A unitary tensor product theory for unitary vertex operator algebra modules

By

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Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August 10, 2018

Nashville, Tennessee

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ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my advisor, Professor Vaughan Jones, who led me to a new fascinating world of mathematics, where different branches of mathematics are unified together. This work would not have been done without his constant support, guidance, and encouragement.

I would like to thank Prof. Yijun Yao, who has helped me so much when I was an undergraduate student.

Thanks to Prof. Yi-Zhi Huang and Prof. James Lepowsky for their hospitality and help during my visit in Rutgers University.

I want to thank the subfactor group in Vanderbilt, especially Prof Dietmar Bisch, Marcel Bischoff, Corey Jones, Zhengwei Liu, and Yunxiang Ren, from whom I have learned a lot. I'm thankful to all my friends in Vanderbilt. I really enjoyed the days that we had in Nashville. Special thanks go to Wenjia Zhang, and my roommates Longxiu Huang and Bin Sun, who have helped me a lot in my daily life.

Last but not least, I want to thank Sci-hub for making a huge amount of literature accessible to me. Without this fantastic and very helpful website I would have been unable to learn many interesting and important ideas from the research papers written by others.

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INTRODUCTION

This work studies quantum symmetries and 2d conformal field theory (CFT) from a unitary point of view. The mathematics lying behind these two things are *unitary modular tensor categories* and *unitary vertex operator algebras* respectively. Our goal in this work is, roughly speaking, to prove that the modular tensor categories constructed from a unitary vertex operator algebra are unitary. Most of this work is adapted from the author's papers [Gui17a, Gui17b].

Quantum symmetries: subfactors, low dimensional topology, and 3d TQFT

The story of quantum symmetries begins with V. Jones' *subfactor* theory in [Jon83]. Due to the Galois correspondence between a finite groups G and its fixed point subfactors M^G under an outer action of G on a hyperfinite II_1 factor M[1], the general subfactor theory is regarded as a quantum Galois theory. The first striking result of quantum symmetries is the non-integer but quantized index values

$$\{4\cos^2\frac{\pi}{n} : n = 3, 4, 5, \dots\} \cup [4, +\infty)$$

of finite-index subfactors proved in [Jon83]. It was soon realized that the Temperley-Lieb algebras and a quantum trace function (the Markov trace) on them, which were extracted from finite index subfactors in [Jon83], produce a non-trivial link invariant: the *Jones polynomial* ([Jon85]). Thus began the interactions between quantum symmetries and low-dimensional topology.

A key observation in [Jon85] is that the Temperley-Lieb algebra obtained from a subfactor gives a unitary *braid group* representation, and the quantum trace of braids gives the link invariant. E.Witten made a breakthrough in [Wit89]. He gave an interpretation of the Jones polynomial via Chern-Simons topological quantum field theory (TQFT). Witten's 3d TQFT not only reproduces the braid group representations and the quantum trace in [Jon85], which therefore reproduces the Jones polynomial, but also gives us new interesting 3d manifold invariants and, more generally, invariants of (framed) links in general closed oriented 3d manifolds.

[1] A *factor* is a von Neumann algebra with trivial center. A von Neumann algebra is a strongly-closed self-adjoint unital subalgebra of the $*$-algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. A factor is called type II_1 if it has a faithful trace.

Quantum symmetries: unitary modular tensor categories

Witten's approach based on path integral quantization of classical Chern-Simons action cannot be adopted directly by mathematicians, so we would like to see a mathematically rigorous construction of Witten's 3d TQFT satisfying the axioms proposed by Atiyah [Ati88]. This was achieved by Reshetikhin and Turaev. In [RT91] and [Tur16], they constructed unitary 3d TQFTs from *unitary modular tensor categories* (UMTCs). A modular tensor category (MTC) is a tensor category with certain structural isomorphisms (associators, braid operators, twists) and non-degeneracy properties (rigidity and modularity); unitarity of a MTC partly means that the structural isomorphisms are unitary operators. Besides 3d TQFTs, one can also construct subfactors from UMTCs. (See, for example, [Wen98] section 4.) So UMTC seems to provide us a (if not the) correct context to understand different aspects of quantum symmetries: subfactors, link and 3d manifold invariants, 3d TQFT. The theory of UMTC looks so powerful. One might doubt if there exist many good examples. Where do UMTCs come from? How can we construct examples of UMTC?

The earliest examples of UMTCs defined rigorously in math are constructed from representations of Hopf algebras (quantum groups at certain roots of unity, quantum doubles of finite groups). (See [BK01] or [Tur16] for a detailed treatment. For general references, we also recommend [Row06].) The quantum group construction produces the same TQFT as Witten's Chern-Simons theory does.

An even broader way to construct UMTCs is through *conformal field theory* (CFT). In fact, it is even conjectured that all (unitary) MTCs can be constructed from (unitary) chiral CFTs. In the physics literature, due to the work of Moore and Seiberg [MS88, MS89], people knew very early how to construct MTCs from chiral CFTs (earlier than the Hopf algebra construction). However, it has always been a big problem to realize this construction in mathematics, since Moore-Seiberg's works are based on some assumptions that are natural in physics but not easy to verify in math. I explain briefly these assumptions in the following.

From vertex operator algebras to modular tensor categories

A CFT is a 2d quantum field theory with local conformal symmetry. Let us take the Euclidean point of view. Then the elements of the local conformal group on $\mathbb{R}^2$ are local holomorphic and antiholomorphic functions on $\mathbb{C}$. This suggests that we first study the chiral (which means holomorphic) and the anti-chiral (antiholomorphic) field operators in a given CFT. The algebra formed by the chiral field operators of a CFT is called a *vertex operator algebra* (VOA) (usually denoted by V).

To study general field operators using chiral ones, one must introduce intertwining operators. Given three representations W_i, W_j, W_k of V, a type $\binom{k}{i\ j}$ *intertwining operator* of V is a *multi-valued* holomorphic function on $\mathbb{C}\backslash\{0\}$ whose values are linear operators mapping $W_i \otimes W_j$ "almost" into W_k. Moreover, these values (linear maps) "intertwine" the actions of V on W_i, W_j, and W_k. In the physics literature, intertwining operators are called conformal blocks, because a general field operator of a CFT can be decomposed as a sum of products of an intertwining operator with the complex conjugate of another one. In the cases that we are interested in, this decomposition is finite, and the CFTs are called *rational*.

Moore and Seiberg's construction of MTCs from rational chiral CFTs is based on the assumption that the intertwining operators of V satisfy braiding, fusion, and modular invariance properties. In short, these properties mean that different ways of constructing correlation functions on a general Riemann surface with holes by "attaching" intertwining operators (regarded as 3-point functions) on genus 0 Riemann surfaces with 3 holes (the pants) are equivalent (see figure 1).

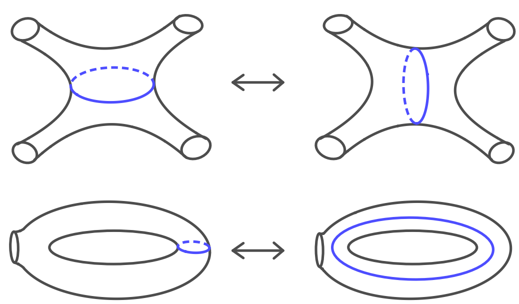

Figure 1

Giving a mathematically rigorous proof of all these assumptions has long been an open problem. Many people have made contributions to this problem, but the first complete and general proof was given by Y.Huang. We refer the reader to [HL13] for a discussion of this issue. Thus, the problem of constructing MTCs from rational chiral CFTs (i.e., from rational VOA) was solved.

The problem of unitarity

We still need to prove that the MTCs constructed from rational unitary VOAs are unitary, which is the goal of our work. To the author's knowledge, previously no one has treated this problem seriously. But why is this problem important? And what should we do to prove this unitarity?

We first answer the first question. We give three reasons for studying the unitarity of MTCs.

(a) To prove that the 3d TQFT constructed from a MTC (in the sense of Reshetikhin-Turaev) is unitary, one must assume that the MTC is unitary. For a general 3d TQFT, the state spaces associated to the boundary 2d manifolds (with holes) do not carry inner products, i.e., we only have vector spaces, but not *Hilbert spaces* for our TQFT. To have inner products, we must require the unitarity of the MTC. For details, see [Tur16] section IV.11.

(b) One can construct subfactors only from unitary MTCs. The reason is plain: an operator algebra is a *-algebra of operators on a *Hilbert space*. Only UMTCs give us Hilbert spaces.

(c) Besides Moore-Seiberg's construction of MTCs from chiral CFTs (i.e., from VOAs), which is by nature geometric, there is a more analytic approach based on the idea of R.Haag's algebraic quantum field theory (AQFT) ([Haag]). In this approach people consider, rather than the vertex algebras of holomorphic field operators localized at points, the nets (precosheafs) of associative *-algebras of bounded operators localized on the open intervals in S^1. These nets of operator algebras are called *conformal nets*. The tensor product of modules of a conformal net is defined using Connes' relative tensor product [Con80], and the tensor categories for conformal nets defined in this way are manifestly unitary. See [Kaw15] and the reference therein. It is an open question whether the MTCs constructed from VOAs and from conformal nets are equivalent. A.Wassermann made the first and very important step toward solving this problem. In ([Was98]) he gave a general strategy to show that the fusion rules in the two tensor categories are equal, the main idea of which also plays a central role in our present work. However, in order to completely solve the equivalence problem, one first needs to prove the unitarity of the MTCs associated to unitary rational VOAs. See [Gui17a] introduction for a detailed discussion.

As for the second question, let us first make it clear the starting point of our theory: we assume that our rational VOA V is unitary, which roughly means that V satisfies reflection positivity. In particular, V has a *-structure. We also assume that any representation (module) W of V is unitarizable, which means that we can equip W with an inner-product, so that the action of V on W preserves the $*$-structure. (Rigorous definitions can be found in chapter 1.) Unitarity of VOAs and their representations are well-studied for many examples (see chapter 8).

But proving the unitarizability of all representations of a given unitary VOA V is far from enough to show that the MTC constructed from the unitary representations of V is unitary. In fact, for any unitary representations W_i, W_j of V, the tensor product $W_i \boxtimes W_j$, which again is a representation of V, cannot be defined in the classical way as the algebraic tensor product of W_i, W_j. It is defined indirectly using the (finite-dimensional) vector spaces of intertwining operators of V. Therefore, even though we know that the representation $W_i \boxtimes W_j$ is unitarizable, it still remains for us to choose a *canonical* unitary structure (or more explicitly, a canonical inner product Λ) on $W_i \boxtimes W_j$. The word "canonical" means that such inner product cannot be chosen arbitrarily: *it should make the structural maps of the MTC (associators, braid operators, twists...) unitary*. Equivalently, we should define an inner product on each vector space of intertwining operators, such that in addition to some small requirements, the braid matrices, the fusion matrices, and the S-matrices (these are the matrices that relate the different ways of attaching intertwining operators indicated in figure 1) are unitary.

As we will see, it is not hard to define such a Λ as a sesquilinear form on $W_i \boxtimes W_j$. Due to the non-degeneracy property of the MTC of V proved by Y.Huang, it is also not hard to prove that Λ is non-degenerate. *The main difficulty*, however, *is to prove the positivity of* Λ, and such a proof will occupy the major part of our work. Once we have shown that the Λ we choose is positive, which means that Λ is an inner product, the unitarity of the MTC of V follows easily.

Outline

In chapter 1 we review some of the basic definitions of unitary VOAs, their unitary representations, and intertwining operators. In chapter 2, we review the braid and fusion relations of intertwining operators proved by Y.Huang. We also review Huang-Lepowsky's construction of MTCs from rational VOAs, the basic idea of which dates back to Moore-Seiberg. Our proof of the positivity of Λ relies on the *energy-bounds condition* for vertex operators and intertwining operators. This condition allows us to connect unitary VOAs with conformal nets. This will be discussed in chapter

3 and 4.

In practice, it is often much easier to show that not all but only a generating set of intertwining operators satisfy energy bounds condition. But then one needs the generalized intertwining operators to bridge a gap between VOAs and conformal nets. We discuss generalized intertwining operators in chapter 5. In chapter 6 we define a non-degenerate sesquilinear form Λ on each $W_i \boxtimes W_j$, and use techniques in conformal nets to show that Λ is positive. In chapter 7 we prove the unitarity of the MTCs. Our theory is applied to unitary minimal models and WZW models of certain types in chapter 8.

Notations.

In this paper, we assume that V is a vertex operator algebra of CFT type. Except in chapter 1, we assume that V also satisfies the following conditions:

(1) V is isomorphic to V'. (0.1)

(2) Every $\mathbb{N}$-gradable weak V-module is completely reducible. (0.2)

(3) V is C_2-cofinite. (0.3)

(See [Hua05b] for the definitions of these terminologies.) The following notations are used throughout this paper.

A^{t}: the transpose of the linear operator A.

$A^\dagger$: the formal adjoint of the linear operator A.

A^*: the ajoint of the possibly unbounded linear operator A.

$\overline{A}$: the closure of the pre-closed linear operator A.

C_i: the antiunitary map $W_i \to W_{\bar{i}}$.

$\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$.

$\mathrm{Conf}_n(\mathbb{C}^\times)$: the n-th configuration space of $\mathbb{C}^\times$.

$\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$: the universal covering space of $\mathrm{Conf}_n(\mathbb{C}^\times)$.

$\mathscr{D}(A)$: the domain of the possibly unbounded operator A.

$\not{d}\theta = \frac{e^{i\theta}}{2\pi} d\theta$.

$e_r(e^{i\theta}) = e^{ir\theta} \quad (-\pi < \theta < \pi)$.

$\mathcal{E}$: a complete list of mutually inequivalent irreducible V-modules.

$\mathcal{E}^{\mathrm{u}}$: the set of unitary V-modules in $\mathcal{E}$.

$\mathrm{Hom}_V(W_i, W_j)$: the vector space of V-module homomorphisms from W_i to W_j.

$\mathcal{H}_i$: the norm completion of the vector space W_i.

$\mathcal{H}_i^r$: the vectors of $\mathcal{H}_i$ that are inside $\mathscr{D}((1+\overline{L_0})^r)$.

$\mathcal{H}_i^\infty = \bigcap_{r\geqslant 0} \mathcal{H}_i^r$.

I^c: the complement of the open interval I.

$I_1 \subset\subset I_2$: $I_1, I_2 \in \mathcal{J}$ and $\overline{I_1} \subset I_2$.

$\mathrm{id}_i = \mathrm{id}_{W_i}$: the identity operator of W_i.

$\mathcal{J}$: the set of (non-empty, non-dense) open intervals of S^1.

$\mathcal{J}(U)$: the set of open intervals of S^1 contained in the open set U.

P_s: the projection operator of W_i onto $W_i(s)$.

$\mathfrak{r}(t) : S^1 \to S^1$: $\mathfrak{r}(t)(e^{i\theta}) = e^{i(\theta+t)}$.

$\mathfrak{r}(t) : C^\infty(S^1) \to C^\infty(S^1)$: $\mathfrak{r}(t)h = h \circ \mathfrak{r}(-t)$.

$\mathrm{Rep}(V)$: the modular tensor category of the representations of V.

$\mathrm{Rep}^{\mathrm{u}}(V)$: the category of the unitary representations of V.

$\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$: When $\mathcal{G}$ is additively closed, it is the subcategory of $\mathrm{Rep}^{\mathrm{u}}(V)$ whose objects are unitary V-modules in $\mathcal{G}$. When $\mathcal{G}$ is multiplicatively closed, then it is furthermore equipped with the structure of a ribbon tensor category.

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

$\mathcal{V}\binom{k}{i\,j}$: the vector space of type $\binom{k}{i\,j}$ intertwining operators.

$W_0 = V$, the vacuum module of V.

W_i: a V-module.

$\widehat{W_i}$: the algebraic completion of W_i.

$W_{\bar{i}} \equiv W_i'$: the contragredient module of W_i.

$W_{ij} \equiv W_i \boxtimes W_j$: the tensor product of W_i, W_j.

$w^{(i)}$: a vector in W_i.

$\overline{w^{(i)}} = C_i w^{(i)}$.

x: a formal variable.

Y_i: the vertex operator of W_i.

$\mathcal{Y}_\alpha$: an intertwining operator of V.

$\mathcal{Y}_{\overline{\alpha}} \equiv \overline{\mathcal{Y}_\alpha}$: the conjugate intertwining operator of $\mathcal{Y}_\alpha$.

$\mathcal{Y}_{\alpha^*} \equiv \mathcal{Y}_\alpha^\dagger$: the adjoint intertwining operator of $\mathcal{Y}_\alpha$.

$\mathcal{Y}_{B_\pm \alpha} \equiv B_\pm \mathcal{Y}_\alpha$: the braided intertwining operators of $\mathcal{Y}_\alpha$.

$\mathcal{Y}_{C\alpha} \equiv C\mathcal{Y}_\alpha$: the contragredient intertwining operator of $\mathcal{Y}_\alpha$.

$\mathcal{Y}_{i0}^i$: the creation operator of W_i.

$\mathcal{Y}_{\bar{i}i}^0$: the annihilation operator of W_i.

Δ_i: the conformal weight of W_i.

Δ_w: the conformal weight (the energy) of the homogeneous vector w.

Θ_{ij}^k: a set of linear basis of $\mathcal{V}\binom{k}{i\ j}$.

$\Theta_{i*}^k = \coprod_{j\in\mathcal{E}} \Theta_{ij}^k, \Theta_{*j}^k = \coprod_{i\in\mathcal{E}} \Theta_{ij}^k, \Theta_{ij}^* = \coprod_{k\in\mathcal{E}} \Theta_{ij}^k$.

θ: the PCT operator of V, or a real variable.

ϑ_i: the twist of W_i.

ν: the conformal vector of V.

$\sigma_{i,j}$: the braid operator $\sigma_{i,j} : W_i \boxtimes W_j \to W_j \boxtimes W_i$.

Ω: the vacuum vector of V.

CHAPTER 1

INTERTWINING OPERATORS OF UNITARY VERTEX OPERATOR ALGEBRAS

We refer the reader to [FHL93] for the general theory of VOAs, their representations, and intertwining operators. Other standard references on VOAs include [FB04, FLM89, Kac98, LL12]. Unitary VOAs were defined by Dong, Lin in [DL14]. Our approach in this article follows [CKLW15].

1.1 Unitary VOAs

Let x be a formal variable. For a complex vector space U, we set

$$U[[x]] = \left\{ \sum_{n\in\mathbb{Z}_{\geqslant 0}} u_n x^n : u_n \in U \right\}, \tag{1.1}$$

$$U((x)) = \left\{ \sum_{n\in\mathbb{Z}} u_n x^n : u_n \in U, u_n = 0 \text{ for sufficiently small } n \right\}, \tag{1.2}$$

$$U[[x^{\pm 1}]] = \left\{ \sum_{n\in\mathbb{Z}} u_n x^n : u_n \in U \right\}, \tag{1.3}$$

$$U\{x\} = \left\{ \sum_{s\in\mathbb{R}} u_s x^s : u_s \in U \right\}. \tag{1.4}$$

We define the formal derivative $\frac{d}{dx}$ to be

$$\frac{d}{dx}\left(\sum_{n\in\mathbb{R}} u_n x^n \right) = \sum_{n\in\mathbb{R}} n u_n x^{n-1}. \tag{1.5}$$

Let V be a complex vector space with grading $V = \bigoplus_{n\in\mathbb{Z}} V(n)$. Assume that $\dim V(n) < \infty$ for each $n \in \mathbb{Z}$, and $\dim V(n) = 0$ for n sufficiently small. We say that V is a **vertex operator algebra** (VOA), if the following conditions are satisfied:

(a) There is a linear map

$$V \to (\text{End } V)[[x^{\pm 1}]]$$

$$u \mapsto Y(u,x) = \sum_{n\in\mathbb{Z}} Y(u,n) x^{-n-1}$$

(where $Y(u,n) \in$End V),

such that for any $v \in V$, $Y(u,n)v = 0$ for n sufficiently large.

(b) (**Jacobi identity**) For any $u, v \in V$ and $m, n, h \in \mathbb{Z}$, we have

$$\sum_{l \in \mathbb{Z}_{\geqslant 0}} \binom{m}{l} Y(Y(u, n+l)v, m+h-l)$$
$$= \sum_{l \in \mathbb{Z}_{\geqslant 0}} (-1)^l \binom{n}{l} Y(u, m+n-l) Y(v, h+l) - \sum_{l \in \mathbb{Z}_{\geqslant 0}} (-1)^{l+n} \binom{n}{l} Y(v, n+h-l) Y(u, m+l). \tag{1.6}$$

(c) There exists a vector $\Omega \in V(0)$ (the **vacuum vector**) such that $Y(\Omega, x) = \mathrm{id}_V$.

(d) For any $v \in V$ and $n \in \mathbb{Z}_{\geqslant 0}$, we have $Y(v,n)\Omega = 0$, and $Y(v,-1)\Omega = v$. This condition is simply written as $\lim_{x\to 0} Y(v,x)\Omega = v$.

(e) There exists a vector $\nu \in V(2)$ (the **conformal vector**) such that the operators $L_n = Y(\nu, n+1)$ ($n \in \mathbb{Z}$) satisfy the Virasoro relation: $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}c$. Here the number $c \in \mathbb{C}$ is called the **central charge** of V.

(f) If $v \in V(n)$ then $L_0 v = nv$. n is called the **conformal weight** (or the **energy**) of v and will be denoted by Δ_v. L_0 is called the **energy operator**.

(g) (**Translation property**) $\frac{d}{dx} Y(v,x) = Y(L_{-1}v, x)$.

Convention 1.1. In this article, we always assume that V is a VOA of **CFT type**, i.e., $V(0) = \mathbb{C}\Omega$, and $\dim V(n) = 0$ when $n < 0$.

Given a (anti)linear bijective map $\phi : V \to V$, we say that ϕ is an **(antilinear) automorphism** of V if the following conditions are satisfied:

$$(a)\phi\Omega = \Omega, \;\; \phi\nu = \nu. \tag{1.7}$$

$$(b)\text{For any } v \in V, \;\; \phi Y(v,x) = Y(\phi v, x)\phi. \tag{1.8}$$

It is easy to deduce from these two conditions that $\phi L_n = L_n \phi$ (for any $n \in \mathbb{Z}$). In particular, since ϕ commutes with L_0, we have $\phi V(n) = V(n)$ for each $n \in \mathbb{Z}$.

Definition 1.2. Suppose that V is equipped with an inner product $\langle\cdot|\cdot\rangle$ (antilinear on the second variable) satisfying $\langle\Omega|\Omega\rangle = 1$. Then we call V a **unitary vertex operator algebra**, if there exists an antilinear automorphism θ, such that for any $v \in V$ we have

$$Y(v,x)^\dagger = Y(e^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1}), \tag{1.9}$$

where † is the formal adjoint operation. More precisely, this equation means that for any $v, v_1, v_2 \in V$ we have

$$\langle Y(v,x)v_1|v_2\rangle = \langle v_1|Y(e^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1})v_2\rangle. \tag{1.10}$$

Remark 1.3. Such θ, if exists, must be unique. Moreover, θ is anti-unitary (i.e. $\langle\theta v_1|\theta v_2\rangle = \langle v_2|v_1\rangle$ for any $v_1, v_2 \in V$), and $\theta^2 = id_V$ (i.e. θ is an involution). We call θ the **PCT operator** of V. (cf. [CKLW15] proposition 5.1.) In this article, θ denotes either the PCT operator of V, or a real variable. These two meanings will be used in different situations. So no confusion will arise.

We say that a vector $v \in V$ is **homogeneous** if $v \in V(n)$ for some $n \in \mathbb{Z}$. If moreover, $L_1 v = 0$, we say that v is **quasi-primary**. It is clear that the vacuum vector Ω is quasi-primary. If we let $u = v = \nu, m = 0, n = 2, h = -1$ in the Jacoby identity (1.6), then we may compute that $L_1\nu = Y(L_1\nu, -1)\Omega = 0$. We conclude that *the conformal vector is quasi-primary*.

Now suppose that V is unitary and $v \in V$ is quasi-primary, then equation (1.9) can be simplified to

$$Y(v,x)^\dagger = (-x^{-2})^{\Delta_v} Y(\theta v, x^{-1}). \tag{1.11}$$

If we take $v = \nu$, then we obtain

$$L_n^\dagger = L_{-n} \quad (n \in \mathbb{Z}). \tag{1.12}$$

In particular, we have $L_0^\dagger = L_0$. This shows that different energy subspaces are orthogonal, i.e., *the grading $V = \bigoplus_{n\geqslant 0} V(n)$ is orthogonal under the inner product $\langle\cdot|\cdot\rangle$.*

Definition 1.4. Let W_i be a complex vector space with grading $W_i = \bigoplus_{s\in\mathbb{R}} W_i(s)$. Assume dim $W_i(s) < \infty$ for each $s \in \mathbb{R}$, and $\dim W_i(s) = 0$ for s sufficiently small. We say that W_i is a **representation of** V (or V**-module**), if the following conditions are satisfied:

(a) There is a linear map

$$V \to (\text{End } W_i)[[x^{\pm 1}]]$$

$$v \mapsto Y_i(v,x) = \sum_{n\in\mathbb{Z}} Y_i(v,n)x^{-n-1}$$

$$(\text{where } Y(v,n) \in \text{End } W_i),$$

such that for any $w^{(i)} \in W_i$, $Y_i(v,n)w^{(i)} = 0$ for n sufficiently large. Y_i is called the **vertex operator** of W_i.

(b) (**Jacobi identity**) For any $u, v \in V$ and $m, n, h \in \mathbb{Z}$, we have

$$\begin{aligned}&\sum_{l\in\mathbb{Z}_{\geqslant 0}} \binom{m}{l} Y_i(Y(u,n+l)v, m+h-l)\\ =&\sum_{l\in\mathbb{Z}_{\geqslant 0}} (-1)^l \binom{n}{l} Y_i(u,m+n-l)Y_i(v,h+l) - \sum_{l\in\mathbb{Z}_{\geqslant 0}} (-1)^{l+n} \binom{n}{l} Y_i(v,n+h-l)Y_i(u,m+l).\end{aligned} \tag{1.13}$$

(c) $Y_i(\Omega, x) = \text{id}_{W_i}$.

(d) The operators $L_n = Y_i(\nu, n+1)$ $(n \in \mathbb{Z})$ satisfy the Virasoro relation: $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}c$, where c is the central charge of V.

(e) If $w^{(i)} \in W_i(s)$ then $L_0 w^{(i)} = s w^{(i)}$. s is called the **conformal weight** (or the **energy**) of $w^{(i)}$ and will be denoted by $\Delta_{w^{(i)}}$, and L_0 is called the **energy operator**.

(f) (**Translation property**) $\frac{d}{dx}Y_i(v,x) = Y_i(L_{-1}v, x)$.

Clearly V itself is a representation of V. We call it the **vacuum module** of V. Modules of V are denoted by $W_i, W_j, W_k, \ldots$, or simply $i, j, k, \ldots$. The vacuum module is sometimes denoted by 0. We let $\text{id}_i = \text{id}_{W_i}$ and $\text{id}_0 = \text{id}_V$ be the identity operators on W_i and V respectively.

A V**-module homomorphism** is, by definition, a linear map $\phi : W_i \to W_j$, such that for any

$v \in V$ we have $\phi Y_i(v,x) = Y_j(v,x)\phi$. It is clear that ϕ preserves the gradings of W_i, W_j, for ϕ intertwines the actions of L_0 on these spaces. The vector space of homomorphisms $W_i \to W_j$ is denoted by $\mathrm{Hom}_V(W_i, W_j)$.

Remark 1.5. If the V-module W_i has a subspace W that is invariant under the action of V, then the restricted action of V on W produces a submodule of W_i. In fact, the only non-trivial thing to check is that W inherits the grading of W_i. But this follows from the fact that L_0, when restriced to W, is diagonalizable on W. (In general, if a linear operator of a complex vector space is diagonalizable, then by polynomial interpolations, it must also be diagonalizable on any invariant subspace.)

From the remark above, we see that a module W_i is irreducible if and only if the vector space W_i has no V-invariant subspace. If W_i is irreducible, we call

$$\Delta_i = \inf\{s : \dim W_i(s) > 0\}$$

the **conformal weight** of W_i. It is easy to show that $W_i = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} W_i(n+\Delta_i)$.

We now review the definition of contragredient modules introduced in [FHL93]. Let again W_i be a V-module. First we note that the dual space W_i^* of W_i has the grading $W_i^* = \prod_{s\in\mathbb{R}} W_i(s)^*$. Here $W_i(s)^*$ is the dual space of the finite dimensional vector space $W(s)$, and if $s \neq t$, the evaluations of $W_i(s)^*$ on $W_i(t)$ are set to be zero. Now we consider the subspace $W_{\bar{i}} \equiv W_i' = \bigoplus_{s\in\mathbb{R}} W(s)^*$ of W^*. We define the action of V on $W_{\bar{i}}$ as follows:

$$Y_{\bar{i}}(v,x) = Y_i(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})^{\mathrm{t}} \tag{1.14}$$

where the superscript "t" stands for the transpose operation. In other words, for any $w^{(\bar{i})} \in W_{\bar{i}} \subset W_i^*$ and $w^{(i)} \in W_i$, we have

$$\langle Y_{\bar{i}}(v,x)w^{(\bar{i})}, w^{(i)}\rangle = \langle w^{(\bar{i})}, Y_i(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})w^{(i)}\rangle. \tag{1.15}$$

We refer the reader to [FHL93] section 5.2 for a proof that $(W_{\bar{i}}, Y_{\bar{i}})$ is a representation of V. This representation is called the **contragredient module** of W_i.

In general, for each V-module W_i, the vector space $\widehat{W}_i = \prod_{s\in\mathbb{R}} W_i(s)$ is called the **algebraic**

completion of W_i. The action Y_i of V on W_i can be clearly extended onto $\widehat{W}_i$. It is clear that $\widehat{W}_i$ can be identified with $W_{\bar{i}}^*$.

Equation (1.14) can be written in terms of modes: if $v \in V$ is a quasi-primary vector with conformal weight Δ_v, then

$$Y_{\bar{i}}(v,n) = \sum_{m \in \mathbb{Z}_{\geqslant 0}} \frac{(-1)^{\Delta_v}}{m!} Y_i(L_1^m v, -n-m-2+2\Delta_v)^{\mathrm{t}}. \tag{1.16}$$

In particular, by letting $v = \nu$, we obtain $L_n^{\mathrm{t}} = L_{-n}$. More precisely, if $w^{(i)} \in W_i, w^{(\bar{i})} \in W_{\bar{i}}$, we have $\langle L_n w^{(i)}, w^{(\bar{i})} \rangle = \langle w^{(i)}, L_{-n} w^{(\bar{i})} \rangle$.

The contragredient operation is an involution: W_i is the contragredient module of $W_{\bar{i}}$. In particular, we have

$$Y_i(v,x) = Y_{\bar{i}}(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})^{\mathrm{t}}. \tag{1.17}$$

Hence we identify i with $\bar{\bar{i}}$, the contragredient module of $\bar{i}$.

Now we turn to the definition of unitary VOA modules.

Definition 1.6. Suppose that V is unitary and W_i is a V-module equipped with an inner product $\langle \cdot | \cdot \rangle$. We call W_i **unitary** if for any $v \in V$ we have

$$Y_i(v,x)^\dagger = Y_i(e^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1}). \tag{1.18}$$

In the remaining part of this section, we assume that V is unitary. Let W_i be a unitary V-module. Then formula (1.18), with $v = \nu$, implies that the action of the Virasoro subalgebras $\{L_n\}$ on W_i satisfies $L_n^\dagger = L_{-n}$. In particular, L_0 is symmetric, and hence the decompsition $W_i = \bigoplus_{s \in \mathbb{R}} W_i(s)$ is orthogonal. If we let P_s be the projection operator of W_i onto $W_i(s)$ (this operator can be defined whether W_i is unitary or not), we have $P_s^\dagger = P_s$.

Proposition 1.7 (Positive energy). *If W_i is unitary, then we have the grading $W_i = \bigoplus_{s \geqslant 0} W_i(s)$. In particular, if W_i is irreducible, then $\Delta_i \geqslant 0$.*

Proof. We choose an arbitrary non-zero homogeneous vector $w^{(i)} \in W_i$ and show that $\Delta_{w^{(i)}} \geqslant 0$.

First, assume that $w^{(i)}$ is **quasi-primary** (i.e., $L_1 w^{(i)} = 0$). Then we have

$$2\Delta_{w^{(i)}}\langle w^{(i)}|w^{(i)}\rangle = 2\langle L_0 w^{(i)}|w^{(i)}\rangle = \langle [L_1, L_{-1}]w^{(i)}|w^{(i)}\rangle = \|L_{-1}w^{(i)}\|^2 \geqslant 0,$$

which implies that $\Delta_{w^{(i)}} \geqslant 0$. In general, we may find $m \in \mathbb{Z}_{\geqslant 0}$ such that $L_1^m w^{(i)} \neq 0$, and $L_1^{m+1} w^{(i)} = 0$. So $\Delta_{L_1^m w^{(i)}} \geqslant 0$, and hence $\Delta_{w^{(i)}} = \Delta_{L_1^m w^{(i)}} + m \geqslant 0$. □

Proposition 1.8. *If W_i is unitary, then its contragredient module $W_{\bar{i}}$ is unitarizable.*

Proof. Assume that W_i has inner product $\langle\cdot|\cdot\rangle$. Define an anti-linear bijective map $C_i : W_i \to W_{\bar{i}}$ such that $\langle C_i w_1^{(i)}, w_2^{(i)}\rangle = \langle w_2^{(i)}|w_1^{(i)}\rangle$ for any $w_1^{(i)}, w_2^{(i)} \in W$. We simply write $C_i w^{(i)} = \overline{w^{(i)}}$. Now we may define the inner product on $W_{\bar{i}}$ such that C_i becomes antiunitary.

For any $v \in V$, we show that $Y_{\bar{i}}(v, x)$ satisfies equation (1.18). Note that for any $A \in \mathrm{End}(W_i)$, if A has a transpose $A^{\mathrm{t}} \in \mathrm{End}(W_{\bar{i}})$, then A also has a formal adjoint $A^\dagger \in \mathrm{End}(W)$, and it satisfies $A^\dagger = C_i^{-1} A^{\mathrm{t}} C_i$. Thus we have

$$\begin{aligned} &Y_{\bar{i}}(v, x) = Y_i(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})^{\mathrm{t}} \\ =&C_i Y_i(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})^\dagger C_i^{-1} = C_i Y_i(\theta v, x) C_i^{-1}, \end{aligned} \tag{1.19}$$

which implies that $Y_{\bar{i}}$ satisfies (1.18). □

From now on, if W_i is a unitary V-module, we fix an inner product on $W_{\bar{i}}$ to be the one constructed in the proof of proposition 1.8. We view $W_{\bar{i}}$ as a unitary V-module under this inner product.

Note that if we let $v = \nu$, then (1.19) implies that $L_n C_i = C_i L_n$ ($n \in \mathbb{Z}$).

Since we use W_0 (or simply 0) to denote the vacuum module V, it is natural to let C_0 represent the conjugation map from V onto its contragredient module $W_{\overline{0}} \equiv V'$. By equation (1.19) (with $i = 0$) and (1.8), we have:

Corollary 1.9. *$C_0\theta : V \to V'$ is a unitary V-module isomorphism.*

Therefore, we identify the vacuum module V with its contragredient module V'. This fact can be simply written as $\overline{0} = 0$. The operators θ and C_0 are also identified. The evaluation map $V \otimes V' \to \mathbb{C}$ is equivalent to the symmetric bilinear form $V \otimes V \to \mathbb{C}$ defined by $\langle v_1, v_2\rangle = \langle v_1|\theta v_2\rangle$, where $v_1, v_2 \in V$.

Recall that we also identify W_i with $W_{\overline{\overline{i}}}$. It is easy to see that the anti-unitary map $C_{\overline{i}} : W_{\overline{i}} \to W_i = W_{\overline{\overline{i}}}$ satisfies $C_{\overline{i}} = C_i^{-1}$.

We now give a criterion for unitary V-modules. First, we say that V is **generated by** a subset E if V is spanned by vectors of the form $Y(v_1, n_1) \cdots Y(v_k, n_m)\Omega$ where $v_1, v_2, \dots, v_m \in E$ and $n_1, \dots, n_m \in \mathbb{Z}$. By the Jacoby identity (1.13) (with $m = 0$), any vertex operator Y_i is determined by its values on E.

Now we have a useful criterion for unitarity of V-modules.

Proposition 1.10. *If V is unitary, W_i is a V-module equipped with an inner product $\langle \cdot | \cdot \rangle$, E is a generating subset of V, and equation* (1.18) *holds under the inner product $\langle \cdot | \cdot \rangle$ for any $v \in E$, then W_i is a unitary V-module.*

Proof. For any $v \in V$ we define $\widetilde{Y}_i(v, x) = Y_i(e^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1})^\dagger$. As in the proof of proposition 1.8, we have $\widetilde{Y}_i(v, x) = C_i^{-1} Y_{\overline{i}}(\theta v, x) C_i$. It follows that $\widetilde{Y}_i$ satisfies the Jacobi identity. Since Y_i also satisfies the Jacobi identity, and since $Y_i(v, x) = \widetilde{Y}_i(v, x)$ for any $v \in E$, we must have $Y_i(v, x) = \widetilde{Y}_i(v, x)$ for all $v \in V$, which proves that W_i is unitary. $\square$

1.3 Intertwining operators of unitary VOAs

Definition 1.11. Let W_i, W_j, W_k be V-modules. A type $\binom{W_k}{W_i W_j}$ (or type $\binom{k}{i\ j}$) **intertwining operator** $\mathcal{Y}_\alpha$ is a linear map

$$W_i \to (\mathrm{Hom}(W_j, W_k))\{x\},$$
$$w^{(i)} \mapsto \mathcal{Y}_\alpha(w^{(i)}, x) = \sum_{s \in \mathbb{R}} \mathcal{Y}_\alpha(w^{(i)}, s) x^{-s-1}$$
$$(\text{where } \mathcal{Y}_\alpha(w^{(i)}, s) \in \mathrm{Hom}(W_j, W_k)),$$

such that for any $w^{(j)} \in W_j$, $\mathcal{Y}_\alpha(w^{(i)}, s) w^{(j)} = 0$ for s sufficiently large.

(b) (**Jacobi identity**) For any $u \in V, w^{(i)} \in W_i, m, n \in \mathbb{Z}, s \in \mathbb{R}$, we have

$$\sum_{l \in \mathbb{Z}_{\geqslant 0}} \binom{m}{l} \mathcal{Y}_\alpha\big(Y_i(u, n+l) w^{(i)}, m+s-l\big)$$
$$= \sum_{l \in \mathbb{Z}_{\geqslant 0}} (-1)^l \binom{n}{l} Y_k(u, m+n-l) \mathcal{Y}_\alpha(w^{(i)}, s+l)$$

$$- \sum_{l \in \mathbb{Z}_{\geqslant 0}} (-1)^{l+n} \binom{n}{l} \mathcal{Y}_\alpha(w^{(i)}, n+s-l) Y_j(u, m+l). \tag{1.20}$$

(c) (**Translation property**) $\frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)}, x) = \mathcal{Y}_\alpha(L_{-1}w^{(i)}, x)$.

Intertwining operators will be denoted by $\mathcal{Y}_\alpha, \mathcal{Y}_\beta, \mathcal{Y}_\gamma, \dots$, or just $\alpha, \beta, \gamma, \dots$.

Note that if we let $n = 0$ and $m = 0$ respectively, (1.20) becomes:

$$\sum_{l \geqslant 0} \binom{m}{l} \mathcal{Y}_\alpha(Y_i(u,l)w^{(i)}, m+s-l) = Y_k(u,m)\mathcal{Y}_\alpha(w^{(i)}, s) - \mathcal{Y}_\alpha(w^{(i)}, s)Y_j(u,m), \tag{1.21}$$

$$\begin{aligned} &\mathcal{Y}_\alpha(Y_i(u,n)w^{(i)}, s) \\ =& \sum_{l \geqslant 0} (-1)^l \binom{n}{l} Y_k(u, n-l)\mathcal{Y}_\alpha(w^{(i)}, s+l) - \sum_{l \geqslant 0} (-1)^{l+n} \binom{n}{l} \mathcal{Y}_\alpha(w^{(i)}, n+s-l)Y_j(u,l). \end{aligned} \tag{1.22}$$

In particular, if we let $u = \nu$ and $m = 0, 1$ respectively, the first equation implies that

$$[L_{-1}, \mathcal{Y}_\alpha(w^{(i)}, x)] = \mathcal{Y}_\alpha(L_{-1}w^{(i)}, x) = \frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)}, x); \tag{1.23}$$

$$[L_0, \mathcal{Y}_\alpha(w^{(i)}, x)] = \mathcal{Y}_\alpha(L_0 w^{(i)}, x) + \frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)}, x). \tag{1.24}$$

The second equation is equivalent to that

$$[L_0, \mathcal{Y}_\alpha(w^{(i)}, s)] = (-s-1+\Delta_{w^{(i)}})\mathcal{Y}_\alpha(w^{(i)}, s) \quad \text{if } w^{(i)} \text{ is homogeneous.} \tag{1.25}$$

Hence $\mathcal{Y}_\alpha(w^{(i)}, s)$ raises the energy by $-s-1+\Delta_{w^{(i)}}$. Equation (1.25) implies the relation

$$z^{L_0}\mathcal{Y}_\alpha(w^{(i)}, x)z^{-L_0} = \mathcal{Y}_\alpha(z^{L_0}w^{(i)}, zx) \tag{1.26}$$

(cf. [FHL93] section 5.4), where z is either a non-zero complex number, or a formal variable which commutes with and is independent of x. In the former case, we need to assign to z an argument, i.e., a real number $\arg z$ such that $z = |z|e^{i \arg z}$. Then, for any $s \in \mathbb{R}$, we let $z^s = |z|^s e^{is \arg z}$, i.e., we let the argument of z^s be $s \arg z$.

Convention 1.12. In this article, unless otherwise stated, we make the following assumptions:

(1) If $t \in \mathbb{R}$ then $\arg e^{it} = t$.

(2) If $z \in \mathbb{C}^\times$ with argument $\arg z$, then $\arg \overline{z} = -\arg z$. If $s \in \mathbb{R}$, then $\arg(z^s) = s \arg z$.

(3) If $z_1, z_2 \in \mathbb{C}^\times$ with arguments $\arg z_1$ and $\arg z_2$ respectively, then $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

Definition 1.13. Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}^\times$ be a continuous function. Suppose that $z_1, z_2 \in U$, and for any $t \in [0,1]$, $tz_1 + (1-t)z_2 \in U$. Then we say that the argument $\arg f(z_2)$ **is close to** $\arg f(z_1)$ **as** $z_2 \to z_1$, if there exists a (unique) continuous function $A : [0,1] \to \mathbb{R}$, such that $A(0) = \arg z_1, A(1) = \arg z_2$, and that for any $t \in [0,1]$, $A(t)$ is an argument of $f(tz_1 + (1-t)z_2)$.

Let $\mathcal{V}\binom{k}{i\,j}$ be the vector space of type $\binom{k}{i\,j}$ intertwining operators. If $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, we say that W_i, W_j and W_k are the **charge space**, the **source space**, and the **target space** of $\mathcal{Y}_\alpha$ respectively. We say that $\mathcal{Y}_\alpha$ is **irreducible** if W_i, W_j, W_k are irreducible V-modules. If $\mathcal{Y}_\alpha$ is irreducible, then by (1.25), it is easy to see that $\mathcal{Y}(w^{(i)}, s) = 0$ except possibly when $s \in \Delta_i + \Delta_j - \Delta_k + \mathbb{Z}$. If V is unitary, and W_i, W_j, W_k are unitary V-modules, then we say that $\mathcal{Y}_\alpha$ is **unitary**.

We have several ways to construct new intertwining operators from old ones. First, for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, we define its **contragredient intertwining operator** (cf. [FHL93]) $C\mathcal{Y}_\alpha \equiv \mathcal{Y}_{C\alpha} \in \mathcal{V}\binom{\overline{j}}{i\,\overline{k}}$ by letting

$$\mathcal{Y}_{C\alpha}(w^{(i)}, x) = \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})^{\mathrm{t}}, \quad w^{(i)} \in W_i. \tag{1.27}$$

In other words, if $w^{(j)} \in W_j$ and $w^{(\overline{k})} \in W_{\overline{k}}$, then

$$\langle \mathcal{Y}_{C\alpha}(w^{(i)}, x)w^{(\overline{k})}, w^{(j)} \rangle = \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})w^{(j)} \rangle. \tag{1.28}$$

We also define, for each $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, an intertwining operator $C^{-1}\mathcal{Y}_\alpha \equiv \mathcal{Y}_{C^{-1}\alpha} \in \mathcal{V}\binom{\overline{j}}{i\,\overline{k}}$ such that

$$\mathcal{Y}_{C^{-1}\alpha}(w^{(i)}, x) = \mathcal{Y}_\alpha(e^{xL_1}(e^{i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})^{\mathrm{t}}, \quad w^{(i)} \in W_i. \tag{1.29}$$

One can show that $C^{-1}C\alpha = CC^{-1}\alpha = \alpha$. (To prove this, we first show that $(xL_1)x_0^{L_0} =$

$x_0^{L_0}(xx_0L_1)$ by checking this relation on any homogeneous vector. We then show that

$$e^{xL_1}x_0^{L_0}=x_0^{L_0}e^{xx_0L_1}, \tag{1.30}$$

where x, x_0 are independent commuting formal variables. Finally, we may use (1.30) to prove the desired relation.)

We now define, for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, a pair of **braided intertwining operators** (cf. [FHL93]) $B_\pm\mathcal{Y}_\alpha \equiv \mathcal{Y}_{B_\pm\alpha} \in \mathcal{V}\binom{k}{j\ i}$ in the following way: If $w^{(i)} \in W_i, w^{(j)} \in W_j$, then

$$\mathcal{Y}_{B_+\alpha}(w^{(j)},x)w^{(i)}=e^{xL_{-1}}\mathcal{Y}_\alpha(w^{(i)},e^{i\pi}x)w^{(j)}, \tag{1.31}$$

$$\mathcal{Y}_{B_-\alpha}(w^{(j)},x)w^{(i)}=e^{xL_{-1}}\mathcal{Y}_\alpha(w^{(i)},e^{-i\pi}x)w^{(j)}. \tag{1.32}$$

It's easy to see that $B_\mp$ is the inverse operation of $B_\pm$. We refer the reader to [FHL93] chapter 5 for a proof that contragredient intertwining operators and braided intertwining operators satisfy the Jacobi identity.

In the remaining part of this section, we assume that V is unitary. Let W_i, W_j, W_k be unitary V-modules with conjugation maps $C_i : W_i \to W_{\overline{i}}, C_j : W_j \to W_{\overline{j}}, C_k : W_k \to W_{\overline{k}}$ respectively. Given $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, we define its **conjugate intertwining operator** $\overline{\mathcal{Y}_\alpha} \equiv \mathcal{Y}_{\overline{\alpha}} \in \mathcal{V}\binom{\overline{k}}{\overline{i}\ \overline{j}}$ by setting

$$\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}},x)=C_k\mathcal{Y}_\alpha(w^{(i)},x)C_j^{-1},\quad w^{(i)}\in W_i. \tag{1.33}$$

It is clear that $\mathcal{Y}_{\overline{\alpha}}$ satisfies the Jacobi identity.

For any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, it is easy to check that

$$\mathcal{Y}_{\overline{B_\pm\alpha}}=\mathcal{Y}_{B_\mp\overline{\alpha}},\qquad \mathcal{Y}_{\overline{C^{\pm1}\alpha}}=\mathcal{Y}_{C^{\mp1}\overline{\alpha}}.$$

We define $\mathcal{Y}_\alpha^\dagger \equiv \mathcal{Y}_{\alpha^*}=\mathcal{Y}_{\overline{C\alpha}} \in \mathcal{V}\binom{j}{\overline{i}\ k}$ and call it the **adjoint intertwining operator** of $\mathcal{Y}_\alpha$. One can easily check, for any $w^{(i)} \in W_i$, that

$$\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},x)=\mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0}w^{(i)},x^{-1})^\dagger. \tag{1.34}$$

where the symbol † on the right hand side means the formal adjoint. In other words, for any $w^{(j)} \in W_j, j, w^{(k)} \in W_k$, we have

$$\langle \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, x)w^{(k)}|w^{(j)}\rangle = \langle w^{(k)}|\mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})w^{(j)}\rangle. \tag{1.35}$$

If $w^{(i)}$ is homogeneous, we can write (1.34) in terms of modes:

$$\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, s) = \sum_{m\in\mathbb{Z}_{\geqslant 0}} \frac{e^{i\pi\Delta_{w^{(i)}}}}{m!}\mathcal{Y}(L_1^m w^{(i)}, -s-m-2+2\Delta_{w^{(i)}})^\dagger \tag{1.36}$$

for all $s \in \mathbb{R}$.

It is also obvious that the adjoint operation is an involution, i.e., $\mathcal{Y}_{\alpha^{**}} = \mathcal{Y}_\alpha$. Hence $* : \mathcal{V}\binom{k}{i\ j} \to \mathcal{V}\binom{j}{\bar{i}\ k}$ is an antiunitary map.

We define the cardinal number N_{ij}^k to be the dimension of the vector space $\mathcal{V}\binom{k}{i\ j}$. N_{ij}^k is called a **fusion rule** of V. The above constructions of intertwining operators imply the following:

$$N_{ij}^k = N_{ik}^{\bar{j}} = N_{ji}^k = N_{\bar{i}\,\bar{j}}^{\bar{k}} = N_{\bar{i}k}^j. \tag{1.37}$$

We now construct several intertwining operators related to a given V-module W_i. First, note that $Y_i \in \mathcal{V}\binom{i}{0\ i}$. It is obvious that $B_+Y_i = B_-Y_i \in \mathcal{V}\binom{i}{i\ 0}$. We define $\mathcal{Y}_{i0}^i = B_\pm Y_i$ and call it the **creation operator** of W_i. Using the definition of $B_\pm$, we have, for any $w^{(i)} \in W_i, v \in V$,

$$\mathcal{Y}_{i0}^i(w^{(i)}, x)v = e^{xL_{-1}}Y_i(v, -x)w^{(i)}. \tag{1.38}$$

In particular, we have

$$\mathcal{Y}_{i0}^i(w^{(i)}, x)\Omega = e^{xL_{-1}}w^{(i)}. \tag{1.39}$$

We define $\mathcal{Y}_{i\bar{i}}^0 := C^{-1}\mathcal{Y}_{i0}^i = C^{-1}B_\pm Y_i \in \mathcal{V}\binom{0}{i\ \bar{i}}$. Thus for any $w_1^{(i)} \in W_i$ and $w_2^{(\bar{i})} \in W_{\bar{i}}$, we may use (1.39) and (1.30) to compute that

$$\langle \mathcal{Y}_{i\bar{i}}^0(w_1^{(i)}, x)w_2^{(\bar{i})}, \Omega\rangle = \langle w_2^{(\bar{i})}, \mathcal{Y}_{i0}^i(e^{xL_1}(e^{i\pi}x^{-2})^{L_0}w_1^{(i)}, x^{-1})\Omega\rangle$$

$$
\begin{aligned}
=&\langle w_2^{(\bar{i})}, e^{x^{-1}L_{-1}} e^{xL_1} (e^{i\pi} x^{-2})^{L_0} w_1^{(i)} \rangle \\
=&\langle e^{x^{-1}L_1} w_2^{(\bar{i})}, e^{xL_1} (e^{i\pi} x^{-2})^{L_0} w_1^{(i)} \rangle \\
=&\langle e^{x^{-1}L_1} w_2^{(\bar{i})}, (e^{i\pi} x^{-2})^{L_0} e^{-x^{-1}L_1} w_1^{(i)} \rangle. \qquad (1.40)
\end{aligned}
$$

Note that by (1.14), $Y_{\bar{i}} = C^{\pm 1} Y_i \in \mathcal{V}\binom{\bar{i}}{0\,\bar{i}}$. $\mathcal{Y}^0_{\bar{i}i} = C^{-1} B_{\pm} Y_{\bar{i}}$ is called the **annihilation operator** of W_i.

Define $\vartheta_i \in \mathrm{End}_V(W_i)$ by setting $\vartheta_i = e^{2i\pi L_0}$. That ϑ_i is a V-module homomorphism follows from (1.26). ϑ_i is called the **twist** of W_i. Then the intertwining operators $\mathcal{Y}^0_{i\bar{i}}$ and $\mathcal{Y}^0_{\bar{i}i}$ can be related in the following way:

Proposition 1.14.

$$
\mathcal{Y}^0_{i\bar{i}}(w^{(i)}, x) = (B_+ \mathcal{Y}^0_{\bar{i}i})(\vartheta_i w^{(i)}, x) = (B_- \mathcal{Y}^0_{\bar{i}i})(\vartheta_i^{-1} w^{(i)}, x), \qquad (1.41)
$$

$$
\mathcal{Y}^0_{i\bar{i}}(w^{(i)}, x) = (B_+ \mathcal{Y}^0_{\bar{i}i})(w^{(i)}, x)\vartheta_i = (B_- \mathcal{Y}^0_{\bar{i}i})(w^{(i)}, x)\vartheta_i^{-1}. \qquad (1.42)
$$

Proof. Using equations (1.30), (1.40), and that $L_1 \Omega = 0$, we see that for any $w_1^{(i)} \in W_i, w_2^{(\bar{i})} \in W_{\bar{i}}$,

$$
\begin{aligned}
&\langle (B_{\pm} \mathcal{Y}^0_{\bar{i}i})(\vartheta_i^{\pm 1} w_1^{(i)}, x) w_2^{(\bar{i})}, \Omega \rangle \\
=&\langle \mathcal{Y}^0_{\bar{i}i}(w_2^{(\bar{i})}, e^{\pm i\pi} x) e^{\pm 2i\pi L_0} w_1^{(i)}, \Omega \rangle \\
=&\langle e^{-x^{-1}L_1} e^{\pm 2i\pi L_0} w_1^{(i)}, (e^{i\pi \mp 2i\pi} x^{-2})^{L_0} e^{x^{-1}L_1} w_2^{(\bar{i})} \rangle \\
=&\langle e^{\pm 2i\pi L_0} e^{-x^{-1}L_1} w_1^{(i)}, (e^{i\pi \mp 2i\pi} x^{-2})^{L_0} e^{x^{-1}L_1} w_2^{(\bar{i})} \rangle \\
=&\langle (e^{i\pi} x^{-2})^{L_0} e^{-x^{-1}L_1} w_1^{(i)}, e^{x^{-1}L_1} w_2^{(\bar{i})} \rangle \\
=&\langle \mathcal{Y}^0_{i\bar{i}}(w_1^{(i)}, x) w_2^{(\bar{i})}, \Omega \rangle. \qquad (1.43)
\end{aligned}
$$

Since V is of CFT type and isomorphic to V' as a V-module, V is a simple VOA, i.e., V is an irreducible V-module (cf., for example, [CKLW15] proposition 4.6-(iv)). Hence Ω is a cyclic vector in V. By (1.21), we have $\langle (B_{\pm} \mathcal{Y}^0_{\bar{i}i})(\vartheta_i^{\pm 1} w_1^{(i)}, x) w_2^{(\bar{i})}, v \rangle = \langle \mathcal{Y}^0_{i\bar{i}}(w_1^{(i)}, x) w_2^{(\bar{i})}, v \rangle$ for any $v \in V$, which proves (1.41). (1.42) can be proved in a similar way. □

When W_i is unitary, we also have

$$\mathcal{Y}_{\bar{i}i}^{0} = (\mathcal{Y}_{i0}^{i})^{\dagger}. \tag{1.44}$$

Indeed, by (1.19), $Y_{\bar{i}} = \overline{Y_i}$. Hence

$$\mathcal{Y}_{\bar{i}i}^{0} = C^{-1}B_{\pm}\overline{Y_i} = \overline{CB_{\mp}Y_i} = (B_{\mp}Y_i)^{\dagger} = (\mathcal{Y}_{i0}^{i})^{\dagger}.$$

CHAPTER 2

BRAIDING AND FUSION OF INTERTWINING OPERATORS

Starting from this chapter, we assume that V satisfies conditions (0.1), (0.2), and (0.3). Recall that, by corollary 1.9, a unitary VOA automatically satisfies condition (0.1).

By [Hua05a] theorem 3.5, *the fusion rules of V are finite numbers, and there are only finitely many equivalence classes of irreducible V-modules.* Let us choose, for each equivalence class $[W_k]$ of irreducible V-module, a representing element W_k, and let these modules form a finite set $\{W_k : k \in \mathcal{E}\}$. (With abuse of notations, we also let $\mathcal{E}$ denote this finite set.) In other words, $\mathcal{E}$ is a complete list of mutually inequivalent irreducible V-modules. We also require that V is inside $\mathcal{E}$. If, moreover, V is unitary, then for any unitarizable W_k ($k \in \mathcal{E}$), we fix a unitary structure on W_k. The unitary structure on V is the standard one. We let $\mathcal{E}^{\mathrm{u}}$ be the set of all unitary V-modules in $\mathcal{E}$.

Let W_i, W_j, W_k be V-modules. Then Θ^k_{ij} will always denote (the index set of) a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta^k_{ij}\}$ of the vector space $\mathcal{V}\binom{k}{i\ j}$. If bases of the vector spaces of intertwining operators are chosen, then for any W_i, W_k, we set $\Theta^k_{i*} = \coprod_{j\in\mathcal{E}} \Theta^k_{ij}$. The notations $\Theta^k_{*j}, \Theta^*_{ij}$ are understood in a similar way.

2.1 Genus 0 correlation functions

In this section, we review the construction of genus 0 correlation functions from intertwining operators. We first give a complex analytic point of view of intertwining operators. Let $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$. For any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(\overline{k})} \in W_{\overline{k}}$,

$$\langle \mathcal{Y}_\alpha(w^{(i)}, z)w^{(j)}, w^{(\overline{k})}\rangle = \langle \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)}, w^{(\overline{k})}\rangle\big|_{x=z} = \sum_{s\in\mathbb{R}} \langle \mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}, w^{(\overline{k})}\rangle z^{-s-1} \tag{2.1}$$

is a finite sum of powers of z. (Indeed, if all the vectors are homogeneous then, by (1.25), the coefficient before each z^{-s-1} is zero, except when $s = \Delta_{w^{(i)}} + \Delta_{w^{(j)}} - \Delta_{w^{(k)}} - 1$.) Since the powers of z are not necessarily integers, (2.1) is a *multivalued holomorphic function* defined for $z \in \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$: the exact value of (2.1) depends not only on z, but also on $\arg z$. We can also regard $\mathcal{Y}_\alpha$ as a multivalued $(W_i \otimes W_j \otimes W_{\overline{k}})^*$-valued holomorphic function on $\mathbb{C}^\times$. Note that by proposition A.1, the transition from the formal series viewpoint to the complex analytic one is faithful.

Convention 2.1. At this point, the notations $\mathcal{Y}_\alpha(w^{(i)}, x)$, $\mathcal{Y}_\alpha(w^{(i)}, z)$, and $\mathcal{Y}_\alpha(w^{(i)}, s)$ seem confusing. We clarify their meanings as follows.

Unless otherwise stated, $\mathcal{Y}_\alpha(w^{(i)}, x)$ is a formal series of the formal variable x. If $z \neq 0$ is a complex number, or if z is a complex variable (possibly taking real values), $\mathcal{Y}_\alpha(w^{(i)}, z)$ is defined by (2.1). If s is a real number, $\mathcal{Y}_\alpha(w^{(i)}, s)$ is a mode of $\mathcal{Y}_\alpha(w^{(i)}, x)$, i.e., the coefficient before x^{-s-1} in $\mathcal{Y}_\alpha(w^{(i)}, x)$.

Intertwining operators are also called 3-point (correlation) functions. In [Hua05a], Y. Z. Huang constructed general n-point functions by taking the products of intertwining operators. His approach can be sketched as follows:

For any $n = 1, 2, 3, \ldots$, we define the **configuration space** $\mathrm{Conf}_n(\mathbb{C}^\times)$ to be the complex sub-manifold of $(\mathbb{C}^\times)^n$ whose points are $(z_1, z_2, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$ satisfying that $z_m \neq z_l$ whenever $1 \leqslant m < l \leqslant n$. We let $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$ be the universal covering space of $\mathrm{Conf}_n(\mathbb{C}^\times)$.

Let $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}, \ldots, \mathcal{Y}_{\alpha_n}$ be intertwining operators V. We say that they form a **chain of intertwining operators**, if for each $1 \leqslant m \leqslant n-1$, the target space of $\mathcal{Y}_{\alpha_m}$ equals the source space of $\mathcal{Y}_{\alpha_{m+1}}$. The following theorem was proved by Huang.

Theorem 2.2 (cf. [Hua05a] theorem 3.5)**.** *Suppose that $\mathcal{Y}_{\alpha_1}, \ldots, \mathcal{Y}_{\alpha_n}$ form a chain of intertwining operators. For each $1 \leqslant m \leqslant n$, we let W_{i_m} be the charge space of $\mathcal{Y}_{\alpha_m}$. We let W_{i_0} be the source space of $\mathcal{Y}_{\alpha_1}$, and let W_k be the target space of $\mathcal{Y}_{\alpha_n}$. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, and $z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that $0 < |z_1| < |z_2| < \cdots < |z_n|$, the expression*

$$\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1}) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) w^{(i_0)}, w^{(\overline{k})} \rangle \tag{2.2}$$

***converges absolutely**, which means that the series*

$$\sum_{s_1, s_2, \ldots, s_{n-1} \in \mathbb{R}} \big| \langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n) P_{s_{n-1}} \mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1}) P_{s_{n-2}}$$

$$\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1)w^{(i_0)},w^{(\overline{k})}\rangle\big| \tag{2.3}$$

converges, where each P_{s_m} ($1\leqslant m\leqslant n-1$) is the projection of the target space of $\mathcal{Y}_{\alpha_m}$ onto its weight-s_m component.

Note that (2.2) also **converges absolutely and locally uniformly**, which means that there exists a neighborhood $U\subset \mathrm{Conf}_n(\mathbb{C}^\times)$ of $(z_1,z_2,\dots,z_n)$, and a finite number $M>0$, such that for any $(\zeta_1,\zeta_2,\dots,\zeta_n)\in U$, (2.3) is bounded by M if we replace each $z_1,z_2,\dots$ with $\zeta_1,\zeta_2,\dots$ in that expression.

To see this, we assume, without loss of generality, that all the vectors in (2.2) are homogeneous, and that all the intertwining operators are irreducible. Consider a new set of coordinates $\omega_1,\omega_2,\dots,\omega_n$ such that $z_m=\omega_m\omega_{m+1}\cdots\omega_n$ ($1\leqslant m\leqslant n$). Then the condition that $0<|z_1|<|z_2|<\cdots<|z_n|$ is equivalent to that $0<|\omega_1|<1,\dots,0<|\omega_{n-1}|<1, 0<|\omega_n|$. By (1.26), expression (2.2) as a formal series also equals

$$\begin{aligned}
&\big\langle\mathcal{Y}_{\alpha_n}\big(w^{(i_n)},\omega_n\big)\mathcal{Y}_{\alpha_{n-1}}\big(w^{(i_{n-1})},\omega_{n-1}\omega_n\big)\cdots\mathcal{Y}_{\alpha_1}\big(w^{(i_1)},\omega_1\omega_2\cdots\omega_n\big)w^{(i_0)},w^{(\overline{k})}\big\rangle\\
=&\big\langle\omega_n^{L_0}\mathcal{Y}_{\alpha_n}\big(\omega_n^{-L_0}w^{(i_n)},1\big)\omega_{n-1}^{L_0}\mathcal{Y}_{\alpha_{n-1}}\big((\omega_{n-1}\omega_n)^{-L_0}w^{(i_{n-1})},1\big)\cdots\\
&\cdot\omega_1^{L_0}\mathcal{Y}_{\alpha_1}\big((\omega_1\omega_2\cdots\omega_n)^{-L_0}w^{(i_1)},1\big)(\omega_1\omega_2\cdots\omega_n)^{-L_0}w^{(i_0)},w^{(\overline{k})}\big\rangle\\
=&\big\langle\omega_n^{L_0}\mathcal{Y}_{\alpha_n}\big(w^{(i_n)},1\big)\omega_{n-1}^{L_0}\mathcal{Y}_{\alpha_{n-1}}\big(w^{(i_{n-1})},1\big)\cdots\omega_1^{L_0}\mathcal{Y}_{\alpha_1}\big(w^{(i_1)},1\big)w^{(i_0)},w^{(\overline{k})}\big\rangle\\
&\cdot\prod_{1\leqslant m\leqslant n}\omega_m^{-\left(\Delta_{w^{(i_0)}}+\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_m)}}\right)},
\end{aligned}\tag{2.4}$$

where $\mathcal{Y}_{\alpha_m}(w^{(i_m)},1)=\mathcal{Y}_{\alpha_m}(w^{(i_m)},x)\big|_{x=1}$. Since the target space of each $\mathcal{Y}_{\alpha_m}$ is irreducible, (2.4) is a **quasi power series of** $\omega_1,\dots,\omega_n$ (i.e., a power series of $\omega_1,\dots,\omega_n$ multiplied by a monomial $\omega_1^{s_1}\cdots\omega_n^{s_n}$, where $s_1,\dots,s_n\in\mathbb{C}$), and the convergence of (2.3) is equivalent to the absolute convergence of the quasi power series (2.4). Therefore, pointwise absolute convergence implies locally uniform absolute convergence.

We see that (2.2) is a multi-valued holomorphic function defined when $0<|z_1|<\cdots<|z_n|$. We let φ be the multi-valued $(W_{i_0}\otimes W_{i_1}\otimes\cdots\otimes W_{i_n}\otimes W_{\overline{k}})^*$-valued holomorphic function on

$\{0 < |z_1| < \cdots < |z_n|\}$ defined by (2.2). φ is called an $(n+2)$**-point (correlation) function**[1] of V, and is denoted by $\mathcal{Y}_{\alpha_n}\mathcal{Y}_{\alpha_{n-1}}\cdots\mathcal{Y}_{\alpha_1}$. We define $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ to be the vector space of $(W_{i_0}\otimes W_{i_1}\otimes\cdots\otimes W_{i_n}\otimes W_{\overline{k}})^*$-valued $n+2$-point functions of V. The following proposition can be used to find a basis of $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$.

Proposition 2.3. *Define a linear map* Φ :

$$\bigoplus_{j_1,\dots,j_{n-1}\in\mathcal{E}}\left(\mathcal{V}\binom{k}{i_n\ j_{n-1}}\otimes\mathcal{V}\binom{j_{n-1}}{i_{n-1}\ j_{n-2}}\otimes\mathcal{V}\binom{j_{n-2}}{i_{n-2}\ j_{n-3}}\otimes\cdots\otimes\mathcal{V}\binom{j_1}{i_1\ i_0}\right)$$
$$\to\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0},$$
$$\mathcal{Y}_{\alpha_n}\otimes\mathcal{Y}_{\alpha_{n-1}}\otimes\mathcal{Y}_{\alpha_{n-2}}\otimes\cdots\otimes\mathcal{Y}_{\alpha_1}\mapsto\mathcal{Y}_{\alpha_n}\mathcal{Y}_{\alpha_{n-1}}\mathcal{Y}_{\alpha_{n-2}}\cdots\mathcal{Y}_{\alpha_1}.$$

Then Φ *is an isomorphism.*

Therefore, if elements in $\{\mathcal{Y}_{\alpha_1}\},\dots,\{\mathcal{Y}_{\alpha_n}\}$ are linearly independent respectively, then the correlation functions $\{\mathcal{Y}_{\alpha_n}\mathcal{Y}_{\alpha_{n-1}}\cdots\mathcal{Y}_{\alpha_1}\}$ are also linearly independent. The proof of this proposition is postponed to section A.2.

It was also shown in [Hua05a] that correlations functions satisfy a system of linear differential equations, the coefficients of which are holomorphic functions defined on $\mathrm{Conf}_n(\mathbb{C}^\times)$. More precisely, we have the following:

Theorem 2.4 (cf. [Hua05a], especially theorem 1.6)**.** *For any* $w^{(i_0)}\in W_{i_0}, w^{(i_1)}\in W_{i_1},\dots,w^{(i_n)}\in W_{i_n}, w^{(\overline{k})}\in W_{\overline{k}}$*, there exist* $h_1,\dots,h_n\in\mathbb{Z}_{\geqslant 0}$*, and single-valued holomorphic functions* $a_{1,m}(z_1,\dots,z_n), a_{2,m}(z_1,\dots,z_n),\dots,a_{h_m,m}(z_1,\dots,z_n)$ *on* $\mathrm{Conf}_n(\mathbb{C}^\times)$*, such that for any* $(W_{i_0}\otimes W_{i_1}\otimes\cdots\otimes W_{i_n}\otimes W_{\overline{k}})^*$*-valued* $(n+2)$*-point correlation function* φ *defined on* $\{0<|z_1|<\cdots<|z_n|\}$*, the function* $\varphi(w^{(i_0)},w^{(i_1)},\dots,w^{(i_n)},w^{(\overline{k})};z_1,z_2,\dots,z_n)$ *of* $(z_1,\dots,z_n)$ *satisfies the following system of differential equations:*

$$\frac{\partial^{h_m}\varphi}{\partial z_m^{h_m}}+a_{1,m}\frac{\partial^{h_m-1}\varphi}{\partial z_m^{h_m-1}}+a_{2,m}\frac{\partial^{h_m-2}\varphi}{\partial z_m^{h_m-2}}+\cdots+a_{h_m,m}\varphi=0\qquad(m=1,\dots,n).\tag{2.5}$$

[1]So far our definition of genus 0 correlation functions is local. We will give a global definition at the end of next section.

Hence, due to elementary ODE theory, φ can be analytically continued to a multivalued holomorphic function on $\mathrm{Conf}_n(\mathbb{C}^\times)$ (or equivalently, a single-valued holomorphic function on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$), which satisfies system (2.5) globally.

Note that *(global) correlation functions are determined by their values at any fixed point in* $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$. Indeed, since φ satisfies (2.5), the function φ is determined by the values of $\{\frac{\partial^l}{\partial z_m^l}\varphi : 1 \leqslant m \leqslant n, 0 \leqslant l \leqslant h_m - 1\}$ at any fixed point. On the other hand, by translation property and the locally uniform absolute convergence of (2.2), we have

$$\begin{aligned}&\frac{\partial}{\partial z_m}\varphi(w^{(i_0)}, w^{(i_1)}, \ldots, w^{(i_n)}, w^{(\overline{k})}; z_1, z_2, \ldots, z_n)\\ =&\varphi(w^{(i_0)}, w^{(i_1)}, \ldots, L_{-1}w^{(i_m)}, \ldots, w^{(i_n)}, w^{(\overline{k})}; z_1, z_2, \ldots, z_n).\end{aligned} \tag{2.6}$$

Hence φ is determined by its value at a point.

2.2 General braiding and fusion relations for intertwining operators

The braid and the fusion relations for two intertwining operators were proved by Huang and Lepowsky in [HL95a, HL95b, HL95c, Hua95, Hua05a]. In this section, we generalize these relations to more than two intertwining operators. We also prove some useful convergence theorems. The proofs are technical, so we leave them to section A.3.

General fusion relations and convergence properties

Theorem 2.5 (Fusion of a chain of intertwining operators)**.** *Let $\mathcal{Y}_{\sigma_2}, \mathcal{Y}_{\sigma_3}, \ldots, \mathcal{Y}_{\sigma_n}$ be a chain of intertwining operators of V with charge spaces $W_{i_2}, W_{i_3}, \ldots, W_{i_n}$ respectively. Let $\mathcal{Y}_\gamma$ be another intertwining operator of V, whose charge space is the same as the target space of $\mathcal{Y}_{\sigma_n}$. Let W_{i_0} be the source space of $\mathcal{Y}_\gamma$, W_{i_1} be the source space of $\mathcal{Y}_{\sigma_2}$, and W_k be the target space of $\mathcal{Y}_\gamma$. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, and any $(z_1, z_2, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$ satisfying*

$$0 < |z_2 - z_1| < |z_3 - z_1| < \cdots < |z_n - z_1| < |z_1|, \tag{2.7}$$

the expression

$$\begin{aligned}&\big\langle \mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\\ &\qquad \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}, w^{(\overline{k})}\big\rangle\end{aligned} \tag{2.8}$$

***converges absolutely and locally uniformly**, which means that there exists a neighborhood* $U \subset \mathrm{Conf}_n(\mathbb{C}^\times)$ *of* $(z_1, z_2, \dots, z_n)$*, and a finite number* $M > 0$*, such that for any* $(z_1, z_2, \dots, z_n) \in U$*,*

$$\begin{aligned}\sum_{s_2,\dots,s_n \in \mathbb{R}} &\big|\langle \mathcal{Y}_\gamma(P_{s_n}\mathcal{Y}_{\sigma_n}(w^{(i_n)}, \zeta_n - \zeta_1)P_{s_{n-1}}\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, \zeta_{n-1} - \zeta_1)\\ &\qquad \cdots P_{s_2}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, \zeta_2 - \zeta_1)w^{(i_1)}, \zeta_1)w^{(i_0)}, w^{(\overline{k})}\rangle\big| < M.\end{aligned} \tag{2.9}$$

Moreover, if $(z_1, z_2, \dots, z_n)$ *satisfies* (2.7) *and*

$$0 < |z_1| < |z_2| < \cdots < |z_n|, \tag{2.10}$$

then (2.8) *as a* $(W_{i_0} \otimes W_{i_1} \otimes \cdots \otimes W_{i_n} \otimes W_{\overline{k}})^*$*-valued holomorphic function defined near* $(z_1, \dots, z_n)$ *is an element in* $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$*, and any element in* $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ *defined near* $(z_1, \dots, z_n)$ *can be written as* (2.8)*.*

The following convergence theorem for products of generalized intertwining operators is necessary for our theory. (See the discussion in the introduction.)

Theorem 2.6. *Let* m *be a positive integer. For each* $a = 1, \dots, m$*, we choose a positive integer* n_a*. Let* $W_{i^1}, \dots, W_{i^m}$ *be* V*-modules, and let* $\mathcal{Y}_{\alpha^1}, \dots, \mathcal{Y}_{\alpha^m}$ *be a chain of intertwining operators with charge spaces* $W_{i^1}, \dots, W_{i^m}$ *respectively. We let* W_i *be the source space of* $\mathcal{Y}_{\alpha^1}$*, and let* W_k *be the target space of* $\mathcal{Y}_{\alpha^m}$*. For each* $a = 1, \dots, m$ *we choose a chain of intertwining operators* $\mathcal{Y}_{\alpha_2^a}, \dots, \mathcal{Y}_{\alpha_{n_a}^a}$ *with charge spaces* $W_{i_2^a}, \dots, W_{i_{n_a}^a}$ *respectively. We let* $W_{i_1^a}$ *be the source space of* $\mathcal{Y}_{\alpha_2^a}$*, and assume that the target space of* $\mathcal{Y}_{\alpha_{n_a}^a}$ *is* W_{i^a}*.*

For any $a = 1, \dots, m$ *and* $b = 1, \dots, n_a$*, we choose a non-zero complex number* z_b^a*. Choose* $w_b^a \in W_{i_b^a}$*. We also choose vectors* $w^i \in W_i, w^{\overline{k}} \in W_{\overline{k}}$*. Suppose that the complex numbers* $\{z_b^a\}$ *satisfy the following conditions:*

(1) For each $a = 1, \dots, m$*,* $0 < |z_2^a - z_1^a| < |z_3^a - z_1^a| < \cdots < |z_{n_a}^a - z_1^a| < |z_1^a|$*;*

(2) For each $a = 1, \dots, m-1$, $|z_1^a| + |z_{n_a}^a - z_1^a| < |z_1^{a+1}| - |z_{n_{a+1}}^{a+1} - z_1^{a+1}|$,

then the expression

$$
\begin{aligned}
&\Big\langle \Big[\prod_{m \geqslant a \geqslant 1} \mathcal{Y}_{\alpha^a} \Big(\Big(\prod_{n_a \geqslant b \geqslant 2} \mathcal{Y}_{\alpha_b^a}(w_b^a, z_b^a - z_1^a) \Big) w_1^a, z_1^a \Big) \Big] w^i, w^{\overline{k}} \Big\rangle \\
\equiv & \big\langle \mathcal{Y}_{\alpha^m} \big(\mathcal{Y}_{\alpha_{n_m}^m}(w_{n_m}^m, z_{n_m}^m - z_1^m) \cdots \mathcal{Y}_{\alpha_2^m}(w_2^m, z_2^m - z_1^m) w_1^m, z_1^m \big) \\
& \vdots \\
& \cdot \mathcal{Y}_{\alpha^1} \big(\mathcal{Y}_{\alpha_{n_1}^1}(w_{n_1}^1, z_{n_1}^1 - z_1^1) \cdots \mathcal{Y}_{\alpha_2^1}(w_2^1, z_2^1 - z_1^1) w_1^1, z_1^1 \big) w^i, w^{\overline{k}} \big\rangle
\end{aligned}
\tag{2.11}
$$

converges absolutely and locally uniformly*, i.e., there exists* $M > 0$ *and a neighborhood* U_b^a *of each* z_b^a*, such that for any* $\zeta_b^a \in U_b^a$ $(1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a)$ *we have:*

$$
\sum_{s_1^a, s_b^a \in \mathbb{R}} \Big| \Big\langle \Big[\prod_{m \geqslant a \geqslant 1} P_{s_1^a} \mathcal{Y}_{\alpha^a} \Big(\Big(\prod_{n_a \geqslant b \geqslant 2} P_{s_b^a} \mathcal{Y}_{\alpha_b^a}(w_b^a, \zeta_b^a - \zeta_1^a) \Big) w_1^a, \zeta_1^a \Big) \Big] w^i, w^{\overline{k}} \Big\rangle \Big| < M. \tag{2.12}
$$

Assume, moreover, that $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ *satisfies the following condition:*

(3) For any $1 \leqslant a, a' \leqslant m, 1 \leqslant b \leqslant n_a, 1 \leqslant b' \leqslant n_{a'}$*, the inequality* $0 < |z_b^a| < |z_{b'}^{a'}|$ *holds when* $a < a'$*, or* $a = a'$ *and* $b < b'$*.*

Then (2.11) *defined near* $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ *is an element in* $\mathcal{V}\binom{k}{i_{n_m}^m \cdots i_1^m \cdots\cdots i_{n_1}^1 \cdots i_1^1 \; i}$.

We need another type of convergence property. The notion of absolute and locally uniform convergence is understood as usual.

Corollary 2.7. *Let* $\mathcal{Y}_{\sigma_2}, \mathcal{Y}_{\sigma_3}, \dots, \mathcal{Y}_{\sigma_m}$ *be a chain of intertwining operators of* V *with charge spaces* $W_{i_2}, W_{i_3}, \dots, W_{i_m}$ *respectively. Let* W_{i_1} *be the source space of* $\mathcal{Y}_{\sigma_2}$ *and* W_i *be the target space of* $\mathcal{Y}_{\sigma_m}$*. Similarly we let* $\mathcal{Y}_{\rho_2}, \mathcal{Y}_{\rho_3}, \dots, \mathcal{Y}_{\rho_m}$ *be a chain of intertwining operators of* V *with charge spaces* $W_{j_2}, W_{j_3}, \dots, W_{j_n}$ *respectively. Let* W_{j_1} *be the source space of* $\mathcal{Y}_{\rho_2}$ *and* W_j *be the target space of* $\mathcal{Y}_{\rho_n}$*. Moreover we choose* V*-modules* $W_{k_1}, W_{k_2}, W_{k_3}$*, a type* $\binom{k_1}{i\ j}$ *intertwining operator* $\mathcal{Y}_\alpha$ *and a type* $\binom{k_2}{k_1\ k_0}$ *intertwining operator* $\mathcal{Y}_\beta$*. Choose* $w^{(i_1)} \in W_{i_1}, w^{(i_2)} \in W_{i_2}, \dots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, w^{(j_2)} \in W_{j_2}, \dots, w^{(j_m)} \in W_{j_m}, w^{(k_0)} \in W_{i_0}, w^{(\overline{k_2})} \in W_{\overline{k_2}}$*. Then for any non-zero complex numbers* $z_1, z_2, \dots, z_m, \zeta_1, \zeta_2, \dots, \zeta_n$*, satisfying* $0 < |\zeta_2 - \zeta_1| < |\zeta_3 - \zeta_1| < \cdots < |\zeta_n - \zeta_1| < |z_1 - \zeta_1| - |z_m - z_1|$ *and* $0 < |z_2 - z_1| < |z_3 - z_1| < \cdots < |z_m - z_1| < |z_1 - \zeta_1| <$

$|\zeta_1| - |z_m - z_1|$, *the expression*

$$\Big\langle \mathcal{Y}_\beta\Big(\mathcal{Y}_\alpha\Big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1 - \zeta_1\Big) \cdot \mathcal{Y}_{\rho_n}(w^{(j_n)}, \zeta_n - \zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \zeta_2 - \zeta_1)w^{(j_1)}, \zeta_1\Big)w^{(k_0)}, w^{(\overline{k_2})}\Big\rangle \tag{2.13}$$

exists and converges absolutely and locally uniformly.

General braid relations

Let $z_1, z_2, \ldots, z_n$ be distinct complex values in $\mathbb{C}^\times$. Assume that $0 < |z_1| = |z_2| = \cdots = |z_n|$, and choose arguments $\arg z_1, \arg z_2, \ldots, \arg z_n$. We define the expression

$$\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1)w^{(i_0)}, w^{(\overline{k})}\rangle \tag{2.14}$$

in the following way: Choose $0 < r_1 < r_2 < \cdots < r_n$. Then the expression

$$\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, r_n z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, r_{n-1}z_{n-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, r_1 z_1)w^{(i_0)}, w^{(\overline{k})}\rangle \tag{2.15}$$

converges absolutely. We define (2.14) to be the limit of (2.15) as $r_1, r_2, \ldots, r_n \to 1$. The existence of this limit is guaranteed by theorem 2.4.

Let S_n be the symmetric group of degree n, and choose any $\varsigma \in S_n$. The general braid relations can be stated in the following way:

Theorem 2.8 (Braiding of intertwining operators)**.** *Choose distinct $z_1, \ldots, z_n \in \mathbb{C}^\times$ satisfying $0 < |z_1| = \cdots = |z_n|$. Let $\mathcal{Y}_{\alpha_{\varsigma(1)}}, \mathcal{Y}_{\alpha_{\varsigma(2)}}, \ldots, \mathcal{Y}_{\alpha_{\varsigma(n)}}$ be a chain of intertwining operators of V. For each $1 \leqslant m \leqslant n$, we let W_{i_m} be the charge space of $\mathcal{Y}_{\alpha_m}$. Let W_{i_0} be the source space of $\mathcal{Y}_{\alpha_{\varsigma(1)}}$, and let W_k be the target space of $\mathcal{Y}_{\alpha_{\varsigma(n)}}$. Then there exists a chain of intertwining operators $\mathcal{Y}_{\beta_1}, \mathcal{Y}_{\beta_2}, \ldots, \mathcal{Y}_{\beta_n}$ with charge spaces $W_{i_1}, W_{i_2}, \ldots, W_{i_n}$ respectively, such that the source space of $\mathcal{Y}_{\beta_1}$ is W_{i_0}, that the target space of $\mathcal{Y}_{\beta_n}$ is W_k, and that for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, the following braid relation holds:*

$$\langle \mathcal{Y}_{\alpha_{\varsigma(n)}}(w^{(i_{\varsigma(n)})}, z_{\varsigma(n)})\cdots\mathcal{Y}_{\alpha_{\varsigma(1)}}(w^{(i_{\varsigma(1)})}, z_{\varsigma(1)})w^{(i_0)}, w^{(\overline{k})}\rangle$$

$$=\langle \mathcal{Y}_{\beta_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\beta_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\beta_1}(w^{(i_1)}, z_1)w^{(i_0)}, w^{(\overline{k})}\rangle. \tag{2.16}$$

We usually omit the vectors $w^{(i_0)}, w^{(\overline{k})}$, and write the above equation as

$$\mathcal{Y}_{\alpha_{\varsigma(n)}}(w^{(i_{\varsigma(n)})}, z_{\varsigma(n)})\cdots\mathcal{Y}_{\alpha_{\varsigma(1)}}(w^{(i_{\varsigma(1)})}, z_{\varsigma(1)}) = \mathcal{Y}_{\beta_n}(w^{(i_n)}, z_n)\cdots\mathcal{Y}_{\beta_1}(w^{(i_1)}, z_1). \tag{2.17}$$

When $n = 2$, the proof of braid relations is based on the following well-known property. For the reader's convenience, we include a proof in section A.3.

Proposition 2.9. *Let $\mathcal{Y}_\gamma, \mathcal{Y}_\delta$ be intertwining operators of V, and assume $\mathcal{Y}_\gamma \in \mathcal{V}\binom{k}{i\ j}$. Choose $z_i, z_j \in \mathbb{C}^\times$ satisfying $0 < |z_j - z_i| < |z_i|, |z_j|$. Choose $\arg(z_j - z_i)$, and let $\arg z_j$ be close to $\arg z_i$ as $z_j \to z_i$. Then for any $w^{(i)} \in W_i, w^{(j)} \in W_j$,*

$$\mathcal{Y}_\delta\big(\mathcal{Y}_{B_\pm\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i\big) = \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}(z_j - z_i))w^{(j)}, z_j\big). \tag{2.18}$$

Remark 2.10. The braid relation (2.17) is unchanged if we scale the norm of the complex variables $z_1, z_2, \ldots, z_n$, or rotate each variable without meeting the others, and change its $\arg$ value continuously. The braid relation might change, however, if $z_1, z_2, \ldots, z_n$ are fixed, but their arguments are changed by 2π multiplied by some integers.

The proof of theorem 2.8 (see section A.3) implies the following:

Proposition 2.11. *Let $\mathcal{Y}_{\gamma_1}, \ldots, \mathcal{Y}_{\gamma_m}, \mathcal{Y}_{\alpha_{\varsigma(1)}}, \ldots, \mathcal{Y}_{\alpha_{\varsigma(n)}}, \mathcal{Y}_{\delta_1}, \ldots, \mathcal{Y}_{\delta_l}$ be a chain of intertwining operator of V with charge spaces $W_{i'_1}, \ldots, W_{i'_m}, W_{i_{\varsigma(1)}}, \ldots, W_{i_{\varsigma(n)}}, W_{i''_1}, \ldots, W_{i''_l}$ respectively. Let W_{j_1} be the source space of $\mathcal{Y}_{\gamma_1}$ and W_{j_2} be the target space of $\mathcal{Y}_{\delta_l}$. Let $z_1, \ldots, z_n, z'_1, \ldots, z'_m, z''_1, \ldots, z''_l$ be distinct complex numbers in S^1 with fixed arguments. Choose vectors $w^{(j_1)} \in W_{j_1}, w^{(i'_1)} \in W_{i'_1}, \ldots, w^{(i'_m)} \in W_{i'_m}, w^{(i''_1)} \in W_{i''_1}, \ldots, w^{(i''_l)} \in W_{i''_l}, w^{(\overline{j_2})} \in W_{\overline{j_2}}$. Let*

$$\mathcal{X}_1 = \mathcal{Y}_{\gamma_m}(w^{(i'_m)}, z'_m)\cdots\mathcal{Y}_{\gamma_1}(w^{(i'_1)}, z'_1),$$
$$\mathcal{X}_2 = \mathcal{Y}_{\delta_l}(w^{(i''_l)}, z''_l)\cdots\mathcal{Y}_{\delta_1}(w^{(i''_1)}, z''_1).$$

Suppose that the braid relation (2.16) holds for all $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in$

$W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$. *Then we also have the braid relation*

$$\begin{aligned}&\langle \mathcal{X}_2 \mathcal{Y}_{\alpha_{\varsigma(n)}}(w^{(i_{\varsigma(n)})}, z_{\varsigma(n)}) \cdots \mathcal{Y}_{\alpha_{\varsigma(1)}}(w^{(i_{\varsigma(1)})}, z_{\varsigma(1)}) \mathcal{X}_1 w^{(j_1)}, w^{(\overline{j_2})} \rangle \\ =&\langle \mathcal{X}_2 \mathcal{Y}_{\beta_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\beta_1}(w^{(i_1)}, z_1) \mathcal{X}_1 w^{(j_1)}, w^{(\overline{j_2})} \rangle. \end{aligned} \tag{2.19}$$

The braiding operators $B_\pm$ and the braid relations of intertwining operators are related in the following way:

Proposition 2.12. *Let $z_i, z_j \in S^1$ and $\arg z_j < \arg z_i < \arg z_j + \pi/3$. Let $\arg(z_i - z_j)$ be close to $\arg z_i$ as $z_j \to 0$, and let $\arg(z_j - z_i)$ be close to $\arg z_j$ as $z_i \to 0$.*

Let $\mathcal{Y}_\beta, \mathcal{Y}_\alpha$ be a chain of intertwining operators with charge spaces W_j, W_i respectively, and let $\mathcal{Y}_{\alpha'}, \mathcal{Y}_{\beta'}$ be a chain of intertwining operators with charge spaces W_i, W_j respectively. Assume that the source spaces of $\mathcal{Y}_\beta$ and $\mathcal{Y}_{\alpha'}$ are W_{k_1}, and that the target spaces of $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\beta'}$ are W_{k_2}.

If there exist a V-module W_k, and $\mathcal{Y}_\gamma \in \binom{k}{i\ j}, \mathcal{Y}_\delta \in \binom{k_2}{k\ k_1}$, such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have the fusion relations:

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \mathcal{Y}_\delta(\mathcal{Y}_\gamma(w^{(i)}, z_i - z_j)w^{(j)}, z_j), \tag{2.20}$$

$$\mathcal{Y}_{\beta'}(w^{(j)}, z_j)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i) = \mathcal{Y}_\delta(\mathcal{Y}_{B_+\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i). \tag{2.21}$$

Then the following braid relation holds:

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \mathcal{Y}_{\beta'}(w^{(j)}, z_j)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i). \tag{2.22}$$

Proof. Clearly we have $\arg(z_i - z_j) = \arg(z_j - z_i) + \pi$. So equation (2.22) follows directly from proposition 2.9. □

Using braid relations, we can give a global description of correlation functions. Consider the covering map $\pi_n : \widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times) \to \mathrm{Conf}_n(\mathbb{C}^\times)$. Choose $\varsigma \in S_n$, let $U_\varsigma = \{(z_1, \dots, z_n) : 0 < |z_{\varsigma(1)}| < |z_{\varsigma(2)}| < \cdots < |z_{\varsigma(n)}|\}$, and choose a connected component $\widetilde{U}_\varsigma$ of $\pi_n^{-1}(U_\varsigma)$. Then a $(W_{i_0} \otimes W_{i_{\varsigma(1)}} \otimes \cdots \otimes W_{i_{\varsigma(n)}} \otimes W_{\overline{k}})^*$-valued correlation function defined when $(z_{\varsigma(1)}, \dots, z_{\varsigma(n)}) \in U_\varsigma$ by the left hand side of equation (2.16) can be lifted through $\pi_n : \widetilde{U}_\varsigma \to U_\varsigma$ and analytically continued to a (single-valued) holomorphic function φ on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$. We define the vector space

$\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ of $(W_{i_0} \otimes W_{i_1} \otimes \cdots \otimes W_{i_n} \otimes W_{\overline{k}})^*$**-valued (genus** 0**) correlation function** to be the vector space of holomorphic functions on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$ of the form φ. This definition does not depend on the choice of ς and $\widetilde{U}_\varsigma$: If $\varsigma' \in S_n$ and $\widetilde{U}'_{\varsigma'}$ is a connected component of $\pi_n^{-1}(U_{\varsigma'})$, then by theorem 2.8, for any $\varphi \in \mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ defined on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$, it is not hard to find a $(W_{i_0} \otimes W_{i_{\varsigma'(1)}} \otimes \cdots \otimes W_{i_{\varsigma'(n)}} \otimes W_{\overline{k}})^*$-valued correlation function defined when $(z_{\varsigma'(1)}, \ldots, z_{\varsigma'(n)}) \in U_{\varsigma'}$ which can be lifted through $\pi_n : \widetilde{U}'_{\varsigma'} \to U_{\varsigma'}$ and analytically continued to the function φ.

2.3 Braiding and fusion with vertex operators and creation operators

In this section, we prove some useful braid and fusion relations. These relations are not only important for constructing a braided tensor category of representations of V, but also necessary for studying generalized intertwining operators.

Braiding and fusion with vertex operators

The Jacobi identity (1.20) can be interpreted in terms of braid and fusion relations:

Proposition 2.13. *Let $\mathcal{Y}_\alpha$ be a type $\binom{k}{i\ j}$ intertwining operator of V. Choose $z, \zeta \in \mathbb{C}^\times$ satisfying $0 < |z - \zeta| < |z| = |\zeta|$. Choose an argument $\arg z$. Then for any $u \in V, w^{(i)} \in W_i$, we have*

$$Y_k(u, \zeta)\mathcal{Y}_\alpha(w^{(i)}, z) = \mathcal{Y}_\alpha(w^{(i)}, z)Y_j(u, \zeta) = \mathcal{Y}_\alpha\big(Y_i(u, \zeta - z)w^{(i)}, z\big). \tag{2.23}$$

Proof. The above braid and fusion relations are equivalent to the following statement: for any $w^{(j)} \in W_j, w^{(\overline{k})} \in W_{\overline{k}}$, and for any $z \in \mathbb{C}^\times$, the functions of ζ:

$$\langle \mathcal{Y}_\alpha(w^{(i)}, z)Y_j(u, \zeta)w^{(j)}, w^{(\overline{k})}\rangle, \tag{2.24}$$

$$\langle \mathcal{Y}_\alpha\big(Y_i(u, \zeta - z)w^{(i)}, z\big)w^{(j)}, w^{(\overline{k})}\rangle, \tag{2.25}$$

$$\langle Y_k(u, \zeta)\mathcal{Y}_\alpha(w^{(i)}, z)w^{(j)}, w^{(\overline{k})}\rangle \tag{2.26}$$

defined respectively near 0, near z, and near ∞ can be analytically continued to the same (single-valued) holomorphic function on $\mathbb{P}^1$. This is equivalent to that for any $f(\zeta, z) \in \mathbb{C}[\zeta^{\pm 1}, (\zeta - z)^{-1}]$,

$$\mathrm{Res}_{\zeta=0}\big(\langle \mathcal{Y}_\alpha(w^{(i)}, z)Y_j(u, \zeta)w^{(j)}, w^{(\overline{k})}\rangle \cdot f(\zeta, z)d\zeta\big)$$

$$+\mathrm{Res}_{\zeta=z}\big(\langle \mathcal{Y}_\alpha\big(Y_i(u,\zeta-z)w^{(i)},z\big)w^{(j)},w^{(\overline{k})}\rangle\cdot f(\zeta,z)d\zeta\big)$$

$$+\mathrm{Res}_{\zeta=\infty}\big(\langle Y_k(u,\zeta)\mathcal{Y}_\alpha(w^{(i)},z)w^{(j)},w^{(\overline{k})}\rangle\cdot f(\zeta,z)d\zeta\big)=0 \tag{2.27}$$

(cf. [Ueno08] theorem 1.22, or [Muk10] theorem 1). It is not hard to compute that equation (2.27) is equivalent to the Jacobi identity (1.20). □

The above intertwining property can be generalized to any correlation function.

Proposition 2.14. [2] *Let $z_0 = 0$, choose $(z_1, z_2, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$, and choose a correlation function $\varphi \in \mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_1\ i_0}$ defined near $(z_1, z_2, \ldots, z_n)$. Then for any $u \in V, w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, and any $0 \leqslant m \leqslant n$, the following formal series in $\mathbb{C}((\zeta - z_m))$:*

$$\begin{aligned}&\psi_{i_m}(\zeta, z_1, z_2, \ldots, z_n)\\ =&\varphi(w^{(i_0)}, \ldots, w^{(i_{m-1})}, Y_{i_m}(u, \zeta - z_m)w^{(i_m)}, w^{(i_{m+1})}, \ldots, w^{(i_n)}, w^{(\overline{k})}; z_1, z_2, \ldots, z_n),\end{aligned} \tag{2.28}$$

and the following formal series in $\mathbb{C}((\zeta^{-1}))$:

$$\begin{aligned}&\psi_k(\zeta, z_1, z_2, \ldots, z_n)\\ =&\varphi(w^{(i_0)}, w^{(i_1)}, w^{(i_2)}, \ldots, w^{(i_n)}, Y_k(u,\zeta)^{\mathrm{t}}w^{(\overline{k})}; z_1, z_2, \ldots, z_n)\end{aligned} \tag{2.29}$$

are expansions of the same (single-valued) holomorphic function on $\mathbb{P}\backslash\{z_0, z_1, z_2, \ldots, z_n, \infty\}$ near the poles $\zeta = z_m$ ($0 \leqslant m \leqslant n$) and $\zeta = \infty$ respectively.

Proof. When $0 < |z_1| < |z_2| < \cdots < |z_n|$, we can prove this property easily using proposition 2.11, proposition 2.13, and theorem 2.6. Note that this property is equivalent to that for any $f(\zeta, z_1, \ldots, z_n) \in \mathbb{C}[\zeta^{\pm 1}, (\zeta - z_1)^{-1}, \ldots, (\zeta - z_n)^{-1}]$,

$$\begin{aligned}&\sum_{0\leqslant m\leqslant n}\mathrm{Res}_{\zeta=z_m}\big(\psi_{i_m}(\zeta, z_1, \ldots, z_n)f(\zeta, z_1, \ldots, z_n)d\zeta\big)\\ =&-\mathrm{Res}_{\zeta=\infty}\big(\psi_k(\zeta, z_1, \ldots, z_n)f(\zeta, z_1, \ldots, z_n)d\zeta\big)\end{aligned} \tag{2.30}$$

[2] One can use proposition 2.14 and the translation property to define correlation functions (parallel sections of conformal blocks). cf. [FB04] chapter 10.

(cf. [Ueno08] theorem 1.22, or [Muk10] theorem 1). If we write down the above equations explicitly, we will find that condition (2.30) is equivalent to a set of linear equations of φ, the coefficients of which are $\mathrm{End}\big((W_{i_0}\otimes W_{i_1}\otimes\cdots\otimes W_{i_n}\otimes W_{\overline{k}})^*\big)$-valued single-valued holomorphic functions on $\mathrm{Conf}_n(\mathbb{C}^\times)$. Since φ satisfies these equations locally, it must satisfy them globally. Therefore φ satisfies the desired property at any point in $\mathrm{Conf}_n(\mathbb{C}^\times)$. $\square$

As an application of this intertwining property, we prove a very useful uniqueness property for correlation functions.

Corollary 2.15. *Fix* $(z_1,z_2,\dots,z_n)\in\mathrm{Conf}_n(\mathbb{C}^\times)$. *Let* $\varphi\in\mathcal{V}\binom{\overline{i_{n+1}}}{i_n\ i_{n-1}\ \cdots\ i_1\ i_0}$ *be a correlation function defined near* $(z_1,z_2,\dots,z_n)$. *Choose* $l\in\{0,1,2,\dots,n+1\}$. *For any* $m\in\{0,1,2,\dots,n+1\}$ *such that* $m\neq l$, *we assume that* W_{i_m} *is irreducible, and choose a nonzero vector* $w_0^{(i_m)}\in W_{i_m}$. *Suppose that for any* $w^{(i_l)}\in W_{i_l}$,

$$\varphi(w_0^{(i_0)},\dots,w_0^{(i_{l-1})},w^{(i_l)},w_0^{(i_{l+1})},\dots,w_0^{(i_{n+1})};z_1,z_2,\dots,z_n)=0,\tag{2.31}$$

then $\varphi=0$.

Proof. We assume that $l\leqslant n$. The case that $l=n+1$ can be proved in a similar way. Suppose that (2.31) holds. Then for any $u\in V$, the formal series in $\mathbb{C}((\zeta-z_l))$:

$$\varphi(w_0^{(i_0)},\dots,w_0^{(i_{l-1})},Y_{i_l}(u,\zeta-z_l)w^{(i_l)},w_0^{(i_{l+1})},\dots,w_0^{(i_{n+1})};z_1,z_2,\dots,z_n)\tag{2.32}$$

equals zero. By proposition 2.14, (2.32) is the expansion of a global holomorphic function (which must be zero) on $\mathbb{P}\backslash\{z_0,z_1,\dots,z_n,\infty\}$, and when ζ is near $z_0=0$, this function becomes

$$\varphi(Y_{i_0}(u,\zeta)w_0^{(i_0)},w_0^{(i_1)}\dots,w_0^{(i_{l-1})},w^{(i_l)},w_0^{(i_{l+1})},\dots,w_0^{(i_{n+1})};z_1,z_2,\dots,z_n),\tag{2.33}$$

which is zero. Therefore, for each mode $Y_{i_1}(u,s)$ ($s\in\mathbb{Z}$), we have

$$\varphi(Y_{i_0}(u,s)w_0^{(i_0)},w_0^{(i_1)}\dots,w_0^{(i_{l-1})},w^{(i_l)},w_0^{(i_{l+1})},\dots,w_0^{(i_{n+1})};z_1,z_2,\dots,z_n)=0.\tag{2.34}$$

Since W_{i_0} is irreducible, for any $w^{(i_0)} \in W_0$ we have

$$\varphi(w^{(i_0)}, w_0^{(i_1)} \ldots, w_0^{(i_{l-1})}, w^{(i_l)}, w_0^{(i_{l+1})}, \ldots, w_0^{(i_{n+1})}; z_1, z_2, \ldots, z_n) = 0. \tag{2.35}$$

If we repeat this argument several times, we see that for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_{n+1})} \in W_{i_{n+1}}$,

$$\varphi(w^{(i_0)}, w^{(i_1)}, \ldots, w^{(i_{n+1})}; z_1, \ldots, z_n) = 0. \tag{2.36}$$

Hence φ equals zero at $(z_1, \ldots, z_n)$. By theorem 2.4 and the translation property, the value of φ equals zero at any point. $\square$

Braiding and fusion with creation operators

Lemma 2.16. *Let $\mathcal{Y}_\alpha$ be a type $\binom{k}{i\,j}$ intertwining operator. Then for any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(\overline{k})} \in W_{\overline{k}}$, $z \in \mathbb{C}^\times$ and $z_0 \in \mathbb{C}$:*

(1) If $0 \leqslant |z_0| < |z|$, and $\arg(z - z_0)$ is close to $\arg z$ as $z_0 \to 0$, then

$$\sum_{s \in \mathbb{R}} \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, z) P_s e^{z_0 L_{-1}} w^{(j)} \rangle \tag{2.37}$$

converges absolutely and equals

$$\langle w^{(\overline{k})}, e^{z_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, z - z_0) w^{(j)} \rangle. \tag{2.38}$$

We simply write

$$e^{z_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, z - z_0) = \mathcal{Y}_\alpha(w^{(i)}, z) e^{z_0 L_{-1}}. \tag{2.39}$$

(2) If $0 \leqslant |z_0| < |z|^{-1}$ and $\arg(1 - z z_0)$ is close to $\arg 1 = 0$ as $z_0 \to 0$, then

$$\sum_{s \in \mathbb{R}} \langle w^{(\overline{k})}, e^{z_0 L_1} P_s \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)} \rangle \tag{2.40}$$

converges absolutely and equals

$$\big\langle w^{(\overline{k})}, \mathcal{Y}_\alpha\big(e^{z_0(1-zz_0)L_1}(1-zz_0)^{-2L_0}w^{(i)}, z/(1-zz_0)\big)e^{z_0L_1}w^{(j)}\big\rangle. \tag{2.41}$$

We simply write

$$e^{z_0L_1}\mathcal{Y}_\alpha(w^{(i)},z) = \mathcal{Y}_\alpha\big(e^{z_0(1-zz_0)L_1}(1-zz_0)^{-2L_0}w^{(i)}, z/(1-zz_0)\big)e^{z_0L_1}. \tag{2.42}$$

Proof. Assume without loss of generality that all the vectors are homogeneous.

(1) Let x, x_0, x_1 be commuting independent formal variables. Note first of all that (2.39) holds in the formal sense:

$$\langle w^{(\overline{k})}, e^{x_0L_{-1}}\mathcal{Y}_\alpha(w^{(i)}, x-x_0)w^{(j)}\rangle = \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)},x)e^{x_0L_{-1}}w^{(j)}\rangle, \tag{2.43}$$

where

$$\mathcal{Y}_\alpha(w^{(i)}, x-x_0) = \sum_{s\in\mathbb{R}}\sum_{r\in\mathbb{Z}_{\geqslant 0}}\mathcal{Y}_\alpha(w^{(i)},s)\binom{-s-1}{r}x^{-s-1-r}(-x_0)^r.$$

Equation (2.43) can be proved using the relation $[L_{-1},\mathcal{Y}_\alpha(w^{(i)},x)] = \frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)},x)$. (See [FHL93] section 5.4 for more details.) Write

$$\langle w^{(\overline{k})}, e^{x_0L_{-1}}\mathcal{Y}_\alpha(w^{(i)}, x_1)w^{(j)}\rangle = \sum_{m\in\mathbb{Z}_{\geqslant 0}} c_m x_0^m x_1^{d-m} \tag{2.44}$$

where $d\in\mathbb{R}$ and $c_m\in\mathbb{C}$. Clearly $c_m = 0$ for all but finitely many m. Then the left hand side of (2.43) equals

$$\sum_{m,l\in\mathbb{Z}_{\geqslant 0}} c_m x_0^m\cdot\binom{d-m}{l}x^{d-m-l}(-x_0)^l.$$

We now substitute z and z_0 for x and x_0 in equation (2.43). For any z_0 satisfying $0\leqslant|z_0|<|z|$, let $\arg(z-z_0)$ be close to $\arg z$ as $z_0\to 0$. Then

$$\begin{aligned}&\langle w^{(\overline{k})}, e^{z_0L_{-1}}\mathcal{Y}_\alpha(w^{(i)}, z-z_0)w^{(j)}\rangle\\ =&\langle w^{(\overline{k})}, e^{x_0L_{-1}}\mathcal{Y}_\alpha(w^{(i)}, x_1)w^{(j)}\rangle\big|_{x_0=z_0,x_1=z-z_0}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m \in \mathbb{Z}_{\geqslant 0}} c_m z_0^m (z - z_0)^{d-m} \\
&= \sum_{m, l \in \mathbb{Z}_{\geqslant 0}} c_m z_0^m \cdot \binom{d-m}{l} z^{d-m-l} (-z_0)^l,
\end{aligned} \tag{2.45}
$$

which converges absolutely and equals

$$
\begin{aligned}
&\langle w^{(\bar{k})}, \mathcal{Y}_\alpha(w^{(i)}, x) e^{x_0 L_{-1}} w^{(j)} \rangle \big|_{x=z, x_0=z_0} \\
=&\langle w^{(\bar{k})}, \mathcal{Y}_\alpha(w^{(i)}, z) e^{z_0 L_{-1}} w^{(j)} \rangle.
\end{aligned} \tag{2.46}
$$

This proves part (1).

(2) Since $\alpha = C^{-1} C\alpha$, we have

$$
\begin{aligned}
&\sum_{s \in \mathbb{R}} \langle w^{(\bar{k})}, e^{z_0 L_1} P_s \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)} \rangle \\
=&\sum_{s \in \mathbb{R}} \langle P_s e^{z_0 L_1} w^{(\bar{k})}, \mathcal{Y}_{C^{-1}C\alpha}(w^{(i)}, z) w^{(j)} \rangle \\
=&\sum_{s \in \mathbb{R}} \big\langle \mathcal{Y}_{C\alpha}\big(e^{zL_1}(e^{i\pi} z^{-2})^{L_0} w^{(i)}, z^{-1}\big) P_s e^{z_0 L_{-1}} w^{(\bar{k})}, w^{(j)} \big\rangle,
\end{aligned}
$$

which, according to part (1), converges absolutely and equals

$$
\big\langle e^{z_0 L_{-1}} \mathcal{Y}_{C\alpha}\big(e^{zL_1}(e^{i\pi} z^{-2})^{L_0} w^{(i)}, z^{-1} - z_0\big) w^{(\bar{k})}, w^{(j)} \big\rangle, \tag{2.47}
$$

where $\arg(z^{-1} - z_0)$ is close to $\arg(z^{-1}) = -\arg z$ as $z_0 \to 0$. This is equivalent to saying that $\arg(1 - zz_0)$ is close to 0 as $z_0 \to 0$.

By the definition of $C\alpha$, (2.47) equals

$$
\begin{aligned}
&\big\langle \mathcal{Y}_{C\alpha}\big(e^{zL_1}(e^{i\pi} z^{-2})^{L_0} w^{(i)}, z^{-1} - z_0\big) w^{(\bar{k})}, e^{z_0 L_1} w^{(j)} \big\rangle \\
=&\big\langle w^{(\bar{k})}, \mathcal{Y}_\alpha\big(e^{(z^{-1}-z_0)L_1}(e^{-i\pi}(z^{-1} - z_0)^{-2})^{L_0} \\
&\cdot e^{zL_1}(e^{i\pi} z^{-2})^{L_0} w^{(i)}, (z^{-1} - z_0)^{-1}\big) e^{z_0 L_1} w^{(j)} \big\rangle.
\end{aligned} \tag{2.48}
$$

Note that (1.30) also holds when $x \in \mathbb{C}, x_0 \in \mathbb{C}^\times$. Therefore, by applying relation (1.30), expression

(2.48) equals (2.41). This finishes the proof of part (2). $\square$

Proposition 2.17. *Let $z_1, \ldots, z_n \in \mathbb{C}^\times$ satisfy $|z_1| < |z_2| < \cdots < |z_n|$ and $|z_2 - z_1| < \cdots < |z_n - z_1| < |z_1|$. Choose arguments $\arg z_1, \arg z_2, \ldots, \arg z_n$. For each $2 \leqslant m \leqslant n$, we let $\arg(z_m - z_1)$ be close to $\arg z_m$ as $z_1 \to 0$. Let $\mathcal{Y}_{\sigma_2}, \ldots, \mathcal{Y}_{\sigma_n}$ be a chain of intertwining operators of V with charge spaces $W_{i_2}, \ldots, W_{i_n}$ respectively. Let W_{i_1} be the source space of $\mathcal{Y}_{\sigma_2}$, and let W_i be the target space of $\mathcal{Y}_{\sigma_n}$. Then for any $w^{(i_1)} \in W_{i_1}, w^{(i_2)} \in W_{i_2}, \ldots, w^{(i_n)} \in W_{i_n}$, we have the fusion relation*

$$\begin{aligned}
&\mathcal{Y}^i_{i0}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1\big) \\
=&\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2) \mathcal{Y}^{i_1}_{i_1 0}(w^{(i_1)}, z_1).
\end{aligned} \tag{2.49}$$

Proof. We assume that $z_1, z_2, \ldots, z_n$ are on the same ray emitting from the origin (e.g. on $\mathbb{R}_{>0}$). (We don't assume, however, that these complex values have the same argument.) Then for each $2 \leqslant m \leqslant n$, these complex numbers satisfy

$$|z_1| + |z_m - z_1| < |z_{m+1}|. \tag{2.50}$$

If (2.49) is proved at these points, then by theorem 2.4 and analytic continuation, (2.49) holds in general.

Choose any $w^{(\bar{i})} \in W_{\bar{i}}$. Using equations (1.39) and (2.39) several times, we have

$$\begin{aligned}
&\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3) \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2) \mathcal{Y}^{i_1}_{i_1 0}(w^{(i_1)}, z_1)\Omega, w^{(\bar{i})} \rangle \\
=&\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3) \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2) e^{z_1 L_{-1}} w^{(i_1)}, w^{(\bar{i})} \rangle \\
=&\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3) e^{z_1 L_{-1}} \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, w^{(\bar{i})} \rangle \\
=&\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) \cdots e^{z_1 L_{-1}} \mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1) \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, w^{(\bar{i})} \rangle \\
&\vdots \\
=&\langle e^{z_1 L_{-1}} \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1) \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, w^{(\bar{i})} \rangle \\
=&\langle \mathcal{Y}^i_{i0}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1\big)\Omega, w^{(\bar{i})} \rangle.
\end{aligned} \tag{2.51}$$

Note that in order to make the above argument valid, we have to check that the expression in each step converges absolutely. To see this, we choose any $m = 1, \ldots, n$, and let W_{j_m} be the target space of $\mathcal{Y}_{\sigma_m}$. Then

$$\begin{aligned}
&\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) \cdots e^{z_1 L_{-1}} \mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, w^{(\bar{i})} \rangle \\
= &\sum_{s_1, \ldots, s_{n-1} \in \mathbb{R}} \langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) P_{s_{n-1}} \cdots P_{s_1} e^{z_1 L_{-1}} P_{s_m} \mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1) P_{s_{m-1}} \\
&\qquad \cdots P_{s_2} \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, w^{(\bar{i})} \rangle \\
= &\sum_{s_1, \ldots, s_{n-1} \in \mathbb{R}} \Big\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n) P_{s_{n-1}} \cdots P_{s_1} \mathcal{Y}^{j_m}_{j_m 0} \big(P_{s_m} \mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1) P_{s_{m-1}} \\
&\qquad \cdots P_{s_2} \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1 \big) \Omega, w^{(\bar{i})} \Big\rangle,
\end{aligned} \tag{2.52}$$

which, by (2.50) and theorem 2.6, converges absolutely. Therefore, equation (2.49) holds when both sides act on the vacuum vector Ω. By (the proof of) corollary 2.15, equation (2.49) holds when acting on any vector $v \in V$. $\square$

Corollary 2.18. *Let* $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$. *Let* $z_i, z_j \in S^1$ *with arguments satisfying* $\arg z_j < \arg z_i < \arg z_j + 2\pi$. *Then for any* $w^{(i)} \in W_i$ *and* $w^{(j)} \in W_j$*, we have the braid relation*

$$\mathcal{Y}_\alpha(w^{(i)}, z_i) \mathcal{Y}^j_{j0}(w^{(j)}, z_j) = \mathcal{Y}_{B_+\alpha}(w^{(j)}, z_j) \mathcal{Y}^i_{i0}(w^{(i)}, z_i). \tag{2.53}$$

Proof. By analytic continuation, we may assume, without loss of generality, that $0 < |z_i - z_j| < 1$. Let $\arg(z_i - z_j)$ be close to $\arg z_i$ as $z_j \to 0$, and let $\arg(z_j - z_i)$ be close to $\arg z_j$ as $z_i \to 0$. Then by propositions 2.17 and 2.12,

$$\begin{aligned}
&\mathcal{Y}_\alpha(w^{(i)}, z_i) \mathcal{Y}^j_{j0}(w^{(j)}, z_j) \\
=&\mathcal{Y}^k_{k0}\big(\mathcal{Y}_\alpha(w^{(i)}, z_i - z_j) w^{(j)}, z_j\big) \\
=&\mathcal{Y}^k_{k0}\big(\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_j - z_i) w^{(i)}, z_i\big) \\
=&\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_j) \mathcal{Y}^i_{i0}(w^{(i)}, z_i).
\end{aligned}$$

$\square$

We refer the reader to [Tur16] for the general theory of ribbon categories and modular tensor categories. See also [BK01, EGNO04]. In this section, we review the construction of the ribbon category $\mathrm{Rep}(V)$ for V by Huang and Lepowspky. (cf. [HL94] and [Hua08b].) As an additive categoy, $\mathrm{Rep}(V)$ is the representation category of V: Objects of $\mathrm{Rep}(V)$ are V-modules, and the vector space of morphisms from W_i to W_j is $\mathrm{Hom}_V(W_i, W_j)$. We now equip with $\mathrm{Rep}(V)$ a structure of a ribbon category.

The **tensor product** of two V-modules W_i, W_j is defined to be

$$\begin{aligned} W_{ij} \equiv W_i \boxtimes W_j &= \bigoplus_{k\in\mathcal{E}} \mathcal{V}\binom{k}{i\ j}^* \otimes W_k, \\ Y_{ij}(v,x) &= \bigoplus_{k\in\mathcal{E}} \mathrm{id} \otimes Y_k(v,x) \qquad (v \in V), \end{aligned} \tag{2.54}$$

where $\mathcal{V}\binom{k}{i\ j}^*$ is the dual space of $\mathcal{V}\binom{k}{i\ j}$. (Recall our notations at the beginning of this chapter.) Thus for any $k \in \mathcal{E}$, we can define an isomorphism

$$\mathcal{V}\binom{k}{i\ j} \to \mathrm{Hom}_V(W_{ij}, W_k), \quad \mathcal{Y} \mapsto R_{\mathcal{Y}},$$

such that if $\check{\mathcal{Y}} \in \mathcal{V}\binom{k}{i\ j}^*$ and $w^{(k)} \in W_k$, then

$$R_{\mathcal{Y}}(\check{\mathcal{Y}} \otimes w^{(k)}) = \langle \check{\mathcal{Y}}, \mathcal{Y} \rangle w^{(k)}. \tag{2.55}$$

$R_{\mathcal{Y}}$ is called the homomorphism represented by $\mathcal{Y}$.

The tensor product of two morphisms are defined as follows: If $F \in \mathrm{Hom}_V(W_{i_1}, W_{i_2}), G \in \mathrm{Hom}_V(W_{j_1}, W_{j_2})$, then for each $k \in \mathcal{E}$ we have a linear map $(F \otimes G)^{\mathrm{t}} : \mathcal{V}\binom{k}{i_2\ j_2} \to \mathcal{V}\binom{k}{i_1\ j_1}$, such that if $\mathcal{Y} \in \mathcal{V}\binom{k}{i_2\ j_2}$, then $(F \otimes G)^{\mathrm{t}}\mathcal{Y} \in \mathcal{V}\binom{k}{i_1\ j_1}$, and for any $w^{(i_1)} \in W_{i_1}, w^{(j_1)} \in W_{j_1}$,

$$\big((F \otimes G)^{\mathrm{t}}\mathcal{Y}\big)(w^{(i_1)}, x)w^{(j_1)} = \mathcal{Y}(Fw^{(i_1)}, x)Gw^{(j_1)}. \tag{2.56}$$

Then $F \otimes G : \mathcal{V}\binom{k}{i_1\ j_1}^* \to \mathcal{V}\binom{k}{i_2\ j_2}^*$ is defined to be the transpose of $(F \otimes G)^{\mathrm{t}}$, and can be extended to a homomorphism

$$F\otimes G=\bigoplus_{k\in\mathcal{E}}(F\otimes G)\otimes\mathrm{id}_k:W_{i_1}\boxtimes W_{j_1}\to W_{i_2}\boxtimes W_{j_2}.$$

Hence we've define the tensor product $F\otimes G$ of F and G.

Let $W_0=V$ be the unit object of $\mathrm{Rep}(V)$. The functorial isomorphisms $\lambda_i:W_0\boxtimes W_i\to W_i$ and $\rho_j:W_i\boxtimes W_0\to W_i$ are defined as follows: If $i\in\mathcal{E}$, then λ_i is represented by the intertwining operator Y_i, and ρ_i is represented by $\mathcal{Y}^i_{i0}$. In general, λ_i (resp. ρ_i) is defined to be the unique isomorphism satisfying that for any $k\in\mathcal{E}$ and any $R\in\mathrm{Hom}_V(W_i,W_k)$, $R\lambda_i=\lambda_k(\mathrm{id}_0\otimes R)$ (resp. $R\rho_i=\rho_k(R\otimes\mathrm{id}_0)$).

We now define the associator. First of all, to simplify our notations, we assume the following:

Convention 2.19. Let $W_i,W_j,W_k,W_{i'},W_{j'},W_{k'}$ be V-modules. Let $\mathcal{Y}_\alpha\in\mathcal{V}\binom{k'}{i'\,j'}$. If either $W_i\neq W_{i'},W_j\neq W_{j'}$, or $W_k\neq W_{k'}$, then for any $w^{(i)}\in W_i,w^{(j)}\in W_j,w^{(\overline{k})}\in W_{\overline{k}},z\in\mathbb{C}^\times$, we let

$$\langle\mathcal{Y}_\alpha(w^{(i)},z)w^{(j)},w^{(\overline{k})}\rangle=0.$$

Therefore, $\mathcal{Y}_\beta(w^{(i)},z_2)\mathcal{Y}_\alpha(w^{(j)},z_1)=0$ if the target space of $\mathcal{Y}_\alpha$ does not equal the source space of $\mathcal{Y}_\beta$; $\mathcal{Y}_\gamma\big(\mathcal{Y}_\delta(w^{(i)},z_1-z_2)w^{(j)},z_2\big)=0$ if the target space of $\mathcal{Y}_\delta$ does not equal the charge space of $\mathcal{Y}_\gamma$.

Given three V-modules W_i,W_j,W_k, we have

$$(W_i\boxtimes W_j)\boxtimes W_k=\bigoplus_{s,t\in\mathcal{E}}\mathcal{V}\begin{pmatrix}t\\ s\,k\end{pmatrix}^*\otimes\mathcal{V}\begin{pmatrix}s\\ i\,j\end{pmatrix}^*\otimes W_t,\tag{2.57}$$

$$W_i\boxtimes (W_j\boxtimes W_k)=\bigoplus_{r,t\in\mathcal{E}}\mathcal{V}\begin{pmatrix}t\\ i\,r\end{pmatrix}^*\otimes\mathcal{V}\begin{pmatrix}r\\ j\,k\end{pmatrix}^*\otimes W_t.\tag{2.58}$$

Choose basis $\Theta^t_{sk},\Theta^s_{ij},\Theta^t_{ir},\Theta^r_{jk}$ of these spaces of intertwining operators. Choose $z_i,z_j\in\mathbb{C}^\times$ satisfying $0<|z_i-z_j|<|z_j|<|z_i|$. Choose $\arg z_i$. Let $\arg z_j$ be close to $\arg z_i$ as $z_i-z_j\to 0$, and let $\arg(z_i-z_j)$ be close to $\arg z_i$ as $z_j\to 0$. For any $t\in\mathcal{E},\alpha\in\Theta^t_{i*},\beta\in\Theta^*_{jk}$, there exist complex numbers $F^{\beta'\alpha'}_{\alpha\beta}$ independent of the choice of z_i,z_j, such that for any $w^{(i)}\in W_i,w^{(j)}\in W_j$, we have the fusion relation

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \sum_{\alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}} F^{\beta'\alpha'}_{\alpha\beta} \mathcal{Y}_{\beta'}(\mathcal{Y}_{\alpha'}(w^{(i)}, z_i - z_j)w^{(j)}, z_j). \tag{2.59}$$

If the source space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_\beta$, or if the target space of $\mathcal{Y}_{\alpha'}$ does not equal the charge space of $\mathcal{Y}_{\beta'}$, we set $F^{\beta'\alpha'}_{\alpha\beta} = 0$. Then, by the proof of proposition 2.3, the numbers $F^{\beta'\alpha'}_{\alpha\beta}$ are uniquely determined by the basis chosen. The matrix $\{F^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}}_{\alpha \in \Theta^t_{i*}, \beta \in \Theta^*_{jk}}$ is called a **fusion matrix**. Define an isomorphism

$$\begin{aligned} A^{\mathrm{t}} : \bigoplus_{r \in \mathcal{E}} \mathcal{V}\binom{t}{i\ r} \otimes \mathcal{V}\binom{r}{j\ k} &\to \bigoplus_{s \in \mathcal{E}} \mathcal{V}\binom{t}{s\ k} \otimes \mathcal{V}\binom{s}{i\ j}, \\ \mathcal{Y}_\alpha \otimes \mathcal{Y}_\beta &\mapsto \sum_{\alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}} F^{\beta'\alpha'}_{\alpha\beta} \mathcal{Y}_{\beta'} \otimes \mathcal{Y}_{\alpha'}. \end{aligned} \tag{2.60}$$

Clearly A^{t} is independent of the basis chosen. Define

$$A : \bigoplus_{s \in \mathcal{E}} \mathcal{V}\binom{t}{s\ k}^* \otimes \mathcal{V}\binom{s}{i\ j}^* \to \bigoplus_{r \in \mathcal{E}} \mathcal{V}\binom{t}{i\ r}^* \otimes \mathcal{V}\binom{r}{j\ k}^* \tag{2.61}$$

to be the transpose of A^{t}, and extend it to

$$A = \sum_{t \in \mathcal{E}} A \otimes \mathrm{id}_t : (W_i \boxtimes W_j) \boxtimes W_k \to W_i \boxtimes (W_j \boxtimes W_k), \tag{2.62}$$

which is an **associator** of $\mathrm{Rep}(V)$. One can prove the pentagon axiom using theorem 2.6 and corollary 2.7, and prove the triangle axiom using propositions 2.13 and 2.9.

Recall the linear isomorphisms

$$B_\pm : \mathcal{V}\binom{k}{j\ i} \to \mathcal{V}\binom{k}{i\ j}, \quad \mathcal{Y} \mapsto B_\pm \mathcal{Y}.$$

We let $\sigma_{i,j} : \mathcal{V}\binom{k}{i\ j}^* \to \mathcal{V}\binom{k}{j\ i}^*$ be the transpose of B_+ and extend it to a morphism

$$\sigma_{i,j} = \sum_{t \in \mathcal{E}} \sigma_{i,j} \otimes \mathrm{id}_t : W_i \boxtimes W_j \to W_j \boxtimes W_i. \tag{2.63}$$

This gives the **braid operator**. The hexagon axiom can be proved using propositions 2.11, 2.12, and theorem 2.6.

For each object i, the twist is just the operator $\vartheta_i = \vartheta_{W_i}$ defined in section 1.3.

With these structural maps, Huang proved in [Hua05b, Hua08a, Hua08b] that $\mathrm{Rep}(V)$ is rigid and in fact a modular tensor category. From his proof, it is clear that $\bar{i}$ is the right dual of i: there exist homomorphisms $\mathrm{coev}_i : V \to W_i \boxtimes W_{\bar{i}}$ and $\mathrm{ev}_i : W_{\bar{i}} \boxtimes W_i \to V$ satisfying

$$(\mathrm{id}_i \otimes \mathrm{ev}_i) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i) = \mathrm{id}_i, \tag{2.64}$$

$$(\mathrm{ev}_i \otimes \mathrm{id}_{\bar{i}}) \circ (\mathrm{id}_{\bar{i}} \otimes \mathrm{coev}_i) = \mathrm{id}_{\bar{i}}. \tag{2.65}$$

Since $i = \bar{\bar{i}}$, $\bar{i}$ is also the left dual of i.

Now assume that V is unitary. The additive category $\mathrm{Rep}^{\mathrm{u}}(V)$ is defined to be the representation category of unitary V-modules. We show that $\mathrm{Rep}^{\mathrm{u}}(V)$ is a C^*-category. First, we need the following easy consequence of Schur's lemma.

Lemma 2.20. *Choose for each $k \in \mathcal{E}^{\mathrm{u}}$ a number $n_k \in \mathbb{Z}_{\geqslant 0}$. Define the unitary V-module*

$$W = \bigoplus_{k \in \mathcal{E}^{\mathrm{u}}}^{\perp} W_k \otimes \mathbb{C}^{n_k} = \bigoplus_{k \in \mathcal{E}^{\mathrm{u}}}^{\perp} \underbrace{W_k \oplus^{\perp} W_k \oplus^{\perp} \cdots \oplus^{\perp} W_k}_{n_k}.$$

Then we have

$$\mathrm{End}_V(W) = \bigoplus_{k \in \mathcal{E}^{\mathrm{u}}} \mathrm{id}_k \otimes \mathrm{End}(\mathbb{C}^{n_k}). \tag{2.66}$$

Theorem 2.21. $\mathrm{Rep}^{\mathrm{u}}(V)$ *is a C^*-category, i.e.,* $\mathrm{Rep}^{\mathrm{u}}(V)$ *is equipped with an involutive antilinear contravariant endofunctor $*$ which is the identity on objects; The positivity condition is satisfied: If W_i, W_j are unitary and $F \in \mathrm{Hom}_V(W_i, W_j)$, then there exists $R \in \mathrm{End}_V(W_i)$ such that $F^*F = R^*R$; The hom-spaces* $\mathrm{Hom}_V(W_i, W_j)$ *are normed spaces and the norms satisfy*

$$\|GF\| \leqslant \|G\|\|F\|, \quad \|F^*F\| = \|F\|^2 \tag{2.67}$$

for all $F \in \mathrm{Hom}(i, j), G \in \mathrm{Hom}(j, k)$.

Proof. For any $F \in \mathrm{Hom}_V(W_i, W_j)$, we let F^* be the formal adjoint of F , i.e. the unique homomorphism $F^* \in \mathrm{Hom}_V(W_j, W_i)$ satisfying $\langle Fw^{(i)}|w^{(j)}\rangle = \langle w^{(i)}|F^*w^{(j)}\rangle$ for all $w^{(i)} \in W_i, w^{(j)} \in W_j$. The existence of F^* follows from lemma 2.20 applied to $W \simeq W_i \oplus^\perp W_j$. Let $\|F\|$ be the operator norm of F, i.e., $\|F\| = \sup_{w^{(i)}\in W_i\backslash\{0\}}(\|Fw^{(i)}\|/\|w^{(i)}\|)$. Using lemma 2.20, it is easy to check that $\mathrm{Rep}^{\mathrm{u}}(V)$ satisfies all the conditions to be a C^*-category. □

It is not clear whether unitarizable V-modules are closed under tensor product. So it may not be a good idea to define a structure of a ribbon category on $\mathrm{Rep}^{\mathrm{u}}(V)$. We consider instead certain subcategories. Let $\mathcal{G}$ be a collection of unitary V-modules. We say that $\mathcal{G}$ is **additively closed**, if the following conditions are satisfied:

(1) If $i \in \mathcal{G}$ and W_j is isomorphic to a submodule of W_i, then $j \in \mathcal{G}$.

(2) If $i_1, i_2, \ldots, i_n \in \mathcal{G}$, then $W_{i_1} \oplus^\perp W_{i_2} \oplus^\perp \cdots \oplus^\perp W_{i_n} \in \mathcal{G}$.

If $\mathcal{G}$ is additively closed, we define the additive category $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ to be the subcategory of $\mathrm{Rep}^{\mathrm{u}}(V)$ whose objects are elements in $\mathcal{G}$.

We say that $\mathcal{G}$ is **multiplicatively closed**, if $\mathcal{G}$ is additively closed, and the following conditions are satisfied:

(a) $0 \in \mathcal{G}$.

(b) If $i \in \mathcal{G}$, then $\bar{i} \in \mathcal{G}$.

(c) If $i, j \in \mathcal{G}$, then $W_{ij} = W_i \boxtimes W_j$ is unitarizable, and any unitarization of W_{ij} is inside $\mathcal{G}$.

Suppose that $\mathcal{G}$ is multiplicatively closed. A **unitary structure** on $\mathcal{G}$ assigns to each triplet $(i, j, k) \in \mathcal{G} \times \mathcal{G} \times \mathcal{E}$ an inner product on $\mathcal{V}{k \choose i\ j}^*$. For each unitary structure on $\mathcal{G}$, we define $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ to be a ribbon category in the following way: If $i, j \in \mathcal{G}$, then W_{ij} as a V-module is defined, as before, to be $\bigoplus_{k\in\mathcal{E}} \mathcal{V}{k \choose i\ j}^* \otimes W_k$. Since $\mathcal{G}$ is multiplicatively closed, each W_k in $\mathcal{E}$ satisfying $N^k_{ij} > 0$ must be equipped with a unitary structure. Hence the inner products on all $\mathcal{V}{k \choose i\ j}^*$'s give rise to a unitary structure on W_{ij}. W_{ij} now becomes a unitary V-module. The other functors and structural maps are defined in the same way as we did for $\mathrm{Rep}(V)$. Clearly $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ is a ribbon fusion category and is equivalent to a ribbon fusion subcategory of $\mathrm{Rep}(V)$.

Our main goal in this two-part series is to define a unitary structure on $\mathcal{G}$, under which $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ becomes a unitary ribbon fusion category. More explicitly, we want to show (cf. [Gal12]) that for any $i_1, i_2, j_1, j_2 \in \mathcal{G}$ and any $F \in \mathrm{Hom}_V(W_{i_1}, W_{i_2}), G \in \mathrm{Hom}_V(W_{j_1}, W_{j_2})$,

$$(F \otimes G)^* = F^* \otimes G^*; \tag{2.68}$$

that the associators, the operators λ_i, ρ_i ($i \in \mathcal{G}$), and the braid operators of $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ are unitary; and that for each $i \in \mathcal{G}$, ϑ_i is unitary, and ev_i and coev_i can be chosen in such a way that the following equations hold:

$$(\mathrm{coev}_i)^* = \mathrm{ev}_i \circ \sigma_{i,\bar{i}} \circ (\vartheta_i \otimes \mathrm{id}_{\bar{i}}), \tag{2.69}$$

$$(\mathrm{ev}_i)^* = (\mathrm{id}_{\bar{i}} \otimes \vartheta_i^{-1}) \circ \sigma_{\bar{i},i}^{-1} \circ \mathrm{coev}_i. \tag{2.70}$$

CHAPTER 3

ANALYTIC ASPECTS OF VERTEX OPERATOR ALGEBRAS

3.1 Intertwining operators with energy bounds

The energy bounds conditions for vertex operators are important when one tries to construct conformal nets/loop groups representations from unitary VOAs/infinite dimensional Lie algebras. This can be seen, for instance, in [GW84], [BS90], and [CKLW15]. In this section, we generalize this notion to intertwining operators of VOAs.

We assume in this chapter that V is unitary. If W_i is a unitary V-module, we let the Hilbert space $\mathcal{H}_i$ be the norm completion of W_i, and view W_i as a norm-dense subspace of $\mathcal{H}_i$. It is clear that the unbounded operator L_0 on $\mathcal{H}_i$ (with domain W_i) is essentially self-adjoint, and its closure $\overline{L_0}$ is positive.

Now for any $r \in \mathbb{R}$, we let $\mathcal{H}_i^r$ be the domain $\mathscr{D}((1+\overline{L_0})^r)$ of $(1+\overline{L_0})^r$. If $\xi \in \mathcal{H}_i^r$, we define the r-th order Sobolev norm to be

$$\|\xi\|_r = \|(1+\overline{L_0})^r\xi\|.$$

Note that the 0-th Sobolev norm is just the vector norm. We let

$$\mathcal{H}_i^\infty = \bigcap_{r \geqslant 0} \mathcal{H}_i^r.$$

Clearly $\mathcal{H}_i^\infty$ contains W_i. Vectors inside $\mathcal{H}_i^\infty$ are said to be **smooth**.

Definition 3.1. Let W_i, W_j, W_k be unitary V-modules, $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, and $w^{(i)} \in W_i$ be homogeneous. Choose $r \geqslant 0$. We say that $\mathcal{Y}_\alpha(w^{(i)}, x)$ satisfies r**-th order energy bounds**, if there exist $M \geqslant 0, t \geqslant 0$, such that for any $s \in \mathbb{R}$ and $w^{(j)} \in W_j$,

$$\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\| \leqslant M(1+|s|)^t \|w^{(j)}\|_r. \tag{3.1}$$

Here $\mathcal{Y}_\alpha(w^{(i)},s)$ is the s-th mode of the intertwining operator $\mathcal{Y}_\alpha(w^{(i)},x)$. It is clear that if $r_1 \leqslant r_2$ and $\mathcal{Y}_\alpha(w^{(i)},x)$ satisfies r_1-th order energy bounds, then $\mathcal{Y}_\alpha(w^{(i)},x)$ also satisfies r_2-th order energy bounds.

1-st order energy bounds are called **linear energy bounds**. We say that $\mathcal{Y}_\alpha(w^{(i)},x)$ is **energy-bounded** if it satisfies energy bounds of some positive order. If for every $w^{(i)} \in W_i$, $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded, we say that $\mathcal{Y}_\alpha$ is **energy-bounded**. A unitary V-module W_i is called **energy-bounded** if Y_i is energy-bounded. The unitary VOA V is called **energy-bounded** if the vacuum module $V = W_0$ is energy-bounded. We now prove some useful properties concerning energy-boundedness.

Proposition 3.2. *If $w^{(i)} \in W_i$ is homogeneous and inequality* (3.1) *holds, then for any $p \in \mathbb{R}$, there exists $M_p > 0$ such that for any $w^{(j)} \in W_j$,*

$$\|\mathcal{Y}_\alpha(w^{(i)},s)w^{(j)}\|_p \leqslant M_p(1+|s|)^{|p|+t}\|w^{(j)}\|_{p+r}. \tag{3.2}$$

Proof. (cf. [TL04] chapter II proposition 1.2.1) We want to show that

$$\|\mathcal{Y}_\alpha(w^{(i)},s)w^{(j)}\|_p^2 \leqslant M_p^2(1+|s|)^{2(|p|+t)}\|w^{(j)}\|_{p+r}^2. \tag{3.3}$$

Since

$$\|\mathcal{Y}_\alpha(w^{(i)},s)w^{(j)}\|_p^2 = \sum_{q\in\mathbb{R}}\|P_{q-s-1+\Delta_{w^{(i)}}}\mathcal{Y}_\alpha(w^{(i)},s)w^{(j)}\|_p^2 = \sum_{q\in\mathbb{R}}\|\mathcal{Y}_\alpha(w^{(i)},s)P_q w^{(j)}\|_p^2,$$
$$\|w^{(j)}\|_{p+r}^2 = \sum_{q\in\mathbb{R}}\|P_q w^{(j)}\|_{p+r}^2,$$

it suffices to assume that $w^{(j)}$ is homogeneous. We also assume that $\mathcal{Y}_\alpha(w^{(i)},s)w^{(j)} \neq 0$. Then by proposition 1.7, $\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - 1 - s \geqslant 0$.

By (3.1) we have

$$\|\mathcal{Y}_\alpha(w^{(i)},s)w^{(j)}\|^2 \leqslant M^2(1+|s|)^{2t}(1+\Delta_{w^{(j)}})^{2r}\|w^{(j)}\|^2. \tag{3.4}$$

Hence

$$
\begin{aligned}
&\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|_p^2\\
=&(\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - s)^{2p}\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|^2\\
\leqslant&(\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - s)^{2p}M^2(1+|s|)^{2t}(1+\Delta_{w^{(j)}})^{2r}\|w^{(j)}\|^2\\
=&M^2\Big(\frac{\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - s}{1+\Delta_{w^{(j)}}}\Big)^{2p}(1+|s|)^{2t}(1+\Delta_{w^{(j)}})^{2(p+r)}\|w^{(j)}\|^2\\
=&M^2\Big(\frac{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}(1+|s|)^{2t}\|w^{(j)}\|_{p+r}^2. \qquad (3.5)
\end{aligned}
$$

If $p \geqslant 0$ then

$$
\begin{aligned}
&\Big(\frac{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}\\
\leqslant&\Big(\frac{1+\Delta_{w^{(i)}} + |s| + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}\\
\leqslant&(1+\Delta_{w^{(i)}} + |s|)^{2p} \leqslant (1+\Delta_{w^{(i)}})^{2p}(1+|s|)^{2p}. \qquad (3.6)
\end{aligned}
$$

If $p < 0$ and $1 \leqslant \Delta_{w^{(i)}} - s$, then

$$
\begin{aligned}
&\Big(\frac{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}\\
=&\Big(\frac{1+\Delta_{w^{(j)}}}{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}\Big)^{2|p|} \leqslant 1. \qquad (3.7)
\end{aligned}
$$

If $p < 0$ and $1 \geqslant \Delta_{w^{(i)}} - s$, then since $\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}} \geqslant 1$,

$$
\begin{aligned}
&\Big(\frac{1+\Delta_{w^{(j)}}}{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}\Big)^{2|p|}\\
=&\Big(1+\frac{1+s-\Delta_{w^{(i)}}}{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}\Big)^{2|p|}\\
\leqslant&(2-\Delta_{w^{(i)}} + s)^{2|p|}\\
\leqslant&(2+2\Delta_{w^{(i)}} + 2|s|)^{2|p|}\\
\leqslant&2^{2|p|}(1+\Delta_{w^{(i)}})^{2|p|}(1+|s|)^{2|p|}. \qquad (3.8)
\end{aligned}
$$

Therefore, if we let $M_p = 2^{|p|}(1+\Delta_{w^{(i)}})^{|p|}$, then (3.3) is always true. $\square$

The next property is obvious.

Proposition 3.3. *If $\mathcal{Y} \in \mathcal{V}\binom{k}{i\ j}$ is unitary, $w^{(i)} \in W_i$ is homogeneous, and $\mathcal{Y}_\alpha(w^{(i)}, x)$ satisfies r-th order energy bounds, then $\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}}, x)$ satisfies r-th order energy bounds.*

Proposition 3.4. *Suppose that $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ is unitary, $w^{(i)} \in W_i$ is homogeneous, $r \geqslant 0$, and for any $m \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{Y}_\alpha(L_1^m w^{(i)}, x)$ satisfies r-th order energy bounds. Then $\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, x)$ and $\mathcal{Y}_{C^{\pm 1}\alpha}(w^{(i)}, x)$ satisfy r-th order energy bounds.*

Proof. First we note that $L_1^m w^{(i)} = 0$ for m large enough. Now suppose that (3.1) holds for all m if we replace $w^{(i)}$ by $L_1^m w^{(i)}$. Then by (1.36), for any $w^{(j)} \in W_j, w^{(k)} \in W_k$ and $s \in \mathbb{R}$,

$$\begin{aligned}
&\Big|\big\langle \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, s)w^{(j)}\big|w^{(k)}\big\rangle\Big| \\
\leqslant &\sum_{m\geqslant 0}\frac{1}{m!}\Big|\big\langle w^{(j)}\big|\mathcal{Y}_\alpha(L_1^m w^{(i)}, -s-k-2+2\Delta_{w^{(i)}})w^{(k)}\big\rangle\Big| \\
= &\sum_{m\geqslant 0}\frac{1}{m!}\Big|\big\langle (1+L_0)^r w^{(j)}\big|(1+L_0)^{-r}\mathcal{Y}_\alpha(L_1^m w^{(i)}, -s-k-2+2\Delta_{w^{(i)}})w^{(k)}\big\rangle\Big| \\
\leqslant &\sum_{m\geqslant 0}\frac{1}{m!}\big\|w^{(j)}\big\|_r\big\|\mathcal{Y}_\alpha(L_1^m w^{(i)}, -s-k-2+2\Delta_{w^{(i)}})w^{(k)}\big\|_{-r}.
\end{aligned}$$

By proposition 3.2, we can find positive numbers C_1, C_2 independent of $w^{(j)}, w^{(k)}$, such that

$$\begin{aligned}
&\big\|\mathcal{Y}_\alpha(L_1^m w^{(i)}, -s-m-2+2\Delta_{w^{(i)}})w^{(k)}\big\|_{-r} \\
\leqslant &C_1\big(1+|s+m+2-2\Delta_{w^{(i)}}|\big)^{r+t}\big\|w^{(k)}\big\| \\
\leqslant &C_2\big(1+|s|\big)^{r+t}\big\|w^{(k)}\big\|.
\end{aligned}$$

Thus there exists $C_3 > 0$ independent of $w^{(j)}, w^{(k)}$, such that

$$\Big|\big\langle \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, s)w^{(j)}\big|w^{(k)}\big\rangle\Big| \leqslant C_3\big(1+|s|\big)^{r+t}\big\|w^{(j)}\big\|_r\big\|w^{(k)}\big\|.$$

This proves that

$$\big\|\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, s)w^{(j)}\big\| \leqslant C_3\big(1+|s|\big)^{r+t}\big\|w^{(j)}\big\|_r. \tag{3.9}$$

Therefore $\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, x)$ satisfies r-th order energy bounds. Since $C\alpha = \overline{\alpha^*}$ and $\mathcal{Y}_{C^{-1}\alpha}(w^{(i)}, x) =$

$e^{2i\pi\Delta_{w^{(i)}}}\mathcal{Y}_{C\alpha}(w^{(i)},x)$, by proposition 3.3, $\mathcal{Y}_{C^{\pm1}\alpha}(w^{(i)},x)$ also satisfy r-th order energy bounds. $\square$

Proposition 3.5. *Let W_i, W_j, W_k be unitary V-modules, $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, and choose homogeneous vectors $w^{(i)} \in W_i, u \in V$. Suppose that $\mathcal{Y}_\alpha(w^{(i)},x), Y_j(u,x), Y_k(u,x)$ are energy-bounded. Then for any $n \in \mathbb{Z}$, $\mathcal{Y}_\alpha\big(Y_i(u,n)w^{(i)},x\big)$ is energy-bounded.*

Proof. By Jacobi Identity, for any $s \in \mathbb{R}$ we have

$$\begin{aligned}&\mathcal{Y}_\alpha(Y_i(u,n)w^{(i)},s)\\=&\sum_{l\in\mathbb{Z}_{\geqslant0}}(-1)^l\binom{n}{l}Y_k(u,n-l)\mathcal{Y}_\alpha(w^{(i)},s+l)-\sum_{l\in\mathbb{Z}_{\geqslant0}}(-1)^{l+n}\binom{n}{l}\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,l).\end{aligned} \tag{3.10}$$

It can be shown by induction on $|n|$ that

$$\limsup_{l\to\infty}\left|\binom{n}{l}\right|l^{-|n|}<+\infty.$$

Choose any homogeneous vector $w^{(j)} \in W_j$ with energy $\Delta_{w^{(j)}}$. Then by energy-boundedness of $\mathcal{Y}_\alpha(w^{(i)},x), Y_j(u,x), Y_k(u,x)$ and proposition 3.2, there exist positive constants $C_1, C_2, \dots, C_8$ and $r_1, t_1, r_2, t_2, r_3, t_3$ independent of $w^{(j)}$ and s, such that

$$\begin{aligned}&\Big\|\sum_{l\geqslant0}(-1)^{l+n}\binom{n}{l}\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,l)w^{(j)}\Big\|\\\leqslant&\sum_{l\geqslant0}C_1l^{|n|}\big\|\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,l)w^{(j)}\big\|\\\leqslant&\sum_{l\geqslant0}C_2l^{|n|}\big(1+|n+s-l|\big)^{t_1}\big\|Y_j(u,l)w^{(j)}\big\|_{r_1}\\\leqslant&\sum_{0\leqslant l\leqslant\Delta_u+\Delta_{w^{(j)}}-1}C_3l^{|n|}(1+|s|)^{t_1}(1+l)^{t_1}\cdot(1+l)^{r_1+t_2}\|w^{(j)}\|_{r_1+r_2}\\\leqslant&\sum_{0\leqslant l\leqslant\Delta_u+\Delta_{w^{(j)}}-1}C_3(1+|s|)^{t_1}(1+l)^{|n|+t_1+r_1+t_2}\|w^{(j)}\|_{r_1+r_2}\\\leqslant&C_4(1+|s|)^{t_1}(1+\Delta_{w^{(j)}})^{1+|n|+t_1+r_1+t_2}\|w^{(j)}\|_{r_1+r_2}\\=&C_4(1+|s|)^{t_1}\|w^{(j)}\|_{1+|n|+t_1+t_2+2r_1+r_2}.\end{aligned} \tag{3.11}$$

Here the inequality $l \leqslant \Delta_u + \Delta_{w^{(j)}} - 1$ comes from the fact that every nonzero $Y_j(u,l)w^{(j)}$ must have non-negative energy. Similarly we have

$$\begin{aligned}
&\Big\| \sum_{l\geqslant 0} (-1)^l \binom{n}{l} Y_k(u, n-l)\mathcal{Y}_\alpha(w^{(i)}, s+l)w^{(j)} \Big\| \\
\leqslant & \sum_{l\geqslant 0} C_5 l^{|n|} \big\| Y_k(u,n-l)\mathcal{Y}_\alpha(w^{(i)}, s+l)w^{(j)} \big\| \\
\leqslant & \sum_{l\geqslant 0} C_6 l^{|n|}(1+|n-l|)^{t_3} \big\| \mathcal{Y}_\alpha(w^{(i)}, s+l)w^{(j)} \big\|_{r_3} \\
\leqslant & \sum_{0\leqslant l\leqslant \Delta_{w^{(i)}}+\Delta_{w^{(j)}}-s-1} C_7 l^{|n|}(1+l)^{t_3}(1+|s+l|)^{r_3+t_2}\|w^{(j)}\|_{r_3+r_2} \\
\leqslant & \sum_{0\leqslant l\leqslant \Delta_{w^{(i)}}+\Delta_{w^{(j)}}-s-1} C_7 (1+|s|)^{r_3+t_2}(1+l)^{|n|+t_3+r_3+t_2}\|w^{(j)}\|_{r_3+r_2} \\
\leqslant & C_8(1+|s|)^{r_3+t_2}(1+\Delta_{w^{(j)}}+|s|)^{1+|n|+t_3+r_3+t_2}\|w^{(j)}\|_{r_3+r_2} \\
\leqslant & C_8(1+|s|)^{2r_3+2t_2+1+|n|+t_3}(1+\Delta_{w^{(j)}})^{1+|n|+t_3+r_3+t_2}\|w^{(j)}\|_{r_3+r_2} \\
= & C_8(1+|s|)^{2r_3+2t_2+1+|n|+t_3}\|w^{(j)}\|_{2r_3+r_2+t_2+1+|n|+t_3}.
\end{aligned} \tag{3.12}$$

The energy-boundedness of $\mathcal{Y}_\alpha(Y_i(u,n)w^{(i)}, x)$ follows from these two inequalities. □

The following proposition is also very useful. One can prove it using the argument in [BS90] section 2.

Proposition 3.6. *If $v = \nu$ or $v \in V(1)$, then for any unitary V-module W_i, $Y_i(v,x)$ satisfies linear energy bounds.*

We summarize the results in this section as follows:

Corollary 3.7. *Let W_i, W_j, W_k be unitary V-modules, and $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$.*
(a) Suppose that V is generated by a set E of homogeneous vectors. If for each $v \in E$, $Y_i(v,x)$ is energy-bounded, then Y_i is energy-bounded.
(b) If W_i is irreducible, Y_j, Y_k are energy-bounded, and there exists a nonzero homogeneous vector $w^{(i)} \in W_i$ such that $\mathcal{Y}_\alpha(w^{(i)}, x)$ is energy-bounded, then $\mathcal{Y}_\alpha$ is energy-bounded.
(c) If $w^{(i)} \in W_i$ is homogeneous, and $\mathcal{Y}_\alpha(w^{(i)}, x)$ is energy-bounded, then $\mathcal{Y}_{C^{\pm 1}\alpha}(w^{(i)}, x)$, $\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}}, x)$, and $\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, x)$ are energy-bounded.

(d) If $w^{(i)} \in W_i$ is quasi-primary, and $\mathcal{Y}_\alpha(w^{(i)},x)$ satisfies r-th order energy bounds. Then $\mathcal{Y}_{C^{\pm 1}\alpha}(w^{(i)},x)$, $\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}},x)$, and $\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},x)$ satisfy r-th order energy bounds.

Proof. (a) and (b) follow from proposition 3.5. (c) follows from propositions 3.3, 3.4, 3.5, and 3.6. (d) follows from propositions 3.3 and 3.4. □

3.2 Smeared intertwining operators

In this section, we construct smeared intertwining operators for energy-bounded intertwining operators, and prove the adjoint relation, the braid relations, the rotation covariance, and the intertwining property for these operators. The proof of the last property requires some knowledge of the strong commutativity of unbounded closed operators on a Hilbert space. We give a brief exposition of this theory in chapter B.

The unbounded operator $\mathcal{Y}_\alpha(w^{(i)},f)$

For any open subset I of S^1, we denote by $C_c^\infty(I)$ the set of all complex smooth functions on S^1 whose supports lie in I. If $I = \{e^{it} : a < t < b\}$ ($a,b \in \mathbb{R}, a < b < a+2\pi$), we say that I is an **open interval** of S^1. We let $\mathcal{J}$ be the set of all open intervals of S^1. In general, if U is an open subset of S^1, we let $\mathcal{J}(U)$ be the set of open intervals of S^1 contained in U. If $I \in \mathcal{J}$, then its **complement** I^c is defined to be $S^1 \backslash \overline{I}$. If $I_1, I_2 \in \mathcal{J}$, we write $I_1 \subset\subset I_2$ if $\overline{I_1} \subset I_2$.

Let $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ be unitary. (Recall that this means that W_i, W_j, W_k are unitary V-modules.) For any $w^{(i)} \in W_i, z \in \mathbb{C}^\times$, $\mathcal{Y}_\alpha(w^{(i)},z)$ is a linear map $W_j \to \widehat{W}_k$. Therefore we can regard $\mathcal{Y}_\alpha(w^{(i)},z)$ as a sesquilinear form $W_j \times W_k \to \mathbb{C}, (w^{(j)},w^{(k)}) \mapsto \langle \mathcal{Y}_\alpha(w^{(i)},z)w^{(j)}|w^{(k)}\rangle$.

We now define the smeared intertwining operators. Let $d\!\!\!/\theta = \frac{e^{i\theta}}{2\pi}d\theta$. For any $f \in C_c^\infty(S^1\backslash\{-1\})$, we define a sesquilinear form

$$\mathcal{Y}_\alpha(w^{(i)},f) : W_j \times W_k \to \mathbb{C}, \quad (w^{(j)},w^{(k)}) \mapsto \langle \mathcal{Y}_\alpha(w^{(i)},f)w^{(j)}|w^{(k)}\rangle$$

satisfying

$$\langle \mathcal{Y}_\alpha(w^{(i)},f)w^{(j)}|w^{(k)}\rangle = \int_{-\pi}^{\pi} \langle \mathcal{Y}_\alpha(w^{(i)},e^{i\theta})w^{(j)}|w^{(k)}\rangle f(e^{i\theta}) d\!\!\!/\theta. \tag{3.13}$$

$\mathcal{Y}_\alpha(w^{(i)},f)$ can be regarded as a linear map $W_j \to \widehat{W}_k$. In the following, we show that when $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded, $\mathcal{Y}_\alpha(w^{(i)},f)$ is a preclosed unbounded operator.

To begin with, we note that for any $f \in C_c^\infty(S^1\backslash\{-1\})$ and any $s \in \mathbb{R}$, the s-th mode of f is

$$\widehat{f}(s) = \int_{-\pi}^{\pi} f(e^{i\theta})e^{-is\theta} \cdot \frac{d\theta}{2\pi}. \tag{3.14}$$

Then we have

$$\mathcal{Y}_\alpha(w^{(i)},f) = \sum_{s\in\mathbb{R}} \mathcal{Y}_\alpha(w^{(i)},s)\widehat{f}(s). \tag{3.15}$$

Define

$$\mathcal{D}_V = \{\Delta_i + \Delta_j - \Delta_k :\ W_i, W_j, W_k \text{ are irreducible } V\text{-modules}\},$$
$$\mathbb{Z}_V = \mathbb{Z} + \mathcal{D}_V.$$

Then $\mathcal{Y}_\alpha(w^{(i)},s) = 0$ except possibly when $s \in \mathbb{Z}_V$. Since V has finitely many equivalence classes of irreducible representations, the set $\mathcal{D}_V$ is finite. Now for any $t \in \mathbb{R}$ we define a norm $|\cdot|_{V,t}$ on $C_c^\infty(S^1\backslash\{-1\})$ to be

$$|f|_{V,t} = \sum_{s\in\mathbb{Z}_V} (1+|s|)^t|\widehat{f}(s)|, \tag{3.16}$$

which is easily seen to be finite. For each $r \in \mathbb{R}$, we define $e_r : S^1\backslash\{-1\} \to \mathbb{C}$ to be

$$e_r(e^{i\theta}) = e^{ir\theta}, \quad (-\pi < \theta < \pi). \tag{3.17}$$

When $r \in \mathbb{Z}$, we regard e_r as a continuous function on S^1.

Lemma 3.8. *Suppose that $w^{(i)} \in W_i$ is homogeneous, and $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded and satisfies condition* (3.1).

(a) Let $p \in \mathbb{R}$. Then there exists $M_p \geqslant 0$, such that for any $f \in C_c^\infty(S^1\backslash\{-1\}), w^{(j)} \in W_j$, we have

$\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)} \in \mathcal{H}_k^\infty$, and

$$\big\|\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)}\big\|_p \leqslant M_p|f|_{V,|p|+t}\big\|w^{(j)}\big\|_{p+r}. \tag{3.18}$$

(b) For any $w^{(j)} \in W_j, w^{(k)} \in W_k$ we have

$$\langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)}\rangle = \sum_{m\geqslant 0}\frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!}\langle\mathcal{Y}_{\alpha^*}(\overline{L_1^m w^{(i)}}, \overline{e_{(m+2-2\Delta_{w^{(i)}})}f})w^{(k)}|w^{(j)}\rangle. \tag{3.19}$$

Proof. (a) We have

$$\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)} = \sum_{s\in\mathbb{Z}_V}\widehat{f}(s)\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}. \tag{3.20}$$

Choose $M_p \geqslant 0$ such that (3.2) always holds. Then

$$\begin{aligned}&\sum_{s\in\mathbb{Z}_V}\big\|\widehat{f}(s)\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\big\|_p\\ \leqslant&\sum_{s\in\mathbb{Z}_V}M_p\big|\widehat{f}(s)\big|(1+|s|)^{|p|+t}\|w^{(j)}\|_{p+r}\\ =&M_p|f|_{V,|p|+t}\big\|w^{(j)}\big\|_{p+r}.\end{aligned} \tag{3.21}$$

In particular, $\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)} \in \mathcal{H}_k^\infty$.

(b) For any $w^{(j)} \in W_j, w^{(k)} \in W_k$, and $z \in \mathbb{C}^\times$ with argument $\arg z$, we have

$$\begin{aligned}&\langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, z)w^{(j)}\rangle\\ =&\langle\mathcal{Y}_{\alpha^*}(e^{\overline{z}L_1}(e^{-i\pi}\overline{z}^{-2})^{L_0}\overline{w^{(i)}}, \overline{z}^{-1})w^{(k)}|w^{(j)}\rangle\\ =&e^{-i\pi\Delta_{w^{(i)}}}\sum_{m\geqslant 0}\frac{\overline{z}^{m-2\Delta_{w^{(i)}}}}{m!}\langle\mathcal{Y}_{\alpha^*}(\overline{L_1^m w^{(i)}}, \overline{z}^{-1})w^{(k)}|w^{(j)}\rangle.\end{aligned} \tag{3.22}$$

Note also that $\overline{d\!\!\!/\theta} = e^{-2i\theta}d\!\!\!/\theta$. Therefore we have

$$\begin{aligned}&\langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)}\rangle\\ =&\int_{-\pi}^{\pi}\langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})w^{(j)}\rangle\overline{f(e^{i\theta})d\!\!\!/\theta}\end{aligned}$$

$$
\begin{aligned}
&=\sum_{m\geqslant 0}\int_{-\pi}^{\pi}\frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!}\langle\mathcal{Y}_{\alpha*}(\overline{L_1^m w^{(i)}},e^{i\theta})w^{(k)}|w^{(j)}\rangle e^{-i(m+2-2\Delta_{w^{(i)}})\theta}\overline{f(e^{i\theta})}d\theta\\
&=\sum_{m\geqslant 0}\frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!}\langle\mathcal{Y}_{\alpha*}(\overline{L_1^m w^{(i)}},\overline{e_{(m+2-2\Delta_{w^{(i)}})}f})w^{(k)}|w^{(j)}\rangle.
\end{aligned}
\tag{3.23}
$$

□

By lemma 3.8, if $w^{(i)}$ is homogeneous and $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded, then $\mathcal{Y}_\alpha(w^{(i)},f)$ can be viewed as an unbounded operator from $\mathcal{H}_j$ to $\mathcal{H}_k$ with domain W_j. Moreover, the domain of $\mathcal{Y}_\alpha(w^{(i)},f)^*$ contains a dense subspace of $\mathcal{H}_k$ (which is W_k). So $\mathcal{Y}_\alpha(w^{(i)},f)$ *is preclosed*. We let $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}$ be its closure . By inequality (3.18), $\mathcal{H}_j^\infty$ is inside $\mathscr{D}(\overline{\mathcal{Y}_\alpha(w^{(i)},f)})$, the domain of $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}$, and $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}\mathcal{H}_j^\infty\subset\mathcal{H}_k^\infty$. In the following, we will always view $\mathcal{Y}_\alpha(w^{(i)},f):\mathcal{H}_j^\infty\to\mathcal{H}_k^\infty$ as the restriction of $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}$ to $\mathcal{H}_j^\infty$. $\mathcal{Y}_\alpha(w^{(i)},f)$ is called a **smeared intertwining operator**. The closed operator $\mathcal{Y}_\alpha(w^{(i)},f)^*=\overline{\mathcal{Y}_\alpha(w^{(i)},f)}^*$ is the adjoint of $\mathcal{Y}_\alpha(w^{(i)},f)$. The **formal adjoint** of $\mathcal{Y}_\alpha(w^{(i)},f)$, which is denoted by $\mathcal{Y}_\alpha(w^{(i)},f)^\dagger$, is the restriction of $\mathcal{Y}_\alpha(w^{(i)},f)^*$ to $\mathcal{H}_k^\infty\to\mathcal{H}_j^\infty$.

The following proposition follows directly from lemma 3.8.

Proposition 3.9. *Suppose that $w^{(i)}\in W_i$ is homogeneous, $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded and satisfies condition* (3.1). *Then for any $f\in C_c(S^1\backslash\{-1\})$, the following statements are true:*
(a) $\mathcal{Y}_\alpha(w^{(i)},f)\mathcal{H}_j^\infty\subset\mathcal{H}_k^\infty$. Moreover, for any $p\in\mathbb{R}$, there exists $M_p\geqslant 0$ independent of f, such that for any $\xi^{(j)}\in\mathcal{H}_j^\infty$, we have

$$
\big\|\mathcal{Y}_\alpha(w^{(i)},f)\xi^{(j)}\big\|_p\leqslant M_p|f|_{V,|p|+t}\big\|\xi^{(j)}\big\|_{p+r}.
\tag{3.24}
$$

(b) $\mathcal{Y}_\alpha(w^{(i)},f):\mathcal{H}_j^\infty\to\mathcal{H}_k^\infty$ has the formal adjoint $\mathcal{Y}_\alpha(w^{(i)},f)^\dagger:\mathcal{H}_k^\infty\to\mathcal{H}_j^\infty$, which satisfies

$$
\mathcal{Y}_\alpha(w^{(i)},f)^\dagger=\sum_{m\geqslant 0}\frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!}\mathcal{Y}_{\alpha*}(\overline{L_1^m w^{(i)}},\overline{e_{(m+2-2\Delta_{w^{(i)}})}f}).
\tag{3.25}
$$

In particular, if $w^{(i)}$ is quasi-primary, then we have the adjoint relation

$$
\mathcal{Y}_\alpha(w^{(i)},f)^\dagger=e^{-i\pi\Delta_{w^{(i)}}}\mathcal{Y}_{\alpha*}(\overline{w^{(i)}},\overline{e_{(2-2\Delta_{w^{(i)}})}f}).
\tag{3.26}
$$

Hence the adjoint relation (3.26) for smeared intertwining operators is established.

Remark 3.10. If $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ is a unitary energy-bounded intertwining operator of V, $w^{(i)} \in W_i$ is not necessarily homogeneous, and $f \in C_c^\infty(S^1\backslash\{-1\})$, then by linearity, we can define a preclosed operator $\mathcal{Y}_\alpha(w^{(i)}, f) : \mathcal{H}_j^\infty \to \mathcal{H}_k^\infty$ to be $\mathcal{Y}_\alpha(w^{(i)}, f) = \sum_{s\in\mathbb{R}} \mathcal{Y}_\alpha(P_s w^{(i)}, f)$. Proposition 3.9-(a) still holds in this case.

Remark 3.11. If W_i is a unitary V-module, then $Y_i \in \mathcal{V}\binom{i}{0\ i}$. Choose any vector $v \in V$. Since the powers of x in $Y(v,x)$ are integers, for each $z \in \mathbb{C}^\times$, $Y_i(v,z)$ does not depend on $\arg z$. Therefore, for any $f \in C_c^\infty(S^1)$, we can defined a smeared vertex operator $Y_i(v,f) : \mathcal{H}_i^\infty \to \mathcal{H}_i^\infty$ using (3.13).

Braiding of smeared intertwining operators

The relation between products of smeared intertwining operators and correlation functions is indicated as follows.

Proposition 3.12. *Let $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}, \dots, \mathcal{Y}_{\alpha_n}$ be a chain of unitary energy-bounded intertwining operators of V with charge spaces $W_{i_1}, W_{i_2}, \dots, W_{i_n}$ respectively. Let W_j be the source space of $\mathcal{Y}_{\alpha_1}$, and let W_k be the target space of $\mathcal{Y}_{\alpha_n}$. Choose mutually disjoint $I_1, I_2, \dots, I_n \in \mathcal{J}(S^1\backslash\{-1\})$. For each $m = 1, 2, \dots, n$ we choose $w^{(i_m)} \in W_{i_m}$ and $f_m \in C_c^\infty(I_m)$. Then for any $w^{(j)} \in W_j$ and $w^{(k)} \in W_k$,*

$$\begin{aligned}&\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle \\ =& \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, e^{i\theta_n}) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, e^{i\theta_1}) w^{(j)} | w^{(k)} \rangle f_1(e^{i\theta_1}) \cdots f_n(e^{i\theta_n}) \cdot d\!\!\!/\theta_1 \cdots d\!\!\!/\theta_n.\end{aligned} \tag{3.27}$$

Proof.

$$\begin{aligned}&\sum_{s_1,\dots,s_n\in\mathbb{R}} \big\| P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) P_{s_{n-1}} \mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, f_{n-1}) P_{s_{n-2}} \cdots P_{s_1} \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} \big\| \\ =& \sum_{t_1,\dots,t_n\in\mathbb{Z}_V} \big\| \mathcal{Y}_{\alpha_n}(w^{(i_n)}, t_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, t_1) w^{(j)} \big\| \cdot \big| \widehat{f}_1(t_1) \cdots \widehat{f}_n(t_n) \big|,\end{aligned} \tag{3.28}$$

which, by proposition 3.2, is finite. Hence, for all $r_1, \ldots, r_n, r_1/r_2, \ldots, r_{n-1}/r_n \in [1/2, 1]$, the following functions of $s_1, \ldots, s_n$:

$$\Big|\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)P_{s_{n-1}}\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)w^{(j)}|w^{(k)}\rangle \\ \cdot r_1^{-\Delta_{w^{(i_1)}}}\cdots r_n^{-\Delta_{w^{(i_n)}}}\left(\frac{r_1}{r_2}\right)^{s_1}\cdots\left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}}\Big| \tag{3.29}$$

are bounded by a constant multiplied by

$$\Big|\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)P_{s_{n-1}}\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)w^{(j)}|w^{(k)}\rangle\Big|, \tag{3.30}$$

the sum of which over $s_1, \ldots, s_n$ is finite. Therefore, if we always assume that $r_1, \ldots, r_n > 0$ and $0 < r_1/r_2 < \cdots < r_{n-1}/r_n \leqslant 1$, then by dominated convergence theorem and relation (1.26),

$$\begin{aligned}
&\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)w^{(j)}|w^{(k)}\rangle \\
=&\sum_{s_1,\ldots,s_n\in\mathbb{R}}\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)P_{s_{n-1}}\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)w^{(j)}|w^{(k)}\rangle \\
=&\sum_{s_1,\ldots,s_n\in\mathbb{R}}\lim_{r_1,\ldots,r_n\to 1}\Big(\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)P_{s_{n-1}}\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)w^{(j)}|w^{(k)}\rangle \\
&\cdot r_1^{-\Delta_{w^{(i_1)}}}\cdots r_n^{-\Delta_{w^{(i_n)}}}\left(\frac{r_1}{r_2}\right)^{s_1}\cdots\left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}}\Big) \\
=&\lim_{r_1,\ldots,r_n\to 1}\sum_{s_1,\ldots,s_n\in\mathbb{R}}\Big(\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)P_{s_{n-1}}\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)w^{(j)}|w^{(k)}\rangle \\
&\cdot r_1^{-\Delta_{w^{(i_1)}}}\cdots r_n^{-\Delta_{w^{(i_n)}}}\left(\frac{r_1}{r_2}\right)^{s_1}\cdots\left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}}\Big) \\
=&\lim_{r_1,\ldots,r_n\to 1}\sum_{s_1,\ldots,s_n\in\mathbb{R}}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, e^{i\theta_n})P_{s_{n-1}} \\
&\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, e^{i\theta_1})w^{(j)}|w^{(k)}\rangle r_1^{-\Delta_{w^{(i_1)}}}\cdots r_n^{-\Delta_{w^{(i_n)}}} \\
&\cdot\left(\frac{r_1}{r_2}\right)^{s_1}\cdots\left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}} f_1(e^{i\theta_1})\cdots f_n(e^{i\theta_n})d\!\!\!/\theta_1\cdots d\!\!\!/\theta_n \\
=&\lim_{r_1,\ldots,r_n\to 1}\sum_{s_1,\ldots,s_n\in\mathbb{R}}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, r_ne^{i\theta_n})P_{s_{n-1}} \\
&\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, r_1e^{i\theta_1})w^{(j)}|w^{(k)}\rangle f_1(e^{i\theta_1})\cdots f_n(e^{i\theta_n})d\!\!\!/\theta_1\cdots d\!\!\!/\theta_n.
\end{aligned} \tag{3.31}$$

By theorem 2.2 and the discussion below, the sum and the integrals in (3.31) commute. Therefore (3.31) equals

$$\begin{aligned}
&\lim_{r_1,\dots,r_n\to 1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\sum_{s_1,\dots,s_n\in\mathbb{R}}\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)},r_ne^{i\theta_n})P_{s_{n-1}}\\
&\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)},r_1e^{i\theta_1})w^{(j)}|w^{(k)}\rangle f_1(e^{i\theta_1})\cdots f_n(e^{i\theta_n})d\!\!\!\!\!\!\not\ \ \theta_1\cdots d\!\!\!\!\!\!\not\ \ \theta_n\\
=&\lim_{r_1,\dots,r_n\to 1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\langle\mathcal{Y}_{\alpha_n}(w^{(i_n)},r_ne^{i\theta_n})\\
&\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},r_1e^{i\theta_1})w^{(j)}|w^{(k)}\rangle f_1(e^{i\theta_1})\cdots f_n(e^{i\theta_n})d\!\!\!\!\!\!\not\ \ \theta_1\cdots d\!\!\!\!\!\!\not\ \ \theta_n.
\end{aligned}\tag{3.32}$$

By continuity of correlation functions, the limit and the integrals in (3.32) commute. So (3.32) equals the right hand side of equation (3.27). Thus the proof is completed. □

Corollary 3.13. *Let $\mathcal{Y}_\alpha,\mathcal{Y}_{\alpha'}$ be unitary energy-bounded intertwining operators of V with common charge space W_i, and $\mathcal{Y}_\beta,\mathcal{Y}_{\beta'}$ be unitary energy-bounded intertwining operators of V with common charge space W_j. Choose $z_i,z_j\in S^1$ and assume that $\arg z_j<\arg z_i<\arg z_j+2\pi$. Choose disjoint open intervals $I,J\in\mathcal{J}(S^1\backslash\{-1\})$ such that I is anticlockwise to J. Suppose that for any $w^{(i)}\in W_i, w^{(j)}\in W_j$, the following braid relation holds:*

$$\mathcal{Y}_\alpha(w^{(i)},z_i)\mathcal{Y}_\beta(w^{(j)},z_j)=\mathcal{Y}_{\beta'}(w^{(j)},z_j)\mathcal{Y}_{\alpha'}(w^{(i)},z_i).\tag{3.33}$$

Then for any $f\in C_c^\infty(I), g\in C_c^\infty(J)$, we have the braid relation for intertwining operators:

$$\mathcal{Y}_\alpha(w^{(i)},f)\mathcal{Y}_\beta(w^{(j)},g)=\mathcal{Y}_{\beta'}(w^{(j)},g)\mathcal{Y}_{\alpha'}(w^{(i)},f).\tag{3.34}$$

Note that if W_k is the source space of $\mathcal{Y}_\beta$, then both sides of equation (3.34) are understood to be acting on $\mathcal{H}_k^\infty$.

Remark 3.14. If $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\alpha'}$ (resp. $\mathcal{Y}_\beta$ and $\mathcal{Y}_{\beta'}$) are the vertex operator Y_k, then the above corollary still holds if we assume that $I\in\mathcal{J}$ (resp. $J\in\mathcal{J}$).

Rotation covariance of smeared intertwining operators

For each $t \in \mathbb{R}$, we define an action

$$\mathfrak{r}(t) : S^1 \to S^1, \quad \mathfrak{r}(t)(e^{i\theta}) = e^{i(\theta+t)}. \tag{3.35}$$

For any $g \in C_c^\infty(S^1)$, we let

$$\mathfrak{r}(t)g = g \circ \mathfrak{r}(-t). \tag{3.36}$$

Therefore, if $J \in \mathcal{J}$, then $\mathfrak{r}(t)C_c^\infty(J) = C_c^\infty(\mathfrak{r}(t)J)$. We also define $g' \in C_c^\infty(S^1)$ to be

$$g'(e^{i\theta}) = \frac{d}{d\theta}g(e^{i\theta}) \tag{3.37}$$

Rotation covariance is stated as follows.

Proposition 3.15. *Suppose that $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ is unitary, $w^{(i)} \in W_j$ is homogeneous, $\mathcal{Y}(w^{(i)}, x)$ is energy bounded, and $J \in \mathcal{J}(S^1 \backslash \{-1\})$. Choose $\varepsilon > 0$ such that $\mathfrak{r}(t)J \subset S^1 \backslash \{-1\}$ for any $t \in (-\varepsilon, \varepsilon)$. Then for any $g \in C_c^\infty(J)$ and $t \in (-\varepsilon, \varepsilon)$, the following equations hold when both sides act on $\mathcal{H}_j^\infty$:*

$$[\overline{L_0}, \mathcal{Y}_\alpha(w^{(i)}, g)] = \mathcal{Y}_\alpha\big(w^{(i)}, (\Delta_{w^{(i)}} - 1)g + ig'\big), \tag{3.38}$$
$$e^{it\overline{L_0}}\mathcal{Y}_\alpha(w^{(i)}, g)e^{-it\overline{L_0}} = \mathcal{Y}_\alpha\big(w^{(i)}, e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g\big). \tag{3.39}$$

Proof. By equation (1.24), for any $z = e^{i\theta} \in J$ we have

$$\begin{aligned}
&[\overline{L_0}, \mathcal{Y}_\alpha(w^{(i)}, z)]\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, z) + z\partial_z\mathcal{Y}_\alpha(w^{(i)}, z)\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta}) - i\partial_\theta\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})
\end{aligned}$$

when evaluated between vectors inside W_j and W_k. Thus we have

$$
\begin{aligned}
&[\overline{L_0},\mathcal{Y}_\alpha(w^{(i)},g)]=\int_{-\pi}^{\pi}[\overline{L_0},\mathcal{Y}_\alpha(w^{(i)},e^{i\theta})]g(e^{i\theta})\text{đ}\theta\\
=&\int_{-\pi}^{\pi}\big(\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)},e^{i\theta})-i\partial_\theta\mathcal{Y}_\alpha(w^{(i)},e^{i\theta})\big)g(e^{i\theta})\text{đ}\theta\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)},g)-i\int_{-\pi}^{\pi}\partial_\theta\mathcal{Y}_\alpha(w^{(i)},e^{i\theta})g(e^{i\theta})\frac{e^{i\theta}}{2\pi}d\theta\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)},g)+i\int_{-\pi}^{\pi}\mathcal{Y}_\alpha(w^{(i)},e^{i\theta})\frac{d}{d\theta}\Big(g(e^{i\theta})\frac{e^{i\theta}}{2\pi}\Big)d\theta\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)},g)+i\int_{-\pi}^{\pi}\mathcal{Y}_\alpha(w^{(i)},e^{i\theta})\big(g'(e^{i\theta})+ig(e^{i\theta})\big)\frac{e^{i\theta}}{2\pi}d\theta\\
=&(\Delta_{w^{(i)}}-1)\mathcal{Y}_\alpha(w^{(i)},g)+i\mathcal{Y}_\alpha(w^{(i)},g').
\end{aligned}
$$

This proves the first equation. To prove the second one, we first note that for any $\tau\geqslant 0$, when $h\in\mathbb{R}$ is small enough, the $|\cdot|_{V,\tau}$-norm of the function

$$
\begin{aligned}
&e^{i(\Delta_{w^{(i)}}-1)(t+h)}r(t+h)g-e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g\\
&-\big(i(\Delta_{w^{(i)}}-1)e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g-e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g'\big)h
\end{aligned}
$$

is $o(h)$. For any $\xi^{(j)}\in\mathcal{H}_j^\infty$, we define a function $\Xi(t)$ for $|t|<\varepsilon$ to be

$$
\Xi(t)=e^{-it\overline{L_0}}\mathcal{Y}_\alpha(w^{(i)},e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g)e^{it\overline{L_0}}\xi^{(j)}.
$$

Now we can apply relation (3.38) and proposition 3.9 to see that the vector norm of $\Xi(t+h)-\Xi(t)$ is $o(h)$ for any $|t|<\varepsilon$. (In fact this is true for any Sobolev norm.) This shows that the derivative of $\Xi(t)$ exists and equals 0. So $\Xi(t)$ is a constant function. In particular, we have $\Xi(0)=\Xi(t)$, which implies (3.39). □

The intertwining property of smeared intertwining operators

Proposition 3.16. *Let $\mathcal{Y}_\alpha\in\mathcal{V}\binom{k}{i\ j}$ be unitary, $w^{(i)}\in W_i$ be homogeneous, and $v\in V$ be quasi-primary. Suppose that $\theta v=v$, $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy bounded, and $Y_j(v,x),Y_k(v,x)$ satisfy linear energy bounds. Let $I\in\mathcal{J},J\in\mathcal{J}(S^1\backslash\{-1\})$ be disjoint. Choose $f\in C_c^\infty(I),g\in C_c^\infty(J)$. Assume* *that f satisfies*

$$e^{i\pi\Delta_v/2}e_{1-\Delta_v}f = \overline{e^{i\pi\Delta_v/2}e_{1-\Delta_v}f}. \tag{3.40}$$

Then $Y_j(v,f)$ and $Y_k(v,f)$ are essentially self-adjoint, and for any $t \in \mathbb{R}$, we have

$$e^{it\overline{Y_j(v,f)}}\mathcal{H}_j^\infty \subset \mathcal{H}_j^\infty, \quad e^{it\overline{Y_k(v,f)}}\mathcal{H}_k^\infty \subset \mathcal{H}_k^\infty, \tag{3.41}$$

$$e^{it\overline{Y_k(v,f)}} \cdot \overline{\mathcal{Y}_\alpha(w^{(i)},g)} = \overline{\mathcal{Y}_\alpha(w^{(i)},g)} \cdot e^{it\overline{Y_j(v,f)}}. \tag{3.42}$$

Proof. Define the direct sum V-module $W_l = W_j \oplus^\perp W_k$ of W_j and W_k. Then $\mathcal{H}_l$ is the norm completion of W_l, $\mathcal{H}_l^\infty$ is the dense subspace of smooth vectors, and $Y_l(v,f) = \mathrm{diag}\big(Y_j(v,f), Y_k(v,f)\big)$. By equations (3.40) and (3.26), $Y_l(v,f)$ is symmetric (i.e., $Y_l(v,f)^\dagger = Y_l(v,f)$). Since $Y_l(v,x)$ satisfies linear energy bounds, by proposition 3.9-(a), relation (3.38), and lemma B.8, $Y_l(v,f)$ is essentially self-adjoint, and $e^{it\overline{Y_l(v,f)}}\mathcal{H}_l^\infty \subset \mathcal{H}_l^\infty$. This is equivalent to saying that $Y_j(v,f)$ and $Y_k(v,f)$ are essentially self-adjoint, and relation (3.41) holds.

Let $A = Y_l(v,f)$. Regard $B = \mathcal{Y}_\alpha(w^{(i)},g)$ as an unbounded operator on $\mathcal{H}_l$, being the original one when acting on $\mathcal{H}_j$, and zero when acting on $\mathcal{H}_k$. (So the domain of B is $\mathcal{H}_j^\infty \oplus^\perp \mathcal{H}_k$.) By propositions 2.13, 3.13, and remark 3.14, $AB = BA$ when both sides of the equation act on $\mathcal{H}_l^\infty$. By theorem B.9, $\overline{A}$ commutes strongly with $\overline{B}$. Therefore $e^{it\overline{A}} \cdot \overline{B} = \overline{B} \cdot e^{it\overline{A}}$, which is equivalent to equation (3.42). □

CHAPTER 4

FROM UNITARY VOAS TO CONFORMAL NETS

In this chapter, we assume that V is unitary and energy-bounded. A net $\mathcal{M}_V$ of von Neumann algebras on the circle can be defined using smeared vertex operators of V. If $\mathcal{M}_V$ is a conformal net, then V is called strongly local. A theorem in [CKLW15] shows that when V is generated by a set of quasi-primary vectors whose field operators satisfy linear energy bounds, then V is strongly local. This is discussed in section 4.1.

Let W_i be an energy-bounded unitary V-module. If this representation of V can be integrated to a representation of the conformal net $\mathcal{M}_V$, we say that W_i is strongly integrable. In section 4.1, we show that the abelian category of energy-bounded strongly-integrable unitary V-modules is equivalent to the category of the corresponding integrated $\mathcal{M}_V$-modules. A similar topic is treated in [CWX].

There are two majors ways to prove the strong integrability of a unitary V-modules W_i. First, if the action of V on W_i is restricted from the inclusion of V in a larger energy-bounded strongly-local unitary VOA, then W_i is strongly local. This result is proved in [CWX], and will not be used in our paper. In section 4.2, we give a different criterion using linear energy bounds of intertwining operators.

4.1 Unitary VOAs, conformal nets, and their representations

We first review the definition of conformal nets. Standard references are [CKLW15, Car04, GF93, GL96, KL04]. Conformal nets are based on the theory of von Neumann algebras. For an outline of this theory, we recommend [Con80] chapter 5. More details can be found in [Jon03, Tak02, Tak13, KR83, KR15].

Let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving diffeomorphisms of S^1. Convergence in $\mathrm{Diff}(S^1)$ means uniform convergence of all derivatives. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{U}(\mathcal{H})$ let the group of unitary operators on $\mathcal{H}$, equipped with the strong (operator) topology. $P\mathcal{U}(\mathcal{H})$ is the quotient topology group of $\mathcal{U}(\mathcal{H})$, defined by identifying x with λx when $x \in \mathcal{U}(\mathcal{H}), \lambda \in S^1$. A

strongly continuous projective representation of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ is, by definition, a continuous homomorphism from $\mathrm{Diff}(S^1)$ into $P\mathcal{U}(\mathcal{H})$.

$\mathrm{Diff}(S^1)$ contains the subgroup $\mathrm{PSU}(1,1)$ of Möbius transformations of S^1. Elements in $\mathrm{PSU}(1,1)$ are of the form

$$z \mapsto \frac{\lambda z + \mu}{\overline{\mu} z + \overline{\lambda}} \quad (z \in S^1), \tag{4.1}$$

where $\lambda, \mu \in \mathbb{C}, |\lambda|^2 - |\mu|^2 = 1$. $\mathrm{PSU}(1,1)$ contains the subgroup $S^1 = \{\mathfrak{r}(t) : t \in \mathbb{R}\}$ of rotations of S^1.

A **conformal net** $\mathcal{M}$ associates to each $I \in \mathcal{J}$ a von Neumann algebra $\mathcal{M}(I)$ acting on a fixed Hilbert space $\mathcal{H}_0$, such that the following conditions hold:

(a) (Isotony) If $I_1 \subset I_2 \in \mathcal{J}$, then $\mathcal{M}(I_1)$ is a von Neumann subalgebra of $\mathcal{M}(I_2)$.

(b) (Locality) If $I_1, I_2 \in \mathcal{J}$ are disjoint, then $\mathcal{M}(I_1)$ and $\mathcal{M}(I_2)$ commute.

(c) (Conformal covariance) We have a strongly continuous projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}_0$, such that for any $g \in \mathrm{Diff}(S^1), I \in \mathcal{J}$,

$$U(g)\mathcal{M}(I)U(g)^* = \mathcal{M}(gI).$$

Moreover, if g fixes the points in I, then for any $x \in \mathcal{M}(I)$,

$$U(g)xU(g)^* = x.$$

(d) (Möbius covariance) The projective unitary representation U of $\mathrm{Diff}(S^1)$ restricts to a unitary representation of the Möbius group $\mathrm{PSU}(1,1)$ on $\mathcal{H}_0$.

(e) (Positivity of energy) The generator of the restriction of U to S^1 is positive.

(f) There exists a vector $\Omega \in \mathcal{H}_0$ (the vacuum vector), such that Ω is fixed by $\mathrm{PSU}(1,1)$, and it is cyclic under the action of $\bigvee_{I \in \mathcal{J}} \mathcal{M}(I)$ (the von Neumann algebra generated by all $\mathcal{M}(I)$).

(g) (Irreducibility) The von Neumann algebra $\bigvee_{I \in \mathcal{J}} \mathcal{M}(I)$ is the algebra of all bounded linear operators on $\mathcal{H}_0$.

The following properties are satisfied by a conformal net, and will be used in our theory:

(1) (Additivity) If $\{I_a : a \in \mathcal{A}\}$ is a collection of open intervals in $\mathcal{J}$, $I \in \mathcal{J}$, and $I = \bigcup_{a \in \mathcal{A}} I_a$, then

$\mathcal{M}(I) = \bigvee_{a\in\mathcal{A}} \mathcal{M}(I_a)$.

(2) (Haag duality) $\mathcal{M}(I)' = \mathcal{M}(I^c)$.

(3) $\mathcal{M}(I)$ is a type III factor. (Indeed, it is of type III_1.)

Properties (2) and (3) are natural consequences of Bisognano-Wichmann theorem, cf. [BGL93, GF93].

Following [CKLW15], we now show how to construct a conformal net $\mathcal{M}_V$ from V. Let the Hilbert space $\mathcal{H}_0$ be the norm completion of V. For any $I \in \mathcal{J}$ we define $\mathcal{M}_V(I)$ to be the von Neumann algebra on $\mathcal{H}_0$ generated by closed operators of the form $\overline{Y(v,f)}$, where $v \in V$ and $f \in C_c^\infty(I)$. Thus we've obtained a net of von Neumann algebras $I \in \mathcal{J} \mapsto \mathcal{M}_V(I)$ and denote it by $\mathcal{M}_V$. The vacuum vector Ω in $\mathcal{H}_0$ is the same as that of V. The projective representation U of $\mathrm{Diff}(S^1)$ is obtained by integrating the action of the real part of the Virasoro algebra on V. The representation of $\mathrm{PSU}(1,1)$ is determined by the action of $L_{\pm 1}, L_0$ on V. *All the axioms of conformal nets, except locality, are satisfied for $\mathcal{M}_V$.*

Locality of $\mathcal{M}_V$, however, is much harder to prove. To be sure, for any disjoint $I, J \in \mathcal{J}$, and any $u, v \in V$, we can use proposition 2.13, corollary 3.13, and proposition 3.9 to show that

$$Y(u,f)Y(v,g) = Y(v,g)Y(u,f), \tag{4.2}$$

$$Y(u,f)^\dagger Y(v,g) = Y(v,g)Y(u,f)^\dagger, \tag{4.3}$$

where both sides act on $\mathcal{H}_0^\infty$. The commutativity of closed operators on a common invariant core, however, does not imply the strong commutativity of these two operators, as indicated by the example of Nelson (cf. [Nel59]). So far, the best result we have for the locality of $\mathcal{M}_V$ is the following:

Theorem 4.1. *Suppose that V is generated by a set E of quasi-primary vectors, and that for any $v \in E$, $Y(v,x)$ satisfies linear energy bounds. Then the net $\mathcal{M}_V$ satisfies the locality condition, and is therefore a conformal net. Moreover, if we let $E_{\mathbb{R}} = \{v + \theta v, i(v - \theta v) : v \in E\}$, then for any $I \in \mathcal{J}$, $\mathcal{M}_V(I)$ is generated by the closed operators $\overline{Y(u,f)}$, where $u \in E_{\mathbb{R}}$, and $f \in C_c^\infty(I)$ satisfies $e^{i\pi\Delta_u/2}e_{1-\Delta_u}f = \overline{e^{i\pi\Delta_u/2}e_{1-\Delta_u}f}$.*

Proof. Clearly $E_{\mathbb{R}}$ generates V. From the proof of [CKLW15] theorem 8.1, it suffices to prove, for any disjoint $I, J \in \mathcal{J}$, $u, v \in E_{\mathbb{R}}$, and $f \in C_c^\infty(I), g \in C_c^\infty(J)$ satisfying $e^{i\pi\Delta_u/2}e_{1-\Delta_u}f =$

$\overline{e^{i\pi\Delta_u/2}e_{1-\Delta_u}f}, e^{i\pi\Delta_v/2}e_{1-\Delta_v}g = \overline{e^{i\pi\Delta_v/2}e_{1-\Delta_v}g}$, that $\overline{Y(u,f)}$ and $\overline{Y(v,g)}$ commute strongly. By proposition 3.9-(b), $Y(u,f)$ and $Y(v,g)$ are symmetric operators. Hence by equation (3.38), proposition 3.9-(a), equation (4.2), Lemma B.8, and theorem B.9, $\overline{Y(u,f)}$ and $\overline{Y(v,g)}$ are self-adjoint operators, and they commute strongly with each other. □

We say that a unitary energy-bounded strongly local VOA V is **strongly local**, if $\mathcal{M}_V$ satisfies the locality condition.

Suppose that V is strongly local. We now discuss representations of the conformal net $\mathcal{M}_V$. Let $\mathcal{H}_i$ be a Hilbert space (currently not yet related to W_i). Suppose that for any $I \in \mathcal{J}$, we have a (normal unital *-) representation $\pi_{i,I} : \mathcal{M}_V(I) \to B(\mathcal{H}_i)$, such that for any $I_1, I_2 \in \mathcal{J}$ satisfying $I_1 \subset I_2$, and any $x \in \mathcal{M}_V(I_1)$, we have $\pi_{i,I_1}(x) = \pi_{i,I_2}(x)$. Then $(\mathcal{H}_i, \pi_i)$ (or simply $\mathcal{H}_i$) is called a (locally normal) **represention** of the $\mathcal{M}_V$ (or a $\mathcal{M}_V$**-module**). We shall abbreviate $\pi_{i,I_1}(x)$ and $\pi_{i,I_2}(x)$ as $\pi_i(x)$, if we do not want to emphasize which von Neumann algebra x belongs to. If $\xi^{(i)} \in \mathcal{H}_i$, we simply write $x\xi^{(i)}$ for $\pi_i(x)\xi^{(i)} = \pi_{i,I}(x)\xi^{(i)}$.

The $\mathcal{M}_V$-modules we are interested in are those arising from unitary V-modules. Let W_i be an energy-bounded unitary V-module, and let $\mathcal{H}_i$ be the norm completion of the inner product space W_i. Assume that we have a representation π_i of $\mathcal{M}_V$ on $\mathcal{H}_i$. Then we say that $(\mathcal{H}_i, \pi_i)$ is **associated with the V-module** (W_i, Y_i), if for any $I \in \mathcal{J}$, $v \in V$, and $f \in C_c^\infty(I)$, we have

$$\pi_{i,I}\big(\overline{Y(v,f)}\big) = \overline{Y_i(v,f)}. \tag{4.4}$$

(See section B.1 for the definition of $\pi_{i,I}$ acting on unbounded closed operators affiliated with $\mathcal{M}_V(I)$.) A $\mathcal{M}_V$-module associated with W_i, if exists, must be unique. We say that an energy-bounded unitary V-module W_i is **strongly integrable** if there exists a $\mathcal{M}_V$-module $(\mathcal{H}_i, \pi_i)$ associated with W_i. Let $\mathcal{S}$ be the collection of strongly integrable energy-bounded unitary V-modules. Obviously $V \in \mathcal{S}$. It is easy to show that S is additively complete.

We now introduce a very useful density property. For any $I \in \mathcal{J}$, we define $\mathcal{M}_V(I)_\infty$ to be the set of **smooth operators** in $\mathcal{M}_V(I)$, i.e., the set of all $x \in \mathcal{M}_V(I)$ satisfying that for any unitary

V-module W_i inside $\mathcal{S}$,

$$x\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty, \quad x^*\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty. \tag{4.5}$$

Proposition 4.2. *If V is unitary, energy-bounded, and strongly local, then $\mathcal{M}_V(I)_\infty$ is a strongly dense self-adjoint subalgebra of $\mathcal{M}_V(I)$.*

Proof. By additivity or by the construction of $\mathcal{M}_V$, we have $\mathcal{M}_V(I) = \bigvee_{J \subset\subset I} \mathcal{M}_V(J)$. ($J \subset\subset I$ means that $J \in \mathcal{J}$ and $\overline{J} \subset I$.) For each $J \subset\subset I$ and $x \in \mathcal{M}_V(J)$, we choose $\epsilon > 0$ such that $\mathfrak{r}(t)J \subset I$ whenever $t \in (-\epsilon, \epsilon)$. For each $h \in C_c^\infty(-\epsilon, \epsilon)$ such that $\int_{-\epsilon}^{\epsilon} h(t)dt = 1$, define

$$x_h = \int_{-\epsilon}^{\epsilon} e^{it\overline{L_0}} x e^{-it\overline{L_0}} h(t)dt.$$

Then by (3.39), $x_h \in \mathcal{M}_V(I)$. For each W_i inside $\mathcal{S}$, equations (3.39) and (4.4) imply that

$$\pi_i(e^{it\overline{L_0}} x e^{-it\overline{L_0}}) = e^{it\overline{L_0}} \pi_i(x) e^{-it\overline{L_0}}. \tag{4.6}$$

So we have

$$\pi_i(x_h) = \int_{-\epsilon}^{\epsilon} e^{it\overline{L_0}} \pi_i(x) e^{-it\overline{L_0}} h(t)dt,$$

which implies that

$$e^{it\overline{L_0}} \pi_i(x_h)\xi^{(i)} = \pi_i(x_{h_t}) e^{it\overline{L_0}} \xi^{(i)}, \tag{4.7}$$

where $h_t(s) = h(s-t)$. From this equation, we see that the derivative of $e^{it\overline{L_0}}\xi^{(i)} \in \mathcal{H}_i^\infty$ at $t = 0$ exists and equals

$$-\pi_i(x_{h'})\xi^{(i)} + i\pi_i(x_h)\overline{L_0}\xi^{(i)}. \tag{4.8}$$

This implies that $\pi_i(x_h)\xi^{(i)} \in \mathcal{H}_i^1$ and $i\overline{L_0}\pi_i(x_h)\xi^{(i)}$ equals (4.8). Using the same argument, we see that for each $n \in \mathbb{Z}_{\geqslant 0}$, the following Leibniz rule holds:

$$\pi_i(x_h)\xi^{(i)} \in \mathscr{D}(\overline{L_0}^n) = \mathcal{H}_i^n,$$

$$\overline{L_0}^n \pi_i(x_h)\xi^{(i)} = \sum_{m=0}^{n} \binom{n}{m} i^m \pi_i(x_{h^{(m)}}) \cdot \overline{L_0}^{n-m} \xi^{(i)},$$

where $h^{(m)}$ is the m-th derivative of h. This proves that $\pi_i(x_h)\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty$.

Since $(x_h)^* = (x^*)_{\overline{h}}$, we also have $x_h^*\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty$. So $x_h \in \mathcal{M}_V(I)_\infty$. Clearly $x_h \to x$ strongly as h converges to the δ-function at 0. We thus conclude that any $x \in \mathcal{M}_V(J)$ can be strongly approximated by elements in $\mathcal{M}_V(I)_\infty$. Hence the proof is finished. □

We study the relation between the representation categories of $\mathcal{M}_V$ and V. Assume, as before, that V is unitary, energy-bounded, and strongly local. We define an additive category $\mathrm{Rep}_{\mathcal{S}}(\mathcal{M}_V)$ as follows: The objects are $\mathcal{M}_V$-modules of the form $\mathcal{H}_i$, where W_i is an element inside $\mathcal{S}$. If W_i, W_j are inside $\mathcal{S}$, then the vector space of morphisms $\mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ consists of bounded linear operators $R : \mathcal{H}_i \to \mathcal{H}_j$, such that for any $I \in \mathcal{J}, x \in \mathcal{M}_V(I)$, the relation $R\pi_i(x) = \pi_j(x)R$ holds.

Define a functor $\mathfrak{F} : \mathrm{Rep}^{\mathrm{u}}_{\mathcal{S}}(V) \to \mathrm{Rep}_{\mathcal{S}}(\mathcal{M}_V)$ in the following way: If W_i is a unitary V-module in $\mathcal{S}$, then we let $\mathfrak{F}(W_i)$ be the $\mathcal{M}_V$-module $\mathcal{H}_i$. If W_i, W_j are in $\mathcal{S}$ and $R \in \mathrm{Hom}_V(W_i, W_j)$, then by lemma 2.20, R is bounded, and hence can be extended to a bounded linear map $R : \mathcal{H}_i \to \mathcal{H}_j$. It is clear that R is an element in $\mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$. We let $\mathfrak{F}(R)$ be this $\mathcal{M}_V$-module homomorphism. Clearly $\mathfrak{F} : \mathrm{Hom}_V(W_i, W_j) \to \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ is linear. We show that $\mathfrak{F}$ is an isomorphism.

Theorem 4.3. [1] *Let V be unitary, energy-bounded, and strongly local. For any W_i, W_j in $\mathcal{S}$, the linear map $\mathfrak{F} : \mathrm{Hom}_V(W_i, W_j) \to \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ is an isomorphism. Therefore, $\mathfrak{F} : \mathrm{Rep}^{\mathrm{u}}_{\mathcal{S}}(V) \to \mathrm{Rep}_{\mathcal{S}}(\mathcal{M}_V)$ is an equivalence of additive categories.*

Proof. The linear map $\mathfrak{F} : \mathrm{Hom}_V(W_i, W_j) \to \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ is clearly injective. We only need to prove that $\mathfrak{F}$ is surjective. Choose $R \in \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$. Define an orthogonal direct sum module $W_k = W_i \oplus^\perp W_j$. Then $\mathcal{H}_k$ is the orthogonal direct sum $\mathcal{M}_V$-module of $\mathcal{H}_i, \mathcal{H}_j$. Regard R as an element in $\mathrm{End}_{\mathcal{M}_V}(\mathcal{H}_k)$, which is the original operator when acting on $\mathcal{H}_i$, and is 0 when acting on $\mathcal{H}_j$. Then for any $I \in \mathcal{J}, x \in \mathcal{M}_V(I)$, R commutes with $\pi_k(x), \pi_k(x^*)$. Therefore, for any homogeneous $v \in V$ and $f \in C_c^\infty(I)$, R commutes strongly with $\pi_k\big(\overline{Y(v,f)}\big) = \overline{Y_k(v,f)}$.

[1] This theorem is also proved in [CWX]. We would like to thank Sebastiano Carpi for letting us know this fact.

We first show that $RW_i \subset W_j$. Choose $I_1, I_2 \in \mathcal{J}$ and $f_1 \in C_c^\infty(I_1, \mathbb{R}), f_2 \in C_c^\infty(I_2, \mathbb{R})$ such that $f_1 + f_2 = 1$. Regard L_0 as an unbounded operator on $\mathcal{H}_k$ with domain W_k. Then L_0 is the restriction of the smeared vertex operator $Y_k(\nu, e_1)$ to W_k. (Recall that by our notation of e_r, $e_1(e^{i\theta}) = e^{i\theta}$.) Therefore,

$$L_0 \subset Y_k(\nu, e_1 f_1) + Y_k(\nu, e_1 f_2),$$

and hence

$$\overline{L_0} \subset \overline{Y_k(\nu, e_1 f_1) + Y_k(\nu, e_1 f_2)} \subset \overline{\overline{Y_k(\nu, e_1 f_1)} + \overline{Y_k(\nu, e_1 f_2)}}.$$

Recall that ν is quasi-primary and $\Delta_\nu = 2$. Therefore, by equation (3.25), $\overline{Y_k(\nu, e_1 f_1)}$ and $\overline{Y_k(\nu, e_1 f_2)}$ are symmetric operators. It follows that $A = \overline{\overline{Y_k(\nu, e_1 f_1)} + \overline{Y_k(\nu, e_1 f_2)}}$ is symmetric. Note that $\overline{L_0}$ is self adjoint. Thus we have

$$\overline{L_0} \subset A \subset A^* \subset \overline{L_0}^* = \overline{L_0},$$

which implies that

$$\overline{L_0} = \overline{\overline{Y_k(\nu, e_1 f_1)} + \overline{Y_k(\nu, e_1 f_2)}}.$$

Therefore, since R commutes strongly with $\overline{Y_k(\nu, e_1 f_1)}$ and $\overline{Y_k(\nu, e_1 f_2)}$, R also commutes strongly with $\overline{L_0}$. In particular, R preserves every eigensubspace of $\overline{L_0}$ in $\mathcal{H}_k$. This implies that $RW_i(s) \subset W_j(s)$ for any $s \in \mathbb{R}$, and hence that $RW_i \subset W_j$.

Now, for any $n \in \mathbb{Z}$, $w^{(i)} \in W_i$, and $v \in V$, we have

$$Y_k(v, n) w^{(i)} = Y_k(v, e_n) w^{(i)} = \overline{Y_k(v, e_n f_1)} w^{(i)} + \overline{Y_k(v, e_n f_2)} w^{(i)}.$$

Since R commutes strongly with $\overline{Y_i(v, e_n f_1)}, \overline{Y_i(v, e_n f_2)}$, we have $RY_k(v, e_n) w^{(i)} = Y_k(v, e_n) R w^{(i)}$, which implies that $RY_i(v, n) w^{(i)} = Y_j(v, n) R w^{(i)}$. Therefore, $R \in \mathrm{Hom}_V(W_i, W_j)$. $\square$

Corollary 4.4. *If W_i is a unitary V-module in $\mathcal{S}$, and $\mathcal{H}_1$ is a (norm-)closed $\mathcal{M}_V$-invariant sub-*

space of $\mathcal{H}_i$, then there exists a V-invariant subspace W_1 of W_i, such that $\mathcal{H}_1$ is the norm closure of W_1.

Proof. Let e_1 be the orthogonal projection of $\mathcal{H}_i$ onto $\mathcal{H}_1$. Then $e_1 \in \mathrm{End}_{\mathcal{M}_V}(\mathcal{H}_i)$. By theorem 4.3, e_1 restricts to an element in $\mathrm{End}_V(W_i)$. So $W_1 = e_1 W_i$ is a V-invariant subspace of W_i, and $e_1 L_0 = L_0 e_1$ when both sides act on W_i. Therefore e_1 commutes strongly with $\overline{L_0}$. Let P_s be the projection of $\mathcal{H}_i$ onto $W_i(s)$. Then P_s is a spectral projection of $\overline{L_0}$. Hence $eP_s = P_s e$ for any $s \geqslant 0$.

Choose any $\xi \in \mathcal{H}_1$. Then $\xi = \sum_{s\geqslant 0} P_s\xi$. Since for any $s \geqslant 0$ we have $P_s\xi = P_s e_1\xi = e_1 P_s \xi \in e_1 W_i = W_1$, we see that ξ can be approximated by vectors in W_1. This proves that $\mathcal{H}_1$ is the norm closure of W_1. □

4.2 A criterion for strong integrability

Assume that V is unitary, energy bounded, and strongly local. In this section, we give a criterion for the strong integrability of energy-bounded unitary V-modules.

Proposition 4.5. *Let W_i be a non-trivial energy-bounded unitary V-module. Then W_i is strongly integrable, if and only if for any $I \in \mathcal{J}$, there exists a unitary operator $U_I : \mathcal{H}_0 \to \mathcal{H}_i$, such that any $v \in V$ and $f \in C_c^\infty(I)$ satisfy*

$$\overline{Y_i(v,f)} = U_I \overline{Y(v,f)} U_I^*. \tag{4.9}$$

Proof. "If part": For any $I \in \mathcal{J}(I)$, we define a representation $\pi_{i,I}$ of $\mathcal{M}_V(I)$ on $\mathcal{H}_i$ to be

$$\pi_{i,I}(x) = U_I x U_I^* \qquad (x \in \mathcal{M}_V(I)). \tag{4.10}$$

If $J \in \mathcal{J}(I)$ and $I \subset J$, then by equation (4.9), $U_J^* U_I$ commutes strongly with every $\overline{Y(v,f)}$ where $v \in V$ and $f \in C_c^\infty(I)$. So $U_J^* U_I$ commutes with $\mathcal{M}_V(I)$, which implies that $\pi_{i,I}$ is the restriction of $\pi_{i,J}$ on $\mathcal{M}_V(I)$. So π_i is a representation of the conformal net $\mathcal{M}_V$. It is obvious that π_i is associated with W_i. So W_i is strongly integrable.

"Only if part": Suppose that W_i is strongly integrable. We let $(\mathcal{H}_i, \pi_i)$ be the $\mathcal{M}_V$-module associated with W_i. For each $I \in \mathcal{M}_I$, $\pi_{i,I}$ is a non-trivial representation of $\mathcal{M}_V(I)$ on $\mathcal{H}_i$. Since the Hilbert spaces $\mathcal{H}_0, \mathcal{H}_i$ are separable, and $\mathcal{M}_V(I)$ is a type III factor, $\pi_{i,I}$ is (unitary) equivalent to the representation $\pi_{0,I}$ of $\mathcal{M}_V(I)$ on $\mathcal{H}_0$. So there exits a unitary $U_I : \mathcal{H}_i \to \mathcal{H}_i$ such that equation (4.9) always holds. □

Remark 4.6. Equation (4.9) is equivalent to the following relations:

$$U_I\overline{Y(v,f)} \subset \overline{Y_i(v,f)}U_I, \tag{4.11}$$

$$U_I^*\overline{Y_i(v,f)} \subset \overline{Y(v,f)}U_I^*. \tag{4.12}$$

Proposition 4.7. *Let W_j, W_k be non-trivial energy-bounded unitary V-modules. Assume that W_j is strongly integrable. If for any $I \in \mathcal{J}$ there exits a collection $\{T_a : a \in \mathcal{A}\}$ of bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_k$, such that $\bigvee_{a\in\mathcal{A}} T_a\mathcal{H}_j$ is dense in $\mathcal{H}_k$, and that for any $a \in \mathcal{A}, v \in V, f \in C_c^\infty(I)$, we have*

$$T_a\overline{Y_j(v,f)} \subset \overline{Y_k(v,f)}T_a, \tag{4.13}$$

$$T_a^*\overline{Y_k(v,f)} \subset \overline{Y_j(v,f)}T_a^*, \tag{4.14}$$

then W_k is strongly integrable.

Proof. Let $W_l = W_j \oplus^\perp W_k$ be the direct sum module of W_j and W_k, and extend each T_a to a bounded linear operator on $\mathcal{H}_l$, such that T_a equals zero on the subspace $\mathcal{H}_k$. Choose any $I \in \mathcal{J}$. Since $\overline{Y_l(v,f)} = \mathrm{diag}(\overline{Y_j(v,f)}, \overline{Y_k(v,f)})$, equations (4.13) and (4.14) are equivalent to that T_a commutes strongly with $\overline{Y_l(v,f)}$ for any $v \in V, f \in C_c^\infty(I)$. We construct a unitary operator $U_I : \mathcal{H}_j \to \mathcal{H}_k$ such that

$$\overline{Y_k(v,f)} = U_I\overline{Y_j(v,f)}U_I^* \tag{4.15}$$

for any $v \in V, f \in C_c^\infty(I)$. Then the strong integrability of W_k will follow immediately from proposition 4.5 and the strong integrability of W_j.

Let $\{U_b : b \in \mathcal{B}\}$ be a maximal collection of non-zero partial isometries from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying the following conditions:

(a) For any $b \in \mathcal{B}, v \in V, f \in C_c^\infty(I)$, U_b commutes strongly with $\overline{Y_l(v,f)}$.

(b) The projections $\{e_b = U_bU_b^* : b \in \mathcal{B}\}$ are orthogonal.

Note that similar to T_a, each U_b is extended to a partial isometry on $\mathcal{H}_l$, being zero when acting on $\mathcal{H}_k$.

Let $e = \sum_{b\in\mathcal{B}} e_b$. We prove that $e = \mathrm{id}_{\mathcal{H}_k}$. Let $e' = \mathrm{id}_{\mathcal{H}_k} - e$. If $e' \neq 0$, then by the density of $\bigvee_{a\in\mathcal{A}} T_a\mathcal{H}_j$ in $\mathcal{H}_k$, there exists $a \in \mathcal{A}$ such that $e'T_a \neq 0$. Take the left polar decomposition $e'T_a = U_aH_a$ of $e'T_a$, where U_a is the partial isometry part. Then $U_aU_a^*$ is the projection of $\mathcal{H}_l$ onto the range of $e'T_a$, which is nonzero and orthogonal to each e_b. For each $v \in V, f \in C_c^\infty(I)$, since e' and T_a commute strongly with $\overline{Y_l(v, f)}$, U_a also commutes strongly with $\overline{Y_l(v, f)}$. Therefore, $\{U_b : b \in \mathcal{B}\} \cup \{U_a\}$ is a collection of partial isometries from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying conditions (a) and (b), and $\{U_b : b \in \mathcal{B}\}$ is its proper sub-collection. This contradicts the fact that $\{U_b : b \in \mathcal{B}\}$ is maximal. So $e' = 0$, and hence $e = \mathrm{id}_{\mathcal{H}_k}$.

For each $b \in \mathcal{B}$ we let $p_b = U_b^*U_b$, which is a non-zero projection on $\mathcal{H}_j$. We now restrict ourselves to operators on $\mathcal{H}_j$. Then p_b commutes strongly with each $\overline{Y_j(v, f)}$, which, by the strong integrability of W_j, is equivalent to that $p_b \in \pi_{j,I}(\mathcal{M}_V(I))'$. Note that $\mathcal{B}$ must be countable. We choose a countable collection $\{q_b : b \in \mathcal{B}\}$ of non-zero orthogonal projections on $\mathcal{H}_j$ satisfying that $\sum_{b\in\mathcal{B}} q_b = \mathrm{id}_{\mathcal{H}_j}$, and that each $q_b \in \pi_{j,I}(\mathcal{M}_V(I))'$. Since $\pi_{j,I}(\mathcal{M}_V(I))'$ is a type III factor, for each b there exists a partial isometry $\widetilde{U}_b \in \pi_{j,I}(\mathcal{M}_V(I))'$ satisfying $\widetilde{U}_b\widetilde{U}_b^* = p_b, \widetilde{U}_b^*\widetilde{U}_b = q_b$.

We turn our attention back to operators on $\mathcal{H}_l$. Since $\widetilde{U}_b \in \pi_{j,I}(\mathcal{M}_V(I))'$, $\widetilde{U}_b$ commutes strongly with each $\overline{Y_l(v, f)}$. Let $U_I = \sum_{b\in\mathcal{B}} U_b\widetilde{U}_b$. Then U_I is a unitary operator from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying relation (4.15) for any $v \in V, f \in C_c^\infty(I)$. Thus our proof is finished. $\square$

We now prove the strong integrability of an energy-bounded unitary V-module using the linear energy-boundedness of intertwining operators.

Theorem 4.8. *Let W_i, W_j, W_k be non-zero unitary irreducible V-modules. Assume that W_j and W_k are energy-bounded, that W_j is strongly integrable, and that there exist a non-zero quasi-primary vector $w_0^{(i)} \in W_i$ and a non-zero intertwining operator $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, such that $\mathcal{Y}_\alpha(w_0^{(i)}, x)$ satisfies linear energy bounds. Then W_k is strongly integrable.*

Proof. Step 1. Fix any $J \in \mathcal{J}(S^1\backslash\{-1\})$, and let $\mathcal{W}_J$ be the subspace of $\mathcal{H}_k$ spanned by the vectors $\mathcal{Y}_\alpha(w_0^{(i)}, g)w^{(j)}$ where $g \in C_c^\infty(J)$ and $w^{(j)} \in W_j$. We show that $\mathcal{W}_J$ is a dense subspace of $\mathcal{H}_k$.

Our proof is similar to that of Reeh-Schlieder theorem (cf. [RS61]). Choose $\xi^{(k)} \in \mathcal{W}_J^{\perp}$. Note that for each $\eta^{(k)} \in \mathcal{H}^k$, the multivalued function

$$z \mapsto z^{\overline{L_0}}\eta^{(k)} = \sum_{s\geqslant 0} z^s P_s \eta^{(k)} \tag{4.16}$$

is continuous on $\overline{D}^{\times}(1) = \{\zeta \in \mathbb{C} : 0 < |\zeta| \leqslant 1\}$ and holomorphic on its interior $D^{\times}(1)$. So we have a multivalued holomorphic function of z:

$$\langle z^{\overline{L_0}}\mathcal{Y}_\alpha(w_0^{(i)}, g)w^{(j)}|\xi^{(k)}\rangle, \tag{4.17}$$

which is continuous on $\overline{D}^{\times}(1)$ and holomorphic on $D^{\times}(1)$. Choose $\varepsilon > 0$ such that the support of $g^t = \exp(it(\Delta_{w_0^{(i)}} - 1))\mathfrak{r}(t)g$ is inside J for any $t \in (-\varepsilon, \varepsilon)$. Then, by proposition 3.15, we have

$$\langle e^{it\overline{L_0}}\mathcal{Y}_\alpha(w_0^{(i)}, g)w^{(j)}|\xi^{(k)}\rangle = \langle \mathcal{Y}_\alpha(w_0^{(i)}, g^t)e^{it\overline{L_0}}w^{(j)}|\xi^{(k)}\rangle, \tag{4.18}$$

which must be zero when $t \in (-\delta, \delta)$.

By Schwarz reflection principle, the value of function (4.17) is zero for any $z \in \overline{D}^{\times}(r)$. In particular, it is zero for any $z \in S^1$. This shows that (4.18) is zero for any $t \in \mathbb{R}$. Here, when we define the smeared intertwining operator, we allow the arguments to exceed the region $(-\pi, \pi)$ under the action of $\mathfrak{r}(t)$. So the right hand side of equation (4.18) becomes

$$\sum_{s\in\mathbb{R}} \int_{t-\pi}^{t+\pi} \langle \mathcal{Y}_\alpha(w_0^{(i)}, e^{i\theta})e^{it\overline{L_0}}w^{(j)}|P_s\xi^{(k)}\rangle \cdot \exp(it(\Delta_{w_0^{(i)}} - 1))g(e^{i(\theta-t)})d\!\!\!/\theta, \tag{4.19}$$

which is 0 for any $t \in \mathbb{R}$. (Recall our notation that $d\!\!\!/\theta = e^{i\theta}d\theta/2\pi$.) Since W_i, W_j, W_k are irreducible, we let $\Delta_i, \Delta_j, \Delta_k$ be their conformal dimensions, and set $\Delta_\alpha = \Delta_i + \Delta_j - \Delta_k$. Then by equation (1.25),

$$\mathcal{Y}_\alpha(w_0^{(i)}, z)z^{\Delta_\alpha} = \sum_{n\in\mathbb{Z}} \mathcal{Y}_\alpha(w_0^{(i)}, \Delta_\alpha - 1 - n)z^n \tag{4.20}$$

is a single valued holomorphic function for $z \in \mathbb{C}^\times$. So the fact that (4.19) always equals 0 implies that

$$\sum_{s\in\mathbb{R}}\int_{-\pi}^{\pi}\langle\mathcal{Y}_\alpha(w_0^{(i)},e^{i\theta})w^{(j)}|P_s\xi^{(k)}\rangle e^{i\Delta_\alpha\theta}\cdot h(e^{i\theta})d\theta=0 \tag{4.21}$$

for any $w^{(j)}\in W_j, I\in\mathcal{J}$ and $h\in C_c^\infty(I)$. By partition of unity on S^1, we see that equation (4.21) holds for any $h\in C^\infty(S^1)$.

For any $m\in\mathbb{Z}$, we choose $h(e^{i\theta})=e^{-im\theta}$. Then the left hand side of equation (4.21) becomes

$$\begin{aligned}
&\sum_{s\in\mathbb{R}}\int_{-\pi}^{\pi}\langle\mathcal{Y}_\alpha(w_0^{(i)},e^{i\theta})w^{(j)}|P_s\xi^{(k)}\rangle e^{i\Delta_\alpha\theta}\cdot e^{-im\theta}d\theta\\
=&\sum_{s\in\mathbb{R}}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\langle\mathcal{Y}_\alpha(w_0^{(i)},\Delta_\alpha-1-n)w^{(j)}|P_s\xi^{(k)}\rangle\cdot e^{i(n-m)\theta}d\theta\\
=&\sum_{s\in\mathbb{R}}\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\langle\mathcal{Y}_\alpha(w_0^{(i)},\Delta_\alpha-1-n)w^{(j)}|P_s\xi^{(k)}\rangle\cdot e^{i(n-m)\theta}d\theta\\
=&2\pi\sum_{s\in\mathbb{R}}\langle\mathcal{Y}_\alpha(w_0^{(i)},\Delta_\alpha-1-m)w^{(j)}|P_s\xi^{(k)}\rangle\\
=&2\pi\langle\mathcal{Y}_\alpha(w_0^{(i)},\Delta_\alpha-1-m)w^{(j)}|\xi^{(k)}\rangle,
\end{aligned} \tag{4.22}$$

which by equation (4.21) must be zero. By corollary 2.15 and the proof of corollary A.4, vectors of the form $\mathcal{Y}_\alpha(w_0^{(i)},s)w^{(j)}$ (where $s\in\mathbb{R},w^{(j)}\in W_j$) span W_k, which is a dense subspace of $\mathcal{H}_j$. So $\xi^{(k)}=0$.

Step 2. Choose any $I\in\mathcal{J}$, and let $J\in\mathcal{J}(I^c\backslash\{-1\})$. Take $W_l=W_j\oplus^\perp W_k$. Then for each $v\in V,f\in C_c^\infty(I)$ we have $\overline{Y_l(v,f)}=\mathrm{diag}(\overline{Y_j(v,f)},\overline{Y_k(v,f)})$. For each $g\in C_c^\infty(J)$, we extend $\mathcal{Y}_\alpha(w_0^{(i)},g)$ to an operator on $\mathcal{H}_l^\infty$ whose restriction to $\mathcal{H}_k^\infty$ is zero. We also regard $A=\mathcal{Y}_\alpha(w_0^{(i)},g)$ as an unbounded operator on $\mathcal{H}_l$ with domain $\mathcal{H}_l^\infty$. Let $\mathcal{N}(I)$ be the von Neumann algebra on $\mathcal{H}_j$ generated by the operators $\overline{Y_l(v,f)}$ where $v\in V,f\in C_c^\infty(I)$, and let $\mathcal{N}(I)_\infty$ be the set of all $x\in\mathcal{N}(I)$ satisfying $x\mathcal{H}_l^\infty\subset\mathcal{H}_l^\infty,x^*\mathcal{H}_l^\infty\subset\mathcal{H}_l^\infty$. Then as in the proof of proposition 4.2, $\mathcal{N}(I)_\infty$ is a strongly dense self-adjoint subalgebra of $\mathcal{N}(I)$. Let $H=(A+A^\dagger)/2$ and $K=(A-A^\dagger)/(2i)$ be symmetric unbounded operators on $\mathcal{H}_l$ with domain $\mathcal{H}_l^\infty$. Then by proposition 2.13, corollary 3.13, remark 3.14, and equation (3.26), for any $v\in V$ and $f\in C_c^\infty(I)$, $Y_l(v,f)$ commutes with H and K when acting on $\mathcal{H}_l^\infty$. By lemma B.8 and relations (3.38), (3.26), $\overline{H}$ and $\overline{K}$ are self adjoint, and by theorem B.9, $\overline{Y_l(v,f)}$ commutes strongly with $\overline{H}$ and $\overline{K}$. Hence any $x \in \mathcal{N}(I)$ commutes strongly with $\overline{H}$ and $\overline{K}$. In particular, if $x \in \mathcal{N}(I)_\infty$, we have $xH = Hx, xK = Kx$ when both sides of the equations act on $\mathcal{H}_l^\infty$. So $x(H+iK) = (H+iK)x$ when acting on $\mathcal{H}_l^\infty$. Therefore, $x\overline{\mathcal{Y}_\alpha(w_0^{(i)},g)} \subset \overline{\mathcal{Y}_\alpha(w_0^{(i)},g)}x$ for any $x \in \mathcal{N}(I)_\infty$, which implies that $\mathcal{N}(I)$ commutes strongly with $\overline{\mathcal{Y}_\alpha(w_0^{(i)},g)}$. Thus $\overline{Y_l(v,f)}$ commutes strongly with $\overline{\mathcal{Y}_\alpha(w_0^{(i)},g)}$.

Let $\overline{\mathcal{Y}_\alpha(w_0^{(i)},g)} = T_g H_g$ be the left polar decomposition of $\overline{\mathcal{Y}_\alpha(w_0^{(i)},g)}$, where T_g is the partial isometry. Then T_g commutes strongly with each $\overline{Y_l(v,f)}$. By step 1, $\{T_g : g \in C_c^\infty(J)\}$ form a collection of bounded operators from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying the conditions in proposition 4.7. Therefore, by that proposition, W_k is strongly integrable. $\square$

CHAPTER 5

GENERALIZED INTERTWINING OPERATORS

The goal of this chapter is to prove (in section 3) the rotation covariance, the intertwining property, the braid relations, and the adjoint relation for generalized smeared intertwining operators. The first two can be derived from the same properties for smeared intertwining operators, so their proofs are easier. To prove the braiding and the adjoint relation for generalized smeared intertwining operators, we first prove them for unsmeared ones, which are the goals of section 1 and 2.

Generalized intertwining operators are nothing but genus 0 correlation functions written in a particular way. Suppose that $\mathcal{Y}_{\sigma_2}, \ldots, \mathcal{Y}_{\sigma_n}$ is a chain of intertwining operators with charge spaces $W_{i_2}, \ldots, W_{i_n}$ respectively, such that the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1}, and the target space of $\mathcal{Y}_{\sigma_n}$ is W_i. Choose $\mathcal{Y}_\alpha \in \mathcal{Y}\binom{k}{i\ j}$. Choose $(z_1, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$, and choose arguments $\arg z_1, \arg(z_2 - z_1), \ldots, \arg(z_n - z_1)$. A **generalized intertwining operator** $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ is defined near $(z_1, \ldots, z_n)$ in the following two situations.

The first case is when $(z_1, \ldots, z_n)$ satisfies $0 < |z_2 - z_1| < \cdots < |z_n - z_1| < |z_1|$. We define a $(W_j \otimes W_{i_1} \otimes \cdots \otimes W_{i_n} \otimes W_{\overline{k}})^*$-valued holomorphic function $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ near $(z_1, \ldots, z_n)$ to satisfy that for any $w^{(j)} \in W_j, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$,

$$\begin{aligned}&\langle \mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, z_n; \ldots; w^{(i_2)}, z_2; w^{(i_1)}, z_1) w^{(j)}, w^{(\overline{k})} \rangle \\ =&\big\langle \mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1\big) w^{(j)}, w^{(\overline{k})} \big\rangle. \end{aligned} \tag{5.1}$$

The V-modules $W_{i_1}, \ldots, W_{i_n}$ are called the **charge spaces** of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}$. W_j is called the **source space** of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}$, and W_k is called the **target space** of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}$. The vector space of generalized intertwining operators with charge spaces $W_{i_1}, \ldots, W_{i_n}$, source space W_j, and target space W_k is also denoted by $\mathcal{V}\binom{k}{i_n\ \ldots\ i_1\ j}$.

In the second case, we choose $I \in \mathcal{J}$, and choose an arbitrary continuous argument function $\arg_I$ on I. We define $\mathscr{O}_n(I)$ to be the set of all $(z_1, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times) \cap I^n$ satisfying that for any $2 \leqslant l < m \leqslant n$, either $\arg_I(z_l z_1^{-1})\arg_I(z_m z_1^{-1}) < 0$, or $|\arg_I(z_l z_1^{-1})| < |\arg_I(z_m z_1^{-1})|$. Our definition is clearly independent of the choice of $\arg_I$, and $\mathscr{O}_n(I)$ is a finite disconnected union of simply-connected sets.

We want to define our generalized intertwining operators near any $(z_1, \ldots, z_n) \in \mathscr{O}_n(I)$. To do this, we rotate $z_1, \ldots, z_n$ along I without meeting each other, until these points satisfy $0 < |z_2 - z_1| < \cdots < |z_n - z_1| < |z_1| = 1$. The arguments of $z_1, z_2 - z_1, \ldots, z_n - z_1$ are changed continuously. We first define $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ near the new point $(z_1, \ldots, z_n)$ using equation (5.1). Then we reverse this process of rotating $z_1, \ldots, z_n$, and change $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ continuously so as to be defined near the original point.

We now define the product of two generalized intertwining operators defined near S^1. Products of more than two generalized intertwining operators are defined in a similar way. Choose disjoint $I, J \in \mathcal{J}$, choose $(z_1, \ldots, z_m) \in \mathscr{O}_m(I), (\zeta_1, \ldots, \zeta_n) \in \mathscr{O}_n(J)$, and choose arguments $\arg z_1, \arg(z_2 - z_1), \ldots, \arg(z_m - z_1), \arg \zeta_1, \arg(\zeta_2 - \zeta_1), \ldots, \arg(\zeta_n - \zeta_1)$. Choose generalized intertwining operators $\mathcal{Y}_{\sigma_m \cdots \sigma_1, \alpha} \in \mathcal{V}\binom{k}{i_m \cdots i_1 i_0}, \mathcal{Y}_{\rho_n \cdots \rho_1, \beta} \in \mathcal{V}\binom{i_0}{j_n \cdots j_1 j_0}$. If we choose $\arg z_2, \ldots, \arg z_m, \arg \zeta_2, \ldots, \arg \zeta_n$, then we can find uniquely chains of intertwining operators $\mathcal{Y}_{\alpha_1}, \ldots, \mathcal{Y}_{\alpha_m}$ with charge spaces $W_{i_1}, \ldots, W_{i_m}$ respectively, and $\mathcal{Y}_{\beta_1}, \ldots, \mathcal{Y}_{\beta_n}$ with charge spaces $W_{j_1}, \ldots, W_{j_n}$ respectively, such that the source space of $\mathcal{Y}_{\beta_1}$ is W_{j_0}, that the source space of $\mathcal{Y}_{\alpha_1}$ and the target space of $\mathcal{Y}_{\beta_n}$ are W_{i_0}, that the target space of $\mathcal{Y}_{\alpha_m}$ is W_k, and that for any $w^{(j_1)} \in W_{j_1}, \ldots, w^{(j_n)} \in W_{j_n}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_m)} \in W_{i_m}$, we have the fusion relations

$$\mathcal{Y}_{\sigma_m \cdots \sigma_2, \alpha}(w^{(i_m)}, z_m; \ldots; w^{(i_1)}, z_1) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1), \tag{5.2}$$

$$\mathcal{Y}_{\rho_n \cdots \rho_2, \beta}(w^{(j_n)}, \zeta_n; \ldots; w^{(j_1)}, \zeta_1) = \mathcal{Y}_{\beta_n}(w^{(j_n)}, \zeta_n) \cdots \mathcal{Y}_{\beta_1}(w^{(j_1)}, \zeta_1). \tag{5.3}$$

We then define a $(W_{j_0} \otimes W_{j_1} \otimes \cdots \otimes W_{j_n} \otimes W_{i_1} \otimes \cdots \otimes W_{i_m} \otimes W_{\overline{k}})^*$-valued holomorphic function $\mathcal{Y}_{\sigma_m \cdots \sigma_1, \alpha} \mathcal{Y}_{\rho_n \cdots \rho_1, \beta}$ near $(\zeta_1, \ldots, \zeta_n, z_1, \ldots, z_m)$ to satisfy that for any $w^{(j_0)} \in W_{j_0}, w^{(j_1)} \in W_{j_1}, \ldots, w^{(j_n)} \in W_{j_n}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_m)} \in W_{i_m}, w^{(\overline{k})} \in W_k$,

$$\begin{aligned} &\langle \mathcal{Y}_{\sigma_m \cdots \sigma_2, \alpha}(w^{(i_m)}, z_m; \ldots; w^{(i_1)}, z_1) \mathcal{Y}_{\rho_n \cdots \rho_2, \beta}(w^{(j_n)}, \zeta_n; \ldots; w^{(j_1)}, \zeta_1) w^{(j_0)}, w^{(\overline{k})} \rangle \\ =&\langle \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) \mathcal{Y}_{\beta_n}(w^{(j_n)}, \zeta_n) \cdots \mathcal{Y}_{\beta_1}(w^{(j_1)}, \zeta_1) w^{(j_0)}, w^{(\overline{k})} \rangle. \end{aligned} \tag{5.4}$$

Remark 5.1. It is clear that our definition does not depend on the choice of $\arg z_2, \dots, \arg z_m, \arg\zeta_2, \dots, \arg\zeta_n$. Moreover, if we choose $\varsigma \in S_m, \varpi \in S_n$, and real variables $\lambda_1, \dots, \lambda_n, r_1, \dots, r_m$ defined near 1 and satisfying $0 < \lambda_{\varpi(1)} < \cdots, \lambda_{\varpi(n)} < r_{\varsigma(1)} < \cdots < r_{\varsigma(m)}$, then the following series

$$\sum_{s\in\mathbb{R}} \langle \mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, r_m z_m; \dots; w^{(i_1)}, r_1 z_1) P_s \mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, \lambda_n\zeta_n; \dots; w^{(j_1)}, \lambda_1\zeta_1) w^{(j_0)}, w^{(\overline{k})}\rangle \tag{5.5}$$

of s converges absolutely, and by proposition 2.11, as $r_1, \dots, r_m, \lambda_1, \dots, \lambda_n \to 1$, the limit of (5.5) exists and equals the left hand side of equation (5.4).

5.1 Braiding of generalized intertwining operators

Theorem 5.2. *Choose disjoint* $I, J \in \mathcal{J}$. *Choose* $(z_1, \dots, z_m) \in \mathcal{O}_m(I), (\zeta_1, \dots, \zeta_n) \in \mathcal{O}_n(J)$. *Choose arguments* $\arg z_1, \arg\zeta_1, \arg(z_2 - z_1), \dots, \arg(z_m - z_1), \arg(\zeta_n - \zeta_1), \dots, \arg(\zeta_n - \zeta_1)$. *Let* $W_i, W_j, W_{i_1}, W_{i_2}, \dots, W_{i_m}, W_{j_1}, W_{j_2}, \dots, W_{j_n}$ *be* V*-modules. Assume that for any* $w^{(i)} \in W_i, w^{(j)} \in W_j$, *the braid relation*

$$\mathcal{Y}_\alpha(w^{(i)}, z_1)\mathcal{Y}_\beta(w^{(j)}, \zeta_1) = \mathcal{Y}_{\beta'}(w^{(j)}, \zeta_1)\mathcal{Y}_{\alpha'}(w^{(i)}, z_1) \tag{5.6}$$

holds. Then for any intertwining operators $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_m}, \mathcal{Y}_{\rho_2}, \dots, \mathcal{Y}_{\rho_n}$, *any* $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \dots, w^{(j_n)} \in W_{j_n}$, *we have the generalized braid relation*

$$\begin{aligned}&\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, z_m; \dots; w^{(i_1)}, z_1)\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, \zeta_n; \dots; w^{(j_1)}, \zeta_1)\\ =&\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)}, \zeta_n; \dots; w^{(j_1)}, \zeta_1)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)}, z_n; \dots; w^{(i_1)}, z_1).\end{aligned} \tag{5.7}$$

(Note that here, as before, we follow convention 2.19 to simplify our statement.)

Proof. By analytic continuation, it suffices to assume that $|z_1 - \zeta_1|$ is small enough with respect to 1, and $|z_2 - z_1|, \dots, |z_m - z_1|, |\zeta_2 - \zeta_1|, \dots, |\zeta_n - \zeta_1|$ are small enough with respect to $|z_1 - \zeta_1|$, such that for any $r, \lambda > 0$ satisfying $\frac{2}{3} < \frac{r}{\lambda} < \frac{3}{2}$, the following inequalities are satisfied:

$$|\zeta_n - \zeta_1| + |z_m - z_1| < 1/4, \tag{5.8}$$

$$0 < |\lambda\zeta_2 - \lambda\zeta_1| < |\lambda\zeta_3 - \lambda\zeta_1| < \cdots < |\lambda\zeta_n - \lambda\zeta_1| < |rz_1 - \lambda\zeta_1| - |rz_m - rz_1|, \tag{5.9}$$

$$0 < |rz_2 - rz_1| < |rz_3 - rz_1| < \cdots < |rz_m - rz_1| < |rz_1 - \lambda\zeta_1| < \lambda - |rz_m - rz_1|. \tag{5.10}$$

Choose $\arg(z_1 - \zeta_1)$. Since $|z_1 - \zeta_1| < 1$, there exist intertwining operators $\mathcal{Y}_\gamma$ and $\mathcal{Y}_\delta$ such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have

$$\mathcal{Y}_\alpha(w^{(i)}, z_1)\mathcal{Y}_\beta(w^{(j)}, \zeta_1) = \mathcal{Y}_\delta(\mathcal{Y}_\gamma(w^{(i)}, z_1 - \zeta_1)w^{(j)}, \zeta_1) = \mathcal{Y}_{\beta'}(w^{(j)}, \zeta_1)\mathcal{Y}_{\alpha'}(w^{(i)}, z_1). \tag{5.11}$$

Choose $r_m > \cdots > r_1 > \lambda_n > \cdots > \lambda_1 > 0$ satisfying $2/3 < r_1/\lambda_1 < 3/2$. When $r_2/r_1, \ldots, r_m/r_1, \lambda_2/\lambda_1, \ldots, \lambda_n/\lambda_1$ are close to 1, by corollary 2.7, the right hand side of the equation

$$\begin{aligned}
&\mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, r_m z_m - r_1 z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, r_2 z_2 - r_1 z_1)w^{(i_1)}, r_1 z_1\big)\\
&\cdot \mathcal{Y}_\beta\big(\mathcal{Y}_{\rho_n}(w^{(j_n)}, \lambda_n\zeta_n - \lambda_1\zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \lambda_2\zeta_2 - \lambda_1\zeta_1)w^{(j_1)}, \lambda_1\zeta_1\big)\\
=&\mathcal{Y}_\delta\bigg(\mathcal{Y}_\gamma\Big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, r_m z_m - r_1 z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, r_2 z_2 - r_1 z_1)w^{(i_1)}, r_1 z_1 - \lambda_1\zeta_1\Big)\\
&\cdot \mathcal{Y}_{\rho_n}(w^{(j_n)}, \lambda_n\zeta_n - \lambda_1\zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \lambda_2\zeta_2 - \lambda_1\zeta_1)w^{(j_1)}, \lambda_1\zeta_1\bigg)
\end{aligned} \tag{5.12}$$

converges absolutely and locally uniformly. If moreover $r_1/\lambda_1 = 4/3$, then by theorem 2.6, the left hand side of equation (5.12) also converges absolutely and locally uniformly, and hence equation (5.12) holds.

Now we let $r_1, \ldots, r_m, \lambda_1, \ldots, \lambda_n \to 1$, then the left hand side of equation (5.12) converges to the left hand side of equation (5.7), and the right hand side of (5.12) converges to

$$\begin{aligned}
&\mathcal{Y}_\delta\bigg(\mathcal{Y}_\gamma\Big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1 - \zeta_1\Big)\\
&\cdot \mathcal{Y}_{\rho_n}(w^{(j_n)}, \zeta_n - \zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \zeta_2 - \zeta_1)w^{(j_1)}, \zeta_1\bigg).
\end{aligned} \tag{5.13}$$

Therefore, the left hand side of equation (5.7) equals (5.13). The same argument shows that the right hand side of (5.7) also equals (5.13). This finishes our proof. $\square$

Note that it is easy to generalize proposition 2.11 to generalized intertwining operators.

5.2 The adjoint relation for generalized intertwining operators

This section is devoted to the proof of the adjoint relation for generalized intertwining operators (5.34). We first recall that if $\mathcal{Y}_\alpha$ is a unitary intertwining operator of a unitary V, $z \in S^1$ with chosen argument, and $w^{(i)} \in W_i$ is quasi-primary, then by relation (1.34),

$$\mathcal{Y}_\alpha(w^{(i)}, z)^\dagger = e^{-i\pi\Delta_{w^{(i)}}} z^{2\Delta_{w^{(i)}}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, z). \tag{5.14}$$

We want to obtain a similar relation for generalized intertwining operators. To achieve this goal, we first need an auxiliary fusion relation. Recall that for any V-module W_i, we have the creation operator $\mathcal{Y}^i_{i0} = B_\pm Y_i$ of W_i, and the annihilation operator $\mathcal{Y}^0_{i\bar{i}} = C^{-1}\mathcal{Y}^i_{i0}$ of $W_{\bar{i}}$. We set $\Upsilon^0_{i\bar{i}} = C\mathcal{Y}^i_{i0}$. Then similar to equation (1.40), for any $w^{(i)}_1 \in W_i, w^{(\bar{i})}_2 \in W_{\bar{i}}$ we have

$$\langle \Upsilon^0_{i\bar{i}}(w^{(i)}_1, x) w^{(\bar{i})}_2, \Omega\rangle = \langle e^{x^{-1}L_1} w^{(\bar{i})}_2, (e^{-i\pi}x^{-2})^{L_0} e^{-x^{-1}L_1} w^{(i)}_1\rangle. \tag{5.15}$$

Proposition 5.3 (Fusion with annihilation operators)**.** *Let $z_1, z_2 \in \mathbb{C}^\times$ satisfy $0 < |z_1|, |z_1 - z_2| < |z_2|$. Choose $\arg z_2$, let $\arg z_1$ be close to $\arg z_2$ as $z_1 \to z_2$, and let $\arg(z_2 - z_1)$ be close to $\arg z_2$ as $z_1 \to 0$. Then for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, $w^{(i)} \in W_i$ and $w^{(j)} \in W_j$, we have the fusion relation*

$$\Upsilon^0_{k\bar{k}}\big(\mathcal{Y}_\alpha(w^{(i)}, e^{i\pi}(z_2 - z_1))w^{(j)}, z_2\big) = \Upsilon^0_{j\bar{j}}(w^{(j)}, z_2)\mathcal{Y}_{C\alpha}(w^{(i)}, z_1). \tag{5.16}$$

Proof. Let us assume that $z_1, z_2 \in \mathbb{R}_{>0}$ and $0 < z_2 - z_1 < z_1 < z_2$. If the proposition is proved for this special case, then by analytic continuation, it also holds in general.

Therefore, we assume that $\arg z_1 = \arg z_2 = \arg(z_2 - z_1) = 0$. Let $\arg(z_1^{-1} - z_2^{-1})$ be close to $\arg(z_1^{-1}) = -\arg z_1$ as $z_2^{-1} \to 0$. Then it is obvious that $\arg(z_1^{-1} - z_2^{-1}) = 0 = \arg(\frac{z_2 - z_1}{z_1 z_2})$.

We now use equation (5.15) and the definition of $C\alpha$ to compute that

$$\langle \Upsilon^0_{j\bar{j}}(w^{(j)}, z_2)\mathcal{Y}_{C\alpha}(w^{(i)}, z_1)w^{(\bar{k})}, \Omega\rangle$$

$$
\begin{aligned}
&= \sum_{s\in\mathbb{R}} \langle \Upsilon^0_{j\bar{j}}(w^{(j)}, z_2) P_s \mathcal{Y}_{C\alpha}(w^{(i)}, z_1) w^{(\bar{k})}, \Omega\rangle \\
&= \sum_{s\in\mathbb{R}} \langle e^{z_2^{-1} L_1} P_s \mathcal{Y}_{C\alpha}(w^{(i)}, z_1) w^{(\bar{k})}, (e^{-i\pi} z_2^{-2})^{L_0} e^{-z_2^{-1} L_1} w^{(j)}\rangle \\
&= \sum_{s\in\mathbb{R}} \big\langle w^{(\bar{k})}, \mathcal{Y}_\alpha\big(e^{z_1 L_1}(e^{-i\pi} z_1^{-2})^{L_0} w^{(i)}, z_1^{-1}\big) P_s e^{z_2^{-1} L_{-1}} (e^{-i\pi} z_2^{-2})^{L_0} e^{-z_2^{-1} L_1} w^{(j)}\big\rangle,
\end{aligned}
\tag{5.17}
$$

which, according to lemma 2.16-(1), converges absolutely and equals

$$
\Big\langle w^{(\bar{k})}, e^{z_2^{-1} L_{-1}} \mathcal{Y}_\alpha\Big(e^{z_1 L_1}(e^{-i\pi} z_1^{-2})^{L_0} w^{(i)}, \frac{z_2 - z_1}{z_1 z_2}\Big)(e^{-i\pi} z_2^{-2})^{L_0} e^{-z_2^{-1} L_1} w^{(j)}\Big\rangle. \tag{5.18}
$$

By (1.26) and (1.30), the above formula equals

$$
\begin{aligned}
&\Big\langle w^{(\bar{k})}, e^{z_2^{-1} L_{-1}} (e^{-i\pi} z_2^{-2})^{L_0} \\
&\qquad \cdot \mathcal{Y}_\alpha\Big((e^{i\pi} z_2^2)^{L_0} e^{z_1 L_1}(e^{-i\pi} z_1^{-2})^{L_0} w^{(i)}, e^{i\pi}(z_2 - z_1)\frac{z_2}{z_1}\Big) e^{-z_2^{-1} L_1} w^{(j)}\Big\rangle \\
=&\Big\langle w^{(\bar{k})}, e^{z_2^{-1} L_{-1}} (e^{-i\pi} z_2^{-2})^{L_0} \\
&\qquad \cdot \mathcal{Y}_\alpha\Big(e^{-z_1 z_2^{-2} L_1} \Big(\frac{z_2}{z_1}\Big)^{2L_0} w^{(i)}, e^{i\pi}(z_2 - z_1)\frac{z_2}{z_1}\Big) e^{-z_2^{-1} L_1} w^{(j)}\Big\rangle.
\end{aligned}
\tag{5.19}
$$

On the other hand, we have

$$
\begin{aligned}
&\big\langle \Upsilon^0_{k\bar{k}}\big(\mathcal{Y}_\alpha(w^{(i)}, e^{i\pi}(z_2 - z_1)) w^{(j)}, z_2\big) w^{(\bar{k})}, \Omega\big\rangle \\
=& \sum_{s\in\mathbb{R}} \big\langle \Upsilon^0_{k\bar{k}}\big(P_s \mathcal{Y}_\alpha(w^{(i)}, e^{i\pi}(z_2 - z_1)) w^{(j)}, z_2\big) w^{(\bar{k})}, \Omega\big\rangle \\
=& \sum_{s\in\mathbb{R}} \langle w^{(\bar{k})}, e^{z_2^{-1} L_{-1}} (e^{-i\pi} z_2^{-2})^{L_0} e^{-z_2^{-1} L_1} P_s \mathcal{Y}_\alpha(w^{(i)}, e^{i\pi}(z_2 - z_1)) w^{(j)}\rangle.
\end{aligned}
\tag{5.20}
$$

Note that $|-z_2^{-1}| < |e^{i\pi}(z_2 - z_1)|^{-1}$. Let $\arg(1 - e^{i\pi}(z_2 - z_1)\cdot(-z_2^{-1}))$ be close to $\arg(1 - e^{i\pi}(z_2 - z_1)\cdot 0) = 0$ as $-z_2^{-1} \to 0$. Then clearly $\arg(1 - e^{i\pi}(z_2 - z_1)\cdot(-z_2^{-1})) = 0 = \arg(\frac{z_1}{z_2})$. We can use lemma 2.16-(2) to compute that (5.20) equals (5.19). This proves equation (5.16) when both sides act on Ω. By the proof of corollary 2.15, equation (5.16) holds when acting on any vector inside V. $\square$

Remark 5.4. By proposition 2.9 and the above property, we have the fusion relation

$$\Upsilon^0_{k\bar{k}}\big(\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_2 - z_1)w^{(i)}, z_1\big) = \Upsilon^0_{j\bar{j}}(w^{(j)}, z_2)\mathcal{Y}_{C\alpha}(w^{(i)}, z_1) \tag{5.21}$$

when $0 < |z_2 - z_1| < |z_1| < |z_2|$, $\arg z_1$ is close to $\arg z_2$ as $z_1 \to z_2$, and $\arg(z_2 - z_1)$ is close to $\arg z_2$ as $z_1 \to 0$. Similarly, we can also show that

$$\mathcal{Y}^0_{k\bar{k}}\big(\mathcal{Y}_{B_-\alpha}(w^{(j)}, z_2 - z_1)w^{(i)}, z_1\big) = \mathcal{Y}^0_{j\bar{j}}(w^{(j)}, z_2)\mathcal{Y}_{C^{-1}\alpha}(w^{(i)}, z_1). \tag{5.22}$$

Theorem 5.5 (Fusion of contragredient intertwining operators)**.** *Let $z_1, \ldots, z_n, z'_1, \ldots, z'_n \in \mathbb{C}^\times$ satisfy the following conditions:*

(1) $0 < |z_1| < |z_2| < \cdots < |z_n|$ and $0 < |z_2 - z_1| < \cdots < |z_n - z_1| < |z_1|$;

(1') $|z'_1| > |z'_2| > \cdots > |z'_n| > 0$ and $0 < |z'_2 - z'_1| < \cdots < |z'_n - z'_1| < |z'_1|$.

Choose arguments $\arg z_1, \arg z'_1$. For each $2 \leqslant m \leqslant n$, we choose arguments $\arg(z_m - z_1), \arg(z'_m - z'_1)$. Let $\arg z_m$ be close to $\arg z_1$ as $z_m \to z_1$, and let $\arg z'_m$ be close to $\arg z'_1$ as $z'_m \to z'_1$.

Let $W_{i_1}, \ldots, W_{i_n}$, and W_i be V-modules, and let $\mathcal{Y}_{\sigma_2}, \ldots, \mathcal{Y}_{\sigma_n}$ be a chain of intertwining operators of V satisfying the following conditions:

(a) for each $2 \leqslant m \leqslant n$, the charge space of $\mathcal{Y}_{\sigma_m}$ is W_{i_m};

(b) the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1};

(c) the target space of $\mathcal{Y}_{\sigma_n}$ is W_i.

Then there exists a chain of intertwining operators $\mathcal{Y}_{\sigma'_2}, \ldots, \mathcal{Y}_{\sigma'_n}$, whose types are the same as those of $\mathcal{Y}_{\sigma_2}, \ldots, \mathcal{Y}_{\sigma_n}$ respectively, such that for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, if $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}, \ldots, \mathcal{Y}_{\alpha_n}$ is a chain of intertwining operators of V satisfying the following conditions:

(i) for each $1 \leqslant m \leqslant n$, the charge space of $\mathcal{Y}_{\alpha_m}$ is W_{i_m};

(ii) the source space of $\mathcal{Y}_{\alpha_1}$ is W_j;

(iii) the target space of $\mathcal{Y}_{\alpha_n}$ is W_k;

(iv) for any $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}$, we have the fusion relation

$$\mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)$$

$$=\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1), \tag{5.23}$$

then the following fusion relation also holds:

$$\begin{aligned}&\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\sigma'_n}(w^{(i_n)},z'_n-z'_1)\cdots\mathcal{Y}_{\sigma'_2}(w^{(i_2)},z'_2-z'_1)w^{(i_1)},z'_1\big)\\ =&\mathcal{Y}_{C\alpha_1}(w^{(i_1)},z'_1)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)},z'_n).\end{aligned} \tag{5.24}$$

Proof. Let $W_{j_1},\dots,W_{j_{n-1}}$ be the target spaces of $\mathcal{Y}_{\alpha_1},\dots,\mathcal{Y}_{\alpha_{n-1}}$ respectively. Choose $\zeta'_0,\zeta'_1,\cdots,\zeta'_n\in\mathbb{R}_{<0}$ satisfying $\zeta'_0<\zeta'_1<\cdots<\zeta'_n<0$ and $|\zeta'_0-\zeta'_1|>|\zeta'_1-\zeta'_n|$. Let $\zeta_1=\zeta'_1-\zeta'_0,\dots,\zeta_n=\zeta'_n-\zeta'_0$. Let $\arg\zeta'_0=\arg\zeta'_1=\cdots=\arg\zeta'_n=-\pi,\arg\zeta_1=\arg(\zeta'_1-\zeta'_0)=0,\dots,\arg\zeta_n=\arg(\zeta'_n-\zeta'_0)=0$. Note that for any $2\leqslant m\leqslant n$, $\zeta_m-\zeta_1=\zeta'_m-\zeta'_1$. We let $\arg(\zeta_m-\zeta_1)=\arg(\zeta'_m-\zeta'_1)=0$.

We now rotate and stretch these points, so that for each $1\leqslant m\leqslant n$, ζ_m is moved to $\widetilde z_m=z_m$, ζ'_m is moved to $\widetilde z'_m=z'_m$, $\arg\zeta_m$ becomes $\arg\widetilde z_m=\arg z_m$, and $\arg\zeta'_m$ becomes $\arg\widetilde z'_m=\arg z'_m$. We assume that during this process, conditions (1) and (1') are always satisfied. (Note that such process might not exist if the choice of $\arg z_2,\arg z_3,\dots$ and $\arg z'_2,\arg z'_3,\dots$ are arbitrary with respect to $\arg z_1$ and $\arg z'_1$.) Denote this process by $(\mathbf{P})$. Then under this process, for each $2\leqslant m\leqslant n$, $\arg(\zeta_m-\zeta_1)$ is changed to an argument $\arg(\widetilde z_m-\widetilde z_1)$ of $\widetilde z_m-\widetilde z_1$, and $\arg(\zeta'_m-\zeta'_1)$ is changed to an argument $\arg(\widetilde z'_m-\widetilde z'_1)$ of $\widetilde z'_m-\widetilde z'_1$ accordingly. Since $\arg(\widetilde z_m-\widetilde z_1)\in\arg(z_m-z_1)+2i\pi\mathbb{Z}$ and $\arg(\widetilde z'_m-\widetilde z'_1)\in\arg(z'_m-z'_1)+2i\pi\mathbb{Z}$, there exist intertwining operators $\mathcal{Y}_{\widetilde\sigma_m},\mathcal{Y}_{\sigma'_m}$ of the same type as that of $\mathcal{Y}_{\sigma_m}$, such that for any $w^{(i_m)}\in W_{i_m}$,

$$\begin{aligned}\mathcal{Y}_{\widetilde\sigma_m}(w^{(i_m)},\widetilde z_m-\widetilde z_1)=\mathcal{Y}_{\sigma_m}(w^{(i_m)},z_m-z_1),\\ \mathcal{Y}_{\sigma'_m}(w^{(i_m)},z'_m-z'_1)=\mathcal{Y}_{\widetilde\sigma_m}(w^{(i_m)},\widetilde z'_m-\widetilde z'_1).\end{aligned}$$

Then equation (5.23) implies that

$$\begin{aligned}&\mathcal{Y}_{\alpha}\big(\mathcal{Y}_{\widetilde\sigma_n}(w^{(i_n)},\widetilde z_n-\widetilde z_1)\cdots\mathcal{Y}_{\widetilde\sigma_2}(w^{(i_2)},\widetilde z_2-\widetilde z_1)w^{(i_1)},\widetilde z_1\big)\\ =&\mathcal{Y}_{\alpha_n}(w^{(i_n)},\widetilde z_n)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},\widetilde z_1).\end{aligned} \tag{5.25}$$

By reversing process $(\mathbf{P})$, the above equation is analytically continued to the equation

$$\begin{aligned}&\mathcal{Y}_{\alpha}\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)},\zeta_n'-\zeta_1')\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)},\zeta_2'-\zeta_1')w^{(i_1)},\zeta_1'-\zeta_0'\big)\\ =&\mathcal{Y}_{\alpha_n}(w^{(i_n)},\zeta_n'-\zeta_0')\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},\zeta_1'-\zeta_0').\end{aligned}\tag{5.26}$$

For any $1\leqslant m\leqslant n$, we let $\arg(\zeta_0'-\zeta_m')$ be close to $\arg\zeta_0'=-\pi$ as $\zeta_m'\to 0$. Then $\arg(\zeta_0'-\zeta_m')=-\pi$, and hence $\zeta_m'-\zeta_0'=e^{i\pi}(\zeta_0-\zeta_m)$. Choose arbitrary $w^{(j)}\in W_j$. Then by lemma 5.3, we have

$$\begin{aligned}&\Upsilon^0_{j\overline{j}}(w^{(j)},\zeta_0')\mathcal{Y}_{C\alpha_1}(w^{(i_1)},\zeta_1')\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)},\zeta_n')\\ =&\Upsilon^0_{j_1\overline{j_1}}\big(\mathcal{Y}_{\alpha_1}(w^{(i_1)},\zeta_1'-\zeta_0')w^{(j)},\zeta_0'\big)\mathcal{Y}_{C\alpha_2}(w^{(i_2)},\zeta_2')\\ &\cdot\mathcal{Y}_{C\alpha_3}(w^{(i_3)},\zeta_3')\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)},\zeta_n')\\ =&\Upsilon^0_{j_2\overline{j_2}}\big(\mathcal{Y}_{\alpha_2}(w^{(i_2)},\zeta_2'-\zeta_0')\mathcal{Y}_{\alpha_1}(w^{(i_1)},\zeta_1'-\zeta_0')w^{(j)},\zeta_0'\big)\\ &\cdot\mathcal{Y}_{C\alpha_3}(w^{(i_3)},\zeta_3')\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)},\zeta_n')\\ &\qquad\vdots\\ =&\Upsilon^0_{k\overline{k}}\big(\mathcal{Y}_{\alpha_n}(w^{(i_n)},\zeta_n'-\zeta_0')\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},\zeta_1'-\zeta_0')w^{(j)},\zeta_0'\big),\end{aligned}\tag{5.27}$$

where, by theorem 2.6, the expression in each step converges absolutely. By (5.26), expression (5.27) equals

$$\Upsilon^0_{k\overline{k}}\Big(\mathcal{Y}_{\alpha}\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)},\zeta_n'-\zeta_1')\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)},\zeta_2'-\zeta_1')w^{(i_1)},\zeta_1'-\zeta_0'\big)w^{(j)},\zeta_0'\Big),\tag{5.28}$$

the absolute convergence of which is guaranteed by corollary 2.7. Again by proposition 5.3, equation (5.28) equals

$$\Upsilon^0_{j\overline{j}}(w^{(j)},\zeta_0')\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)},\zeta_n'-\zeta_1')\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)},\zeta_2'-\zeta_1')w^{(i_1)},\zeta_1'\big),\tag{5.29}$$

the absolute convergence of which follows from theorem 2.6. Therefore, the left hand side of equation (5.27) equals (5.29). By proposition 2.3, we obtain

$$\begin{aligned}
&\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\tilde{\sigma}_n}(w^{(i_n)},\zeta'_n-\zeta'_1)\cdots\mathcal{Y}_{\tilde{\sigma}_2}(w^{(i_2)},\zeta'_2-\zeta'_1)w^{(i_1)},\zeta'_1\big)\\
=&\mathcal{Y}_{C\alpha_1}(w^{(i_1)},\zeta'_1)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)},\zeta'_n).
\end{aligned}\tag{5.30}$$

Now we do process $(\mathbf{P})$. Then (5.30) is analytically continued to the equation

$$\begin{aligned}
&\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\tilde{\sigma}_n}(w^{(i_n)},\bar{z}'_n-\bar{z}'_1)\cdots\mathcal{Y}_{\tilde{\sigma}_2}(w^{(i_2)},\bar{z}'_2-\bar{z}'_1)w^{(i_1)},\bar{z}'_1\big)\\
=&\mathcal{Y}_{C\alpha_1}(w^{(i_1)},\bar{z}'_1)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)},\bar{z}'_n),
\end{aligned}\tag{5.31}$$

which implies (5.24). Thus the proof is completed. □

Remark 5.6. Choose (not necessarily disjoint) $I,J\in\mathcal{J}$, and choose $(z_1,\ldots,z_n)\in\mathscr{O}_n(I),(z'_1,\ldots,z'_n)\in\mathscr{O}_n(J)$. Choose continuous argument functions $\arg_I,\arg_J$ on I,J respectively, and let $\arg z_1=\arg_I(z_1),\ldots,\arg z_n=\arg_I(z_n),\arg z'_1=\arg_J(z'_1),\ldots,\arg z'_n=\arg_J(z'_n)$. For each $2\leqslant m\leqslant n$ we choose arguments $\arg(z_m-z_1)$ and $\arg(z'_m-z'_1)$. Then by theorem 5.5, for any chain of intertwining operators $\mathcal{Y}_{\sigma_2},\ldots,\mathcal{Y}_{\sigma_n}$ satisfying conditions (a), (b), and (c) of theorem 5.5, there exists a chain of intertwining operators $\mathcal{Y}_{\sigma'_2},\ldots,\mathcal{Y}_{\sigma'_n}$ whose types are the same as those of $\mathcal{Y}_{\sigma_2},\ldots,\mathcal{Y}_{\sigma_n}$ respectively, such that

$$\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},z_n;\ldots,w^{(i_1)},z_1)=\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1)\tag{5.32}$$

always implies

$$\mathcal{Y}_{\sigma'_n\cdots\sigma'_2,C\alpha}(w^{(i_n)},z'_n;\ldots,w^{(i_1)},z'_1)=\mathcal{Y}_{C\alpha_1}(w^{(i_1)},z'_1)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)},z'_n).\tag{5.33}$$

Corollary 5.7 (Adjoint of generalized intertwining operators)**.** *Let V be unitary. Let $I\in\mathcal{J}$, choose $(z_1,\ldots,z_n)\in\mathscr{O}_n(I)$, and choose arguments $\arg z_1,\arg(z_2-z_1),\ldots,\arg(z_n-z_1)$. Let $W_{i_1},\ldots,W_{i_n}$, and W_i be unitary V-modules, and let $\mathcal{Y}_{\sigma_2},\ldots,\mathcal{Y}_{\sigma_n}$ be a chain of unitary intertwining operators of V satisfying the following conditions:*
(a) for each $2\leqslant m\leqslant n$, the charge space of $\mathcal{Y}_{\sigma_m}$ is W_{i_m};

(b) the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1};

(c) the target space of $\mathcal{Y}_{\sigma_n}$ is W_i.

Then for each $2 \leqslant m \leqslant n$, there exists a unitary intertwining operator $\mathcal{Y}_{\widetilde{\sigma}_m}$ whose type is the same as that of $\mathcal{Y}_{\overline{\sigma_m}}$, such that for any unitary $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, and any nonzero quasi-primary vectors $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}$, we have

$$
\begin{aligned}
&\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\big(w^{(i_n)}, z_n; \dots; w^{(i_1)}, z_1\big)^\dagger \\
=&e^{-i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)} z_1^{2\Delta_{w^{(i_1)}}} \cdots z_n^{2\Delta_{w^{(i_n)}}} \cdot \mathcal{Y}_{\widetilde{\sigma}_n\cdots\widetilde{\sigma}_2,\alpha^*}\big(\overline{w^{(i_n)}}, z_n; \dots; \overline{w^{(i_1)}}, z_1\big),
\end{aligned}
\tag{5.34}
$$

where the formal adjoint is defined for evaluations of the operators between the vectors inside W_j and W_k.

Proof. Let $\arg_I$ be the continuous argument function on I satisfying $\arg_I(z_1) = \arg z_1$. We let $\arg z_2 = \arg_I(z_2), \dots, \arg z_n = \arg_I(z_n)$. Recall that by convention 1.12, we have $\arg \overline{z_1} = -\arg z_1, \arg \overline{z_2} = -\arg z_2, \dots, \arg \overline{z_n} = -\arg z_n$. Let $\arg(\overline{z_2} - \overline{z_1}) = -\arg(z_2 - z_1), \dots, \arg(\overline{z_n} - \overline{z_1}) = -\arg(z_n - z_1)$. By remark 5.6, we can find a chain of unitary intertwining operators $\mathcal{Y}_{\sigma_2'}, \dots, \mathcal{Y}_{\sigma_n'}$ whose types are the same as those of $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ respectively, such that for any chain of intertwining operators $\mathcal{Y}_{\alpha_1}, \dots, \mathcal{Y}_{\alpha_n}$ and any unitary $\mathcal{Y}_\alpha$, if equation (5.32) holds for any $w^{(i_1)} \in W_1, \dots, w^{(i_n)} \in W_{i_n}$, then

$$
\mathcal{Y}_{\sigma_n'\cdots\sigma_2',C\alpha}(w^{(i_n)}, \overline{z_n}; \dots, w^{(i_1)}, \overline{z_1}) = \mathcal{Y}_{C\alpha_1}(w^{(i_1)}, \overline{z_1}) \cdots \mathcal{Y}_{C\alpha_n}(w^{(i_n)}, \overline{z_n}). \tag{5.35}
$$

Now assume that $w^{(i_1)}, \dots, w^{(i_n)}$ are quasi-primary. By equation (1.27), for any $1 \leqslant m \leqslant n$, we have

$$
\mathcal{Y}_{C\alpha_m}(w^{(i_m)}, \overline{z_m}) = e^{-i\pi\Delta_{w^{(i_m)}}} z_m^{2\Delta_{w^{(i_m)}}} \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m)^{\mathrm{t}}. \tag{5.36}
$$

Therefore, by equation (5.32), we see that (5.35) equals

$$
\begin{aligned}
&e^{-i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)} z_1^{2\Delta_{w^{(i_1)}}} \cdots z_n^{2\Delta_{w^{(i_n)}}} \big(\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1)\big)^{\mathrm{t}} \\
=&e^{-i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)} z_1^{2\Delta_{w^{(i_1)}}} \cdots z_n^{2\Delta_{w^{(i_n)}}} \mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\big(w^{(i_n)}, z_n; \dots; w^{(i_1)}, z_1\big)^{\mathrm{t}}.
\end{aligned}
\tag{5.37}
$$

Recall that $\alpha^* = \overline{C\alpha}$. It is obvious that equation

$$C_j^{-1}\mathcal{Y}_{\sigma_n'\cdots\sigma_2',C\alpha}(w^{(i_n)},\overline{z_n};\ldots,w^{(i_1)},\overline{z_1})C_k = \mathcal{Y}_{\overline{\sigma_n'}\cdots\overline{\sigma_2'},\alpha^*}(\overline{w^{(i_n)}},z_n;\ldots;\overline{w^{(i_1)}},z_1) \tag{5.38}$$

holds when $z_1,\ldots,z_n$ also satisfy $0<|z_2-z_1|<\cdots<|z_n-z_1|<|z_1|$. By analytic continuation, it holds for general $(z_1,\ldots,z_n)\in\mathscr{O}_n(I)$. Therefore, if we apply $C_j^{-1}(\cdot)C_k$ to the left hand side of equation (5.35) and the right hand side of equation (5.37), we obtain

$$\begin{aligned}&\mathcal{Y}_{\overline{\sigma_n'}\cdots\overline{\sigma_n'},\alpha^*}(\overline{w^{(i_n)}},z_n;\ldots;\overline{w^{(i_1)}},z_1)\\ =&e^{i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)}z_1^{-2\Delta_{w^{(i_1)}}}\cdots z_n^{-2\Delta_{w^{(i_n)}}}\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\big(w^{(i_n)},z_n;\ldots;w^{(i_1)},z_1\big)^\dagger.\end{aligned} \tag{5.39}$$

So if we let $\mathcal{Y}_{\tilde{\sigma}_2}=\mathcal{Y}_{\overline{\sigma_2'}},\ldots,\mathcal{Y}_{\tilde{\sigma}_2}=\mathcal{Y}_{\overline{\sigma_2'}}$, then equation (5.34) is proved. □

5.3 Generalized smeared intertwining operators

In this section, we assume that V is unitary, energy-bounded, and strongly local. Let $\mathcal{F}$ be a non-empty set of non-zero irreducible unitary V-modules, and let $\overline{\mathcal{F}}=\{W_{\bar{i}}:i\in\mathcal{F}\}$. Let $\mathcal{F}^\boxtimes$ be the collection of unitary V-modules W_i, where W_i is equivalent to a finite direct sum of submodules of tensor products of some V-modules in $\mathcal{F}\cup\overline{\mathcal{F}}$. So $\mathcal{F}^\boxtimes$ is additively closed, and any irreducible element in $\mathcal{F}^\boxtimes$ is equivalent to a submodule of $W_{i_1}\boxtimes\cdots\boxtimes W_{i_n}$, where $i_1,\ldots,i_n\in\mathcal{F}\cup\overline{\mathcal{F}}$. If $i\in\mathcal{F}$, we let $E^1(W_i)$ be the vector space of all quasi-primary vectors $w^{(i)}\in W_i$ satisfying the condition that for any $j,k\in\mathcal{F}^\boxtimes$ and any $\mathcal{Y}_\alpha\in\mathcal{V}\binom{k}{i\ j}$, $\mathcal{Y}_\alpha(w^{(i)},x)$ satisfies linear energy bounds. $E^1(V)$ is defined in a similar way to be the set of all quasi-primary vectors $v\in V$, such that for any $k\in\mathcal{F}^\boxtimes$, $Y_k(v,x)$ satisfies linear energy bounds.

In this section, we always assume, unless otherwise stated, that $\mathcal{F}$ satisfies one of the following two conditions.

Condition A.

(a) Every irreducible submodule of a tensor product of V-modules in $\mathcal{F}\cup\overline{\mathcal{F}}$ is unitarizable.

(b) V is generated by $E^1(V)$.

(c) If $i\in\mathcal{F}$, $j,k\in\mathcal{F}^\boxtimes$, and $\mathcal{Y}_\alpha\in\mathcal{V}\binom{k}{i\ j}$, then $\mathcal{Y}_\alpha$ is energy-bounded.

Condition B.

(a) Every irreducible submodule of a tensor product of V-modules in $\mathcal{F} \cup \overline{\mathcal{F}}$ is unitarizable and energy-bounded.

(b) For any $i \in \mathcal{F}$, $E^1(W_i)$ contains at least one non-zero vector.

Note that if V is unitary and $\mathcal{F}$ satisfies condition A-(b), then by corollary 3.7 and theorem 4.1, V is energy bounded and strongly local. By corollary 3.7, Conditions A-(a),(b) $\Rightarrow$ condition B-(a), and condition B-(b) $\Rightarrow$ A-(c).

Remark 5.8. If $\mathcal{F}$ satisfies condition B, then by theorem 4.8, any unitary V-module W_i in $\mathcal{F}^{\boxtimes}$ is strongly integrable. Now we suppose that $\mathcal{F}$ satisfies condition A. Then, using the same argument as in the proof of theorem 4.8, one can show that any W_i in $\mathcal{F}^{\boxtimes}$ is **almost strongly integrable**, which means the following: Define a real vector subspace $E^1(V)_{\mathbb{R}} = \{v + \theta v, i(v - \theta v) : v \in E^1(V)\}$ of $E^1(V)$. Then there exists a representation π_i of the conformal net $\mathcal{M}_V$ on the $\mathcal{H}_i$, such that for any $I \in \mathcal{J}, v \in E^1(V)_{\mathbb{R}}$, and $f \in C_c^{\infty}(I)$ satisfying that

$$e^{i\pi\Delta_v/2}e_{1-\Delta_v}f = \overline{e^{i\pi\Delta_v/2}e_{1-\Delta_v}f}, \tag{5.40}$$

we have

$$\pi_{i,I}\big(\overline{Y(v,f)}\big) = \overline{Y_i(v,f)}. \tag{5.41}$$

Note that by theorem 4.1, the von Neumann algebra $\mathcal{M}_V(I)$ is generated by these $\overline{Y(v,f)}$'s. Therefore, such representation π_i, if exists, must be unique. In this way, we have a functor $\mathfrak{F} : \mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^{\boxtimes}}(V) \to \mathrm{Rep}_{\mathcal{F}^{\boxtimes}}(\mathcal{M}_V)$ sending the object (W_i, Y_i) to $(\mathcal{H}_i, \pi)$. By proposition 3.6, the conformal vector ν is inside $E^1(V)_{\mathbb{R}}$. Therefore, from their proof we see that theorem 4.3 and corollary 4.4 still hold, with $\mathcal{S}$ replaced by $\mathcal{F}^{\boxtimes}$.

We define $\mathcal{M}_V(I)_\infty$ to be the set of all $x \in \mathcal{M}_V(I)$ satisfying relation (4.5) for any $i \in \mathcal{F}^{\boxtimes}$. We can conclude that $\mathcal{M}_V(I)_\infty$ is a strongly dense self-adjoint subalgebra of $\mathcal{M}_V(I)$, either by using the same argument as in the proof of proposition 4.2, or by observing that every $e^{it\overline{Y(v,f)}}$ is inside $\mathcal{M}_V(I)_\infty$ (by Lemma B.8-(1)), where $t \in \mathbb{R}, v \in E^1(V)_{\mathbb{R}}$, and $f \in C_c^{\infty}(I)$ satisfies equation (5.40).

We now define generalized smeared intertwining operators. First, for any $I \in \mathcal{J}, n = 1, 2, \ldots,$

we choose an arbitrary continuous argument function $\arg_I$ on I, and define $\mathfrak{O}_n(I)$ to be the set of all $(I_1,\dots,I_n)$, where $I_1,\dots,I_n\in\mathcal{J}(I)$ are mutually disjoint, and for any $2\leqslant l<m\leqslant n$, either $\arg_I(z_lz_1^{-1})\arg_I(z_mz_1^{-1})<0$ for all $z_m\in I_m, z_l\in I_l$, or $|\arg_I(z_lz_1^{-1})|<|\arg_I(z_mz_1^{-1})|$ for all $z_m\in I_m, z_l\in I_l$.

Let $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$ be a generalized intertwining operator in $\mathcal{V}\binom{k}{i_n\ \cdots\ i_1\ j}$. We say that $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$ is **controlled by** $\mathcal{F}$ if $i_1,\dots,i_n\in\mathcal{F}\cup\overline{\mathcal{F}}$, and $j,k\in\mathcal{F}^{\boxtimes}$. Choose $I\in\mathcal{J}(S^1\backslash\{-1\})$, $(I_1,\dots,I_n)\in\mathfrak{O}_n(I)$ and $f_1\in C_c^\infty(I_1),\dots,f_n\in C_c^\infty(I_n)$. For any $w^{(i_1)}\in W_{i_1},\dots,w^{(i_n)}\in W_{i_n}$, we define a sesquilinear form

$$\begin{gathered}\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},f_n;\dots;w^{(i_1)},f_1):W_j\times W_k\to\mathbb{C},\\ (w^{(j)},w^{(k)})\mapsto\langle\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},f_n;\dots;w^{(i_1)},f_1)w^{(j)}|w^{(k)}\rangle\end{gathered}$$

using the equation

$$\begin{aligned}&\langle\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},f_n;\dots;w^{(i_1)},f_1)w^{(j)}|w^{(k)}\rangle\\ =&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\langle\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},e^{i\theta_n};\dots;w^{(i_1)},e^{i\theta_1})w^{(j)}|w^{(k)}\rangle\cdot f_1(e^{i\theta_1})\cdots f_n(e^{i\theta_n})d\!\!\!{}^-\theta_1\cdots d\!\!\!{}^-\theta_n,\end{aligned}\tag{5.42}$$

where, for each $l=2,3,\dots,n$, $\arg(e^{i\theta_l}-e^{i\theta_1})$ is close to $\theta_l=\arg e^{i\theta_l}$ as $e^{i\theta_1}\to 0$.

Proposition 5.9. *Assume that $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$ is controlled by $\mathcal{F}$. Then the linear operator $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},f_n;\dots;w^{(i_1)},f_1):W_j\to\widehat{W}_k$ maps W_j into $\mathcal{H}_k^\infty$. If we regard it as an unbounded operator $\mathcal{H}_j\to\mathcal{H}_k$ with domain W_j, then it is preclosed. The closure $\overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},f_n;\dots;w^{(i_1)},f_1)}$ maps $\mathcal{H}_j^\infty$ into $\mathcal{H}_k^\infty$, and its adjoint maps $\mathcal{H}_k^\infty$ into $\mathcal{H}_j^\infty$. Moreover, there exists $p\in\mathbb{Z}_{\geqslant 0}$, such that for any $l\in\mathbb{Z}_{\geqslant 0}$, we can find $C_{l+p}>0$, such that the inequality*

$$\big\|\overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},f_n;\dots;w^{(i_1)},f_1)}\xi^{(j)}\big\|_l\leqslant C_{l+p}\|\xi^{(j)}\|_{l+p}\tag{5.43}$$

holds for any $\xi^{(j)}\in\mathcal{H}_j^\infty$.

Proof. Choose any $z_1\in I_1,\dots,z_n\in I_n$. Choose arguments $\arg z_1,\dots,\arg z_n\in(-\pi,\pi)$. For each $l=2,3,\dots,n$, we let $\arg(z_l-z_1)$ be close to $\arg z_l$ as $z_1\to 0$. Suppose that for any

$w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}$ we have the fusion relation

$$\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, z_n; \ldots; w^{(i_1)}, z_1) = \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) \tag{5.44}$$

for a chain of intertwining operators $\mathcal{Y}_{\alpha_1}, \ldots, \mathcal{Y}_{\alpha_n}$. Then the source spaces and the charge spaces of these intertwining operators are unitary V-modules in $\mathcal{F}^{\boxtimes}$. By condition A-(c) and proposition 3.3, these intertwining operators are energy-bounded. It follows from proposition 3.12 that

$$\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) = \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) \tag{5.45}$$

when both sides act on W_j. Therefore, by equation (3.25), the adjoint of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)$ has a dense domain containing $\mathcal{H}_k^\infty$, which proves that $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)$ is preclosed. By proposition 3.9, there exists $p \in \mathbb{Z}_{\geqslant 0}$, such that for any $l \in \mathbb{Z}_{\geqslant 0}$, there exists $C_{l+p} > 0$, such that inequality (5.43) holds for any $\xi^{(j)} \in W_j$. From this we know that $\mathcal{H}_j^\infty$ is inside the domain of $\overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)}$, that this closed operator maps $\mathcal{H}_j^\infty$ into $\mathcal{H}_k^\infty$, and that inequality (5.43) holds for any $\xi^{(j)} \in \mathcal{H}_j^\infty$. Clearly we have

$$\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)^* \supset \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)^\dagger \cdots \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)^\dagger.$$

So $\overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)}^*$ maps $\mathcal{H}_k^\infty$ into $\mathcal{H}_j^\infty$. $\square$

We regard the linear operator $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) : \mathcal{H}_j^\infty \to \mathcal{H}_k^\infty$ as the restriction of $\overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)}$ to $\mathcal{H}_j^\infty$, and call it a **generalized smeared intertwining operator**. Then, if the fusion relation (5.44) holds, relation (5.45) holds when both sides act on $\mathcal{H}_j^\infty$. The **formal adjoint** $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)^\dagger : \mathcal{H}_k^\infty \to \mathcal{H}_j^\infty$ of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)$ is defined to be the restriction of the closed operator $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)^*$ to $\mathcal{H}_k^\infty$.

Proposition 5.10 (Intertwining property)**.** *Let $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha} \in \mathcal{V}\binom{k}{i_n \, \cdots \, i_1 \; j}$ be controlled by $\mathcal{F}$, $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}$, $I \in \mathcal{J}, J \in \mathcal{J}(S^1 \backslash \{-1\})$ be disjoint, and $(J_1, \ldots, J_n) \in \mathfrak{O}_n(J)$. If $\mathcal{F}$ satisfies condition A, then for any $x \in \mathcal{M}_V(I), w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, g_1 \in C_c^\infty(J_1), \ldots, g_n \in$*

$C_c^\infty(J_n)$*, we have*

$$\pi_k(x) \cdot \overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, g_n; \ldots; w^{(i_1)}, g_1)} \subset \overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, g_n; \ldots; w^{(i_1)}, g_1)} \cdot \pi_j(x). \quad (5.46)$$

Relation (5.46) *still holds if we assume that $\mathcal{F}$ satisfies condition B, and that $x \in \mathcal{M}_V(I), w^{(i_1)} \in E^1(W_{i_1}), \ldots, w^{(i_n)} \in E^1(W_{i_n}), g_1 \in C_c^\infty(J_1), \ldots, g_n \in C_c^\infty(J_n)$.*

Proof. We assume that the fusion relation (5.44) holds when $z_1 \in J_1, \ldots, z_n \in J_n$ and the arguments are chosen as in the proof of proposition 5.9.

First, suppose that $\mathcal{F}$ satisfies condition A. By theorem 4.1, the von Neumann algebra $\mathcal{M}_V(I)$ is generated by the bounded operators $e^{it\overline{Y(v,f)}}$, where $t \in \mathbb{R}, v \in E^1(V)_{\mathbb{R}}, f \in C_c^\infty(I)$, and $e^{i\pi\Delta_v/2} e_{1-\Delta_v} f = \overline{e^{i\pi\Delta_v/2} e_{1-\Delta_v} f}$. Now for $m = 1, 2, \ldots, n$ we let $W_{j_{m-1}}$ and W_{j_m} be the source space and the target space of $\mathcal{Y}_{\alpha_m}$ respectively. Then by proposition 3.16 (and proposition B.1), for any $x \in \mathcal{M}_V(I), w^{(i_m)} \in W_{i_m}, g_m \in C_c^\infty(J_m)$, we have

$$\pi_{j_m}(x)\overline{\mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m)} \subset \overline{\mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m)}\pi_{j_{m-1}}(x). \quad (5.47)$$

Therefore, if $x \in \mathcal{M}_V(I)_\infty$, then equation

$$\pi_{j_m}(x)\mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m)\pi_{j_{m-1}}(x). \quad (5.48)$$

holds when both sides act on $\mathcal{H}_{j_{m-1}}^\infty$. Thus, by (5.45), for any $x \in \mathcal{M}_V(I)_\infty$, equation

$$\pi_k(x) \cdot \mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, g_n; \ldots; w^{(i_1)}, g_1) = \mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, g_n; \ldots; w^{(i_1)}, g_1) \cdot \pi_j(x) \quad (5.49)$$

also holds when both sides act on $\mathcal{H}_j^\infty$. This proves relation (5.46) for any $x \in \mathcal{M}_V(I)_\infty$, and hence for any $x \in \mathcal{M}_V(I)$.

Now we assume that $\mathcal{F}$ satisfies condition B. Then from step 2 of the proof of theorem 4.8, relation (5.47) holds for any $x \in \mathcal{M}_V(I)$. This again implies relation (5.46). Thus we are done with the proofs for both cases. $\square$

Proposition 5.11 (Rotation covariance). *Let $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha} \in \mathcal{V}\binom{k}{i_n \, \cdots \, i_1 \, j}$ be controlled by $\mathcal{F}$, $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}$ be homogeneous, $J \in S^1 \backslash \{-1\}$, and $(J_1, \ldots, J_n) \in \mathfrak{O}_n(J)$. Choose $\varepsilon > 0$*

such that $\mathfrak{r}(t)J \subset S^1\backslash\{-1\}$. *Then for any* $g_1 \in C_c^\infty(J_1), \ldots, g_n \in C_c^\infty(J_n)$, *and* $t \in (-\varepsilon, \varepsilon)$, *we have*

$$
\begin{aligned}
&e^{it\overline{L_0}} \cdot \overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, g_n; \ldots; w^{(i_1)}, g_1)} \cdot e^{-it\overline{L_0}}\\
=&\overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\big(w^{(i_n)}, e^{i(\Delta_{w^{(i_n)}}-1)t}\mathfrak{r}(t)g_n; \ldots; w^{(i_1)}, e^{i(\Delta_{w^{(i_1)}}-1)t}\mathfrak{r}(t)g_1\big)}.
\end{aligned} \tag{5.50}
$$

Proof. This follows from relations (5.45) and (3.39). □

Theorem 5.12 (Braiding). *Let* $I, J \in \mathcal{J}(S^1\backslash\{-1\})$ *be disjoint. Choose* $(I_1, \ldots, I_m) \in \mathfrak{O}_m(I), (J_1, \ldots, J_n) \in \mathfrak{O}_n(J)$. *Choose* $z \in I, \zeta \in J$, *and let* $-\pi < \arg z, \arg\zeta < \pi$. *Let* $\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha} \in \mathcal{V}\binom{k'}{i_m\ \ldots\ i_1\ k_1}, \mathcal{Y}_{\rho_n\cdots\rho_2,\beta} \in \mathcal{V}\binom{k_1}{j_n\ \ldots\ j_1\ k}, \mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'} \in \mathcal{V}\binom{k_2}{i_m\ \ldots\ i_1\ k}, \mathcal{Y}_{\rho_n\cdots\rho_2,\beta'} \in \mathcal{V}\binom{k'}{j_n\ \ldots\ j_1\ k_2}$ *be generalized intertwining operators of* V *controlled by* $\mathcal{F}$. *Suppose that* W_i *is the charge spaces of* $\mathcal{Y}_\alpha$ *and* $\mathcal{Y}_{\alpha'}$, W_j *is the charge space of* $\mathcal{Y}_\beta$ *and* $\mathcal{Y}_{\beta'}$, *and for any* $w^{(i)} \in W_i, w^{(j)} \in W_j$, *we have the braid relation*

$$
\mathcal{Y}_\alpha(w^{(i)}, z)\mathcal{Y}_\beta(w^{(j)}, \zeta) = \mathcal{Y}_{\beta'}(w^{(j)}, \zeta)\mathcal{Y}_{\alpha'}(w^{(i)}, z). \tag{5.51}
$$

Then for any $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \ldots, w^{(j_n)} \in W_{j_n}, f_1 \in C_c^\infty(I_1), \ldots f_m \in C_c^\infty(I_m), g_1 \in C_c^\infty(J_1), \ldots, g_n \in C_c^\infty(J_n)$, *we have the braid relation*

$$
\begin{aligned}
&\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1)\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\\
=&\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1).
\end{aligned} \tag{5.52}
$$

Proof. Choose $z_1 \in I_1, \ldots, z_m \in I_m, \zeta_1 \in J_1, \ldots, \zeta_n \in J_n$. Let $-\pi < \arg z_1, \ldots, \arg z_m, \arg\zeta_1, \ldots, \arg\zeta_n < \pi$, and let $\arg(z_2 - z_1), \ldots, \arg(z_m - z_1), \arg(\zeta_2 - \zeta_1), \ldots, \arg(\zeta_n - \zeta_1)$ be close to $\arg z_2, \ldots, \arg z_m, \arg\zeta_2, \ldots, \arg\zeta_n$ as $z_1, \ldots, z_1, \zeta_1, \ldots, \zeta_1 \to 0$ respectively. Suppose that for any $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \ldots, w^{(j_n)} \in W_{j_n}$, we have the fusion relations

$$
\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, z_m; \ldots; w^{(i_1)}, z_1) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1), \tag{5.53}
$$

$$
\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, \zeta_n; \ldots; w^{(j_1)}, \zeta_1) = \mathcal{Y}_{\beta_n}(w^{(j_n)}, \zeta_n)\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, \zeta_1). \tag{5.54}
$$

Then the source spaces and the target spaces of $\mathcal{Y}_{\alpha_1}, \ldots \mathcal{Y}_{\alpha_m}, \mathcal{Y}_{\beta_1}, \ldots, \mathcal{Y}_{\beta_n}$ are unitary V-modules inside $\mathcal{F}^{\boxtimes}$. So these intertwining operators are energy-bounded. By relation (5.45), we have

$$\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, f_m)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1), \tag{5.55}$$

$$\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1) = \mathcal{Y}_{\beta_n}(w^{(j_n)}, g_n)\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, g_1). \tag{5.56}$$

Therefore, by proposition 3.12,

$$\begin{aligned}
&\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1)\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\\
=&\mathcal{Y}_{\alpha_m}(w^{(i_m)}, f_m)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)\mathcal{Y}_{\beta_n}(w^{(j_n)}, g_n)\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, g_1)\\
=&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdot\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathcal{Y}_{\alpha_m}(w^{(i_m)}, e^{i\theta_m})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, e^{i\theta_1})\\
&\cdot\mathcal{Y}_{\beta_n}(w^{(j_n)}, e^{i\vartheta_n})\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, e^{i\vartheta_1})f_1(e^{i\theta_1})\cdots f_m(e^{i\theta_m})\\
&\cdot g_1(e^{i\vartheta_1})\cdots g_n(e^{i\vartheta_n})d\!\!\!/\theta_1\cdots d\!\!\!/\theta_m d\!\!\!/\vartheta_1\cdots d\!\!\!/\vartheta_n\\
=&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdot\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, e^{i\theta_m};\cdots;w^{(i_1)}, e^{i\theta_1})\\
&\cdot\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, e^{i\vartheta_n};\cdots;w^{(j_1)}, e^{i\vartheta_1})f_1(e^{i\theta_1})\cdots f_m(e^{i\theta_m})\\
&\cdot g_1(e^{i\vartheta_1})\cdots g_n(e^{i\vartheta_n})d\!\!\!/\theta_1\cdots d\!\!\!/\theta_m d\!\!\!/\vartheta_1\cdots d\!\!\!/\vartheta_n.
\end{aligned}\tag{5.57}$$

The same argument shows that

$$\begin{aligned}
&\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1)\\
=&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdot\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)}, e^{i\vartheta_n};\cdots;w^{(j_1)}, e^{i\vartheta_1})\\
&\cdot\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)}, e^{i\theta_m};\cdots;w^{(i_1)}, e^{i\theta_1})f_1(e^{i\theta_1})\cdots f_m(e^{i\theta_m})\\
&\cdot g_1(e^{i\vartheta_1})\cdots g_n(e^{i\vartheta_n})d\!\!\!/\theta_1\cdots d\!\!\!/\theta_m d\!\!\!/\vartheta_1\cdots d\!\!\!/\vartheta_n.
\end{aligned}\tag{5.58}$$

By theorem 5.2, the right hand sides of equations (5.57) and (5.58) are equal, which proves equation (5.52). □

Theorem 5.13 (Adjoint relation)**.** *Choose $I \in \mathcal{J}(S^1\backslash\{-1\})$ and $(I_1, \ldots, I_n) \in \mathfrak{O}_n(I)$. Let $W_{i_1}, W_{i_2}, \ldots, W_{i_n}$ be unitary V-modules in $\mathcal{F} \cup \overline{\mathcal{F}}$, and let $\mathcal{Y}_{\sigma_2}, \ldots, \mathcal{Y}_{\sigma_n}$ be a chain of unitary* *intertwining operators of V with charge spaces $W_{i_2}, \ldots, W_{i_n}$ respectively, such that the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1}. We let $W_i \in \mathcal{F}^{\boxtimes}$ be the target space of $\mathcal{Y}_{\sigma_n}$. Then for each $2 \leqslant m \leqslant n$, there exists a unitary intertwining operator $\mathcal{Y}_{\tilde{\sigma}_m}$ whose type is the same as that of $\mathcal{Y}_{\overline{\sigma_m}}$, such that for any unitary V-modules W_j, W_k in $\mathcal{F}^{\boxtimes}, \mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}$ being quasi-primary, and $f_1 \in C_c^\infty(I_1), \ldots, f_n \in C_c^\infty(I_n)$, we have*

$$\begin{aligned}&\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}\big(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1\big)^\dagger \\ =&e^{-i\pi\left(\Delta_{w^{(i_1)}} + \cdots + \Delta_{w^{(i_n)}}\right)} \cdot \mathcal{Y}_{\tilde{\sigma}_n \cdots \tilde{\sigma}_2, \alpha^*}\big(\overline{w^{(i_n)}}, \overline{e_{(2-2\Delta_{w^{(i_n)}})} f_n}; \ldots; \overline{w^{(i_1)}}, \overline{e_{(2-2\Delta_{w^{(i_1)}})} f_1}\big).\end{aligned} \tag{5.59}$$

Proof. This is obtained by multiplying both sides of equation (5.34) by the expression

$$\overline{f_1(e^{i\theta_1})} \cdots \overline{f_n(e^{i\theta_n})} e^{-2i(\theta_1 + \cdots + \theta_n)} d\!\!\!{}^{-}\theta_1 \cdots d\!\!\!{}^{-}\theta_n,$$

and then taking the integral. We leave the details to the reader. $\square$

CHAPTER 6

DEFINING AN INNER PRODUCT Λ ON $W_i \boxtimes W_j$

In this chapter, we define (in section 2) a sesquilinear form Λ on $W_{ij} = W_i \boxtimes W_j$ using transport matrices, and prove (in section 3) that these forms are inner products. As discussed in the introduction of part I, our strategy for proving the positivity of Λ is to identify the form Λ on a dense subspace of $\mathcal{H}_{ij}$ with the inner product on a subspace of the Connes fusion product $\mathcal{H}_i \boxtimes \mathcal{H}_j$ of the conformal net modules $\mathcal{H}_i$ and $\mathcal{H}_j$. In section 1, we prove a density property for constructing such a dense subspace.

Note that the Connes fusion product (Connes relative tensor product) is a motivation rather than a logistic background of our theory. So we don't assume the reader has any previous knowledge on this topic, nor shall we give a formal definition on Connes fusion in this paper. Those who are interested in this topic can read [Was98] section 30 for a brief introduction, or read [Con80] or [Tak13] section IX.3 for more details.

6.1 Density of the range of fusion product

Recall from section 3.2 that $W_{ij} = W_i \boxtimes W_j = \bigoplus_{k\in\mathcal{E}} \mathcal{V}{k \choose i\ j}^* \otimes W_k$ is the tensor product module of W_i, W_j. We now define a type ${ij \choose i\ j}$ intertwining operator $\mathcal{Y}_{i\boxtimes j} : W_i \otimes W_j \to W_{ij}\{x\}$ in the following way: If $\mathcal{Y}_\alpha \in \mathcal{V}{k \choose i\ j}$, $w^{(i)} \in W_i$, $w^{(j)} \in W_j$ and $w^{(\overline{k})} \in W_{\overline{k}}$, then

$$\langle \mathcal{Y}_\alpha \otimes w^{(\overline{k})}, \mathcal{Y}_{i\boxtimes j}(w^{(i)}, x)w^{(j)} \rangle = \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)} \rangle. \tag{6.1}$$

For any $k \in \mathcal{E}$, we choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta^k_{ij}\}$ of $\mathcal{V}{k \choose i\ j}$, and let $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Theta^k_{ij}\} \subset \mathcal{V}{k \choose i\ j}^*$ be the dual basis of Θ^k_{ij}. (i.e., if $\alpha, \beta \in \Theta^k_{ij}$, then $\langle \mathcal{Y}_\alpha, \check{\mathcal{Y}}^\beta \rangle = \delta_{\alpha,\beta}$.) Then for any $w^{(i)} \in W_i$ and $w^{(j)} \in W_j$ we have

$$\mathcal{Y}_{i\boxtimes j}(w^{(i)}, x)w^{(j)} = \sum_{k\in\mathcal{E}} \sum_{\alpha\in\Theta^k_{ij}} \check{\mathcal{Y}}^\alpha \otimes \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)} = \sum_{\alpha\in\Theta^*_{ij}} \check{\mathcal{Y}}^\alpha \otimes \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)}. \tag{6.2}$$

(See the beginning of section 2 for notations.)

The following density property generalizes proposition A.3.

Proposition 6.1. *Let* $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ *be a chain of non-zero irreducible intertwining operators of* V *with charge spaces* $W_{i_2}, \dots, W_{i_n}$ *respectively. Let* W_{i_1} *be the source space of* $\mathcal{Y}_{\sigma_2}$*, and let* W_i *be the target space of* $\mathcal{Y}_{\sigma_n}$*. Choose a* V*-module* W_j*, non-zero vectors* $w_0^{(i_1)} \in W_{i_1}, \dots, w_0^{(i_n)} \in W_{i_n}$*,* $I \in \mathcal{J}, (z_1, \dots, z_n) \in \mathscr{O}_n(I)$*, and choose arguments* $\arg z_1, \arg(z_2 - z_1), \dots, \arg(z_n - z_1)$*. Fix* $w^{(\overline{ij})} \in W_{\overline{ij}}$*. Suppose that for any* $w^{(j)} \in W_j$*,*

$$\langle w^{(\overline{ij})}, \mathcal{Y}_{\sigma_n \cdots \sigma_2, i \boxtimes j}(w_0^{(i_n)}, z_n; \dots ; w_0^{(i_1)}, z_1) w^{(j)} \rangle = 0, \tag{6.3}$$

then $w^{(\overline{ij})} = 0$*.*

Proof. Suppose that equation (6.3) holds. From the proof of corollary 2.15, we see that

$$\langle w^{(\overline{ij})}, \mathcal{Y}_{\sigma_n \cdots \sigma_2, i \boxtimes j}(w^{(i_n)}, z_n; \dots ; w^{(i_1)}, z_1) w^{(j)} \rangle = 0 \tag{6.4}$$

for all $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}, w^{(i)} \in W_i$. By theorem 2.4 and the discussion below, equation (6.4) holds for all $(z_1, \dots, z_n) \in \mathscr{O}_n(I)$ (the arguments $\arg z_1, \arg(z_2 - z_1), \dots, \arg(z_n - z_1)$ are changed continuously). In particular, for any $(z_1, \dots, z_n) \in \mathscr{O}_n(I)$ satisfying $0 < |z_2 - z_1| < |z_3 - z_1| < \cdots < |z_n - z_1| < |z_1|$, equation (6.4) reads

$$\Big\langle w^{(\overline{ij})}, \mathcal{Y}_{i \boxtimes j}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1\big) w^{(j)} \Big\rangle = 0. \tag{6.5}$$

If we let z_2 be close to z_1, then by proposition A.1, for any $s_2 \in \mathbb{R}$, we have

$$\Big\langle w^{(\overline{ij})}, \mathcal{Y}_{i \boxtimes j}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1) \mathcal{Y}_{\sigma_2}(w^{(i_2)}, s_2) w^{(i_1)}, z_1\big) w^{(j)} \Big\rangle = 0, \tag{6.6}$$

where $\mathcal{Y}_{\sigma_2}(w^{(i_2)}, s_2)$ is a mode of the intertwining operator $\mathcal{Y}_{\sigma_2}(w^{(i_2)}, x)$. Let W_{j_2} be the target space of $\mathcal{Y}_{\sigma_2}$ (which is also the source space of $\mathcal{Y}_{\sigma_3}$). Then by corollary A.4, vectors of the form

$\mathcal{Y}_{\sigma_2}(w^{(i_2)}, s_2)w^{(i_1)}$ span the vector space W_{j_2}. Therefore, for any $w^{(j_2)} \in W_{j_2}$, we have

$$\big\langle w^{(\overline{ij})}, \mathcal{Y}_{i\boxtimes j}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1)w^{(j_2)}, z_1\big)w^{(j)}\big\rangle = 0. \tag{6.7}$$

If we apply the same argument several times, then for any $w^{(i)} \in W_i, w^{(j)} \in W_j$,

$$\langle w^{(\overline{ij})}, \mathcal{Y}_{i\boxtimes j}(w^{(i)}, z_1)w^{(j)}\rangle = 0. \tag{6.8}$$

So by proposition A.3, $w^{(\overline{ij})}$ must be zero. $\square$

A smeared version of the above proposition can be stated as follows.

Proposition 6.2. *Let V be unitary, energy-bounded, and strongly local. Let $\mathcal{F}$ be a non-empty set of non-zero irreducible unitary V-modules satisfying condition A or B. Let W_i, W_j be unitary V-modules in $\mathcal{F}^{\boxtimes}$, and assume that W_i is irreducible. Fix an arbitrary unitary structure on W_{ij}.*

Let $W_{i_1}, \dots, W_{i_n}$ be irreducible unitary V-modules in $\mathcal{F} \cup \overline{\mathcal{F}}$. Let $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ be a chain of non-zero irreducible unitary intertwining operators of V with charge spaces $W_{i_2}, \dots, W_{i_n}$ respectively, such that W_{i_1} is the source space of $\mathcal{Y}_{\sigma_2}$, and W_i is the target space of $\mathcal{Y}_{\sigma_n}$. Choose $I \in \mathcal{J}(S^1\backslash\{-1\}), (I_1, \dots, I_n) \in \mathfrak{O}_n(I)$. Fix non-zero homogeneous vectors $w_0^{(i_1)} \in W_{i_1}, \dots, w_0^{(i_n)} \in W_{i_n}$. Then for any $l \in \mathbb{Z}_{\geqslant 0}$, vectors of the form

$$\pi_{ij}(x)\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n; \dots; w_0^{(i_1)}, f_1)w^{(j)}, \tag{6.9}$$

span a core for $\overline{L_0}^l$, where $x \in \mathcal{M}_V(I)_\infty, f_1 \in C_c^\infty(I_1), \dots, f_n \in C_c^\infty(I_n), w^{(j)} \in W_j$.

Proof. Let $\mathcal{W}_1$ be the subspace of $\mathcal{H}_{ij}^\infty$ spanned by vectors of the form (6.9). We first show that $\mathcal{W}_1$ is a dense subspace of $\mathcal{H}_{ij}$.

The first step is to show that $\mathcal{W}_1^\perp$ is invariant under the action of the conformal net $\mathcal{M}_V$. Choose an open interval $J \subset\subset I$, and choose $\delta > 0$ such that $\mathfrak{r}(t)J \subset I$ for any $t \in (-\delta, \delta)$. Fix $\xi^{(ij)} \in \mathcal{W}_1^\perp$. Then for any $w^{(j)} \in W_j, m \in \mathbb{Z}_{>0}, x_1, \dots, x_m \in \mathcal{M}_V(J)_\infty, f_1 \in C_c^\infty(I_1), \dots, f_n \in C_c^\infty(I_n)$, we have

$$\langle x_m \cdots x_2 x_1 \mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n; \dots; w_0^{(i_1)}, f_1)w^{(j)}|\xi^{(ij)}\rangle = 0. \tag{6.10}$$

Choose $\varepsilon > 0$ such that the support of

$$f_a^t = \exp(it(\Delta_{w_0^{(i_a)}} - 1))\mathfrak{r}(t)f_a$$

is inside I_a for any $t \in (-\varepsilon, \varepsilon)$ and any $a = 1, 2, \ldots, n$. Then, by proposition 5.11, for any $t \in \mathbb{R}$ we have

$$\begin{aligned}&\langle x_m \cdots x_1 \cdot e^{it\overline{L_0}}\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1)w^{(j)}|\xi^{(ij)}\rangle\\ =&\langle x_m \cdots x_1\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n^t; \ldots; w_0^{(i_1)}, f_1^t)e^{it\overline{L_0}}w^{(j)}|\xi^{(ij)}\rangle,\end{aligned} \tag{6.11}$$

which must be zero when $t \in (-\varepsilon, \varepsilon)$. Therefore, as in step 1 of the proof of theorem 4.8, the Schwarz reflection principle implies that (6.11) equals zero for any $t \in \mathbb{R}$. (Note that when we define generalized smeared intertwining operators, the arguments are restricted to $(-\pi, \pi)$. Here we allow the arguments to exceed $(-\pi, \pi)$ and change continuously according to the action of $\mathfrak{r}(t)$.) Hence we conclude that equation (6.10) holds for any $t \in \mathbb{R}, w^{(j)} \in W_j, x_1, \ldots, x_m \in \mathcal{M}_V(J)_\infty, f_1 \in C_c^\infty(\mathfrak{r}(t)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t)I_n)$.

We use similar argument once more. Choose any $w^{(j)} \in W_j, t_0, t \in \mathbb{R}, x_1, \ldots, x_m \in \mathcal{M}_V(J)_\infty, f_1 \in C_c^\infty(\mathfrak{r}(t_0)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t_0)I_n)$. Then by proposition 5.11 and equation (4.6), we have

$$\begin{aligned}&\langle x_m \cdots x_2 \cdot e^{it\overline{L_0}}\pi_{ij}(x_1)\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1)w^{(j)}|\xi^{(ij)}\rangle\\ =&\langle x_m \cdots x_2 \cdot e^{it\overline{L_0}}\pi_{ij}(x_1)e^{-it\overline{L_0}}\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n^t; \ldots; w_0^{(i_1)}, f_1^t)e^{it\overline{L_0}}w^{(j)}|\xi^{(ij)}\rangle\\ =&\langle x_m \cdots x_2 \cdot \pi_{ij}(e^{it\overline{L_0}}x_1e^{-it\overline{L_0}})\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n^t; \ldots; w_0^{(i_1)}, f_1^t)e^{it\overline{L_0}}w^{(j)}|\xi^{(ij)}\rangle.\end{aligned} \tag{6.12}$$

If $t \in (-\delta, \delta)$, then $e^{it\overline{L_0}}x_1e^{-it\overline{L_0}} \in \mathcal{M}_V(\mathfrak{r}(t)J)_\infty \subset \mathcal{M}_V(I)_\infty$, and hence (6.12) must be zero. So the value of (6.12) equals zero when $t \in (-\delta, \delta)$. By Schwarz reflection principle, (6.12) equals zero for any $t \in \mathbb{R}$. Since the choice of t_0 is arbitrary, we conclude that equation (6.10) holds for any $t_0, t_1 \in \mathbb{R}, x_1 \in \mathcal{M}_V(\mathfrak{r}(t_1)J)_\infty, x_2 \in \mathcal{M}_V(J)_\infty, \ldots, x_m \in \mathcal{M}_V(J)_\infty, w^{(j)} \in W_j, f_1 \in C_c^\infty(\mathfrak{r}(t_0)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t_0)I_n)$. The same argument shows that equation 6.10 holds for any $t_0, t_1, t_2, \ldots, t_m \in \mathbb{R}, w^{(j)} \in W_j, x_1 \in \mathcal{M}_V(\mathfrak{r}(t_1)J)_\infty, x_2 \in \mathcal{M}_V(\mathfrak{r}(t_2)J)_\infty, \ldots, x_m \in$

$\mathcal{M}_V(\mathfrak{r}(t_m)J)_\infty, f_1 \in C_c^\infty(\mathfrak{r}(t_0)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t_0)I_n)$. Hence, by proposition 4.2 and the additivity of $\mathcal{M}_V$, the equation

$$\langle x_m \cdots x_1 \mathcal{Y}_{\sigma_n\cdots\sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1)w^{(j)}|\xi^{(ij)}\rangle = 0 \tag{6.13}$$

holds for any $m \in \mathbb{Z}_{\geqslant 0}, J_1, \ldots, J_m \in \mathcal{J}, x_1 \in \mathcal{M}_V(J_1), \ldots, x_m \in \mathcal{M}_V(J_m), f_1 \in C_c^\infty(I_1), \ldots, f_n \in C_c^\infty(I_n), w^{(j)} \in W_j, \xi^{(ij)} \in \mathcal{W}_1^\perp$. This proves that $\mathcal{W}_1^\perp$ is $\mathcal{M}_V$-invariant.

Now suppose that $\mathcal{W}_1^\perp$ is non-trivial. By corollary 4.4 and remark 5.8, $\mathcal{W}_1^\perp$ is the closure of a non-trivial V-submodule of W_{ij}. Thus there exists a non-zero vector $w^{(ij)} \in W_{ij} \cap \mathcal{W}_1^\perp$. For any $f_1 \in C_c^\infty(I_1), \ldots, f_n \in C_c^\infty(I_n), w^{(j)} \in W_j$, we have

$$\langle \mathcal{Y}_{\sigma_n\cdots\sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1)w^{(j)}|w^{(ij)}\rangle = 0. \tag{6.14}$$

Fix $z_1 \in I_1, \ldots, z_n \in I_n$. For each $1 \leqslant m \leqslant n$ we let f_m converge to the δ-function at z_m. Then we have

$$\langle \mathcal{Y}_{\sigma_n\cdots\sigma_2, i\boxtimes j}(w_0^{(i_n)}, z_n; \ldots; w_0^{(i_1)}, z_1)w^{(j)}|w^{(ij)}\rangle = 0 \tag{6.15}$$

for any $w^{(j)} \in W_j$. By proposition 6.1, $w^{(ij)}$ equals zero, which is impossible. So $\mathcal{W}_1$ must be dense.

Now we show that $\mathcal{W}_1$ is a core for $\overline{L_0}^l$. Choose an open interval $K \subset\subset I$, and $(K_1, \ldots, K_n) \in \mathfrak{O}_n(K)$, such that $K_1 \subset\subset I_1, \ldots, K_n \subset\subset I_n$. Let $\mathcal{W}_2$ be the subspace of $\mathcal{H}_{ij}^\infty$ spanned by vectors of the form

$$\pi_{ij}(x)\mathcal{Y}_{\sigma_n\cdots\sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1)w^{(j)},$$

where $x \in \mathcal{M}_V(K)_\infty, f_1 \in C_c^\infty(K_1), \ldots, f_n \in C_c^\infty(K_n), w^{(j)} \in W_j$. Then clearly $\mathcal{W}_2$ is also dense in $\mathcal{H}_{ij}$. Choose $\epsilon > 0$ such that for any $t \in (-\epsilon, \epsilon)$, $\mathfrak{r}(t)K \subset I, \mathfrak{r}(t)K_1 \subset I_1, \ldots, \mathfrak{r}(t)K_n \subset I_n$. Then by proposition 5.11, $e^{it\overline{L_0}}\mathcal{W}_2 \subset \mathcal{W}_1$. Hence, by the next lemma, $\mathcal{W}_1$ is a core for $\overline{L_0}^l$. □

Lemma 6.3 (cf. [CKLW15] lemma 7.2.)**.** *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $U(t) = e^{itA}$, $t \in \mathbb{R}$ be the corresponding strongly-continuous one-parameter group of unitary* *operators on $\mathcal{H}$. For any $k \in \mathbb{Z}_{\geqslant 0}$, let $\mathcal{H}^k$ denote the domain of A^k, and let $\mathcal{H}^\infty = \bigcap_{k \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^k$. Assume that there exists a real number $\epsilon > 0$ and two dense linear subspaces $\mathscr{D}_\epsilon$ and $\mathscr{D}$ of $\mathcal{H}^\infty$ such that $U(t)\mathscr{D}_\epsilon \subset \mathscr{D}$ for any $t \in (-\epsilon, \epsilon)$. Then, for every positive integer k, $\mathscr{D}$ is a core for A^k.*

6.2 The sesquilinear form Λ on $W_i \boxtimes W_j$

Beginning with this section, we assume that V is unitary, energy bounded, and strongly local, and that there exists a non-empty set $\mathcal{F}$ of non-zero irreducible unitary V-modules satisfying condition A or B.

Choose unitary V-modules W_i, W_j in $\mathcal{F}^{\boxtimes}$. We now define, for any $k \in \mathcal{E}$, a sesquilinear form $\Lambda = \Lambda(\cdot|\cdot)$ on $\mathcal{V}\binom{k}{i\,j}^*$ (antilinear on the second variable). Choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta^k_{ij}\}$ of $\mathcal{V}\binom{k}{i\,j}$. Choose $z_1, z_2 \in \mathbb{C}^\times$ satisfying $0 < |z_2 - z_1| < |z_1| < |z_2|$. Choose $\arg z_2$, let $\arg z_1$ be close to $\arg z_2$ as $z_2 - z_1 \to 0$, and let $\arg(z_2 - z_1)$ be close to $\arg z_2$ as $z_1 \to 0$. By fusion of intertwining operators, there exists a complex $N^k_{ij} \times N^k_{ij}$ matrix $\Lambda = \{\Lambda^{\alpha\beta}\}_{\alpha,\beta \in \Theta^k_{ij}}$, such that for any $w^{(i)}_1, w^{(i)}_2 \in W_i$ we have the following **transport formula** (version 1):

$$\begin{aligned}
&Y_j\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2 - z_1) w^{(i)}_1, z_1\big)\\
=&\sum_{k \in \mathcal{E}} \sum_{\alpha,\beta \in \Theta^k_{ij}} \Lambda^{\alpha\beta} \mathcal{Y}_{\beta*}(\overline{w^{(i)}_2}, z_2) \mathcal{Y}_\alpha(w^{(i)}_1, z_1)\\
=&\sum_{\alpha,\beta \in \Theta^*_{ij}} \Lambda^{\alpha\beta} \mathcal{Y}_{\beta*}(\overline{w^{(i)}_2}, z_2) \mathcal{Y}_\alpha(w^{(i)}_1, z_1).
\end{aligned} \tag{6.16}$$

The matrix Λ is called a **transport matrix** of V. Let $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Theta^k_{ij}\}$ be the dual basis of Θ^k_{ij}. We then define a sesquilinear form $\Lambda(\cdot|\cdot)$ on $\mathcal{V}\binom{k}{i\,j}^*$ by setting

$$\Lambda(\check{\mathcal{Y}}^\alpha | \check{\mathcal{Y}}^\beta) = \Lambda^{\alpha\beta}. \tag{6.17}$$

It is easy to see that this definition does not depend on the basis chosen. These sesquilinear forms induce one on the vector space $W_i \boxtimes W_j = \bigoplus_{k \in \mathcal{E}} \mathcal{V}\binom{k}{i\,j}^* \otimes W_k$: if $k_1, k_2 \in \mathcal{E} \cap \mathcal{F}^{\boxtimes}, \check{\mathcal{Y}}_1 \in$

$\mathcal{V}\binom{k_1}{i\ j}^*, \check{\mathcal{Y}}_2 \in \mathcal{V}\binom{k_2}{i\ j}^*, w^{(k_1)} \in W_{k_1}, w^{(k_2)} \in W_{k_2}$, then

$$\Lambda\big(\check{\mathcal{Y}}_1 \otimes w^{(k_1)} \big| \check{\mathcal{Y}}_2 \otimes w^{(k_2)}\big) = \begin{cases} \Lambda(\check{\mathcal{Y}}_1|\check{\mathcal{Y}}_2)\langle w^{(k_1)}|w^{(k_2)}\rangle & \text{if } k_1 = k_2, \\ 0 & \text{if } k_1 \neq k_2. \end{cases} \tag{6.18}$$

In the next section, we will prove that Λ is an inner product.

Remark 6.4. Our definition of transport formulas is motivated by A.Wassermann (cf. [Was98] section 31). In that paper, transport formulas were defined, using smeared intertwining operators, only when the fusion rules are at most 1. So transport matrices become transport coefficients. Proving the strict positivity of these coefficients is one of the key steps to compute the *Connes* fusion rules of representations of type A_n unitary WZW models in [Was98]. Unfortunately, it is not easy to generalize this proof to other examples.

The non-negativity of the transport coefficients in [Was98] was proved by computing explicitly the monodromy coefficients of the solutions of differential equations (2.5) (in the case of WZW models, the Knizhnik-Zamolodchikov (KZ) equations). In the case of [Was98], these equations reduce to a generalized hypergeometric equation, the manipulation of which is still possible. For other examples, say type G_2 WZW models, these differential equations are so complicated that computing the exact values of transport coefficients becomes very hard.

On the other hand, one might think of showing the nonzeroness of transport coefficients (or the non-degeneracy of transport matrices) *without* computing their exact values, and it turns out that this task is directly related to the problem of proving the (weak) *rigidity* of the braided tensor category $\mathrm{Rep}(V)$ (see step 3 of the proof of theorem 6.7). A general proof of the rigidity of $\mathrm{Rep}(V)$ does not exist until the significant works of Y.Z.Huang [Hua05b], [Hua08a], [Hua08b]. In these works, Huang proved the modular invariance of genus 1 (chiral) correlation functions of V, generalizing the partial result of Y.C.Zhu [Zhu96], and used this theorem to solve the rigidity problem. It was Huang who first noticed that the rigidity of $\mathrm{Rep}(V)$, a purely *genus 0* phenomenon, is related to the modular invariance of *genus 1* correlation functions. We refer the reader to [HL13] for a discussion of this issue.

Generalizing the positivity result of [Was98] is no easier. Wassermann's argument can be used to prove the positivity of Λ on $\mathcal{V}\binom{k}{i\ j}^*$ when $i \in \mathcal{F}$, i.e., when all intertwining operators with charge space W_i are energy-bounded. The reason was explained in the introduction of part I: when intertwining operators are energy-bounded, the smeared ones are thus preclosed, so they can be approximated by *bounded* operators intertwining the actions of $\mathcal{M}_V(I)$. The problem is that one cannot always show the energy-boundedness of all intertwining operators. In the case of type A_n unitary WZW models, only those whose charge spaces have the lowest conformal dimension (the vector representation) were proved energy-bounded. It is for proving the general positivity of transport matrices that we introduce and study generalized (smeared) intertwining operators in our papers.

The sesquilinear form Λ is also closely related to the non-degenerate bilinear form constructed in [HK07]. This will be explained in section 8.3.

For any $k \in \mathcal{E} \cap \mathcal{F}^{\boxtimes}$, since W_k is irreducible, we have $N_{0k}^k = N_{k0}^k = 1$. That the sesquilinear forms Λ on $\mathcal{V}\binom{k}{0\,k}^*$ and on $\mathcal{V}\binom{k}{k\,0}^*$ are positive definite can be seen from the following two fusion relations:

$$Y_k\big(Y(u, z_2 - z_1)v, z_1\big) = Y_k(u, z_2)Y_k(v, z_1), \tag{6.19}$$

$$Y\big(\mathcal{Y}_{\overline{k}k}^0(\overline{w_2^{(k)}}, z_2 - z_1)w_1^{(k)}, z_1\big) = \mathcal{Y}_{\overline{k}k}^0(\overline{w_2^{(k)}}, z_2)\mathcal{Y}_{k0}^k(w_1^{(k)}, z_1), \tag{6.20}$$

where $u, v \in V$, and $w_1^{(k)}, w_2^{(k)} \in W_k$. The first equation follows from proposition 2.13, and the second one follows from proposition 2.17. (Note that these two fusion relations hold for any V-module W_k.) Moreover, the dual element of Y_k is an orthonormal basis of $\mathcal{V}\binom{k}{0\,k}^*$, and the dual element of $\mathcal{Y}_{k0}^k$ is an orthonormal basis of $\mathcal{V}\binom{k}{k\,0}^*$.

We derive now some variants of transport formulas.

Proposition 6.5. *Let $I \in \mathcal{J}$. Choose distinct complex numbers $z_1, z_2 \in I$. Choose $z_0 \in I^c$ with argument $\arg z_0$. Define a continuous argument function $\arg_I$ on I, and let $\arg z_1 = \arg_I(z_1), \arg z_2 = \arg_I(z_2)$. Let W_i, W_j be unitary V-modules in $\mathcal{F}^{\boxtimes}$.*

(1) Let W_s, W_r be unitary V-modules in $\mathcal{F}^{\boxtimes}$, and choose $\mathcal{Y}_\gamma \in \mathcal{V}\binom{r}{j\,s}$. Then for any $w_1^{(i)}, w_2^{(i)} \in W_i, w^{(j)} \in W_j$, we have the braid relation

$$\mathcal{Y}_\gamma(w^{(j)}, z_0)\Big(\sum_{\alpha,\beta\in\Theta_{is}^*}\Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\Big)$$

$$=\Big(\sum_{\alpha,\beta\in\Theta^*_{ir}}\Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}},z_2)\mathcal{Y}_\alpha(w_1^{(i)},z_1)\Big)\mathcal{Y}_\gamma(w^{(j)},z_0). \tag{6.21}$$

(2) For any $w_1^{(i)},w_2^{(i)}\in W_i$ *and* $w^{(j)}\in W_j$*, we have the transport formula (version 2)*

$$\begin{aligned}&\mathcal{Y}^j_{j0}(w^{(j)},z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}},z_2)\mathcal{Y}^i_{i0}(w_1^{(i)},z_1)\\ =&\Big(\sum_{\alpha,\beta\in\Theta^*_{ij}}\Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}},z_2)\mathcal{Y}_\alpha(w_1^{(i)},z_1)\Big)\mathcal{Y}^j_{j0}(w^{(j)},z_0). \end{aligned}\tag{6.22}$$

(3) If $\arg z_0<\arg z_2<\arg z_0+2\pi$*, then for any* $w_2^{(i)}\in W_i,w^{(j)}\in W_j$*, we have the transport formula (version 3)*

$$\mathcal{Y}^j_{j0}(w^{(j)},z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}},z_2)=\sum_{\alpha,\beta\in\Theta^*_{ij}}\Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}},z_2)\mathcal{Y}_{B_+\alpha}(w^{(j)},z_0). \tag{6.23}$$

If $\arg z_2<\arg z_0<\arg z_2+2\pi$*, then equation* (6.23) *still holds, with* $B_+\alpha$ *replaced by* $B_-\alpha$*.*

Proof. (1) By rotating z_1,z_2 along I and changing their arguments continuously, we can assume that $0<|z_1-z_2|<1$. Then clearly $\arg z_1$ is close to $\arg z_2$ as $z_2-z_1\to 0$. We also let $\arg(z_2-z_1)$ be close to $\arg z_2$ as $z_1\to 0$. Then by equation (6.16), proposition 2.13, and theorem 5.2, we have

$$\begin{aligned}&\mathcal{Y}_\gamma(w^{(j)},z_0)\Big(\sum_{\alpha,\beta\in\Theta^*_{is}}\Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}},z_2)\mathcal{Y}_\alpha(w_1^{(i)},z_1)\Big)\\ =&\mathcal{Y}_\gamma(w^{(j)},z_0)Y_s\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}},z_2-z_1)w_1^{(i)},z_1\big) &(6.24)\\ =&Y_r\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}},z_2-z_1)w_1^{(i)},z_1\big)\mathcal{Y}_\gamma(w^{(j)},z_0) &(6.25)\\ =&\Big(\sum_{\alpha,\beta\in\Theta^*_{ir}}\Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}},z_2)\mathcal{Y}_\alpha(w_1^{(i)},z_1)\Big)\mathcal{Y}_\gamma(w^{(j)},z_0),\end{aligned}$$

where (6.24) and (6.25) are understood as products of two generalized intertwining operators (see the beginning of chapter 5). This proves equation (6.21).

(2) Equation (6.22) is a special case of equation (6.21).

(3) If $\arg z_0<\arg z_2<\arg z_0+2\pi$, we choose $z_1\in S^1\backslash\{-1\}$ close to z_2 and let $\arg z_1$ be

close to $\arg z_2$ as $z_1 \to z_2$. Then by equation (6.22), corollary 2.18, and proposition 2.11, we have

$$\begin{aligned}
&\mathcal{Y}^j_{j0}(w^{(j)}, z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}^i_{i0}(w^{(i)}_1, z_1)\\
=&\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_0)\mathcal{Y}^i_{i0}(w^{(i)}_1, z_1).
\end{aligned}$$

By proposition 2.3, we obtain equation (6.23). The other case is proved in a similar way. $\square$

6.3 Positive definiteness of Λ

Let W_i, W_j be unitary V-modules in $\mathcal{F}^\boxtimes$, and let W_k be in $\mathcal{E} \cap \mathcal{F}^\boxtimes$ as before. We prove in this section that the sesquilinear form Λ on $\mathcal{V}{k \choose i\ j}^*$ is positive definite. One suffices to prove this when W_i, W_j are irreducible. Indeed, if W_i, W_j are not necessarily irreducible, and have orthogonal decompositions $W_i = W_{i_1} \oplus W_{i_2} \oplus \cdots \oplus W_{i_m}, W_j = W_{j_1} \oplus W_{j_2} \oplus \cdots \oplus W_{j_n}$. Then clearly the unitary V-modules $W_{i_1}, \dots, W_{i_m}, W_{j_1}, \dots, W_{j_n}$ are in $\mathcal{F}^\boxtimes$. It is easy to see that the transport matrix for $\mathcal{V}{k \choose i\ j}^*$ can be diagonalized into the mn blocks of the transport matrices for $\mathcal{V}{k \choose i_a\ j_b}^*$ ($1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n$). Therefore, if we choose $W_{i_1}, \dots, W_{i_m}, W_{j_1}, \dots, W_{j_n}$ to be irreducible, and if we can prove that the transport matrix for every $\mathcal{V}{k \choose i_a\ j_b}^*$ is positive definite, then the one for $\mathcal{V}{k \choose i\ j}^*$ is also positive definite.

So let us assume that W_i, W_j are irreducible. We let $\mathcal{Y}_{\kappa(i)} = \mathcal{Y}^i_{i0}$ and $\mathcal{Y}_{\kappa(j)} = \mathcal{Y}^j_{j0}$. Then $\mathcal{Y}_{\kappa(i)*} = \mathcal{Y}^0_{\bar{i}i}, \mathcal{Y}_{\kappa(j)*} = \mathcal{Y}^0_{\bar{j}j}$. Since W_i (resp. W_j) is in $\mathcal{F}^\boxtimes$, there exits unitary V-modules $W_{i_1}, \dots, W_{i_m}$ (resp. $W_{j_1}, \dots, W_{j_n}$) in $\mathcal{F} \cup \overline{\mathcal{F}}$, such that W_i (resp. W_j) is equivalent to a submodule of $W_{i_m \cdots i_1} = W_{i_m} \boxtimes \cdots \boxtimes W_{i_1}$ (resp. $W_{j_n \cdots j_1}$). Therefore, we can choose a chain of non-zero irreducible unitary intertwining operators $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_m}$ (resp. $\mathcal{Y}_{\rho_2}, \dots, \mathcal{Y}_{\rho_n}$) with charge spaces $W_{i_2}, \dots, W_{i_m}$ (resp. $W_{j_2}, \dots, W_{j_n}$) respectively, such that W_{i_1} (resp. W_{j_1}) is the source space of $\mathcal{Y}_{\sigma_2}$ (resp. $\mathcal{Y}_{\rho_2}$), and that W_i (resp. W_j) is the target space of $\mathcal{Y}_{\sigma_m}$ (resp. $\mathcal{Y}_{\rho_n}$).

Fix non-zero quasi-primary vectors $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \dots, w^{(j_n)} \in W_{j_n}$. If $\mathcal{F}$ satisfies condition B, we assume moreover that $w^{(i_1)} \in E^1(W_{i_1}), \dots, w^{(i_m)} \in E^1(W_{i_m}), w^{(j_1)} \in E^1(W_{j_1}), \dots, w^{(j_n)} \in E^1(W_{j_n})$. Choose disjoint open intervals $I, J \in \mathcal{J}(S^1 \backslash \{-1\})$, and choose $(I_1, \dots, I_m) \in \mathfrak{O}_m(I), (J_1, \dots, J_n) \in \mathfrak{O}_n(J)$. We define two sets $\mathcal{A} = \mathcal{M}_V(I)_\infty \times C^\infty_c(I_1) \times \cdots \times C^\infty_c(I_m)$ and $\mathcal{B} = \mathcal{M}_V(J)_\infty \times C^\infty_c(J_1) \times \cdots \times C^\infty_c(J_n)$.

For any $a = (x, f_1, \dots, f_m) \in \mathcal{A}$ and $b = (y, g_1, \dots, g_n) \in \mathcal{B}$, we define two linear operators $A(a) : \mathcal{H}_0^\infty \to \mathcal{H}_i^\infty$ and $B(b) : \mathcal{H}_0^\infty \to \mathcal{H}_j^\infty$ as follows: if $\xi^{(0)} \in \mathcal{H}_0^\infty$ then

$$A(a)\xi^{(0)} = \pi_i(x)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\kappa(i)}(w^{(i_m)}, f_m; \dots, w^{(i_1)}, f_1)\xi^{(0)}, \tag{6.26}$$

$$B(b)\xi^{(0)} = \pi_j(y)\mathcal{Y}_{\rho_n\cdots\rho_2,\kappa(j)}(w^{(j_n)}, g_n; \dots, w^{(j_1)}, g_1)\xi^{(0)}. \tag{6.27}$$

By proposition 5.9, the formal adjoints of these two linear operators exist.

Lemma 6.6. *For any $N \in \mathbb{Z}_{>0}$, $a_1, \dots, a_N \in \mathcal{A}$, $b_1, \dots, b_N \in \mathcal{B}$ and $\xi_1^{(0)}, \dots, \xi_N^{(0)} \in \mathcal{H}_0^\infty$, we have*

$$\sum_{s,t=1,\dots,N} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | B(b_t)\xi_t^{(0)}\rangle \geqslant 0. \tag{6.28}$$

Proof. Suppose that

$$\sum_{s,t=1,\dots,N} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | B(b_t)\xi_t^{(0)}\rangle \notin [0, +\infty). \tag{6.29}$$

Then we can find $\varepsilon > 0$, such that for any $\tau \in [0, +\infty)$,

$$\left| \sum_{s,t=1,\dots,N} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | B(b_t)\xi_t^{(0)}\rangle - \tau \right| \geqslant \varepsilon. \tag{6.30}$$

By proposition 5.10, for any $x \in \mathcal{M}_V(J^c)$ and $r = 1, \dots, N$, we have $\pi_j(x)\overline{B(b_r)} \subset \overline{B(b_r)}\pi_0(x)$. We also regard $\overline{B(b_r)}$ as an unbounded operator on $\mathcal{H}_0 \oplus \mathcal{H}_j$, being the original operator when restrict to $\mathcal{H}_0$, and the zero map when restricted to $\mathcal{H}_j$. We let x act on $\mathcal{H}_0 \bigoplus \mathcal{H}_j$ diagonally (i.e., $x = \mathrm{diag}(\pi_0(x), \pi_I(x))$). Then $x\overline{B(b_r)} \subset \overline{B(b_r)}x$. Since x^* also satisfies this relation, elements in $\mathcal{M}_V(J^c)$ commute strongly with $\overline{B(b_r)}$. Therefore, if we take the right polar decomposition $\overline{B(b_r)} = K_rV_r$ (where K_r is self-adjoint and V_r is an partial isometry), then $\mathcal{M}_V(J^c)$ commutes strongly with V_r and K_r. We let $K_r = \int_{-\infty}^{+\infty} \lambda dQ_r(\lambda)$ be the spectral decomposition of K_r. Then for each $\lambda \geqslant 0$, $Q_r(\lambda) = \int_{-\infty}^{\lambda} dQ_r(\mu)$ commutes with $\mathcal{M}_V(J^c)$. Therefore, the *bounded* operator $\overline{Q_r(\lambda)\overline{B(b_r)}}$ commutes with $\mathcal{M}_V(J^c)$, i.e., $\overline{Q_r(\lambda)\overline{B(b_r)}} \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0, \mathcal{H}_j)$.

Now we choose a real number $M > 0$, such that for any $s, t = 1, \dots, N$,

$$\|B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}\| \leqslant M, \ \ \|B(b_t)\xi_t^{(0)}\| \leqslant M. \tag{6.31}$$

For each $r = 1, \dots, N$, since the projection $Q_r(\lambda)$ converges strongly to 1 as $\lambda \to +\infty$, there exists $\lambda_r > 0$, such that for any $t = 1, \dots, N$,

$$\|B(b_r)A(a_t)^\dagger A(a_r)\xi_r^{(0)} - Q_r(\lambda_r)B(b_r)A(a_t)^\dagger A(a_r)\xi_r^{(0)}\| < \frac{\varepsilon}{4MN^2}, \tag{6.32}$$

$$\|B(b_r)\xi_r^{(0)} - Q_r(\lambda_r)B(b_r)\xi_r^{(0)}\| < \frac{\varepsilon}{4MN^2}. \tag{6.33}$$

We let $\mathfrak{B}(b_r) = \overline{Q_r(\lambda_r)\overline{B(b_r)}} \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0, \mathcal{H}_j)$, then the above inequalities imply that

$$\left| \sum_{s,t} \langle \mathfrak{B}(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | \mathfrak{B}(b_t)\xi_t^{(0)} \rangle - \sum_{s,t} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | B(b_t)\xi_t^{(0)} \rangle \right| < \frac{\varepsilon}{2}. \tag{6.34}$$

Now, for any $1 \leqslant r \leqslant N$, since $\mathfrak{B}(b_r) \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0, \mathcal{H}_j)$, we also have $\mathfrak{B}(b_r)^* \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_j, \mathcal{H}_0)$. Thus, for any $1 \leqslant s, t \leqslant N$, we have $\mathfrak{B}(b_s)^*\mathfrak{B}(b_t) \in \mathrm{End}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0) = \mathcal{M}_V(J^c)'$. By Haag duality, $\mathfrak{B}(b_s)^*\mathfrak{B}(b_t) \in \mathcal{M}_V(J)$. By proposition 5.10, $\pi_i\big(\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\big)\overline{A(a_t)} \subset \overline{A(a_t)}\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)$. In particular, $\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\mathscr{D}(\overline{A(a_t)}) \subset \mathscr{D}(\overline{A(a_t)})$. Since $\xi_t^{(0)} \in \mathcal{H}_0^\infty \subset \mathscr{D}(\overline{A(a_t)})$,

$$\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)} \in \mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\mathscr{D}(\overline{A(a_t)}) \subset \mathscr{D}(\overline{A(a_t)}). \tag{6.35}$$

Therefore,

$$\begin{aligned}
&\langle \mathfrak{B}(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | \mathfrak{B}(b_t)\xi_t^{(0)} \rangle \\
=&\langle A(a_t)^\dagger A(a_s)\xi_s^{(0)} | \mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)} \rangle \\
=&\langle \overline{A(a_t)}^* \cdot \overline{A(a_s)}\xi_s^{(0)} | \mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)} \rangle \\
=&\langle \overline{A(a_s)}\xi_s^{(0)} | \overline{A(a_t)}\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)} \rangle.
\end{aligned} \tag{6.36}$$

Let $\overline{A(a_s)} = H_s U_s$ be the right polar decomposition of $\overline{A(a_s)}$, and take the spectral decomposition

$H_s = \int_{-\infty}^{+\infty} \kappa dP_s(\kappa)$. Then for each s, we can find $\kappa_s > 0$ such that

$$\left| \sum_{s,t} \langle \overline{A(a_s)} \xi_s^{(0)} | \overline{A(a_t)} \mathfrak{B}(b_s)^* \mathfrak{B}(b_t) \xi_t^{(0)} \rangle - \sum_{s,t} \langle \mathfrak{A}(a_s) \xi_s^{(0)} | \mathfrak{A}(a_t) \mathfrak{B}(b_s)^* \mathfrak{B}(b_t) \xi_t^{(0)} \rangle \right| < \frac{\varepsilon}{2}, \quad (6.37)$$

where $\mathfrak{A}(a_s) = \overline{P_s(\kappa_s)\overline{A(a_s)}} \in \mathrm{Hom}_{\mathcal{M}(I^c)}(\mathcal{H}_0, \mathcal{H}_i)$. Note that $\mathfrak{A}(a_s)$ and $\mathfrak{B}(b_t)$ are bounded operators. Set

$$\tau = \sum_{s,t} \langle \mathfrak{A}(a_s) \xi_s^{(0)} | \mathfrak{A}(a_t) \mathfrak{B}(b_s)^* \mathfrak{B}(b_t) \xi_t^{(0)} \rangle = \sum_{s,t} \langle \mathfrak{B}(b_s) \mathfrak{A}(a_t)^* \mathfrak{A}(a_s) \xi_s^{(0)} | \mathfrak{B}(b_t) \xi_t^{(0)} \rangle. \quad (6.38)$$

Then by inequalities (6.34), (6.37), and equation (6.36),

$$\left| \sum_{s,t} \langle B(b_s) A(a_t)^\dagger A(a_s) \xi_s^{(0)} | B(b_t) \xi_t^{(0)} \rangle - \tau \right| < \varepsilon. \quad (6.39)$$

We now show that $\tau \geqslant 0$, which will contradict condition (6.30) and thus prove inequality (6.28). Let $M(N, \mathbb{C})$ be the complex valued $N \times N$ matrix algebra. By evaluating between vectors in $\mathcal{H}_0^{\oplus N}$, we find that the $\mathcal{M}_V(I)$-valued matrix $[\mathfrak{A}(a_t)^* \mathfrak{A}(a_s)]_{N \times N}$ is a positive element in the von Naumann algebra $\mathcal{M}_V(I) \otimes M(N, \mathbb{C})$. So $[\pi_{j,I}(\mathfrak{A}(a_t)^* \mathfrak{A}(a_s))]_{N \times N} \in \pi_{j,I}(\mathcal{M}_V(I)) \otimes M(N, \mathbb{C})$ is also positive. Therefore, if for each s we define a vector $\eta_s = \mathfrak{B}(b_s) \xi_s^{(0)}$, then

$$\sum_{s,t} (\pi_{j,I}(\mathfrak{A}(a_t)^* \mathfrak{A}(a_s)) \eta_s | \eta_t) \geqslant 0. \quad (6.40)$$

Since $\mathfrak{B}(b_s) \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0, \mathcal{H}_j) \subset \mathrm{Hom}_{\mathcal{M}_V(I)}(\mathcal{H}_0, \mathcal{H}_j)$, we have

$$\mathfrak{B}(b_s) \mathfrak{A}(a_t)^* \mathfrak{A}(a_s) \xi_s^{(0)} = \pi_{j,I}\big(\mathfrak{A}(a_t)^* \mathfrak{A}(a_s)\big) \mathfrak{B}(b_s) \xi_s^{(0)} = \pi_{j,I}(\mathfrak{A}(a_t)^* \mathfrak{A}(a_s)) \eta_s. \quad (6.41)$$

Hence

$$\tau = \sum_{s,t} \langle \mathfrak{B}(b_s) \mathfrak{A}(a_t)^* \mathfrak{A}(a_s) \xi_s^{(0)} | \mathfrak{B}(b_t) \xi_t^{(0)} \rangle = \sum_{s,t} (\pi_{j,I}(\mathfrak{A}(a_t)^* \mathfrak{A}(a_s)) \eta_s | \eta_t) \geqslant 0. \quad (6.42)$$

□

Theorem 6.7. *Suppose that V is unitary, energy bounded, and strongly local, and $\mathcal{F}$ is a non-empty* *set of non-zero irreducible unitary V-modules satisfying condition A or B. Let W_i, W_j be unitary V-modules in $\mathcal{F}^{\boxtimes}$. Then the sesquilinear form Λ on $W_i \boxtimes W_j$ is an inner product. Equivalently, for any irreducible unitary V-module W_k in $\mathcal{E} \cap \mathcal{F}^{\boxtimes}$, the sesquilinear form Λ on $\mathcal{V}\binom{k}{i\ j}^*$ is positive definite.*

Proof. As argued at the beginning of this section, we can assume, without loss of generality, that W_i, W_j are irreducible.

Step 1. We first show that Λ is positive. For each $k \in \mathcal{E} \cap \mathcal{F}^{\boxtimes}$, we choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{ij}^k\}$ of $\mathcal{V}\binom{k}{i\ j}$, let $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Theta_{ij}^k\}$ be its dual basis in $\mathcal{V}\binom{k}{i\ j}^*$, and define an inner product on $\mathcal{V}\binom{k}{i\ j}^*$ under which $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Theta_{ij}^k\}$ becomes orthonormal. We extend these inner products to a unitary structure on $W_{ij} = \bigoplus_k \mathcal{V}\binom{k}{i\ j}^* \otimes W_k$, just as we extend Λ using (6.18). As usual, we let $\mathcal{H}_{ij}$ be the corresponding $\mathcal{M}_V$-module. The sesquilinear form Λ on W_{ij} defined by (6.18) can be extended uniquely to a *continuous* sesquilinear form Λ on the Hilbert space $\mathcal{H}_{ij}$.

Choose intertwining operators $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_m}, \mathcal{Y}_{\rho_2}, \dots, \mathcal{Y}_{\rho_n}$, disjoint open intervals $I, J, (I_1, \dots, I_m) \in \mathfrak{O}_m(I), (J_1, \dots, J_n) \in \mathfrak{O}_n(J)$, and non-zero quasi-primary vectors $w^{(i_1)}, \dots, w^{(i_m)}, w^{(j_1)}, \dots, w^{(j_n)}$ as at the beginning of this section. By proposition 6.2, for each $l \in \mathbb{Z}_{\geqslant 0}$, vectors of the form

$$B(b)\xi^{(0)} = \pi_j(y)\mathcal{Y}_{\rho_n\cdots\rho_2,\kappa(j)}(w^{(j_n)}, g_n; \dots; w^{(j_1)}, g_1)\xi^{(0)} \tag{6.43}$$

span a core for $\overline{L_0}^l$ in $\mathcal{H}_j^\infty$, where $b = (y, g_1, \dots, g_n) \in \mathcal{B}$, and $\xi^{(0)} \in \mathcal{H}_0^\infty$. For any $a = (x, f_1, \dots, f_m) \in \mathcal{A}$, we define an unbounded operator $\widetilde{A}(a) : \mathcal{H}_j \to \mathcal{H}_{ij}$ with domain $\mathcal{H}_j^\infty$ to satisfy

$$\widetilde{A}(a) = \pi_{ij}(x)\mathcal{Y}_{\sigma_m\cdots\sigma_2,i\boxtimes j}(w^{(i_m)}, f_m; \dots; w^{(i_1)}, f_1). \tag{6.44}$$

Then, by inequality (5.43), vectors of the form (6.43) span a core for $\widetilde{A}(a)$. Therefore, by proposition 6.2, vectors of the form

$$\xi^{(ij)} = \sum_{s=1,\dots,N} \widetilde{A}(a_s)B(b_s)\xi_s^{(0)} \tag{6.45}$$

form a dense subspace of $\mathcal{H}_{ij}$, where $N = 1, 2, \ldots$, and for each s, $a_s = (x_s, f_{s,1}, \ldots, f_{s,m}) \in \mathcal{A}, b_s = (y_s, g_{s,1}, \ldots, g_{s,n}) \in \mathcal{B}$, and $\xi_s^{(0)} \in \mathcal{H}_0$. If we can prove, for any $\xi^{(ij)} \in \mathcal{H}_{ij}$ of the form (6.45), that $\Lambda(\xi^{(ij)}|\xi^{(ij)}) \geqslant 0$, then Λ is positive on $W_i \boxtimes W_j$.

Step 2. We show that $\Lambda(\xi^{(ij)}|\xi^{(ij)}) \geqslant 0$. Let us simplify the notations a little bit. Let $\vec{w}^{(\vec{i})} = (w^{(i_1)}, \ldots, w^{(i_m)}), \vec{\sigma} = (\sigma_2, \ldots, \sigma_m), \vec{f_s} = (f_{s,1}, \ldots, f_{s,m})$. If $\mathcal{Y}_\alpha$ is an intertwining operator whose charge space, source space, and target space are inside $\mathcal{F}^{\boxtimes}$, then we set

$$\mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s}) = \mathcal{Y}_{\sigma_m \cdots \sigma_2, \alpha}(w^{(i_m)}, f_{s,m}; \ldots; w^{(i_1)}, f_{s,1}). \tag{6.46}$$

Similarly, we let $\vec{w}^{(\vec{j})} = (w^{(j_1)}, \ldots, w^{(j_n)}), \vec{\rho} = (\rho_2, \ldots, \rho_n), \vec{g}_s = (g_{s,1}, \ldots, g_{s,n})$. $\mathcal{Y}_{\vec{\rho},\kappa(j)}(w^{(\vec{j})}, \vec{g}_s)$ is defined in a similar way.

Assume, without loss of generality, that I is anti-clockwise to J, i.e., for any $z \in I, \zeta \in J$, we have $-\pi < \arg \zeta < \arg z < \pi$. By proposition 5.10, for any $s = 1, \ldots, N$,

$$\begin{aligned}
&\widetilde{A}(a_s)B(b_s) = x_s \mathcal{Y}_{\vec{\sigma}, i \boxtimes j}(\vec{w}^{(\vec{i})}, \vec{f_s}) y_s \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s) \\
&= \sum_{\alpha \in \Theta_{ij}^*} \check{\mathcal{Y}}^\alpha \otimes x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s}) \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s).
\end{aligned} \tag{6.47}$$

So for any $s, t = 1, \ldots, N$,

$$\begin{aligned}
&\Lambda\big(\widetilde{A}(a_s)B(b_s)\xi_s^{(0)} \big| \widetilde{A}(a_t)B(b_t)\xi_t^{(0)}\big) \\
=& \sum_{\alpha,\beta \in \Theta_{ij}^*} \Lambda^{\alpha\beta} \big\langle x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s}) \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s)\xi_s^{(0)} \big| x_t y_t \mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})}, \vec{f_t}) \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_t)\xi_t^{(0)} \big\rangle \\
=& \sum_{\alpha,\beta \in \Theta_{ij}^*} \Lambda^{\alpha\beta} \big\langle \mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})}, \vec{f_t})^\dagger x_t^* x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s}) \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s)\xi_s^{(0)} \big| y_t \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_t)\xi_t^{(0)} \big\rangle \\
=& \sum_{\alpha,\beta \in \Theta_{ij}^*} \Lambda^{\alpha\beta} \big\langle \mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})}, \vec{f_t})^\dagger x_t^* x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s}) \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s)\xi_s^{(0)} \big| B(b_t)\xi_t^{(0)} \big\rangle.
\end{aligned} \tag{6.48}$$

By corollary 2.18 and theorem 5.12,

$$\sum_{\alpha,\beta \in \Theta_{ij}^*} \Lambda^{\alpha\beta} \mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})}, \vec{f_t})^\dagger x_t^* x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s}) \mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s)$$

$$
\begin{aligned}
&= \sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger x_t^* x_s y_s \mathcal{Y}_{\vec{\rho},B_+\alpha}(\vec{w}^{(\vec{j})},\vec{g}_s)\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s)\\
&= \sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta} y_s\mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger \mathcal{Y}_{\vec{\rho},B_+\alpha}(\vec{w}^{(\vec{j})},\vec{g}_s)x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s). \qquad (6.49)
\end{aligned}
$$

By theorem 5.13, for each $l = 2,\dots,m$, there exists an intertwining operators $\widetilde{\sigma_l}$ having the same type as that of σ_l, such that (5.59) holds for all $\mathcal{Y}_\alpha$ whose charge space, source space, and target space are unitary V-modules in $\mathcal{F}^{\boxtimes}$. Let $h_{t,1} = e^{i\pi\Delta_{w^{(i_1)}}}(e_{2-2\Delta_{w^{(i_1)}}}f_{t,1}),\dots,h_{t,m} = e^{i\pi\Delta_{w^{(i_m)}}}(e_{2-2\Delta_{w^{(i_m)}}}f_{t,m})$. Set $\vec{h}_t = (h_{t,1},\dots,h_{t,m}), \overline{\vec{h}_t} = (\overline{h_{t,1}},\dots,\overline{h_{t,m}}), \overline{\vec{w}^{(\vec{i})}} = (\overline{w^{(i_1)}},\dots,\overline{w^{(i_m)}})$. Then (6.49) equals

$$
\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta} y_s\mathcal{Y}_{\vec{\tilde{\sigma}},\beta^*}(\overline{\vec{w}^{(\vec{i})}},\overline{\vec{h}_t})\mathcal{Y}_{\vec{\rho},B_+\alpha}(\vec{w}^{(\vec{j})},\vec{g}_s)x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s). \qquad (6.50)
$$

By equation (6.23) and theorem 5.12, (6.50) equals

$$
y_s\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})},\vec{g}_s)\mathcal{Y}_{\vec{\tilde{\sigma}},\kappa(i)^*}(\overline{\vec{w}^{(\vec{i})}},\overline{\vec{h}_t})x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s), \qquad (6.51)
$$

which, due to equation (5.59), also equals

$$
\begin{aligned}
&y_s\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})},\vec{g}_s)\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s)\\
=&B(b_s)A(a_t)^\dagger A(a_s). \qquad (6.52)
\end{aligned}
$$

Substitute this expression into equation (6.48), we see that

$$
\Lambda\big(\widetilde{A}(a_s)B(b_s)\xi_s^{(0)}\big|\widetilde{A}(a_t)B(b_t)\xi_t^{(0)}\big) = \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|B(b_t)\xi_t^{(0)}\rangle. \qquad (6.53)
$$

Therefore, by lemma 6.6,

$$
\begin{aligned}
\Lambda(\xi^{(ij)}|\xi^{(ij)}) &= \sum_{s,t=1,\dots,N} \Lambda\big(\widetilde{A}(a_s)B(b_s)\xi_s^{(0)}\big|\widetilde{A}(a_t)B(b_t)\xi_t^{(0)}\big)\\
&= \sum_{s,t=1,\dots,N} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|B(b_t)\xi_t^{(0)}\rangle \geqslant 0. \qquad (6.54)
\end{aligned}
$$

Step 3 (See also [HK07] theorem 3.4). We prove the non-degeneracy of Λ using the rigidity of $\mathrm{Rep}(V)$. Since Λ is positive, for each $k \in \mathcal{E}$, we can choose a basis Θ_{ij}^k, such that the transport matrix Λ is a diagonal, and that the entries are either 1 or 0. Thus, we have the transport formula

$$Y_j(\mathcal{Y}_{\bar{i}i}^0(\overline{w_2^{(i)}}, z_2 - z_1)w_1^{(i)}, z_1) = \sum_{\alpha \in \Theta_{ij}^*} \lambda_\alpha \mathcal{Y}_{\alpha^*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1), \tag{6.55}$$

where each λ_α is either 1 or 0. For each $k \in \mathcal{E}$, we let n_{ij}^k be the number of $\alpha \in \Theta_{ij}^k$ satisfying $\lambda_\alpha = 1$. Then clearly $n_{ij}^k \leqslant N_{ij}^k$. If we can show that $n_{ij}^k = N_{ij}^k$, then the non-degeneracy of Λ follows.

Since W_i is irreducible, we have $N_{\bar{i}i}^0 = N_{0i}^i = 1$. So there exists a complex number $\mu_i \neq 0$ such that $\mathcal{Y}_{\bar{i}i}^0$ represents the morphism $\mu_i \mathrm{ev}_i : W_{\bar{i}} \boxtimes W_i \to V$. We also regard $\mathcal{Y}_\alpha$ as a morphism $W_i \boxtimes W_j \to W_k$, and $\mathcal{Y}_{\alpha^*}$ a morphism $W_{\bar{i}} \boxtimes W_k \to W_j$ (see section 2.4). Then equation (6.55) is equivalent to the following relation for morphisms:

$$\mu_i(\mathrm{ev}_i \otimes \mathrm{id}_j) = \sum_{k \in \mathcal{E}} \sum_{\alpha \in \Theta_{ij}^k} \lambda_\alpha \mathcal{Y}_{\alpha^*} \circ (\mathrm{id}_{\bar{i}} \otimes \mathcal{Y}_\alpha). \tag{6.56}$$

By equation (2.64),

$$\begin{aligned}
\mu_i(\mathrm{id}_i \otimes \mathrm{id}_j) =& \mu_i[(\mathrm{id}_i \otimes \mathrm{ev}_i) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i)] \otimes \mathrm{id}_j \\
=& \mu_i(\mathrm{id}_i \otimes \mathrm{ev}_i \otimes \mathrm{id}_j) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i \otimes \mathrm{id}_j) \\
=& \sum_{k \in \mathcal{E}} \sum_{\alpha \in \Theta_{ij}^k} \lambda_\alpha \big(\mathrm{id}_i \otimes (\mathcal{Y}_{\alpha^*} \circ (\mathrm{id}_{\bar{i}} \otimes \mathcal{Y}_\alpha))\big) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i \otimes \mathrm{id}_j) \\
=& \sum_{k \in \mathcal{E}} \sum_{\alpha \in \Theta_{ij}^k} \lambda_\alpha (\mathrm{id}_i \otimes \mathcal{Y}_{\alpha^*}) \circ (\mathrm{id}_i \otimes \mathrm{id}_{\bar{i}} \otimes \mathcal{Y}_\alpha) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i \otimes \mathrm{id}_j) \\
=& \sum_{k \in \mathcal{E}} \sum_{\alpha \in \Theta_{ij}^k} \lambda_\alpha (\mathrm{id}_i \otimes \mathcal{Y}_{\alpha^*}) \circ (\mathrm{coev}_i \otimes \mathcal{Y}_\alpha) \\
=& \sum_{k \in \mathcal{E}} \sum_{\alpha \in \Theta_{ij}^k} \lambda_\alpha (\mathrm{id}_i \otimes \mathcal{Y}_{\alpha^*}) \circ (\mathrm{coev}_i \otimes \mathrm{id}_k) \circ (\mathrm{id}_0 \otimes \mathcal{Y}_\alpha).
\end{aligned}$$

This equation implies that the isomorphism $\mu_i(\mathrm{id}_i \otimes \mathrm{id}_j) : W_i \boxtimes W_j \to W_i \boxtimes W_j$ factors through the homomorphism

$$\Phi:\sum_{k\in\mathcal{E}}^{\oplus}\sum_{\alpha\in\Theta_{ij}^k,\lambda_\alpha\neq0}^{\oplus}\mathrm{id}_0\otimes\mathcal{Y}_\alpha:W_i\boxtimes W_j\to W=\bigoplus_{k\in\mathcal{E}}\bigoplus_{\alpha\in\Theta_{ij}^k,\lambda_\alpha\neq0}W_k.$$

So Φ must be injective, which implies that $W_i\boxtimes W_j$ can be embedded as a submodule of W. Note that $W_i\boxtimes W_j\simeq\bigoplus_{k\in\mathcal{E}}W_k^{\oplus N_{ij}^k}$ and $W\simeq\bigoplus_{k\in\mathcal{E}}W_k^{\oplus n_{ij}^k}$. So we must have $n_{ij}^k\geqslant N_{ij}^k$. $\square$

Remark 6.8. The problem of the positivity of Λ is a stated in the pure language of vertex operator algebras. However, as we have seen, its proof relies heavily on conformal nets and operator algebras. It is interesting to notice that a problem in algebra is solved using ideas and techniques in functional analysis.

CHAPTER 7

UNITARITY OF THE RIBBON FUSION CATEGORIES

In this chapter, we still assume that V is unitary, energy bounded, and strongly local, and that $\mathcal{F}$ is a non-empty set of non-zero irreducible unitary V-modules satisfying condition A or B. If W_i, W_j are unitary V-modules in $\mathcal{F}^\boxtimes$, then by theorem 6.7, for each $k \in \mathcal{E}$, the sesquilinear form Λ on $\mathcal{V}{k \choose i\,j}^*$ defined by the transport matrix is an inner product. Therefore, we have a unitary structure on $\mathcal{F}^\boxtimes$ defined by Λ (see section 2.4). We fix this unitary structure, and show that the ribbon fusion category $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^\boxtimes}(V)$ is unitary.

We first note that the inner product Λ on $\mathcal{V}{k \choose i\,j}^*$ induces naturally an antilinear isomorphism map $\mathcal{V}{k \choose i\,j} \to \mathcal{V}{k \choose i\,j}^*$. We then define the inner product Λ on $\mathcal{V}{k \choose i\,j}$ so that this map becomes anti-unitary. Then a basis $\Theta^k_{ij} \subset \mathcal{V}{k \choose i\,j}$ is orthonormal if and only if its dual basis is an orthonormal basis of $\mathcal{V}{k \choose i\,j}^*$. Therefore, if for each $k \in \mathcal{E} \cap \mathcal{F}^\boxtimes$, Θ^k_{ij} is an orthonormal basis of $\mathcal{V}{k \choose i\,j}$, then the transport formulas (6.16), (6.21) and (6.23) become

$$Y_j\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2 - z_1)w^{(i)}_1, z_1\big) = \sum_{\alpha\in\Theta^*_{ij}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1), \tag{7.1}$$

$$\mathcal{Y}_\gamma(w^{(j)}, z_0)\bigg(\sum_{\alpha\in\Theta^*_{is}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1)\bigg) = \bigg(\sum_{\alpha\in\Theta^*_{ir}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1)\bigg)\mathcal{Y}_\gamma(w^{(j)}, z_0), \tag{7.2}$$

$$\mathcal{Y}^j_{j0}(w^{(j)}, z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2) = \sum_{\alpha\in\Theta^*_{ij}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_0). \tag{7.3}$$

7.1 Unitarity of braid matrices

For any unitary V-modules W_i, W_j in $\mathcal{F}^\boxtimes$, and any $s, t \in \mathcal{E} \cap \mathcal{F}^\boxtimes$, we choose bases $\Theta^t_{is}, \Theta^t_{sj}$ of $\mathcal{V}{t \choose i\,s}, \mathcal{V}{t \choose s\,j}$ respectively. Now fix $i, j \in \mathcal{F}^\boxtimes$, we also define

$$\Theta^*_{i*} = \coprod_{s,t\in\mathcal{E}\cap\mathcal{F}^\boxtimes} \Theta^t_{is}, \Theta^*_{*j} = \coprod_{s,t\in\mathcal{E}\cap\mathcal{F}^\boxtimes} \Theta^t_{sj}.$$

Choose distinct $z_i, z_j \in S^1$, and let $\arg z_j < \arg z_i < \arg z_j + 2\pi$. For any $\alpha, \alpha' \in \Theta^*_{i*}, \beta, \beta' \in \Theta^*_{j*}$, if either the source space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_\beta$, or the target space of $\mathcal{Y}_{\alpha'}$ does not equal the source space of $\mathcal{Y}_{\beta'}$, or the target space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_{\beta'}$, or the source space of $\mathcal{Y}_\beta$ does not equal the source space of $\mathcal{Y}_{\alpha'}$, then we set $(B_+)^{\beta'\alpha'}_{\alpha\beta} = 0$; otherwise the values $(B_+)^{\beta'\alpha'}_{\alpha\beta}$ are determined by the following braid relation: for any $w^{(i)} \in W_i, w^{(j)} \in W_j$,

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \sum_{\alpha' \in \Theta^*_{i*}, \beta' \in \Theta^*_{j*}} (B_+)^{\beta'\alpha'}_{\alpha\beta} \mathcal{Y}_{\beta'}(w^{(j)}, z_j)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i). \tag{7.4}$$

The matrix $(B_+)_{ij} = \{(B_+)^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha' \in \Theta^*_{i*}, \beta' \in \Theta^*_{j*}}_{\alpha \in \Theta^*_{i*}, \beta \in \Theta^*_{j*}}$ is called a **braid matrix**. The matrix $(B_-)_{ij} = \{(B_-)^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha' \in \Theta^*_{i*}, \beta' \in \Theta^*_{j*}}_{\alpha \in \Theta^*_{i*}, \beta \in \Theta^*_{j*}}$ is defined in a similar way by assuming $\arg z_i < \arg z_j < \arg z_i + 2\pi$. Clearly $(B_\pm)_{ij}$ is the inverse matrix of $(B_\mp)_{ji}$.

Proposition 7.1. *For any $\alpha, \alpha' \in \Theta^*_{i*}, \beta, \beta' \in \Theta^*_{j*}$, we have*

$$\overline{(B_\pm)^{\beta'\alpha'}_{\alpha\beta}} = (B_\mp)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}. \tag{7.5}$$

Proof. Choose distinct $z_i, z_j \in S^1$, and let $\arg z_j < \arg z_i < \arg z_j + 2\pi$. Then for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, the braid relation (7.4) holds. Taking the formal adjoint of (7.4), we have

$$\mathcal{Y}_\beta(w^{(j)}, z_j)^\dagger \mathcal{Y}_\alpha(w^{(i)}, z_i)^\dagger = \sum_{\alpha', \beta'} \overline{(B_+)^{\beta'\alpha'}_{\alpha\beta}} \mathcal{Y}_{\alpha'}(w^{(i)}, z_i)^\dagger \mathcal{Y}_{\beta'}(w^{(j)}, z_j)^\dagger. \tag{7.6}$$

By equation (1.34), for any $w^{(i)} \in W_i, w^{(j)} \in W_j$ we have

$$\mathcal{Y}_{\beta^*}(\overline{w^{(j)}}, z_j)\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, z_i) = \sum_{\alpha', \beta'} \overline{(B_+)^{\beta'\alpha'}_{\alpha\beta}} \mathcal{Y}_{\alpha'^*}(\overline{w^{(i)}}, z_i)\mathcal{Y}_{\beta'^*}(\overline{w^{(j)}}, z_j). \tag{7.7}$$

But $\{(B_-)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}\}$ is also the braid matrix for the braid relation (7.7). So we must have $\overline{(B_+)^{\beta'\alpha'}_{\alpha\beta}} = (B_-)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}$. If we let $\arg z_i < \arg z_j < \arg z_i + 2\pi$, then we obtain $\overline{(B_-)^{\beta'\alpha'}_{\alpha\beta}} = (B_+)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}$. □

Proposition 7.2. *If the bases $\Theta^*_{i*}, \Theta^*_{j*}$ are orthonormal under the inner product Λ, then for any*

$\alpha, \alpha' \in \Theta^*_{i*}, \beta, \beta' \in \Theta^*_{j*}$, *we have*

$$(B_\pm)^{\beta'\alpha'}_{\alpha\beta} = (B_\mp)^{\alpha^*\beta'}_{\beta\alpha'^*} = (B_\pm)^{\beta^*\alpha^*}_{\alpha'^*\beta'^*}. \tag{7.8}$$

Proof. Choose distinct $z_1, z_2, z_3, z_4 \in S^1$ with arguments $\arg z_1 < \arg z_2 < \arg z_3 < \arg z_4 < \arg z_1 + 2\pi$. By relation (7.2), for any $k \in \mathcal{E} \cap \mathcal{F}^{\boxtimes}, w_0, w_5 \in W_k, w_1, w_2 \in W_i, w_3, w_4 \in W_j$, we have, following convention 2.19,

$$\begin{aligned}&\sum_{\substack{\alpha'\in\Theta^*_{i*}\\ \beta\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\beta^*}(\overline{w_4}, z_4)\mathcal{Y}_\beta(w_3, z_3)\mathcal{Y}_{\alpha'^*}(\overline{w_2}, z_2)\mathcal{Y}_{\alpha'}(w_1, z_1)w_0\Big| w_5\Big\rangle\\ =&\sum_{\substack{\alpha\in\Theta^*_{i*}\\ \beta\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\beta^*}(\overline{w_4}, z_4)\mathcal{Y}_{\alpha^*}(\overline{w_2}, z_2)\mathcal{Y}_{\alpha}(w_1, z_1)\mathcal{Y}_\beta(w_3, z_3)w_0\Big| w_5\Big\rangle.\end{aligned} \tag{7.9}$$

By exchanging $\mathcal{Y}_\alpha$ and $\mathcal{Y}_\beta$, (7.9) equals

$$\sum_{\substack{\alpha,\alpha'\in\Theta^*_{i*}\\ \beta,\beta'\in\Theta^*_{j*}}} (B_-)^{\beta'\alpha'}_{\alpha\beta}\Big\langle \mathcal{Y}_{\beta^*}(\overline{w_4}, z_4)\mathcal{Y}_{\alpha^*}(\overline{w_2}, z_2)\mathcal{Y}_{\beta'}(w_3, z_3)\mathcal{Y}_{\alpha'}(w_1, z_1)w_0\Big| w_5\Big\rangle. \tag{7.10}$$

By proposition 2.3, we have

$$\mathcal{Y}_\beta(w_3, z_3)\mathcal{Y}_{\alpha'^*}(\overline{w_2}, z_2) = \sum_{\substack{\alpha,\alpha'\in\Theta^*_{i*}\\ \beta,\beta'\in\Theta^*_{j*}}} (B_-)^{\beta'\alpha'}_{\alpha\beta}\mathcal{Y}_{\alpha^*}(\overline{w_2}, z_2)\mathcal{Y}_{\beta'}(w_3, z_3). \tag{7.11}$$

This proves that $(B_+)^{\alpha^*\beta'}_{\beta\alpha'^*} = (B_-)^{\beta'\alpha'}_{\alpha\beta}$.

Similarly, we also have

$$\begin{aligned}&\sum_{\substack{\alpha'\in\Theta^*_{i*}\\ \beta\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\beta^*}(\overline{w_4}, z_4)\mathcal{Y}_\beta(w_3, z_3)\mathcal{Y}_{\alpha'^*}(\overline{w_2}, z_2)\mathcal{Y}_{\alpha'}(w_1, z_1)w_0\Big| w_5\Big\rangle\\ =&\sum_{\substack{\alpha'\in\Theta^*_{i*}\\ \beta'\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\alpha'^*}(\overline{w_2}, z_2)\mathcal{Y}_{\beta'^*}(\overline{w_4}, z_4)\mathcal{Y}_{\beta'}(w_3, z_3)\mathcal{Y}_{\alpha'}(w_1, z_1)w_0\Big| w_5\Big\rangle\end{aligned} \tag{7.12}$$

$$= \sum_{\substack{\alpha,\alpha'\in\Theta_{i*}^*\\ \beta,\beta'\in\Theta_{j*}^*}} (B_-)_{\alpha'^*\beta'^*}^{\beta^*\alpha^*} \Big\langle \mathcal{Y}_{\beta^*}(\overline{w_4}, z_4)\mathcal{Y}_{\alpha^*}(\overline{w_2}, z_2)\mathcal{Y}_{\beta'}(w_3, z_3)\mathcal{Y}_{\alpha'}(w_1, z_1)w_0 \Big| w_5 \Big\rangle, \tag{7.13}$$

which implies that $(B_+)_{\beta\alpha'^*}^{\alpha^*\beta'} = (B_-)_{\alpha'^*\beta'^*}^{\beta^*\alpha^*}$.

If $z_1, z_2, z_3, z_4 \in S^1$ and their arguments are chosen such that $\arg z_4 < \arg z_3 < \arg z_2 < \arg z_1 < \arg z_4 + 2\pi$, then the same argument implies that $(B_+)_{\alpha\beta}^{\beta'\alpha'} = (B_-)_{\beta\alpha'^*}^{\alpha^*\beta'} = (B_+)_{\alpha'^*\beta'^*}^{\beta^*\alpha^*}$. □

Corollary 7.3. *If the bases $\Theta_{i*}^*, \Theta_{j*}^*$ are orthonormal under the inner product Λ, then the braid matrix $(B_\pm)_{ij}$ is unitary.*

Proof. If we apply propositions 7.1 and 7.2, then for any $\alpha, \alpha' \in \Theta_{i*}^*, \beta, \beta' \in \Theta_{j*}^*$, we have

$$(B_\pm)_{\alpha\beta}^{\beta'\alpha'} = (B_\pm)_{\alpha'^*\beta'^*}^{\beta^*\alpha^*} = \overline{(B_\mp)_{\beta'\alpha'}^{\alpha\beta}}, \tag{7.14}$$

which shows that $(B_\pm)_{ij}$ is the adjoint of $(B_\mp)_{ji}$. But we know that $(B_\pm)_{ij}$ is also the inverse matrix of $(B_\mp)_{ji}$. So $(B_\pm)_{ij}$ is unitary. □

7.2 Unitarity of fusion matrices

Recall from section 2.4 that for any W_i, W_j, W_k, W_t in $\mathcal{F}^{\boxtimes}$, we have a fusion matrix $\{F_{\alpha\beta}^{\beta'\alpha'}\}_{\alpha\in\Theta_{i*}^t, \beta\in\Theta_{jk}^*}^{\alpha'\in\Theta_{ij}^*, \beta'\in\Theta_{*k}^t}$ defined by the fusion relation

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \sum_{\alpha'\in\Theta_{ij}^*, \beta'\in\Theta_{*k}^t} F_{\alpha\beta}^{\beta'\alpha'} \mathcal{Y}_{\beta'}(\mathcal{Y}_{\alpha'}(w^{(i)}, z_i - z_j)w^{(j)}, z_j), \tag{7.15}$$

where $z_i, z_j \in \mathbb{C}^\times, 0 < |z_i - z_j| < |z_j| < |z_i|$, $\arg z_j$ is close to $\arg z_i$ as $z_j \to z_i$, and $\arg(z_i - z_j)$ is close to $\arg z_i$ as $z_j \to 0$. We let $F_{\alpha\beta}^{\beta'\alpha'} = 0$ if the source space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_\beta$, or if the target space of $\mathcal{Y}_{\alpha'}$ does not equal the charge space of $\mathcal{Y}_{\beta'}$. In this section, we show that fusion matrices are unitary.

Proposition 7.4. *Choose unitary V-modules W_i, W_k in $\mathcal{F}^{\boxtimes}$, W_j, W_t in $\mathcal{E} \cap \mathcal{F}^{\boxtimes}$. Then for any for* *any* $\alpha \in \Theta^t_{i*}, \beta \in \Theta^*_{jk}, \alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}$*, we have*

$$F^{\beta'\alpha'}_{\alpha\beta} = (B_+)^{B_+\beta',\alpha'}_{\alpha,B_+\beta} = (B_-)^{B_-\beta',\alpha'}_{\alpha,B_-\beta}. \tag{7.16}$$

Proof. Choose distinct $z_i, z_j, z_k \in S^1$ with arguments $\arg z_k < \arg z_j < \arg z_i < \arg z_k + 2\pi$, and assume that $0 < |z_i - z_j| < 1$. Choose $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(k)} \in W_k$. By corollary 2.18, we have

$$\begin{aligned}
&\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j)\mathcal{Y}^k_{k0}(w^{(k)}, z_k)\\
=&\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_{B_+\beta}(w^{(k)}, z_k)\mathcal{Y}^j_{j0}(w^{(j)}, z_j)\\
=&\sum_{\substack{\alpha'\in\Theta^*_{i*}\\ \beta'\in\Theta^*_{*k}}} (B_+)^{B_+\beta',\alpha'}_{\alpha,B_+\beta}\mathcal{Y}_{B_+\beta'}(w^{(k)}, z_k)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i)\mathcal{Y}^j_{j0}(w^{(j)}, z_j). \qquad (7.17)
\end{aligned}$$

On the other hand, by corollary 2.18 and theorem 5.2,

$$\begin{aligned}
&\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j)\mathcal{Y}^k_{k0}(w^{(k)}, z_k)\\
=&\sum_{s\in\mathcal{E}}\sum_{\substack{\alpha'\in\Theta^s_{ij}\\ \beta'\in\Theta^t_{sk}}} F^{\beta'\alpha'}_{\alpha\beta}\mathcal{Y}_{\beta'}\big(\mathcal{Y}_{\alpha'}(w^{(i)}, z_i - z_j)w^{(j)}, z_j\big)\mathcal{Y}^k_{k0}(w^{(k)}, z_k) \qquad (7.18)\\
=&\sum_{s\in\mathcal{E}}\sum_{\substack{\alpha'\in\Theta^s_{ij}\\ \beta'\in\Theta^t_{sk}}} F^{\beta'\alpha'}_{\alpha\beta}\mathcal{Y}_{B_+\beta'}(w^{(k)}, z_k)\mathcal{Y}^s_{s0}\big(\mathcal{Y}_{\alpha'}(w^{(i)}, z_i - z_j)w^{(j)}, z_j\big), \qquad (7.19)
\end{aligned}$$

where (7.18) and (7.19) are understood as products of two generalized intertwining operators (see the beginning of chapter 5). By proposition 2.17, (7.19) equals

$$\sum_{s\in\mathcal{E}}\sum_{\substack{\alpha'\in\Theta^s_{ij}\\ \beta'\in\Theta^t_{sk}}} F^{\beta'\alpha'}_{\alpha\beta}\mathcal{Y}_{B_+\beta'}(w^{(k)}, z_k)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i)\mathcal{Y}^j_{j0}(w^{(j)}, z_j). \tag{7.20}$$

Comparing this result with (7.17), we see immediately that $F^{\beta'\alpha'}_{\alpha\beta} = (B_+)^{B_+\beta',\alpha'}_{\alpha,B_+\beta}$. If we assume at the beginning that $\arg z_i < \arg z_j < \arg z_k < \arg z_i + 2\pi$, then we obtain $F^{\beta'\alpha'}_{\alpha\beta} = (B_-)^{B_-\beta',\alpha'}_{\alpha,B_-\beta}$. □

Proposition 7.5. *Let* W_i, W_j *be unitary* V*-modules in* $\mathcal{F}^\boxtimes$*. For each* $k \in \mathcal{E} \cap \mathcal{F}^\boxtimes$*, we let* $\{\mathcal{Y}_\alpha : \alpha \in$

$\Theta^k_{ij}\}$ *be a set of orthonormal basis of* $\mathcal{V}\binom{k}{i\ j}$ *under the inner product* Λ. *Then* $B_+\Theta^k_{ij} = \{\mathcal{Y}_{B_+\alpha} : \alpha \in \Theta^k_{ij}\}$ *and* $B_-\Theta^k_{ij} = \{\mathcal{Y}_{B_-\alpha} : \alpha \in \Theta^k_{ij}\}$ *are orthonormal bases of* $\mathcal{V}\binom{k}{j\ i}$.

Proof. Choose distinct $z_i, z_j \in S^1$ with arguments satisfying $\arg z_i < \arg z_j < \arg z_i + 2\pi$. By proposition 6.5-(3), for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have

$$\mathcal{Y}^j_{j0}(w^{(j)}, z_j)\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}}, z_i) = \sum_{\alpha\in\Theta^*_{ij}} \mathcal{Y}_{\alpha*}(\overline{w^{(i)}}, z_i)\mathcal{Y}_{B_-\alpha}(w^{(j)}, z_j). \tag{7.21}$$

Take the formal adjoint of both sides, we obtain

$$\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}}, z_i)^\dagger\mathcal{Y}^j_{j0}(w^{(j)}, z_j)^\dagger = \sum_{\alpha\in\Theta^*_{ij}} \mathcal{Y}_{B_-\alpha}(w^{(j)}, z_j)^\dagger\mathcal{Y}_{\alpha*}(\overline{w^{(i)}}, z_i)^\dagger. \tag{7.22}$$

Recall that $(\mathcal{Y}^j_{j0})^\dagger = \mathcal{Y}^0_{\bar{j}j}$ and $(\mathcal{Y}^0_{\bar{i}i})^\dagger = \mathcal{Y}^i_{i0}$. Thus, by equation (1.34), equation (7.22) shows that

$$\begin{aligned}\mathcal{Y}^i_{i0}(w^{(i)}, z_i)\mathcal{Y}^0_{\bar{j}j}(\overline{w^{(j)}}, z_j) &= \sum_{\alpha\in\Theta^*_{ij}} \mathcal{Y}_{(B_-\alpha)*}(\overline{w^{(j)}}, z_j)\mathcal{Y}_{\alpha}(w^{(i)}, z_i)\\ &= \sum_{\beta\in B_-\Theta^*_{ij}} \mathcal{Y}_{\beta*}(\overline{w^{(j)}}, z_j)\mathcal{Y}_{B_+\beta}(w^{(i)}, z_i),\end{aligned} \tag{7.23}$$

which, by proposition 6.5-(3), shows that $B_-\Theta^k_{ij}$ is an orthonormal basis of $\mathcal{V}\binom{k}{i\ j}$ for any $k \in \mathcal{E}$. The other case is treated in a similar way. □

Corollary 7.6. *For any* W_i, W_j, W_k *in* $\mathcal{F}^{\boxtimes}$ *and* W_t *in* $\mathcal{E}$, *the fusion matrix* $\{F^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha'\in\Theta^*_{ij},\beta'\in\Theta^t_{*k}}_{\alpha\in\Theta^t_{i*},\beta\in\Theta^*_{jk}}$ *is unitary.*

Proof. If W_j is irreducible, then W_j is unitarily equivalent to a unitary V-module in $\mathcal{E} \cap \mathcal{F}^{\boxtimes}$. The unitarity of the fusion matrix follows then from propositions 7.4, 7.5, and the unitarity of braid matrices proved in the last section. In general, the fusion matrix is diagonalized according to the orthogonal decomposition of W_j into irreducible submodules. Thus the unitarity can be proved easily. □

In this section, we prove that $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^{\boxtimes}}(V)$ is unitary when the unitary structure on $\mathcal{F}^{\boxtimes}$ is defined by Λ. By corollary 7.6, the associators are unitary. By proposition 7.5, the braid operators are unitary. That $\lambda_i : V \boxtimes W_i \to W_i$ and $\rho_i : W_i \boxtimes V \to W_i$ are unitary follows from equations (6.19) and (6.20).

Choose $W_{i_1}, W_{i_2}, W_{j_1}, W_{j_2}$ in $\mathcal{F}^{\boxtimes}$. We show, for any $F \in \mathrm{Hom}_V(W_{i_1}, W_{i_2}), G \in \mathrm{Hom}_V(W_{j_1}, W_{j_2})$, that

$$(F \otimes G)^* = F^* \otimes G^*. \tag{7.24}$$

Consider direct sum modules $W_i = W_{i_1} \oplus^{\perp} W_{i_2}, W_j = W_{j_1} \oplus^{\perp} W_{j_2}$. For each $k \in \mathcal{E}$, it is easy to see that $\mathcal{V}\binom{k}{i\,j}$ has the natural orthogonal decomposition

$$\mathcal{V}\begin{pmatrix} k \\ i\ j \end{pmatrix} = \bigoplus_{a,b=1,2}^{\perp} \mathcal{V}\begin{pmatrix} k \\ i_a\ j_b \end{pmatrix}, \tag{7.25}$$

which induces the natural decomposition

$$W_i \boxtimes W_j = \bigoplus_{a,b=1,2}^{\perp} W_{i_a} \boxtimes W_{j_b}. \tag{7.26}$$

Therefore, if we regard F, G as endomorphisms of the modules W_i, W_j respectively, then $F \otimes G$ and $F^* \otimes G^*$ can be regarded as endomorphisms of $W_i \boxtimes W_j$. Thus, it suffices to prove equation (7.24) for any $F \in \mathrm{End}_V(W_i), G \in \mathrm{End}_V(W_j)$.

Since $\mathrm{End}_V(W_i)$ and $\mathrm{End}_V(W_j)$ are C^*-algebras (see theorem 2.21), they are spanned by unitary elements inside them. Therefore, by linearity, it suffices to prove (7.24) when $F \in \mathrm{End}_V(W_i), G \in \mathrm{End}_V(W_j)$ are unitary operators. By equation (2.56), it is easy to see that $F \otimes G$ is unitary. Hence we have

$$(F^* \otimes G^*)(F \otimes G) = F^* F \otimes G^* G = \mathrm{id}_i \otimes \mathrm{id}_j = \mathrm{id}_{ij}, \tag{7.27}$$

which implies that $F^* \otimes G^* = (F \otimes G)^{-1} = (F \otimes G)^*$. This proves relation (7.24).

For each W_i in $\mathcal{F}^{\boxtimes}$, the twist $\vartheta_i = e^{2i\pi L_0}$ is clearly unitary. Hence, in order to prove the unitarity of $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^{\boxtimes}}(V)$, it remains to find $\mathrm{ev}_i, \mathrm{coev}_i$, such that equations (2.69) and (2.70) hold.

To prove this, we let $\mathrm{ev}_{i,\bar{i}} \in \mathrm{Hom}_V(W_i \boxtimes W_{\bar{i}}, V)$ be the homomorphism represented by the intertwining operator $\mathcal{Y}^0_{i\bar{i}}$, and let $\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}^*_{i\bar{i}}$. Since i and $\bar{\bar{i}}$ are identified, we can define $\mathrm{ev}_{\bar{i},i}$ and $\mathrm{coev}_{\bar{i},i}$ in a similar way. Set $\mathrm{ev}_i = \mathrm{ev}_{\bar{i},i}, \mathrm{coev}_i = \mathrm{coev}_{i,\bar{i}}$. If we can verify, for all W_i in $\mathcal{F}^{\boxtimes}$, the following relations:

$$(\mathrm{id}_i \otimes \mathrm{ev}_{\bar{i},i}) \circ (\mathrm{coev}_{i,\bar{i}} \otimes \mathrm{id}_i) = \mathrm{id}_i, \tag{7.28}$$

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathrm{id}_i) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i}) = \mathrm{id}_i, \tag{7.29}$$

$$\mathrm{ev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i} \circ \sigma_{i,\bar{i}} \circ (\vartheta_i \otimes \mathrm{id}_{\bar{i}}), \tag{7.30}$$

$$\mathrm{coev}_{i,\bar{i}} = (\mathrm{id}_i \otimes \vartheta_{\bar{i}}^{-1}) \circ \sigma_{i,\bar{i}}^{-1} \circ \mathrm{coev}_{\bar{i},i}, \tag{7.31}$$

then equations (2.64), (2.65), (2.69), and (2.70) are true for all W_i, and our modular tensor category is unitary.

To begin with, we define the positive number d_i to be the norm square of the vector $\mathcal{Y}^0_{i\bar{i}}$ inside $\mathcal{V}\binom{0}{i\ \bar{i}}$, i.e.,

$$d_i = \|\mathcal{Y}^0_{i\bar{i}}\|^2. \tag{7.32}$$

By propositions 1.14 and 7.5, $d_{\bar{i}} = d_i$. The following property will indicate that d_i is the quantum dimension of W_i.

Proposition 7.7.

$$\mathrm{ev}_{i,\bar{i}} \circ \mathrm{coev}_{i,\bar{i}} = d_i. \tag{7.33}$$

Proof. First we assume that W_i is irreducible. Then $\{\mathcal{Y}^0_{i\bar{i}}\}$ is a basis of $\mathcal{V}\binom{0}{i\ \bar{i}}$. Let $\{\check{\mathcal{Y}}^0_{i\bar{i}}\}$ be its dual basis. Then $\check{\mathcal{Y}}^\alpha = d_i^{\frac{1}{2}}\check{\mathcal{Y}}^0_{i\bar{i}}$ has unit length. Now, for any $v \in V$, $\mathrm{ev}_{i,\bar{i}}$ maps $\check{\mathcal{Y}}^\alpha \otimes v \in W_i \boxtimes W_{\bar{i}}$ to $\langle \check{\mathcal{Y}}^\alpha, \mathcal{Y}^0_{i\bar{i}}\rangle v = d_i^{\frac{1}{2}}\langle \check{\mathcal{Y}}^0_{i\bar{i}}, \mathcal{Y}^0_{i\bar{i}}\rangle v = d_i^{\frac{1}{2}} v$. It follows that its adjoint $\mathrm{coev}_{i,\bar{i}}$ maps each $v \in V$ to $d_i^{\frac{1}{2}}\check{\mathcal{Y}}^\alpha \otimes v$. Hence $\mathrm{ev}_{i,\bar{i}} \circ \mathrm{coev}_{i,\bar{i}}(v) = d_i v$.

In general, W_i has decomposition $W_i = \bigoplus^{\perp}_a W_{i_a}$, where each W_{i_a} is irreducible. Let p_a be the projection of W_i on W_{i_a}. Then the projection $\overline{p_a}$ of $W_{\bar{i}}$ on $W_{\overline{i_a}}$ satisfies $\overline{p_a}\overline{w^{(i)}} = \overline{p_a w^{(i)}}$ ($w^{(i)} \in W_i$). It is easy to check that

$$\mathrm{ev}_{i,\bar{i}} = \sum_a \mathrm{ev}_{i,\bar{i}} \circ (p_a \otimes \overline{p_a}) = \sum_a \mathrm{ev}_{i_a,\overline{i_a}}, \tag{7.34}$$

$$\mathrm{coev}_{i,\bar{i}} = \sum_a (p_a \otimes \overline{p_a}) \circ \mathrm{coev}_{i,\bar{i}} = \sum_a \mathrm{coev}_{i_a,\overline{i_a}}, \tag{7.35}$$

and $d_i = \sum_a d_{i_a}$. The general case can be proved using these relations. $\square$

Now we are ready to prove equations (7.28)-(7.31).

Proof of equation (7.29). By equations (7.34) and (7.35), it suffices to prove (7.29) when W_i is irreducible. Choose $w_1^{(i)}, w_2^{(i)} \in W_i$. Choose $z_1, z_2 \in \mathbb{C}^\times$ satisfying $0 < |z_2 - z_1| < |z_1| < |z_2|$. Choose $\arg z_2$, let $\arg z_1$ be close to $\arg z_2$ as $z_2 - z_1 \to 0$, and let $\arg(z_2 - z_1)$ be close to $\arg z_2$ as $z_1 \to 0$. Since $\|\check{\mathcal{Y}}^0_{\bar{i}i}\|^2 = d_i^{-1}$, by transport formula we have

$$\begin{aligned}
&Y_i\big(\mathcal{Y}^0_{i\bar{i}}(w_2^{(i)}, z_2 - z_1)\overline{w_1^{(i)}}, z_1\big)\\
=&d_i^{-1}(\mathcal{Y}^0_{\bar{i}i})^\dagger(w_2^{(i)}, z_2)\mathcal{Y}^0_{\bar{i}i}(\overline{w_1^{(i)}}, z_1) + \mathcal{Y}_\gamma(w_2^{(i)}, z_2)\mathcal{Y}_\beta(\overline{w_1^{(i)}}, z_1)\\
=&d_i^{-1}\mathcal{Y}^i_{i0}(w_2^{(i)}, z_2)\mathcal{Y}^0_{\bar{i}i}(\overline{w_1^{(i)}}, z_1) + \mathcal{Y}_\gamma(w_2^{(i)}, z_2)\mathcal{Y}_\beta(\overline{w_1^{(i)}}, z_1)
\end{aligned} \tag{7.36}$$

where $\mathcal{Y}_\beta, \mathcal{Y}_\gamma$ are a chain of intertwining operators, and the target space of $\mathcal{Y}_\beta$ does not contain any submodule equivalent to the vacuum module V. Equation (7.36) is equivalent to the relation

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathrm{id}_i) = d_i^{-1}(\mathrm{id}_i \otimes \mathrm{ev}_{\bar{i},i}) + \mathcal{Y}_\gamma \circ (\mathrm{id}_i \otimes \mathcal{Y}_\beta), \tag{7.37}$$

where $\mathcal{Y}_\gamma$ and $\mathcal{Y}_\beta$ denote the corresponding morphisms. By proposition 7.7,

$$\begin{aligned}
&(\mathrm{ev}_{i,\bar{i}} \otimes \mathrm{id}_i) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i})\\
=&d_i^{-1}(\mathrm{id}_i \otimes \mathrm{ev}_{\bar{i},i}) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i}) + \mathcal{Y}_\gamma \circ (\mathrm{id}_i \otimes \mathcal{Y}_\beta) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i})\\
=&\mathrm{id}_i + \mathcal{Y}_\gamma \circ \big(\mathrm{id}_i \otimes (\mathcal{Y}_\beta \circ \mathrm{coev}_{\bar{i},i})\big).
\end{aligned} \tag{7.38}$$

Since $\mathcal{Y}_\beta \circ \mathrm{coev}_{\bar{i},i}$ is a morphism from the vacuum module V to a V-module with no irreducible submodule equivalent to V, $\mathcal{Y}_\beta \circ \mathrm{coev}_{\bar{i},i}$ must be zero. So (7.38) equals id_i, and equation (7.29) is proved. □

Proof of equations (7.28)*,* (7.30)*, and* (7.31). Take the adjoint of equation (7.29), we immediately obtain equation (7.28). Equation (7.30) follows from equation (1.41). Equation (1.42) indicates that

$$\mathrm{ev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i} \circ \sigma_{i,\bar{i}} \circ (\mathrm{id}_i \otimes \vartheta_{\bar{i}}), \tag{7.39}$$

the adjoint of which is (7.31). □

Thus we've proved the unitarity of our ribbon fusion category.

Theorem 7.8. *Let V be unitary, energy bounded, and strongly local, and let $\mathcal{F}$ be a non-empty set of non-zero irreducible unitary V-modules satisfying condition A or B. If we define a unitary structure on $\mathcal{F}^{\boxtimes}$ using Λ, then the ribbon fusion category $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^{\boxtimes}}(V)$ is unitary.*

Note that the proof of this theorem uses only the positive definiteness of Λ.

CHAPTER 8

APPLICATIONS

8.1 APPLICATION TO UNITARY VIRASORO VOAs

Let $\mathrm{Vir} = \mathrm{Span}_{\mathbb{C}}\{C, L_n : n \in \mathbb{Z}\}$ be the Virasoro Lie algebra satisfying the relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}C \qquad (m, n \in \mathbb{Z}),$$
$$[C, L_n] = 0 \qquad (n \in \mathbb{Z}).$$

If W is a Vir-module, and the vector space W is equipped with an inner product $\langle \cdot | \cdot \rangle$, we say that W is a unitary Vir-module, if $L_n^\dagger = L_{-n}$ holds for any $n \in \mathbb{Z}$. More precisely, this means that for any $w_1, w_2 \in W$, we have

$$\langle L_n w_1 | w_2 \rangle = \langle w_1 | L_{-n} w_2 \rangle. \tag{8.1}$$

Choose Lie subalgebras $\mathrm{Vir}_+ = \mathrm{Span}_{\mathbb{C}}\{L_n : n \in \mathbb{Z}_{>0}\}$ and $\mathrm{Vir}_- = \mathrm{Span}_{\mathbb{C}}\{L_n : n \in \mathbb{Z}_{<0}\}$ of Vir, and let $U(\mathrm{Vir})$ be the universal enveloping algebra of Vir. For each $c, h \in \mathbb{C}$, the Verma module $M(c, h)$ for Vir is the free $U(\mathrm{Vir}_-)$-module generated by a distinguished vector (the highest weight vector) $v_{c,h}$, subject to the relation

$$U(\mathrm{Vir}_+)v_{c,h} = 0, \qquad Cv_{c,h} = cv_{c,h}, \qquad L_0 v_{c,h} = hv_{c,h}. \tag{8.2}$$

Then there exists a unique maximal proper submodule $J(c, h)$ of $M(c, h)$. We let $L(c, h) = M(c, h)/J(c, h)$. It was proved in [FQS84] and [GKO86] that the Vir-module $L(c, h)$ is unitarizable if and only if there exist $m, r, s \in \mathbb{Z}$ satisfying $2 \leqslant m, 1 \leqslant r \leqslant m-1, 1 \leqslant s \leqslant m$, such that

$$c = 1 - \frac{6}{m(m+1)}, \tag{8.3}$$

$$h = h_{r,s} = \frac{\big((m+1)r - ms\big)^2 - 1}{4m(m+1)}. \tag{8.4}$$

For such a module $L(c,h)$, we fix a unitary structure such that $\langle v_{c,h}|v_{c,h}\rangle$=1.

Let $\Omega = v_{c,0}, \nu = L_{-2}\Omega$. Then there exists a unique VOA structure on $L(c,0)$, such that Ω is the vacuum vector, and ν is the conformal vector (cf.[FZ92]). Let $E = \{\Omega, \nu\}$, then E is a set of quasi-primary vectors generating $L(c,0)$.

We now assume that c satisfies relation (8.3). Then by [DL14] theorem 4.2 or [CKLW15] proposition 5.17, $L(c,0)$ is a unitary VOA. The PCT operator θ is determined by the fact that θ fixes vectors in E. $L(c,0)$ satisfies conditions $(\alpha), (\beta)$, and (γ) in the introduction. (See the introduction of [Hua08b], and the reference therein.)

Since $Y(\nu, n) = L_{n-1}$, representations of $L(c,0)$ are determined by their restrictions to Vir. By [Wang93] theorem 4.2, irreducible representations of $L(c,0)$ are precisely those that can be restricted to irreducible Vir-modules of the form $L(c,h_{r,s})$, where the highest weight $h_{r,s}$ satisfies relation (8.4). By proposition 1.10, $L(c,h_{r,s})$ is a unitary $L(c,0)$-module. It follows that any $L(c,0)$-module is unitarizable. Clearly the conformal dimension of $L(c,h_{r,s})$ is $h_{r,s}$.

Let $\mathcal{F} = \{L(c,h_{1,2}), L(c,h_{2,2})\}$. The fusion rules of $L(c,0)$ (see [Wang93] theorem 4.3) indicate that $\mathcal{F}$ is **generating**, i.e., any unitary $L(c,0)$-module is in $\mathcal{F}^{\boxtimes}$. We check that $\mathcal{F}$ satisfies condition A:

Condition A-(a): Since we know that any $L(c,0)$-module is unitarizable, condition A-(a) is obvious.

Condition A-(b): Since $E \subset E^1(L(c,0))$, $E^1(L(c,0))$ is generating.

Condition A-(c): If $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ is unitary and irreducible (hence W_i, W_j, W_k restrict to irreducible highest weight Vir-modules), we choose a non-zero highest weight vector $v^{(i)} \in W_i$. We then define a linear map

$$\phi_\alpha : W_j \to W_k\{x\},$$
$$w^{(j)} \mapsto \phi_\alpha(x)w^{(j)} = \mathcal{Y}_\alpha(v^{(i)}, x)w^{(j)}.$$

Then ϕ_α is a primary field in the sense of [Loke94] chapter II. By [Loke94] proposition IV.1.3, if $W_i \in \mathcal{F}$, then ϕ_α satisfies 0-th order energy bounds. This proves condition A-(c). Theorem 7.8 now implies the following:

Theorem 8.1. *Let $c = 1 - \frac{6}{m(m+1)}$ where $m = 2, 3, 4, \ldots$, and let $L(c, 0)$ be the unitary Virasoro VOA with central charge c. Then any $L(c, 0)$-module is unitarizable, the transport matrices of $L(c, 0)$ are positive definite, and the modular tensor category $\mathrm{Rep}^{\mathrm{u}}(L(c, 0))$ of the unitary representations of $L(c, 0)$ is unitary.*

8.2 Application to unitary affine VOAs

Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$, and let $L(\lambda)$ be the irreducible highest weight module of $\mathfrak{g}$ with highest weight λ and a distinguished highest (non-zero) vector $v_\lambda \in L(\lambda)$.

Choose the normalized invariant bilinear form $(\cdot, \cdot)$ satisfying $(\theta, \theta) = 2$, where θ is the highest root of $\mathfrak{g}$. Let $\widehat{\mathfrak{g}} = \mathrm{Span}_{\mathbb{C}}\{K, X(n) : X \in \mathfrak{g}, n \in \mathbb{Z}\}$ be the affine Lie algebra satisfying

$$
\begin{aligned}
&[X(m), Y(n)] = [X, Y](m+n) + m(X, Y)\delta_{m,-n}K \qquad (X, Y \in \mathfrak{g}, m, n \in \mathbb{Z}), \\
&[K, X(n)] = 0 \qquad (X \in \mathfrak{g}, n \in \mathbb{Z}).
\end{aligned}
$$

Let $\mathfrak{g}_{\mathbb{R}}$ be a compact real form of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus_{\mathbb{R}} i\mathfrak{g}_{\mathbb{R}}$. If W is a $\widehat{\mathfrak{g}}$-module, and the vector space W is equipped with an inner product $\langle \cdot | \cdot \rangle$, we say that W is a unitary $\widehat{\mathfrak{g}}$-module, if for any $X \in \mathfrak{g}_{\mathbb{R}}$ and $n \in \mathbb{Z}$, we have

$$
X(n)^\dagger = -X(-n), \qquad K^\dagger = K. \tag{8.5}
$$

Let $U(\widehat{\mathfrak{g}})$ be the universal enveloping algebra of $\widehat{\mathfrak{g}}$. Choose Lie subalgebras $\widehat{\mathfrak{g}}_+ = \mathrm{Span}_{\mathbb{C}}\{X(n) : X \in \mathfrak{g}, n > 0\}, \widehat{\mathfrak{g}}_- = \mathrm{Span}_{\mathbb{C}}\{X(n) : X \in \mathfrak{g}, n < 0\}$ of $\widehat{\mathfrak{g}}$. We regard $\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathfrak{g}}$ by identifying $X \in \mathfrak{g}$ with $X(0) \in \widehat{\mathfrak{g}}$. For any $k \in \mathbb{C}, \lambda \in \mathfrak{h}^*$, the Verma module $M(k, \lambda)$ for $\widehat{\mathfrak{g}}$ is the free $U(\widehat{\mathfrak{g}}_-)$-module generated by $L(\lambda)$ and subject to the conditions

$$
U(\widehat{\mathfrak{g}}_+)L(\lambda) = 0, \qquad K|_{L(\lambda)} = k \cdot \mathrm{id}|_{L(\lambda)}. \tag{8.6}
$$

We let $M(k, \lambda)$ be graded by $\mathbb{Z}_{\geqslant 0}$: For any $X_1, \ldots, X_m \in \mathfrak{g}, n_1, \ldots, n_m > 0, v \in L(\lambda)$, the weight of $X_1(-n_1)\cdots X_m(-n_m)v$ equals $n_1+\cdots+n_m$. There exits a unique maximal proper graded submodule $J(k,\lambda)$ of $M(k,\lambda)$. We let $L(k,\lambda)=M(k,\lambda)/J(k,\lambda)$. Then by [Kac94] theorem 11.7, the $\widehat{\mathfrak{g}}$-module $L(k,\lambda)$ is unitarizable if and only if

$$k=0,1,2,\ldots, \tag{8.7}$$

$$\lambda \text{ is a dominant integral weight of } \mathfrak{g}, \text{ and } (\lambda,\theta)\leqslant k. \tag{8.8}$$

For such a $\widehat{\mathfrak{g}}$-module $L(k,\lambda)$, we fix a unitary structure.

Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. Let Ω be a highest weight vector of $L(k,0)$. It was proved in [FZ92] that when $k\neq -h^\vee$, there exists a unique VOA structure on $L(k,0)$, such that Ω is the vacuum vector, that for any $X\in\mathfrak{g}$ we have

$$Y\big(X(-1)\Omega,x\big)=\sum_{n\in\mathbb{Z}}X(n)x^{-n-1}, \tag{8.9}$$

and that the conformal vector ν is defined by

$$\nu=\frac{1}{2(k+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}}X_i(-1)^2\Omega, \tag{8.10}$$

where $\{X_i\}$ is an orthonormal basis of $i\mathfrak{g}_{\mathbb{R}}$ under the inner product $(\cdot,\cdot)$. The set $E=\{\Omega,X(-1)\Omega: X\in\mathfrak{g}_{\mathbb{R}}\}$ generates $L(k,0)$. By writing the operator $L_1=Y(\nu,2)$ in terms of $X(n)$'s using Jacobi identity, one can show that the vectors in E are quasi-primary.

We now assume that $k\in\mathbb{Z}_{\geqslant 0}$. Then $L(k,0)$ satisfies conditions $(\alpha),(\beta)$, and (γ) in the introduction. (See the introduction of [Hua08b], and the reference therein.) By [DL14] theorem 4.7 or [CKLW15] proposition 5.17, $L(k,0)$ is a unitary VOA, and the PCT operator θ is determined by the fact that it fixes the vectors in E.

Representations of $L(k,0)$ are determined by their restrictions to $\widehat{\mathfrak{g}}$. By [FZ92] theorem 3.1.3, irreducible $L(k,0)$-modules are precisely those which can be restricted to the $\widehat{\mathfrak{g}}$-modules of the form $L(k,\lambda)$, where $\lambda\in\mathfrak{h}^*$ satisfies condition (8.8). By proposition 1.10, these $L(k,0)$-modules are unitary. Hence all $L(k,0)$-modules are unitarizable, and any set $\mathcal{F}$ of irreducible unitary $L(k,0)$-module satisfies condition A-(a).

By proposition 3.6, $E \subset E^1(L(k,0))$. Since E generates $L(k,0)$, any $\mathcal{F}$ also satisfies condition A-(b). Checking condition A-(c) is much harder, and requires case by case studies. Note that given the set $\mathcal{F}$, finding out which irreducible modules are inside $\mathcal{F}^{\boxtimes}$ requires the knowledge of fusion rules. A very practical way of calculating fusion rules for a unitary affine VOA is to calculate the dimensions of the spaces of primary fields.

Primary fields

Fix $k \in \mathbb{Z}_{>0}$. For each $\lambda \in \mathfrak{h}^*$ satisfying condition (8.8), we write $U_\lambda = L(\lambda), W_\lambda = L(k,\lambda)$. Let Δ_λ be the conformal dimension of the $L(k,0)$-module W_λ. We define the normalized energy operator on W_λ to be $D = L_0 - \Delta_\lambda$.

Assume that $\lambda, \mu, \nu \in \mathfrak{h}^*$ satisfy condition (8.8). We let $\Delta^\nu_{\lambda\mu} = \Delta_\lambda + \Delta_\mu - \Delta_\nu$. A **type** ${\nu \choose \lambda\ \mu}$ **primary field** ϕ_α is a linear map

$$\phi_\alpha : U_\lambda \otimes W_\mu \to W_\nu[[x^{\pm 1}]]x^{-\Delta^\nu_{\lambda\mu}},$$
$$u^{(\lambda)} \otimes w^{(\mu)} \mapsto \phi_\alpha(u^{(\lambda)}, x)w^{(\mu)} = \sum_{n\in\mathbb{Z}} \phi_\alpha(u^{(\lambda)}, n)w^{(\mu)}x^{-\Delta^\nu_{\lambda\mu}-n}$$
$$(\text{where } \phi_\alpha(u^\lambda, n) \in \mathrm{Hom}(W_\mu, W_\nu)),$$

such that for any $u^{(\lambda)} \in U_\lambda, X \in \mathfrak{g}, m \in \mathbb{Z}$, we have

$$[X(m), \phi_\alpha(u^{(\lambda)}, x)] = \phi_\alpha(Xu^{(\lambda)}, x)x^m, \tag{8.11}$$
$$[L_0, \phi_\alpha(u^{(\lambda)}, x)] = \Big(x\frac{d}{dx} + \Delta_\lambda\Big)\phi_\alpha(u^{(\lambda)}, x). \tag{8.12}$$

We say that U_λ is the **charge space** of ϕ_α.

Note that the above two conditions are equivalent to that for any $m, n \in \mathbb{Z}, u^{(\lambda)} \in U_\lambda, X \in \mathfrak{g}$,

$$[X(m), \phi_\alpha(u^{(\lambda)}, n)] = \phi_\alpha(Xu^{(\lambda)}, n+m), \tag{8.13}$$
$$[D, \phi_\alpha(u^{(\lambda)}, n)] = -n\phi_\alpha(u^{(\lambda)}, n). \tag{8.14}$$

Primary fields and intertwining operators are related in the following way: Let $\mathcal{V}_{\mathrm{p}}{\nu \choose \lambda\ \mu}$ be the vector space of type ${\nu \choose \lambda\ \mu}$ primary fields. If $\mathcal{Y}_\alpha \in \mathcal{V}{\nu \choose \lambda\ \mu}$ is a type ${\nu \choose \lambda\ \mu}$ intertwining operator of

$L(k,0)$, then by relation (1.26), for any $w^{(\lambda)} \in W_\lambda$ we have,

$$\mathcal{Y}_\alpha(w^{(\lambda)}, x) = x^{L_0}\mathcal{Y}_\alpha(x^{-L_0}w^{(\lambda)}, 1)x^{-L_0} \in \mathrm{End}(W_\mu, W_\nu)[[x^{\pm 1}]]x^{-\Delta^\nu_{\lambda\mu}} \tag{8.15}$$

where $\mathcal{Y}_\alpha(\cdot, 1) = \mathcal{Y}_\alpha(\cdot, x)\big|_{x=1}$. We define a linear map $\phi_\alpha : U_\lambda \otimes W_\mu \to W_\nu[[x^{\pm 1}]]x^{-\Delta^\nu_{\lambda\mu}}$ to be the restriction of $\mathcal{Y}_\alpha$ to $U_\lambda \otimes W_\mu$. Then the Jacobi identity and the translation property for $\mathcal{Y}_\alpha$ implies that $\mathcal{Y}_\alpha$ satisfies equations (8.11) and (8.12). Therefore, we have a linear map

$$\Phi : \mathcal{V}\binom{\nu}{\lambda\ \mu} \to \mathcal{V}_{\mathrm{p}}\binom{\nu}{\lambda\ \mu}, \qquad \mathcal{Y}_\alpha \mapsto \phi_\alpha. \tag{8.16}$$

The injectivity of Φ follows immediately from relation (1.22) or from corollary 2.15. Φ is also surjective. Indeed, if we fix any $z \in \mathbb{C}^\times$ and define another linear map

$$\Psi_z : \mathcal{V}_{\mathrm{p}}\binom{\nu}{\lambda\ \mu} \to (W_\lambda \otimes W_\mu \otimes W_{\overline{\nu}})^*,$$
$$\phi_\alpha \mapsto \phi_\alpha(\cdot, z) = \phi_\alpha(\cdot, x)\big|_{x=z},$$

then by equation (8.12), Ψ_z is injective. By equation (8.11) and [Ueno08] theorem 3.18, the dimension of the image of Ψ_z is no greater than that of "the space of vacua" $\mathcal{V}^\dagger_{\mu\lambda\overline{\nu}}(\mathbb{P}^1; 0, z, \infty)$ defined in [TUY89] and [Ueno08]. The later can be calculated using the Verlinde formula proved in [Bea94], [Fal94], and [Tel95]. The same Verlinde formula for $N^\nu_{\lambda\mu}$ proved in [Hua08a] shows that the dimension of the vector space $\mathcal{V}\binom{\nu}{\lambda\ \mu}$ (which is the fusion rule $N^\nu_{\lambda\mu}$) equals that of $\mathcal{V}^\dagger_{\mu\lambda\overline{\nu}}(\mathbb{P}^1; 0, z, \infty)$. So $\dim \mathcal{V}_{\mathrm{p}}\binom{\nu}{\lambda\ \mu} \leqslant N^\nu_{\lambda\mu}$, and hence Φ must be surjective. We conclude the following:

Proposition 8.2. *The linear map Φ defined in* (8.16) *is an isomorphism. In particular, the fusion rule $N^\nu_{\lambda\mu}$ of $L(k,0)$ equals the dimension of the vector space of type $\binom{\nu}{\lambda\ \mu}$ primary fields of $L(k,0)$.*

Theorem 8.3. *Let $k = 0, 1, 2, \dots$, and let $L(k,0)$ be the level k unitary affine VOA associated to $\mathfrak{g}$. Then any $L(k,0)$-module is unitarizable. Suppose that $\mathcal{F}$ is a* generating *set of irreducible unitary $L(k,0)$-modules (i.e., $\mathcal{F}^{\boxtimes}$ contains any unitary $L(k,0)$-module), and that for any $\lambda \in \mathcal{F}$, all primary fields of $L(k,0)$ with charge spaces U_λ are energy-bounded. Then the transport matrices of $L(k,0)$ are positive definite, and the modular tensor category $\mathrm{Rep}^{\mathrm{u}}(L(k,0))$ is unitary.*

We now show that theorem 8.3 can be applied to the unitary affine VOAs of type A_n and D_n.

The case $\mathfrak{g} = \mathfrak{sl}_n$ $(n \geqslant 2)$

Let $L(\square)$ be the (n-dimensional) vector representation of $\mathfrak{sl}_n$, and let

$$\mathcal{F} = \{L(k, \square)\}.$$

In [Was98], especially in section 25, it was proved that if $\lambda = \square$ and the weights μ, ν of $\mathfrak{sl}_n$ satisfy condition (8.8), then

$$\dim \mathcal{V}_{\mathrm{p}}\binom{\nu}{\lambda \; \mu} = \dim \Big(\mathrm{Hom}_{\mathfrak{g}}\big(L(\lambda) \otimes L(\mu), L(\nu)\big)\Big). \tag{8.17}$$

(Note that this relation is not true for general $L(\lambda)$.) Using this relation, one can show that $\mathcal{F}$ is generating. In the same section, it was proved that any $\phi_\alpha \in \mathcal{V}_{\mathrm{p}}\binom{\nu}{\square \; \mu}$ satisfies 0-th order energy bounds.

The case $\mathfrak{g} = \mathfrak{so}_{2n}$ $(n \geqslant 3)$

Let $L(\square)$ be the vector representation of $\mathfrak{so}_{2n}$, and let $L(s_+)$ and $L(s_-)$ be the two half-spin representations of $\mathfrak{so}_{2n}$. In [TL04] chapter IV, it was proved that if λ equals $\square$ or $s_\pm$, and the weights μ, ν of $\mathfrak{so}_{2n}$ satisfy condition (8.8), then relation (8.17) holds. This shows that the set

$$\mathcal{F} = \{L(k, \square), L(k, s_+), L(k, s_-)\}$$

is generating. By [TL04] theorem VI.3.1, any primary field whose charge space is $L(k, \square)$, $L(k, s_+)$, or $L(k, s_-)$ is energy-bounded.

We conclude the following.

Theorem 8.4. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$ $(n \geqslant 2)$ or $\mathfrak{so}_{2n}$ $(n \geqslant 3)$, let $k = 0, 1, 2, \ldots$, and let $L(k, 0)$ be the unitary affine VOA associated to $\mathfrak{g}$. Then the transport matrices of $L(k, 0)$ are positive definite, and the modular tensor category $\mathrm{Rep}^{\mathrm{u}}(L(k, 0))$ of the unitary representations of $L(k, 0)$ is unitary.*

Other examples

As we see in theorem 8.3, to finish proving the unitarity of the modular tensor categories associated to unitary affine VOAs, one has to show, for the remaining types, that a "generating" set of primary fields are energy-bounded. Our success in type A_n and D_n unitary WZW models, as well as in unitary minimal models, shows that achieving this goal is promising. Indeed, the main idea of proving the energy-boundedness of a primary field in [Was98], [Loke94], and [TL04] is to embedd the original VOA V in a larger (super) VOA $\widetilde{V}$, the energy-boundedness of the field operators of which is easy to show, and realize the primary field as the compression of a field operator in $\widetilde{V}$. This strategy is proved to work for some other types of affine VOAs, say type G_2 (cf. [Gui]). We expect that it works for all examples.

8.3 Full conformal field theory with reflection positivity

In this section, we give an interpretation of our unitarity results from the perspective of full conformal field theory. In [HK07], Y.Z.Huang and L.Kong constructed (genus 0) full conformal field theory for V. This construction relies on the non-degeneracy of a bilinear form on each pair $\mathcal{V}\binom{k}{i\,j}\otimes\mathcal{V}\binom{\overline{k}}{\overline{i}\,\overline{j}}$, which follows from the rigidity of $\mathrm{Rep}(V)$. These bilinear forms $(\cdot,\cdot)$ are directly related to our sesquilinear forms $\Lambda(\cdot|\cdot)$ on each $\mathcal{V}\binom{k}{i\,j}$:

$$(\mathcal{Y}_\alpha,\mathcal{Y}_{\overline{\beta}})=\Lambda(\mathcal{Y}_\alpha|\mathcal{Y}_\beta)\qquad(\alpha,\beta\in\mathcal{V}\binom{k}{i\,j}).\tag{8.18}$$

In light of this relation, we sketch the construction of full field algebras in [HK07] from a unitary point of view.

Let us assume that V is unitary, all V-modules are unitarizable, and all transport matrices are positive definite. (This last condition holds for V if there exists a generating set $\mathcal{F}$ of irreducible unitary V-modules satisfying condition A of B.) We define a vector space

$$F=\bigoplus_{i\in\mathcal{E}}W_i\otimes W_{\overline{i}}.\tag{8.19}$$

Its algebraic completion is $\widehat{F}=\bigoplus_{i\in\mathcal{E}}\widehat{W}_i\otimes\widehat{W}_{\overline{i}}$.

For each $i, j, k \in \mathcal{E}$, we choose an orthonormal basis Θ^k_{ij} of $\mathcal{V}\binom{k}{i\,j}$ under the inner product Λ. The full field operator $\mathbb{Y}$ is defined to be an $\mathrm{End}(F \otimes F, \widehat{F})$-valued continuous function on $\mathbb{C}^\times$, such that for any $w_L^{(i)} \otimes \overline{w_R^{(i)}} \in W_i \otimes W_{\overline{i}} \subset F, w_L^{(j)} \otimes \overline{w_R^{(j)}} \in W_j \otimes W_{\overline{j}} \subset F$,

$$\mathbb{Y}(w_L^{(i)} \otimes \overline{w_R^{(i)}}; z, \overline{z})(w_L^{(j)} \otimes \overline{w_R^{(j)}}) = \sum_{k \in \mathcal{E}} \sum_{\alpha \in \Theta^k_{ij}} \mathcal{Y}_\alpha(w_L^{(i)}, z) w_L^{(j)} \otimes \mathcal{Y}_{\overline{\alpha}}(\overline{w_R^{(i)}}, \overline{z}) \overline{w_R^{(j)}}. \tag{8.20}$$

Then $(F, \mathbb{Y})$ is a full field algebra of V satisfying certain important properties, including the commutativity ([HK07] proposition 1.5) and the associativity ([HK07] proposition 1.4). In fact, in our unitarity context, it is not hard for the reader to check that these two properties are equivalent to the unitarity of braid matrices and fusion matrices respectively.

Let us equip the vector space F with an *inner product* $\langle\cdot|\cdot\rangle$, such that the decomposition (8.19) is orthogonal, and for any $i \in \mathcal{E}, w_{L,1}^{(i)}, w_{R,1}^{(i)}, w_{L,2}^{(i)}, w_{R,2}^{(i)} \in W_i$,

$$\langle w_{L,1}^{(i)} \otimes \overline{w_{R,1}^{(i)}} | w_{L,2}^{(i)} \otimes \overline{w_{R,2}^{(i)}} \rangle = d_i^{-1} \langle w_{L,1}^{(i)} | w_{L,2}^{(i)} \rangle \langle w_{R,2}^{(i)} | w_{R,1}^{(i)} \rangle. \tag{8.21}$$

We also define an antilinear operator $\theta : F \to F$ sending each $w_L^{(i)} \otimes \overline{w_R^{(i)}}$ to $w_R^{(i)} \otimes \overline{w_L^{(i)}}$, which is easily checked to be an anti-automorphism:

$$\theta Y(w; z, \overline{z}) = Y(\theta w; z, \overline{z})\theta \qquad (w \in F). \tag{8.22}$$

We call θ the **PCT** operator of $(F, \mathbb{Y})$.

Note that when V is non-unitary, we can only define a non-degenerate bilinear form on F, and show that under this bilinear form, the full field algebra $(F, \mathbb{Y})$ satisfies the *invariance property* ([HK07] definition 3.9). But in our case, this invariance property should be replaced by the **reflection positivity**:

$$\mathbb{Y}(w; z, \overline{z})^\dagger = \mathbb{Y}\big(e^{zL_1^L + \overline{z}L_1^R} (e^{-i\pi} z^{-2})^{L_0^L} \big(\overline{e^{-i\pi} z^{-2}}\big)^{L_0^R} \theta w; z^{-1}, \overline{z^{-1}}\big) \qquad (w \in F), \tag{8.23}$$

where for each $n \in \mathbb{Z}$,, the linear operators $L_n^L = L_n \otimes 1, L_n^R = 1 \otimes L_n$ are defined on F. The factor $e^{-i\pi}$ in equation (8.23) can be replaced by any $e^{i(2n+1)\pi}$, where $n \in \mathbb{Z}$. The reflection positivity is equivalent to the fact that for any $i, j, k \in \mathcal{E}, \mathcal{Y}_\alpha, \mathcal{Y}_\beta \in \mathcal{V}\binom{k}{i\ j}$,

$$\langle \mathcal{Y}_{C\alpha} | \mathcal{Y}_{C\beta} \rangle = \frac{d_k}{d_j} \langle \mathcal{Y}_\alpha | \mathcal{Y}_\beta \rangle. \tag{8.24}$$

This relation is essentially proved in [HK07] using properties of the fusion matrices of intertwining operators. We remark that it can also be proved using graphical calculations for ribbon fusion categories.

A final remark. The positivity of Λ is not used in full power to prove the reflection positivity of F. One only uses the positivity of quantum dimensions d_i and the fact that Λ is Hermitian (i.e., $\Lambda(\mathcal{Y}_\alpha | \mathcal{Y}_\beta) = \overline{\Lambda(\mathcal{Y}_\beta | \mathcal{Y}_\alpha)}$), which can be checked more directly without doing long and tedious analysis as in our papers. So unlike the non-degeneracy of Λ, which is of significant importance in constructing full field algebras, the positivity of Λ only plays a marginal role.

Appendix A

APPENDIX FOR CHAPTER 2

A.1 Uniqueness of formal series expansions

Using Cauchy's integral formula, the coefficients of a Laurent series $\sum_{n\geqslant N} a_n z^n$ are determined by the values of this series when z is near 0. This uniqueness property can be generalized to formal series, as we now see.

Let $\mathscr{G}_0$ be a *finite* subset of $\mathbb{R}$, and let $\mathscr{G} = \mathscr{G}_0 + \mathbb{Z}_{\geqslant 0} = \{\mu + m : \mu \in \mathscr{G}_0, m \in \mathbb{Z}_{\geqslant 0}\}$. It is clear that the series

$$f(z_1, \ldots, z_n) = \sum_{\mu_1,\ldots,\mu_n\in\mathscr{G}} c_{\mu_1,\ldots,\mu_n} z_1^{\mu_1} \cdots z_n^{\mu_n} \tag{A.1}$$

converges absolutely if and only if for any $\mu_1, \ldots, \mu_n \in \mathscr{G}_0$, the power series

$$\sum_{m_1,\ldots,m_n\in\mathbb{Z}_{\geqslant 0}} c_{\mu_1+m_1,\ldots,\mu_n+m_n} z_1^{\mu_1+m_1} \cdots z_n^{\mu_n+m_n}$$

converges absolutely. Hence, by root test, if $f(z_1, \ldots, z_n)$ converges absolutely for some $z_1, \ldots, z_n \neq 0$, then $f(\zeta_1, \ldots, \zeta_n)$ converges absolutely whenever $0 < |\zeta_1| < |z_1|, \ldots, 0 < |\zeta_n| < |z_n|$.

The uniqueness property is stated as follows:

Proposition A.1. *Let $r_1, \ldots, r_n > 0$. For any $1 \leqslant l \leqslant n$, we choose a sequence of complex numbers $\{z_l(m_l) : 0 < |z_l(m_l)| < r_l\}_{m_l\in\mathbb{Z}_{>0}}$ such that $\lim_{m_l\to\infty} z_l(m_l) = 0$. Suppose that* (A.1) *converges absolutely when $0 < |z_1| < r_1, \ldots, 0 < |z_n| < r_n$, and that for any $m_1, \ldots, m_n$, we have $f\big(z_1(m_1), \ldots, z_n(m_n)\big) = 0$. Then for any $\mu_1, \ldots, \mu_n \in \mathscr{G}$, the coefficient $c_{\mu_1,\ldots,\mu_n} = 0$.*

Proof. (cf. [Hua95] section 15.4) By induction, it suffices to prove the case when $n = 1$. Then the series can be written as $f(z) = \sum_{k\in\mathbb{Z}_{\geqslant 1}} c_{\mu_k} z^{\mu_k}$, where $\mu_{k+1} > \mu_k$ for any k, and we have a sequence of complex values $\{z_m\}$ converging to zero, on which the values of f vanish. Define a series $g(z) = \sum_{k\in\mathbb{Z}_{\geqslant 2}} c_{\mu_k} z^{\mu_k-\mu_2}$. Then the series $g(z)$ converges absolutely when $0 < |z| < r$, and $\limsup_{z\to 0} |g(z)| < +\infty$. Since $f(z)z^{-\mu_1} = c_{\mu_1} + z^{\mu_2-\mu_1} g(z)$, we have $c_{\mu_1} = \lim_{m\to\infty} f\big(z(m)\big) z(m)^{-\mu_1} = 0$. This proves that $c_{\mu_1} = 0$. Repeat the same argument, we see that $c_{\mu_k} = 0$ for any k. □

A.2 Linear independence of products of intertwining operators

This section is devoted to the proof of proposition 2.3. First, we need the following lemma, the proof of which is an easy exercise.

Lemma A.2. *Let W_i be an irreducible V-module. Let $n = 1, 2, \ldots$. Consider the V-module $W_i^{\oplus n} = \underbrace{W_i \oplus W_i \oplus \cdots \oplus W_i}_{n}$. Then for any V-module homomorphism $R : W_i \to W_i^{\oplus n}$, there exist complex numbers $\lambda_1, \ldots, \lambda_n$ such that*

$$R(w^{(i)}) = (\lambda_1 w^{(i)}, \lambda_2 w^{(i)}, \ldots, \lambda_n w^{(i)}) \qquad (w^{(i)} \in V). \tag{A.2}$$

Proof. For any $1 \leqslant m \leqslant n$, let p_m be the projection of $W_i^{\oplus n}$ onto its m-th component. Then $p_m R \in \mathrm{End}_V(W_i)$. Since W_i is irreducible, there exists $\lambda_m \in \mathbb{C}$ such that $p_m R = \lambda_m \mathrm{id}_{W_i}$. (A.2) now follows immediately. □

Let W_i, W_j be two V-modules. For any $k \in \mathcal{E}$ we choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{ij}^k\}$ of $\mathcal{V}\binom{k}{i\,j}$. Consider the V-module $W_l = \bigoplus_{k\in\mathcal{E}} \big(\bigoplus_{\alpha\in\Theta_{ij}^k} W_k^\alpha\big)$, where each W_k^α is a V-module equivalent to W_k. It's contragredient module is $W_{\bar{l}} = \bigoplus_{k\in\mathcal{E}} \big(\bigoplus_{\alpha\in\Theta_{ij}^k} W_{\bar{k}}^\alpha\big)$, where $W_{\bar{k}}^\alpha$ is the contragredient module of W_k^α. Consider a type $\binom{l}{i\,j}$ intertwining operator $\mathcal{Y}$ defined as follows: for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we let

$$\mathcal{Y}(w^{(i)}, x) w^{(j)} = \bigoplus_{k\in\mathcal{E}} \Bigg(\bigoplus_{\alpha\in\Theta_{ij}^k} \mathcal{Y}_\alpha(w^{(i)}, x) w^{(j)} \Bigg), \tag{A.3}$$

i.e., the projection of $\mathcal{Y}(w^{(i)}, x) w^{(j)}$ to W_k^α is $\mathcal{Y}_\alpha(w^{(i)}, x) w^{(j)}$.

The following property is due to Huang. See [Hua95] lemma 14.9. The notations and terminologies in that article are different from ours, so we include a proof here.

Proposition A.3. *Choose $z \in \mathbb{C}^\times$ with argument $\arg z$. Let $w^{(\bar{l})} \in W_{\bar{l}}$. If for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have*

$$\langle w^{(\bar{l})}, \mathcal{Y}(w^{(i)}, z)w^{(j)} \rangle = 0, \tag{A.4}$$

then $w^{(\bar{l})} = 0$.

Proof. Let W_1 be the subspace of all $w^{(\bar{l})} \in W_{\bar{l}}$ satisfying (A.4). We show that $W_1 = 0$.

Note that by relation (1.21), for any $u \in V, m \in \mathbb{Z}$ we have

$$Y_l(u,m)\mathcal{Y}(w^{(i)}, z) - \mathcal{Y}(w^{(i)}, z)Y_k(u,m) = \sum_{h \in \mathbb{Z}_{\geqslant 0}} \binom{m}{h} \mathcal{Y}(Y_i(u,h)w^{(i)}, z)z^{m-h}. \tag{A.5}$$

From this we see that W_1 is a V-submodule of $W_{\bar{l}}$. If $W_1 \neq 0$, then W_1 contains an irreducible submodule equivalent to $W_{\overline{k}}$ for some $k \in \mathcal{E}$. This implies that we have a non-zero V-module homomorphism $R : W_{\overline{k}} \to \bigoplus_{\alpha \in \Theta_{ij}^k} W_{\overline{k}}^\alpha \subset W_{\bar{l}}$, and that the image of R is inside W_1.

By lemma A.2, we can choose complex numbers $\{\lambda_\alpha : \alpha \in \Theta_{ij}^k\}$, not all of which are zero, such that for any $w^{(\overline{k})}$, $Rw^{(\overline{k})} = \bigoplus_{\alpha \in \Theta_{ij}^k} \lambda_\alpha w^{(\overline{k})}$. Hence for any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(\overline{k})} \in W_k$, we have

$$\sum_{\alpha \in \Theta_{ij}^k} \lambda_\alpha \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, z)w^{(j)} \rangle = 0.$$

Since 3-point correlation functions are determined by their values at the point z, we have

$$\sum_{\alpha \in \Theta_{ij}^k} \lambda_\alpha \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)} \rangle = 0,$$

where x is a formal variable. But we know that $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{ij}^k\}$ are linearly independent, which forces all the coefficients λ_α to be zero. Hence we have a contradiction. □

Corollary A.4. *Vectors of the form $\mathcal{Y}(w^{(i)}, s)w^{(j)}$ ($w^{(i)} \in W_i, w^{(j)} \in W_j, s \in \mathbb{R}$) span the vector space W_l.*

Proof. Choose any $w^{(\bar{l})} \in W_{\bar{l}}$ satisfying that for any $w^{(i)} \in W_i, w^{(j)} \in W_j, s \in \mathbb{R}$,

$$\langle w^{(\bar{l})}, \mathcal{Y}(w^{(i)}, s) w^{(j)} \rangle = 0. \tag{A.6}$$

Then for any $z \in \mathbb{C}^\times$, equation (A.4) holds. So $w^{(\bar{l})}$ must be zero. $\square$

Proof of proposition 2.3. It is clear that Φ is surjective. So we only need to prove that Φ is injective. By induction, it suffices to prove that the linear map Ψ:

$$\bigoplus_{j \in \mathcal{E}} \left(\mathcal{V}\binom{k}{i_n\, i_{n-1} \,\cdots\, i_2\, j} \otimes \mathcal{V}\binom{j}{i_1\, i_0} \right) \to \mathcal{V}\binom{k}{i_n\, i_{n-1} \,\cdots\, i_1\, i_0},$$
$$\mathcal{X} \otimes \mathcal{Y}_\alpha \mapsto \mathcal{X}\mathcal{Y}_\alpha$$

is injective. To prove this, we choose, for any $j \in \mathcal{E}$, a linear basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta^j_{i_1 i_0}\}$ of $\mathcal{V}\binom{j}{i_1\, i_0}$. If we can prove, for any $j \in \mathcal{E}, \alpha \in \Theta^j_{i_1 i_0}, \mathcal{X}_\alpha \in \mathcal{V}\binom{k}{i_n\, i_{n-1} \,\cdots\, i_2\, j}$, that

$$\sum_{j \in \mathcal{E}} \sum_{\alpha \in \Theta^j_{i_1 i_0}} \mathcal{X}_\alpha \mathcal{Y}_\alpha = 0 \tag{A.7}$$

always implies that $\mathcal{X}_\alpha = 0$ for all α, then the injectivity of Ψ follows immediately.

Now suppose that (A.7) is true. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, s \in \mathbb{R}$, and $z_2, \ldots, z_n$ satisfying $0 < |z_2| < \cdots < |z_n|$, we have, by proposition A.1,

$$\sum_{j \in \mathcal{E}} \sum_{\alpha \in \Theta^j_{i_1 i_0}} \mathcal{X}_\alpha(w^{(i_n)}, \ldots, w^{(i_2)}; z_n, \ldots, z_2) \mathcal{Y}_\alpha(w^{(i_1)}, s) w^{(i_0)} = 0. \tag{A.8}$$

By corollary A.4, for any $j \in \mathcal{E}, w^{(j)} \in W_j$ and $\alpha \in \Theta^j_{i_1 i_0}$, there exist $w_1^{(i_0)}, \ldots, w_m^{(i_0)} \in W_{i_0}, w_1^{(i_1)}, \ldots, w_m^{(i_1)} \in W_{i_1}, s_1, \ldots, s_m \in \mathbb{R}$, such that

$$\mathcal{Y}_\alpha(w_1^{(i_1)}, s_1) w_1^{(i_0)} + \cdots + \mathcal{Y}_\alpha(w_m^{(i_1)}, s_m) w_m^{(i_0)} = w^{(j)},$$

and that for any $\beta \neq \alpha$,

$$\mathcal{Y}_\beta(w_1^{(i_1)}, s_1) w_1^{(i_0)} + \cdots + \mathcal{Y}_\beta(w_m^{(i_1)}, s_m) w_m^{(i_0)} = 0.$$

Hence $\mathcal{X}_\alpha(w^{(i_n)}, \ldots, w^{(i_2)}; z_n, \ldots, z_2)w^{(j)} = 0$. □

A.3 General braiding and fusion relations

In this section, we prove all the results claimed in section 2.2. Our proof of the convergence properties is motivated by that of [HLZ11] proposition 12.7. To prove these properties, we need a technical lemma, for which we now introduce some temporary notations.

For any $r > 0$, let $D(r) = \{z \in \mathbb{C} : |z| < r\}$, $D^\times(r) = D(r)\backslash\{0\}$, and $E(r) = D(r) \cap (0, +\infty)$. Then we have the following:

Lemma A.5. *Given a power series*

$$\sum_{n_0, n_1, \ldots, n_l \in \mathbb{Z}_{\geqslant 0}} c_{n_0 n_1 \ldots n_l} z_0^{n_0} z_1^{n_1} \cdots z_l^{n_l} \tag{A.9}$$

of the complex variables $z_0, z_1, \ldots, z_l$, where $l \in \mathbb{Z}_{>0}$ and each $c_{n_0 n_1 \ldots n_l} \in \mathbb{C}$. Suppose that there exist $r_0, r_1, \ldots, r_l > 0$, such that for any n_0, the power series

$$g_{n_0}(z_1, \ldots, z_l) = \sum_{n_1, \ldots, n_l \in \mathbb{Z}_{\geqslant 0}} c_{n_0 n_1 \ldots n_l} z_1^{n_1} \cdots z_l^{n_l} \tag{A.10}$$

converges absolutely on $D(r_1) \times \cdots \times D(r_l)$; that for any $z_1 \in E(r_1), \ldots, z_l \in E(r_l)$,

$$f(z_0, z_1, \ldots, z_l) = \sum_{n_0 \in \mathbb{Z}_{\geqslant 0}} g_{n_0}(z_1, \ldots, z_n) z_0^n, \tag{A.11}$$

converges absolutely as a power series of z_0 on $D(r_0)$; and that f can be analytically continued to a multivalued holomorphic function on $D^\times(r_0) \times D^\times(r_1) \times \cdots \times D^\times(r_l)$. Then the power series (A.9) *converges absolutely on $D(r_0) \times D(r_1) \times \cdots \times D(r_l)$.*

Proof. Consider the multivalued holomorphic function f. From (A.11), we know that for any $z_1 \in E(r_1), \ldots, z_l \in E(r_l)$, f is single-valued for $z_0 \in D^\times(r_0)$. So f is single-valued on z_0 for any $z_1 \in D^\times(r_1), \ldots, z_l \in D^\times(r_l)$.

Now, for any $n_0 \in \mathbb{Z}$,

$$\widetilde{g}_{n_0}(z_1, \ldots, z_n) = \oint_0 f(z_0, z_1, \ldots, z_l) z_0^{-n-1} \frac{dz_0}{2i\pi} \tag{A.12}$$

is a multivalued holomorphic function on $D^\times(r_1) \times \cdots \times D^\times(r_l)$. If $n_0 \geqslant 0$, then by (A.11), we must have $\widetilde{g}_{n_0} = g_{n_0}$ on $E(r_1) \times \cdots \times E(r_l)$. Since g_{n_0} is holomorphic, $\widetilde{g}_{n_0} = g_{n_0}$ on $D^\times(r_1) \times \cdots \times D^\times(r_l)$. Hence $\widetilde{g}_{n_0}$ is single-valued. Similarly, when $n_0 < 0$, we have $\widetilde{g}_{n_0}(z_1, \ldots, z_n) = 0$ on $E(r_1) \times \cdots \times E(r_l)$, and hence on $D^\times(r_1) \times \cdots \times D^\times(r_l)$. Therefore, $f(z_0, z_1, \ldots, z_n) = \sum_{n_0 \in \mathbb{Z}} \widetilde{g}_{n_0}(z_1, \ldots, z_n) z_0^{n_0}$ is single-valued on $D^\times(r_0) \times D^\times(r_1) \times \cdots \times D^\times(r_n)$, and the Laurant series expansion of f near the origin has no negative powers of $z_0, z_1, \ldots, z_n$. So f is a single-valued holomorphic function on $D(r_0) \times D(r_1) \times \cdots \times D(r_l)$ with power series expansion (A.9). We can thus conclude that (A.9) converges absolutely on $D(r_0) \times D(r_1) \times \cdots \times D(r_l)$. □

Recall that a series $f(z_1, \ldots, z_n) = \sum_{s_1, \ldots, s_n \in \mathbb{R}} c_{s_1 \ldots s_n} z_1^{s_1} \cdots z_n^{s_n}$ is called a **quasi power series** of $z_1, \ldots, z_n$, if f equals a power series multiplied by a monomial of $z_1, \ldots, z_n$, i.e., if there exist $t_1, \ldots, t_n \in \mathbb{C}$ such that $f(z_1, \ldots, z_n) z_1^{t_1} \cdots z_n^{t_n} \in \mathbb{C}[[z_1, \ldots, z_n]]$.

Proof of theorem 2.5. Step 1. We first prove the convergence. Let W_i be the charge space of $\mathcal{Y}_\gamma$. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i)} \in W_i$, we have

$$\begin{aligned}
&\mathcal{Y}_\gamma(w^{(i)}, x) w^{(i_0)} \\
=&\mathcal{Y}_{B_+B_-\gamma}(w^{(i)}, x) w^{(i_0)} \\
=&e^{xL_{-1}} \mathcal{Y}_{B_-\gamma}(w^{(i_0)}, e^{i\pi} x) w^{(i)},
\end{aligned}$$

where x is a formal variable. Then for any $w^{(\overline{k})} \in W_{\overline{k}}$, we have

$$\begin{aligned}
&\langle \mathcal{Y}_\gamma(w^{(i)}, z_1) w^{(i_0)}, w^{(\overline{k})} \rangle \\
=&\langle \mathcal{Y}_\gamma(w^{(i)}, x) w^{(i_0)}, w^{(\overline{k})} \rangle \big|_{x=z_1} \\
=&\langle e^{xL_{-1}} \mathcal{Y}_{B_-\gamma}(w^{(i_0)}, e^{i\pi} x) w^{(i)}, w^{(\overline{k})} \rangle \big|_{x=z_1} \\
=&\langle \mathcal{Y}_{B_-\gamma}(w^{(i_0)}, e^{i\pi} x) w^{(i)}, e^{xL_1} w^{(\overline{k})} \rangle \big|_{x=z_1} \\
=&\langle \mathcal{Y}_{B_-\gamma}(w^{(i_0)}, e^{i\pi} z_1) w^{(i)}, e^{z_1 L_1} w^{(\overline{k})} \rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\big\langle \mathcal{Y}_\gamma\big(P_{s_n}\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)P_{s_{n-1}}\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\\
&\qquad \cdots P_{s_2}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}, w^{(\overline{k})}\big\rangle\\
=&\langle \mathcal{Y}_{B_-\gamma}(w^{(i_0)}, e^{i\pi}z_1)P_{s_n}\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)P_{s_{n-1}}\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\\
&\qquad \cdots P_{s_2}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, e^{z_1L_1}w^{(\overline{k})}\rangle. \qquad (A.13)
\end{aligned}$$

Hence, by theorem 2.2 and the discussion below, the sum of (A.13) over $s_2, s_3, \ldots, s_n \in \mathbb{R}$ converges absolutely and locally uniformly.

Step 2. Assume that

$$\begin{aligned}
&0 < |z_1| < |z_2| < \cdots < |z_n|,\\
&0 < |z_2 - z_1| < |z_3 - z_1| \cdots < |z_n - z_1| < |z_1|, \qquad (A.14)
\end{aligned}$$

and choose arguments $\arg z_1, \arg z_2, \ldots, \arg z_n, \arg(z_2 - z_1), \ldots, \arg(z_n - z_1)$. We prove, by induction on n, that (2.8) defined near the point $(z_1, z_2, \ldots, z_n)$ is a correlation function, i.e., it can be written as a product of a chain of intertwining operators. The case $n = 2$ was proved in [Hua95] and [Hua05a]. Suppose this theorem holds for $n - 1$, we now prove it for n. By analytic continuation, it suffices to assume also that

$$|z_1| + |z_2 - z_1| < |z_3|. \qquad (A.15)$$

Let W_{j_2} be the target space of $\mathcal{Y}_{\sigma_2}$. By induction, there exists a chain of intertwining operators $\mathcal{Y}_\delta, \mathcal{Y}_{\alpha_3}, \mathcal{Y}_{\alpha_4}, \ldots, \mathcal{Y}_{\alpha_n}$ with charge spaces $W_{j_2}, W_{i_3}, W_{i_4}, \ldots, W_{i_n}$ respectively, such that W_{i_0} is the source space of $\mathcal{Y}_\delta$, that W_k is the target space of $\mathcal{Y}_{\alpha_n}$, and that for any $w^{(i_0)} \in W_{i_0}, w^{(j_2)} \in W_{j_2}, w^{(i_3)} \in W_{i_3}, w^{(i_4)} \in W_{i_4}, \ldots, w^{(i_n)} \in W_{i_n}$, we have the fusion relation

$$\begin{aligned}
&\mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1)w^{(j_2)}, z_1\big)w^{(i_0)}\\
=&\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)\mathcal{Y}_\delta(w^{(j_2)}, z_1)w^{(i_0)} \qquad (A.16)
\end{aligned}$$

near the point $(z_1, z_3, z_4, \dots, z_n)$.

There also exists a chain of intertwining operator $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}$ with charge spaces W_{i_1}, W_{i_2}, such that the source space of $\mathcal{Y}_{\alpha_1}$ is W_{i_0}, that the target space of $\mathcal{Y}_{\alpha_2}$ equals that of $\mathcal{Y}_\delta$, and that the fusion relation

$$\mathcal{Y}_\delta\big(\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big) = \mathcal{Y}_{\alpha_2}(w^{(i_2)}, z_2)\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) \tag{A.17}$$

holds near the point (z_1, z_2). Now we compute, omitting the evaluation under any $w^{(\overline{k})} \in W_{\overline{k}}$, that

$$\begin{aligned}
&\mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}\\
=&\sum_{s_1\in\mathbb{R}}\mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\cdots P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}\\
=&\sum_{s_1\in\mathbb{R}}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)\\
&\qquad\cdot\mathcal{Y}_\delta\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}\\
=&\sum_{s_1\in\mathbb{R}}\sum_{s_2,\dots,s_{n-1}\in\mathbb{R}}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})P_{s_{n-2}}\\
&\qquad\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)P_{s_2}\mathcal{Y}_\delta\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}.
\end{aligned}\tag{A.18}$$

If we can prove, for any $w^{(\overline{k})} \in W_{\overline{k}}$, and any $z_1, z_2, \dots, z_n$ satisfying

$$\begin{aligned}
0 < |z_2 - z_1| < |z_1| < |z_3| < |z_4| < \cdots < |z_n|,\\
|z_1| + |z_2 - z_1| < |z_3|,
\end{aligned}\tag{A.19}$$

that the expression

$$\begin{aligned}
&\langle\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)\\
&\cdot\mathcal{Y}_\delta\big(\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}, w^{(\overline{k})}\rangle
\end{aligned}\tag{A.20}$$

converges absolutely, i.e., the sum of the absolute values of

$$\big\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})P_{s_{n-2}}$$

$$\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)P_{s_2}\mathcal{Y}_{\delta}\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}, w^{(\overline{k})}\big\rangle \tag{A.21}$$

over $s_1, s_2, \ldots, s_n \in \mathbb{R}$ is a finite number, then the two sums on the right hand side of (A.18) commute. Hence (A.18) equals

$$\begin{aligned}
&\sum_{s_2,\ldots,s_n\in\mathbb{R}}\sum_{s_1\in\mathbb{R}} P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})P_{s_{n-2}}\\
&\qquad\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)P_{s_2}\mathcal{Y}_{\delta}\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}\\
=&\sum_{s_2,\ldots,s_n\in\mathbb{R}}\sum_{s_1\in\mathbb{R}} P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})P_{s_{n-2}}\\
&\qquad\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)P_{s_2}\mathcal{Y}_{\alpha_2}(w^{(i_2)}, z_2)P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1)w^{(i_0)}\\
=&\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1)w^{(i_0)}.
\end{aligned} \tag{A.22}$$

Therefore, if the series (A.20) converges absolutely, then (2.8) defines an $(n+2)$-point correlation function of V. The converse statement (every $(n+2)$-point function can be written in the form (A.20)) can be proved in a similar way.

Step 3. We show that when (A.19) holds, (A.20) converges absolutely. Assume, without loss of generality, that all the intertwining operators in (A.20) are irreducible, and that all the vectors in (A.20) are homogeneous. Define a new set of variables $\omega_1, \omega_2, \ldots, \omega_n$ by setting

$$\begin{gathered}
z_m = \omega_m\omega_{m+1}\cdots\omega_n \quad (3 \leqslant m \leqslant n),\\
z_1 = \omega_2\omega_3\cdots\omega_n,\\
z_2 - z_1 = \omega_1\omega_2\cdots\omega_n.
\end{gathered}$$

Then condition (A.19) is equivalent to the condition

$$\begin{gathered}
0 < |\omega_m| < 1 \quad (1 \leqslant m \leqslant n-1),\\
0 < |\omega_n|,\\
|\omega_2|(1 + |\omega_1|) < 1.
\end{gathered} \tag{A.23}$$

It is clear that if $\mathring{\omega}_1, \mathring{\omega}_2, \dots \mathring{\omega}_n$ are complex numbers satisfying condition (A.23), then there exist positive numbers $r_1 > |\mathring{\omega}_1|, r_2 > |\mathring{\omega}_2|, \dots, r_n > |\mathring{\omega}_n|$, such that whenever $0 < |\omega_m| < r_m$ $(1 \leqslant m \leqslant n)$, condition (A.23) is satisfied. We now prove that the sum of (A.21) over $s_1, \dots, s_n$ converges absolutely on $\{0 < |\omega_1| < r_1, \dots, 0 < |\omega_n| < r_n\}$.

Let

$$\begin{aligned}
&c_{s_1 s_2 \dots s_n}\\
=&\langle P_{s_n} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, 1) P_{s_{n-1}} \mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, 1) P_{s_{n-2}} \cdots\\
&\cdot P_{s_3} \mathcal{Y}_{\alpha_3}(w^{(i_3)}, 1) P_{s_2} \mathcal{Y}_{\delta}\big(P_{s_1} \mathcal{Y}_{\sigma_2}(w^{(i_2)}, 1) w^{(i_1)}, 1\big) w^{(i_0)}, w^{(\overline{k})}\rangle,
\end{aligned} \tag{A.24}$$

where each $\mathcal{Y}_\cdot(\cdot, 1) = \mathcal{Y}_\cdot(\cdot, x)\big|_{x=1}$. By relation (1.26), it is easy to see that (A.21) equals

$$\begin{aligned}
&\langle P_{s_n} \omega_n^{L_0} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, 1) P_{s_{n-1}} \omega_{n-1}^{L_0} \mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, 1) P_{s_{n-2}} \cdots\\
&\cdot P_{s_3} \omega_3^{L_0} \mathcal{Y}_{\alpha_3}(w^{(i_3)}, 1) P_{s_2} \omega_2^{L_0} \mathcal{Y}_{\delta}\big(P_{s_1} \omega_1^{L_0} \mathcal{Y}_{\sigma_2}(w^{(i_2)}, 1) w^{(i_1)}, 1\big) w^{(i_0)}, w^{(\overline{k})}\rangle\\
=&c_{s_1 s_2 \dots s_n} \omega_1^{s_1} \omega_2^{s_2} \cdots \omega_n^{s_n}
\end{aligned} \tag{A.25}$$

multiplied by a monomial $\omega_1^{r_1} \omega_2^{r_2} \cdots \omega_n^{r_n}$, where the powers $r_1, r_2, \dots, r_n \in \mathbb{R}$ are independent of $s_1, s_2, \dots, s_n$. Therefore, the absolute convergence of (A.20) is equivalent to the absolute convergence of the series

$$\sum_{s_1, s_2, \dots, s_n \in \mathbb{R}} c_{s_1 s_2 \dots s_n} \omega_1^{s_1} \omega_2^{s_2} \cdots \omega_n^{s_n} \tag{A.26}$$

on $\{0 < |\omega_1| < r_1, 0 < |\omega_2| < r_2, \dots, 0 < |\omega_n| < r_n\}$. Note that by irreducibility of the intertwining operators, (A.26) is a quasi power series of $\omega_1, \omega_2, \dots, \omega_n$. So we are going to prove the absolute convergence of (A.26) by checking that (A.26) satisfies all the conditions in lemma A.5.

Since (A.21) equals (A.25) multiplied by $\omega_1^{r_1} \omega_2^{r_2} \cdots \omega_n^{r_n}$, for each $s_2 \in \mathbb{R}$, step 1 and theorem 2.2 imply that the series

$$\sum_{s_1, s_3, s_4, \dots, s_n \in \mathbb{R}} c_{s_1 s_2 s_3 \dots s_n} \omega_1^{s_1} \omega_3^{s_3} \omega_4^{s_4} \cdots \omega_n^{s_n} \tag{A.27}$$

converges absolutely on $\{0 < |\omega_1| < r_1, 0 < |\omega_3| < r_3, 0 < |\omega_4| < r_4, \dots, 0 < |\omega_n| < r_n\}$. If we assume moreover that $0 < \omega_1 < r_1$, then $0 < |\omega_2| < r_2$ clearly implies $0 < |z_1| < |z_2| < \cdots < |z_n|$ and $0 < |z_2 - z_1| < |z_1|$. Hence, the following quasi power series of ω_2

$$\begin{aligned}
&\omega_1^{r_1}\omega_2^{r_2}\cdots\omega_n^{r_n}\cdot\bigg(\sum_{s_2\in\mathbb{R}}\bigg(\sum_{s_1,s_3,\dots,s_n\in\mathbb{R}} c_{s_1s_2s_3\dots s_n}\omega_1^{s_1}\omega_3^{s_3}\cdots\omega_n^{s_n}\bigg)\omega_2^{s_2}\bigg)\\
=&\sum_{s_2\in\mathbb{R}}\langle\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)},z_3)\\
&\qquad\cdot P_{s_2}\mathcal{Y}_\delta\big(\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},z_1\big)w^{(i_0)},w^{(\overline{k})}\rangle\\
=&\sum_{s_2\in\mathbb{R}}\langle\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)},z_3)\\
&\qquad\cdot P_{s_2}\mathcal{Y}_{\alpha_2}(w^{(i_2)},z_2)\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1)w^{(i_0)},w^{(\overline{k})}\rangle
\end{aligned}\tag{A.28}$$

must converge absolutely on $\{0 < |\omega_2| < r_2\}$. By theorem 2.4, the function (A.28) defined on $\{0 < \omega_1 < r_1, 0 < |\omega_2| < r_2, \dots, 0 < |\omega_n| < r_n\}$ can be analytically continued to a multivalued holomorphic function on $\{0 < |\omega_1| < r_1, 0 < |\omega_2| < r_2, \dots, 0 < |\omega_n| < r_n\}$. Hence by lemma A.5, the quasi power series (A.26) converges absolutely on $\{0 < |\omega_1| < r_1, \dots, 0 < |\omega_n| < r_n\}$. □

Proof of theorem 2.6. The argument here is similar to step 3 of the proof of theorem 2.5. Assume, without loss of generality, that all the intertwining operators in (2.11) are irreducible, and all the vectors in it are homogeneous. We prove this theorem by induction on m. The case that $m = 1$ is proved in theorem 2.5. Suppose that the theorem holds for $m-1$, we prove this for m.

Define a new set of variables $\{\omega_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ in the following way: For any $1 \leqslant a \leqslant m$, we set

$$z_1^a = \omega_1^a\omega_1^{a+1}\cdots\omega_1^m,\tag{A.29}$$

and if $2 \leqslant b \leqslant n_a$, we set

$$z_b^a - z_1^a = \omega_1^a\omega_1^{a+1}\cdots\omega_1^m\cdot\omega_b^a\omega_{b+1}^a\cdots\omega_{n_a}^a.\tag{A.30}$$

Then the condition (1) and (2) on $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ is equivalent to the condition

$$0 < |\omega_b^a| < 1 \quad (1 \leqslant a \leqslant m, 2 \leqslant b \leqslant n_a),$$
$$0 < |\omega_1^m|,$$
$$0 < |\omega_1^a|\big(1 + (1-\delta_{n_a,1})|\omega_{n_a}^a|\big) < 1 - \big(1-\delta_{n_{a+1},1}\big)|\omega_{n_{a+1}}^{a+1}| \quad (1 \leqslant a \leqslant m-1). \tag{A.31}$$

It is clear that if $\{\mathring{\omega}_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ are complex numbers satisfying condition (A.31), then there exist positive numbers $\{r_b^a > |\mathring{\omega}_b^a|\}$, such that whenever $0 < |\omega_b^a| < r_b^a$ for all a and b, then (A.31) is true. If, moreover, any ω_b^a except ω_1^1 satisfies $0 < \omega_b^a < r_b^a$, then condition (3) also also holds for $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$.

Let $\vec{s}$ be the sequence $\{s_b^a\}$, $\vec{\omega}$ be $\{\omega_b^a\}$, $\vec{s}\backslash s_1^1$ be $\{$all s_b^a except $s_1^1\}$, and $\vec{\omega}\backslash\omega_1^1$ be $\{$all ω_b^a except $\omega_1^1\}$. We let $\vec{\omega}^{\vec{s}} = \prod_{1\leqslant a\leqslant m, 1\leqslant b\leqslant n_a} (\omega_b^a)^{s_b^a}$. For each $\vec{s}$, we define

$$c_{\vec{s}} = \Big\langle \Big[\prod_{m\geqslant a\geqslant 1} P_{s_1^a}\mathcal{Y}_{\alpha^a}\Big(\Big(\prod_{n_a\geqslant b\geqslant 2} P_{s_b^a}\mathcal{Y}_{\alpha_b^a}(w_b^a, 1)\Big) w_1^a, 1\Big)\Big] w^i, w^{\overline{k}} \Big\rangle, \tag{A.32}$$

where each $\mathcal{Y}_\cdot(\cdot, 1)$ means $\mathcal{Y}_\cdot(\cdot, x)|_{x=1}$. Then by (1.26), the expression

$$\Big\langle \Big[\prod_{m\geqslant a\geqslant 1} P_{s_1^a}\mathcal{Y}_{\alpha^a}\Big(\Big(\prod_{n_a\geqslant b\geqslant 2} P_{s_b^a}\mathcal{Y}_{\alpha_b^a}(w_b^a, z_b^a - z_1^a)\Big) w_1^a, z_1^a\Big)\Big] w^i, w^{\overline{k}} \Big\rangle \tag{A.33}$$

equals $c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}}$ multiplied by a monomial of $\vec{\omega}$ whose power is independent of $\vec{s}$. By induction, we can show that for each $s_1^1 \in \mathbb{R}$, the series $\sum_{\vec{s}\backslash s_1^1} c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}} \cdot (\omega_1^1)^{-s_1^1}$ of $\vec{\omega}\backslash\omega_1^1$ converges absolutely on $\{\vec{\omega}\backslash\omega_1^1 : 0 < |\omega_b^a| < r_b^a\}$; that for all $\vec{\omega}\backslash\omega_1^1$ satisfying $0 < \omega_b^a < r_b^a$,

$$\sum_{s_1^1\in\mathbb{R}} \sum_{\vec{s}\backslash s_1^1} c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}}, \tag{A.34}$$

as a series of ω_1^1, converges absolutely on $\{\omega_1^1 : 0 < |\omega_1^1| < r_1^1\}$; and that as a function of $\vec{\omega}$, (A.34) can be analytically continued to a multivalued holomorphic function on $\{\vec{\omega} : 0 < |\omega_b^a| < r_b^a\}$. Hence, by lemma A.5, the quasi power series $\sum_{\vec{s}} c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}}$ converges absolutely on $\{\vec{\omega} : 0 < |\omega_b^a| < r_b^a\}$. If, moreover, $\{z_b^a\}$ satisfy condition (3), then by induction and the argument in step 2 of the proof of theorem 2.5, (2.11) can be written as a product of a chain of intertwining operators. So it is a correlation function defined near $\{z_b^a\}$. $\square$

Proof of corollary 2.7. One can prove this corollary, either by theorem 2.6 and the argument in step 1 of the proof of theorem 2.5, or by induction and the argument in step 3 of the proof of theorem 2.5. We leave the details to the reader. $\square$

Proof of proposition 2.9. Fix $z_i \in \mathbb{C}^\times$. Let w_1 (resp. w_2) be a vector in the source space (resp. in the contragredient module of the target space) of $\mathcal{Y}_\delta$. Let $x_i, x_{ji}, \widetilde{x}_{ji}$ be commuting independent formal variables. It is easy to check that for any $w^{(k)} \in W_k$,

$$\begin{aligned} &\Big\langle \mathcal{Y}_\delta\big(e^{\widetilde{x}_{ji}L_{-1}}w^{(k)}, x_i\big)w_1, w_2\Big\rangle = \Big\langle \mathcal{Y}_\delta\big(w^{(k)}, x_i + \widetilde{x}_{ji}\big)w_1, w_2\Big\rangle \\ :=& \sum_{s\in\mathbb{R}, l\in\mathbb{Z}_{\geqslant 0}} \Big\langle \mathcal{Y}_\delta(w^{(k)}, s)w_1, w_2\Big\rangle \binom{-s-1}{l} x_i^{-s-1-l}\widetilde{x}_{ji}^l. \end{aligned} \tag{A.35}$$

Put $x_i = z_i$, we have

$$\begin{aligned} &\Big\langle \mathcal{Y}_\delta\big(e^{\widetilde{x}_{ji}L_{-1}}w^{(k)}, z_i\big)w_1, w_2\Big\rangle = \Big\langle \mathcal{Y}_\delta\big(w^{(k)}, z_i + \widetilde{x}_{ji}\big)w_1, w_2\Big\rangle \\ :=& \sum_{s\in\mathbb{R}, l\in\mathbb{Z}_{\geqslant 0}} \Big\langle \mathcal{Y}_\delta(w^{(k)}, s)w_1, w_2\Big\rangle \binom{-s-1}{l} z_i^{-s-1-l}\widetilde{x}_{ji}^l. \end{aligned} \tag{A.36}$$

Clearly

$$\Big\langle \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}z_{ji})w^{(j)}, z_i + \widetilde{z}_{ji}\big)w_1, w_2\Big\rangle \tag{A.37}$$

is a multivalued holomorphic function of $z_{ji}, \widetilde{z}_{ji}$ when $0 < |z_{ji}|, |\widetilde{z}_{ji}| < \frac{1}{2}|z_i|$. Since the series

$$\sum_{s\in\mathbb{R}} \Big\langle \mathcal{Y}_\delta\big(P_s\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}z_{ji})w^{(j)}, z_i + \widetilde{z}_{ji}\big)w_1, w_2\Big\rangle \tag{A.38}$$

converges absolutely and locally uniformly, the infinite sum commutes with Cauchy's integrals around the pole $\widetilde{z}_{ji} = 0$. From this we see that (A.37) has the series expansion

$$\Big\langle \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}x_{ji})w^{(j)}, z_i + \widetilde{x}_{ji}\big)w_1, w_2\Big\rangle\Big|_{x_{ji}=z_{ji}, \widetilde{x}_{ji}=\widetilde{z}_{ji}}, \tag{A.39}$$

which must be absolute convergent, and also equals

$$\left\langle \mathcal{Y}_\delta\big(e^{\widetilde{x}_{ji}L_{-1}}\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}x_{ji})w^{(j)}, z_i\big)w_1, w_2\right\rangle\Big|_{x_{ji}=z_{ji},\widetilde{x}_{ji}=\widetilde{z}_{ji}}. \tag{A.40}$$

Therefore, when $0 < |z_j - z_i| < \frac{1}{2}|z_i|$, the series

$$\sum_{r,s\in\mathbb{R}} \left\langle \mathcal{Y}_\delta\big(P_r e^{(z_j-z_i)L_{-1}} P_s \mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}(z_j - z_i))w^{(j)}, z_i\big)w_1, w_2\right\rangle \tag{A.41}$$

converges absolutely and equals (A.37) with $z_{ji} = \widetilde{z}_{ji} = z_j - z_i$.

One the other hand,

$$\begin{aligned}
&\left\langle \mathcal{Y}_\delta\big(\mathcal{Y}_{B_\pm\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i\big)w_1, w_2\right\rangle\\
=&\sum_{r\in\mathbb{R}} \left\langle \mathcal{Y}_\delta\big(P_r\mathcal{Y}_{B_\pm\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i\big)w_1, w_2\right\rangle\\
=&\sum_{r\in\mathbb{R}} \left\langle \mathcal{Y}_\delta\big(P_r e^{(z_j-z_i)L_{-1}}\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}(z_j - z_i))w^{(j)}, z_i\big)w_1, w_2\right\rangle,
\end{aligned}$$

which is just (A.41). So it also equals (A.37) with $z_{ji} = \widetilde{z}_{ji} = z_j - z_i$. This proves relation (2.18) when $0 < |z_j - z_i| < \frac{1}{2}|z_i|$. The general case follows from analytic continuation. $\square$

Proof of theorem 2.8. The case $n = 2$ follows immediately from proposition 2.9 and the fusion relations of two intertwining operators. We now prove the general case.

Since S_n is generated by adjacent transpositions, we can assume that ς exchanges $m, m+1$ and fixes the other elements in $\{1, 2, \dots, n\}$. Write

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{Y}_{\alpha_{m-1}}(w^{(i_{m-1})}, z_{m-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1),\\
\mathcal{X}_2 &= \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\cdots\mathcal{Y}_{\alpha_{m+2}}(w^{(i_{m+2})}, z_{m+2}).
\end{aligned}$$

To proof the braid relation in this case, it is equivalent to showing that if $0 < |z_1| < \cdots < |z_{m-1}| < |z_{m+1}| < |z_m| < |z_{m+2}| < \cdots < |z_n|$, and if we move z_m, z_{m+1} to satisfy $0 < |z_1| < \cdots < |z_{m-1}| < |z_m| < |z_{m+1}| < |z_{m+2}| < \cdots < |z_n|$ by scaling the norms of z_m, z_{m+1}, then we can find intertwining operators $\mathcal{Y}_{\beta_m}, \mathcal{Y}_{\beta_{m+1}}$ independent of the choice of vectors, such that

$$\langle \mathcal{X}_2 \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m) \mathcal{Y}_{\alpha_{m+1}}(w^{(i_{m+1})}, z_{m+1}) \mathcal{X}_1 w^{(i_0)}, w^{(\overline{k})} \rangle \tag{A.42}$$

can be analytically continued to

$$\langle \mathcal{X}_2 \mathcal{Y}_{\beta_{m+1}}(w^{(i_{m+1})}, z_{m+1}) \mathcal{Y}_{\beta_m}(w^{(i_m)}, z_m) \mathcal{X}_1 w^{(i_0)}, w^{(\overline{k})} \rangle. \tag{A.43}$$

By analytic continuation, we can also assume that during the process of moving z_m, z_{m+1}, conditions $0 < |z_1| < \cdots < |z_{m-1}| < |z_m|, |z_{m+1}| < |z_{m+2}| < \cdots < |z_n|$ and $0 < |z_m - z_{m+1}| < |z_{m+1}|$ are always satisfied.

Let W_{j_1} be the source space of $\mathcal{Y}_{\alpha_{m+1}}$ and W_{j_2} be the target space of $\mathcal{Y}_{\alpha_m}$. By braiding of two intertwining operators, there exists a chain of intertwining operators $\mathcal{Y}_{\beta_m}, \mathcal{Y}_{\beta_{m+1}}$ with charge spaces $W_{i_m}, W_{i_{m+1}}$ respectively, such that the source space of $\mathcal{Y}_{\beta_m}$ is W_{j_1}, that the target space of $\mathcal{Y}_{\beta_{m+1}}$ is W_{j_2}, and that for any $w^{(j_1)} \in W_{j_1}, w^{(i_m)} \in W_{i_m}, w^{(i_{m+1})} \in W_{i_{m+1}}, w^{(\overline{j_2})} \in W_{\overline{j_2}}$, the expression

$$\langle \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m) \mathcal{Y}_{\alpha_{m+1}}(w^{(i_{m+1})}, z_{m+1}) w^{(j_1)}, w^{(\overline{j_2})} \rangle \tag{A.44}$$

defined on $0 < |z_{m+1}| < |z_m|$ can be analytically continued to

$$\langle \mathcal{Y}_{\beta_{m+1}}(w^{(i_{m+1})}, z_{m+1}) \mathcal{Y}_{\beta_m}(w^{(i_m)}, z_m) w^{(j_1)}, w^{(\overline{j_2})} \rangle \tag{A.45}$$

defined on $0 < |z_m| < |z_{m+1}|$ by scaling the norms of z_m and z_{m+1}.

Now, by fusion of intertwining operators, there exist intertwining operators $\mathcal{Y}_\delta, \mathcal{Y}_\gamma$ with suitable charge spaces, source spaces, and target spaces, such that (A.44) equals

$$\langle \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i_m)}, z_m - z_{m+1}) w^{(i_{m+1})}, z_{m+1}\big) w^{(j_1)}, w^{(\overline{j_2})} \rangle \tag{A.46}$$

when $|z_{m+1}| < |z_m|$. Then (A.45) equals (A.46) when $|z_m| < |z_{m+1}|$. By theorem 2.6, the expression

$$\langle \mathcal{X}_2 \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i_m)}, z_m - z_{m+1}) w^{(i_{m+1})}, z_{m+1}\big) \mathcal{X}_1 w^{(i_0)}, w^{(\overline{k})} \rangle \tag{A.47}$$

converges absolutely and locally uniformly. Hence it is a locally defined holomorphic function when $0 < |z_1| < \cdots < |z_{m-1}| < |z_m|, |z_{m+1}| < |z_{m+2}| < \cdots < |z_n|$. Therefore (A.42) can be analytically continued to (A.43) from $\{0 < |z_{m+1}| < |z_m|\}$ to $\{0 < |z_m| < |z_{m+1}|\}$. $\square$

Appendix B

APPENDIX FOR CHAPTER 3

B.1 von Neumann algebras generated by closed operators

Let A be a (densely defined) unbounded operator on $\mathcal{H}$ with domain $\mathscr{D}(A)$. Choose $x \in B(\mathcal{H})$, i.e., let x be a bounded operator on $\mathcal{H}$. Recall that the notation $xA \subset Ax$ means that $x\mathscr{D}(A) \subset \mathscr{D}(A)$, and $xA\xi = Ax\xi$ for any $\xi \in \mathscr{D}(A)$. The following proposition is easy to show.

Proposition B.1. *Let A be a preclosed operator on $\mathcal{H}$ with closure $\overline{A}$.*

(1) If $x \in B(\mathcal{H})$ and $xA \subset Ax$, then we have $x^*A^* \subset A^*x^*$ and $x\overline{A} \subset \overline{A}x$.

(2) If A is closed, then the set of all $x \in B(\mathcal{H})$ satisfying $xA \subset Ax$ form a strongly closed subalgebra of $B(\mathcal{H})$.

Proof. If $xA \subset Ax$ then $(Ax)^* \subset (xA)^*$. Recall that in general, if A, B are two densely defined unbounded operators on $\mathcal{H}$, and if AB has dense domain, then $B^*A^* \subset (AB)^*$. If A is bounded, then $B^*A^* = (AB)^*$. Thus we have $x^*A^* \subset (Ax)^* \subset (xA)^* = A^*x^*$. Apply this relation to x^*, A^*, and note that $A^{**} = \overline{A}$, then we have $x\overline{A} \subset \overline{A}x$. This proves part (1). Part (2) is a routine check. □

Definition B.2. Let A be a closed operator on a Hilbert space $\mathcal{H}$ with domain $\mathscr{D}(A)$, and let $x \in B(\mathcal{H})$. We say that A and x **commute strongly**[1], if the following relations hold:

$$xA \subset Ax, \quad x^*A \subset Ax^*. \tag{B.1}$$

Corollary B.3. *Suppose that $\mathfrak{S}$ is a collection of closed operators on $\mathcal{H}$. We define its* ***commutant*** *$\mathfrak{S}'$ to be the set of all bounded operators on $\mathcal{H}$ which commute strongly with any element of $\mathfrak{S}$. Then $\mathfrak{S}'$ is a von Neumann algebra. It's double commutant $\mathfrak{S}''$, which is the commutant of $\mathfrak{S}'$, is called the* ***von Neumann algebra generated by*** *$\mathfrak{S}$.*

[1] Our definition follows [Neu16] chapter XIV, in which the strong commutativity of an unbounded operator with a bounded one is called adjoint commutativity.

Lemma B.4. *Suppose that A is a closed operator on $\mathcal{H}$, and $v \in B(\mathcal{H})$ is a unitary operator. Let $A = uH$ (resp. Hu) be the left (resp. right) polar decomposition of A, such that u the partial isometry and H the self adjoint opertor. Then the following conditions are equivalent:*

(a) v commutes strongly with A.

(b) $vA = Av$. (B.2)

(c) $[u, v] = 0,$ and $[e^{itH}, v] = 0$ for any $t \in \mathbb{R}$. (B.3)

Proof. We prove this for the left polar decomposition. The other case can be proved in the same way.

(a)⇒(b): Since v commutes strongly with A, we have $vA \subset Av$ and $v^{-1}A \subset Av^{-1}$. Therefore, $v\mathscr{D}(A) \subset \mathscr{D}(A)$ and $v^{-1}\mathscr{D}(A) \subset \mathscr{D}(A)$. So we must have $v\mathscr{D}(A) = \mathscr{D}(A)$, and hence $vA = Av$.

(b)⇒(a): If $vA = Av$, then $vAv^{-1} = A$. So $Av^{-1} = v^{-1}A$, which proves (a).

(b)⇒(c): We have $vAv^{-1} = A$. Thus by uniqueness of left polar decompositions, we have $vuv^{-1} = u$ and $vHv^{-1} = H$. Hence for any $t \in \mathbb{R}$ we have

$$ve^{itH}v^{-1} = e^{iv(tH)v^{-1}} = e^{itH}.$$

This proves (c).

(c)⇒(b): Suppose that we have (B.3). Then $vuv^{-1} = u$ and $ve^{itH}v^{-1} = e^{itH}$. On the other hand, we always have $ve^{itH}v^{-1} = e^{itvHv^{-1}}$ in general. So vHv^{-1} and H are both generators of the one parameter unitary group $ve^{itH}v^{-1}$. Hence we must have $vHv^{-1} = H$. This implies that $vA = Av$. Therefore (b) is true. □

Proposition B.5. *Let $\mathfrak{S}$ be a set of closed operators on $\mathcal{H}$. For each $A \in \mathfrak{S}$, we either let $A = u_A H_A$ be the left polar decomposition of A, or let $A = H_A u_A$ be the right polar decomposition of A. Then $\mathfrak{S}''$ is the von Neumann algebra generated by the bounded operators $\{u_A, e^{itH_A} : t \in \mathbb{R}, A \in \mathfrak{S}\}$.*

Proof. Let $\mathcal{M}$ be the von Neumann algebras generated by those u_A and e^{itH_A}. We show that $\mathcal{M} = \mathfrak{S}''$.

Let $\mathcal{U}(\mathfrak{S}')$ be the set of unitary operators in $\mathfrak{S}'$. We know that $\mathcal{U}(\mathfrak{S}')$ generates $\mathfrak{S}'$. So $\mathfrak{S}'' = \mathcal{U}(\mathfrak{S}')'$. By lemma B.4 (a)$\Rightarrow$(c) we see that $\mathcal{M}$ commutes with $\mathcal{U}(\mathfrak{S}')$. Hence $\mathcal{M} \subset \mathcal{U}(\mathfrak{S}')' = \mathfrak{S}''$.

Let $\mathcal{U}(\mathcal{M}')$ be the set of unitary operators in $\mathcal{M}'$, the commutant of $\mathcal{M}$. Then by lemma B.4 (c)$\Rightarrow$(a) we also have $\mathcal{U}(\mathcal{M}') \subset \mathfrak{S}'$. Hence $\mathcal{M}' \subset \mathfrak{S}'$, which implies that $\mathcal{M} \supset \mathfrak{S}''$. Thus we've proved that $\mathcal{M} = \mathfrak{S}''$. □

Corollary B.6. *Assume that A is a closed operator on $\mathcal{H}$ and $x \in B(\mathcal{H})$. Let $A = uH$ (resp. Hu) be the left (resp. right) polar decomposition of A with u the partial isometry and H the self adjoint opertor. Then x commutes strongly with A if and only if $[u, x] = 0$ and $[e^{itH}, x] = 0$ for any $t \in \mathbb{R}$.*

Proof. Let $\mathfrak{S} = \{A\}$. Then by proposition B.5, $\mathfrak{S}''$ is generated by u and all e^{itH}. Thus $x \in \mathfrak{S}'$ if and only if x commutes with u and all e^{itH}. □

Definition B.7. Let A and B be two closed operators on a Hilbert space $\mathcal{H}$. We say that A and B **commute strongly**, if the von Neumann algebra generated by A commutes with the one generated by B.

If $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ and A is a closed operator on $\mathcal{H}$. We say that A is **affiliated with** $\mathcal{M}$, if the von Neumann algebra generated by the single operator A is inside $\mathcal{M}$. Now suppose that $\mathcal{N}$ is another von Neumann algebra on a Hilbert space $\mathcal{K}$, and $\pi : \mathcal{M} \to \mathcal{N}$ is a normal (i.e. $\sigma-$weakly continuous) unital *-homomorphism. We define $\pi(A)$ to be a closed operator on $\mathcal{K}$ affiliated with $\mathcal{N}$ in the following way: Let $A = uH$ be its left polar decomposition. Define $\pi(H)$ to be the generator of the one parameter unitary group $\pi(e^{itH})$ acting on $\mathcal{H}$, i.e., the unique self-adjoint operator on $\mathcal{K}$ satisfying

$$e^{it\pi(H)} = \pi(e^{itH}) \quad (t \in \mathbb{R}). \tag{B.4}$$

We then define

$$\pi(A) = \pi(u)\pi(H). \tag{B.5}$$

We can also define $\pi(A)$ using the right polar decomposition of A. It is easy to show that these two definitions are the same.

B.2 A criterion for strong commutativity

A famous example of Nelson (cf. [Nel59]) shows that two unbounded self-adjoint operators commuting on a common invariant core might not commute strongly. In this section, we give a criterion on the strong commutativity of unbounded closed operators. Our approach follows [TL99] and [TL04]. See also [GJ12] section 19.4 for related materials.

Suppose that D is a self-adjoint positive operator on a Hilbert space $\mathcal{H}$. For any $r \in \mathbb{R}$, we let $\mathcal{H}^r$ be the domain of $(1 + D)^r$. It is clear that $\mathcal{H}^{r_1} \supset \mathcal{H}^{r_2}$ if $r_1 < r_2$. We let $\mathcal{H}^\infty = \bigcap_{r \geqslant 0} \mathcal{H}^r$. Define a norm $\|\cdot\|_r$ on $\mathcal{H}_r$ to be $\|\xi\|_r = \|(1 + D)^r \xi\|$. Suppose that K is an unbounded operator on $\mathcal{H}$ with invariant domain $\mathcal{H}^\infty$ ("invariant" means that $K\mathcal{H}^\infty \subset \mathcal{H}^\infty$), that K is symmetric, i.e., for any $\xi, \eta \in \mathcal{H}^\infty$ we have

$$\langle K\xi|\eta\rangle = \langle \xi|K\eta\rangle, \tag{B.6}$$

and that for any $n \in \mathbb{Z}_{\geqslant 0}$ there exist positive numbers $|K|_{n+1}$ and $|K|_{D,n+1}$, such that for any $\xi \in \mathcal{H}^\infty$ we have

$$\|K\xi\|_n \leqslant |K|_{n+1} \|\xi\|_{n+1}, \tag{B.7}$$

$$\|[D, K]\xi\|_n \leqslant |K|_{D,n+1} \|\xi\|_{n+1}. \tag{B.8}$$

Since K is symmetric, it is obviously preclosed. We let $\overline{K}$ denote the closure K. The following lemma is due to Toledano-Laredo (cf. [TL99] proposition 2.1[2] and corollary 2.2).

Lemma B.8. *$\overline{K}$ is self-adjoint. Moreover, the following statements are true:*

(1) For any $n \in \mathbb{Z}_{\geqslant 0}$ and $t \in \mathbb{R}$, the unitary operator $e^{it\overline{K}}$ restricts to a bounded linear map $\mathcal{H}^n \to \mathcal{H}^n$ with

$$\|e^{it\overline{K}}\xi\|_n \leqslant e^{2nt|K|_{D,n}} \|\xi\|_n, \quad \xi \in \mathcal{H}^n. \tag{B.9}$$

(2) For any $\xi \in \mathcal{H}^\infty$, $h \in \mathbb{R}$ and $k = 1, 2, \ldots$, we have

$$e^{i(t+h)\overline{K}}\xi = e^{it\overline{K}}\xi + \cdots + \frac{h^k}{k!} K^k e^{it\overline{K}}\xi + R(h), \tag{B.10}$$

where all terms are in $\mathcal{H}^\infty$ and $R(h) = o(h^k)$ in each $\|\cdot\|_n$ norm, i.e., $\|R(h)\|_n h^{-k} \to 0$ as $h \to 0$.

[2]Toledano-Laredo's proof of this proposition was based on a trick in [FL74] theorem 2.

This lemma may help us prove the following important criterion for strong commutativity of unbounded closed operators.

Theorem B.9. *Let T be another unbounded operator on $\mathcal{H}$ with invariant domain $\mathcal{H}^\infty$. Suppose that T satisfies the following conditions:*

(1) There exists $m \in \mathbb{Z}_{\geqslant 0}$, such that for any $n \in \mathbb{Z}_{\geqslant 0}$, we can find a positive number $|T|_{n+m}$, such that

$$\|T\xi\|_n \leqslant |T|_{n+m}\|\xi\|_{n+m} \quad (\xi \in \mathcal{H}^\infty). \tag{B.11}$$

(2) T is a preclosed operator on $\mathcal{H}$.

(3) $KT\xi = TK\xi$ for any $\xi \in \mathcal{H}^\infty$.

Then the self-adjoint operator$\overline{K}$ commutes strongly with $\overline{T}$, the closure of T.

Proof. By lemma B.8, for each $t \in \mathbb{R}$, $e^{it\overline{K}}$ leaves $\mathcal{H}^\infty$ invariant. We want to show that

$$e^{it\overline{K}}Te^{-it\overline{K}} = T \quad \text{on } \mathcal{H}^\infty. \tag{B.12}$$

For any $\xi \in \mathcal{H}^\infty$ we define a $\mathcal{H}^\infty$-valued function Ξ on $\mathbb{R}$ by

$$\Xi(t) = e^{it\overline{K}}Te^{-it\overline{K}}\xi. \tag{B.13}$$

If we can show that this function is constant, then we have $\Xi(t) = \Xi(0)$, which proves (B.12). To prove this, it suffices to show that the derivative of this function is always 0.

For any $t \in \mathbb{R}$, if $0 \neq h \in \mathbb{R}$, then

$$\begin{aligned}
\Xi(t+h) =& e^{i(t+h)\overline{K}}Te^{-i(t+h)\overline{K}}\xi && \text{(B.14)}\\
=& e^{i(t+h)\overline{K}}T\big((1-ihK)e^{-it\overline{K}}\xi + o(h)\big) && \text{(B.15)}\\
=& e^{i(t+h)\overline{K}}T(1-ihK)e^{-it\overline{K}}\xi + o(h) && \text{(B.16)}\\
=& e^{i(t+h)\overline{K}}Te^{-it\overline{K}}\xi - ihe^{i(t+h)\overline{K}}KTe^{-it\overline{K}}\xi + o(h) && \text{(B.17)}\\
=& [e^{it\overline{K}}(1+ihK)Te^{-it\overline{K}}\xi + o(h)]\\
&- ih[e^{it\overline{K}}(1+ihK)KTe^{-it\overline{K}}\xi + o(h)] + o(h) && \text{(B.18)}\\
=& e^{it\overline{K}}Te^{-it\overline{K}}\xi + o(h) = \Xi(t) + o(h), && \text{(B.19)}
\end{aligned}$$

where (B.15) and (B.18) follow from (B.10), and (B.17) follows from the relation $KT = TK$ on $\mathcal{H}^\infty$. We also used the fact that $To(h) = o(h)$ (which follows from (B.11)) in (B.16). Here the meaning of $o(h)$ is same as that in lemma B.8.

Hence we have shown that $\Xi'(t) = 0$ for any $t \in \mathbb{R}$, which proves (B.12). Now we regard T as an unbounded operator on $\mathcal{H}$. By passing to the closure, we have $e^{it\overline{K}}\overline{T}e^{-it\overline{K}} = \overline{T}$. This shows that $\overline{T}$ commutes strongly with $\overline{K}$. $\square$

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