\mu_i$, and

$$(T_1WT_2)^{-1}U''(T_1WT_2) =_{G_i} V'' \tag{4.15}$$

for some cyclic shifts U'', V'' of U' and V', respectively. Therefore, in order to check whether $U' \sim_{conj} V'$ in G_i, it is enough to check equality (4.15) for all mentioned collection of words (T_1, T_2, W, U'', V''). Clearly, this checking can be done in polynomial time, provided that the standard parameters are sparse enough and $f(n) = \mathcal{C}_{{}_n\mathcal{R}_{\mathcal{I}(n)}}$ is bounded by a polynomial.

□

4.4.1 The condition $C'\big(\mathcal{TM}, (g_i)_{i=1}^{\infty}, (\rho_i)_{i=1}^{\infty}\big)$

Definition 4.7. If for fixed sequence $(g_i)_{i=1}^{\infty}$, fixed function $f_{\bar{\rho}} : \mathbb{N} \to \mathbb{N}$ and fixed Turing machine $\mathcal{TM}$ (all are defined is Subsection 4.3), elements of the sequence $(\rho_i)_{i=1}^{\infty}$ are large enough so that Theorem 12 and Theorem 14 hold, then we say that the presentation (4.6) of $\bar{G}$ satisfies the condition

$$C'\big(\mathcal{TM}, (g_i)_{i=1}^{\infty}, (\rho_i)_{i=1}^{\infty}\big).$$

Property 2. *As it follows from the proof of Theorems 12 and 14, there exist a linear time computable function $f_\rho : \mathbb{N}^6 \to \mathbb{N}$ such that in order Theorem 12 and Theorem 14 to hold it is enough to require*

$\rho_i \geqslant f_\rho(\delta_i', \lambda_i, c_i, \epsilon_i, \mu_i, \Phi(i))$ *for all* $i \in \mathbb{N}$.

Chapter 5

Proof of Theorems 3, 4, 5 and 6

5.1 The general scheme for group constructions of Sections 5.2, 5.3, 5.4 and 5.5

The proofs of Theorems 3, 4, 5 and 6, given in Sections 5.2, 5.3, 5.4 and 5.5, respectively, are constructive and the corresponding groups constructed in these sections are inductive limits of presentations of type (4.6) satisfying the condition

$$C'\big(\mathcal{TM}, (g_i)_{i=1}^{\infty}, (\rho_i)_{i=1}^{\infty}\big)$$

for $g_i(n) = n^{\frac{1}{i}}$. Moreover, the presentation

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \ldots, \tag{5.1}$$

for the corresponding constructions is such that if the group $G_i = \langle X \rangle / \ll \mathcal{R}_i \gg$ is already constructed, then the group $H_i = G_{i-1} * F(Y_i) / \ll \mathcal{S}_i \gg$ is defined uniformly, in the sense that the definition of H_i does not depend on specific values of i. The standard parameters $(\delta_{i-1}, \delta_i', \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)_{i=1}^{\infty}$ are different in the corresponding constructions only in terms of their "sparseness", however, since we are not interested in their specific values, we will not go into the details of defining them, instead we will assume that they are sparse enough.

The groups G_i, $i = 1, 2, \ldots$ in all those constructions will be defined uniformly as $G_i = H_i / \ll \mathcal{R}_i \gg$, where

$$\mathcal{R}_i = \mathcal{R}\big(Z_i, U_i, V_i, \delta_i', \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i\big), \tag{5.2}$$

according to the definition (2.29) in Subsection 2.5.3. Also Z_i is a set of elements of H_i with a "natural" order such that $\cup_{j=1}^{i} Y_i \subseteq Z_i$ and $(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ are sparse enough so that Lemma 18 guarantees that $\mathcal{R}_i$ satisfies the small-cancellation condition $C'(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ and the chain (5.1) satisfies the condition $C'\big(\mathcal{TM}, (g_i)_{i=1}^{\infty}, (\rho_i)_{i=1}^{\infty}\big)$ for $g_i(n) = n^{\frac{1}{i}}$ for some fixed Turing machine $\mathcal{TM}$ computing the i-th level data for (5.1). Note that we will note specify the details about $\mathcal{TM}$ in the constructions of Sections 5.2, 5.3 and 5.4 since what we need is actually the only fact that such a Turing machine $\mathcal{TM}$ indeed exists.

As a conclusion of what is said in this section, in Sections 5.2, 5.3 and 5.4 we will only specify description of the following:

- G_0;
- How does H_i relate to G_{i-1};
- Precise definitions of Z_i, U_i and V_i from (5.2).

Theorem 15. *The groups constructed according to the above described scheme have word problem decidable in almost linear time when for all $i \in \mathbb{N}$, $\mathcal{R}_i$ contains one element up to cyclic shift, and in almost quadratic time otherwise. Also, the G-conjugacy problem with respect to* (5.1) *is polynomial time decidable.*

Proof. For sparse enough standard parameters, for all large enough $n \in \mathbb{N}$, n is much larger than $\max\{\lambda_i, \epsilon_i, c_i, (1-23\mu_i)^{-1}\}$ where $i = \mathcal{I}(n)$, hence the function

$$n \mapsto \left\lceil \frac{\lambda_{\mathcal{I}(n)}(n + 2\epsilon_{\mathcal{I}(n)}) + c_{\mathcal{I}(n)}}{1 - 23\mu_{\mathcal{I}(n)}} \right\rceil \tag{5.3}$$

can be computed in time $\mathcal{O}(n)$. Therefore, the first statement of the theorem follows directly from Property 1 and Theorem 12 and the second statement follows from Property 1 and Theorem 14. □

5.2 Proof of Theorem 3

In this section we are going to show that for any given non-elementary, torsion-free δ_0-hyperbolic group G_0, there exists a lacunary hyperbolic quotient of G_0, denoted by $\check{G}$, which satisfies the conditions of Theorem 3.

Our approach is constructive and will be based on the scheme described in Section 5.1. First of all, this means that $\check{G}$ will be constructed as an inductive limit of a chain of hyperbolic groups of type (4.6), that is

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \dots . \tag{5.4}$$

In our construction below we will inductively show that the groups H_i and G_i, $i = 1, \dots$ are non-elementary torsion-free hyperbolic groups. In this section, the limit of (5.4) we denote by $\check{G}$.

Let the finite symmetric set $X = \{x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_{n_0}^{\pm 1}\}$ be a generating set of G_0 such that $\Gamma(G_0, X)$ is δ-hyperbolic for some $\delta \in \mathbb{N}$. Let us denote $X^- = \{x_1^{-1}, x_2^{-1}, \dots, x_{n_0}^{-1}\}$ and $X^+ = \{x_1^{+1}, x_2^{+1}, \dots, x_{n_0}^{+1}\}$. Also let us order X in the following natural way: $x_i^{-1} < x_j^{-1} < x_i^1 < x_j^1$ if $i < j$, and the elements of X^- precede the elements of X^+, i.e., for all $i, j \in \mathbb{N}$, $x_i^{-1} < x_j$. Hereafter whenever we consider an indexed alphabet X', the order of the set $(X')^{\pm 1}$ will be defined just like it was done for $X = X^- \cup X^+$.

Let us consider the free group $F_1 = F(Y)$ of infinite rank, where $Y = \{y_1, y_2, \dots\}$ and let us introduce an order on the set of reduced words from F in the following natural way: for reduced words $u, v \in F$, we define $u > v$ if either $\|u\| = \|v\|$ and $u > v$ lexicographically (here we regard words as vectors of letters from $Y^{\pm 1}$) or $\|u\| > \|v\|$. In the analogous way, we order elements of the free group $F_2 = F(X)$. For a reduced word $u \in F(Y)$, we say that u is a *dense* word, if there exists $i \in \mathbb{N}$ such that u contains at least one letter from each of the following sets $\{y_1^{\pm 1}\}, \dots, \{y_i^{\pm 1}\}$ and does not contain any other letters.

Finally, let us introduce a partial linear order on the set $F_1 \times F_2$ in the following way: let u, u' be reduced words in $F_1 = F(X)\backslash\{1\}$ and v, v' be dense words in $F_2 = F(Y)\backslash\{1\}$, then we define $(u, v) < (u', v')$ if either $\|u\| + \|v\| < \|u'\| + \|v'\|$ or

$$\|u\| + \|v\| = \|u'\| + \|v'\| \text{ and } u < u'.$$

Denote the i^{th} element of the set $F_1 \times F_2$ with respect to this partial order by (u_i, v_i). The reason for considering only the dense words from F_2 (and also the partial order) is that for dense words, balls of finite radius with respect to the word metric have finite volume and hence, by the introduced partial order, we will be able to effectively enumerate all the aforementioned pairs $(u, v) \in F_1 \times F_2$. (Also note that the map $i \mapsto (u_i, v_i)$ is not bijective). As it will be clear from what is discussed below, this is important for the construction of machines TM_1 and TM_2. Without loss of generality we assume that x_1 and x_2 are different elements of infinite order in G_0.

5.2.0.1 Definition of H_{i+1} ($i \geqslant 0$) for $\check{G}$

Suppose that the non-elementary torsion-free δ_i-hyperbolic group G_i is already constructed. Let $(u_{i+1}, v_{i+1}) \in F_1 \times F_2$ be a pair of dense word as defined above.

Case 1. ($u_{i+1} =_{G_i} 1$). In this case define $H_{i+1} = G_i$;

Case 2. ($u_{i+1} \neq_{G_i} 1$). In this case let us define n_{i+1} to be the least positive integer such that $v_{i+1} \in F(y_1, \ldots, y_{n_{i+1}})$ (and consequently, $v_{i+1} \notin F(y_1, \ldots, y_{n_{i+1}-1})$). Note that the existence of n_{i+1} follows from the fact that v_{i+1} is dense by definition.

Let us define $G'_i = G_i * F(y_1, \ldots, y_{n_{i+1}})$. By Corollary 5, there exists a Turing machine which for input $(G_i, \delta_i; u_{i+1}, v_{i+1})$ outputs the pairs (u'_{i+1}, k) and (v'_{i+1}, l), where $u'_{i+1}, v'_{i+1} \in (X \cup \{(y_1, \ldots, y_{n_{i+1}}\})^*$, $k, l \in \mathbb{N}$ are such that $u_{i+1} =_{G'_i} (u'_{i+1})^k$ and $v_{i+1} =_{G'_i} (v'_{i+1})^l$ and u'_{i+1}, v'_{i+1} represent root elements of u_{i+1} and v_{i+1} in G'_i (i.e. $E(u_{i+1}) = \langle u'_{i+1} \rangle$ and $E(v_{i+1}) = \langle v'_{i+1} \rangle$ in G'_i). We will use the standard notation $v_{i+1} = v_{i+1}(y_1, \ldots, y_{n_{i+1}})$ to emphasize that v_{i+1} is formed by the letters (or, in the context of diophantine equations, by variables) $y_1, \ldots, y_{n_{i+1}}$.

Let $\langle z_{i+1} \rangle$ be an infinite cyclic group disjoint from G'_i. Define the group H^0_{i+1} as an HNN-extension of G_i as follows.

$$H^0_{i+1} = \langle G_i * \langle z_{i+1} \rangle, t_{i+1} \mid t^{-1}_{i+1} u'_{i+1} t_{i+1} = z^l_{i+1} \rangle.$$

Now define H as an HNN-extension of H^0 as follows.

$$H_{i+1} = \langle H^0_{i+1} * F(y_1, \ldots, y_{n_{i+1}}), s_{i+1} \mid s^{-1}_{i+1} v'_{i+1} s_{i+1} = z^k_{i+1} \rangle. \tag{5.5}$$

Finally, define $Y_{i+1} = \{y_1, \ldots, y_{n_{i+1}}\} \cup \{z_{i+1}\} \cup \{t_{i+1}\} \cup \{s_{i+1}\}$.

Proposition 5. *The group H_{i+1} is a torsion-free non-elementary hyperbolic group and the identity map $id : X \to X$ induces an embedding of G_i into H_{i+1}.*

Proof. First of all, if $H_{i+1} = G_i$ then the statement follow from the inductive assumption that G_i

is a torsion-free non-elementary hyperbolic group.

Now assume that H_{i+1} is defined by (5.5). Then, since $\langle u'_{i+1} \rangle$ is a maximal elementary subgroup of $G_i * \langle z_{i+1} \rangle$ and since for all $g \in G_i * \langle z_{i+1} \rangle$ we have $g^{-1} \langle u'_{i+1} \rangle g \cap \langle z_{i+1} \rangle = \{1\}$, by Theorem 8, H^0_{i+1} is a hyperbolic group.

Now, since $\langle v'_{i+1} \rangle$ is a maximal elementary subgroup in $F(y_1, \ldots, y_{n_{i+1}})$, we get that $\langle v'_{i+1} \rangle$ is a maximal elementary subgroup in $H^0_{i+1} * F(y_1, \ldots, y_{n_{i+1}})$ and for all $h \in H^0_{i+1} * F(y_1, \ldots, y_{n_{i+1}})$, we have $h^{-1} \langle v'_{i+1} \rangle h \cap \langle z_{i+1} \rangle = \{1\}$. Therefore, by Theorem 8, H_{i+1} is a hyperbolic group.

The fact that H_{i+1} is torsion free follows from the fact that G_i is torsion free and from Lemma 14.

The part of the statement that the identity map $id : X \to X$ induces an embedding of G_i into H_{i+1} follows from the basic properties of HNN-extensions. See [40].

Finally, since H_{i+1} contains an isomorphic copy of G_i and G_i is non-elementary, it follows that H_{i+1} is non-elementary as well. □

Proposition 6. *The equation* $v_{i+1}(y_1^{s_{i+1}t_{i+1}^{-1}}, \ldots, y_{n_{i+1}}^{s_{i+1}t_{i+1}^{-1}}) = u_{i+1}$ *holds in* H_{i+1}. *In other words,* $y_1 \mapsto y_1^{s_{i+1}t_{i+1}^{-1}}, \ldots, y_{n_{i+1}} \mapsto y_{ni+1}^{s_{i+1}t_{i+1}^{-1}}$ *is a solution to the diophantine equation*

$$v_{i+1}(y_1, \ldots, y_{n_{i+1}}) = u_{i+1}$$

in H_{i+1}.

Proof. Indeed, first of all, the relations $t_{i+1}^{-1} u'_{i+1} t_{i+1} = z_{i+1}^l$ and $s_{i+1}^{-1} v'_{i+1} z_{i+1}^k$ imply that

$$(t_{i+1}^{-1} u'_{i+1} t_{i+1})^k = t_{i+1}^{-1} u_{i+1} t_{i+1} = z_{i+1}^{lk} = (z_{i+1}^k)^l = (s_{i+1}^{-1} v s_{i+1}) l = s_{i+1}^{-1} v_{i+1} s_{i+1}.$$

Therefore, $t_{i+1} s_{i+1}^{-1} v_{i+1} s_{i+1} t_{i+1}^{-1} = v_{i+1}^{s_{i+1}t_{i+1}^{-1}} = u_{i+1}$. Now, since

$$v_{i+1}^{s_{i+1}t_{i+1}^{-1}} = v_{i+1}(y_1^{s_{i+1}t_{i+1}^{-1}}, \ldots, y_{n_{i+1}}^{s_{i+1}t_{i+1}^{-1}}),$$

we get that $y_1 \mapsto y_1^{s_{i+1}t_{i+1}^{-1}}, \ldots, y_{n_{i+1}} \mapsto y_{n_{i+1}}^{s_{i+1}t_{i+1}^{-1}}$ is a solution of the diophantine equation $v_{i+1}(y_1, \ldots, y_{n_{i+1}}) = u_{i+1}$. □

Proposition 7. *If $x_2 \notin E(x_1)$ in G, then $x_2 \notin E(x_1)$ in H_{i+1}. Also, for all $y \in Y_{i+1}$, $y \notin E(x_1)$ in H_{i+1}.*

Proof. This fact immediately follows from Lemma 14. □

Proposition 8. *Let $U, V \in X^*$ be such that $U \sim_{conj} V$ in H_{i+1}. Then $U \sim_{conj} V$ in G_i.*

Proof. Suppose that $U \nsim_{conj} V$ in G_i. Then we want to show that $U \nsim_{conj} V$ in H.

By contradiction let us assume that $U \sim_{conj} V$ in H_{i+1}. Then there exists a minimal (U, V)-conjugacy diagram Δ over the HNN-extension H_{i+1} with boundary $ABCD$, $lab(AD) = V$, $lab(BC) = U$. Note that since $U \nsim_{conj} V$ in G_i, Δ must contain at least one t_{i+1}- or s_{i+1}-band which has its ends on different sides of $ABCD$. Also, since U and V do not contain edges with labels from $\{s_{i+1}^{\pm 1}, t_{i+1}^{\pm 1}\}$, it must be that all these bands are horizontal, i.e., have their ends on AB and DC.

Next, we will show that Δ cannot contain horizontal bands. By contradiction let us assume that it contains horizontal bands.

First, suppose that Δ contains more than one horizontal bands. In this case, let us choose edges $e_1, e_2 \in AB$ and $e_1', e_2' \in CD$ such that they have labels from $\{s_{i+1}^{\pm 1}, t_{i+1}^{\pm 1}\}$ and e_1 and e_2 are connected by horizontal bands to e_1' and e_2', respectively. Additionally, without loss of generality let us assume that there is no horizontal band between these two bands. See Figure 5.1.

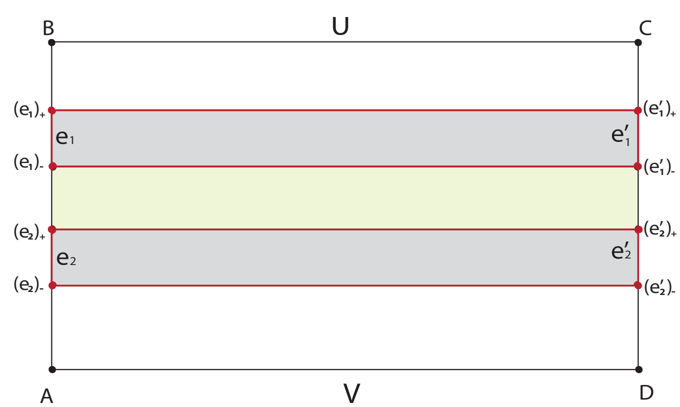

Figure 5.1: Δ with the two horizontal bands, depicted as grey areas.

Note that then $(e_1)_-$, $(e_1')_-$ and $(e_2)_+$, $(e_2')_+$ are pairs of mirroring point, respectively. Therefore, the subdiagram of Δ bounded between $(e_2)_+$, $(e_1)_-$, $(e_1')_-$ and $(e_2')_+$ is a $\Big(lab\big([(e_1)_-, (e_1')_-]\big), lab\big([(e_2)_+, (e_2')_+]\big)\Big)$-conjugacy diagram over G_i'. In particular,

$lab\big([(e_1)_-,(e_1')_-]\big)$ and $lab\big([(e_2)_+,(e_2')_+]\big)$ are conjugate in G_i'. Now, since $lab([(e_1)_-,(e_1')_-])$ and $lab([(e_2)_+,(e_2')_+])$ are (free) powers of elements from $\{u_{i+1}',v_{i+1}',z_{i+1}\}$, it follows that, in fact, they must be freely equal. This means that $lab([(e_1)_+,(e_1')_+])$ and $lab([(e_2)_-,(e_2')_-])$ are also freely equal. But, since $\big((e_1)_+,(e_1')_+\big)$ and $\big((e_2)_-,(e_2')_-\big)$ are pairs of mirroring points on $\partial\Delta$, by Lemma 25, this contradicts the assumption that Δ is chosen to be slender. Therefore, Δ cannot contain two horizontal bands, hence, since by our assumptions it contains at least one horizontal $\{s_{i+1}^{\pm 1},t_{i+1}^{\pm 1}\}$-band, it means that the number of such bands is exactly one.

Now suppose that Δ contains only one horizontal band and that only horizontal band of Δ has its ends on edges e_1 and e_1', i.e. in Figure 5.1 just neglect the bottom band. Since one of $lab\big([(e_1)_-,(e_1')_-]\big)$ and $lab\big([(e_1')_+,(e_1')_+]\big)$ belongs to Y_{i+1}^*, without loss of generality assume that $lab\big([(e_1)_+,(e_1')_+]\big)\in Y_{i+1}^*$. Then, since $lab([B,(e_1)_+])\equiv lab([C,(e_1')_+])$, we get that $u_{i+1}'\sim_{conj} lab\big([(e_1)_+,(e_1')_+]\big)$ in G_i', which is impossible, since $G_i'=G_i*F(Y_{i+1})$ and $u_{i+1}'\in X^*$. A contradiction.

□

5.2.0.2 Definition of G_{i+1} ($i\geqslant 0$) for $\check{G}$

Assuming that the torsion-free non-elementary hyperbolic group H_{i+1} is already defined, G_{i+1} we define as

$$G_{i+1}=H_{i+1}/\ll\mathcal{R}\big(Y_{i+1},x_1,x_2,\delta_{i+1}',\lambda_{i+1},c_{i+1},\epsilon_{i+1},\mu_{i+1},\rho_{i+1}\big)\gg.$$

Note that, if $x_1\notin E(x_2)$ in G_i, then, by Proposition 7, $x_1\notin E(x_2)$ in H_{i+1}, hence for sparse enough standard parameters $\delta_{i+1}',\lambda_{i+1},c_{i+1},\epsilon_{i+1},\mu_{i+1},\rho_{i+1}$, the set of words

$$\mathcal{R}_{i+1}=\mathcal{R}\big(Y_{i+1},x_1,x_2,\delta_{i+1}',\lambda_{i+1},c_{i+1},\epsilon_{i+1},\mu_{i+1},\rho_{i+1}\big)$$

satisfies the small-cancellation condition $C'(\lambda_{i+1},c_{i+1},\epsilon_{i+1},\mu_{i+1},\rho_{i+1})$. Therefore,

1. By Lemma 20, G_{i+1} will be non-elementary torsion-free hyperbolic group;

2. By Lemma 29, x_1 is not a proper power in G_{i+1}. Therefore, since G_{i+1} is a torsion-free hyperbolic group, we get $x_1\notin E(x_2)$ in G_{i+1}. Thus, by inductive hypothesis, G_{i+1} is well-

defined non-elementary torsion-free hyperbolic group.

5.2.1 Main properties of $\check{G}$

Note that since the groups G_i are torsion-free non-elementary hyperbolic groups, the group $\check{G}$ is torsion-free infinite lacunary hyperbolic group (recall that we assume that the standard parameters are sparse enough).

From Proposition 6 if follows that $\check{G}$ is verbally complete.

From Theorem 15, it follows that for sparse enough standard parameters the word problem in $\check{G}$ is decidable in almost linear time and the conjugacy problem is decidable in polynomial time.

Thus Theorem 3 is proved.

5.3 Proof of Theorem 4

Let $G_0 = \langle X \rangle$, $X = \{x_1, x_2, \ldots, x_n\}$, be a torsion-free non-elementary δ-hyperbolic group with respect to X. Without loss of generality we assume that $E(x_i) \cap E(x_j) = \{1\}$ if $i \neq j$ and that $x_1, \ldots, x_n$ are root elements (i.e. $E(x_i) = \langle x_i \rangle$ for $1 \leqslant i \leqslant n$).

Let X be linearly ordered such that $x_i^{-1} < x_j^{-1} < x_i < x_j$ if $i < j$. We denote the set of reduced non-empty words of X^* by $F'(X)$. Let us enumerate the set $F'(X)$ as $F'(X) = \{w_1, w_2, \ldots\}$ where for $i < j$, $w_i < w_j$ according to the lexicographical order induced from the order on X. Then clearly $w_1 = x_1$, $w_2 = x_2$. Now, based on this order of $F'(X)$ let us lexicographically order the set $F'(X) \times F'(X) \backslash \{(w, w) \mid w \in F'(X)\}$ and enumerate it according to that order. Let

$$F'(X) \times F'(X) \backslash \{(w, w) \mid w \in F'(X)\} = \{(u_1, v_1), (u_2, v_2) \ldots\},$$

where for $i < j$, we have $(u_i, v_i) < (u_j, v_j)$.

As it was mentioned in Section 5.1, in this section we are going to construct the group $\hat{G}$ from Theorem 4 which will be a direct limit of a chain of non-elementary torsion-free hyperbolic groups of the form (4.6), that is

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \ldots . \tag{5.6}$$

In this section we define $H_i = G_{i-1}$ for all $i \in \mathbb{N}$ and the map $\beta_{i-1} : G_{i-1} \to H_i$ is the identity map.

Now let us describe how G_i is obtained from G_{i-1} for $i \in \mathbb{N}$. For that purpose by induction let us assume that G_{i-1} is a non-elementary torsion-free hyperbolic group (below we will show that for G_i the same property holds as well). Then, there exists smallest index $j_i \geqslant i$ such that $v_{j_i} \notin E(u_{j_i})$ in G_{i-1}, and the set

$$Z_i \stackrel{\text{def}}{=} \{x \in X \mid x \notin E(u_{j_i})\}$$

is non-empty. By Corollary 5, (u_{j_i}, v_{j_i}) and Z_i can be found algorithmically.

Now define

$$G_i = G_{i-1}/ \ll \mathcal{R}\big(Z_i, u_{j_i}, v_{j_i}, \delta'_i, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i\big) \gg .$$

Note that by Lemma 18, if the standard parameters $\delta'_i, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i$ are sparse enough, then the set of words

$$\mathcal{R}_i \stackrel{\text{def}}{=} \mathcal{R}\big(Z_i, u_{j_i}, v_{j_i}, \delta'_i, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i\big)$$

satisfies the small-cancellation condition $C'(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$. Also note that, by the definition of G_i, $G_i = \langle u_{j_i}, v_{j_i} \rangle$.

Lemma 40. *The following are true about G_i.*

1. *G_i is a torsion-free non-elementary hyperbolic group;*

2. *Either $v_i \in E(u_i)$ in G_i or $\langle u_i, v_i \rangle = G_i$.*

3. *For each $x \in X$, $E(x) = \langle x \rangle$ in G_i (we assume that for G_{i-1} this is already shown).*

Proof. Part (1) of the statement follows from Lemma 20.

For part (2) simply notice that, by our definition of j_i if $j_i > i$ then $u_i \in E(v_i)$ in G_i, otherwise if $j_i = i$ then $v_i \notin E(u_i)$ in G_i and $G_i = \langle u_i, v_i \rangle$.

For Part (3), first, note that it immediately follows from Lemma 29 that x is not a proper power in G_i. Therefore, since by Part (1) of the current lemma, G_i is a torsion-free abelian group, we get that $E(x) = \langle x \rangle$. $\square$

Proposition 9. *The group $\hat{G}$, which is defined as inductive limit of $(G_i)_{i=1}^{\infty}$, satisfies the statement of Theorem 4. That is* *(i). Every proper subgroup of $\hat{G}$ is an infinite cyclic group, while $\hat{G}$ is not cyclic;*

(ii). The word problem in $\hat{G}$ is decidable in almost quadratic time and the conjugacy problem in $\hat{G}$ is decidable in polynomial time.

Proof. (i). First of all, notice that $\hat{G}$ is not cyclic, because otherwise, for some $i \in \mathbb{N}$, G_i would be cyclic, which is impossible, since $G_1, G_2, \dots$ are non-elementary hyperbolic groups by Lemma 40 .

Now, by contradiction let us assume that $\hat{G}$ contains a proper non-cyclic subgroup K. Then, since K is a proper subgroup, by part (2) of Lemma 40, K is abelian (even more, each finitely generated subgroup of K is cyclic). Let us fix any non-trivial element $x \in K$. Then in each of the groups G_i the centralizer of x coincides with $E(x)$, hence it is cyclic. This means that in the inductive limit $\hat{G}$ the centralizer of x is again cyclic. Therefore, since K is contained in the centralizer of x, K is cyclic as well. A contradiction.

(ii). Second part of Proposition follows from Theorem 15. As for conjugacy problem, let us notice that two elements of $\hat{G}$ are conjugate if and only if they are G-conjugate, hence Theorem 15 implies that conjugacy problem in $\hat{G}$ is polynomial.

□

Thus Theorem 4 is proved.

5.4 Proof of Theorem 5

Let $\mathcal{A}$ be any finite alphabet, and let $\mathcal{L} \subseteq \mathcal{A}^*$ be any recursively enumerable subset of $\mathcal{A}^*$.

For the two generated free group $F(x_1, x_2)$, let us denote by $F^+(x_1, x_2)$ the set of words from $F(x_1, x_2)$ which do not contain the letters x_1^{-1} and x_2^{-1}.

Let us also fix a bijective map $\Lambda_0 : \mathcal{A}^* \to F^+(x_1, x_2)$ such that Λ_0 and Λ_0^{-1} are computable in linear time. Construction of such a map can be easily achieved through a standard binary encoding of the set $\mathcal{A}^*$.

Let us define $G_0 = F_1 * F_2 * F_3$, where $F_1 = F(x_1, x_2, x_3)$, $F_2 = F(y_1, y_2, y_3)$, $F_3 = F(z_1, z_2)$ are free groups with freely generating sets $X_0 = \{x_1, x_2, x_3\}$, $Y_0 = \{y_1, y_2, y_3\}$ and $Z_0 = \{z_1, z_2\}$, respectively. For the convenience in the further exposition, let us also introduce the following notations: $G_{0,1} = F_1$, $G_{0,2} = F_2$ and $G_{0,3} = F_3$.

Let $\varsigma : F_1 \to F_2$ be the isomorphism between F_1 and F_2 induced by the map $x_1 \mapsto y_1, x_2 \mapsto y_2$, $x_3 \mapsto y_3$.

Define $\Lambda : \mathcal{L} \to X_0^* \times Y_0^*$ as follows: For all $\omega \in \mathcal{L}$,

$$\Lambda(\omega) = (\ \Lambda_0(\omega)x_3,\ \varsigma(\Lambda_0(\omega))y_3).$$

Clearly, Λ in an injection. Let

$$\Lambda(\mathcal{L}) = \{(u_1, v_1), (u_2, v_2), \ldots\},$$

where the enumeration is with respect to some fixed Turing machine M_Λ which for input $i \in \mathbb{N}$ outputs (u_i, v_i). Note that such an enumeration exists since $\mathcal{L}$ is recursively enumerable.

As it was mentioned in Section 5.1, in this section we are going to construct the group $G_{\mathcal{L}}$ from Theorem 4 which will be a direct limit of a chain of non-elementary torsion-free hyperbolic groups of the form (4.6), that is

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \ldots. \tag{5.7}$$

More specifically, $G_0 \overset{def}{=} F(X)$, where

$$X \overset{def}{=} X_0 \cup Y_0 \cup Z_0 = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2\}.$$

5.4.0.1 Definition of H_{i+1} ($i \geqslant 0$) for $\check{G}$

Assuming that G_i is already constructed we define H_{i+1} as an HNN-extension of G_i. More precisely,

$$H_{i+1} = \langle G_i, t_{i+1} \mid u_{i+1} = t_{i+1}^{-1} v_{i+1} t_{i+1} \rangle. \tag{5.8}$$

Then, clearly the identity map $id : X \to X$ induces an embedding $\beta_i : G_i \hookrightarrow H_{i+1}$. Define

$$Y_{i+1} = \{t_{i+1}\}.$$

We will show by induction that for all $i \geqslant 0$, H_{i+1} is a torsion-free, non-elementary δ'_{i+1}-hyperbolic group (for some $\delta'_{i+1} \in \mathbb{N}$ such that the map $i + 1 \mapsto \delta'_{i+1}$ is computable) with respect to the generating sets $X \cup \{t_{i+1}\}$ and $X \cup \{t_1, \ldots, t_{i+1}\}$.

5.4.0.2 Definition of G_{i+1} ($i \geqslant 0$) for $G_{\mathcal{L}}$

Suppose that H_{i+1} is already constructed and it is non-elementary, torsion-free δ'_{i+1}-hyperbolic with respect to the generating set $X \cup \{t_1, \ldots, t_i\}$ for $\delta'_{i+1} \in \mathbb{N}$. Then, we define G_{i+1} as follows

$$G_{i+1} \stackrel{\text{def}}{=} H_{i+1}/ \ll \mathcal{R}\big(\{t_{i+1}\}, z_1, z_2, \delta'_{i+1}, \lambda_{i+1}, c_{i+1}, \epsilon_{i+1}, \mu_{i+1}, \rho_{i+1}\big) \gg,$$

where $\lambda_{i+1} \succ c_{i+1} \succ \epsilon_{i+1} \succ \mu_{i+1} \succ \rho_{i+1}$ are sparse enough standard parameters. Denote $\mathcal{R}_{i+1} = \mathcal{R}\big(\{t_{i+1}\}, z_1, z_2, \delta'_{i+1}, \lambda_{i+1}, c_{i+1}, \epsilon_{i+1}, \mu_{i+1}, \rho_{i+1}\big)$, and let $R_{i+1} \in \mathcal{R}_{i+1}$ be any fixed representative of $\mathcal{R}_{i+1}$ (i.e. $\mathcal{R}_{i+1}$ is the set of cyclic shifts of R_{i+1}).

Note that G_{i+1} is generated by the image of X (which we denote by X too) under the natural homomorphism from G_i to G_{i+1} induced by the identity map $id : X \to X$. We will show by induction that for all $i \geqslant 0$, G_{i+1} is a torsion-free, non-elementary δ_{i+1}-hyperbolic group (for some $\delta_{i+1} \in \mathbb{N}$ such that the map $i + 1 \mapsto \delta_{i+1}$ is computable) with respect to the generating set X.

For the further exposition let us define the concept of *truncated* contiguity diagrams as follows: In a van Kampen diagram Δ over $G_i = H_i/ \ll \mathcal{R}_i \gg$ which contains an essential cell Π and an outer contiguity diagram Γ connecting an arc $\check{q}_\Gamma$ of Π to an arc $\hat{q}_\Gamma$ of $\partial\Delta$, we say that Γ is *truncated* if p_Γ and p'_Γ are the shortest paths in $Proj(\Delta)$ joining, respectively, $(\check{q}_\Gamma)_-$ and $(\check{q}_\Gamma)_+$ to $\partial\Delta$.

Note that truncated contiguity diagrams are truncated diagrams according to Definition 2.8.

5.4.0.3 Main properties of the chain (5.7)

(a$_i$). Let $W \in (X_0 \cup Y_0 \cup Z_0)^*$ and for some $i \geqslant 1$, $W =_{H_i} W'$, where W' is a geodesic word in $\Gamma\big(H_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_i\}\big)$. Then W' does not contain the letter $t_i^{\pm 1}$, i.e. $W' \in (X_0 \cup Y_0 \cup Z_0)^*$. Also, if $W \in X_0^* \cup Y_0^*$ is a freely reduced word, then W is geodesic in $\Gamma(H_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_i\})$;

(b$_i$). There is no ϵ_i-contiguity subdiagram Γ of rank i such that $lab(\hat{q}_\Gamma) \in X_0^* \cup Y_0^*$ and $\|\check{q}_\Gamma\| \geqslant \mu_i \|R_i\|$. Moreover, if $\check{q}_\Gamma$ does not contain an edge labeled by $t_i^{\pm 1}$, then it is enough to require $\|\check{q}_\Gamma\| \geqslant \mu_i \|R_i\|/2$;

(b$'_i$). If a truncated ϵ_i-contiguity subdiagram Γ of rank i is such that $\check{q}_\Gamma$ is geodesic in $\Gamma\big(H_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_1, \ldots, t_i\}\big)$ and $\|\check{q}_\Gamma\| \geqslant \mu_i \|R_i\|$, then $Area(\Gamma) = 0$;

(c$_i$). If $w \in X_0^* \cup Y_0^*$ is a reduced word, then it is a geodesic word in $\Gamma\big(G_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_i\}\big)$. Moreover, if for some word $u \in (X_0 \cup Y_0 \cup Z_0 \cup \{t_i\})^*$, u is geodesic in $\Gamma\big(G_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_i\}\big)$ and $u =_{G_i} w$, then $u \equiv w$ (i.e. u is freely equal to w);

(d$_i$). If $U \in X_0^* \cup Y_0^*$ is a reduced word which is not a proper power of another word from G_0, then it represents an element in G_i which is not a proper power of another element from G_i;

(e$_i$). $G_{i,1} \cap G_{i,2} = \{1\}$;

(f$_i$). Assuming that G_{i-1} is a non-elementary torsion-free δ_{i-1}-hyperbolic group with respect to the generating set $X_0 \cup Y_0 \cup Z_0$, we have that H_i is a non-elementary torsion-free δ'_i-hyperbolic group with respect to the generating set $X \cup \{t_1, \ldots, t_i\}$, where δ'_i is some (computable) positive integer. Also, the group G_i is non-elementary, torsion-free hyperbolic group.

Clearly this properties are true for $i = 0$. Next, based on induction on i we will prove that they are true for every i.

5.4.0.4 Proof of the properties (a$_{i+1}$)-(f$_{i+1}$).

Lemma 41. *Assuming that the statements (a_i)-(f_i) are true, the following properties hold.*

(a_{i+1}). Let $W \in (X_0 \cup Y_0 \cup Z_0)^*$ and for some $i \geqslant 1$, $W =_{H_{i+1}} W'$, where W' is a geodesic word in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Then W' does not contain the letter $t_{i+1}^{\pm 1}$, i.e. $W' \in (X_0 \cup Y_0 \cup Z_0)^*$. Also, if $W \in X_0^* \cup Y_0^*$ is a freely reduced word, then W is geodesic in $\Gamma(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})$;

(b_{i+1}). There is no ϵ_{i+1}-contiguity subdiagram Γ of rank $i+1$ such that $lab(\hat{q}_\Gamma) \in X_0^* \cup Y_0^*$ and $\|\check{q}_\Gamma\| \geqslant \mu_{i+1}\|R_{i+1}\|$. Moreover, if $\check{q}_\Gamma$ does not contain an edge labeled by $t_{i+1}^{\pm 1}$, then it is enough to require $\|\check{q}_\Gamma\| \geqslant \mu_{i+1}\|R_{i+1}\|/2$;

(b'_{i+1}). If a truncated ϵ_{i+1}-contiguity diagram Γ of rank $i+1$ is such that $\|\check{q}_\Gamma\| \geqslant \mu_{i+1}\|R_{i+1}\|$, then $Area(\Gamma) = 0$;

(c_{i+1}). If $w \in X_0^* \cup Y_0^*$ is a reduced word, then it is a geodesic word in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Moreover, if for some word $u \in (X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})^*$, u is geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$ and $u =_{G_{i+1}} w$, then $u \equiv w$ (i.e. u is freely equal to w);

(d_{i+1}). If $U \in X_0^* \cup Y_0^*$ is a reduced word which is not a proper power of another word G_0, then it represents an element in G_{i+1} which is not a proper power of another element from G_{i+1};

(e_{i+1}). $G_{i+1,1} \cap G_{i+1,2} = \{1\}$;

(f_{i+1}). Assuming that G_i is a non-elementary torsion-free δ_i-hyperbolic group with respect to the generating set $X_0 \cup Y_0 \cup Z_0$, we have that H_{i+1} is a non-elementary torsion-free δ'_{i+1}-hyperbolic group with respect to the generating set $X \cup \{t_1, \ldots, t_{i+1}\}$, where δ'_{i+1} is some (computable) positive integer. Also, the group G_{i+1} is non-elementary, torsion-free hyperbolic group.

Proof. Based on the inductive assumption we will prove Lemma 41 using the following scheme: $the\ inductive\ hypothesis \implies (a_{i+1}) \implies (b_{i+1}) \implies (b'_{i+1}),\ (c_{i+1}) \implies (d_{i+1}) \implies (e_{i+1}) \implies (f_{i+1})$.

(a$_{i+1}$). If $W =_{H_{i+1}} W'$ and W' is a geodesic word in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$, then there is a minimal van Kampen diagram Δ over H_{i+1} such that $\partial\Delta = pq^{-1}$, where $lab(p) = W$ and $lab(q) = W'$.

If W' contains a letter from $t_{i+1}^{\pm 1}$, then q contains an edge with label from $t_{i+1}^{\pm 1}$, hence Δ contains a t_{i+1}-band. Therefore, since W does not contain $t_{i+1}^{\pm 1}$ (or equivalently, p does not contain edges with labels from $t_{i+1}^{\pm 1}$) we get that the t_{i+1}-bands of Δ must start and end on q. Let us consider edges e and e' on q such that they are connected by a t_{i+1}-band and between them there is no other edge labeled by $t_{i+1}^{\pm 1}$. Let us denote the sides of this t_{i+1}-band which are not on q by q_1 and q_2 as in Figure 5.2. Note that since in the definition (5.8) of H_{i+1} the words u_{i+1} and v_{i+1} are freely cyclically reduced and $\|u_{i+1}\| = \|v_{i+1}\|$, we get $\|q_1\| = \|q_2\|$. Let us also denote by q' the subpath of q between e_+ and $(e')_-$ as in Figure 5.2.

By our assumptions, there is no edge on q' labeled by $t_{i+1}^{\pm 1}$. Therefore, since $lab(q_2(q')^{-1})$ does not contain edges with labels from $t_{i+1}^{\pm 1}$, we get that the subdiagram of Δ with the boundary $q_2(q')^{-1}$ is a diagram over G_i (see Figure 5.2). Therefore, since by our assumptions q', as a subpath of the geodesic path q, is geodesic in $\Gamma(H_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})$, it is also geodesic in $\Gamma(G_i, X_0 \cup Y_0 \cup Z_0)$. Also, since by the statement of (c$_i$), q_2 is geodesic in $\Gamma(G_i, X_0 \cup Y_0 \cup Z_0)$ too, we get that $\|q_2\| = \|q'\|$. Also, since $\|q_1\| = \|q_2\|$, we get $\|q_1\| = \|q'\|$. Therefore, if we replace the subpath $eq'e'$ of q with q_1, then q will be shortened by 2. The last observation contradicts the assumption that q is geodesic in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Therefore, it must be that W' does not contain $t_{i+1}^{\pm 1}$, i.e. $W' \in (X_0 \cup Y_0 \cup Z_0)^*$.

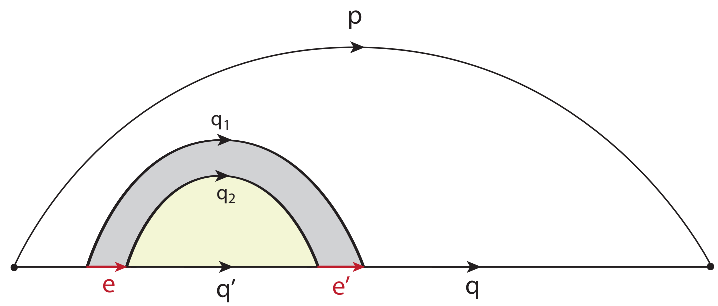

Figure 5.2: Δ: $lab(p) = W$, $lab(q) = W'$, $lab(e) \in \{t_{i+1}^{\pm 1}\}$, $lab(e) \in \{t_{i+1}^{\mp 1}\}$.

Now let us turn to the last statement of part (a$_{i+1}$). Namely, if $W \in X_0^* \cup Y_0^*$, then W is geodesic in $\Gamma(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})$.

Suppose that $W' \in (X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})^*$ is a geodesic word in $\Gamma(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})$ such that $W' =_{H_{i+1}} W$. Then, by the first part of (a$_{i+1}$), W' does not contain $t_{i+1}^{\pm 1}$, which implies that $W' =_{G_i} W$. By inductive hypothesis (more precisely, by (c$_i$)), since $W \in X_0^* \cup Y_0^*$, we get that W is geodesic in $\Gamma(G_i, X_0 \cup Y_0 \cup Z_0)$. Therefore, $W =_{G_i} W'$ implies $\|W\| = \|W'\|$ and since W' is geodesic in $\Gamma(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})$, we get that W is geodesic in $\Gamma(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})$ as well.

(b$_{i+1}$). Suppose Γ is a contiguity subdiagram satisfying the conditions described in the statement of (b$_{i+1}$), which, in particular, means that $lab(\hat{q}_\Gamma) \in X_0^* \cup Y_0^*$.

First of all, let us notice that since $lab(\hat{q}_\Gamma) \in X_0^* \cup Y_0^*$, by (a$_{i+1}$) we get that $\hat{q}_\Gamma$ is geodesic in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$.

Now, let $\partial\Gamma = ABCD$, where $AB = p_\Gamma$, $BC = \hat{q}_\Gamma$, $DC = p'_\Gamma$ and $AD = \check{q}_\Gamma$. Without loss of generality assume that $\|AB\| + \|DC\|$ is minimal among all contiguity subdiagrams satisfying the conditions stated in (b$_{i+1}$).

Now we are going to show that Γ does not contain any t_{i+1}-bands with both ends on $AB \cup BC \cup CD$. For that purpose, let us notice that since by definition AB and DC are geodesics, there is no t_{i+1}-band with both ends on AB or on DC. Also, since BC does not contain an edge with a label from $\{t_{i+1}^{\pm 1}\}$, there is no t_{i+1}-band which ends on BC. Also, since AB and DC are geodesics, there is no t_{i+1}-band with both of its ends on AB or on DC (the impossibility of such scenario is explained in the proof of part (a$_{i+1}$)). Thus the only possible way for a t_{i+1}-band to have both of its ends on $AB \cup BC \cup CD$ is when one end is on AB and the other one is on DC.

Now assume that there are edges e and e' on AB and DC, respectively, such that their labels belong to $\{t_{i+1}^{\pm 1}\}$ and they are connected by a t_{i+1}-band. Suppose e belongs to $[A, e_+]$ and e' belongs to $[D, e'_+]$. Then denote $B' = e_-$ and $C' = e'_-$. See Figure 5.3. Then, since the labels of sides of t_{i+1}-bands belong to X_0^* or Y_0^*, we get that the subdiagram $AB'C'D$ is another ϵ_{i+1}-contiguity subdiagram which satisfies all the conditions put on Γ in (b$_{i+1}$). But since $\|AB'\| + \|DC'\| < \|AB\| + \|DC\|$, this contradicts the minimality assumption on $\|AB\| + \|DC\|$. Therefore, there is no t_{i+1}-band with both of its ends on $AB \cup BC \cup CD$.

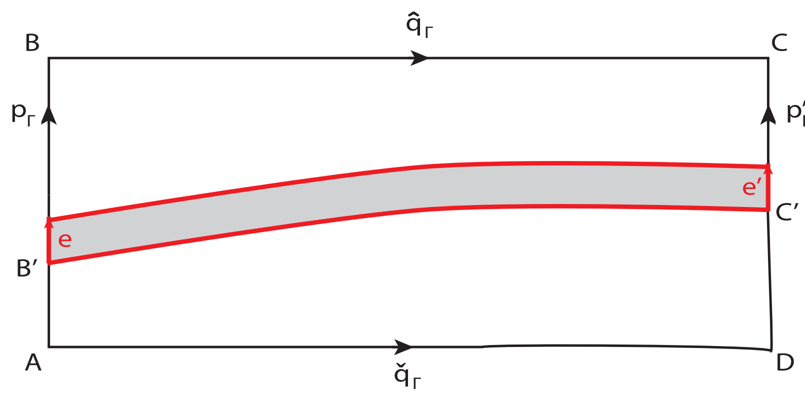

Figure 5.3

Now let us consider the cases when Γ contains a t_{i+1}-band with one of its ends on AD and the other one on $AB \cup DC$ and when it does not contain any t_{i+1}-band.

From the structure of the words from $\mathcal{R}_{i+1}$ it follows that these words contain exactly one letter from $t_{i+1}^{\pm 1}$. Therefore, since $lab(AD)$ is a subword of some word from $\mathcal{R}_{i+1}$, we get that AD contains maximum one edge with a label from $\{t_{i+1}^{\pm 1}\}$. Hence in Γ there is no t_{i+1}-band with both of its ends on AD. Thus the only possible t_{i+1}-band in Γ starts on AD and ends on $AB \cup DC$ as it is depicted in Figure 5.4.

Below we discuss in more details the only two possible cases: Case 1 – when $\check{q}_\Gamma$ does not contain an edge with a label from $\{t_{i+1}^{\pm 1}\}$ and respectively Γ does not contain a t_{i+1}-band, and Case 2 – when $\check{q}_\Gamma$ contains exactly one edge with label from $\{t_{i+1}^{\pm 1}\}$.

Case 1. ($\check{q}_\Gamma$ does not contain an edge with a label from $\{t_{i+1}^{\pm 1}\}$). In this case, clearly there is no t_{i+1}-band in Γ which starts on AD and ends on $AB \cup DC$. Therefore, Γ does not contain any t_{i+1}-band and $lab(\partial\Gamma) \in (X_0 \cup Y_0 \cup Z_0)^*$. Now, since the boundary of Γ does not contain an edge with label $t_{i+1}^{\pm 1}$, clearly, for some $0 < j \leqslant i$, Γ is a diagram over G_j. Let us assume that j is chosen to be minimal (since $G_0 = F_1 * F_2 * F_3$, j cannot be 0). Then, clearly there exists a reduced diagram over the quotient $G_j = H_j / \ll \mathcal{R}_j \gg$ with the boundary $\partial\Gamma$. Therefore, let us regard Γ as a reduced diagram over $G_j = H_j / \ll \mathcal{R}_j \gg$. From Lemma 18 and from the structure of the words $\bigcup_k \mathcal{R}_k$, it follows that there is no $\mathcal{R}_j$-cell Π_0 in Γ connected to $[A, D]$ by a ϵ_j-contiguity subdiagram Γ_0 such that $(\Pi_0, \Gamma_0, [A, D]) \geqslant \rho_j$.

Let us choose $B', C' \in [B, C]$ such that $d(A, B') = dist(A, [B, C])$ and $d(D, C') = dist(D, [B, C])$ in $\Gamma(H_j, (X_0 \cup Y_0 \cup Z_0 \cup \{t_j\})^*$. Let $[A, B']$ and $[D, C']$ be geodesics in $\Gamma(H_j, (X_0 \cup Y_0 \cup Z_0 \cup \{t_j\})^*$ joining A to B' and D to C', respectively. Note that, since $lab([A, B]), lab([B, B']) \in (X_0 \cup Y_0 \cup Z_0)^*$, by the property (a$_j$), $lab([A, B']) \in (X_0 \cup Y_0 \cup Z_0)^*$. The same way we get $lab([D, C']) \in (X_0 \cup Y_0 \cup Z_0)^*$. Therefore, from the minimality assumption on $\|[A, B]\| + \|[C, D]\| (= \|p_\Gamma\| + \|p'_\Gamma\|)$ it follows that $\|[A, B]\| = \|[A, B']\|$ and $\|[D, C]\| = \|[D, C']\|$, which means that we can simply assume that $B = B'$ and $C = C'$. Consequently, combining this observation with Lemma 23 (note that since $\|\check{q}_\Gamma\| \geqslant \mu_{i+1}\|R_{i+1}\|/2$, by LPP, we can assume that $\|\check{q}_\Gamma\| \geqslant \lambda_{i+1}(2\epsilon_{i+1} + 2\epsilon_i + 24\mu_i\|R_i\|) + c_{i+1}$, so that Lemma 23 can be applied) and with the observation that there is no $\mathcal{R}_j$-cell Π_0 in Γ connected to $[A, D]$ by a ϵ_j-contiguity subdiagram Γ_0 such that $(\Pi_0, \Gamma_0, [A, D]) \geqslant \rho_j$, we conclude that Γ does not contain an $\mathcal{R}_j$-cell. Therefore, Γ is a diagram over $H_j = \langle X \cup \{t_j\} \rangle$. But since $\partial\Gamma$ does not contain an edge with a label from $\{t_j^{\pm 1}\}$, we conclude that, in fact, Γ is a diagram over G_{j-1}, which contradicts the minimality assumption on j. Since Γ cannot be a diagram over G_0, we conclude that such a Γ does not exist. Thus Case 1 is proved.

Case 2. ($\check{q}_\Gamma$ contains exactly one edge with label from $\{t_{i+1}^{\pm 1}\}$). In this case, there exists exactly one t_{i+1}-band joining AD to AB or to DC. Without loss of generality let us assume that there is an edge e on AD and an edge e' on DC labeled by $t_{i+1}^{\pm 1}$ such that they are connected by a t_{i+1}-band. Let us denote the side $[e_+, e'_+]$ of this t_{i+1}-band by q_2. Also let us denote the diagram between q_2, $\check{q}_\Gamma$ and p'_Γ by Γ'. See Figure 5.4.

Then, since $\partial\Gamma'$ does not contain an edge with label $t_{i+1}^{\pm 1}$, from Case 1 it follows that $\big\|[e_+, D]\big\| < \mu_{i+1}\|R_{i+1}\|/2$. Otherwise, since Γ' is ϵ_{i+1}-contiguity subdiagram as well and $\partial\Gamma'$ does not contain edges labeled by $t_{i+1}^{\pm 1}$, its existence is impossible as it is shown in Case 1.

Therefore, since $\|AD\| \geqslant \mu_{i+1}\|R_{i+1}\|$, for the point $D'' \in AD$ such that $\|AD''\| = \lceil \mu_{i+1}\|R_{i+1}\|/2 \rceil$, we get that D'' is between A and e_-, i.e., AD'' does not contain an edge with label $t_{i+1}^{\pm 1}$, and by Corollary 3, we get that there is a point $C'' \in BC$ such that $d(D'', C'') \leqslant 2R_{\lambda_{i+1}, c_{i+1}} + 2\delta'_{i+1} \leqslant^{\text{by LPP}} \epsilon_{i+1}$. This means that the ϵ_{i+1}-contiguity subdiagram $ABC''D''$ satisfies all the conditions put on Γ, and since AD'' does not contain an edge with label $t_{i+1}^{\pm 1}$, we already showed that this cannot happen. See Figure 5.4 for visual description.

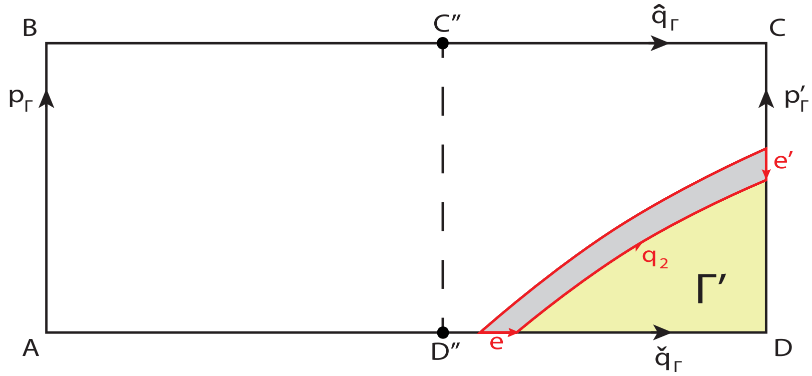

Figure 5.4: The case when $\check{q}_\Gamma$ contains an edge e with a label from $t_{i+1}^{\pm 1}$ joined by a t_{i+1}-band to p'_Γ.

(b$'_{i+1}$). Suppose that Γ is a truncated ϵ_{i+1}-contiguity subdiagram satisfying the conditions from the statement of (b$'_{i+1}$). Now let $\partial\Gamma = ABCD$, where $AB = p_\Gamma$, $BC = \hat{q}_\Gamma$, $DC = p'_\Gamma$ and $AD = \check{q}_\Gamma$ as it was in in (b$_{i+1}$) (see Figure 5.4).

Assume that Γ contains $t_{i'}$-bands with both ends on $\partial\Gamma$ for some $1 \leqslant i' \leqslant i+1$. By (b$_{i+1}$), there is no t_j-band in Γ with both of its ends on $AB \cup DC$. Also, since $lab(\check{q}_\Gamma)$ is a subword of a word $R_i \in \mathcal{R}_i$, we get that it can contain maximum one end of t-bands, where $t \in \{t_1, \ldots, t_{i+1}\}$ (more precisely, it must be that $t = t_{i+1}$).

First, let us assume that there is no t_{i+1}-band with one of its ends on $AD = \check{q}_\Gamma$. Then all t-bands of Γ have their ends on $AB \cup BC \cup DC$, and no band has its sides on the same edge. Let e_1 and e_2 be edges on AB and DC, respectively, such that they are ends of some t-bands and $[A, A']$ and $[D, D']$ do not contain ends of t-bands, where $A' = (e_1)_-$ and $D' = (e_2)_-$. Let $e'_1, e'_2 \in BC$ be the other ends of these bands, respectively. Denote $B' = (e'_1)_-$ and $D' = (e_2)_-$. Also denote the subdiagram $AA'B'C'D'D$ by Γ'. See Figure 5.5.

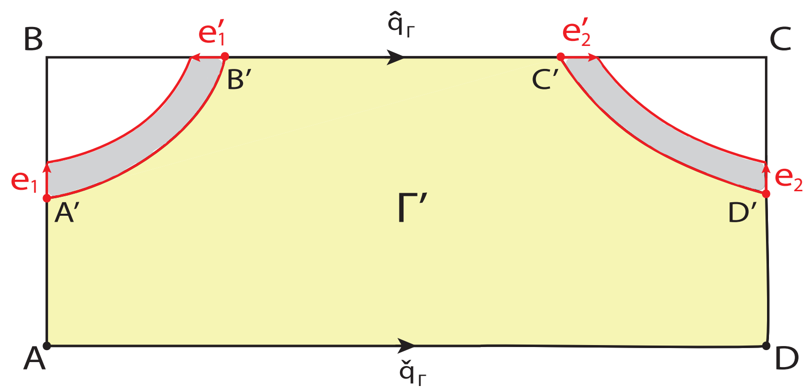

Figure 5.5

Since Γ' does not contain t-bands, it is either a diagram over G_0 or Γ' contains an $\mathcal{R}_j$-cell for some $1 \leqslant j \leqslant i$. Let us consider these two cases separately.

Case 1. If the first case holds, then, since G_0 is a free group, we get $Area(\Gamma') = 0$, in which case, since $lab([A', B']), lab([D', C']) \in X_0^* \cup Y_0^*$ and $lab([A, D]) \in Z_0^*$, we get that $A, D \in [B', C']$, but this contradicts the assumption that Γ is truncated.

Case 2. Now assume that Γ' contains an $\mathcal{R}_j$-cell for some $1 \leqslant j \leqslant i$ and j is chosen to be maximal. Then, since the sides $[A, A']$, $[A', B']$, $[B', C']$, $[C', D']$, $[D', D]$ and $[A, D]$ of Γ' are (λ_j, c_j)-quasi-geodesic in $\Gamma(H_j, X_0 \cup Y_0 \cup Z_0 \cup \{t_1, \ldots, t_j\})$, by Lemma 19, we get that Γ' contains an essential $\mathcal{R}_j$-cell Π, connected to $[A, A']$, $[A', B']$, $[B', C']$, $[C', D']$, $[D', D]$ and $[A, D]$ by essential ϵ_j-contiguity subdiagrams Γ_1, Γ_2, Γ_3, Γ_4, Γ_5 and Γ_6, respectively. See Figure 5.6.

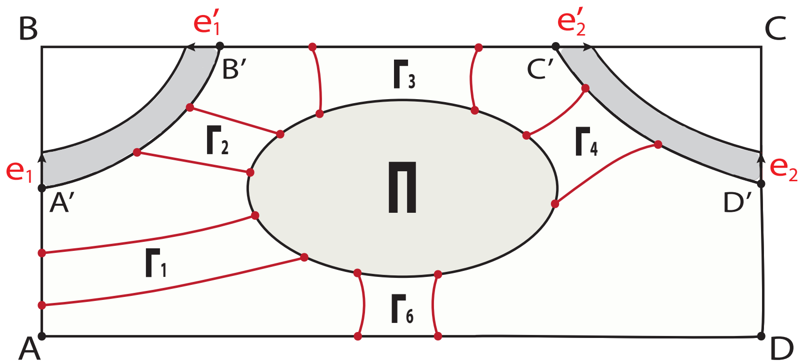

Figure 5.6

Since $\|AD\| \geqslant \mu_{i+1}\rho_{i+1}$, for sparse enough standard parameters we get that at least one of Γ_1 and Γ_5 must be empty, because, otherwise, by the triangle inequality it would be that

$$\|AD\| \geqslant \mu_{i+1}\rho_{i+1} \geqslant 2\epsilon_{i+1} + 2\epsilon_j + \|\Pi\|,$$

which we can assume to be wrong by LPP.

Therefore, without loss of generality we can assume that Γ_5 is empty. Now, from Lemma 18 it follows that $(\Pi, \Gamma_6, [A, D]) \leqslant \rho_j$. Also, because of the fact that $lab([A', B']), lab([D', C']) \in X_0^* \cup Y_0^*$, by (b$_{i+1}$), we get that

$$(\Pi, \Gamma_2, [A', B']), (\Pi, \Gamma_4, [C', D']) \leqslant \rho_j.$$

Therefore, we get

$$(\Pi, \Gamma_1, [A, A']) + (\Pi, \Gamma_3, [B', C']) > 1 - 26\rho_j,$$

which is impossible because of Lemma 23. Thus we showed that $Area(\Gamma) = 0$ when $\check{q}_\Gamma$ does not contain an edge which is an end of a t-band for $t \in \{t_1, \ldots, t_{i+1}\}$. The case when $\check{q}_\Gamma$ contains such an edge can be treated in a similar way.

(c$_{i+1}$). Now let us turn to the part (c$_{i+1}$) of the statement. By contradiction assume that there exists a reduced word $w \in X_0^*$ which is not geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Without loss of generality we can assume that w is the shortest one among such words. Then, since w is not geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$, there exists a word $w' \in \big(X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)^*$ which is geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$ and $\|w'\| < \|w\|$ and $w' =_{G_{i+1}} w$. Since $w^{-1}w' =_{G_{i+1}} 1$, there exists a reduced diagram Δ over G_{i+1} with the boundary label $w^{-1}w'$. Let us denote $\partial\Delta = pq$, where $lab(p) = w'$, $lab(q) = w$. By (c$_i$), w is geodesic in $\Gamma\big(G_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_i\}\big)$, hence the inequality $\|w'\| < \|w\|$ implies $w' \notin (X_0 \cup Y_0 \cup Z_0)^*$, i.e. w' contains a letter from $\{t_{i+1}^{\pm 1}\}$. In particular, this means that $w \neq_{G_i} w'$.

On the other hand, since w' is geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$, there is no t_{i+1}-band in Δ which starts and ends on p (otherwise, we will obtain a contradiction as in the proof of part (a$_{i+1}$)). Also, since w does not contain any letter from $\{t_{i+1}^{\pm 1}\}$, by (a$_{i+1}$) it follows that Δ does not contain t_{i+1}-bands at all. Therefore, $w \neq_{H_{i+1}} w'$, because, since $w \neq_{G_i} w'$, if $w =_{H_{i+1}} w'$ then Δ would contain a t_{i+1}-band. Therefore, Δ contains an $\mathcal{R}_{i+1}$-cell.

Let $w = w_0 x$, where $x \in X_0$. Denote the subword of q with the label w_0 by q_0 and the one with the label x by q_1. Since we chose w to be of minimal length with the mentioned properties, it must be that w_0 is a geodesic word in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Therefore, $\partial\Delta$ is a geodesic triangle in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$ with geodesic sides p, q_0 and q_1. Therefore, by Lemma 19, Δ contains an essential $\mathcal{R}_{i+1}$-cell Π connected to p, q_0 and q_1 by a system of essential ϵ_{i+1}-contiguity subdiagrams Γ_1, Γ_2 and Γ_3, respectively. See Figure 5.7.

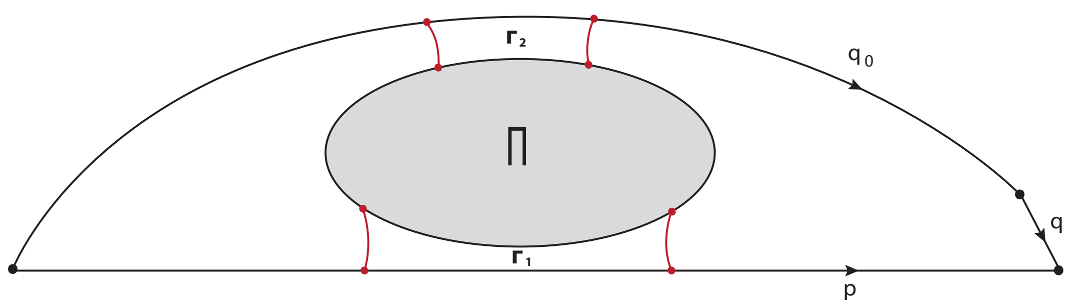

Figure 5.7: $\Delta : lab(q_0 q_1) = w$, $lab(p) = w'$.

From (b$_{i+1}$) it follows that $(\Pi, \Gamma_2, q_0) < \mu_{i+1}$ and $(\Pi, \Gamma_3, q_1) < \mu_{i+1}$. Therefore, $(\Pi, \Gamma_1, p) > (1 - 23\mu_{i+1}) - 2\mu_{i+1} = 1 - 25\mu_{i+1}$. But since p is geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$, by Lemma 22, for sparse enough standard parameters, this is impossible. A contradiction.

(d$_{i+1}$). Let $U \in X_0^*$ be a reduced word which is not a proper power of any other word from G_0. Without loss of generality assume that U is a freely cyclically reduced word. By contradiction let us assume that for some $k \geqslant 2$ and $W \in (X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\})^*$, $U =_{G_{i+1}} W^k$. First of all, it directly follows from Lemma 14 and from the inductive hypothesis (i.e. by the statement (d$_i$)) that U is not a proper power in H_{i+1}.

Now, let $W' \in \big(X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)^*$ be a cyclically minimal representative of W in G_{i+1}. This means that there exists $T \in \big(X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)^*$ such that $W =_{G_{i+1}} TW'T^{-1}$ and W' has minimal length among such words. In particular, this means that $U =_{G_{i+1}} T(W')^kT^{-1}$, and W' is cyclically geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Note that, since G_{i+1} is a quotient of H_{i+1}, we get that W' is cyclically geodesic in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$ as well. Therefore, by Lemma 8 and by LPP, $(W')^k$ is cyclically (λ_{i+1}, c_{i+1})-quasi-geodesic in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$.

Since W' is conjugate to W in G_{i+1} and $U =_{G_{i+1}} W^k$, there exists a $(U, (W')^k)$-conjugacy diagram over G_i. Hence there exists a cyclically slender $(U, (W')^k)$-conjugacy diagram over G_i. Let Δ be such a diagram. As before, let us denote $\partial\Delta = ABCD$, where $lab(BC), lab(AD)$ are cyclic shifts of $(W')^k$ and U, respectively, and $lab(AB) = lab(DC)$ are geodesic words in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Note that by (c$_{i+1}$), U is also cyclically geodesic in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$. Therefore, by Lemma 19, Δ contains an essential $\mathcal{R}_{i+1}$-cell, Π. Let Γ_1, Γ_2, Γ_3 and Γ_4 be essential ϵ_{i+1}-contiguity subdiagrams connecting Π to AB, BC, CD and DA, respectively. Since we chose Δ to be cyclically slender, by Lemma 28, Γ_2 and Γ_4 are non-empty and

$$(\Pi, \Gamma_2, BC) + (\Pi, \Gamma_4, DA) \geqslant 1 - 121\lambda_{i+1}\mu_{i+1}. \tag{5.9}$$

Also, by statement (b$_{i+1}$) and (c$_{i+1}$) of the current lemma and by LPP, since $lab(\hat{q}_{\Gamma_4}) \in X_0^*$, it follows that

$$(\Pi, \Gamma_4, DA) < \mu_{i+1}.$$

Combining this with (5.9), we get

$$(\Pi, \Gamma_2, BC) > (1 - 121\lambda_{i+1}\mu_{i+1}) - \mu_{i+1} > 1 - 122\lambda_{i+1}\mu_{i+1}. \tag{5.10}$$

Therefore, since W' is cyclically geodesic in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$, by LPP and by Lemma 22, we get that $lab(\hat{q}_{\Gamma_2})$ is not a subword of a cyclic shift of W'. This means that $lab(\hat{q}_{\Gamma_2})$ is of the form

$$lab(\hat{q}_{\Gamma_2}) = (W'')^n Q,$$

where W'' is a cyclic shift of W', $n \geqslant 1$, and Q is a prefix of W''.

Let us separately consider the cases when $n = 1$ and when $n > 1$.

Before that, let us notice that by Corollary 3, the Hausdorff distance between $\check{q}_{\Gamma_2}$ and $\hat{q}_{\Gamma_2}$ is bounded from above by $\epsilon_{i+1} + 2R_{\lambda_{i+1},c_{i+1}} + 2\delta_{i+1} <^{\text{by LPP}} 2\epsilon_{i+1}$.

Case 1. ($n = 1$). For this case, let us partition $\hat{q}_{\Gamma_2} = \hat{q}_1\hat{q}_2\hat{q}_3$, where $lab(\hat{q}_1) = lab(\hat{q}_3) = Q$. Let us also partition $\check{q}_{\Gamma_2} = \check{q}_1\check{q}_2\check{q}_3$ such that $(\check{q}_1)_+$ and $(\check{q}_2)_+$ are the closest to $(\hat{q}_1)_+$ and to $(\hat{q}_2)_+$ points on $\check{q}_{\Gamma_2}$, respectively. Since $lab(\hat{q}_1) = lab(\hat{q}_3)$, from the observation right above Case 1 and from Lemma 24, it follows that $\|\check{q}_1\|, \|\check{q}_3\| \leqslant 2\mu_{i+1}\|\Pi\| < \mu_{i+1}\lambda_{i+1}\|\Pi\|$. Then combining this with (5.10), we get that $\|\check{q}_1\check{q}_2\| > (1 - 23\lambda_{i+1}\mu_{i+1})\|\Pi\|$. But, since $\|W''\| = \big\|\check{q}_1\check{q}_2\big\|$ and W'' is a geodesic word in $\Gamma\big(G_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$, this is impossible for sparse enough standard parameters. Thus we are done with Case 1.

Case 2. ($n > 1$). For this case, again we partition $\hat{q}_{\Gamma_2}$ into three parts $\hat{q}_{\Gamma_2} = \hat{q}_1\hat{q}_2\hat{q}_3$ such that $lab(\hat{q}_1) = lab(\hat{q}_3)$ and $lab(\hat{q}_2)$ is a suffix of W''. Then, since $lab(\hat{q}_{\Gamma_2}) = (W'')^n Q$ and $n \geqslant 2$, we get that $\|\hat{q}_1\| = \|\hat{q}_3\| > \frac{1}{3}\|\hat{q}_2\|$, hence $\|\hat{q}_{\Gamma_2}\| < 3\|\hat{q}_1\|$. Also, just like we showed in case $n = 1$, by Lemma 24, in this case also $\|\hat{q}_1\|, \|\hat{q}_3\| \leqslant 2\mu_{i+1}\|\Pi\|$. Therefore, $\|\hat{q}_{\Gamma_2}\| < 6\mu_{i+1}\|\Pi\|$. But, since by LPP $1 - 122\lambda_{i+1}\mu_{i+1} > 6\mu_{i+1}$, we get a contradiction with (5.10). The case when $U \in Y_0^*$ can be dealt in the same way. Thus we are done with this case as well.

($\mathbf{e}_{i+1}$). By contradiction, let us assume that for some non-trivial reduced words $U \in X_0^*$, $V \in Y_0^*$ we have $U =_{G_{i+1}} V$. Then there exists a reduced van Kampen diagram Δ such that $\partial\Delta = q_1 q_2^{-1}$ and $lab(q_1) = U$, $lab(q_2) = V$. Since U and V do not contain letters from $\{t_{i+1}^{\pm 1}\}$ and $U \neq_{G_i} V$, by (e$_i$) we have that Δ contains an $\mathcal{R}_{i+1}$-cell. Therefore, since by (c$_{i+1}$) U and V

are geodesic word in $\Gamma\big(H_{i+1}, X_0 \cup Y_0 \cup Z_0 \cup \{t_{i+1}\}\big)$, by Lemma 19, Δ contains an essential $\mathcal{R}_{i+1}$-cell, Π. Let us assume that Π is connected to q_1 and q_2 by ϵ_{i+1}-contiguity subdiagrams Γ_1 and Γ_2, respectively. Then we have $(\Pi, \Gamma_1, q_1) + (\Pi, \Gamma_2, q_2) \geqslant 1 - 23\mu_{i+1}$. But, on the other hand, by (b_{i+1}) we have that $(\Pi, \Gamma_1, q_1) + (\Pi, \Gamma_2, q_2) < 2\mu_{i+1}$. But since by LPP we can assume

$$2\mu_{i+1} < 1 - 23\mu_i,$$

we get a contradiction.

($\mathbf{f}_{i+1}$). The fact that H_i is a hyperbolic group follows from Theorem 8 and parts (d_{i+1}) and (e_{i+1}) of the current lemma. □

Corollary 13. *Suppose $U \in X^*$ is a $(\lambda_i, c_i, \epsilon_i, 1 - 122\lambda_i\mu_i)$-cyclic-reduced word for $i = \mathcal{I}(\|U\|)$, and $U =_{G_{\mathcal{L}}} V$ for some $V \in X_0^* \cup Y_0^*$. Then $U \in X_0^* \cup Y_0^*$.*

Proof. This directly follows from properties (a_i),(b_i) and Lemma 19. □

5.4.1 The conjugacy problem in $G_{\mathcal{L}}$

Lemma 42. *If $U \in X_0^* \backslash \{1\}$, $V \in Y_0^* \backslash \{1\}$, then U is not G-conjugate to V in $G_{\mathcal{L}}$.*

Proof. This follows immediately from Lemma 26 and properties (b_i), (c_i). □

Definition 5.1 (Λ-pairs of words)**.** We say that a pair of words $(U, V) \in X^* \times X^*$ is a Λ-pair if either U is a cyclic shift of V or for some $k \in \mathbb{N}, l \in \mathbb{Z}$, U is a cyclic shift of u_k^l and V is a cyclic shift of v_k^l or vice versa (i.e. V is a cyclic shift of u_k^l and U is a cyclic shift of v_k^l).

Note that the Λ-pair relation is an equivalence relation.

Lemma 43. *Two cyclically reduced non-empty words $U, V \in X^* \cup Y^*$ are conjugate one to each other in $G_{\mathcal{L}}$ if and only if (U, V) is a Λ-pair. Moreover, if U is not a cyclic shift of V, then U is not G-conjugate to V in $G_{\mathcal{L}}$.*

Proof. Assume that U and V freely reduced words which are conjugate in $G_{\mathcal{L}}$. The case when U is a cyclic shift of V is trivial. Therefore, without loss of generality assume that U is not a cyclic shift of V. Then, by the definition of G_0, it is clear that U and V are not conjugate in G_0. Therefore, there exists a minimal index $i \geqslant 1$ such that U and V are conjugate in G_i. This means that there exists a minimal cyclically slender (U, V)-conjugacy diagram Δ of rank i. If $i = 0$ then the statement of the lemma follows from basic properties of free groups. Suppose that U, V are chosen so that the corresponding index $i \in \mathbb{N}$ is minimal. Now, let us assume that $i \geqslant 1$ and apply induction on i.

As usual, let us denote the boundary $\partial\Delta$ of Δ by $ABCD$. Let $U' = lab(BC)$ and $V' = lab(AD)$, where U' and V' are some cyclic shifts of U and V, respectively.

Claim 1. U is not G-conjugate to V in $G_{\mathcal{L}}$.

Proof of the claim. Indeed, assume that U is G-conjugate to V in $G_{\mathcal{L}}$. Then, since by the property (c$_i$), U and V are geodesic words in $\Gamma(H_i, X \cup \{t_1, \ldots, t_j\})$, according to Lemma 28, we get that at least one of U' and V' must contain a $(\epsilon_i, (1 - 121\lambda_i\mu_i)/2)$-subword, which contradicts to property (b$_i$). □

By Claim 1, we get that Δ is a slender (U, V)-conjugacy diagram over H_i. Therefore, since we chose the index i to be minimal, Δ contains t_i-bands. Since $lab(AB)$ and $lab(DC)$ do not contain letters from $\{t_i^{\pm 1}\}$, we get that the t_i-bands of Δ must be horizontal, i.e. their ends belong to $[A, B]$ and $[D, C]$.

Now let us choose an edge e_1 on the side AB such that $lab(e_1) \in \{t_i^{\pm 1}\}$ and $lab([(e_1)_+, B])$ does not contain $t_i^{\pm 1}$. From the basic properties of HNN-extensions, it it follows that there exists an edge e_1' on DC such that $lab(e_1) \in \{t_i^{\pm 1}\}$ and e_1 is connected to e_1' by a t_i-band. Moreover, $lab([(e_1)_+, B]) = lab([(e_1')_+, C])$.

Let us denote the side of the t_i-band connecting $(e_1)_+$ to $(e'_1)_+$ by p_1 and the side connecting $(e_1)_-$ to $(e'_1)_-$ by q_1. See Figure 5.8. Then $lab(p_1)$ belongs to either X_0^* or Y_0^*. Denote $U'' = lab(p_1)$.

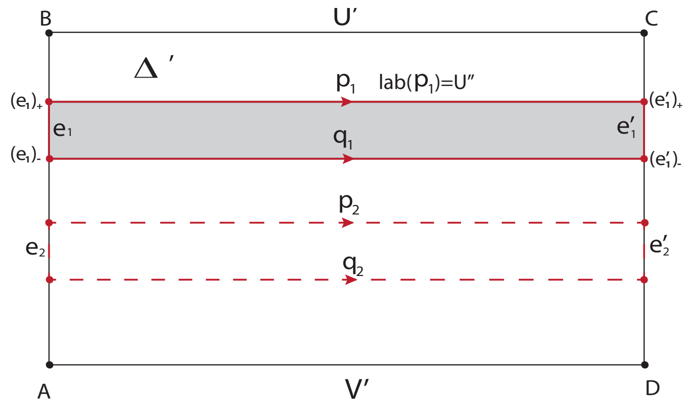

Figure 5.8: The cyclically slender (U, V)-conjugacy diagram Δ. Below it is shown that the second t_i-band, joining e_2 to e'_2, actually, cannot exist.

Let us denote by Δ' the (U', U'')-conjugacy subdiagram of Δ which is bounded between $(e_1)_+$, B, C and $(e_1)'_+$. See Figure 5.8. Since Δ' does not contain R_i-cells and t_i-bands, we conclude that it is a (U', U'')-conjugacy diagram of rank j where $0 \leqslant j < i$, hence U' is conjugate to U'' in G_j. On the other hand, since $U'' \in X^*$ or Y^* and since i was chosen to be minimal, by inductive argument on i we conclude that either U'' is a cyclic shift of U' (hence also of U) or U' is a cyclic shift of $lab(q_1)$ (this means that $(U, lab(p_1))$ is a Λ-pair).

Now we are going to show that besides the considered t_i-band, Δ does not contain any other t_i-band. Assume that this is not true. Then there exist edges e_2 and e'_2 on AB and DC, respectively, such that they have a label from $\{t_i^{\pm 1}\}$, and between e_2 and e_1 (also between e'_2 and e'_1, respectively) there is no other edge with label from $\{t_i^{\pm 1}\}$. See Figure 5.8. Then it must be that e_2 is connected to e'_2 by a t_i-band. Correspondingly, define p_2 and q_2 as we defined p_1 and q_1. Then repeating the above stated arguments we get that $\big(lab(q_1), lab(q_1)\big)$ is a Λ-pair. The last observation implies that either $lab(p_1) \equiv lab(p_2)$ or $lab(p_1) \equiv lab(2_2)$, which is impossible by Lemma 25. Thus Lemma 43 is proved.

$\square$

The next lemma is a stronger version of Lemma 43.

Lemma 44. *Let $U \in (X_0 \cup Y_0 \cup Z_0)^*$ and $V \in X_0^* \cup Y_0^*$. Then $U \sim_{conj} V$ in $G_{\mathcal{L}}$ if and only if (U', V) is a Λ-pair, where U' is any $(\lambda_i, c_i, \epsilon_i, 1 - 122\lambda_i\mu_i)$-cyclic reduction of U for $i = \mathcal{I}(\|U\|)$.*

Proof. Suppose that $U \sim_{conj} V$ in $G_{\mathcal{L}}$. Let us fix a $(\lambda_i, c_i, \epsilon_i, 1 - 122\lambda_i\mu_i)$-cyclic reduction U' of U. Then, clearly $U' \sim_{conj} V$ in $G_{\mathcal{L}}$. Let us separately consider two cases: Case 1 – U' is G-conjugate to V in $G_{\mathcal{L}}$, and Case 2 – U' is H-conjugate to V in $G_{\mathcal{L}}$.

Case 1. (U' is G-conjugate to V in $G_{\mathcal{L}}$). In case $U \sim_{conj} V$ in G_0 the statement of the lemma is trivial. Now suppose that $U \not\sim_{conj} V$ in G_0. Then, by Lemma 38, there exists an index $j \in \mathbb{N}$ such that $j \leqslant i$ and $U \sim_{conj} V$ in G_j but $U \not\sim_{conj} V$ in H_j. Therefore, by Lemma 28, U' and V contain (ϵ_j, κ_1)- and (ϵ_j, κ_2)-arcs, respectively, such that $\kappa_1 + \kappa_2 \geqslant 1 - 122\lambda_j\mu_j$. Also, since $V \in X_0^* \cup Y_0^*$, by property (b_j), $\kappa_2 \leqslant \rho_j$. Therefore, U' contains a $(\epsilon_j, 1 - 122\lambda_j\mu_j)$-subword, which contradicts to the fact that U' is cyclically $(\lambda_j, c_j, \epsilon_j, 1 - 122\lambda_j\mu_j)$-reduced. So we are done with Case 1.

Case 2. (U' is H-conjugate to V in $G_{\mathcal{L}}$). Then for some $k \in \mathbb{N}$, $U' \sim_{conj} V$ in H_k and $U' \not\sim_{conj} V$ in G_{k-1}, and consequently, there exists a slender (U', V)-conjugacy diagram Δ over $H_k = \langle X \cup \{t_1, \ldots, t_k\}\rangle$ which contains at least one t_k-band. Note that, since V does not contain a letter from $\{t_k^{\pm 1}\}$, Δ must contain only horizontal t_k-bands. Without loss of generality assume that $V \in X_0^* \cup Y_0^*$ is chosen so that Δ contains minimal number of horizontal t-bands, $t \in \{t_1, t_2, \ldots\}$. Therefore, by a standard inductive argument and by Lemma 43, we get that U' and V form a Λ-pair with a label of a side of any (horizontal) t_k-band from Δ, hence, since Λ-pair relation is an equivalence relation, we get that (U', V) is a Λ-pair. $\square$

Lemma 45. *Let $U, V \in (X_0 \cup Y_0 \cup Z_0)^*$. Suppose that $U', V' \in (X \cup \{t_1, \ldots, t_j\})^*$ are any $(\lambda_i, c_i, \epsilon_i, 1 - 122\lambda_i\mu_i)$-cyclic-reductions of U and V, respectively, where $i = \mathcal{I}(\|U\| + \|V\|)$. Then U is H-conjugate to V in $G_{\mathcal{L}}$ if and only if (U', V') is a Λ-pair and U' is not a cyclic shift of V'.*

Proof. First of all, if (U, V) is a Λ-pair, then, clearly, $U \sim_{conj} V$ in $G_{\mathcal{L}}$. Moreover, by Lemma 43, if U is not a cyclic shift of V, then U is not G-conjugate to V, hence U is H-conjugate to V in $G_{\mathcal{L}}$.

Now let us assume that U is H-conjugate to V in $G_{\mathcal{L}}$. Then there exists an index $j \in \mathbb{N}$ such that U is conjugate to V in H_j, but U is not conjugate to V in G_{j-1}. This means that there exists a slender (U, V)-conjugacy diagram Δ over H_j which contains a horizontal t_j-band. Therefore, U and V are conjugate to conjugate words from $X_0^* \cup Y_0^*$. Hence, by Lemma 44 and by the fact that Λ-pair relation is an equivalence relation, we get that (U', V') is a Λ-pair. $\square$

The next lemma is an obvious corollary from the structure of the words of $\Lambda(\mathcal{A}^*)$ and definition of Λ-pairs.

Lemma 46. *The decision problem which for any pair of words $U, V \in (X_0 \cup Y_0 \cup Z_0)^*$ asks whether or not (U, V) is a Λ-pair can be strongly reduced to the membership problem for $\mathcal{L} \subseteq \mathcal{A}^*$ in $\mathcal{O}(\|U\| + \|V\|)$ time.*

The combination of Lemma 43 and Lemma 45 implies the following proposition.

Proposition 10. *Suppose that $U, V \in (X_0 \cup Y_0 \cup Z_0)^*$. Then, $U \sim_{conj} V$ if and only if exactly one of the following holds.*

1. *U is G-conjugate to V in $G_{\mathcal{L}}$;*
2. *(U', V') is a Λ-pair and U' is not a cyclic shift of V', where $U', V' \in (X_0 \cup Y_0 \cup Z_0 \cup \{t_i\})^*$ are $(\lambda_i, c_i, \epsilon_i, 1 - 122\lambda_i\mu_i)$-cyclic reductions of U and V, respectively, for $i = \mathcal{I}(\|U\| + \|V\|)$.*

Proof. This proposition directly follows from Lemma 43 and Lemma 45. $\square$

Proposition 11. *The membership problem for $\mathcal{L} \subseteq \mathcal{A}^*$ can be strongly reduced to the conjugacy problem in $G_{\mathcal{L}}$ in linear time; and the H-conjugacy problem in $G_{\mathcal{L}}$ can be strongly reduced to the membership problem for $\mathcal{L} \subseteq \mathcal{A}^*$ in almost linear time.*

Proof. Indeed, it follows from the definition of Λ-pairs that for any $\omega \in \mathcal{A}^*$, $\omega \in \mathcal{L}$ if and only if the pair of words $\Lambda(\omega)$ is a Λ-pair. Therefore, since $\Lambda(\omega) \in Y_0^* \times Z_0^*$, by Lemma 43, $\omega \in \mathcal{L}$ if and only if $\Lambda(\omega)$ is a pair of words conjugate in $G_{\mathcal{L}}$.

Now let us show the opposite side. For that let us consider a pair of words $(U, V) \in X^* \times X^*$. Then one can find $(\lambda_i, c_i, \epsilon_i, 1 - 122\lambda_i\mu_i)$-cyclic-reductions U' and V' of U and V, respectively, in almost linear time, where $i = \mathcal{I}(\|U\| + \|V\|)$ (see Remark 11). Therefore, since by Lemma 45 U' is H-conjugate to V' in $G_{\mathcal{L}}$ if and only if (U', V') is a Λ-pair, by Lemma 46, the H-conjugacy problem in $G_{\mathcal{L}}$ can be strongly reduced to the membership problem for $\mathcal{L} \subseteq \mathcal{A}^*$ in almost linear time. $\square$

5.4.1.1 Geometry of slender G-conjugacy diagrams and time complexity of the G-conjugacy problem in $G_{\mathcal{L}}$

Lemma 47. *Let for some $i \in \mathbb{N}$, $U, V \in (X_0 \cup Y_0 \cup Z_0 \cup \{t_1, \ldots, t_i\})^*$ be $(\lambda_i, c_i, \epsilon_i, 1 - 122\lambda_i\mu_i)$-cyclic-reduced words in $\Gamma\big(H_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_1, \ldots, t_i\}\big)$ and suppose $U \sim_{conj} V$ in G_i but $U \nsim_{conj} V$ in H_i. Then there exists a (U, V)-conjugacy diagram Δ over*

$$G_i = \langle X_0 \cup Y_0 \cup Z_0 \cup \{t_1, \ldots, t_i\} \mid \mathcal{R}_j, t_j^{-1}u_jt_jv_j^{-1}, 1 \leqslant j \leqslant i\rangle$$

such that $\partial\Delta = ABCD$, $lab(AD) \equiv U$, $lab(BC) \equiv V$, $lab(AB) \equiv lab(DC)$ and for every cell Π in Δ, $\partial\Pi \cap AD$, $\partial\Pi \cap BC \neq \varnothing$. Moreover, if Π is an $\mathcal{R}_j$-cell for some $1 \leqslant j \leqslant i$, then $\|\Pi \cap AD\|, \|\Pi \cap BC\| \geqslant \rho_j\|\Pi\|$. Also, if Π is a cell with label of the form $t_j^{-1}u_jt_jv_j^{-1}$, then u_j is contained either in $lab(\Pi \cap AD)$ or in $lab(Pi \cap BC)$, and the same is true about v_j

Proof. Let Δ_0 be a reduced cyclically slender (U, V)-conjugacy diagram over G_i. Let $\partial\Delta_0 = A_0B_0C_0D_0$ be such that $lab([A_0, D_0]) = U'$ and $lab([B_0, C_0]) = V'$ for some cyclic shifts U' and V' of U and V, respectively. Then, by Lemma 28, there exists an $\mathcal{R}_i$-cell Π connected by non-empty ϵ_i-conjugacy subdiagrams Γ and Γ' to A_0D_0 and B_0C_0 such that $(\Pi, \Gamma, A_0D_0) + (\Pi, \Gamma', B_0C_0) \geqslant 1 - 121\lambda_i\mu_i$. Without loss of generality assume that Γ and Γ' are truncated. Now, since U' and V' are $(\lambda_i, c_i, \epsilon_i, 1-122\lambda_i\mu_i)$-reduced, we get $(\Pi, \Gamma, A_0D_0), (\Pi, \Gamma', B_0C_0) < 1-122\lambda_i\mu_i$. Therefore,

$$(\Pi, \Gamma, A_0D_0), (\Pi, \Gamma', B_0C_0) > \mu_i.$$

Hence, by property (b$'_i$), we get that $Area(\Gamma) = Area(\Gamma') = 0$.

$$\|W\|, \|W'\| \geqslant \mu\|R'_i\|.$$

Now the proof of Lemma 47 follows after applying some standard inductive arguments. $\square$

Visually, Lemma 47 tells us that if for some $i \in \mathbb{N}$, $U, V \in (X_0 \cup Y_0 \cup Z_0 \cup \{t_1, \ldots, t_i\})^*$ are

$(\lambda_i, c_i, \epsilon_i, 1-122\lambda_i\mu_i)$-cyclic-reduced words in $\Gamma\big(H_i, X_0 \cup Y_0 \cup Z_0 \cup \{t_1, \ldots, t_i\}\big)$ and $U \sim_{conj} V$ in G_i but $U \nsim_{conj} V$ in H_i, then there exists a (U,V)-conjugacy diagram Δ which looks like in Figure 5.9, where by $\Pi_1, \ldots \Pi_k$ we denoted the cells of Δ.

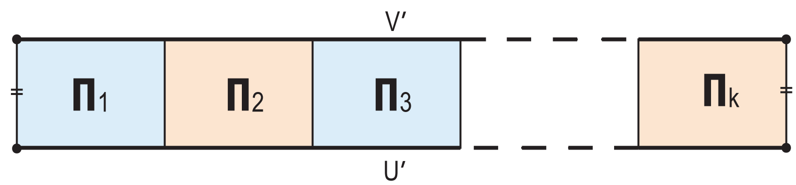

Figure 5.9

Lemma 48. *The G-conjugacy problem in $G_{\mathcal{L}}$ is decidable in almost linear time.*

Proof. Indeed, in order to check for a given pair of words $(U,V) \in X^* \times X^*$ whether or not U is G-conjugate to V in $G_{\mathcal{L}}$, one can first compute $(\lambda_i, c_i, \epsilon_i, 1-122\lambda_i\mu_i)$-cyclic-reductions U' and V' of U and V, respectively, in almost linear time (see Remark 11), then check whether there exists a (U',V')-conjugacy diagram satisfying the properties described in Lemma 47. Notice that the last checking can be done in $\mathcal{O}(\|U\| + \|V\|)$ time. Therefore, the whole checking can be done in almost linear time. □

5.4.1.2 Main properties of $G_{\mathcal{L}}$

Combining Proposition 11 with Lemma 48 one immediately gets the statements $(II.i)$ and $(II.ii)$ of Theorem 5.

Corollary 14. *The membership problem for $\mathcal{L} \subseteq \mathcal{A}^*$ can be strongly reduced to the conjugacy problem in $G_{\mathcal{L}}$ in linear time; and the conjugacy problem in $G_{\mathcal{L}}$ can be strongly reduced to the membership problem for $\mathcal{L} \subseteq \mathcal{A}^*$ in almost linear time.*

Another corollary is the following.

Corollary 15. *The individual conjugacy problems in $G_{\mathcal{L}}$ are decidable in almost linear time.*

Proof. Let us fix an element $g \in G_{\mathcal{L}}$ and let $U \in X^*$ be a reduced word representing g. The key observation is that since there are only finitely many words W such that (U,W) form a Λ-pair, it can be checked in a fixed time whether or not (U,V') form a Λ-pair. Therefore, without loss of generality we can assume that (U, V') do not form a Λ-pair. Hence, from Lemmas 43 and 45 it follows that $U \sim_{conj} V'$ in $G_{\mathcal{L}}$ if and only if U is G-conjugate to V' in $G_{\mathcal{L}}$, which can be checked in almost linear time according to Lemma 48.

□

5.5 Proof of Theorem 6

In this section we are going to construct a lacunary hyperbolic group $\tilde{G}$ which satisfies the properties of Theorem 6. $\tilde{G}$ will be constructed as a direct limit of a chain

$$G_0 \overset{\beta_0}{\hookrightarrow} H_1 \overset{\gamma_1}{\twoheadrightarrow} G_1 \overset{\beta_1}{\hookrightarrow} H_2 \overset{\gamma_2}{\twoheadrightarrow} \dots \tag{5.11}$$

of non-elementary torsion-free hyperbolic groups of the form (4.6) according to the scheme described in Section 5.1. More specifically, $G_0 \overset{def}{=} F(X)$, where $X = \{x_1, x_2\}$.

Let $\mathcal{N} = \{n_1, n_2, \dots\} \subset \mathbb{N}$ be a fixed recursively enumerable but not recursive subset of positive integers. Let us enumerate elements of G_0 according to their lexicographical order as $G_0 = \{1 = u_0, u_1, u_2, \dots\}$ and denote $\mathcal{U} = \{u_1, u_2, \dots\}$. Let $\mathcal{V} = \sqcup_{i=1}^{\infty}\mathcal{U} = \{v_1, v_2, \dots\}$ be a disjoint union of copies of $\mathcal{U}$ with recursive enumeration such that before the next copy of a given element $u \in \mathcal{U}$ appears in $\mathcal{V}$, all the elements preceding u already appeared in $\mathcal{V}$ at least once.

Denote by $\mathcal{P} = \{p_1, p_2, \dots\}$ the set of prime numbers indexed in correspondence with their natural order.

5.5.0.1 Definition of H_{i+1} for $\tilde{G}$

Suppose that for $i \geqslant 0$, G_i is already constructed and it satisfies the following properties.

A1. G_i is a non-elementary torsion-free δ_i-hyperbolic group for $\delta_i \in \mathbb{N}$,

A2. Words of the form $x_1^n x_2$, $n \in \mathbb{Z}$, are not proper powers in G_i.

Below we show how to construct G_{i+1} from G_i which, in particular, preserves properties A1 and A2.

Suppose that the set $\{\tilde{v}_{j_1}, \tilde{v}_{j_2}, \ldots, \tilde{v}_{j_i}\}$ of words from X^* is such that its elements are not proper powers in G_i, i.e. $E(\tilde{v}_{j_k}) = \langle \tilde{v}_{j_k} \rangle$ in G_i for $1 \leqslant k \leqslant i$. Then define $v_{j_{i+1}}$ as the element from $\{v_{j_i+1}, v_{j_i+2}, \ldots\}$ of minimal index such that $v_{j_{i+1}}$ does not represent the trivial element in G_i. Now define $\tilde{v}_{j_{i+1}}$ as follows.

- If $v_{j_{i+1}}$ commensurates with any element from $\{\tilde{v}_{j_1}, \ldots, \tilde{v}_{j_i}\}$ in G_i, then define $\tilde{v}_{j_{i+1}} \equiv \tilde{v}_{j_k}$, where $1 \leqslant k \leqslant i$ is the smallest index such that $v_{j_{i+1}}$ commensurates with $\tilde{v}_{j_k}$ in G_i;
- Otherwise, if $v_{j_{i+1}}$ commensurates in G_i with an element of the form $x_1^n x_2$, then define $\tilde{v}_{j_{i+1}} \equiv x_1^{n_0} x_2$, where $|n_0|$ is the smallest positive number such that $v_{j_{i+1}}$ commensurates in G_i with $x_1^{n_0} x_2$;
- Otherwise, if $v_{j_{i+1}}$ is not a proper power in G_i, then define $\tilde{v}_{j_{i+1}} \equiv v_{j_{i+1}}$;
- Otherwise, if $v_{j_{i+1}}$ is a proper power in G_i, then define $\tilde{v}_{j_{i+1}}$ to be a cyclically geodesic word in $\Gamma(G_i, X)$ such that $E(v_{j_{i+1}}) = \langle \tilde{v}_{j_{i+1}} \rangle$ in G_i.

Define $q_1 = p_1$ and suppose that the set $\{q_1, \ldots, q_i\}$ of prime numbers is already defined such that for $1 \leqslant k, l \leqslant i$, $q_k = q_l$ whenever $\tilde{v}_{j_k} \equiv \tilde{v}_{j_l}$. Then, define $q_{i+1} \in \mathbb{N}$ as follows.

- If $\tilde{v}_{j_{i+1}} \equiv \tilde{v}_{j_k}$ for some $1 \leqslant k \leqslant i$, then define $q_{i+1} = q_k$;
- Otherwise, define $q_{i+1} = p_{i+1}$.

Define $\tilde{n}_{i+1} = n_{s+1} \in \mathcal{N}$, where $s = \#\{1 \leqslant k \leqslant i \mid \tilde{v}_{j_k} \equiv \tilde{v}_{j_{i+1}}\}$. Now define $\tilde{w}_{i+1} = x_1^{q_{i+1}^{\tilde{n}_{i+1}}} x_2$. Define H_{i+1} as follows: If $\tilde{w}_{i+1}$ commensurates with $\tilde{v}_{j_{i+1}}$ in G_i, then $H_{i+1} = G_i$, otherwise H_{i+1} is an HNN-extension of G_i. More precisely,

$$H_{i+1} = \langle G_i, t_{i+1} \mid t_{i+1}^{-1} \tilde{v}_{j_{i+1}} t_{i+1} = \tilde{w}_{i+1} \rangle. \tag{5.12}$$

Lemma 49. *H_{i+1} is non-elementary torsion-free δ'_{i+1}-hyperbolic group for some $\delta'_{i+1} \in \mathbb{N}$.*

Proof. Hyperbolicity of H_{i+1} follows directly from the above mentioned assumption A2 when combined with Theorem 8, because by definition $\tilde{w}_{i+1}$ does not commensurate with $\tilde{v}_{j_{i+1}}$.

The fact that H_{i+1} is non-elementary and torsion-free follows from the basic properties of HNN-extensions, namely, by the fact that G_i embeds in H_{i+1} and by Lemma 14.

□

Lemma 50. *The words* $\{\tilde{v}_{j_1}, \ldots, \tilde{v}_{j_{i+1}}\}$ *and the words of the form* $x_1^m x_2$, $m \in \mathbb{Z}$, *are not proper powers in* H_{i+1} *(provided that this statement is true for* G_i*).*

Proof. Directly follows from Lemma 14. □

Lemma 51. $t_{i+1} \notin E(x_1) \cup E(x_2)$, $x_1 \notin E(x_2)$ *and* $x_2 \notin E(x_1)$ *in* H_{i+1}.

Proof. Indeed, it follows from the basic properties of HNN-extensions and from the inductive assumption that G_i is a non-elementary group. □

5.5.0.2 Definition of G_{i+1}

Suppose that H_{i+1} is already constructed and it is a non-elementary torsion free δ'_{i+1}-hyperbolic group for $\delta'_{i+1} \in \mathbb{N}$ such that the map $i+1 \mapsto \delta'_{i+1}$ is computable. Define G_{i+1} as follows: If $H_{i+1} = G_i$, then $G_{i+1} = G_i$, otherwise

$$G_{i+1} = H_{i+1} / \ll \mathcal{R}\big(\{t_{i+1}\}, x_1, x_2, \delta'_{i+1}, \lambda_{i+1}, c_{i+1}, \epsilon_{i+1}, \mu_{i+1}, \rho_{i+1}\big) \gg . \tag{5.13}$$

Denote $\mathcal{R}\big(\{t_{i+1}\}, x_1, x_2, \delta'_{i+1}, \lambda_{i+1}, c_{i+1}, \epsilon_{i+1}, \mu_{i+1}, \rho_{i+1}\big)$ by $\mathcal{R}_{i+1}$.

Lemma 52. *For sparse enough standard parameters* $\lambda_{i+1}, c_{i+1}, \epsilon_{i+1}, \mu_{i+1}, \rho_{i+1}$*, the group* G_{i+1} *is torsion-free non-elementary* δ_{i+1}*-hyperbolic for some* $\delta_{i+1} \in \mathbb{N}$.

Proof. Follows directly from Lemmas 51 and 20. □

Lemma 53. *For sparse enough standard parameters* λ_{i+1}, c_{i+1}*, words of the form* $x_1^m x_2$ *are cyclically* (λ_{i+1}, c_{i+1})*-quasi-geodesic in* $\Gamma(H_{i+1}, X \cup \{t_{i+1}\})$.

Proof. Let $x_1^{m_1}x_2x_1^{m_2}$ be an arbitrary subword of a cyclic shift of the word $x_1^m x_2$, where $|m_1| + |m_2| \leqslant |m|$. Then, since by Lemma 51, $x_2 \notin E(x_1)$ in H_{i+1}, by Lemma 10, either $\max\{m_1, m_2\} < f(|x|, \delta_{i+1}, \lambda'_{i+1}, c'_{i+1}, 1)$ (f is defined as in Lemma 51) or

$$|x_1^{m_1}x_2x_1^{m_2}|_{H_{i+1}} > \frac{1}{12\lambda'_{i+1}} \max\{m_1, m_2\} > \frac{1}{24\lambda'_{i+1}} \|x_1^{m_1}x_2x_1^{m_2}\| - 1,$$

where λ'_{i+1} and c'_{i+1} are such that for each $n \in \mathbb{Z}$, x^n is $(\lambda'_{i+1}, c'_{i+1})$-quasi-geodesic in $\Gamma(H_{i+1}, X \cup \{t_{i+1}\})$ (they can be computed according to Lemma 7). Therefore, since $x_1^{m_1}x_2x_1^{m_2}$ was taken arbitrarily, we get that $x_1^m x_2$ is cyclically $(24\lambda_{i+1}, c_{i+1})$-quasi-geodesic. Consequently, in order Lemma 53 to be true, it is enough to require that $\lambda_{i+1} \geqslant 24\lambda'_{i+1}$ and $c_{i+1} \geqslant 1$. □

Lemma 54. *For sparse enough standard parameters $\lambda_{i+1}, c_{i+1}, \epsilon_{i+1}, \mu_{i+1}, \rho_{i+1}$, no word of the form $(x_1^{m_1}x_2x_1^{m_2})^{\pm 1}$, $m_1, m_2 \in \mathbb{Z}$, has a $(\epsilon_{i+1}, \lambda_{i+1}\mu_{i+1})$-subword with respect to the quotient $G_{i+1} = H_{i+1}/ \ll \mathcal{R}_{i+1} \gg$, and (λ_{i+1}, c_{i+1})-quasi-geodesic words in $\Gamma(G_{i+1}, X \cup \{t_{i+1}\})$ do not contain $(\epsilon_{i+1}, 1 - 122\lambda_{i+1}\mu_{i+1})$-subwords.*

Proof. Indeed, the first statement follows from the fact that the words $\mathcal{R}_{i+1}$ satisfy the small cancellation condition $C'(\lambda_{i+1}, c_{i+1}, \epsilon_{i+1}, \mu_{i+1}, \rho_{i+1})$ (see conditions (2.1) and (2.2) in the definition of $C'(\lambda, c, \epsilon, \mu, \rho)$ condition). The second statement follows from Lemma 22. □

Lemma 55. *For sparse enough standard parameters λ_{i+1}, c_{i+1}, ϵ_{i+1}, μ_{i+1}, ρ_{i+1}, the words $\{\tilde{v}_{j_1}, \ldots, \tilde{v}_{j_{i+1}}\}$ and the words of the form $x_1^m x_2$ are not proper powers in G_{i+1}, provide that these statements hold in H_{i+1}.*

Proof. The part about the words $\{\tilde{v}_{j_1}, \ldots, \tilde{v}_{j_{i+1}}\}$ immediately follows from Lemma 29, because, since the standard parameters are assumed to be sparse enough, in particular, we can assume that ρ_{i+1} is sufficiently larger than $\max\{\|\tilde{v}_{j_1}\|, \ldots, \|\tilde{v}_{j_i}\|\}$ and then apply Lemma 29.

Now, by contradiction, assume that for some fixed $m \in \mathbb{N}$, the word $x_1^m x_2$ is a proper power in G_{i+1}. Then, there exists $k > 1$ and $u \in (X \cup \{t_{i+1}\})^*$ such that u is cyclically minimal in $\Gamma(G_{i+1}, X \cup \{t_{i+1}\})$ and

$$x_1^m x_2 \sim_{conj} u^k$$

in G_{i+1}. By Lemma 7, for sparse enough standard parameters λ_{i+1}, c_{i+1}, ϵ_{i+1}, μ_{i+1}, ρ_{i+1}, the word u^k is (λ_{i+1}, c_{i+1})-quasi-geodesic in $\Gamma(H_{i+1}, X \cup \{t_{i+1}\})$. Therefore, since by Lemma 54, any cyclic shift of $x_1^m x_2$ does not contain a $(\epsilon_{i+1}, \mu_{i+1}\lambda_{i+1})$-subwords with respect to the quotient $G_{i+1} = H_{i+1}/ \ll \mathcal{R}_{i+1} \gg$, by Lemma 28, we get that for some cyclic shift u' of u, $(u')^k$ must contain a $(\epsilon_{i+1}, 1 - 122\lambda_{i+1}\mu_{i+1})$-subword with respect to $G_{i+1} = H_{i+1}/ \ll \mathcal{R}_{i+1} \gg$, which is impossible because of Lemma 29. A contradiction.

□

5.5.1 Properties of $\tilde{G}$

Define

$$\tilde{\mathcal{V}} = \{\tilde{v}_{j_1}, \tilde{v}_{j_2}, \ldots\}$$

and for all $i \in \mathbb{N}$, define

$$\tilde{\mathcal{V}}_i = \{\tilde{v}_{j_k} \in \tilde{\mathcal{V}} \mid \tilde{v}_{j_k} \equiv \tilde{v}_{j_i}\}$$

and

$$\tilde{\mathcal{N}}_i = \{n \in \mathbb{N} \mid \tilde{v}_{j_i} \sim_{conj} x_1^{q_i^n} x_2 \text{ in } \tilde{G}\}.$$

Lemma 56. *For all $i, m \in \mathbb{N}$, words of the form $\tilde{v}_{j_i}$ and $x_1^m x_2$ are not proper powers in $\tilde{G}$.*

Proof. Follows immediately from Lemmas 50 and 55. □

Lemma 57. *Words of the form $(x_1^{m_1}x_2)^{\tau_1}$ and $(x_1^{m_2}x_2)^{\tau_2}$, where $m_1, m_2 \in \mathbb{Z}$, $\tau_1, \tau_2 \in \{\pm 1\}$, are G-conjugate in $\tilde{G}$ if and only if $m_1 = m_2$ and $\tau_1 = \tau_2$.*

Proof. If $(x_1^{m_1}x_2)^{\tau_1}$ and $(x_1^{m_2}x_2)^{\tau_2}$ are conjugate in G_0, then clearly $m_1 = m_2$ and $\tau_1 = \tau_2$. Now suppose that $(x_1^{m_1}x_2)^{\tau_1}$ and $(x_1^{m_2}x_2)^{\tau_2}$ are G-conjugate in $\tilde{G}$, but $(x_1^{m_1}x_2)^{\tau_1} \nsim_{conj} (x_1^{m_2}x_2)^{\tau_2}$ in

G_0. Then there exists $s \in \mathbb{N}$ such that $(x_1^{m_1}x_2)^{\tau_1} \sim_{conj} (x_1^{m_2}x_2)^{\tau_2}$ in G_s but $(x_1^{m_1}x_2)^{\tau_1} \nsim_{conj} (x_1^{m_2}x_2)^{\tau_2}$ in H_s. Since by Lemma 53, the words $(x_1^{m_1}x_2)^{\tau_1}$ and $(x_1^{m_2}x_2)^{\tau_2}$ are cyclically (λ_s, c_s)-quasi-geodesic in $\Gamma(H_s, X \cup \{t_s\})$, and since by Lemma 54, cyclic shifts of $(x_1^{m_1}x_2)^{\tau_1}$ and $(x_1^{m_2}x_2)^{\tau_2}$ do not contain $(\epsilon_s, \lambda_s\mu_s)$-subwords with respect to the quotient $G_s = H_s/ \ll \mathcal{R}_s \gg$, by Lemma 28, we get a contradiction, because Lemma 28 tells us that in case $(x_1^{m_1}x_2)^{\tau_1} \sim_{conj} (x_1^{m_2}x_2)^{\tau_2}$ in G_s but $(x_1^{m_1}x_2)^{\tau_1} \nsim_{conj} (x_1^{m_2}x_2)^{\tau_2}$ in H_s, a cyclic shift of at least one of the words $(x_1^{m_1}x_2)^{\tau_1}$ and $(x_1^{m_2}x_2)^{\tau_2}$ contains a $(1 - 121\lambda_s\mu_s)/2$-subword with respect to the quotient $G_s = H_s/ \ll \mathcal{R}_s \gg$, which contradicts to the assertion of Lemma 54.

□

Lemma 58. *For all $i, k \in \mathbb{N}$, $\tau \in \{\pm 1\}$, $\tilde{v}_{j_i}$ is G-conjugate with $(\tilde{v}_{j_k})^\tau$ in $\tilde{G}$ if and only if $\tilde{v}_{j_i} \equiv \tilde{v}_{j_k}$ and $\tau = 1$.*

Proof. If $\tilde{v}_{j_i} \sim_{conj} \tilde{v}_{j_k}^\tau$ in G_0, then clearly $\tilde{v}_{j_i} \equiv \tilde{v}_{j_k}$ and $\tau = 1$. In G_0 the inverse is true as well. Now assume that $\tilde{v}_{j_i}$ is G-conjugate with $\tilde{v}_{j_k}^\tau$ in $\tilde{G}$, but $\tilde{v}_{j_i} \nsim_{conj} \tilde{v}_{j_k}^\tau$ in G_0. Then there exists $s \in \mathbb{N}$ such that $\tilde{v}_{j_i} \sim_{conj} \tilde{v}_{j_k}^\tau$ in G_s, but $\tilde{v}_{j_i} \nsim_{conj} \tilde{v}_{j_k}^\tau$ in H_s.

Without loss of generality assume that $i \leqslant k$. By the definition of $\tilde{v}_{j_k}$, $\tilde{v}_{j_k}^\tau$ is not conjugate in G_{k-1} with any element from

$$\{\tilde{v}_{j_l} \mid 1 \leqslant l < k,\ \tilde{v}_{j_k}^\tau \not\equiv \tilde{v}_{j_l}\},$$

hence $s \geqslant k$. However, by Lemma 28, if the standard parameters $\lambda_s, c_s, \epsilon_s, \mu_s, \rho_s$ are sparse enough, in particular, if ρ_s is much larger than $\|\tilde{v}_{j_i}\|$ and $\|\tilde{v}_{j_k}\|$, then it cannot happen that $\tilde{v}_{j_i} \sim_{conj} (\tilde{v}_{j_k})^\tau$ in G_s but $\tilde{v}_{j_i} \nsim_{conj} (\tilde{v}_{j_k})^\tau$ in H_s.

□

Lemma 59. *If for some $i \in \mathbb{N}$, $n \in \mathbb{Z}$, $\tau \in \{\pm 1\}$, $\tilde{v}_{j_i}$ is G-conjugate to $(x_1^n x_2)^\tau$ in $\tilde{G}$, then $\tilde{v}_{j_i} \equiv (x_1^n x_2)^\tau$.*

Proof. Without loss of generality assume that $i = \min\{k \mid \tilde{v}_j \equiv \tilde{v}_i\}$.

If $\tilde{v}_{j_i} \sim_{conj} (x_1^n x_2)^\tau$ in G_0, then clearly $\tilde{v}_{j_i} \equiv (x_1^n x_2)^\tau$. Now assume that $\tilde{v}_{j_i}$ is G-conjugate with $(x_1^n x_2)^\tau$ in $\tilde{G}$, but $\tilde{v}_{j_i} \nsim_{conj} (x_1^n x_2)^\tau$ in G_0. Then there exists $s \in \mathbb{N}$ such that $\tilde{v}_{j_i} \sim_{conj} (x_1^n x_2)^\tau$ in G_s but $\tilde{v}_{j_i} \nsim_{conj} (x_1^n x_2)^\tau$ in H_s.

If $s < i$, then by the definition of $\tilde{v}_{j_i}$, the fact that $\tilde{v}_{j_i} \sim_{conj} (x_1^n x_2)^\tau$ in G_s implies that $\tilde{v}_{j_i} \equiv x_1^{n'} x_2$ for some $n' \in \mathbb{Z}$. Therefore, by Lemma 57, $x_1^{n'} x_2 \equiv (x_1^n x_2)^\tau$, which implies that $\tilde{v}_{j_i} \sim_{conj} (x_1^n x_2)^\tau$ in G_0. A contradiction.

If $s \geqslant i$, then since by Lemma 54, any cyclic shift of the word $(x_1^n x_2)^\tau$ does not contain a $(\epsilon_s, \lambda_s \mu_s)$-subword with respect to the quotient $G_s = H_s / \ll \mathcal{R}_s \gg$, by Lemma 28, some cyclic shift of the word $\tilde{v}_{j_i}$ must contain $(\epsilon_s, 1 - 122\lambda_s \mu_s)$-subwords with respect to the quotient $G_s = H_s / \ll \mathcal{R}_s \gg$, which is impossible provided that the standard parameters $\lambda_s, c_s, \epsilon_s, \mu_s, \rho_s$ are sparse enough (in particular, if ρ_s is much larger than $\|\tilde{v}_{j_i}\|$). $\square$

Lemma 60. *Let*

$$w_1, w_2 \in \left\{ \tilde{v}_{j_i}, (x_1^n x_2)^\tau \mid i \in \mathbb{N}, n \in \mathbb{Z}, \tau \in \{\pm 1\} \right\}$$

and $w_1 \not\equiv w_2$ such that $w_1 \sim_{conj} w_2$ in $\tilde{G}$. Then, for the group

$$H'_s = \langle X, t_1, t_2, \ldots, t_s \mid t_1^{-1} \tilde{v}_{j_1} t_1 = x_1^{q_1^{\tilde{n}_1}} x_2, \ldots, t_s^{-1} \tilde{v}_{j_s} t_s = x_1^{q_s^{\tilde{n}_s}} x_2 \rangle,$$

there exists $T \in \{t_1, t_2, \ldots, t_s\}^*$ such that

$$T^{-1} w_1 T = w_2 \text{ in } H'_s,$$

where s is such that $w_1 \sim_{conj} w_2$ in H_s, but $w_1 \nsim_{conj} w_2$ in G_{s-1}

Proof. Suppose that $w_1 \not\equiv w_2$, then by Lemmas 57, 58 and 59, the fact that $w_1 \sim_{conj} w_2$ in $\tilde{G}$, implies that w_1 is H-conjugate to w_2 in $\tilde{G}$. Therefore, there exists $s \in \mathbb{N}$ such that $w_1 \sim_{conj} w_2$ in H_s, but $w_1 \nsim_{conj} w_2$ in G_{s-1}. Therefore, by Collins' Lemma, there exists $h \in H_s$ such that

$$h^{-1} w_1 h = w_2 \text{ in } H_s$$

such that $\theta(h)$ is minimal for all possible conjugators h. Moreover, by Britton's lemma, $\theta(h) > 0$, where θ is defined in Subsection 2.3.

We will prove the lemma by induction on $(s, \theta(h))$, where we define $(s_1, \theta(h_1)) < (s_2, \theta(h_2))$ if either $s_1 < s_2$ or $s_1 = s_2$ and $\theta(h_1) < \theta(h_2)$.

If $s = 1$, then note that H'_s coincides with H_s, and the statement of the lemma follows from the combination of Collins' Lemma (see Lemma 13) with the fact that G_0 is a free group.

Now assume that $s > 1$ and for all smaller pairs $(s', \theta(h'))$ the statement is true. Note that, by Collins' Lemma, there exist $m \in \mathbb{Z}$, $h_1, h_2 \in H_s$ such that $h_1 t_s h_2 = h$ and, in particular, $\theta(h_1), \theta(h_2) < \theta(h)$, and either

$$h_1^{-1} w_1 h_1 =_{H_s} (\tilde{v}_{j_s})^m \text{ and } h_2^{-1} (x_1^{q_s^{\tilde{n}_s}} x_2)^m h_2 =_{H_s} w_2$$

or

$$h_1^{-1} w_1 h_1 =_{H_s} (x_1^{q_s^{\tilde{n}_s}} x_2)^m \text{ and } h_2^{-1} (\tilde{v}_{j_s})^m h_2 =_{H_s} w_2.$$

Since by Lemma 56, the words w_1 and w_2 are not proper powers, we get that $m \in \{\pm 1\}$. Also, note that since by Lemmas 57, 58 and 59, $w_1, (\tilde{v}_{j_s})^m$ and $w_1, (x_1^{q_s^{\tilde{n}_s}} x_2)^m$ are not G-conjugate in $\tilde{G}$, in case $\theta(h_1) = 0$, we get that either w_1 is conjugate to $(\tilde{v}_{j_s})^m$ in H_{s-1} or w_1 is conjugate to $(x_1^{q_s^{\tilde{n}_s}} x_2)^m$ in H_{s-1}. Analogous statement is true for the pairs $(w_2, (\tilde{v}_{j_s})^m)$ and $(w_2, (x_1^{q_s^{\tilde{n}_s}} x_2)^m)$ if $\theta(h_2) = 0$. Therefore, the statement of the lemma follows from the inductive hypothesis. $\square$

Lemma 61. *Let $i \in \mathbb{N}$. Then for all but finitely many $m \in \mathbb{N}$, if the word $x_1^{q_i^m} x_2$ is conjugate with $(\tilde{v}_{j_i})^\tau$, $\tau \in \{\pm 1\}$, in $\tilde{G}$, then $m \in \mathcal{N}$ and $\tau = 1$.*

Proof. Assume that $m \in \mathbb{N}$ is such that $x_1^{q_i^m} x_2 \sim_{conj} \tilde{v}_{j_i}$ in $\tilde{G}$ and also without loss of generality assume that $x_1^{q_i^m} x_2 \not\equiv \tilde{v}_{j_i}$. Then, because of Lemma 57, we get $x_1^{q_i^m} x_2$ is H-conjugate with $(\tilde{v}_{j_i})^\tau$ in $\tilde{G}$. Therefore, there exists $s \in \mathbb{N}$ such that $x_1^{q_i^m} x_2 \sim_{conj} (\tilde{v}_{j_i})^\tau$ in H_s, but $x_1^{q_i^m} x_2 \not\sim_{conj} (\tilde{v}_{j_i})^\tau$ in G_{s-1}. Then, by Lemma 60, for the group

$$H'_s = \langle X, t_1, t_2, \ldots, t_s \mid t_1^{-1} \tilde{v}_{j_1} t_1 = x_1^{q_1^{\tilde{n}_1}} x_2, \ldots, t_s^{-1} \tilde{v}_{j_s} t_s = x_1^{q_s^{\tilde{n}_s}} x_2 \rangle,$$

there exists $T \in \{t_1, t_2, \ldots, t_s\}^*$ such that T is of the minimal length for which

$$T^{-1}(\tilde{v}_{j_i})^{\tau} T = x_1^{q_i^m} x_2 \text{ in } H'_s.$$

Now it is an easy exercise to see that $\tau = 1$ from the last inequality.

Claim 1. T does not contain subwords of the form $t_{s_1} t_{s_2}^{-1}$, where $1 \leqslant s_1, s_2 \leqslant s$.

Proof of Claim 1. By contradiction suppose that $T = T_1 t_{s_1} t_{s_2}^{-1} T_2$. Then, by Britton's Lemma (see Lemma 66), the identities

$$T^{-1} \tilde{v}_{j_i} T (x_1^{q_i^m} x_2)^{-1} =_{H'_s} 1$$

and

$$\tilde{v}_{j_i} T (x_1^{q_i^m} x_2)^{-1} T^{-1} =_{H'_s} 1$$

imply that

$$(T_1 t_{s_1})^{-1} \tilde{v}_{j_i} T_1 t_{s_1} \in \langle x_1^{q_{s_1}^{\tilde{n}_{s_1}}} x_2 \rangle$$

and

$$t_{s_2}^{-1} T_2 x_1^{q_i^m} x_2 (t_{s_2}^{-1} T_2)^{-1} \in \langle x_1^{q_{s_2}^{\tilde{n}_{s_2}}} x_2 \rangle$$

in H'_s, and on the other hand

$$(T_1 t_{s_1})^{-1} \tilde{v}_{j_i} T_1 t_{s_1} =_{H'_s} t_{s_2}^{-1} T_2 x_1^{q_i^m} x_2 (t_{s_2}^{-1} T_2)^{-1}.$$

Therefore, we get $s_1 = s_2$ and hence $T =_{H'_s} T_1 T_2$, which contradicts the assumption that T was chosen to be of minimal length. $\square$

Claim 2. If $\|T\| \geqslant 2$, then T is of the form $T_1 t_{s_0}^{-1}$, for some $1 \leqslant s_0 \leqslant s$.

Proof of Claim 2. Indeed, if T was of the form $T_1 t_{s_0}$ for some $1 \leqslant s_0 \leqslant s$, then by Britton's Lemma, the identity $\tilde{v}_{j_i} T (x_1^{q_i^m} x_2)^{-1} T^{-1} =_{H'_s} 1$ would imply that $t_{s_0} x_1^{q_i^m} x_2 t_{s_0}^{-1} \in \langle \tilde{v}_{j_i} \rangle$ in H'_s, which implies that $t_{s_0} x_1^{q_i^m} x_2 t_{s_0}^{-1} = \tilde{v}_{j_i}$. However, the last identity contradicts the assumption that $\|T\| \geqslant 1$ and T was chosen of minimal length. $\square$

Note that if $\|T\| = 1$, then the identity $T^{-1}\tilde{v}_{j_i}T(x_1^{q_i^m}x_2)^{-1} =_{H'_s} 1$ can hold only for finitely many values of $m \notin \mathcal{N}$. Hence without loss of generality, let us assume that $\|T\| > 2$. Then, by Claims 1 and 2, T is of the form $T = t_{s_1}^{-1} \ldots t_{s_k}^{-1}$, where $k \geqslant 2$ and $1 \leqslant s_1, \ldots, s_k \leqslant s$.

Now, note that by Britton's Lemma, for some $n \in \mathbb{N}$, $t_{s_1}\tilde{v}_{j_n}t_{s_1}^{-1} = \tilde{v}_{j_i}$, and hence $\tilde{v}_{j_i} = x_1^{q_{s_1}^{\tilde{n}_{s_1}}}x_2$, which implies that s_1 is defined uniquely. The same way $s_2, \ldots, s_k$ are defined uniquely. Therefore, if for some $m_1 \neq m$, we have

$$(T')^{-1}\tilde{v}_{j_i}T'(x_1^{q_i^{m_1}}x_2)^{-1} =_{H'_s} 1$$

and $\|T'\| \geqslant 1$ also $\|T'\|$ is minimal, then either T' is a prefix of T or T is a prefix of T' and T' is of the same form as T. However, an application of Britton's Lemma shows that this cannot happen. Indeed, if without loss of generality we assume that $\|T'\| > \|T\|$, then $T' = Tt_{s_{k+1}}^{-1} \ldots t_{s_{k+l}}^{-1}$. Then, since $T^{-1}\tilde{v}_{j_i}T =_{H'_s} x_1^{q_i^{m_1}}x_2$, we would have

$$t_{s_{k+1}}T^{-1}\tilde{v}_{j_i}Tt_{s_{k+1}}^{-1} = t_{s_{k+1}}x_1^{q_i^{m_1}}x_2t_{s_{k+1}}^{-1} \in \langle \tilde{v}_{j_{s_{k+1}}} \rangle$$

and also $q_i = q_{s_{k+1}}$. However, from the definition of the elements $q_1, q_2, \ldots$, and from the last identities, we get $\tilde{v}_{j_{s_{k+1}}} = \tilde{v}_{j_i}$ and $t_{s_{k+1}}x_1^{q_i^{m-1}}x_2t_{s_{k+1}}^{-1} =_{H'_s} \tilde{v}_{j_i}$. The last identity contradicts the assumptions that $\|T'\| \geqslant 2$ and that T' was chosen to be of minimal length.

Thus the lemma is proved.

□

Lemma 62. *Let $i \in \mathbb{N}$ and let i_0 be the smallest index such that $\tilde{v}_{i_0} =_{\tilde{G}} \tilde{v}_i$. Then, the set $\tilde{\mathcal{V}}_{i_0}$ is infinite and the set $\mathcal{N} \triangle \tilde{\mathcal{N}}_{i_0} = (\mathcal{N} \backslash \tilde{\mathcal{N}}_{i_0}) \cup (\tilde{\mathcal{N}}_{i_0} \backslash \mathcal{N})$ is finite.*

Proof. The first statement follows from the definitions of $\tilde{\mathcal{V}}_{i_0}$ and of the elements $\{\tilde{v}_{j_1}, \tilde{v}_{j_2}, \ldots\}$.

As for the second statement, first of all, note that Lemmas 50, 55 and 61 imply that $\tilde{\mathcal{N}}_{i_0} \backslash \mathcal{N}$ is finite. Indeed, by Lemmas 50, 55 and 61, words of the set $\{\tilde{v}_{j_i}, x_1^n x_2\}$ are not proper powers in $\tilde{G}$. Therefore, $\tilde{v}_{j_{i_0}}$ commensurates with $x_1^n x_2$ if and only if $\tilde{v}_{j_{i_0}} \sim_{conj} x_1^n x_2$ in $\tilde{G}$.

Also, since the set $\mathcal{V}_{i_0}$ is infinite, by the definition of the set $\{\tilde{n}_1, \tilde{n}_2 \ldots\} \subseteq \mathcal{N}$ we get $\{\tilde{n}_1, \tilde{n}_2 \ldots\} = \mathcal{N}$. Therefore, $\mathcal{N} \backslash \tilde{\mathcal{N}}_{i_0} = \varnothing$. Thus $\mathcal{N} \triangle \tilde{\mathcal{N}}_{i_0}$ is finite.

$\square$

Lemma 63. *Let $i \in \mathbb{N}$, $m \in \mathbb{Z}\backslash\{0\}$. Let i_0 be the smallest index such that $\tilde{v}_{j_{i_0}}$ is conjugate to $\tilde{v}_{j_i}$ in $\tilde{G}$. Then for all but finitely many positive integers n, $(\tilde{v}_{j_{i_0}})^m$ is conjugate to $\left(x_1^{q_{i_0}^n} x_2\right)^{\tau m}$ in $\tilde{G}$, where $\tau \in \{\pm 1\}$, if and only if $n \in \mathcal{N}$ and $\tau = 1$.*

Proof. Indeed, suppose that $(\tilde{v}_{j_{i_0}})^m$ is conjugate with $(\tilde{v}_{j_{i_0}})^{\tau m}$ in $\tilde{G}$. Then, there exists $s \in \mathbb{N}$, such that $(\tilde{v}_{j_{i_0}})^m$ is conjugate with $\left(x_1^{q_{i_0}^n} x_2\right)^{\tau m}$ in G_s. Therefore, $E\big((\tilde{v}_{j_{i_0}})^m\big)$ is conjugate with $E\big(\big(x_1^{q_{i_0}^n} x_2\big)^m\big)$ in G_s. But since G_s is a torsion-free hyperbolic group and by Lemma 56, $\tilde{v}_{j_{i_0}}$ and $x_1^{q_i^n} x_2$ are not proper powers, we get that $\langle \tilde{v}_{j_{i_0}} \rangle$ is conjugate with $\langle x_1^{q_{i_0}^n} x_2 \rangle$ in G_s. Consequently, $\tilde{v}_{j_{i_0}}$ is conjugate with $\left(x_1^{q_{i_0}^n} x_2\right)^{\tau}$ in $\tilde{G}$. Therefore, by Lemma 61, for all but finitely many n, we get $n \in \mathcal{N}$.

The inverse statement follows immediately from Lemma 62. $\square$

Lemma 64. *For any word $u \in X^*$ representing a non-trivial element of $\tilde{G}$, there exists an element $\tilde{v}_{j_i} \in \tilde{\mathcal{V}}$ and $m \in \mathbb{Z}$ such that $u \sim_{conj} \tilde{v}_{j_i}^m$ in $\tilde{G}$.*

Proof. Indeed, by the definition of the words $\{\tilde{v}_{j_1}, \tilde{v}_{j_2}, \ldots\}$, for each $u \in X^*$, there exists $i \in \mathbb{N}$ and $v \in X^*$ such that u commensurates with $\tilde{v}_{j_i}$ in G_i. In other words, $v^{-1}uv \in E(\tilde{v}_{j_i})$ in G_i. But since G_i is a torsion-free hyperbolic group and, by Lemma 56, $\tilde{v}_{j_i}$ is not a proper power, we get that $v^{-1}uv \in E(\tilde{v}_{j_i})$ is equivalent to $v^{-1}uv \in \langle \tilde{v}_{j_i} \rangle$ in G_i. Therefore, for some $m \in \mathbb{Z}$, $u \sim_{conj} \tilde{v}_{j_i}^m$ in $\tilde{G}$.

$\square$

Lemma 65. *The word problem in $\tilde{G}$ is decidable in almost linear time, however, for each $g \in \tilde{G}\backslash\{1\}$, the individual conjugacy problem $ICP(g)$ is undecidable.*

Proof. The decidability of the word problem in almost linear time follows from Theorem 15.

Now suppose that $g \neq 1$ in $\tilde{G}$. Then, by Lemma 64, there exist $m \in \mathbb{Z}$ and minimal index i such that $\tilde{v}_{j_i}^m$ is conjugate to g in $\tilde{G}$. Therefore, by Lemma 63, for all but finitely many $n \in \mathbb{N}$, the question of whether or not $(x_1^{q_i^n} x_2)^m$ is conjugate to g is equivalent to the question of whether or not $n \in \mathcal{N}$. Therefore, since $\mathcal{N}$ is not recursive, we get that the decision problem which for each input $n \in \mathbb{N}$ asks whether or not $(\tilde{v}_{j_i}^n x)^m$ is conjugate to g in $\tilde{G}$ is undecidable. In particular, this implies that $ICP(g)$ is undecidable. $\square$

Chapter 6

The question of Collins

This chapter is dedicated to answering a question of Collins about the existence of finitely presented torsion-free groups with decidable word problem that cannot be embedded into groups with decidable conjugacy problem. We show that, in general, such groups exist.

6.1 Overview

As we know from the previous chapters, if the conjugacy problem is decidable for a finitely generated group G, then the word problem is decidable as well. However, in general, the inverse is far from being true (See Theorems 5, 6, or [46, 20, 45, 43]).

Groups on which the word and conjugacy problems have significantly different behavior attracted the attention of group theorists for at least several decades. For example, if G is a finitely generated group and $H \leqslant G$ is a subgroup of finite index, then the word problem in G is decidable if and only if it is decidable for H. However, it is shown by Goryaga-Kirkinskii, [33], and independently by Collins-Miller, [21], that subgroups of index 2 of some specific finitely generated groups have decidable (respectively, undecidable) conjugacy problem, while the groups themselves have undecidable (respectively, decidable) conjugacy problem.

Another important type of questions about word and conjugacy problems in groups is the following: Is it true that every finitely generated group with decidable word problem (respectively, conjugacy problem) embeds in a finitely presented group with decidable word problem (respectively, conjugacy problem)? Both of these questions have positive answer. The answer for the word problem is obtained by Clapham in 1967, [18], based on the classical embedding theorem of Higman (see [40]), while the analogous question for the conjugacy problem was a long-standing open problem until it got positive answer in 2004 by a work of Olshanskii and Sapir. See [58] and also [59].

In light of the aforementioned, a natural question about the connection of word and conjugacy problems in groups is the following question, asked by Donald Collins in the early 1970s.

Question 1. Can every torsion-free group with solvable word problem be embedded in a group with solvable conjugacy problem?

This question appears in the 1976 edition of The Kourovka Notebook as Problem 5.21, [38]. Probably, the first source where this problem was posed in a written form is [15]. For yet another source, see [7].

It was mentioned by Collins in [38] that due to an example by A. Macintyre, there exists a group with torsions which cannot be embedded into a finitely generated group with decidable conjugacy problem. However, the case for torsion-free groups remained open until now. Indeed, one of the reasons why the torsion and torsion-free cases are different is based on the observation that conjugate elements in a group must have the same order, and since in a torsion-free group all non trivial elements have the same (infinite) order, in case of torsion-free groups, one cannot make use of this observation in order to answer Question 1.

In [59], Olshanskii and Sapir showed the following theorem which gives a positive answer to Question 1 under the stronger assumption of decidability of the power problem.

Theorem 16 (Olshanskii-Sapir, [59])**.** *Every countable group with solvable power and order problems is embeddable into a 2-generated finitely presented group with solvable conjugacy and power problems.*

Note that as it is defined in [59], for a given group G the power problem is said to be decidable, if there exists an algorithm which for any given pair (g, h) of elements from G decides whether or not g is equal to some power of h in G. The order problem is decidable in G if there exists an algorithm which for each input $g \in G$ computes the order of g.

The main result of the current work is the negative answer to Question 1 in the general case.

Theorem 17. *There exists a finitely presented torsion-free group $\mathcal{G}$ with decidable word problem such that $\mathcal{G}$ cannot be embedded into a group with decidable conjugacy problem.*

A remarkable theorem of Osin (see [60]) says that every torsion-free countable group can be embedded into a two generated group with exactly two conjugacy classes. In the context of this theorem, it is very natural to ask whether or not every torsion-free countable group with decidable word problem (= computable group) can be embedded into a group with exactly two conjugacy classes and with decidable word problem. A more relaxed version of this question would be whether or not every torsion-free countable group with decidable word problem can be embedded in a finitely generated recursively presented group with finitely many conjugacy classes.

It turns out that a direct consequence of Theorem 17 gives negative answer to both of these questions.

In fact, the decidability of the conjugacy problem for groups with exactly two conjugacy classes is equivalent to the decidability of the word problem. Even more, as it is shown in a recent paper of Miasnikov and Schupp [43], a finitely generated recursively presented group with finitely many conjugacy classes has decidable conjugacy problem. Therefore, a direct corollary from Theorem 17 is the following.

Theorem 18. *There exists a torsion-free finitely presented group with decidable word problem that cannot be embedded into a finitely generated recursively presented group with finitely many conjugacy classes.*

Proof. Just take the group $\mathcal{G}$ from Theorem 17. □

Remark 12. In fact, the mentioned result of Miasnikov and Schupp is true not only for finitely generated recursively presented groups, but for all recursively presented groups in general. Therefore, Theorem 18 stays true after dropping the assumption that the group in which the initial group is embedded is finitely generated. (The exact definition of recursive presentations of groups is given in the next section.)

6.2 Preliminaries

6.2.1 Groups with decidable word problem

Recall that a countable group G is said to have *recursive presentation*, if G can be presented as $G = \langle X \mid R \rangle$ such that X and R are enumerable by some algorithm (i.e. Turing machine). See

[34]. If in addition, there is an algorithm which for each pair of words (w, w') from $(X \cup X^{-1})^*$ verifies whether or not w and w' represent the same element of G, then the presentation $G = \langle X \mid R \rangle$ is called *computable* and in case G possesses such a presentation, the group G itself is called *computable* as well. Modulo some slight variances, the original definition of the concept of computable groups is due to Rabin [63] and Mal'cev [42].

In case the group G is finitely generated (i.e. $|X| < \infty$) computability property of G is the same as saying that G has decidable *word problem*. It is not hard to notice that decidability of the word problem does not depend on the finite generating sets. From the computability perspective, the last observation is one of the main advantages of finitely generated groups over countably generated ones, because in case of finitely generated groups decidability of the word problem is an intrinsic property of a group, rather than of its presentation.

However, in this paper, to keep the notations as uniform as possible, we say that G has decidable word problem if it is given by a computable presentation.

Let $G = \langle x_1, x_2, \ldots \mid r_1, r_2, \ldots \rangle$, where $\{x_1, x_2, \ldots\}$ and $\{r_1, r_2, \ldots\}$ are recursive enumerations of X and R, respectively. Then, the embedding constructions of [23] and [59] imply the following theorem.

Theorem 19. *If $G = \langle x_1, x_2, \ldots \mid r_1, r_2, \ldots \rangle$ has decidable word problem, then there exists an embedding $\Phi : G \to H$ of G into a two generated group H such that the following holds.*

(1). The word problem is decidable in H;

(2). The map $i \mapsto \Phi(x_i)$ is computable;

(3). An element of H is of finite order if and only if it is conjugate to an image under Φ of an element of finite order in G.

6.2.2 HNN-extensions

In the proof of the existence of the group $\mathcal{G}$ from Theorem 17 we use some group theoretical constructions based on HNN-extensions. Therefore, in this subsection we would like to recall some well-known basic facts about HNN-extensions. The basics of the theory of HNN-extensions can also be found in [40].

Suppose that $A, B \leqslant H$ and $\phi : A \to B$ is a group isomorphism from A to B. Then the *HNN-extension* H' of H with respect to A and B (and ϕ) and with *stable letter* t is defined as

$$H' = \langle H, t \mid t^{-1}at = \phi(a), a \in A\rangle.$$

In the current text, the isomorphism ϕ will be clear from the context, hence we will simply use the notation $H' = \langle H, t \mid t^{-1}At = B\rangle$.

Clearly, every element h' of H' can be decomposed as a product

$$h' = h_0 t^{\epsilon_1} h_1 \dots t^{\epsilon_n} h_n, \tag{6.1}$$

where $\epsilon_i \in \{\pm 1\}$, $h_j \in H$ for $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n$.

The decomposition (6.1) is said to be in *reduced form*, if it does not contain subproduct of one of the forms $t^{-1}g_i t$, $g_i \in A$ or $tg_i t^{-1}$, $g_i \in B$, for $1 \leqslant i \leqslant n$.
Analogously, if $H = \langle X\rangle$, then the word $u' \in (X \cup X^{-1} \cup \{t^{\pm 1}\})^*$ given by

$$u' = u_0 t^{\epsilon_1} u_1 t^{\epsilon_2} \dots t^{\epsilon_n} u_n,$$

where $\epsilon_i \in \{\pm 1\}$, $u_j \in (X \cup X^{-1})^*$, is said to be a *reduced word* with respect to the HNN-extension H' if the decomposition $h_0 t^{\epsilon_1} h_1 \dots t^{\epsilon_n} h_n$ is in reduced form, where h_i corresponds to the word u_i in H.

The following well-known lemma is attributed to Britton in [40].

Lemma 66 (Britton's Lemma)**.** *If the decomposition* (6.1) *is reduced and* $n \geqslant 1$*, then* $h' \neq 1$ *in* H'.

Lemma 67 (See Theorem 2.1 in [40])**.** *Let* $H' = \langle H, t \mid t^{-1}At = B\rangle$ *be an HNN-extension with respect to isomorphic subgroups* A *and* B*. Then* H *embeds in* H' *by the maps* $h \mapsto h$*,* $h \in H$.

Lemma 68 (The Torsion Theorem for HNN-extensions. See Theorem 2.4 in [40])**.** *Let* $H' = \langle H, t \mid t^{-1}At = B\rangle$ *be an HNN-extension. Then every element of finite order in* H' *is a conjugate of an* *element of finite order in the base H. Thus H' has elements of finite order n if an only if H has elements of order n.*

6.3 Proof of Theorem 17

In order to show the existence of $\mathcal{G}$ from Theorem 17, first, we will construct a special countable group $\dot{G}$ with decidable word problem, then $\mathcal{G}$ will be defined as a group in which $\dot{G}$ embeds in a certain way.

Two disjoint sets of natural numbers $S_1, S_2 \subset \mathbb{N}$ are called *recursively inseparable* if there is no recursive set $T \subset \mathbb{N}$ such that $S_1 \subseteq T$ and $S_2 \subseteq \mathbb{N}\backslash T$. The set T is called *separating set*. Clearly, if two disjoint sets are recursively inseparable, then they cannot be recursive. Indeed, if, say, S_1 and S_2 are disjoint and, say, S_1 is recursive, then as a recursive separating set one could simply take S_1. Nevertheless, it is a well-known fact that there exist disjoint recursively enumerable and recursively inseparable sets. See, for example, [66].

Let us fix two disjoint recursively enumerable and recursively inseparable sets $\mathcal{N} = \{n_1, n_2, \ldots\} \subset \mathbb{N}$ and $\mathcal{M} = \{m_1, m_2, \ldots\} \subset \mathbb{N}$ such that the maps $i \mapsto n_i$ and $i \mapsto m_i$ are computable.

Now, for all $n \in \mathbb{N}$, define A_n as a torsion-free abelian additive group of rank two with basis $\{a_{n,0}, a_{n,1}\}$, i.e.

$$A_n = \langle a_{n,0} \rangle \bigoplus \langle a_{n,1} \rangle$$

and such that the groups $A_1, A_2, \ldots$ are disjoint.

For all $n \in \mathbb{N}$, define the groups $\dot{A}_n$ as follows.

$$\dot{A}_n = \begin{cases} A_n & \text{if } n \notin \mathcal{N} \cup \mathcal{M}\,, \\ A_n / \ll a_{n,1} = 2^i a_{n,0} \gg & \text{if } n = n_i \in \mathcal{N}\,, \\ A_n / \ll a_{n,1} = 3^i a_{n,0} \gg & \text{if } n = m_i \in \mathcal{M}. \end{cases}$$

For all $n \in \mathbb{N}$ and $m \in \{0, 1\}$, let us denote the images of $a_{n,m}$ under the natural homomorphisms $A_n \to \dot{A}_n$ by $\dot{a}_{n,m}$.

Convention. In this text, whenever we deal with an additive group, say, A, with finite generating set, say, $\{a_1, \ldots, a_k\}$, by $\{\pm a_1, \ldots, \pm a_k\}^*$ we denote the set of formal finite sums of the form $w = \sum \lambda_i a_{j_i}$, where $\lambda_i \in \mathbb{Z}$ and $a_{j_i} \in \{a_1, \ldots, a_k\}$, and we say that w is a word formed by letters $\{\pm a_1, \ldots, \pm a_k\}$. Note that this is the additive analogue of the central in combinatorial group theory concept of words, where the alphabet composing the words is a set of group generators. This is why the finite formal sums $w = \sum \lambda_i a_{j_i}$ we call words from $\{\pm a_1, \ldots, \pm a_k\}^*$.

Before moving forward, we prove the following important lemma.

Lemma 69. *There exists an algorithm such that for each input $n \in \mathbb{N}$ and $w \in \{\pm \dot{a}_{n,0}, \pm \dot{a}_{n,1}\}^*$, it decides whether or not w represents the trivial element in the group $\dot{A}_n$.*

Proof. Indeed, since $\dot{A}_n$ is abelian with generating set $\{\dot{a}_{n,0}, \dot{a}_{n,1}\}$, each word w from $\{\pm \dot{a}_{n,0}, \pm \dot{a}_{n,1}\}^*$ can be effectively transformed to a word of the form

$$w' = \lambda_0 \dot{a}_{n,0} + \lambda_1 \dot{a}_{n,1}$$

which represents the same element in $\dot{A}_n$ as the initial word w, where $\lambda_0, \lambda_1 \in \mathbb{Z}$.

Now, assuming that $\lambda_0 \neq 0, \lambda_1 \neq 0$, in order w' to represent the trivial element in $\dot{A}_n$ it must be that $n \in \mathcal{N} \cup \mathcal{M}$, because otherwise, by definition, the group $\dot{A}_n$ is torsion-free abelian of rank 2 with basis $\{\dot{a}_{n,0}, \dot{a}_{n,1}\}$.

In case $n \in \mathcal{N}$, by definition we have that $\dot{a}_{n,1} = 2^x \dot{a}_{n,0}$, where x is the index of n in $\mathcal{N}$, i.e. $n = n_x$.

Similarly, in case $n \in \mathcal{M}$, by definition we have that $\dot{a}_{n,1} = 3^x \dot{a}_{n,0}$, where x is the index of n in $\mathcal{M}$, i.e. $n = m_x$.

Now, if $\lambda_0 = 0$ and $\lambda_1 = 0$, then clearly w' (hence also w) represents the trivial element in $\dot{A}_n$. Therefore, without loss of generality we can assume that at least one of λ_0 and λ_1 is not 0. Then, if we treat x as an unknown variable, depending on whether $n = n_x$ or $n = m_x$, the equality $w' = 0$

would imply one of the following equations:

$$\lambda_0 + \lambda_1 2^x = 0 \tag{6.2}$$

or

$$\lambda_0 + \lambda_1 3^x = 0, \tag{6.3}$$

respectively.

This observation suggests that in case $\lambda_0 \neq 0$ or $\lambda_1 \neq 0$, in order to verify whether or not $w' = 0$ in $\dot{A}_n$, we can first try to find x satisfying (6.2) or (6.3), and in case such an x does not exist, conclude that w' (hence, also w) does not represent the trivial element in $\dot{A}_n$. Otherwise, if x is the root of the equation (6.2), we can check whether or not $n = n_x$ (since $\mathcal{N}$ is recursively enumerable, this checking can be done algorithmically). Similarly, if x is the root of the equation (6.3), we can check whether or not $n = m_x$.

If as a result of this checking, we get $n = n_x$ (respectively, $n = m_x$), then the conclusion will be that w' (hence, also w) represents the trivial element in $\dot{A}_n$, otherwise, if $n \neq n_x$ (respectively, $n \neq m_x$), then the conclusion will be that w' (hence, also w) does not represent the trivial element in $\dot{A}_n$.

□

Now, for all $n \in \mathbb{N}$, define the group B_n as a torsion-free additive abelian group of rank 2, that is

$$B_n = \langle b_{n,0} \rangle \oplus \langle b_{n,1} \rangle$$

such that B_1, B_2, . . . are disjoint.

Now, for all $n \in \mathbb{N}$, define the groups $\dot{B}_n$ as follows.

$$\dot{B}_n = \begin{cases} B_n & \text{if } n \notin \mathcal{N} \cup \mathcal{M}\,, \\ B_n / \ll b_{n,1} = 2^i b_{n,0} \gg & \text{if } n = n_i \in \mathcal{N} \text{ or } n = m_i \in \mathcal{M}. \end{cases}$$

For all $n \in \mathbb{N}$, $m \in \{0, 1\}$, let us denote the images of $b_{n,m}$ under the natural homomorphism $B_n \to \dot{B}_n$ by $\dot{b}_{n,m}$.

It follows from the definitions of $\dot{A}_n$ and $\dot{B}_n$ that for all $n \in \mathbb{N}$, these groups are infinite and torsion free.

Lemma 70. *There exists an algorithm such that for each input $n \in \mathbb{N}$ and $w \in \{\pm\dot{b}_{n,0}, \pm\dot{b}_{n,1}\}^*$, it decides whether or not w represents the trivial element in the group $\dot{B}_n$.*

Proof. Follows from the repetition of arguments of the proof of Lemma 69. □

Lemma 71. *The map $\dot{a}_{n,0} \mapsto \dot{b}_{n,0}$, $\dot{a}_{n,1} \mapsto \dot{b}_{n,1}$ induces a group isomorphism between the groups $\langle \dot{a}_{n,0}, \dot{a}_{n,1} \rangle = \dot{A}_n$ and $\langle \dot{b}_{n,0}, \dot{b}_{n,1} \rangle = \dot{B}_n$ if and only if $n \in \mathbb{N} \backslash \mathcal{M}$.*

Proof. Indeed, in case $n \in \mathcal{N}$, by definition, $\langle \dot{a}_{n,0}, \dot{a}_{n,1} \rangle = \langle \dot{a}_{n,0} \rangle$ and $\dot{a}_{n,1} = 2^i \dot{a}_{n,0}$, where i is the index of n in $\mathcal{N}$. Also $\langle \dot{b}_{n,0}, \dot{b}_{n,1} \rangle = \langle \dot{b}_{n,0} \rangle$ and $\dot{b}_{n,1} = 2^i \dot{b}_{n,0}$. Therefore, in case $n \in \mathcal{N}$, the map $\dot{a}_{n,0} \mapsto \dot{b}_{n,0}$, $\dot{a}_{n,1} \mapsto \dot{b}_{n,1}$ induces a group isomorphism between the groups $\langle \dot{a}_{n,0}, \dot{a}_{n,1} \rangle$ and $\langle \dot{b}_{n,0}, \dot{b}_{n,1} \rangle$.

In case $n \in \mathbb{N} \backslash (\mathcal{N} \cup \mathcal{M})$, the groups $\dot{A}_n$ and $\dot{B}_n$ are torsion-free and abelian of rank 2 with generating sets $\{\dot{a}_{n,0}, \dot{a}_{n,1}\}$ and $\{\dot{b}_{n,0}, \dot{b}_{n,1}\}$, respectively. Therefore, if $n \in \mathbb{N} \backslash (\mathcal{N} \cup \mathcal{M})$, the map $\dot{a}_{n,0} \mapsto \dot{b}_{n,0}$, $\dot{a}_{n,1} \mapsto \dot{b}_{n,1}$ induces a group isomorphism between the groups $\langle \dot{a}_{n,0}, \dot{a}_{n,1} \rangle$ and $\langle \dot{b}_{n,0}, \dot{b}_{n,1} \rangle$ as well.

Now suppose that $n \in \mathcal{M}$. Then, $\langle \dot{a}_{n,0}, \dot{a}_{n,1} \rangle = \langle \dot{a}_{n,0} \rangle$ and $\langle \dot{b}_{n,0}, \dot{b}_{n,1} \rangle = \langle \dot{b}_{n,0} \rangle$, however, by definition, $\dot{a}_{n,1} = 3^i \dot{a}_{n,0}$ while $\dot{b}_{n,1} = 2^i \dot{b}_{n,0}$. Therefore, the map $\dot{a}_{n,0} \mapsto \dot{b}_{n,0}$, $\dot{a}_{n,1} \mapsto \dot{b}_{n,1}$ does not induce a group isomorphism between the groups $\langle \dot{a}_{n,0}, \dot{a}_{n,1} \rangle$ and $\langle \dot{b}_{n,0}, \dot{b}_{n,1} \rangle$ when $n \in \mathcal{M}$.

□

Now, let $T = F(t_1, t_2, \ldots)$ be a free group with countable free basis $\{t_1, t_2, \ldots\}$.

Denote the infinite free products $\dot{A}_1 * \dot{A}_1 * \ldots$ and $\dot{B}_1 * \dot{B}_1 * \ldots$ by $*_{n=1}^{\infty}\dot{A}_n$ and $*_{n=1}^{\infty}\dot{B}_n$, respectively. Then define

$$\dot{G} = (*_{n=1}^{\infty}\dot{A}_n) * (*_{n=1}^{\infty}\dot{B}_n) * T/ \ll \mathcal{R} \gg, \tag{6.4}$$

where the set of defining relators $\mathcal{R}$ is defined as

$$\mathcal{R} = \left\{t_i^{-1}\dot{a}_{n_i,0}t_i = \dot{b}_{n_i,0} \mid i \in \mathbb{N}\right\}.$$

Define

$$\dot{G}_0 = (*_{n=1}^{\infty}\dot{A}_n) * (*_{n=1}^{\infty}\dot{B}_n),$$

and for all $k \in \mathbb{N}$, define $\dot{G}_k$ as

$$\dot{G}_k = (*_{n=1}^{\infty}\dot{A}_n) * (*_{n=1}^{\infty}\dot{B}_n) * F(t_1, \ldots, t_k)/ \ll \mathcal{R}_k \gg,$$

where the set of defining relators $\mathcal{R}_k$ is defined as

$$\mathcal{R}_k = \left\{t_i^{-1}\dot{a}_{n_i,0}t_i = \dot{b}_{n_i,0} \mid 1 \leqslant i \leqslant k\right\}.$$

Then, clearly the group $\dot{G}$ is the direct limit of the sequence of group $\{\dot{G}_k\}_{k=0}^{\infty}$ connected by homomorphisms $\epsilon_k : \dot{G}_k \to \dot{G}_{k+1}$ such that ϵ_k are the homomorphisms induced by the identity maps from $\left\{\dot{a}_{n,0}, \dot{a}_{n,1}, \dot{b}_{n,0}, \dot{b}_{n,1},\ t_i \mid n \in \mathbb{N}, i \in \{1, 2, \ldots, k\}\right\}$ to themselfs for all $k \in \mathbb{N}$.

Let us denote

$$\mathcal{S}_0 = \left\{ \pm\dot{a}_{n,m},\ \pm\dot{b}_{n,m}\ \mid n \in \mathbb{N},\ m \in \{0, 1\} \right\}$$

and for $k \in \mathbb{N}$,

$$\mathcal{S}_k = \left\{ \pm\dot{a}_{n,m},\ \pm\dot{b}_{n,m},\ t_1^{\pm 1}, \ldots, t_k^{\pm 1}\ \mid n \in \mathbb{N},\ m \in \{0, 1\} \right\}.$$

Note that since the sets $\mathcal{N}$ and $\mathcal{M}$ are recursively enumerable, the groups $\dot{G}$ and $\dot{G}_k$ have recursive presentations with respect to the generating sets $\mathcal{S}_0 \cup \{t_1, t_2, \ldots\}$ and $\mathcal{S}_k$, $k \in \mathbb{N} \cup \{0\}$, respectively.

Lemma 72. *There exists an algorithm such that for each input $w \in \mathcal{S}_0^*$ it decides whether or not $w = 1$ in $\dot{G}_0$.*

Moreover, there exists an algorithm such that for each input (w, i), $w \in \mathcal{S}_0^*$, $i \in \mathbb{N}$, it decides whether or not w represents an element from $\langle \dot{a}_{n_i,0} \rangle$, and in case it represents such an element, the algorithm returns $\lambda \dot{a}_{n_i,0}$, $\lambda \in \mathbb{Z}$, such that $w = \lambda \dot{a}_{n_i,0}$ in $\dot{G}_0$. Analogous statement remains true when we replace $\dot{a}_{n_i,0}$ with $\dot{b}_{n_i,0}$.

Proof. Indeed, these properties immediately follow from the basic properties of the direct products of groups combined with Lemmas 69 and 70.

□

Lemma 73. *For all $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, the following holds.*

(i). The groups $\dot{A}_n$ and $\dot{B}_n$ embed into $\dot{G}_k$ under the maps induced by $\dot{a}_{n,m} \mapsto \dot{a}_{n,m}$ and $\dot{b}_{n,m} \mapsto \dot{b}_{n,m}$ for $m \in \{0, 1\}$, respectivley;

(ii). The group $\dot{G}_{k+1}$ is an HNN-extension of the group $\dot{G}_k$. More precisely, $\dot{G}_{k+1} = \langle \dot{G}_k,\ t_{k+1} \mid t_{k+1}^{-1} \dot{a}_{n_{k+1},0} t_{k+1} = \dot{b}_{n_{k+1},0} \rangle$.

Proof. Indeed, if $k = 0$, then (i) and (ii) are obvious. Now, let us apply induction with respect to k.

Suppose that for all $0 \leqslant l < k$, the statements of (i) and (ii) are true. Then, since by the inductive assumption, $\dot{G}_k$ is obtained from $\dot{G}_{k-1}$ as an HNN-extension with respect to the isomorphic subgroups $\langle \dot{a}_{n_k,0} \rangle \backsimeq \langle \dot{b}_{n_k,0} \rangle$, by the basic properties of HNN-extensions (see Lemma 67), we get that the statement of (i) holds for $\dot{G}_k$. Therefore, since the subgroups $\langle \dot{a}_{n_{k+1},0} \rangle \leqslant \dot{G}_k$ and $\langle \dot{b}_{n_{k+1},0} \rangle \leqslant \dot{G}_k$ are isomorphic, and in the definition of $\dot{G}_{k+1}$ the only defining relation which involves the letters $t_{k+1}^{\pm 1}$ is the relation $t_{k+1}^{-1}\dot{a}_{n_{k+1},0}t_{k+1} = \dot{b}_{n_{k+1},0}$, we get that the statement of (ii) holds as well. $\square$

Corollary 16. *If $k < l$, then the group $\dot{G}_k$ embeds into the group $\dot{G}_l$ under the map induced by*

$$\dot{a}_{n,m} \mapsto \dot{a}_{n,m},\, \dot{b}_{n,m} \mapsto \dot{b}_{n,m} \text{ for } n \in \mathbb{N} \text{ and } m \in \{0,1\}$$

and

$$t_1 \mapsto t_1, \ldots, t_k \mapsto t_k.$$

Proof. Indeed, by Lemma 73, the group $\dot{G}_l$ is obtained from the group $\dot{G}_k$ by (multiple) HNN-extensions. Therefore, the statement follows from the basic properties of HNN-extensions, namely, by Lemma 67.

$\square$

Corollary 17. *The map $\dot{a}_{n,0} \mapsto \dot{b}_{n,0}$, $\dot{a}_{n,1} \mapsto \dot{b}_{n,1}$ induces a group isomorphism between the subgroups $\langle \dot{a}_{n,0}, \dot{a}_{n,1}\rangle = \dot{A}_n$ and $\langle \dot{b}_{n,0}, \dot{b}_{n,1}\rangle = \dot{B}_n$ of $\dot{G}$ if and only if $n \in \mathbb{N}\backslash\mathcal{M}$.*

Proof. By Corollary 16, $\dot{G}_0$ embeds in $\dot{G}$ by the map induced by $\dot{a}_{n,0} \mapsto \dot{a}_{n,0}$, $\dot{a}_{n,1} \mapsto \dot{a}_{n,1}$, $\dot{b}_{n,0} \mapsto \dot{b}_{n,0}$, $\dot{b}_{n,1} \mapsto \dot{b}_{n,1}$ for $n \in \mathbb{N}$. Therefore, the statement of the corollary follows from Lemma 71. $\square$

Definition 6.1 (Reduced words over $\mathcal{S}_k^*$)**.** Let $k \in \mathbb{N}$. Then, for a given word $w \in \mathcal{S}_k^*$, we say that w is a reduced word over $\mathcal{S}_k^*$ if the following properties hold.

(0). w is freely reduced, i.e. w does not contain subwords of the form xx^{-1}, $x \in \mathcal{S}_k$;

(1). For all $1 \leqslant i \leqslant k$, w does not contain subwords of the form $t_i^{-1}ut_i$, where $u \in \mathcal{S}_0^*$ is such that $u = \lambda\dot{a}_{n_i,0}$ in $\dot{G}_0$ for some $\lambda \in \mathbb{Z}$;

(2). For all $1 \leqslant i \leqslant k$, w does not contain subwords of the form $t_ivt_i^{-1}$, where $v \in \mathcal{S}_0^*$ is such that $v = \lambda\dot{b}_{n_i,0}$ in $\dot{G}_0$ for some $\lambda \in \mathbb{Z}$.

Lemma 74. *For all $k \in \mathbb{N}$, if $w \in \mathcal{S}_k^* \backslash \mathcal{S}_{k-1}^*$ is a reduced word over $\mathcal{S}_k^*$, then $w \neq 1$ in $\dot{G}_k$. Moreover, $w \neq u$ in $\dot{G}_k$ for any word $u \in \mathcal{S}_{k-1}^*$.*

Proof. Let us prove by induction on k. If $k = 1$, then the group $\dot{G}_1 = \langle \dot{G}_0, t_1 \mid t_1^{-1}\dot{a}_{n_1,0}t_1 = \dot{b}_{n_1,0}\rangle$ is an HNN-extension of $\dot{G}_0$ with respect to the isomorphic subgroups $\langle \dot{a}_{n_1,0}\rangle \leqslant \dot{G}_0$ and $\langle \dot{b}_{n_1,0}\rangle \leqslant \dot{G}_0$. Therefore, by Britton's Lemma (see Lemma 66), $w \neq 1$ in $\dot{G}_1$ provided that it is a reduced word over $\mathcal{S}_1^*$.

Also for any $u \in \mathcal{S}_0^*$, the word wu^{-1} is a reduced word with respect to the HNN-extension $\dot{G}_1 = \langle \dot{G}_0, t_1 \mid t_1^{-1}\dot{a}_{n_1,0}t_1 = \dot{b}_{n_1,0}\rangle$. Therefore, by Britton's Lemma (see Lemma 66), $wu^{-1} \neq 1$ in $\dot{G}_1$ or, in other words, $w \neq u$ in $\dot{G}_1$.

Now assume that $k > 1$ and $w \in \mathcal{S}_k^* \backslash \mathcal{S}_{k-1}^*$ is a reduced word over $\mathcal{S}_k^*$. Also, suppose that the statement of the lemma is true for all $l < k$. Then, first of all, note that from the definition of the reduced words over $\mathcal{S}_k^*$ it follows that if v is a subword of w such that $v \in \mathcal{S}_{k-1}^*$, then v is a reduced word over $\mathcal{S}_{k-1}^*$. Consequently, by the inductive hypothesis, if $t_k^{-1}ut_k$ (or $t_kut_k^{-1}$) is a subword of w such that $u \in \mathcal{S}_{k-1}^*$ and u represents an element from the image of $\dot{A}_{n_k}$ (or $\dot{B}_{n_k}$) in $\dot{G}_k$, then $u \in \mathcal{S}_0^*$. However, this contradicts the assumption that w is a reduced word over $\mathcal{S}_k^*$. Therefore, since $\dot{G}_k = \langle \dot{G}_{k-1}, t_k \mid t_k^{-1}\dot{a}_{n_k,0}t_k = \dot{b}_{n_k,0}\rangle$ is an HNN-extension of $\dot{G}_{k-1}$ with respect to the isomorphic subgroups $\langle a_{n_k,0}\rangle = \dot{A}_{n_k} \leqslant \dot{G}_{k-1}$ and $\langle b_{n_k,0}\rangle = \dot{B}_{n_k} \leqslant \dot{G}_{k-1}$, we get that if w is a reduced word over $\mathcal{S}_k^*$, then w is a reduced word over this HNN-extension. Hence, by Britton's Lemma, we get that $w \neq 1$ in $\dot{G}_k$. Similarly, for any $u \in \mathcal{S}_0^*$, again by Britton's Lemma, we get that $wu^{-1} \neq 1$ in $\dot{G}_k$ or, in other words, $w \neq u$ in $\dot{G}_k$. $\square$

Lemma 75. *There exists an algorithm such that for each input (k, w), $k \in \mathbb{N} \cup \{0\}$, $w \in \mathcal{S}_k^*$, it decides whether or not $w = 1$ in $\dot{G}_k$.*

Proof. Let (k, w) be a fixed input. Without loss of generality assume that w is a freely reduced word in $\mathcal{S}_k^*$.

If $k = 0$, then one can apply the word problem algorithm for the group $\dot{G}_0 = \langle \mathcal{S}_0^* \rangle$. See Lemma 72.

Otherwise, if $k \geqslant 1$, for each $k_1 \leqslant k$ such that w contains a letter from $\{t_{k_1}, t_{k_1}^{-1}\}$, do the following: Find all subwords of w which are of one of the forms $t_{k_1}^{-1}ut_{k_1}$ or $t_{k_1}vt_{k_1}^{-1}$, where $u, v \in$

$\mathcal{S}_0^*$ and $u = \lambda\dot{a}_{n_{k_1},0}$, $v = \lambda\dot{b}_{n_{k_1},0}$ in $\dot{G}_0$ for some $\lambda \in \mathbb{Z}$. (By Lemma 72, subwords of these form can be found algorithmically.) Then, if, say, a subword of the form $t_{k_1}^{-1}ut_{k_1}$ is found, replace it with $\lambda\dot{b}_{n_{k_1},0}$. Thanks to the identity $t_{k_1}^{-1}\lambda\dot{a}_{n_{k_1},0}t_{k_1} = \lambda\dot{b}_{n_{k_1},0}$, the newly obtained word is equal to w in $\dot{G}_k$. Then repeat this procedure on the newly obtained word until there is no more subwords of the mentioned forms. Let w_1 be the word obtained as a result of this procedure. Then, by Lemma 74, either $w_1 \in \mathcal{S}_0^*$ or for some $k_0 \geqslant 1$, $w_1 \in \mathcal{S}_{k_0}^* \backslash \mathcal{S}_{k_0-1}^*$. Then, in the last case, by Lemma 74, w_1 is a reduced word over $\mathcal{S}_{k_0}^*$. Also in the first case (i.e. when $w_1 \in \mathcal{S}_0^*$), $w_1 = 1$ in $\dot{G}_k$ if and only if $w_1 = 1$ in $\dot{G}_0$, hence by Lemma 72, in this case, the identity $w_1 = 1$ can be checked algorithmically. In the second case, by Lemma 74, $w_1 \neq 1$ in $\dot{G}_k$.

□

Lemma 76. *The word problem in $\dot{G}$ is decidable with respect to the presentation* (6.4).

Proof. Suppose that w is a finite word with letters from

$$\mathcal{S}_k = \left\{ \pm \dot{a}_{n,m},\ \pm \dot{b}_{n,m},\ t_1^{\pm 1}, \ldots, t_k^{\pm 1} \ \mid n \in \mathbb{N},\ m \in \{0,1\} \right\},$$

where k is some natural number. Also suppose that w represents the trivial element in $\dot{G}$. Then, since $\dot{G}$ is a direct limit of the groups $\{\dot{G}_i\}_{i=1}^{\infty}$, there exists a minimal integer $N \geqslant 0$ such that w represents the trivial element in $\dot{G}_N$.

We claim that $N \leqslant k$. Indeed, if $N > k$, then since N was chosen as the minimal index such that $w = 1$ in $\dot{G}_N$, we get $w \neq 1$ in $\dot{G}_k$. However, by Corollary 16, $\dot{G}_k$ embeds into $\dot{G}_N$ under the map induces by

$$\dot{a}_{n,m} \mapsto \dot{a}_{n,m} \text{ and } t_1 \mapsto t_1, \ldots, t_k \mapsto t_k, \text{ for } n \in \mathbb{N}, m \in \{0,1\},$$

which implies that if $w \neq 1$ in $\dot{G}_k$, then $w \neq 1$ in $\dot{G}_N$. A contradiction.

Thus, if $w \in \mathcal{S}_k^*$ represents the trivial element in $\dot{G}$, then it represents the trivial element in $\dot{G}_k$ as well. In other words, in order to check whether or not w represents the trivial element in $\dot{G}$ it is enough to check its triviality in $\dot{G}_k$. Therefore, since for each $w \in \mathcal{S}^*$ one can algorithmically find (the minimal) $k \in \mathbb{N}$ such that $w \in \mathcal{S}_k^*$, the decidability of the word problem in $\dot{G}$ follows from Lemma 75. □

Lemma 77. *The group $\dot{G}$ is torsion-free.*

Proof. First of all, notice that by the properties of the groups $\dot{A}_k, \dot{B}_k,\ k \in \mathbb{N}$, and by the basic properties of direct products, the group $\dot{G}_0$ is torsion free.

Now, suppose that $u \in \mathcal{S}^*$ is such that it represents a torsion element of $\dot{G}$. Then, since $\dot{G}$ is a direct limit of the groups $\{\dot{G}_i\}_{i=1}^{\infty}$, there exists $k \in \mathbb{N}$ such that $u \in \mathcal{S}_k^*$ and u represents a torsion element in $\dot{G}_k$ as well. Since $\dot{G}_k$ is obtained from $\dot{G}_0$ by multiple HNN-extensions, then, by Lemma 68, $\dot{G}_k$ is a torsion free group. Therefore, u represents the trivial element in $\dot{G}_k$ as well as in $\dot{G}$. $\square$

Now suppose that $\Phi : \dot{G} \hookrightarrow \ddot{G}$ is an embedding of the group $\dot{G}$ into a finitely generated torsion-free group $\ddot{G}$ such that the maps

$$\phi_1 : (n, m) \mapsto \Phi(\dot{a}_{n,m}),\ \phi_2 : (n, m) \mapsto \Phi(\dot{b}_{n,m}),\ \text{and } \phi_3 : n \mapsto \Phi(t_n),$$

$$\text{where } n \in \mathbb{N}, m \in \{0, 1\},$$

are computable, and $\ddot{G}$ has decidable word problem. Then the next lemma shows that the group $\ddot{G}$ has the desirable properties we were looking for.

Lemma 78. *The group $\ddot{G}$ cannot be embedded in a group with decidable conjugacy problem.*

Proof. By contradiction, let us assume that $\ddot{G}$ embeds in a group $\bar{G}$ which has decidable conjugacy problem. Then, for the purpose of convenience, without loss of generality let us assume that $\ddot{G}$ is a subgroup of the group $\bar{G}$.

Below we show that the decidability of the conjugacy problem in $\bar{G}$ contradicts the assumption that $\mathcal{N}$ and $\mathcal{M}$ are disjoint and recursively inseparable.

Let us define $\mathcal{C} \subseteq \mathbb{N}$ as

$$\mathcal{C} = \{n \in \mathbb{N} \mid \Phi(\dot{a}_{n,0}) \text{ is conjugate to } \Phi(\dot{b}_{n,0}) \text{ in } \bar{G}\}.$$

Then, the decidability of the conjugacy problem in $\bar{G}$ implies that the set $\mathcal{C}$ is recursive, because, since the group $\bar{G}$ has decidable conjugacy problem, and since by our assumptions, the above mentioned maps ϕ_1, ϕ_2 and ϕ_3 are computable, for any input $n \in \mathbb{N}$ one can algorithmically verify whether or not $\Phi(\dot{a}_{n,0})$ is conjugate to $\Phi(\dot{b}_{n,0})$ in $\bar{G}$.

Therefore, since for groups with decidable conjugacy problem one can algorithmically find conjugator element for each pair of conjugate elements of the group, we also get that there exists a computable map

$$f : \mathcal{C} \to \bar{G}$$

such that for all $n \in \mathcal{C}$ we have

$$f(n)^{-1}\Phi(\dot{a}_{n,0})f(n) = \Phi(\dot{b}_{n,0}).$$

For $n \in \mathcal{C}$, let us denote

$$f(n) = g_n \in \bar{G}.$$

Now let us define

$$\mathcal{A} = \{n \in \mathcal{C} \mid g_n^{-1}\Phi(\dot{a}_{n,1})g_n = \Phi(\dot{b}_{n,1})\} \subseteq \mathbb{N}.$$

Since the word problem in $\bar{G}$ is decidable, the sets $\mathcal{C}$ is recursive and the maps Φ and f are computable, we get that the set $\mathcal{A}$ is a recursive subset of $\mathbb{N}$. Also since the following identities

$$\dot{a}_{n_i,1} = 2^i\dot{a}_{n_i,0}, \dot{b}_{n_i,1} = 2^i\dot{b}_{n_i,0} \text{ and } t_i^{-1}\dot{a}_{n_i,0}t_i = \dot{b}_{n_i,0}, \text{ for } i \in \mathbb{N},$$

hold in $\dot{G}$, we get that in $\bar{G}$ the following identities hold

$$\Phi(\dot{a}_{n_i,1}) = \Phi(\dot{a}_{n_i,0})^{2^i}, \Phi(\dot{b}_{n_i,1}) = \Phi(\dot{b}_{n_i,0})^{2^i}$$

and

$$\Phi(t_i)^{-1}\Phi(\dot{a}_{n_i,0})\Phi(t_i) = \Phi(\dot{b}_{n_i,0}) \text{ for all } n_i \in \mathcal{N}.$$

Therefore, we get that

$$\mathcal{N} \subseteq \mathcal{A}.$$

On the other hand, Corollary 17 implies that for any $n \in \mathcal{M}$, the pairs of elements

$$\big(\Phi(\dot{a}_{n,0}),\ \Phi(\dot{b}_{n,0})\big) \text{ and } \big(\Phi(\dot{a}_{n,1}),\ \Phi(\dot{b}_{n,1})\big)$$

cannot be conjugate in $\bar{G}$ by the same conjugator. Therefore, we get that

$$\mathcal{A} \cap \mathcal{M} = \emptyset.$$

Thus we get that $\mathcal{N} \subseteq \mathcal{A}$ and $\mathcal{A} \cap \mathcal{M} = \emptyset$, which implies that $\mathcal{A} \subset \mathbb{N}$ is a recursive separating set for $\mathcal{N}$ and $\mathcal{M}$, which contradicts the assumption that $\mathcal{N}$ and $\mathcal{M}$ are recursively inseparable.

$\square$

Finally, the embedding $\Phi : \dot{G} \hookrightarrow \ddot{G}$ with the prescribed properties exists, thanks to Theorem 19. Therefore, the group $\ddot{G}$ with the above mentioned properties exists. Also by a version of Higman's embedding theorem described by Aanderaa and Cohen in [1], the group $\ddot{G}$ can be embedded into a finitely presented group $\mathcal{G}$ with decidable word problem. By a recent result of Chiodo and Vyas, [16], the group $\mathcal{G}$ defined this way will also inherit the property of torsion-freeness from the group $\ddot{G}$.

Clearly, since $\ddot{G}$ cannot be embedded into a group with decidable conjugacy problem, this property will be inherited by $\mathcal{G}$. Thus Theorem 17 is proved.

Appendix A

Appendix

A.0.1 Proof of Lemma 10

Let $U, V, T_1, T_2, L, \lambda, c, m, n$ be defined as in the statement of Lemma 10.

Let us assume that

$$L \leqslant \frac{\|U\|}{12\lambda} m. \tag{A.1}$$

Then in the Cayley graph $\Gamma(G, X)$ there exists a rectangle $ABCD$ such that $lab(AB) = T_1$, $lab(BC) = U^m$, $lab(CD) = T_2$ and $lab(AD) = V^n$. Since the sides BC and AD are (λ, c)-quasi-geodesic and $\|T_1\|, \|T_2\| \leqslant L$, by Corollary 3, we get that the Hausdorff distance between BC and AD is bounded from above by $L + 2R_{\lambda,c} + 2\delta$. Moreover, by Corollary 4, for any point $o \in BC$ such that its distance from B and C is more than $L + R_{\lambda,c} + 2\delta$, we have $dist(o, AD) \leqslant 2R_{\lambda,c} + 2\delta$.

Let us fix the points $B', C' \in BC$ such that

$$lab(BB') = lab(C'C) = U^{2\lfloor(\lambda(L+R_{\lambda,c}+2\delta))+c)/\|U\|\rfloor+1}. \tag{A.2}$$

Note that then $d(B, B'), d(C, C') > L + R_{\lambda,c} + 2\delta$ and

$$\begin{aligned} & 2\big(2\lfloor(\lambda(L + R_{\lambda,c} + 2\delta)) + c)/\|U\|\rfloor + 1\big) \\ & \leqslant \frac{4\lambda L}{\|U\|} + \frac{4\lambda R_{\lambda,c} + 8\delta + 2c}{\|U\|} + 2 < \frac{4\lambda L}{\|U\|} + |X|^{2R_{\lambda,c}+2\delta+\|V\|} \\ \text{by (A.1) },\ & \leqslant \frac{m}{3} + |X|^{2R_{\lambda,c}+2\delta+\|V\|} \leqslant \frac{2m}{3}. \end{aligned} \tag{A.3}$$

Following Olshanskii, [55], we call a point on CD a phase vertex, , say O, if $lab(BO)$ is a power of U. Correspondingly, we call a point on AD, say O', a phase vertex, if $lab(AO')$ is a power of V. Since $B'C'$ is contained in the $(2R_{\lambda,c} + 2\delta)$-neighborhood of AD, for each phase vertex $O \in B'C'$ there exists a phase vertex $O' \in AD$ such that $d(O, O') \leqslant 2R_{\lambda,c} + 2\delta + \|V\|$. This follows from

Lemma 4 and from the simple observation that the set of phase vertices on AD is a $\|V\|$-net.

By (A.2) and (A.3) we get that the number of phase vertices on $B'C'$ is greater than $|X|^{2R_{\lambda,c}+2\delta+\|V\|}$ (recall that X is a symmetric set). Therefore, by the pigeonhole principle, there exist at least two phase vertices $O_1, O_2 \in BC$ and two phase vertices $O_1', O_2' \in AD$ such that $d(O_1, O_1'), d(O_2, O_2') \leqslant 2R_{\lambda,c} + 2\delta + \|V\|$ and $lab(O_1O_1') \equiv lab(O_2O_2')$, where by $lab(O_1O_1')$ and $lab(O_2O_2')$ we mean the labels of some geodesic paths joining O_1 to O_1' and O_2 to O_2', respectively.

Denote $Q = lab(O_1O_1') = lab(O_2O_2')$. Then we have that for some integers m_0 and n_0, $Q^{-1}U^{m_0}Q =_G V^{n_0}$. On the other hand, $T_1 =_G U^{m_1}QV^{n_1}$, where the integers m_1, n_1 are such that $lab(BO_1) = U^{m_1}$ and $lab(O_1'A) = V^{n_1}$. But this means that $T_1U^{m_0}T_1^{-1} =_G V^{n_0}$. Therefore, since every element of a hyperbolic group is contained in a unique maximal elementary subgroup (see [55]), $T_1UT_1^{-1}$ and V are contained in the same subgroup $E(V)$. The same way $T_2UT_2^{-1} \in E(V)$.

In case $U =_G V$, by the properties described in the beginning of Section 2.2, the fact that $T_1UT_1^{-1} \in E(V)(= E(U))$ implies that $T_1 \in E(V)(= E(T_2))$. The same way $T_2 \in E(V)(= E(U))$. Also, since V^{n_0} is a label of a subpath of DA, as it follows from the above described, we get that the sign of n_0 coincides with the sign of n. Therefore, $T_1, T_2 \in E^+(U)$ for $n > 0$ and $T_1, T_2 \in E^-(U)$ for $n \leqslant 0$.

. $\square$

A.0.2 Proof of Lemma 28

Since Δ is minimal and contains an $\mathcal{R}$-cell, by Lemma 19, it must contain an essential $\mathcal{R}$-cell. Let us consider an essential $\mathcal{R}$-cell Π in Δ, connected to AB, BC, CD and DA by contiguity subdiagrams Γ_1, Γ_2, Γ_3 and Γ_4, respectively. Then, in general, our diagram Δ looks like in Figure A.1, with a possibility that some of the contiguity subdiagrams Γ_1, Γ_2, Γ_3 and Γ_4, in fact, are empty (i.e. do not exist).

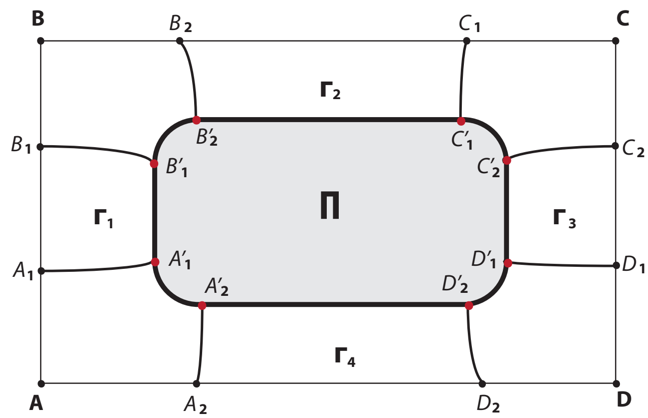

Figure A.1: $lab(AB) = lab(DC)$ are geodesic words and $lab(BC)$, $lab(AD)$ are cyclic shifts of U and V, respectively.

Proof of Part (1) of Lemma 28.

First of all, by contradiction assume that at least one of Γ_2 and Γ_4 is empty. First we will consider the case when just one of them is empty and then, separately, the case when both of them are empty.

Case 1.1. (Exactly one of Γ_2 and Γ_4 is empty).

For this case, without loss of generality assume that Γ_4 is empty. Then our conjugacy diagram Δ would look like in Figure A.2.

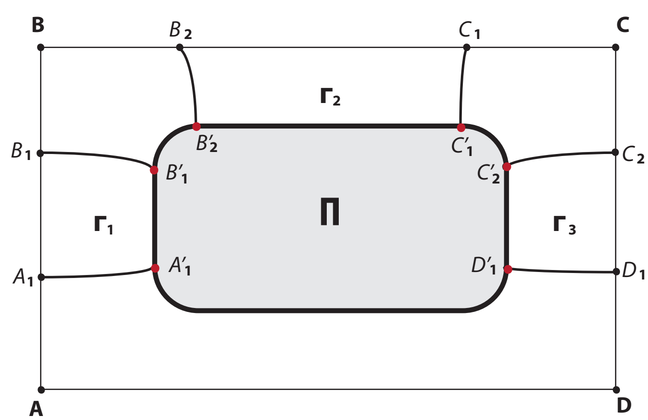

Figure A.2: Γ_4 is empty.

Since $lab(BC) = U$ is a cyclically $(\lambda, c, \epsilon, 1 - 121\lambda\mu)$-reduced word, we get that

$$(\Pi, \Gamma_2, BC) < 1 - 121\lambda\mu. \tag{A.4}$$

Therefore, since Π is an essential cell, meaning that $\sum_{i=1}^{4}(\Pi, \Gamma_i, \partial\Delta) > 1 - 23\mu$, it must be that

$$(\Pi, \Gamma_1, AB) + (\Pi, \Gamma_3, CD) > (121\lambda - 23)\mu > 98\lambda\mu. \tag{A.5}$$

In particular, at least one of Γ_1 and Γ_3 is non-empty. In fact, we claim that neither one of Γ_1 and Γ_3 is empty.

Claim. Neither one of Γ_1 and Γ_3 is empty.

Proof of the claim. First of all, without loss of generality assume that Γ_1 is non-empty.

Now since Δ is a slender (U, V)-conjugacy diagram, it must be that $d(A, B) \leqslant d(A, C_1)$. For the next chain of inequalities, in case Γ_3 is empty, we will simply assume $d(D_1', C_2') = 0$. Thus we have

$$\begin{aligned} d(A, B) =& d(A, A_1) + d(A_1, B) \leqslant d(A, C_1) \\ \leqslant& d(A, A_1) + d(A_1, C_1) \\ \leqslant& d(A, A_1) + d(A_1, A_1') \\ &+ d(A_1', D_1') + d(D_1', C_2') + d(C_2', C_1') + d(C_1', C_1), \end{aligned} \tag{A.6}$$

consequently, since $d(A_1', D_1') + d(C_2', C_1') \leqslant 23\mu\|\Pi\|$, we have

$$d(A, B) \leqslant d(A, A_1) + \epsilon + d(D_1', C_2') + 23\mu\|\Pi\| + \epsilon.$$

Therefore, $d(A_1, B_1) \leqslant d(A_1, B) \leqslant d(D_1', C_2') + 23\mu\|\Pi\| + 2\epsilon$. Combining this with the inequality $\|[A_1', B_1']\| \leqslant \lambda(d(A_1, B_1) + 2\epsilon) + c$, we get

$$\frac{\|[A_1', B_1']\| - c}{\lambda} - 2\epsilon \leqslant d(A_1, B_1) \leqslant d(D_1', C_2') + 23\mu\|\Pi\| + 2\epsilon.$$

Now, in case Γ_3 is empty, i.e. if $d(D_1', C_2') = 0$, we also have

$$\begin{aligned}\big\|[A_1', B_1']\big\| &\geqslant \|\Pi\| - 23\mu\|\Pi\| - \big\|[B_2', C_1']\big\| \\ &> (1 - 23\lambda\mu)\|\Pi\| - \big\|[B_2', C_1']\big\| \\ &> (1 - 23\lambda\mu)\|\Pi\| - (1 - 121\lambda\mu)\|\Pi\|, \text{ by (A.4)} \\ &= 98\lambda\mu\|\Pi\| \\ &> \lambda(4\epsilon + 23\mu\|\Pi\|) + c, \text{ by LPP.}\end{aligned} \tag{A.7}$$

From (A.7) it follows that $d(A_1, B_1) \geqslant 23\mu\|\Pi\| + 2\epsilon$. Therefore, $d(A, B) \geqslant d(A, A_1) + d(A_1, B_1) \geqslant d(A, A_1) + \epsilon + 23\mu\|\Pi\| + \epsilon$, but this contradicts (A.6). Therefore, in order not to have contradictions, Γ_3 have to be non-empty. $\square$

Note that

$$\begin{aligned}d(A_1, D_1) &\leqslant d(A_1, A_1') + d(A_1', D_1') + d(D_1', D_1) \\ &\leqslant 2\epsilon + 23\mu\|\Pi\|.\end{aligned}$$

Therefore, since $d(A, B) = d(D, C)$ and since by the property of cyclically slenderness, $d(A, B) \leqslant d(A, C)$, $d(D, C) \leqslant d(D, B)$, we get

$$\big|d(B, A_1) - d(C, D_1)\big| \leqslant d(A_1, D_1) \leqslant 2\epsilon + 23\mu\|\Pi\|. \tag{A.8}$$

Also, since $d(A, B) \leqslant d(A, B_2)$, we get

$$d(B_1, B) \leqslant d(B_1, B_2) \leqslant d(B_1, B_1') + d(B_1', B_2') + d(B_2', B_2) \leqslant 2\epsilon + 23\mu\|\Pi\|. \tag{A.9}$$

Analogously, we get

$$d(C_2, C) \leqslant d(C_2, C_1) \leqslant 2\epsilon + 23\mu\|\Pi\|. \tag{A.10}$$

After combining inequalities (A.8), (A.9) and (A.10), we get that

$$|d(A_1, B_1) - d(D_1, C_2)| \leqslant 2(2\epsilon + 23\mu\|\Pi\|) = 4\epsilon + 46\mu\|\Pi\|.$$

Moreover, since $lab(AB) = lab(DC)$, we get that $lab(A_1B_1)$ and $lab(D_1C_2)$ have a common subword of length at least $\max\left\{\big\|[A_1, B_1]\big\|, \big\|[D_1, C_1]\big\|\right\} - (4\epsilon + 46\mu\|\Pi\|)$. We will show that this is impossible.

Assume that it is possible. Then there exist $O_1, O_2 \in [A_1, B_1]$ such that $lab(O_1O_2)$ is also a subword of $lab(D_1C_2)$ and

$$\big\|[O_1, O_2]\big\| \geqslant \max\left\{\big\|[A_1, B_1]\big\|, \big\|[D_1, C_1]\big\|\right\} - (4\epsilon + 46\mu\|\Pi\|). \tag{A.11}$$

In light of (A.5), without loss of generality we can assume that $\big\|[A_1', B_1']\big\| \geqslant 49\lambda\mu\|\Pi\|$, which, by (A.11), implies that

$$\big\|[O_1, O_2]\big\| \geqslant 49\lambda\mu\|\Pi\| - (4\epsilon + 46\mu\|\Pi\|). \tag{A.12}$$

Now note that, by Corollary 3, there exist $O_1', O_2' \in [A_1', B_1']$ such that $d(O_1, O_1'), d(O_2, O_2') \leqslant \epsilon + R_{\lambda,c} + 2\delta \leqslant 2\epsilon$. Therefore, by the triangle inequality, we have

$$\begin{aligned}
\big\|[O_1', O_2']\big\| &\geqslant \big\|[O_1, O_2]\big\| - 2(\epsilon + R_{\lambda,c} + 2\delta) \\
&\geqslant 49\lambda\mu\|\Pi\| - (4\epsilon + 46\mu\|\Pi\|) - 2(2\epsilon) \\
\text{by (A.12)} &> 2\mu\|\Pi\|, \text{ by LPP.}
\end{aligned}$$

The last inequality contradicts Lemma 24. Therefore, we got a contradiction, which means that we are done with Case 1.1.

Illustration. For the sake of clarity of the above arguments, let us consider the following diagram: let us consider a (U, V)-conjugacy-diagram $\bar{\Delta}$ which is a copy of Δ with $\bar{\Delta} = \bar{A}\bar{B}\bar{C}\bar{D}$ and all points and subdiagrams inside have the same notations but with $\bar{bar}$ and let us attach this diagram to δ along the sides DC and $\bar{A}\bar{B}$. Let us denote the new diagram obtained this way by $\bar{\Delta}'$.

See Figure A.3.

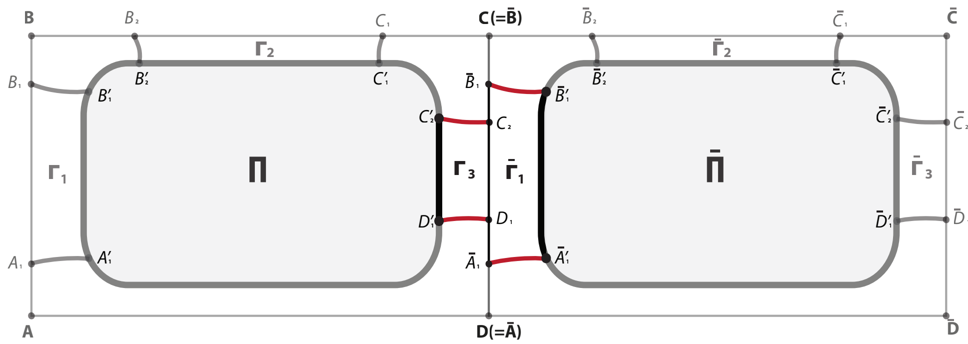

Figure A.3: $\bar{\Delta}'$: in the figure depicted the case when $O_1 = D_1$ and $O_2 = C_2$.

<u>*Case 1.2.*</u> (Both Γ_2 and Γ_4 are empty).

In this case the (U, V)-conjugacy diagram Δ looks like in Figure A.4.

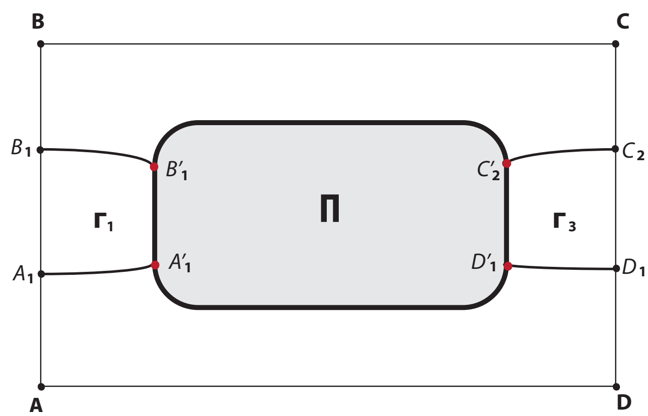

Figure A.4: Γ_2 and Γ_4 are empty.

The emptiness of Γ_2 and Γ_4 implies the following estimation of the lengths of arcs $[B_1', C_2']$ and $[D_1', A_1']$: $\|[B_1', C_2']\|, \|[D_1', A_1']\| \leqslant 23\mu\|\Pi\|$. Therefore, from the cyclically slenderness of Δ, it follows that

$$\begin{aligned} d(A,B) &= d(A,A_1) + d(A_1,B) \leqslant d(A,C) \\ &\leqslant d(A,A_1) + d(A_1,A_1') + d(A_1',D_1') + d(D_1',D_1) + d(D_1,C) \\ &\leqslant 23\mu\|\Pi\| + 2\epsilon + d(D_1,C). \end{aligned}$$

Therefore, we get that $d(A_1, B) - d(D_1, C) = d(D, D_1) - d(A, A_1) \leqslant 23\mu\|\Pi\|$. And from the symmetric arguments, we obtain $|d(A, A_1) - d(D, D_1)| \leqslant 23\mu\|\Pi\|$. Analogously, $|d(B, B_1) - d(C, C_2)| \leqslant 23\mu\|\Pi\|$. The rest is just a repetition of arguments of Case 1.

Thus the conclusion from Case 1 and Case 2 is that, in fact, Γ_2 and Γ_4 are non-empty.

At this point we already showed that Γ_1 and Γ_4 must be non-empty, i.e. we are done with the first part of the lemma. Thus the part (1) of the lemma is proved.

Now we are in a position to show the parts (2) and (3).

Proof of Parts (2) and (3) of Lemma 28.

First of all, note that since Π is an essential cell, i.e. $\sum_{i=1}^{4}(\Pi, \Gamma_i, \partial\Delta) > 1 - 23\mu$, part (2) immediately follows from part (3). Therefore, it is enought= to prove the statement of part (3).

To that end, let us first consider the case when at least one of Γ_1 and Γ_3 is empty. If both of Γ_1 and Γ_3 are empty, then there is nothing to prove for part (3), and part (2) is also true in that case, because Π is an essential cell. Therefore, let us separately consider two cases: when exactly one of Γ_1 and Γ_3 is empty and when both of them are non-empty.

Case 2.1. (Exactly one of Γ_1 and Γ_3 is empty).

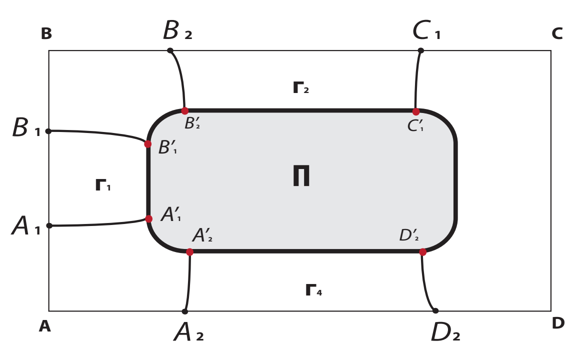

Figure A.5: Γ_3 is empty, but Γ_1 is not.

For this case, without loos of generality let us assume that Γ_1 is non-empty and Γ_3 is empty.

See Figure A.5. Then, since, by cyclic slenderness property, we have

$$d(D,C) \leqslant d(D_2,C_1) \leqslant d(D_2,D_2') + d(D_2',C_1') + d(C_1',C_1) \leqslant 2\epsilon + 23\mu\|\Pi\|$$

and $d(A_1,B_1) \leqslant d(A,B) = d(D,C)$, we get that $d(A_1,B_1) \leqslant 2\epsilon + 23\mu\|\Pi\|$. But also, since $lab[A_1',B_1']$ is a (λ,c)-quasi-geodesic word in $\Gamma(H,X)$, we have that

$$\begin{aligned}\left\|[A_1',B_1']\right\| \leqslant & \lambda d(A_1',B_1') + c \leqslant \lambda(d(A_1,B_1) + 2\epsilon) + c \\ \leqslant & \lambda(23\mu\|\Pi\| + 2\epsilon) + c < 29\lambda\mu\|\Pi\| \text{ by LPP.}\end{aligned}$$

Thus we are done in the case when at least one of Γ_1 and Γ_3 is empty, i.e with Case 2.1.

Case 2.2. (Both Γ_1 and Γ_3 are non-empty).

Since we already showed that Γ_2 and Γ_4 are non-empty, this case is equivalent of saying that all Γ_i, $i = 1,2,3,4$, are non-empty, that is the case depicted in Figure A.1.

For this case, by contradiction, assume that $\max\{(\Pi,\Gamma_1,AB),(\Pi,\Gamma_3,CD)\} > 49\lambda\mu$.

Now, since Δ is cyclically slender, we get that $d(B,A) \leqslant d(B,A_2)$. Therefore,

$$d(A_1,A) \leqslant d(A_1,A_2) \leqslant d(A_1,A_1') + d(A_1',A_2') + d(A_2',A_2) \leqslant 2\epsilon + 23\mu\|\Pi\|.$$

The same way we get that $d(B,B_1), d(C,C_2), d(D,D_1) \leqslant 2\epsilon + 23\mu\|\Pi\|$. Therefore, since $d(A,B) = d(D,C)$, we get that $|d(A_1,B_1) - d(D_1,C_2)| \leqslant 2(2\epsilon + 23\mu\|\Pi\|)$. Moreover, this observation, combined with the fact that $lab(AB) = lab(DC)$, implies that $lab([A_1,B_1])$ and $lab([D_1,C_2])$ have a common subword of length at least $\max\{\|[A_1,B_1]\|,\|[D_1,C_2]\|\} - 2(2\epsilon + 23\mu\|\Pi\|)$. But this is exactly a situation which we discussed while dealing with Case 1.2. Moreover, there we showed that this case is impossible if $\max\{(\Pi,\Gamma_1,AB),(\Pi,\Gamma_3,CD)\} > 49\lambda\mu$, hence we get a contradiction. This finishes the discussion of Case 2.2.

Thus part (3) of the lemma is proved too.

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