AN INVESTIGATION OF CONTENT KNOWLEDGE FOR TEACHING: UNDERSTANDING ITS DEVELOPMENT AND ITS INFLUENCE ON PEDAGOGY

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To my wife, Molly, for her never-ending support,
encouragement, and love. And for giving me
the greatest gift possible…

To my daughter, Audrey Allen: welcome.
It has been wonderful getting to know you
these past few weeks. There's a lot we've
got to show you and we look forward
to every moment.

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PART ONE

PRELIMINARIES

CHAPTER I

INTRODUCTION

This study investigates conditions that enable teachers to teach mathematics for understanding. It is not well understood how pedagogical or mathematical knowledge developed in university courses exhibits itself in pre-service teachers' (PSTs') classroom practices.

My aim is to understand the intricacies of this "transfer" from a university setting to school-based teaching practices. In doing this, I focus first on PSTs' understandings of mathematics as the primary resource upon which they draw while teaching. The importance of teachers' knowledge of content has been acknowledged by a variety of scholars (Ball, 1993; Ball & McDiarmid, 1989; Bransford, Brown, & Cocking, 2000; Grossman, 1990; Grossman, Wilson, & Shulman, 1989; Ma, 1999; Schifter, 1990, 1995; Shulman, 1986). However, it is axiomatic that a teacher's knowledge of mathematics alone is insufficient to support his or her attempts to teach for understanding. In that vein, Shulman (1986) coined the phrase *pedagogical content knowledge* [PCK], or specific content knowledge as applied to teaching, to address what at that time had become increasingly evident – that content knowledge itself is not sufficient for teachers to be successful. Ma (1999) and Stigler and Hiebert (Stigler & Hiebert, 1999) further refined the idea of PCK by arguing that teachers need a *profound* understanding of mathematics – knowledge having the characteristics of breadth, depth, and thoroughness: "Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics" (p. 124).

My first research question builds from Ma's construct of profound understanding of mathematics:

Research Question 1:

What understandings of function will a group of PSTs have after participating in instruction that employs simultaneous covariation of quantities as a pathway to their development of profound understandings of function?

My previous work with student teachers (Silverman, 2004a) led me to believe that PST's naïve conceptions of "profound" understandings of mathematics are inconsistent with teaching mathematics for understanding. Since teachers' understandings of mathematics enable or constrain their ability to orchestrate mathematical discussions that provide students with opportunities to make sense of advanced mathematical ideas, it is important for teacher educators to understand both the understandings with which PSTs enter our programs and ways in which those understandings can be productively influenced. By *teachers' understandings of mathematics* I mean "the loose ensembles of actions, operations, and ways of thinking that come to mind unawarely – of what they wish their students to learn, and the language in which they have captured those images" (Thompson & Thompson, 1996, p. 16). It is against the background of the images that teachers hold with regard to their own understandings and of the understandings they hope students will have that they select tasks, pose questions, and make other pedagogical decisions.

Thus, in this study, I am extending Ma's (1999) notion of profound mathematical understanding in two key ways. First, I argue that teachers must develop explicit images of (1) the mathematics that they want their students to understand, (2) an understanding of the pedagogical importance of these understandings, and (3) a sense of how these understandings might develop in students. Second, I will build upon Simon's (2002) notion of a *key developmental understanding* of a mathematical idea to propose the idea of *key pedagogical understanding* of a mathematical idea as a threshold for teaching for mathematical understanding. A *key developmental understanding* is a particular understanding of a mathematical idea that facilitates understanding a variety of additional mathematical topics. A *key pedagogical understanding* involves an individual's awareness of the pedagogical implications of those key developmental understandings of important mathematical ideas. While teacher education research locates notions such as profound understanding of fundamental mathematics, key developmental understandings, and pedagogical content knowledge as particular states along a developmental trajectory, I will argue that focusing on the idea of key pedagogical understanding addresses how one might one come to develop such understandings. As such, I propose a second broad research question:

> *Research Question 2:*
>
> *How do PSTs' understandings of covariation impact their image of instruction and their engagement with students when teaching concepts of function? Put another way, in what ways can a profound understanding of covariation serve as a key pedagogical idea in teaching for understanding of functions?*

CHAPTER II

THEORETICAL BACKGROUND

In this chapter, I will make explicit the theoretical perspectives employed in this study's design and analysis. This chapter will consist of a discussion of (a) teaching and learning mathematics with understanding; (b) genetic epistemology, reflecting abstraction, and radical constructivism (theories of knowledge development); (c) key developmental and pedagogical understandings; (d) different understandings and conceptions of the concept of function; and (e) didactic objects and didactic models.

Teaching and Learning Mathematics with Understanding

Teaching and learning mathematics with understanding is the cornerstone of the National Council of Teachers of Mathematics' [NCTM] *Principles and Standards of School Mathematics* (National Council of Teachers of Mathematics, 2000). The idea of teaching mathematics with understanding is not new– it was also the focus of the three prior NCTM standards documents (National Council of Teachers of Mathematics, 1989, 1991, 1995) and has its roots in the work of Brownell (1928, 1932), Dewey (Finken, 2001), and constructivism (Stiff, 2001). These reform initiatives attempted to fundamentally change what it means to learn mathematics. Instead of memorizing techniques and algorithms, they envisioned that students develop mathematical power: "[the] ability to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve non-routine problems" (National Council of Teachers of Mathematics, 1989, p.5). Mathematics, therefore, is about sense-making and problem solving as opposed to thoughtless applications of algorithms and procedures.

A similar image of teaching and learning mathematics with understanding is expressed by NCTM mathematics reform organizations (Committee of Inquiry into the Teaching of Mathematics in Schools, 1993; Mathematical Sciences Education Board and National Research Council, 1989) and research in mathematics education and psychology. For example, Bransford, Brown and Cocking (2000) speak of competence in an area in a way that is consistent with learning mathematics with understanding. They claim that this understanding requires "a deep factual knowledge," understanding the "facts and ideas in a context of a conceptual framework," and an organization of the knowledge "in ways that facilitate retrieval and application" (p. 16). Their definition is consistent with learning mathematics "with understanding," which is described by the NCTM as "the ability to use knowledge flexibly, applying what is learned in one setting appropriately in another" (National Council of Teachers of Mathematics, 2000, p. 20) or as understanding mathematics in a way that allows one to apply mathematical concepts in a variety of situations and applications (McDiarmid, Ball, & Anderson, 1989). My definition of learning mathematics with understanding follows Bransford, Brown & Cocking's description, but my focus will be on the first two aspects – deep knowledge of mathematics and a conceptual framework within which those facts reside. I also agree with Kahan, Cooper, & Bethea (2003), who believe in the importance of factual knowledge, but "value it most when it is coordinated with deeper understanding and ready for application" (p. 225).

Thompson & Saldanha (2003) define "to understand" as "to assimilate to a scheme," relying on Piaget's notion of assimilation (Piaget, 1970/1971, 1976a; von Glasersfeld, 1995), which Glasersfeld (1995) describes as "[coming] about when a cognizing organism fits an experience within a conceptual structure it already has" (p. 62). In this case, the conceptual structure is analogous to Bransford et al's notion of conceptual framework. This conceptual structure is the web of connections that is developed by an individual that allows them to act and enact within situations he or she encounters (Carpenter, 1986; Hiebert, 1986; Thompson & Saldanha, 2003). In this way, understanding is not a "store" of facts, algorithms, procedures, etc. to be obtained. Rather, it is a lens through which an individual organizes the world. This notion is a key aspect of Piaget's view that knowledge "cannot have any iconic correspondence with an ontological reality. … [Thus,] the cognitive organism shapes and coordinates its experience and, in doing so, transforms it into a structural world" (Glasersfeld, 1995, p. 57). This bold notion is the basis of radical constructivism, which traces its origins to Piaget's theory, and can be summarized by the following three "tenets:"

> 1. Mathematics is created through human activity. Humans have no access to a mathematics that is independent of our ways of knowing it.
>
> 2. What individuals currently know affords and constrains what they can assimilate (perceive and understand).
>
> 3. Learning mathematics is a process of transforming one's ways of knowing and acting (Heinz, Kinzel, Simon, & Tzur, 2000).

Noting that (1) students have no direct access to mathematical content and (2) that what students learn is dependent on what they already "know," and (3) that learning is an internal process of construction from previous constructions, begs the question of how any new knowledge or understandings are developed.

Comments on Teaching Mathematics with Understanding

The push for teaching and learning mathematics "with understanding" began as a response to widespread dissatisfaction with teaching and learning of mathematics (Ebert, 1993).

In fact, the NCTM, in the *Principles and Standards of School Mathematics* [PSSM], notes that research has shown that today's mathematics students are not learning the mathematics they need in order to be active contributors in a mathematics intensive society (Mathematical Sciences Education Board and National Research Council, 1989).

The PSSM claims that "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (National Council of Teachers of Mathematics, 2000, p. 20). This belief addresses two issues. First, it addresses the need for developing a coherent conceptual framework within which the factual knowledge might lie. Thus, learning mathematics with understanding is about relationships between concepts. Second, it brings to the fore the idea that mathematical objects are constructed and re-constructed within each individual student – mathematical objects do not exist outside of these constructions.

Reflecting Abstraction: A Theory of the Development of Knowledge

Piaget (1980) proposed reflecting abstraction as the process by which new, more refined conceptions develop from an individual's current conceptions. This process is described by Piaget (1977/2001) as a process where more advanced cognitive structures are developed from "lower level" cognitive structures. Dubinsky (1991) claims that this process is at the heart of the development of mathematical thought. In this section, I will discuss reflecting abstraction for two reasons. First, it responds to the question posed in the previous section about how new understandings are developed. Second, it highlights a theory of knowledge as a dynamic web of meanings that is in a state of continual flux (though seeking equilibrium) and, at the same time, is the lens through which we view the world as well as the means through which we attempt to organize the world and make sense of it.

The idea that knowing is always a dynamic process, involving mental operations, and that these mental operations are always part of a larger system of operating, was central to Piaget's work. It is this knowing as a dynamic, related system of mental operations that will be the focus of this study. For example, when I speak of a PST's knowledge, what I am referring to is the lens through which she interprets her task: enabling the mathematical development of her students (by some metric). The "knowledge" the PSTs bring to the task both enables and constrains not only how they plan to achieve their instructional goal, but also how they envision their task and how they plan to achieve it. Knowledge, or knowing, is not an endpoint: "Little by little, there has to be a constant equilibrium established between the parts of the subject's knowledge and the totality of his knowledge at any given moment. There is a constant differentiation of the totality of knowledge into the parts and an integration of the parts back into the whole" (Piaget, 1977, p. 11). Rather than a fixed goal state, or static equilibrium, Piaget's idea of knowledge is akin to dynamic equilibrium, where the forces exerted (or change in quantities of particular elements or compounds) all balance each other out. To Piaget, it was not *equilibrium* that was important, but rather *the process of equilibration*.

Equilibration is the process through which an individual "organizes the world" that they have no direct access to (Piaget, 1937/1971). Equilibration consists of two sub-processes: assimilation and accommodation. Assimilation involves an individual "[fitting] an experience into a conceptual structure [they] already have" (Glasersfeld, 1995, p. 62). Piaget believed that "… no behavior, even if it is new to the individual, constitutes an absolute beginning. It is always grafted onto previous schemes and therefore amounts to assimilating new elements to already constructed structures (innate, as reflexes are, or previously acquired)" (Piaget, 1976b, p. 17). At times, the experience may not fit within a conceptual structure – it may not act similar to the individual's expectations. In cases such as this, Piaget believed that in order to equilibrate, the individual must modify his conceptual structures, or schemes, in order to better organize his experiences (and thus better "organize" the world). This modification of conceptual structures is referred to as accommodation. It is the process of equilibration, via accommodation and assimilation, that enables the development of knowledge. Glasersfeld (1995) details a theory where learning "in a specific direction take[s] place when a scheme, instead of producing the expected result, leads to a perturbation [something that is not easily assimilated], and perturbation, in turn, to an accommodation that maintains or reestablishes equilibrium" (p. 68). While I must comment that perturbations cannot dictate what an individual will learn, Piaget's theory is grounded on the fact that all learning is "triggered" by perturbations.

The theory of equilibration is not sufficient for explaining the development of new, more advanced conceptions out of existing ones – cognitive structures can be accommodated in order to "fit" with experiences, but how can that explain the development of new cognitive structures that differ almost entirely from existing structures? Piaget calls upon abstraction as the mechanism whereby new cognitive structures are developed, and abstraction should also be thought of as a (more advanced) means of equilibration. Abstraction involves abstracting properties of coordinations of actions from the actions themselves. "It does so in two phases. The first phase projects a structure at a lower developmental level (such as the action *coordination of interest*) onto a higher level (where the coordination may now be understood concisely and explicitly). The second phase reorganizes the structure and higher level; an explicit understanding of something about our knowledge or our actions is not a mere copy of the previous cognitive structure, and it needs to be able to be integrated with other new structures at the higher level" (Piaget, 1977/2001, p. 4).

Piaget distinguished among three types of abstraction: empirical, pseudo-empirical, and reflecting. For my purposes, I will focus this discussion on empirical and reflecting abstraction.

Empirical abstraction involves looking for similarities and differences among the objects under consideration, or "draw[ing] … information from objects and from the material or observable characteristics of actions," (Piaget, 1977/2001, p. 317). Empirical, or simple, abstraction is the extraction of characteristics from an object or set of objects and the classification on the bases of these characteristics alone. Empirical abstraction, though it may be a transformation of a previous cognitive structure, focuses on classifying based on characteristics already in one's conception of that object. This does not result in the creation of a new cognitive structure, but rather involves focusing attention on particular attributes of the object (Piaget, 1977/2001). For example, Saldanha (2003) notes that a child may abstract from his or her daily experiences that all apples are green and smooth. This child would likely be surprised when presented with a red delicious or russet apple.

Consider, briefly, the following diagram:

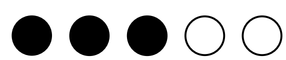

Figure 2-1: Colored Marbles

Thompson (2002) notes that diagrams like the one shown in Figure 2-1, are often presented within the context of a discussion of additive reasoning. As an example, Thompson notes that, if viewed additively, the collection might be viewed as one of the following: 3 of five disks are dark, there are 2 more disks than dark disks, etc. In contrast, Thompson notes that, if viewed multiplicatively, a person might understand Figure 3 as *the number of disks is 5 times as large as one-third the number of dark disks* (i.e., the number of disks is five-thirds the number of dark disks) (Thompson, 2002)[1]. It is well accepted that reasoning additively is conceptually simpler than reasoning multiplicatively. In the case of the additive reasoning discussed above, the student is simply looking for similarities and difference between the objects themselves, or "draw[ing] … information from objects and from the material or observable characteristics of actions," (Piaget, 1977/2001, p. 317) where, in this case, the action is understanding the characteristics of the collection. Piaget refers to this as empirical abstraction. Knowledge obtained through empirical abstraction, though it may be a transformation of previous knowledge, is not viewed as the development of new knowledge, simply focusing on characteristics already in that object (Piaget, 1977/2001).

A question to ask might be how might someone develop multiplicative reasoning skills if additive reasoning is all that is currently available to them? As previously discussed, Piaget proposed Reflecting Abstraction as the answer to this question (Piaget, 1977/2001). Reflecting abstraction, as described previously, is a process by which new, more advanced conceptions develop out of existing conceptions and involves abstracting properties of our action coordinations in order to develop new cognitive structures. In the case of the student who reasons multiplicatively, the student needs to separate the action of understanding the collection from the actual collection: in essence, multiplicative reasoning involves invariant relationships between quantities (i.e. the number of disks is 5 times as large as one-third the number of dark disks) as opposed to particular relationships between sets (three of the five marbles are dark). In the case of multiplicative reasoning, the student has transcended the objects themselves and discovered a characteristic of "five times as large" by using the collection of marbles as tools for

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[1] The difference between additive and multiplicative reasoning is not of interest in this discussion, except to note that the multiplicative reasoning is the more complex of the two.

his exploration. The abstracted knowledge is no longer in terms of marbles, but characteristics of the group of marbles (Piaget, 1970/1971).

Reflecting abstraction involves developing new cognitive structures by abstracting invariant features in our coordinations of action ensembles (Piaget, 1980) and therefore enriching an object with characteristics previously not present in it. This statement makes sense, of course, only with the understanding that the "objects" of which Piaget spoke were known objects, not objects existing independently of a knower (Piaget, 1970/1971). In another classic example, Piaget described the development of how a child might develop the notion of a quantity. In his example, when placing ten pebbles in various configurations (lines, circles, etc), the child discovers that he always got ten, regardless of how the pebbles were arranged or how he counted them:

> It is true that the pebbles, as it were, let him arrange them in various ways; he could not have done the same with drops of water. So in this sense there was a physical aspect to his knowledge. But the order was not in the pebbles; it was he, the subject, who put the pebbles in a line and then in a circle. Moreover, the sum was not in the pebbles themselves; it was he who united them. The knowledge . . . was drawn [sic] not from the physical properties of the pebbles, but from the actions that he carried out on the pebbles (Piaget, 1970/1971, p. 17).

Key Developmental Understandings and Key Pedagogical Understandings in Mathematics

Simon (1995) introduces the idea of a key developmental understanding in mathematics as a way to think about understandings that can be useful goals of mathematics instruction. He describes two characteristics of a key developmental understanding. First, a key developmental understanding involves a conceptual advance or a "change in the learner's ability to think about and/or perceive particular mathematical relationships" (Simon, 2002, p. 993). Students who possess a key developmental understanding tend to find different, yet conceptually related ideas and problems understandable, solvable and sometimes even trivial. I describe this understanding as a *powerful* understanding of mathematics[2]. Second, Simon claims that key developmental understandings are not developed through explanation and/or demonstration of the concepts to be understood. In this way, key developmental understandings are related to the previous discussion of reflecting abstraction – when developing a key developmental understanding, the learner must actually imbue new characteristics on her knowledge and therefore fundamentally transform her understanding. Thompson & Thompson (1996) describe such an understanding where the students' ability to solve problems is *as a consequence* of their understanding and is distinct from explicitly teaching how to solve the same problems.

It is unclear whether teachers who develop or have developed key developmental understandings are thereby able to orchestrate instructional environments within which students are positioned to develop a robust understanding of the particular mathematical ideas. My hypothesis is that key developmental understandings are not, by themselves, sufficient for a teacher to teach for understanding. In the hands of particular teachers, a key developmental understanding might aid the teacher's goals for instruction and the means for achieving them. However, there is nothing in a key developmental understanding that indicates that one who possesses this understanding is aware of its utility. It is for this reason that I propose the concept of a *key pedagogical understanding* [KPU]. A key pedagogical understanding also involves a conceptual advance that is not developed through telling or explaining. A key pedagogical understanding is a person's transformation of a key developmental understanding, from a way of personally understanding *a particular mathematical concept,* to a way of understanding how this

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[2] Though it seems that a *key developmental understanding* and a *powerful mathematics* understanding are synonymous, I argue that the difference is an important one. In short, the understanding may be developmental but a developmental understanding may or may not be seen by the teacher as *powerful* – i.e. something that it is worthwhile for the students to attain.

key developmental understanding could empower students' learning of related ideas were they to have it.

The Development of Key Pedagogical Understanding

In this section, I discuss one possible mechanism for the development of a key pedagogical understanding from a key developmental understanding. I use reflecting abstraction as a construct to explain the development of key pedagogical understandings.

A key developmental understanding might be viewed as a pedagogical action, where *action* is used in the Piagetian sense[3]. Teachers are engaged in pedagogical actions when they wonder, "What might I do to help students think like what I have in mind?" Their question is posed in a domain specific manner, such as "How might I help my students think about logarithms as an accelerated condensing and recoding of the number line?

The development of a key pedagogical understanding involves separating one's own understanding from the hypothetical understanding of the learner (Steffe, 1994). When a person views a pedagogical action as if she is not an actor in the situation (even though she is), and when the person can separate herself from the action (and thereby reflect on it), the pedagogical action has been transformed into a pedagogical understanding. It is this understanding that is capable of being reflected upon, for the teacher now sees various alternatives that could have happened and has developed agency over the process. The teacher is also now able to see the "pedagogical power" of a key developmental understanding.

When a teacher develops a key developmental understanding, his content knowledge becomes "related" to other content knowledge and extends his web of connections (Thompson &

[3] Piaget defined the word broadly as any change to the perceptual input (Piaget, 1967).

Saldanha, 2003). A key developmental understanding could then be viewed as knowledge that is assimilated to a scheme. This new understanding (and thus new knowledge) cannot be PCK because this transformed knowledge is not in and of itself pedagogical[4]. At this point, this new knowledge is mathematical knowledge that has pedagogical potential. It is not until the teacher transforms this knowledge into knowledge that is pedagogically powerful that the teacher has developed PCK. Thus, rather than content knowledge for teaching, PCK is the transformation of content knowledge into a form that is recognized by the teacher as pedagogically powerful, and that transformation entails the teachers' creation of key developmental understandings, becoming reflectively aware of them, and placing them within a model of students' learning in the context of instruction.

PCK and Instructional Environments

Thompson (2002) proposed the construct of a didactic object to make sense of instructional design from a constructivist perspective. By instructional design, however, Thompson envisioned something quite different from a typical lesson plan consisting of materials, objectives, procedures, etc. He described the process of instructional design as creating "a particular dynamical space, one that will be propitious for individual growth in some intended direction, but will also allow for a variety of understandings that will fit with where individual students are at that moment of time" (p. 194). With regards to this dynamical space, Thompson (1985) notes that when conceptualized in this way, instructional design requires that the objectives of instruction be stated in cognitive terms and that images of instruction be of a

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[4] Thus there is a transformation in the development of further refined and developed mathematical knowledge. It is, however, new mathematical knowledge, not pedagogical knowledge.

teacher choreographing conversations which have the possibility of stimulating reflective discourse around the desired mathematical idea. Instructional design, therefore, is not about teaching particular content; rather it is about understanding the content and how one who might come to know the content *as conceptualized by the teacher as powerful.* It is also about designing instructional environments that take into account the varied understandings the students bring to the class within which they have a significant chance of understanding the content that way.

Instructional design of this type is consistent with learning mathematics with understanding as discussed previously. First, by prescribing the instructional environment, as opposed to the content per se, natural variations among students' histories and existing knowledge increases the likelihood that they will bring different perspectives to any discussion. Additionally, these different perspectives lead to different solutions or views on the topics at hand and different means of justification of these solutions or views. Both of these facts will provide the teacher with opportunities to highlight the relationships between the different mathematical ideas that are at play in any particular situation. Second, by not *prescribing* instruction, this mode of instructional design allows for students to take part in the act of constructing the mathematics as opposed to simply learning *the mathematics* that is "out there."

Didactic Objects

Thompson describes a *didactic object*, as "a thing to talk about" that is designed with the intention of supporting reflective mathematical discourse (p. 198). The didactic object, along with the ensuing discourse, is envisioned by the instructor to provide an environment within which students can be exposed to rich mathematical concepts, take part in reflexive discourse, and ultimately be provided with opportunities within which to construct their own images of the mathematics at play. Thompson stresses that an object is not didactic in itself. Rather, it is didactic only to the extent that the instructional designer using it (e.g., a teacher) has conceived of it as such. I would push this issue a bit to say that it is didactic when the teacher has designed it to be; a didactic object is a *tool* that the teacher envisions as having the possibility of engendering productive discourse. One designs the object to be didactic by envisioning three things: Conversations regarding it, ways to support those conversations, and students' interactions with it so that the students are afforded many opportunities to construct an image of mathematics that is consistent with both the teachers' mathematics and the standard mathematical canon. This requires that the teacher have deep understanding of the mathematics at play and its role within the mathematical curriculum and mathematics at large, as well as a working understanding of the students' possible interactions with the object and the ensuing discussions.

Though it appears that Thompson has clarified an environment in which one might learn mathematics for understanding, we must still answer the question, "What knowledge must teachers possess in order to conceive of such an environment?"

An example might help clear up a few issues. One example of a didactic object discussed in Thompson (2002) is a common diagram used as a didactic object. Figure 2-2 below depicts a collection of different colored marbles.

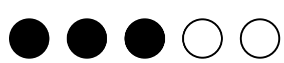
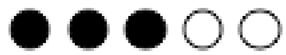


Figure 2-2: Colored Marbles

As noted previously, diagrams like this can be found in any school mathematics textbook and are likely to be discussed in a way that emphasizes additive reasoning. Thompson proposes the use of Figure 2-2 and Figure 2-3 as a didactic object (see Figure 2-3, below).



- Raise your hand when you can see 3/5 of something.
  (What do you see? What is the something of which you see three-fifths?
  Why is it [hard or easy] to see that?)
- Raise your hand when you can see 5/3 of something.
  (What do you see? What is the something of which you see three-fifths?
  Why is it [hard or easy] to see that?)
- Raise your hand when you can see 3/5 of 5/3.
- (What do you see? How much is three-fifths of five-thirds? Why is it [hard or easy] to see that?)
- Raise your hand when you can see 5/3 of 3/5.
  (What do you see? How much is five-thirds of three-fifths? Why is it [hard or easy] to see that?)

Figure 2-3: Figure 2-2 as a Didactic Object
(Thompson, 2002, p. 199)

The conversations had with his or her students by a teacher who envisions Figure 2-2 as a didactic object are ones in which reflection on both the characteristics of the figure and the contributions of other students is encouraged. The parenthetical questions (in Figure 2-3) are designed to provide the students with an environment in which they are likely to construct a multiplicative understanding of the collection. What the example does for us is allow us now to ask the question, “What knowledge might one need in order to conceive of such an object?”

Clearly, knowledge of multiplicative reasoning alone is not sufficient knowledge for a teacher to conceive of such an object, regardless of how robust and connected that knowledge is. Pedagogical knowledge is also undoubtedly insufficient. The knowledge necessary to conceive of objects as didactic is knowledge of the pedagogical power and utility of a particular key understanding of mathematics. The knowledge necessary to conceive of a didactic object is a key pedagogical understanding.

Didactic Models

A key pedagogical understanding is similar to Thompson's notion of a didactic model: "A scheme of meanings, actions, and interpretation that constitute the instructor's or instructional designer's image of all that needs to be understood for someone to make sense of the didactic object in the way he or she intends" (Thompson, 2002, p. 211). Thompson notes the similarity between a didactic model and Simon's (1995) idea of a learning trajectory, but distinguishes the two by highlighting the clear distinction between the instructional activities and what it is envisioned that the students will come to understand in a didactic model.

Another way to phrase the purpose of this study is to better understand the development of didactic models and the relationship between didactic models and key pedagogical understandings. My second research question, *How do the PSTs' understandings of covariation impact their image of instruction and their engagement with students when teaching concepts of function?,* focuses precisely on this issue. My initial hypothesis is that KPU's are necessary but not sufficient for one to develop a didactic model.

The Concept of Function

In the previous section, I discussed Thompson's notion of instructional design from a constructivist perspective. One of the primary notions discussed was that the objectives of instruction be in cognitive terms. In other words, developing instruction must begin with answering the question *How do we want students to come to think about particular mathematical* *ideas.* The process of answering this question is similar to Glasersfeld's (1995) Conceptual Analysis, an analytic method whose aim is to answer the question, "What mental operations must be carried out to see the presented situation in the way one is seeing it?" (p. 78). The main difference between conceptual analysis and instructional design is that conceptual analysis is a method which attempts to explain how one might know something in light of how they act and interact; instructional design, from a constructivist perspective, begins with an explanation of a desired "way of knowing" and attempts to design instructional situations within which one is likely to have productive mathematical conversations towards this end. In this section, I discuss a particular "way of knowing" functions.

The predominant conception of function in mathematics today can be described as functions as correspondence, or "a rule that assigns each element x in a set A exactly one element, y, called $f(x)$, in a set B" (Stewart, 1999). For more than 100 years, the field of mathematics has accepted this conception of function as *the* definition of a function. As a result, this is virtually the only way functions are presented in school mathematics, regardless of level. This presentation is not consistent with the historical development of the function concept, for it was little more than 100 years ago that the correspondence definition of a function was introduced, largely because it was helpful for those who wished to define functions by a limiting process (Kleiner, 1989; Thompson, 1994). The correspondence conception of function was also consistent with the push within the mathematics community for formalization of mathematics in response to the number paradoxes that arise from imprecise mathematical definitions (Burton, 1999). This current definition of functions persists despite the fact that many mathematicians and mathematics educators (Eisenberg, 1991; Thompson, 1994; Wilder, 1967) criticize this conception on pedagogical grounds.

In response to the criticisms of the correspondence conception of a function a number of researchers (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Thompson, 1994; Thompson & Thompson, 1994) have proposed a covariational conception of a function[5]. The covariational conception of a function is consistent with "reform" mathematics which calls for a shift in attention in the mathematics curriculum from functions as rules and formulas to functional relationships, understood in both mathematical settings and in related applications. A covariational conception of a function highlights two key aspects of the functional relationship. First, that a function is a relationship between quantities, which can be represented by an ordered pair whose coordinates represent *values of the two quantities simultaneously*. Second, a covariational conception entails the notion that *the two quantities' values can, in fact, vary* (Saldanha & Thompson, 1998).

Saldanha & Thompson (1998) discuss the development of covariational reasoning as emerging from focusing on a quantity of variable magnitude (henceforth variable) and tracking its variation. I find it important to reiterate the fact that a variable has two important characteristics. First, it is a measurable quantity (it has a magnitude); second, the measure of that quantity can vary. Covariational reasoning involves the coordination of two variables, each of which can be envisioned as varying independently. Ultimately, this way of thinking allows students to (a) envision a graph as a collection of points; (b) envision the collection of points as being generated by keeping track, simultaneously, of two quantities whose values vary; and (c) envisioning that every point in a graph represents, at once, simultaneous values of two quantities.

[5] Historically, the covariational conception of function pre-dated the correspondence conception.

Covariation as a Candidate for a Key Developmental Understanding of Functions

The basis of this study is grounded in the fact that a covariational conception of a function can be a key developmental understanding of the concept of function. Ultimately, whether or not a student possesses a KDU is an empirical question that will be investigated through analysis of how the students' understandings allow them to make sense of more conceptually challenging problems that it is unlikely that they would have been able to solve without the KDU.

A covariational conception can be thought of as a conceptual precursor to a fully-developed correspondence conception of a function. At an appropriate time, as Thompson suggests, I believe that the correspondence (set-theoretic) conception of function should be introduced to students of mathematics. This notion is consistent with the NCTM standards.

With regard to the development of the function concept, Thompson (1994) notes that many elementary mathematics students tend to see a function as a "command to calculate" and that early algebra students are no more likely to see the expression $x\ (12\ (x - 5))$ as representing a number as elementary students are to see that the expression 4 (12 (4 – 5)) represents anything other than *something to do*. Researchers (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thompson, 1996; Briedenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992) have labeled this such a conception of a mathematical concept as an *action* conception. When a learner is at this stage, they are able to perform the prescribed actions when induced by external stimuli – in the case of functions, a number and an expression to be evaluated. A *process* conception of a function involves the learner automating lengthy sequences of operations into an expression that, in his or her image of it, "evaluates itself" (Thompson, 1994). When a student possesses a process conception of function, he or she can imagine the function as something that performs the sequences of operations but no longer needs to actually think about the chain of operations when envisioning the result of the evaluation. In the case of the process conception, the function is no longer tied to external stimuli but rather under the individual's control. Whereas a student with an action conception of a function tends to struggle with piece-wise defined functions, inverse functions, students with a process conception can begin to understand these advanced ideas (Briedenbach et al., 1992; Sfard, 1987, 1992). Moreover, a process conception of function is necessary to understand trigonometric functions such as sin x since no explicit instructions are given for obtaining an output for a given input (Asiala et al., 1996).

The covariational conception is related to the progression of a function from an action conception to a process conception (Zandieh, 2000). For example, Zandieh notes that once a person conceives of a function as the covariation of quantities, they "can begin to imagine 'running through' a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number" (Thompson, 1994, p. 26). It is this notion of "quantities varying" that is a conceptual precursor to a fully-developed correspondence conception of a function. For example, consider students' understanding of the concept of derivative. If the student's conception of function is simply a relationship between quantities (for example, when $x = 2$ and $f'(x) = 4$), it is unlikely that the student truly understands the concept of derivative. Understanding the concept of derivative <u>requires</u> that students think of some change in x and the corresponding change in y. Cottrill, Dubinsky, Nichols, Shcwingendorf, Thomas and Vidakovic (1996) found that for students to understand the derivative concept, which by definition is directly related to the concept of a limit, the students must consider the function as involving dynamic aspects rather than as a static entity. Thus, without that background image of the derivative, the slope of the tangent line has no meaning with respect to the characteristics of the original function[6]. Furthermore, Dubinsky and Harel (1992) remind us that the process conception of a function, in addition to a formula not necessarily being needed to conceive of a function value, involves the notion of "a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity" (p. 85). It is in this way that a dynamic covariational conception is a conceptual precursor to a fully-developed correspondence conception. The notion of an invariant functional relationship, as opposed to a formulaic recipe, is abstracted from the students' experiences with the functions. Also, note that this abstraction is not empirical abstraction, for the *relationship* is not in the realm of the observables.

Additionally, the covariational conception of a function allows the student to make sense of traditionally more advanced mathematics. For example, for a student who possesses a covariational conception of function, it is not a big conceptual leap to make sense of the behavior of polynomials, piecewise defined functions "mod" (modulus after division) functions, trigonometric functions, functions in polar coordinates, functions defined parametrically, and functions of more than one variable.

Summary and Comments

The perspectives presented in this chapter form a framework that served to guide the analysis that I present in Part II of this dissertation. *Teaching and Learning Mathematics with Understanding* and *Radical Constructivism* provided a background theory that helps focus the

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[6] Note that this problem is further compounded when students study multivariable calculus. Without the notion of quantities varying the partial derivatives, directional derivatives, gradient, etc have no meaning other than either (a) new equations you can evaluate at values of (*x,y*) or possibly (b) a memorized definition like *the partial of f with respect to x is the derivative of the function f(x,y) with the y's treated as constant.*

kinds of questions that are asked. As Thompson (2002) noted "[A background theory] constrains the types of explanations we give, [helps] to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (p. 192). Key developmental and pedagogical understandings, didactic objects and models, and functions as covariation of quantities provide the content for the analysis that follows.

CHAPTER III

LITERATURE REVIEW

While research on the relationships between teachers' content knowledge and their pedagogy has a long history, it has received increased attention in the education literature in the past 20 years. In this chapter, I discuss this "recent" research to locate this study within the literature.

Recent Research on Teacher Education and Teacher Knowledge

Teachers' knowledge of content specific to teaching has been dubbed *pedagogical content knowledge* [PCK] by Shulman and his colleagues (1986). In this section, I will discuss past and current research on teacher knowledge, with particular emphasis being placed on pedagogical content knowledge. The purpose of this discussion will be to substantiate the claim for the need to re-conceptualize PCK. It is this re-conceptualization that is key to understanding the knowledge necessary for teachers to have the possibility of designing instructional environments within which students can learn mathematics with understanding. I will propose a re-conceptualization of PCK that is consistent with the notions of learning mathematics with understanding and the development of knowledge. Unlike the current conceptualization of PCK, which involves teaching of particular strands of mathematical content and the "best" or most effective ways of teaching them as endpoints of mathematics teacher preparation, I will speak of PCK as *a way of knowing* particular mathematical ideas that allows teachers to conceive of and enact environments within which students have the likelihood of learning mathematics with understanding. Though the necessity of students constructing their own relationships abounds in the literature on the learning of mathematics (Simon, 1995; Thompson, 1996, 2002; von Glasersfeld, 1995), it is strikingly absent from the literature on the teaching of mathematics.

The Principles and Standards for School Mathematics [PSSM], notes that teaching for understanding "requires knowing and understanding mathematics, students as learners, and pedagogical strategies" (National Council of Teachers of Mathematics, 2000, p. 17). The knowledge noted by the PSSM, as well as other factors not explicitly considered, such as classroom norms and socio-mathematical norms (Cobb, 1999; Cobb & Yackel, 1996; McClain & Cobb, 2001) and general pedagogical knowledge (Shulman, 1986), are all key aspects of teaching and worthy of study in their own right. However, research indicates that the knowledge and skills a teacher draws upon are interrelated (Colton & Sparks-Langer, 1993). Thus, by deepening our understanding of the mathematics-specific knowledge and its relationship to pre-service teachers' emerging pedagogy, I am actually further developing and articulating one aspect of the complex tapestry of teaching. Further understanding this complex tapestry is essential for the improvement of mathematics teacher education.

Research on Teacher Knowledge

Early study of teacher knowledge was grounded in the search for statistical relationships between teacher knowledge and student achievement (Grossman et al., 1989)[7]. One view of teacher knowledge is that it can be at least roughly quantified by evaluating teachers' subject matter preparation. For instance, Monk (1994) found a positive correlation between teacher

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[7] This strand of research continues to be encouraged by modern day legislation, including The Glenn Commission (United States Department of Education, 2001) and Conference Board of the Mathematical Sciences (Conference Board of the Mathematical Sciences, 2001).

knowledge and student achievement. Ashton & Crocker (1987), in their review of studies on teacher preparation, found studies that documented a positive relationship between courses taken in the academic field and student achievement. Additionally, in Druva and Anderson's (1983) meta-analysis, both pre-service teacher's education and science courses were positively associated with successful teaching.

A great deal of research, however, contradicts these findings. The majority of the research of this genre has failed to establish a clear relationship between student achievement and teacher knowledge (Ashton & Crocker, 1987; Grossman et al., 1989). For example, a 1990 study conducted by the National Center for Educational Statistics found that there was no relationship between student achievement and the academic preparation of science teachers as measured by the number of content courses taken (National Center for Educational Statistics (NCES), 1992). These results, combined with the common-sense notion that there *must* be some relationship between teacher knowledge and student achievement, suggests the need for further exploration of this issue. Grossman, Wilson, et al. (1989) propose three possible reasons for the lack of definitive evidence backing our common sense notions. First, there may be no relation between teacher knowledge and student achievement, but they note that "the belief that teachers who know more about the content can probably teach more about the content is too appealing a notion to be cast off lightly" (p. 25). Second, they propose Begle's (1972) suggestion of a "threshold effect:" teachers need a certain amount of subject matter knowledge and more subject matter knowledge results in small changes in student achievement. Third, they propose that the relationship may be inadequately conceptualized. I will follow up on this third reason; however, I do so noting that I believe the last two reasons may be related: If the relationship between teacher subject matter knowledge and student achievement is inadequately conceptualized, then Begle's suggestion must then be re-examined in that light.

Researchers have noted that teacher knowledge and student achievement are extremely hard to quantify (Byrne, 1983; Heaertel, 1986). For example, a teacher's subject matter knowledge is most commonly quantified by scores on standardized tests, or by whether they have a major or have advanced coursework in an academic discipline ("The no child left behind act of 2001", 2001). McDiarmid, Ball, and Anderson (1989) note that teachers require flexible understanding of the subject matter, which requires not only specific content area knowledge, but knowledge that bridges both content within the subject area and across disciplines. This type of understanding is not often tested on standardized tests. With respect to the amount of preparation for teachers, it "may differ both quantitatively, in the number of units teachers have taken in the subject and qualitatively, as in the relative coherence of [the] subject matter coursework" (McDiarmid et al., 1989, p. 24).

Researchers within alternative traditions in research on teaching have begun to answer these criticisms by studying how teachers think *in action,* examining the decision making that actually takes place (Johnson & Whitenack, 1992). As a result, researchers have begun to explore how teachers come to view teaching and learning in the way that they do (Ball, 1988; Lederman, 1992). In other words, rather than trying to quantify *how much teachers know* and its result on student achievement, this new strand of research focuses on the development of knowledge and its use in action. This research has, at its core, the assumption that teacher knowledge is not easily quantifiable and that the relationship between teacher thinking, teacher knowledge, and student achievement is not a simple cause-effect relationship as previously assumed. Rather, the relationship between teacher knowledge, teachers' instructional practices, and student achievement is complex and interconnected (Barnes, 1989; Fennema & Franke, 1992) and therefore not easily understood through statistical analysis.

Research on teacher thinking has pushed researchers to seek out new conceptualizations of teacher knowledge. For example, in their review of the literature on teacher thinking, Clark and Peterson note that

> [the research on teachers in action] shows that thinking plays an important part in teaching… Teachers do plan in a rich variety of ways, and these plans do have real consequences in the classroom. Teachers do have thoughts and make decisions frequently (one every two minutes) during interactive teaching. Teachers do have theories and belief systems that influence their perceptions, plans, and actions. [Reviewing the literature on teacher thinking] has given us the opportunity to broaden our appreciation for what teaching is by adding rich descriptions of the mental activities of teachers to the existing body of work that describes the visible behavior of teachers. (Clark & Peterson, 1986, p. 292)

Thus, Clark and Peterson envision teacher thinking and teacher knowledge as the focus of further study. They go on to claim that in order to do so, a much finer-grained characterizations of teacher thinking and teacher knowledge is needed. The need for a finer-grained measure of knowledge for teaching is supported in the literature by the work of Ball & McDiarmid (1989), Ball & Wilson (1990), and Ma (1999). For example, Ma notes that elementary school teachers in China, when asked about mathematical teaching scenarios, produce far richer instructional plans and justifications for them than university trained elementary school teachers in the United States, despite the fact that the Chinese teachers have less than an undergraduate degree in mathematics or mathematics education (Ma, 1999).

Pedagogical Content Knowledge: Knowledge for Teaching

Shulman and his colleagues (Grossman et al., 1989; Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987) proposed a number of finer-grained characterization of teacher knowledge, each of which were an attempt to “probe the complexities of teacher understanding” and answer the question, "How might we think about the knowledge that grows in the minds of teachers?" (Shulman, 1986, p. 9). Shulman describes multiple models of a knowledge base for teaching[8]. In each model, Shulman distinguishes among three (in some cases more) broad categories of teacher knowledge: (a) subject matter knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. He distinguishes the categories as follows: content knowledge refers to both the subject matter knowledge and its organization or lack thereof, while pedagogical content knowledge involves content knowledge as it is directly related to the teaching of specific subject matter. Finally, curricular knowledge deals with knowledge of the programs of study and instructional materials, including texts, software, visual aides, manipulatives, etc. available to the teachers.

Pedagogical content knowledge [PCK] is of primary concern for two main reasons. First, the importance of PCK has been accepted by educators in fields from mathematics to science (Tobin & Garnett, 1988), English (Grossman, 1989), and social studies (Gudmundsdottir & Shulman, 1987). Regardless of discipline, educators are interested in PCK despite the fact that there is little research backing the claim that it is an important knowledge base for prospective teachers (Wilson, Floden and Ferrini-Mundi 2001). Second, it is important to note that PCK, which lies at the confluence of the content and the curriculum, is the knowledge base of mathematics education: PCK is the knowledge and skills required for one to transform his knowledge into a set of experiences, activities, and environments that optimize the likelihood of students learning.

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[8] Shulman and his colleagues propose multiple models for teacher knowledge for the simple reason that there was no consensus on an appropriate model for the necessary knowledge for teaching. To date, there still is no consensus.

Validation of PCK as an important aspect of teacher development is most strongly grounded in common-sense: It “just makes sense” that teachers need some amount of PCK in order to be successful in their endeavors with students. In Shulman’s definition of PCK, he mentions subject matter knowledge *for teaching*, a phrase that is highly laden and potentially problematic. By teaching, does PCK refer to knowledge required for teaching traditionally, as in the majority of American mathematics classrooms (Stevenson & Stigler, 1992) or for teaching mathematics as espoused in the recent mathematics reform documents (National Council of Teachers of Mathematics, 1991, 2000)? Though one might assume the latter, there is nothing inherent in PCK that demands this. In fact, it could be argued that Shulman and his colleagues were not especially concerned with reform modes of education, as is evidenced by their use of language. For example, they speak of the “goals of instruction including the transmission of knowledge and understanding to students” (Wilson et al., 1987, p. 104) and of PCK as the knowledge that enables this “transmission of content knowledge” (Shulman, 1986, p. 9), which appears to be at odds with the current mathematics reform movement.

Additionally, there is no consensus about exactly what PCK is. It is commonly defined component-wise as an amalgam of pedagogical and content based “stuff.” Most definitions of PCK within mathematics education are simply mappings of Shulman’s definition onto the domain of mathematics. For example, Ball and Bass (2000) define pedagogical content as knowledge “[that] bundles mathematical knowledge with knowledge of learners, learning and pedagogy” (p. 88). Wilson, et al. (1987) describe PCK as “specialized understanding of the subject matter, one that permits them to foster understanding in most of their students” (p. 104). These examples of attempts to define PCK have one factor in common: they speak of pedagogical content knowledge as a blend of different kinds of knowledge, as content knowledge as it is applied to pedagogy or pedagogical knowledge as it is applied to content, not as a type of knowledge itself. Pedagogical content "knowing" in the way described by the teacher education literature would not likely allow a pre-service teacher to conceive of the variety of possible student conceptions, the value of particular student conceptions or the development of instructional agendas centered on powerful conceptions of mathematics content. Though in the literature, PCK is *grounded* in particular mathematics, it is often not guided by precisely explicated conceptions of mathematics.

Some mathematics teacher education programs have as their goal the development of additional pedagogical knowledge that could supplement and possibly revise the teachers' understanding of particular aspects of mathematical knowledge (see, for example, Carpenter, Fennema, & Franke, 1996). This knowledge might consist of reflecting on the students' thinking en route to developing a richer understanding of the students' mathematics (Steffe, 1994) at play or how one might help a student who understands the mathematics in a certain way develop a deeper understanding. This knowledge can be contrasted with developing more sophisticated understanding of the mathematics in a way that was pedagogical. These two forms of knowledge are qualitatively different. While the former focuses on how teachers might help students develop particular understandings of mathematical concepts, the latter focuses on how teachers might reconceptualize the mathematics in a way that is pedagogical (i.e. unpack their own knowledge and develop a conceptual framework within which their knowledge of mathematics may lie).

For example, in a mathematics methods class, pre-service teachers (PSTs) often learn different ways to help students answer the question, "What is x?" as posed in typical mathematics textbook, where it is tacitly understood that x stands for a single number. In this case, the PSTs' beliefs about algebra as *finding the unknown* and the implications of such a belief are left unchanged. PSTs are likely to teach in a way that emphasizes what x is. Students will be prone to imagine that x is the number and that their activity should end in something like "$x = 4$." In contrast, if through a methods class the PST comes to understand that an equation implies the question, "among all the values that can be substituted for x, which values make the equation true?", then the PST's students are more likely to imagine that answering the question will produce two sets of values – those that make the equation false and those that make the equation true. Thus, the former PST's students will think that "solve $2(x+1) = 2x +2$" has no answer (it cannot be reduced to the form "x = number"), whereas the latter PST's students will be more likely to see, by inspection, that x can be any value in the variable's domain.

Thus, another finer-grained measure for teacher knowledge is not just the *amount* of knowledge that PSTs possess, but rather *how they understand particular mathematical content.* It is this understanding that mediates both how PSTs, as students, learn additional mathematics and how they, as teachers, conceive of the mathematics to be taught. The work of Ma (1999), Thompson & Thompson (1996, 1994), and Silverman (2004a) call for more attention to this aspect of teacher knowledge.

Re-Conceptualizing PCK

Pedagogical content knowledge has become a "catch phrase" in educational research and in policy debates regarding mathematics teaching and mathematics teacher education. It would be hard to question *the spirit* of the construct, but as currently conceptualized, its utility must be questioned. The spirit of PCK does raise interesting questions about the relationship between teachers' knowledge and their teaching practices, but the contradictions described above present the possibility that the questions being asked are "ill-defined." In an ill-defined problem, it is unclear from the beginning about what the problem is and thus, what a solution is. Thus, it is necessary to first clarify the questions being related before one can even begin to seek a solution.

With regard to PSTs' understandings of mathematics, this study will focus on their image of what they wish their students to learn, and the language in which they have captured those images. With regard to PSTs' teaching practices, this study will focus on their pedagogical decision making (identifying key "big ideas" in understanding a mathematical idea, planning for and selecting appropriate tasks that have the likelihood of eliciting productive mathematical discourse, orchestrating this mathematical conversation, etc.).

Thompson & Thompson (1996) note that how a teacher conceived of particular mathematical content, not just *whether* they knew it, had a significant impact on the interactions the teacher *could conceive of having* with a student. In their report of one teacher's interactions with one student, it became evident that his understandings of mathematics prevented him from being aware of key aspects of the students' reasoning, and thus he was forced to remediate the problem *as he saw it*. Rather than helping the student develop more powerful ways of understanding the mathematics, the teacher resorted to a deficit model of instruction, telling and questioning in an attempt to help the student "see" aspects of the mathematics at hand. How might mathematics teacher education be designed to position future teachers to develop mathematical and pedagogical understandings that would allow them to conceive of environments within which students are likely to develop powerful mathematical understandings? At an even more basic level, what might these pedagogical and mathematical understandings look like and how might we study their development? I examine these ideas in the following section.

An Example of Teacher Knowledge: The Concept of Area

In this section, I discuss an example of research that focuses on the transformation of teachers' knowledge en route to the development of PCK (conceived of as the mathematical and pedagogical understandings described in the previous section). The purpose behind sharing this example is to analyze the development of a group of PSTs' understandings of mathematics as they interact with particular mathematics content as a way to make further sense of the question of how to position future teachers to develop mathematical and pedagogical understandings that allow them to conceive of environments within which students are likely to develop powerful mathematical understandings. This example will also help to gain insight into the question of what these pedagogical and mathematical understandings look like and how we might study their development.

I will briefly discuss the structural differences between a group of PSTs prior and transformed mathematical knowledge. These examples are not to be read simply as examples of "good teacher education," but rather as a case of teacher education designed to help PSTs develop PCK – content knowledge transformed into deep, organized knowledge that is pedagogically useful and necessary for the development of educative environments.

As part of the Construction of Elementary Mathematics program (Simon, 1995), PSTs were studied as they took part in a course whose goal was to increase their mathematical knowledge. Simon (1995) documents one segment of the course that centered on the multiplicative relationships and employing the concept of area as a vehicle for the development of multiplicative reasoning. Simon began the segment of instruction with the following problem:

> RECTANGLES PROBLEM 1: Determine how many rectangles, of the size and shape that you were given, could fit on the top surface of your table. Rectangles cannot be overlapped, cannot be cut, nor can they overlap the edges of the table. Be prepared to describe to the class how you solved the problem (Simon, 1995, p 123).

As groups of students worked through the problem, the question of whether the orientation of the rectangle should be maintained for the second measurement became an issue for (at least some) of the PSTs. They wondered whether the rectangle should be maintained (see Figure 3-1a) or rotated 90 degrees (Figure 3-1b) in order to “do the measuring.” Though they all recognized the need to multiply, questions such as the one about the orientation of the rectangle indicate that the PSTs lacked a fundamental understanding of the relationship between the product of length and width and the rectangle’s area. Simon pushed this issue, not to improve the students’ ability to calculate the area, at which they were already proficient, but to help them understand why multiplication was appropriate in this situation and to help them see the multiplication of length and width as a logical consequence of their understanding.

Figure 3-1: (a) Maintaining the orientation of the rectangle; (b) Rotating the rectangle to measure the adjacent side

Rectangles problem 1 was followed by a similar problem that asked the PSTs to explain why, when maintaining the orientation of the rectangle, and multiplying the “width” of the table by the “length,” the corner rectangle was not counted twice. In the ensuing whole class discussion, the topic of conversation shifts from “counting quicker” to speaking of “groups:”

Simon: And how is that connected to the issue about the corner?
Molly: Because it … the corner not only represents a one, it’s just one numbering of a group, or it’s also numbering a part of that unit – a unit in that group – so it’s not, it is two different things … it’s a unit and also representing a group.

Candy: … it makes it confusing to try to look at the length times width. … You should really treat it as so many sets or so many groups, like nine groups …, thirteen groups of nine. That way, you're not even going to deal with the corner and you won't even have that problem (Simon, 1995, p. 126)

In this interchange we see the beginning of a shift from thinking of measurement of area as "multiplying length times width" to measurement of area as a certain number of groups of a given size. Simon then returned to the question of rotating the rectangles in order to measure the area with a particular side of the rectangle, and the class eventually decided that this calculation of area (see Figure 3-2 below) had no meaning because the rectangles overlapped. They did not recognize that they were measuring the lengths of the larger rectangles' sides in units of the length of one side of the smaller rectangle, thereby *creating a square* whose area is taken as "one", and measuring the area of the larger rectangle in units of this new "one". Simon ended his discussion of the teaching experiment noting that his efforts to help his students deeply understand the multiplicative relationship at play were less than successful, for in the end, only half of the small groups were able to explain why this method of "overlapping rectangles" (see Figure 3-2) was useful for finding the areas of rectangles in general. Additionally, of the students who were able to explain their thinking, some wondered how one would know when to use the different units of measurement.

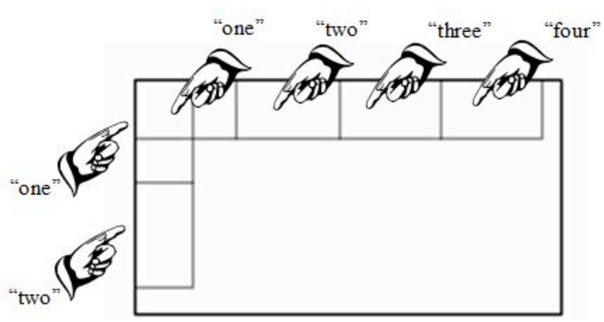


Figure 3-2: Students believed that the calculation 4 times 2 (=8) had no meaning in this scenario.

It was clear that, prior to this instruction, the students believed that calculating the area of a rectangle was about multiplication. When asked why they multiply, students responded "… 'cause that's the way we've been taught," or "… it's a mathematical law" (Simon, 1995, p. 124). Though Simon notes that these explanations were a product of both cognitive and social factors, I wish to focus on the cognitive. Throughout multiple iterations of the mathematics teaching cycle (Simon, 1995), the students came to think of this multiplication as a number of rectangles of a given size. This realization enabled some of the students to conceive of the idea of a unit (which was previously just "the rectangle") and ultimately make sense of what the square units were measuring. Thus, the students had developed a conceptual link between the procedure that they were employing and the measuring of the area.

The students in Simon's study had developed knowledge that, conceivably, will assist them in teaching area and multiplication. With respect to the development of mathematics teachers, an important question must be asked: what were the qualities of this new knowledge? This question is not asked to reduce their knowledge to a new "file folder" containing new retrievable information, but rather as a new cognitive structure that will allow (and possibly necessitate) them to organize and interpret the world differently. This "new" way of knowing would ideally allow them to see different aspects of their interactions with Simon, and thus enabling them to conceive of the problems of instruction in a different way.

PCK as a "New" Way of Knowing

I will argue that the knowledge that PSTs developed in discussing Simon's rectangle problem was truly a new way of knowing. By demonstrating that instead of simply adding to and reorganizing current knowledge, the PSTs developed a way of knowing in which they actually imbued upon the objects some characteristics that were not previously present in the object. Before discussing the example of Simon and his pre-service teachers, I will briefly discuss an example that will highlight issues critical to understanding the development of the PSTs' knowledge.

As noted in the previous section, the class did develop a means of understanding the relationship between the larger rectangle's area and "multiplying the length times the width". With respect to this, Candy notes: "   … it makes it confusing to try to look at the length times width. … You should really treat it as so many sets or so many groups, like nine groups …, thirteen groups of nine" (Simon, 1995, p. 126). This realization seemed to be the result of students visualizing a number of copies of their rectangle covering the surface to be measured. Much like the case of additive reasoning (see Chapter 2, *Reflecting Abstraction*), the students were prone to using empirical abstraction, or drawing on observable characteristics of the situation. In this case, students might reason something like this: "If I cover the surface with copies of my rectangle and try to count all the rectangles, it would be easier if I counted the number in a row [or column] and then see how many rows [columns] there are. The total area will be the same as the number of rows times how many rectangles there are in a row." In contrast, consider the PSTs who developed the ability to explain why measuring the lengths of sides to create square units was useful in describing areas. These PSTs realized that they were not measuring the larger rectangle's area by covering it with smaller rectangles. They realized that they were using *the length of one side of the smaller rectangle to measure the length of the sides of the larger rectangle, and thus creating a unit of area out of units of length.*

The latter PSTs developed new knowledge through reflecting, not empirical, abstraction. Rather than conceiving of the smaller rectangle as something to use to "cover" the larger rectangle, they abstracted from their actions the notion that *a side* of the smaller rectangle can be used to measure *the sides* of the larger rectangle. In essence, the side of the smaller rectangle is now a tool for measuring length. Moreover, what they are measuring is not how many smaller rectangles fill up the larger rectangle, but how many *sides* of the smaller rectangle can cover one side of the larger rectangle. It was this realization that allowed them to make sense of the area of the larger rectangles in units of "square sides." It is thus my claim that the PSTs who came to recognize the square as a *derived* unit of measurement developed new knowledge that was transformed, or abstracted, from their prior knowledge. Those PSTs who simply developed an understanding of why they multiply when calculating area did not fundamentally transform their knowledge, but simply augmented it with additional characteristics.

An understanding of area being measured by these new units allows a teacher to conceive of the problems of teaching area in a different way. When encountering a student who is struggling with the notion of area, rather than relying on many different ways of saying, essentially, "multiply length times width," a teacher could also focus on the development of the idea of area as "covering" and the relationship between linear measurements and area. This understanding of area as an n+1-dimensional, derived unit requiring the coordination of two n-dimensional quantities, in turn could help students make sense of commonly problematic areas such as the relationship between area and volume.

CHAPTER IV

BACKGROUND OF THE STUDY AND ANALYTICAL PROCEDURES

The previous chapter served to explicate an empirically testable conceptualization of PCK. I began by noting that mathematics content knowledge alone is not PCK. Moreover, any intervention in which PSTs are expected to develop new mathematics content knowledge will not necessarily assist PSTs in developing PCK. As described above, PSTs must be presented with opportunities within which they can transform their (new) content knowledge into a form that is recognized by the PST as pedagogically powerful. In addition, the PSTs must develop a sense of the ways in which such mathematical understandings might develop. It is my conjecture that mathematical and pedagogical knowledge of this sort is necessary, but not sufficient for a teacher to have the possibility of developing didactic models around mathematical topics.

The Setting for the Study

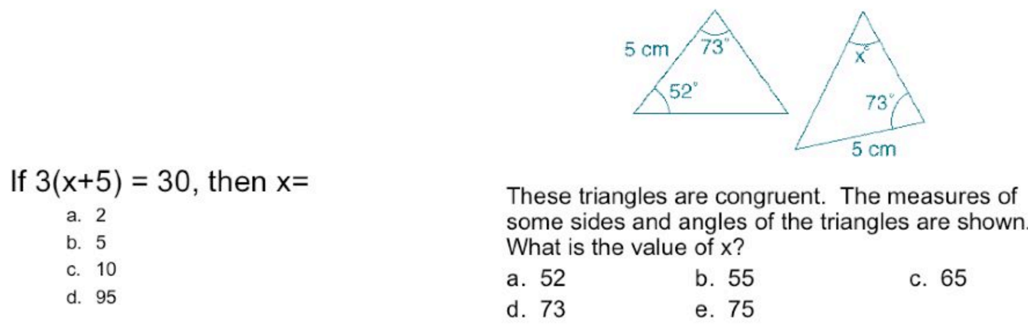
This study took place in a course titled *Computers, Teaching and Mathematical Visualization*, a required course for sophomore and junior mathematics majors pursuing secondary mathematics licensure (henceforth the pre-service teachers or PSTs). Table 4-1 describes the overall organization of the course. I will briefly discuss each segment of the course.

Table 4-1: Course Overview: Computers, Teaching and Mathematical Visualization

Weeks	Topics
1	Problematizing the teaching and learning of mathematics
2-3	Variables and rate Introduction to *understanding* mathematics
3-5	Introduction to Graphing and Covariation
5-12	Applications and Extensions of Covariation
13-16	Geometry and Proof

Problematizing the Teaching and Learning of Mathematics

The PSTs read and discussed *The Teaching Gap* (Stigler & Hiebert, 1999). In the book, Stigler and Hiebert present characterizations of teaching in Japan, Germany, and the United States that are drawn from their work on the Third International Mathematics and Science Study [TIMMS]. The book begins by noting the staggering statistics about the performance of American 8th grade students in mathematics. For example, on a basic algebra problem (Figure 4-1a) less than 75% of US students answered the question correctly while 90% of Japanese students answered it correctly. On the geometry problem shown in Figure 4-1b, 17% of US students correctly answered the question, while 69% of the Japanese students answered it correctly.



(a) (b)

Figure 4-1: (a) Algebra Problem and (b) Geometry Problem from 8th Grade TIMMS Study

As a way to investigate results such as these, the TIMSS study set out to better understand what teaching in these countries might look like. Based on analysis of videotapes of hundreds of classrooms from the three countries, the researchers developed generalizations of the kinds of teaching that take place in each country. They characterize US classrooms as focusing on learning terms and practicing procedures and note that, for the most part, content coverage in US classrooms consist of a great deal of "review," with little student participation beyond giving short answers to questions. In contrast, Japanese classrooms consist of structured problem solving with more active involvement in the learning process. Through reading *The Teaching Gap* and the ensuing class discussions, the PSTs were positioned to reconsider their images of teaching and learning mathematics. It is worthwhile mentioning that the characterization of teaching in Japan is largely consistent with teaching mathematics for understanding as discussed previously.

Variables and Rate: Introduction to Understanding Mathematics

Thompson & Thompson (1994) give a detailed analysis of the interactions of Bill, an experienced middle school teacher who took part in 6 months of professional development focusing on supporting students' development of quantitative reasoning, and Ann, a sixth-grade student in another teacher's class. Bill spent three sessions working one-on-one with Ann, teaching her about the concept of rates, which had been a recent focus of development sessions. Videotapes for the three one-on-one sessions between Bill and Ann, a high-achieving middle school student, were the focus of two university class sessions and one writing assignment.

Though Bill was an experienced mathematics teacher with an impressive "grasp of curricular goals and pedagogical principles" (Thompson & Thompson, 1994), he had significant difficulty speaking conceptually about rates. Thompson & Thompson (1994) suggest that this difficulty was a result of Bill's lack of understanding of the subtleties involved in understanding the concept of rate conceptually. Bill had a "packed" understanding of division and proportionality (Thompson & Thompson, 1996). In the third video, an experienced mathematics educator takes over instruction of Ann. The educator conceives of the purpose of the instruction as to assist Ann in developing a "speed schema" that is detailed by the following "conceptual curriculum for speed," an image of speed that entailed conceiving of the following:

> Speed is the quantification of motion;
>
> Completed motion involves two completed quantities – distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation);
>
> Speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance;
>
> There is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go m distance units in s time units at a constant speed, then at this speed you will go a/b ·m distance units in a/b · s time units (Thompson & Thompson, 1994, p. 5).

In simplest terms, this understanding of speed can be summarized as *when given information about any two of distance, speed, or time on a given interval, we also know something about the third quantity*. The educator's interaction with Ann was therefore guided by his knowledge of the following three (complementary) ideas:

> Division is an appropriate calculation to evaluate the size of a whole piece when a quantity is partitioned into a number of equal sized pieces;
>
> Constant speed implies a bi-directional, proportional correspondence between segments of accumulated distance and accumulated time;
>
> Total time as a number of seconds can be imagined also as a partition of total time into a number of equal-sized partitions (Thompson & Thompson, 1996, p. 17).

The PSTs viewed the two videos from Bill's work with Ann and one video of the mathematics teacher educator's work with Ann and compared and contrasted the instructional actions of Bill (and Ann's responses) with the instructional actions of the experienced mathematics educator (and Ann's responses). The PSTs then reflected on and analyzed Ann's learning and the conceptual development that was supported by the instructional materials.

Introduction to Graphing and Covariation

Approximately three weeks of the course were devoted to an introduction to graphing and covariation. The purpose of this instructional unit was to help the PSTs develop a more structural understanding of the concept of function as covariation of quantities, as discussed in the Chapter II (see *The Concept of Function*). This segment of the course and the majority of the next section will be the focus of this study and discussed in detail in the following chapters.

Applications of the Concept of Function

The longest segment of the course involved the instructor presenting the PSTs with opportunities to further develop their conceptions of functions and to experience the utility of the covariational conception of function. This phase of the course consisted of 11 problem sets covering a wide array of mathematical topics, including mathematical modeling, families of functions, trigonometry, polar coordinates, rates of change, and calculus. Each of the problem sets consisted of 6-10 problems. The problems were discussed briefly and ways of thinking that might be helpful in solving the problem were discussed. The PSTs then worked at home, either individually or in small groups, on a subset of the problems. At the following class session, PST solutions to assigned problems were the topic of a class discussion. The instructor envisioned each of the problems as didactic objects; the purpose of the problem was not to see if the PSTs could or could not answer it correctly. Each problem was thought to be something that the students could engage with and that a discussion about their thinking would result in a productive mathematical conversation. This study analyzes the PSTs' work and classroom discussions regarding many of the problems assigned in this section.

Geometry and Proof

The course ended with a 3-week discussion of geometry and proof. The mode of instruction was similar; there were three problem sets that focused on constructions and highlighted relationships and dependencies within the constructions. These constructions were assigned for multiple purposes. First, they allowed the PSTs to explore and understand the importance of relationships and dependencies using the dynamic features of Geometer's Sketchpad. Second, they were designed to present an opportunity for the students to begin to learn *to understand* geometry rather than *to do* geometry. As was true in the entire course, the focus was on the students understanding the problems and proceeding logically through sensible steps.

Assessment in the Course

There were three types of written assignments that served to focus instruction and to evaluate the PSTs' progress in the course. For each of the problem sets, a write-up was submitted to the instructor following the class discussion of the problems. In addition, there were two in-class exams. Finally, there were three out-of-class projects, each with a write-up that was submitted to the instructor. These projects consisted of (1) designing instruction on polar coordinates and teaching a school student; (2) designing a plan for instruction on linear functions by identifying cognitive objectives and designing activities that help students develop the identified ways of thinking; and (3) revising a textbook section.

Overview of the Study and the Participants

This study focused primarily on the class sessions that took place during weeks 3-10 of the course (Introduction to Graphing and Covariation and Applications and Extensions of

Covariation). The author was a participant-observer in the class for the duration of the study. The course was taught by an experienced mathematics educator who was also a graduate faculty member with whom the author had a prior working relationship. In an effort to maintain the integrity of the study and to reduce any "contamination" of the data as a result of this academic relationship, communication between the course instructor and the author was avoided throughout the data gathering process – communication was kept to brief discussions regarding administrative issues (what was planned for the day, when particular items would be due, etc). The author and the instructor met and discussed the analysis of the data approximately once per month after data collection.

Three of the four PSTs enrolled in the class participated in the study (data from the PST who did not participate was not analyzed as a result of inconsistencies in the PST's participation in the course)[9]. The three participants were each mathematics and secondary education double majors. Two were juniors and one was a sophomore. Each of the participants described him or herself as liking mathematics and considered himself or herself successful or very successful in mathematics.

Each of the participants also had a pre-existing association with the instructor – in addition to teaching this course, the instructor was also active in the secondary mathematics teacher education program. Though this factor added a level of complexity to any analysis of the interactions between the PSTs and the instructor, it is not believed that the PSTs felt any additional pressure to succeed or to respond in any particular way (that is, any more than they would with any classroom teacher). The analysis of students' mathematical and pedagogical

[9] The small number of participants in this study was not by choice. For a number of years, the course that this study was conducted in routinely had an enrollment of 8-12.

understandings that follows will cite multiple data sources to ensure that the effect of this pre-existing association on the results is minimized.

Data and Methodology

Data Corpus

Data collected for this study came from a wide variety of sources. I briefly describe each below.

Initial Assessment. At the beginning of the course, the PSTs completed an initial assessment, which was analyzed to better understand the participants' conceptions of function at the beginning of the study.

Videotaped class sessions. Each class session from weeks 3 through 8 was videotaped (a total of 16 class sessions).

Interviews. PSTs were interviewed once at approximately week 8 of the course and then again after they had taught their lesson to their high school student.

Video of Instruction. The PSTs' teaching of their lesson to their high school student (as part of Project #1) was videotaped.

Assignment Write-Ups. For each of the problem sets, each PST submitted write-ups of specified problems.

Instructional Plan and Reflective Essay. As part of Project #1 the PSTs designed a lesson on polar coordinates. In addition, once they had taught their lesson, they were assigned the task of writing a reflective essay that highlighted what happened during their instruction, what their high school student learned, and why each was important.

Methodology

The methodology for this study was based on the Simon's Teacher Development Experiment [TDE] (Simon, 2000). This methodology has its roots in the constructivist teaching experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000; Thompson, 1979), which involves researchers "working at the edge of their [and their participants] evolving knowledge" (Simon, 2000, p. 336). The TDE is conducted from (and builds on) the "emergent perspective," from which learning can be described as both individual psychological development and the development of the social practices of a group:

> The basic relation posited between students' constructive activities and the social processes in which they participate in the classroom is one of reflexivity, in which neither is given preeminence over the other. In this view, students are considered to contribute to the evolving classroom mathematical practices as they reorganize their individual mathematical activities. Conversely, the ways in which then make these reorganizations are constrained by their participation in the evolving classroom practices (Cobb, 2000).

Though my focus was on the individual development of the PSTs, the emergent perspective reminded me that a good part of this learning takes place in the social setting of the classroom, and understanding and explaining these social interactions that result in the (either desired or unexpected) individual development was vital to this study. Additionally, the emergent perspective gives this study the theoretical ammunition to examine the relationships between the PSTs participation in the classroom intervention (the classroom intervention) and their design and enactment of their own lessons and interactions with students. Consistent with the emergent perspective, I did not focus specifically on the individual development of particular PSTs or the development of social processes of the group of PSTs. Understanding the relationship between the participation in particular social practices and the individual development of PCK was a primary focus of this study.

Secondly, the TDE is concerned not only with the mathematical development of the participants. It is also specifically concerned with better understanding teachers' professional development. Simon describes this aspect of the TDE as a "whole-class teaching experiment in the context of teacher development" (p. 345). As a result, the TDE is an attempt to understand the relationship between the development of PSTs in a university course setting coupled with the learning and development that takes place in their work with students in school settings.

Analytical Techniques

The analysis for this study took place in two major phases: ongoing and retrospective. Analysis of each of the phases of the study ultimately had as its goal the development of characterizations of the PSTs' understandings of mathematics and mathematics teaching in the context of a particular event. These characterizations can be described as understandings (or "models") of how the students might be thinking. These models were created by observing students' actions, individually and collectively, and by examining the artifacts they generated in order to develop hypotheses about the meanings and understandings they had so as to explain why each student acted as he or she did. The retrospective analysis provided the additional level of analysis by focusing not only on verifying the hypotheses developed throughout the study but for developing and verifying or refuting hypotheses about the relationship between these two aspects of mathematics teacher understanding.

The development of such models of student understanding is a theory-building activity — a process of examining the available data and interpreting students' actions and their generated artifacts in order to generate coherent and viable hypotheses (Clement, 2000; Glaser & Strauss, 1967; Steffe & Thompson, 2000). Ultimately, the goal of this study was to develop a *grounded theory*:

> A grounded theory is one what is inductively derived from the study of the phenomenon it represents. That is, it is discovered, developed, and provisionally verified through systematic data collection and analysis of data pertaining to that phenomenon. Therefore, data collection, analysis, and theory stand in reciprocal relationship with each other. One does not begin with a theory, then prove it. Rather, one begins with an area of study and what is relevant to that area is allowed to emerge (Strauss & Corbin, 1990).

Ongoing Analysis

Ongoing analysis took place throughout the duration of the study. The classroom instruction segment of the study entailed the creation of field notes for each of the classroom sessions. These field notes included detailed descriptions of the events and interactions that took place in the classroom and also included methodological notes, which indicated evaluations of the data collection methods, any changes to the plan for the study and data collection, and the rationale for such changes. In addition, the field notes included theoretical notes[10], or attempts to tie the observations to the relevant theoretical constructs as well as patterns in the data indicative of issues of possible theoretical importance. Finally, the field notes included any relevant personal notes, including my personal reactions to the experiences in the field. Immediately following each class session, the video from the session was viewed, the field notes annotated, and a theoretical memo for the session was generated. This theoretical memo included (i) a

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[10] I would like to thank Dr. Deborah Rowe for her contribution to the development of my methods of analysis. This came, in large part, from her course notes for EDUC 3912: Methods of Qualitative Research, Spring 2002.

discussion of what happened in the session; (ii) events or interactions of theoretical importance; (iii) examination of previously generated hypotheses; (iv) new hypotheses; (v) changes to the study design or data gathering procedure and a rationale for such a change. Ongoing analyses for the other phases of the study were done similarly. Though field notes were not generated, I viewed each piece of data (interview video or student generated artifact) and created a theoretical memo for it. The recursive nature of the theoretical memos both organized the data and is consistent with grounded theory (Glaser & Strauss, 1967).

Retrospective Analysis

The retrospective analysis allowed me to further understand the relationships between the results of the first and second research questions – of ultimate interest for the field of mathematics education is the relationship between how teachers (might) understand a mathematical concept and the impact of this understanding on their instruction. The analysis took place in three levels.

Analysis from a global perspective. At the end of the data generation, I viewed the entire data corpus chronologically, further annotating the theoretical memos and revising the hypotheses generated throughout the data collection. This phase of analysis served to help me get a sense of the data corpus in its entirety. Additionally, segments of theoretical importance with respect to the development of teachers' understandings of functions, covariation, and pedagogy were identified.

Line-by-line analysis. Segments identified in the previous phase were coded for issues of theoretical importance. Particular attention was paid to (a) the rationale for instructional activity and (b) students' activity and understandings. Previously generated hypotheses were revised and refined.

Pattern-seeking. This level of analysis is an adaptation of the constant comparative method described by Glaser & Strauss (1967) and refined by Cobb and Whitenack (1996). Initially, the retrospective analysis was guided by the tentative conjectures from (a) the review of the literature discussed in the previous section and (b) the revisable conjectures from the analyses conducted during the ongoing analysis. This analysis involved comparing the previously generated conjectures about the development of study participants with the entire data corpus (in chronological order). Throughout this phase of retrospective analysis, conjectures made about ways the PSTs have come to understand functions and its implications on the way they conceive of and enact mathematics instruction were tested and, as necessary, revised while analyzing subsequent episodes. Much like the recursive nature of the analyses conducted during the experiment, this phase of analyses further tested and revised conjectures against the entire data corpus. With respect to this phase of analysis, McClain notes that "this constant comparison of conjectures with data results in the formulation of claims or assertions that span the data set but yet remain empirically grounded in the details of specific episodes" (McClain, 2002a, p. 1545-1550).

PART TWO

PRE-SERVICE TEACHERS' EMERGENT UNDERSTANDINGS OF COVARIATION

Introduction

In the chapters of Part II, I analyze the development of a particular understanding of the concept of function among the PSTs in this study. The PSTs took part in instruction designed with the intent that they develop a coherent understanding of function as covariation of quantities and that they develop an understanding of the implications of that understanding on the teaching and learning of the majority of the concepts in secondary and college mathematics. Part II begins with an analysis of an initial assessment, which gave insight to how the PSTs thought about functions and functional situations before instruction began. I then present an analysis of the PSTs' participation in the instructional sessions that took place throughout weeks 3 – 8 of the course. These class sessions were centered around three "problem sets" that were both the foci for classroom discussions and homework for the students to work on independently or in groups. The chapters of Part II will consist of an activity-by-activity analysis of the PSTs' understandings of covariation; I use the instructional trajectory to organize the narrative account of the PSTs' developing understandings of covariation. In order to see the PSTs' growth, it is necessary to look at the details of how they engaged with the instructional activities.

The two segments of the MTED 2800 instructional trajectory that this study focused on were an introductory phase and an application phase. Both of those phases were further subdivided into two activities. This organization is depicted in Table II-1 below. Interviews with the PSTs took place near the end of the application phase.

Table II-1: Introductory and Application Phases of Instruction

4. Tues 9/7	5. Thurs 9/9	6. Tues 9/14	7. Thurs 9/16	8. Tues 9/21	9. Thurs 9/23	10. Tues 9/28	11. Thurs 9/30
Introductory Phase 1. Introduction to Graphing 2. Introduction to Covariation			**Application Phase** 1. Functions and Models 2. Graphs and Graphing				

CHAPTER V

PRE-SERVICE TEACHERS' INITIAL CONCEPTIONS OF FUNCTION

Initial Assessment

It was conjectured prior to instruction that the PSTs would have some form of a correspondence conception of a function, though how entrenched these conceptions would be was not known. In this section, I discuss the PSTs' initial conceptions of function, as evidenced from an initial questionnaire they completed prior to instruction. A selection of questions from the initial questionnaire is shown in Figure 5-1.

1. Give your own "personal" definition of a function. This definition might include words, mathematical symbols, pictures or a combination of the above. Be sure to explain your use of symbols and pictures.

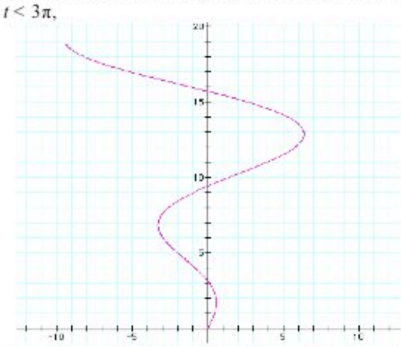
2. For each of the following, decide if each is (a) a function (b) not a function or (c) you're not sure. Regardless of your choice, explain your answer and what is special or unique about the given equation (i.e. how is it different than the rest of the given equations or statements).

a. $y = \frac{3x^2 + 2x}{x - 1}$

b. $x^2 + y^2 = 4$

c. The ages of 500 randomly selected people in relation to their weights.

3. The following graph is of the function that is defined parametrically by $x = t \cos t$ and $y = 2t$, with $0 < t < 3\pi$,



a. Does the graph represent a function? Explain why or why not?

b. What is the independent variable? What is/are the dependent variable(s)?

c. Explain, using your knowledge of functions and the given equation, why the graph looks as it does.

d. How does t show up in the graph? Is knowing how t "shows up in the graph" important for graphing the function? Is it important for understanding the behavior of the function?

e. How is this function related to $f(x) = x\cos(x)$? How are they the same? How are they different?

Figure 5-1: Selection of Items from Initial Assessment

Based on the initial questionnaire, the PSTs' understandings and conceptions of function can be described as fitting within two broad categories: (1) an attempt to recite a memorized definition (akin to the set theoretic, correspondence definition of a function as a mapping) and (2) function as something you "plug a number into to get the output." In each of these categories, the PSTs believed that key characteristic of a function is that it is one-to-one.

Memorized Definitions of an "Association"

Each of PSTs' responses to the question about their personal definition of a function indicated that they knew that there were two numbers involved, but could not clearly explain how what one number had to do with the other. Though a number of different words describing the association were used (relationship, correspondence, equation) this was not the key aspect of their definition. Rather than focus on the relationship, their definitions focused on the "one-to-oneness." Below are examples of the PSTs personal definition[11] of function (Question 1):

> KN: A function is a relationship between two sets of variables such that every variable in the first set corresponds to only one variable in the second set.
>
> DH: To me, a function is an equation that for every "y" value only has one associated "x" value.

Though SS does not specify the one-to-oneness in her definition of function, she does rely on it throughout the questionnaire. For example, in answering Question #2, which deals with judging the validity of various definitions of functions, she refers to a linear function and notes that the function allows her to "plug in *x* and get out exactly one *y*."

[11] Though SS does not specify the one-to-one*ness* in her "definition" of function, she does rely on it throughout the questionnaire.

These definitions seem to rely on the notion of a mapping, a rule, or a correspondence. These ideas of mappings can be imagined as the act of "pairing" one element from a set with one element from a second (the same or possibly a different) set. There is no sense that the variables represent values of quantities, nor that the elements in the set are particular values of that quantity. In addition, there is no sense that the quantities can vary, and the function, rather than "doing something" to a number, describes how one quantity behaves over a subset of sequential values of the second quantity – in short, for these PSTs, with respect to functions, nothing is varying.

Functions as "Things You Plug Numbers In To"

Two of the three PSTs (SS & DG) had an underlying understanding of functions as something you "plug numbers in to". SS highlighted this aspect of a function in her explanation of her personal definition of a function (Question 1):

> SS: A function is an equation that has variables in it. These variables (*x, t, v, …*) can be substituted with data to produce another set of data.

DH, in an attempt to explain whether something "whose function values are all equal to teach other" was a function (question 4) noted:

> DH: [It can be a function, for example] $f(x) = 1$ is a function. No matter what *x*-value you plug in, all of the *y*-values, or function values, will be the same, 1.

This notion of a function fits with traditional instruction, where function problems are often of the form "given a function $f(x)$, find $f(a)$," where a is some number or some algebraic expression containing numbers and variables.

Embedded Assumptions about "Function" Problems

A second aspect of the participants' conceptions of function, which emerged from analysis of the data, was the fact that, regardless of the particulars of the problem given, PSTs had a tendency to answer the questions in a particular way. First, they would describe key characteristics of the graph, function, equation, etc. They would then call on the "vertical line test" or some analytic version of it to determine functionness. Though this way of thinking was apparent throughout the initial questionnaire, it was most apparent in Question 3 (Given various descriptions of "relationships," participants were asked to determine if each was or was not a function). For example, for $y = \frac{3x^2 + 2x}{x - 1}$ (Question 3a) the PSTs concluded that it was a function:

> KN: This is an asymptotic function
>
> DH: This equation is not continuous at *x*= 1 (there will be a hole in the graph).

Pictures indicating the asymptote and the application of the "vertical line test" accompanied their explanations. Similarly, for Question 3c, a piecewise defined function, KN noted

> KN: [It is a] function. This is piecewise. For every odd integer, there is only one y-value. For every integer there is only one y-value.

For Question 3d, given the equation $x^2 + y^2 = 4$, DH drew a picture of a circle and commented

> DH: [It is] not a function. This is the equation of a circle with radius 4. As I said before, there cannot be multiple *y*-values [when] given one *x*-value, and a circle has 2 *y*-values for every *x*-value.

For the same problem, SS solved the equation for both x $\left(x = \sqrt{4 - y^2}\right)$ and y $\left(y = \sqrt{4 - x^2}\right)$, as shown below (incorrectly) in Figure 5-2.

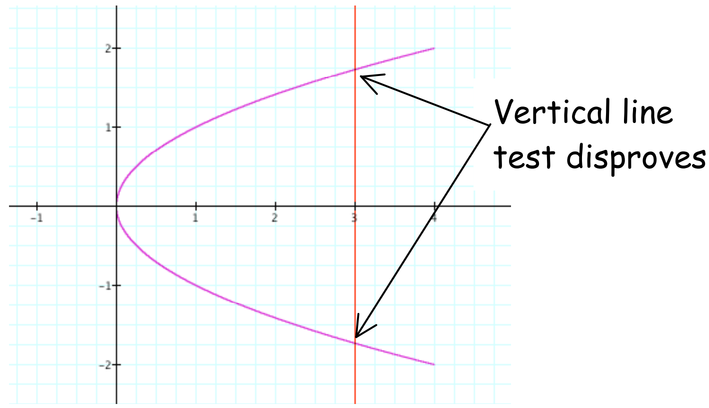


Figure 5-2: SS's (incorrect) figure for Problem 3d, $x^2 + y^2 = 4$

Discussion of Initial Conceptions

PSTs' responses to the initial questionnaire indicate that their understanding of the idea of function was very poorly structured. First, they knew that a function is some sort of an association between two numbers, however they did not have any sense of the significance of that association. Second, they knew that if given a value they could use the function to find the corresponding function value. Evidence indicates that their initial conceptions were more advanced than an action conception, because they talk about evaluating a function "in general" (see SS's comment on Page 61). To see why this understanding is "more advanced," it is helpful to compare the SS's comments with a student who when presented with a function, needs to be given an x-value (or a number of x-values) to make sense of the function; the PSTs regularly spoke of plugging in "some x-value," not a specific ones. In addition, KN, in explaining why a function as a "computational process" was not a good definition of function, noted that a function is not a "producer of values," but a relationship between quantities. Despite this, there is evidence that the PSTs were not reasoning covariationally. First, the PSTs showed no clear understanding as to what the variables were; KN (in (1)) defined a function as a relationship between two sets of variables, not sets of elements. Further there is no notion of the fact that the values of the variables vary. There is no indication that the PSTs had developed an image of variables as either values of a quantity or a variable magnitude. The relationship they described was solely an input-output relationship between elements in the domain and elements in the range. Though they may have been able to "connect the dots" in a graph, there was no evidence that they were conscious of the fact that the quantity represented by the variable *x* varies, say from 1 to 2. It should be noted that these PSTs' understandings of functions is consistent with the K-14 mathematics curriculum, which places no emphasis on covariation (Thompson, 1994b).

Traditional Images of Functions

Despite the fact that PSTs did possess understandings of functions as associations and functions as things you plug numbers in to, these ideas were not the focus of the PSTs' responses to the items on the questionnaire. Rather, they focused on the fact that for something to be a function, it must be one-to-one. Each of the PSTs, when asked about their personal definition of a function, highlighted the fact that there can be no more than one output for every input and drew pictures of functions, some of which passed the vertical line test and some of which did not. Thus for these PSTs, the question dealing with their understanding of the concept of function conjured up an image of particular problems or tasks associated with functions.

This notion is consistent with the two major images that dominate high school and university discussions of function: a mapping definition and a "function machine" definition. The "mapping" image of a function involves two sets of elements, the domain and the range, and lines connecting an element in the domain to exactly one element in the range (each line is to represent the functional relationship or the "mapping" between a particular element in the domain and the range). An example of mapping diagram of a function that maps 1 to 2, 2 to 3, 3 to 4, 4 to 5, and 5 to 1 is shown below in Figure 5-3.

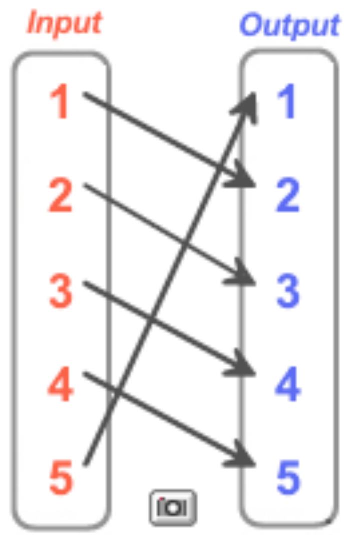


Figure 5-3: Mapping diagram of a function.

A function machine image of a function involves imagining a "machine" that converts elements from the domain into an element in the range. The following is a typical description of a function machine:

> Another way to understand a function is as a machine. A machine has an input and an output. There is a relationship that exists between the input and output. The output depends on the input. The machine receives the input and transforms it into the output. For example, a toaster is a machine. When bread is input in the machine the output is toast. A washer is a machine. When dirty clothes are input into the machine the output is clean clothes. An oven is a machine. When raw meat is input into the machine the output is cooked meat. Some machines are complex. The human body, for example, is the most complex and sophisticated machine known. Think of the myriad of physical, emotional, mental, social, and spiritual inputs needed to have healthy persons. The output is nothing less than the whole of civilization across time and geographical boundaries![12]

[12] Quoted from "Function Machine" handout, available at http://score.kings.k12.ca.us/lessons/ functions.html.

It comes as no surprise that the PSTs were not attuned to the relationships between varying quantities, for these images depict functions as a “black box” that transforms inputs to outputs via some (known or unknown) process. I am aware that it is likely that curriculum designers, textbook authors, and classroom teachers see mathematical significance in these “black box” images of function, but we cannot assume that PSTs do. Our sophisticated meanings and understandings of inscriptions can complicate the task of making sense of our students understandings (McClain, 2002b). I am not claiming that these inscriptions cannot embody deep mathematical ideas. Rather I am claiming that these common inscriptions do not highlight the aspects of functions that I am focusing on and therefore it makes sense that the PSTs have not developed understanding of these aspects.

PSTs' Assumptions About Tasks

The second observation regarding the PSTs’ understanding of functions was the fact that they seemed to have a routine approach for answering questions that dealt with functions. First, they described key aspects of the graph of the function and then used the vertical line test to determine if the graph of the function was a function. This indicates two things. First, the PSTs, as a result of their experiences, have developed a notion of the types of problems asked about functions and as a result, think of functions as things to describe the shape of or to things to classify. A result of this is the PSTs’ assumption that functions are graphs, for the aspects of the functions that are being organized are visible from their graph and not from their equation. Second, there is little thought given to the particulars of the function in question. It is as if the PSTs’ primary agenda is to classify the graph (function or not function; linear or non-linear, etc.). This notion will arise throughout this study, for it is taken as evidence of the PSTs *not* focusing on the variable quantities and how they covary but rather on broad characteristics of the graph.

CHAPTER VI

INTRODUCTION TO GRAPHING AND COVARIATION

The intent of this segment of instruction was that the PSTs begin to develop an articulated, imagistic understanding of functions as covariation of quantities. It was not the intent that the PSTs slightly modify their current conceptions of function, but rather for them to take part in activities that would allow them to develop a variety of experiences that would (i) serve as an impetus for the development of a new or significant refinement of their current conception of function and (ii) serve as an experiential base from which the PSTs might abstract aspects of a covariational conception of function.

As indicated previously, the introductory phase consisted of two activities: *Introduction to Graphing* and *Introduction to Covariation*.

Introduction to Graphing

TI[13] held the stance that inherent in the desired understanding of graphing was the idea of recording values of quantities that are varying simultaneously. This was the broad goal of the instructional activity that was introduced briefly at the end of session 3 and continued through session 4. An overview of this set of activities is shown below in Table 6-1.

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[13] Henceforth, I will use TI to refer to the instructor.

Table 6-1: Overview of Activity 1: Introduction to Graphing

Lesson/Date	Activity 1: Introduction to Graphing	Approx. Duration
3 9/7	Part 1: Keeping track of One Quantity Part 2: Keeping track of Two Quantities & Coordination of the Quantities	11 min 28 min
4 9/9	Part 3: Reflection on Introduction to Graphing Activities	12 min

This activity was designed to help the PSTs (a) envision a graph as being a collection of points; (b) envision the collection of points as being generated by keeping track, simultaneously, of two quantities whose values vary; and (c) envision that the coordinates of every point in a graph represents, at once, values of two quantities. The progression began with TI walking a distance and the PSTs keeping track of particular attributes of TI's motion. The activity involved the PSTs modeling the time that elapsed during TI's walk and the total distance that he had walked first separately, then together, and then finally in a coordinated manner. TI believed that the PSTs would develop the notion of a variable quantity (either distance walked or elapsed time) through the actual physical activity of modeling each quantity. This modeling served two purposes. First, it foregrounded the fact that a variable is, in fact, a measurable quantity. Second, the PSTs were also physically modeling a measurable quantity that is varying (or at least *can* vary) and thus they were developing experiences with variable quantities. Finally, through coordinating the behaviors of the two variable quantities, the PSTs were developing experiences that would enable them to begin to develop a covariational conception of function. Pictorial depictions of these activities are shown below in Figure 6-1 through Figure 6-3.

Activity 1, Part 1: Keeping Track of One Quantity

The first activity involved the PSTs using the length of a vertical segment to model the distance traveled by TI. The PSTs defined their segment by locating a starting point on their desk as one endpoint and using the location of their left forefinger finger as the other endpoint (Figure 6-1). The PSTs were told to move their finger such that this vertical distance would represent the total distance traveled by TI during his walk. In the second activity, the PSTs were to use the length of a horizontal segment to model the time that TI had been walking. They defined their segment by their original starting point as one endpoint and the location of their right forefinger as the other, with the distance between the starting point and right finger representing the elapsed time at every moment of the teacher's walk (Figure 6-2). The purpose of this activity was to help the PSTs develop an experiential sense of variables (what the PSTs were keeping track of) as quantities that can vary.

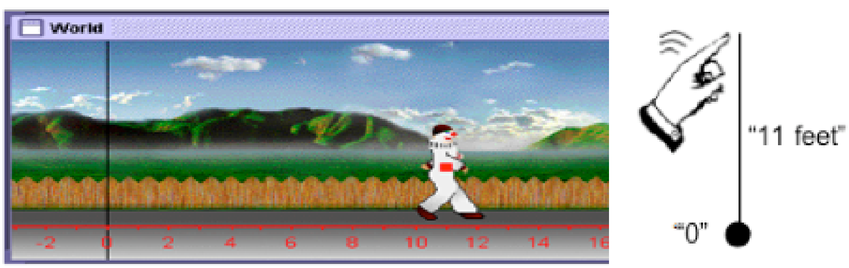


Figure 6-1: Keeping track of distance traveled

Figure 6-2: Keeping track of elapsed time

Summary of Instruction

The only difficulty the PSTs experienced in this activity was when TI reversed direction. Every student failed to note that his *total distance traveled* was still increasing. Instead, they adjusted their vertical segments to indicate that his total distance traveled began to decrease. TI clarified what the PSTs were supposed to track and noted the fact that he purposely made the quantity that they were tracking one that they could not perceive directly – they had to conceptualize it.

A similar difficulty occurred when the PSTs were asked to keep track of the elapsed time: During his walk TI stopped momentarily, standing stationary. One PST stopped moving his finger, indicating that he was using that finger to follow TI's movement instead of using it to track time. TI pointed this out, and noted that the elapsed time had not stopped.

Discussion of Part 1

At the beginning of each of the segments of Part 1, the PSTs' attention was focused on a limited view of the motion that they were tracking. They were initially focused on the aspects of the motion that were visually perceptible – for example when tracking the distance TI had walked, the PSTs were tracking the movement of TI not the distance that he had walked. In order to keep track of and model these variable quantities, the PSTs needed to focus on non-visible attributes of the motion, not the actual movement of TI. After a few tries, the PSTs were able to model the behavior of each of the two quantities, indicating that the PSTs were at least unconsciously paying attention to a variable magnitude quantity – precisely the understanding of the idea of variable that is one of the main aims of the introduction to graphing activities.

Activity 1, Part 2: Keeping Track of Two Quantities

Part 2 served as an opportunity for the PSTs to coordinate the behavior of the two quantities they had been tracking in Part 1 by keeping track of total distance with their left hand and elapsed time with their right hand simultaneously. This was done in two steps: at first simply letting both fingers model the varying quantities (Figure 6-3a) as they had done in Part 1, but moving both fingers at the same time; and second, coordinating the two by keeping the "distance" finger directly above the "time" finger (Figure 6-3b). The intent of Part 2 was twofold: (1) that the PSTs develop an experiential base for thinking about simultaneous variation of two quantities, and (2) that the PSTs actions would give TI an opportunity to relate the idea of covariation and the idea of a graph. TI anticipated accomplishing this by (a) asking the PSTs to imagine their distance finger covered in fairy dust, so that it left a trace of its locations at every moment of tracking (see callout in Figure 6-4) and (b) orchestrating a discussion about what each particle of fairy dust represented.

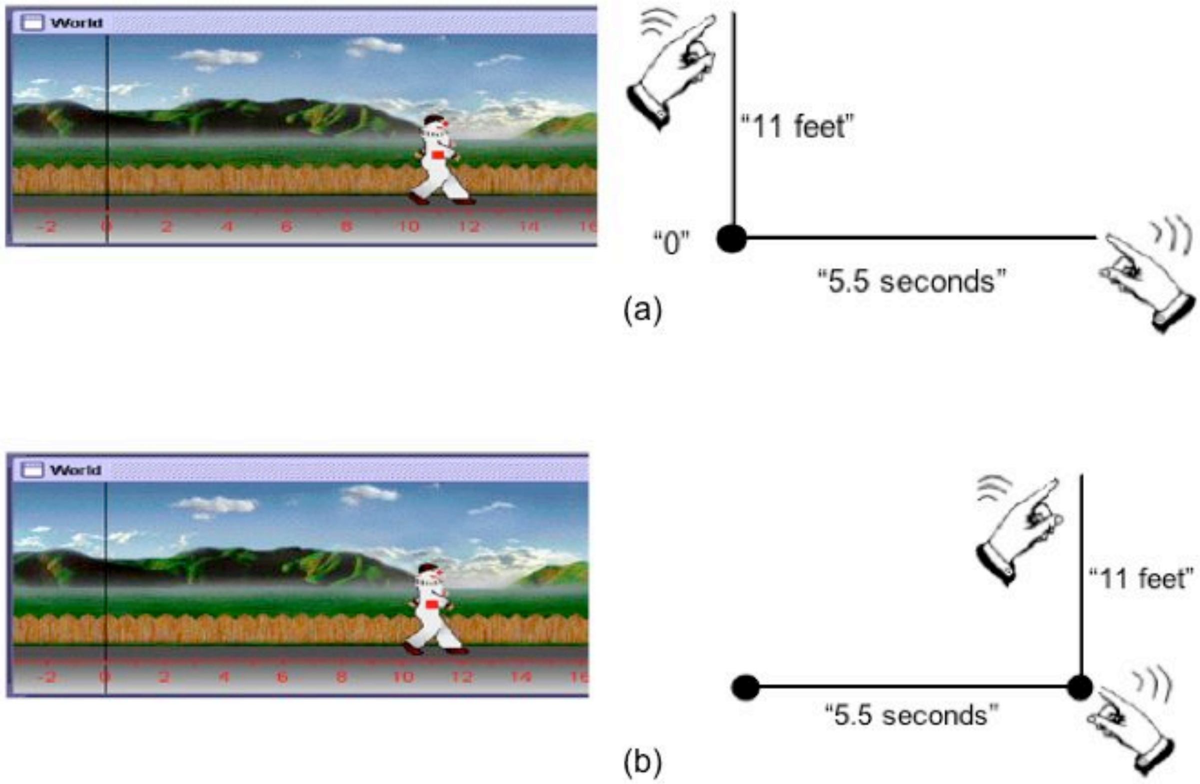


Figure 6-3: (a) Keeping track of elapsed time and distance traveled together; (b) Keeping track of elapsed time and distance together in a coordinated manner

Summary of Instruction

After a few attempts, each of the PSTs was able to trace out the covariation of the distance traveled and the time elapsed (Figure 6-4). TI then related their previous activities and the PSTs' developing notion of variables to the idea of a graph. In Excerpt 6-1, we see TI helping the PSTs develop an image of the graph as emerging from keeping track of the varying quantities. He then used this image of the graph as a didactic object to support reflective discourse on what composes a graph (points) and what the coordinates of each point on a graph represent (values of each quantity at the same moment).

Figure 6-4: PSTs modeling the covariation

Excerpt 6-1 (Session 3, 09/07/04)

1. TI: OK, are you getting a sense of where this is going? What have you just made?

2. SS: A graph.

3. DH: A graph.

4. TI: You've made a graph, didn't you? … By keeping track of how much time that I've traveled simultaneously with how far I've traveled. Now let's make it more concrete that you've made a graph. … Unbeknownst to you I have put a little cup of fairy dust in front of each of your places. Now you know what tinker bell does when she flies through the air. … She leaves a trail, right. (pause) What's that trail made of?

5. DH: Lots of points

6. TI: Yeah. … Lots of little "fairy dust" particles. Yes. (pause) What does each one of those particles represent?

7. SS: How far she has gone in a certain time.

8. TI: Yes. Where she was at a certain moment in time. It just stays right there [points to a particular location on the imaginary graph] … and you come up and you get your I eyeglass out and you look at it and can say that tinker bell was right here at whatever moment of time it took her to get right here. … So now you reach out and put your

distance finger into the fairy dust bowl. So that you have fairy dust on the end of your finger–

9. DH: – Are we still doing the time finger too?

10. TI: Now do that with both fingers together. Distance always staying above the time finger, all right? Here goes. [TI walks]

All right. Now can you imagine the graph you just made out of fairy dust. … So you see this line, this line that we normally thought about being swept out. And being solid. … When we think about it as being composed of fairy dust particles, it's clear that that line won't hold any weight. You can't put something on top of it. It's ephemeral. (pause) What does each particle of fairy dust represent … In the graph that you've just made?

11. DH: The total distance you've traveled at that point in time.

12. TI: Yes. The total distance I've traveled at that particular moment in time. So each particle is two-dimensional. Each particle records two pieces of information. How far I travel, and the amount of time I've taken to travel that far.

In the above excerpt, we see TI shifting the discussion from modeling the scenario to a discussion about what the modeling has to do with the idea of a graph. The PSTs (lines 2 & 3) were quick to notice that their fingers were tracing a graph of TI's distance traveled as a function of the elapsed time. In lines 4-8, TI returned to the notion of creating a graph and proceeded through an activity where a graph was created from fairy dust as the PSTs tracked values of distance and time simultaneously. The PSTs appeared to make this shift fairly easily, however DH's comment brings this notion into question. By asking if she is to move her "time" finger as well (line 9), DH was indicating that though she may be aware of the fact that the scenario involves two quantities, in this activity her attention is focused on only one quantity (the distance traveled).

Activity 1, Part 3: Reflections on Introduction to Graphing

As homework, the PSTs reviewed a lesson description and rationale that TI had created for the activities in which they had participated (Activity 1, Parts 1 & 2). Class session 4 began with a discussion of the lesson summary.

Summary of Discussion

The discussion centered on two main characteristics of the lesson (as planned *and* as enacted – the PSTs all felt that the lesson plan accurately depicted the activities and discussions they had taken part in in the previous lesson). The first topic of discussion was whether the PSTs' future students would "like" the activities and whether middle or high school students would see the activities as too childish. The second topic was about the PST's interpretations of the importance of the lesson. Excerpt 6-2 contains a segment of this discussion.

Excerpt 6-2 (Session 4, 09/09/04)

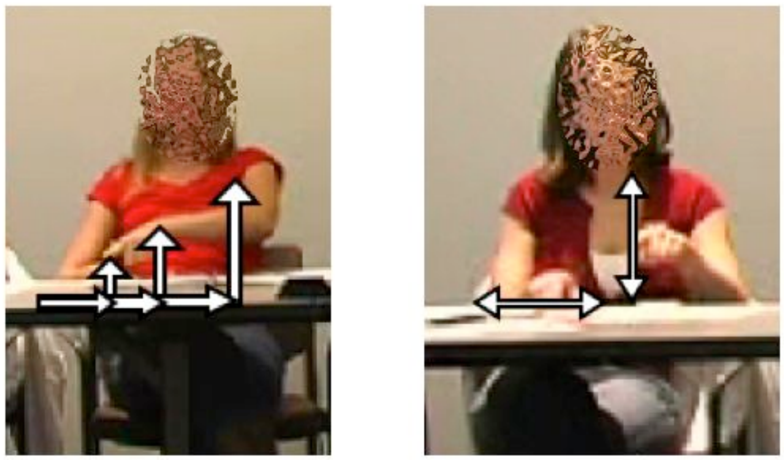
[TI poses a question about the importance of the lesson]

1. SS: I mean it's a good visual with like moving your fingers and, pretty much any kid can do that. [SS moving her fingers, coordinated in the air – the vertical finger staying directly above the horizontal finger (Figure 6-5a)]. … And like see how it's changing.

… (discussion about the ages of students)

2. DH: Well, I think [it's important to] … draw this graph in the air. [DH is moving two fingers in front of her in an uncoordinated manner (Figure 6-5b)] You know, having … drawing it on paper is a more concrete version. Kids are like, "math is done on paper". Whereas, if they're doing it in the air and trying … and trying to just make these relationships, it's not like you're doing a graph.
3. TI: OK, anything else? (pause) Where do you see this going? Do you see any long-term payoff?
4. SS: You mean like for the student who's taught this?
5. TI: Right … any long term payoff in terms of what you as a teacher could leverage and use thematically in your instruction.

SS: I fell like you wouldn't have to … I mean with something like this … and then being able to grasp a lot deeper what's going on, I mean what's really going on, not just like "ooh, I can draw the graph" but truly understanding the graph. Then you wouldn't have to explain it as much.



(a) (b)

Figure 6-5: Modeling quantities (a) in coordinated manner (2-dimensions) and (b) in an uncoordinated manner (each in 1-dimension)

In the above excerpt, we see further evidence of DH not thinking in terms of the covariation of quantities. In line 2, we get a glimpse of her mental image of the activity involving moving her fingers to "try to make the relationships." Her movement of her fingers in one dimension is evidence that she has not yet developed a means of coordinating the variations of the two quantities. She envisioned two varying quantities, not two quantities covarying.

Excerpt 6-2 also indicates that the PSTs' were focusing on "visualizing" the graph (line 1) and "understanding" graphs (line 6). Of interest is the question of what it was that they were visualizing and understanding.

Discussion of Introduction to Graphing

Thus far, I have claimed that the PSTs' attention in the *Introduction to Graphing* activities was on the behavior of individual quantities. At least one of the PSTs had yet to develop a mechanism to coordinate the behavior of the quantities (or – at a minimum – the coordination of the quantities was not the salient aspect of the activity); in order to reason covariationally, the PSTs must develop imagery that involves the coordination of the variable quantities. This coordination is difficult to develop, for the PSTs likely have two competing images of variables, functions, and graphs. The first involves varying quantities and the graph as a collection of correspondence points, each representing particular coordinated values of the two varying quantities (consistent with the *Introduction to Graphing* instruction). The second involves the notion of a point moving along a graph much like a wooden bead moves along sculpted metal wires in children's toys. In the latter, the point is moving along a specific path and understanding the graph involves understanding the shape of the path, which does not rely on developing a scheme for coordinating the two variable quantities. This latter image, one which the PSTs have developed over years of traditional school mathematics instruction, appeared to dominate DH's reasoning about variable quantities: her attention was on trying to fit her fingers to the envisioned curve and not on trying to understand the curve as a result of the variation of the quantities that are modeled by the movements of her fingers. This notion of competing images and a graph defining a point (as opposed to the graph being a locus of points) will arise repeatedly in subsequent sections.

§

Introduction to Covariation

The instructional activity that followed the introduction to graphing was titled *Introduction to Covariation*. This segment of instruction was designed to provide experiences that would assist the PSTs in their transition to covariational reasoning. The activities called upon a relationship that does not fit the traditional "function" characteristics of having an independent and a dependent variable[14]. The PSTs related two distances that varied independently of each other (i.e. neither was "dependent" on the other). The overall instructional objective was to have the PSTs interact with a computer generated environment designed to focus attention on (i) varying quantities and (ii) the way those quantities vary together (covariation). It was believed that this set of instructional activities would foreground issues of variable quantities and graphs as a record of the covariation. The introduction to covariation was the focus in parts of three classroom sessions and one written assignment. An overview of this instructional segment is presented in Table 6-2 below.

[14] Though there was, in reality, an independent variable and two dependent variables, the quantities that were being tracked were independent of each other.

Table 6-2: Overview of Activity 2: Introduction to Covariation

Lesson/Date	Activity 2	Approx. Duration
3 9/7	Part 1: Introduction to Cities A&B Part 2: Discussion of *Introduction to Covariation* Homework	11 min 5 min
4 9/9	Part 3: Imagining behavior of varying quantities: Arrangement 1 Part 4: Imagining behavior of varying quantities: Arrangement 2 Part 5: Imagining the arrangement: Arrangement (b)	8 min 6 min 5 min

Overview of Cities A&B[15]

A simple Geometer's Sketchpad [GSP] sketch, titled Cities A & B, was used as the setting for the PSTs to explore variables and covariation. The dynamic sketch involved a model of a car on a straight road that passes near two cities (see Figure 6-6a). PSTs could choose to make the sketch generate a horizontal line segment to represent the distance between the car and City B and a vertical line segment (with the same origin as the horizontal one) to represent the distance from the car to City A (Figure 6-6a). Additionally, the PSTs could make the sketch show the path traced out by the *correspondence point* – the point whose coordinates represent the distance from the car to City A and the car to City B – for all possible locations of the car (Figure 6-6a). This sketch was created specifically so that the resulting "graph" was created by keeping track simultaneously of values of the varying quantities. TI believed that this would help alleviate the PSTs of their tendency to think of graphs as simply points that are found by substituting a value of x to get a corresponding value of y.

[15] The reader is encouraged to see Saldanha & Thompson (1998) for more information regarding these instructional activities.

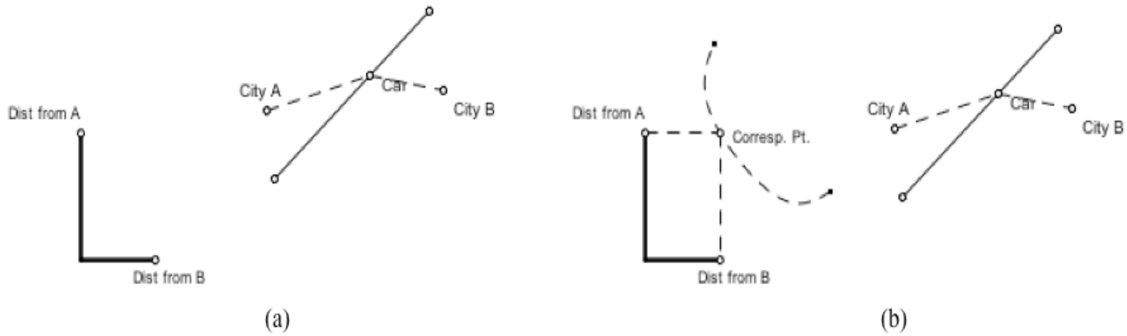


Figure 6-6: Cities A & B Geometer's Sketchpad Sketch, (a) with horizontal and vertical line segments representing quantities and (b) with a locus of "correspondence points"

The plan for the Cities A & B instruction consisted of four levels (Saldanha & Thompson, 1998). The first level, called *Engagement*, was designed to allow the PSTs to "begin 'making sense' of the simulation and building the idea of covariation" (Saldanha & Thompson, 1998). The second level, called *Move to Representation* was designed to help focus the PSTs' attention on the behavior of an individual quantity (this was really two activities – one for the distance from the car to City A and one for the distance from the car to City B). This was accomplished through tasks that presented the PSTs with configurations of the car and cities and asking them to use enactive modeling to predict the behaviors of the distances from the car to each of the cities (Figure 6-8a and Figure 6-8b) as the car moved along the road and then to predict the graph they would make by tracking the distances simultaneously (as in *Introduction to Graphing*). As part of these tasks, the PSTs were to imagine the behavior of the quantities, conjecture about the graph of the covariation, test the conjecture, and finally to revise their conjecture. The third level involved predicting the arrangement of the cities given a graph of the correspondence point, and again asked the PSTs to conjecture, test, and revise their conjectures. The fourth level was aimed at helping the PSTs examine general properties of the graph of the correspondence point to help “internalize the simulation, so that they can engage in the simulation by thought experiments, and later so that they can think about properties of covarying quantities” (Saldanha & Thompson, 1998).

The activity’s design highlighted the problematic aspects of the *Introduction to Graphing* activities: rather than focus on the movement of the car along the road, the PSTs needed to focus their attention on “imaginary” segments that represented the distance between the car and the cities. Again, the PSTs had to pay attention to an attribute not initially obvious or visible in the setting.

Activity 2, Part 1: Introduction to Cities A & B.

At the end of class session 3, approximately 10 minutes was spent orienting the PSTs towards the Cities A & B activities. This activity was to serve two purposes. First, the activity was envisioned to help the PSTs become familiar with the computer application. Second, it would allow TI to begin to focus the PSTs’ attention on the variable quantities and thus provide the PSTs with a firm background to work on the activities at home. Figure 6-7, below, shows the activities that were projected onto a screen at the front of the room.

1. Move the point labeled "Car."

- Watch the distance from City A and from City B as you vary the car's location.
- Imagine a segment from Car to City A and observe how its length changes as the car's position varies.
- Imagine a segment from Car to City A and observe how its length changes as the car's position varies.

2. Click Show Axes.

This is the set of axes on which the distance between the car and the two cities will be plotted.

3. Click Show A.

- Move the car. Watch Distance from A's behavior relative to the car's distance from an end as you move the car along the road. Keep moving the car until you understand why the distance changes as it do.
- Describe Distance from A's behavior as the car moves away from or toward an end (you pick the end).

4. Click Hide A; then click Show B

- Move the car. Watch Distance from A's behavior relative to the car's distance from an end as you move the car along the road. Keep moving the car until you understand why the distance changes as it does.
- Describe Distance from A's behavior as the car moves away from or toward an end (you pick the end).

5. Click Show A (so that both A and B are showing)

- Move the car.
- Describe how Distance from A and Distance from B change in relation to each other. Write your description.

6. Click Show Correspondence.

- What does any location of the of Correspondence Point represent?
- Move the car; Watch the correspondence point.
- Explain the Correspondence Point's behavior with respect tot eh car's location relative to the two cities as it moves along the road.

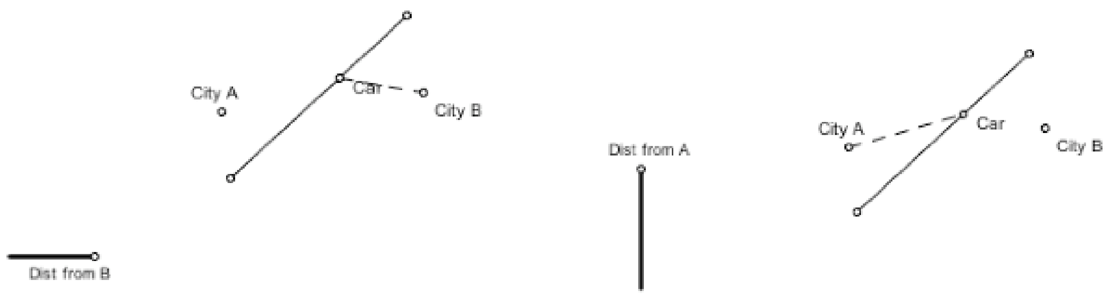
7. Click Show Graph.

- What does the graph represent?
- What information does each point on the graph give?
- What is the relationship between Correspondence Point and the graph?

Figure 6-7: Excerpt from Activity 2, Cities A&B

Summary of Instruction

Instruction began with the PSTs imagining and describing the behavior of the length of the segment from the car and City A (#1-4 in Figure 6-7). In response to questions 1 through 4, the PSTs' first inclination was to discuss global characteristics of the situation. For example, when explaining the behavior of the "distance to City A" (Figure 6-8b), KN noted that the distance is "going to get shorter then longer." To see why this makes sense, the reader should imagine the pictures in Figure 6-8(a & b) below as dynamic sketches. In particular the point labeled “car” is free to move along the “road” and as it does, the length of the dashed line, representing the distance between the car and the city is changing. The horizontal and vertical lines are constructed so that they are exactly the same length as the dashed line, but in a particular location and orientation. As the car moves along the road, the vertical (and horizontal) segment does get shorter and then get longer.



(a) (b)

Figure 6-8: (a) Sketch showing horizontal bar representing the distance to B and (b) showing vertical bar representing the distance to A

TI probed the PSTs about *when* the distance ceased getting shorter and began getting longer, an attempt to shift the PSTs’ attention from global characteristics (bigness and smallness) to a specific description of the situation. The following excerpt describes that interchange and what followed.

Excerpt 6-3 (Session 4, 09/09/04)

[On the screen is the cities A&B sketch and only the vertical bar labeled City B is shown (Figure 6-9, right). TI is moving the car along the road and the height of the vertical bar is changing as the distance from the car to City A is changing.]

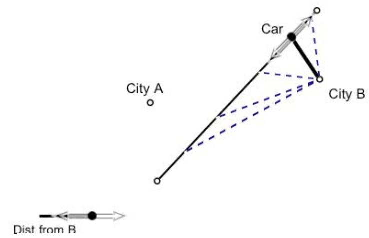


Figure 6-9: Distance from B Varies

1. TI: It's going to get shorter and longer. Where will it stop getting shorter and start getting longer?

2. KN: When the line connecting the car and the city is perpendicular.

3. TI: Okay. And is the same true for the distance between the car and City B? (pause) All right, so I'll go ahead and … [TI hides the horizontal bar and shows the vertical bar, which represents the distance between the car and City A. He then moves the car along the road and the length of the bar changes accordingly. See Figure 6-10.]. Now is that distance varying in the way you anticipated?

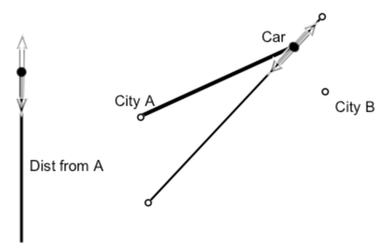


Figure 6-10: Distance from A varies

4. DH: Well yes, but why is one horizontal and vertical –

5. SS: – If you graph them together it's just like – one is horizontal and one is vertical.

6. DH: Oh

7. TI: You could put it together as well. OK, now I'm going to show A again. … [TI shows both vertical bar – representing the distance between the car and city A – and the horizontal bar – representing the distance between the car and City B simultaneously. He then moves the car along the road and the lengths of the bars change accordingly. See Figure 6-11] Now, you see, now what do you see about the distances?

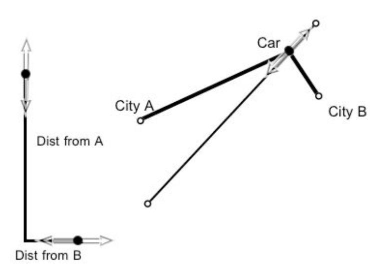


Figure 6-11: Both distances vary

8. SS: How they relate to each other?

9. TI: OK. (pause) Prior to even how they relate … is the fact that they're even varying simultaneously. … Now if you talk about how they relate, SS, how would you talk about how they relate?

10. SS: I don't know. It just seems like problems I've seen where they ask questions like that what point are you the same distance from A and B. … At what point are you the farthest from A and the closest to B? … Which is like – (pause) That's what I recall. Problems like that. This is a much more visual presentation.

11. TI: Now that's in the context of getting an answer to a question … now have I asked any questions? … So just what do you notice? What do you notice about the way they vary?

12. SS: They both decrease and increase again

13. TI: OK, they both decrease and increase again. Anything else? [TI animates car moving along road. As a result, the vertical segment (Distance from A) and the horizontal segment (Distance from B) vary accordingly.] What do you notice about the way those distances are varying together? … Are they getting longer and shorter in the same way? … DH? Try describing what you see.

14. DH: I see that both of them at different times, but when they turn around to go the other direction it slows down. They're not doing it at the same time, but when the distance from A, right about there, it's changing directions it slows down and then speeds up. And then B is doing the same thing. It's going away and then coming back. And it's almost topped it goes back a little bit, but when it turns around its slowing down.

15. TI: Anybody else?

16. DH: Are they doing the same things at different times?

17. SS: You put the cities so they're doing exactly the same things, but switched. But … at this moment [each is] like just a mirror image of the [other] distance.

18. DH: It looks like they're going through the exact same pattern just starting at a different time.

All of the PSTs were in agreement with KN's utterance in line 2 that the distance between the car and a city changes from decreasing to increasing when the car is at the intersection of the perpendicular from the city to the road. (In fact, line 10, SS recognized this as a “traditional” problem and before thinking about the situation, indicated that the answer was going to be “when it was perpendicular.”) The PSTs were aware of the location along the road where one characteristic of the quantity in question changes. For purposes of clarity, I will call this point a one-dimensional landmark (the need for “one-dimensional” as a modifier will be explained later). In line 7, TI moved the discussion to the relationship between the quantities, making the logical assumption that the PSTs would be able to make sense of the one-dimensional landmark for the second quantity (distance to City B). The PSTs’ initial inclination, as evidenced in lines 13-18, was to compare the behaviors of individual quantities, rather than to focus on the covariation of quantities.

TI then intervened and proposed the following example of how one might begin to think about the way the distances vary together:

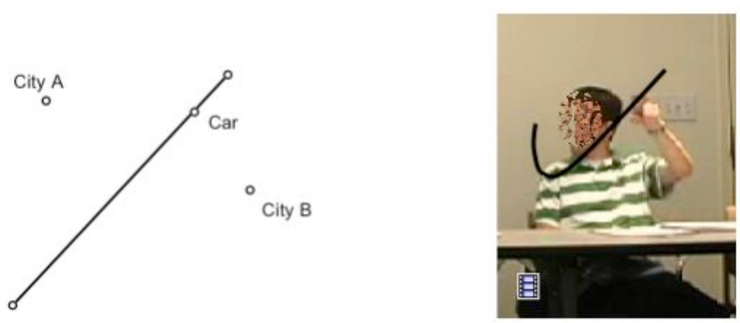
> As the car starts from that end, while the distance to City B gets shorter, the distance to City A gets shorter faster. And as it passes perpendicular, the distance from City B increases while the distance from City A continues to decrease, until the car gets directly opposite City A and then they both increase simultaneously. (TI, 09/07/04)

The reader should notice the qualitative differences between TI’s description of how the distances vary together and the PSTs’ explanations in the above excerpt. Rather than being focused on how the behavior of the two quantities compare, TI focused on key landmarks (where the line segment between the car and the city is perpendicular to the road) and then on what happened to one quantity as the values of the second quantity vary within the particular sub-interval of the road.

The class continued with the PSTs trying to explain the behavior of the quantities, however their explanations were still grounded in the individual behavior of the quantities, rather than the way they covary. For example, SS described the way the distances vary together:

> OK, so for instance, both City A City B distances are decreasing, once [the car] hits city B perpendicular, then it starts, umm, [increasing] for City B and City A … decreasing all the way until it's perpendicular to City A and then they both increase again the rest of the distance. (SS, 9/7/04)

The PSTs' attention at this point, as evidenced in Excerpt 6-3 by SS's quote above, was on the behavior of the individual quantities – the PSTs are still comparing them. The class session ended with TI asking the PSTs to track the distance between the car and City A with the horizontal finger and the distance between the car and City B with the vertical finger, as they had done in the introduction to graphing activities. After some practicing, the PSTs were able to model (in the air, with their fingers) the behavior of the two quantities. However, it was quite obvious that their attention was not on the covariation of quantities, but rather on the overall shape of the graph. For example, in response to the arrangement of the cities and the road (Figure 6-12a), the PSTs had generated (with their fingers) a graph similar to the one shown in Figure 6-12b.



(a) (b)

Figure 6-12: A "U-Shaped" Graph

TI the asked the PSTs what they had just created. The PSTs' replies were focused on the "shape of the graph", as indicated by responses like "looked like a parabola" or "kind of V shaped." Their attention was no longer focused on the fact that the curve was the result of tracking the quantities' values as they varied simultaneously. Rather, their focus had returned to the "shape" of the graph.

Discussion of Part 1

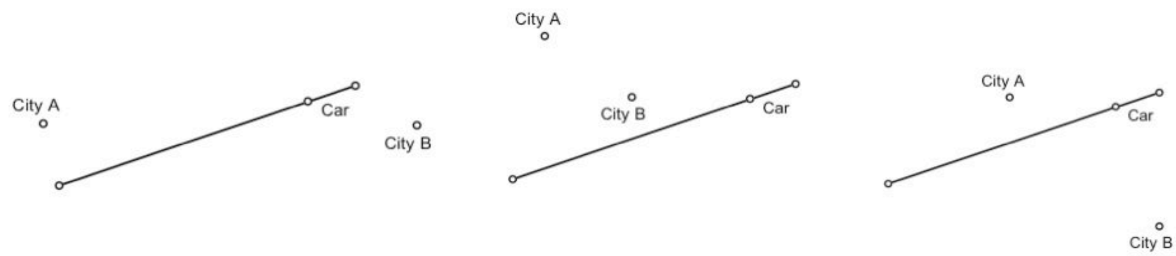
In this activity, the PSTs first explored the Cities A & B GSP sketch. They were able to describe the behavior of the two distances individually and, with a bit of help from TI, were able to describe the covariation of the quantities. Their description was consistent with the graph that the GSP sketch generated as the collection of correspondence points. Once the graph was displayed, however, the PSTs' attention returned to the graph's shape and was no longer focused on the behavior of the quantities whose covariation produced the graph. It must be noted that the fact that it was or was not a parabola is not my purpose in discussing this classroom interaction. More important was the fact that their attention had shifted from the quantities that were varying to global characteristics of a graph. This indicates that the PSTs were not thinking of "shape" as an emergent property of the coordination. At this point, the PSTs' inclination was not to understand the behavior, but rather was to still to classify the graph – this inclination had not changed from their initial assessment.

Part 2: Class Discussion of Introduction to Covariation Homework

Overview of Class Discussion

The PSTs indicated that they did not have any problems with Level 1, but that they had "a lot of trouble" with the rest of the assignment. When asked by TI to work through the first arrangement from Level 2 of the introduction to covariation activities (Figure 6-13), the PSTs' inclination was to use their fingers to model the behavior of each of the quantities. Though they were able to model the behavior of the quantities individually (they were able to model the distance from the car to City A, similar to the behavior of the vertical line segment in Figure 6-8b), they were unable to coordinate the two and thus trace out the curve with their fingers.

Here are three arrangements of the road, City A, and City B.



For each arrangement:

- Imagine the behavior of Distance from A relative to the car's position as you move the car from one end of the road to the other.
- Imagine the behavior of Distance from B relative to the car's position as you move the car from one end of the road to the other.
- Imagine the behavior of Correspondence Point as you move the car from one end of the road to the other.
- Sketch your prediction of the graph that Correspondence Point will make with this arrangement.
- Test your prediction

Make up arrangements that you will give to someone else. Try to make them as "surprising" as you can.

Figure 6-13: Excerpt from Introduction to Covariation Assignment (Level 2)

The PSTs believed that their main problem with this assignment was the fact that they were not “coordinated enough” to move the two fingers at once. SS even likened this activity to “the old pat your head and rub your tummy thing” (9/9/04), indicating the fact that she thought she knew what she was supposed to be doing, but could not physically do the two things at once. DH described her similar experiences:

> Level one was fine but the same problem I was encountering in class, when we were trying to … figure out like, whatever it was … trace the movement, I forget whether it was … trace the movement … but when I got to level two, I know what was supposed to happen but I kind of have a lot of trouble physically making my fingers do it (DH, 09/09/04).

Discussion of Part 2

Analysis of the PSTs’ engagement with Activity 1, Part 1 and Part 2 indicated that DH could, in fact, describe the behavior of each of the quantities separately, but had yet to make sense of how one might describe how the quantities covary. The traditional interpretation of the PSTs' trouble would be that they have the prerequisite knowledge; they just need to learn to put the two together. In light of the common conception of mathematics as "building blocks" – that once one possesses the requisite building blocks, they should be able to understand the next topic – it is notable that the PSTs had significant trouble taking that next step. TI accepted their explanations of "rub tummy-pat head" difficulties, but also offered that he suspected that their problem was conceptual, not physical.

Activity 2, Part 3: Imagining the Behavior of Varying Quantities (Arrangement 1)

TI had expected that the PSTs would not be completely successful in working through the *Introduction to Covariation* activities at home and TI had expected to devote a significant portion of the class-time to the PSTs working through the activities as a group. The instructional purpose of this activity was to have the PSTs engage as a group with the GSP sketch and to have them use each other (and TI) as a resource and to question and discuss each other's developing understanding of variables and covariation in relation to the presented scenario. As a first activity, the PSTs were presented with the task of describing the graph generated by the correspondence point from Arrangement 1 (Figure 6-14). A static picture (not dynamic sketch) of the arrangement was projected at the front of the classroom.

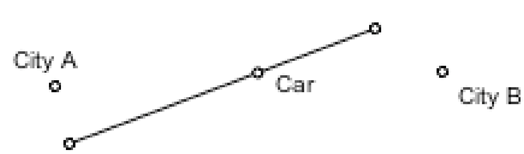


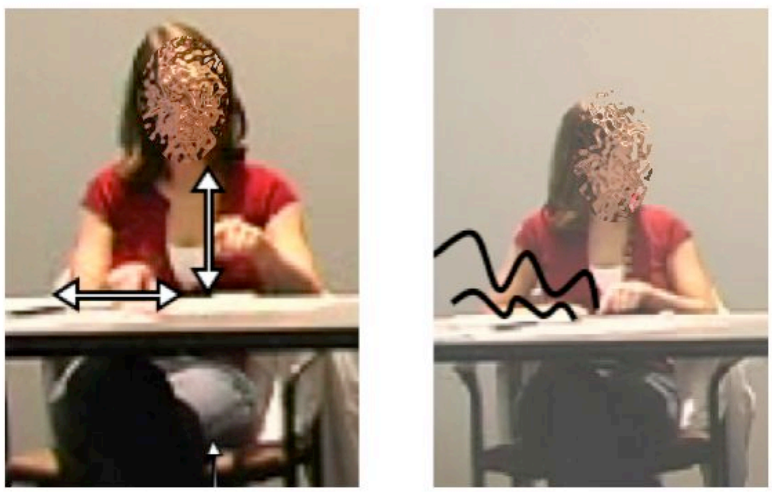
Figure 6-14: Level 2, Arrangement 1

Summary of Instruction

DH began the conversation:

> Well I think the first step is … is how we did it. You know A goes this way [moves left hand vertically] and B goes this way [moving right hand along table]. I found that I was able to do these things [moves her fingers as in Figure 6-15a] and the second step was putting them on top of each other [moves her fingers as in Figure 6-15b] (DH, 09/09/04).

Two things are evident from this brief extract. First, at this point, DH has not made the connection between the modeling of the quantities and covariation – the task was not as much about covariation as it was about "putting them on top of each other." In addition, her gestures, which are depicted in Figure 6-15a and Figure 6-15b, indicate that she had yet to make full sense of the connection between graphing in the Cartesian plane and the variable quantities.



(a) (b)

Figure 6-15: DH's Attempts to Model the Covariation

KN then initiated a shift in the discourse while attempting to describe the first of the three arrangements (Figure 6-14). Rather than comparing the varying quantities, he began to speak of the way one quantity behaved with respect to another:

> Well it seems like they would just [moves hands up and down; left and right, apparently uncoordinated, similar to DH in Figure 6-15b]. As it's getting closer to A, it's getting farther away from the … almost in all cases and vice versa (KN, 09/09/04).

Though he was unable to coordinate his fingers with respect to the motion he was describing, there was a definite shift in the nature of his description. The previous day, aside from following TI's description, the PSTs' descriptions all hinged on comparing qualitative features of the behavior of the two quantities – they described the graph, not the covariation. In KN's comment, we see the beginning of the PSTs being attuned to how one quantity behaves with respect to the other. The following interchange ensued in response to KN's comment:

Excerpt 6-4 (Session 4, 09/09/04)

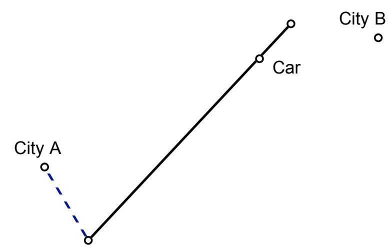


Figure 6-16: Is City A Perpendicular at the Road?

1. DH: But there's like it looks like when you're exactly perpendicular to the line of the car and you're at city A, it's not exactly at the end [See Figure 6-16]. Doesn't it look that way to you?
2. KN: Yes.
3. DH: So, I mean like it's … at … if the car's moving downward that way at first [from upper-right to lower-left], like it's getting … it's moving farther and farther away from B and closer to A until right when it hits perpendicular and then it gets a little farther from A and it's still farther away from B.
4. KN: Right.
5. DH: But if it was perpendicular, it would never get farther from A, so it must be a little bit in from the end.
6. KN: Yes.

The activity ended with the PSTs using KN's reasoning to predict what the graph would look like and TI using the actual Geometer's Sketchpad sketch to generate the graph.

Discussion of Part 3.

In this activity, and especially in Excerpt 6-4, we see DH focusing on specifying the relationship that KN had described previously. In line 1, we see her attention on whether City A is located "at the end of the road." By this, she means whether a line segment drawn from City A to the end of the road will be perpendicular to the road. The significance of this statement is that she is focused on the implications of whether there is a "hook" at the end of the graph or not (Figure 6-17). Her attention, thus, was not on the behavior of either quantity, but on a characteristic of the relationship between the two quantities. In particular, she was aware that if one focused on the car moving from the upper-right endpoint of the road towards the lower-left endpoint of the road, the distance from the car to City B would always be decreasing. She was attempting to convince the class that City A was "not exactly at the end of the road," because as the car approached the end of the road, the distance from the car to City B would still be increasing, but the distance from the car to City A would get smaller until it reaches perpendicular and then it will begin to increase, with the distance from City B *still* increasing (line 3). She claims that City A cannot be at the end of the road, because if it were, as the car approached the end of the road, the distance from the car to City B would be increasing, while the distance from the car to City A will get smaller and smaller until it reaches a minimum value when the car is located at the end of the road – it would never increase.

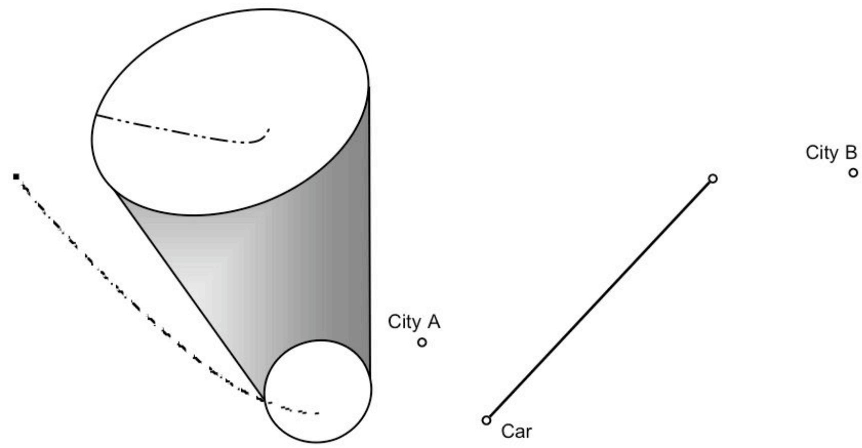


Figure 6-17: A "Hook" at the End of the Graph

Activity 2, Part 4: Imagining the Behavior of Varying Quantities (Arrangement 2)

Part 4 was very similar to Part 3, except that it involved the second arrangement from level 2, shown below in Figure 6-18. An image of this arrangement was projected at the front of the class throughout the discussion[16].

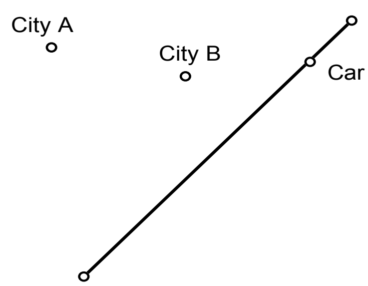


Figure 6-18: Arrangement 2

[16] Again, the figures projected on the screen at the front of the classroom were static pictures, not dynamic sketches which allow quantities to be varied within the sketch.

Summary of Instruction

Part 4 began with the PSTs trying to predict and explain the graph resulting from the second arrangement. In the excerpt below, we see the development of a more advanced notion of covariational reasoning:

Excerpt 6-5 (Session 4, 09/09/04)

1. DH: So … at the same time, if B is starting like … far away [at the top right of Figure 6-18] and it goes in and out … it looks like it changes at about halfway [DH is referring to a location about halfway along the road – see arrow in Figure 6-19].

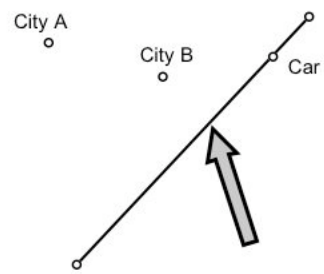


Figure 6-19: Right About Halfway

2. KN: Right.
3. DH: Like it takes halfway … And so when B goes down and [then starts going] up … A's going, it goes, it still goes up.
4. KN: It just goes down and up … uh oh [He has a problem locating when A turns around].
5. SS: B goes …
6. DH: What I'm confused about is the … OK so B's still moving farther even after A turns around and is starting to go away from.
7. SS: You've got to go there and back [indicates far from starting point, moving closer and then farther away with her fingers on the table]. One changes about halfway and the other changes a little after that [see Figure 6-20].
8. DH: So you've got to go like this [changes direction of B finger, decreasing then increasing] and then like this [changes direction of A finger, decreasing then increasing].

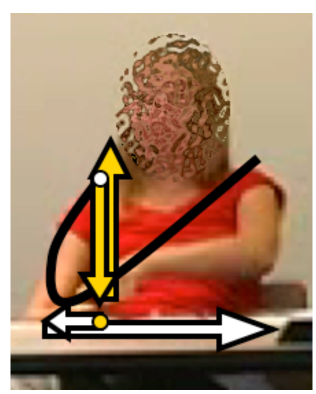


Figure 6-20: SS: They both go "down than up."

9. SS: So when it's closest to A, it's almost the closest to B, so your finger better get over there.

10. DH: So A turns around before B turns around.

11. KN: Yes.

12. DH: So, it's like A turns around and then B turns around… Oh. [The PSTs are all generating fairly accurate graphs with their fingers.]

In line 1, we see a shift in the discourse from talking about distances or lengths of line segments to speaking of a "location" on the road, which corresponds to an instance where there was a change in the coordinated behavior of the quantities. Though the PSTs had previously referred to this point as being "where the city is perpendicular to the road," it appears this location had taken on more significance than simply where one quantity, a particular distance, stopped decreasing and started increasing. DH introduced this notion that something important happens "at about halfway" (line 1). This halfway point was essential for her to coordinate the covariation of the two quantities. For example, her thought process seemed to go something like this: *If the car started at the upper-right end of the road, as it moved towards the other end of the road, both the distance from the car to City B and the distance from the car to City A are decreasing. Once the car reaches the halfway point, about where City B is perpendicular to the road, the distance between the car and city B began to increase while the distance between the car and city A still decreased. As the car reached the point on the road where City A is perpendicular, the distance between the car and City B remained increasing while the distance between the car and City A began to increase.* Figure 6-21(a-c) below depicts the three regions she seemed to be envisioning.

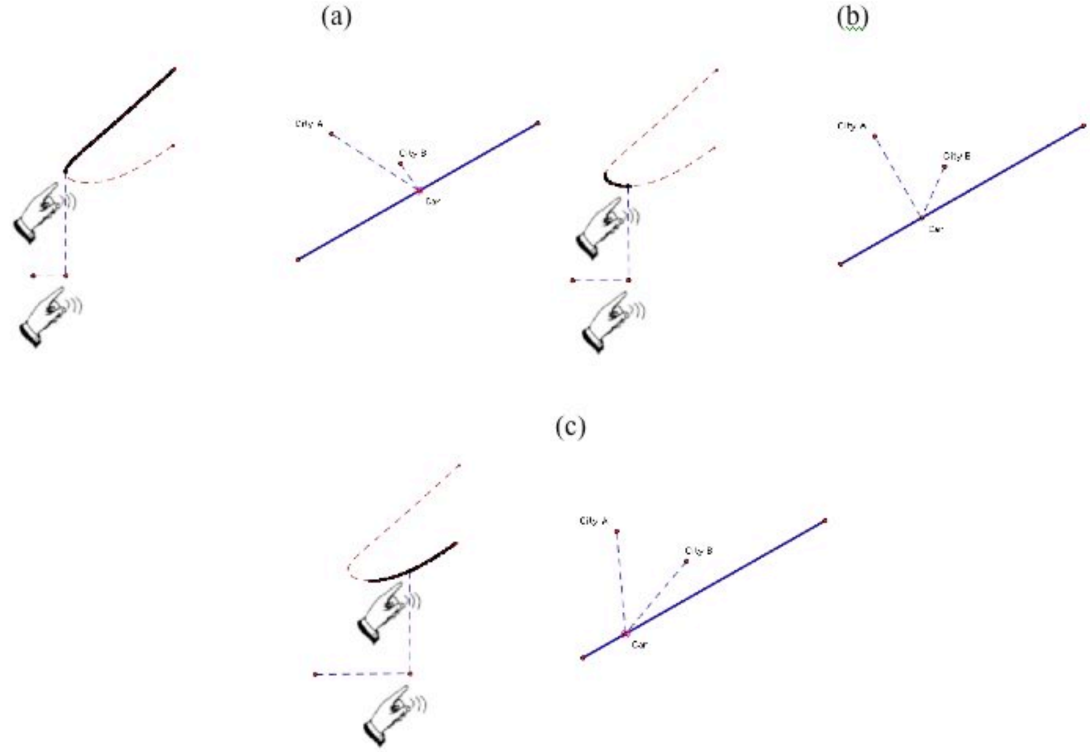


**Figure 6-21: Three regions. In (a), both distances are decreasing. In (b) the distance to city B is increasing while the distance to city A is still decreasing.
In (c) both distances are increasing.**

KN described his similar thought process:

> As the car passes the point in which the line from the car to City B is perpendicular to the road, the distance from [the car to] City B stops decreasing and starts increasing. The distance from City B continues to increase, and the distance from City A continues to decrease until the car reaches the point where the line connecting it to City A is perpendicular to the road. After that point, both distances increase until the car reaches the end of the road (KN, *Introduction to Covariation* Assignment Write-up).

Discussion of Part 4

The reasoning that the PSTs displayed was significant because it was no longer in terms of the behavior of one quantity. Each region was significant because of the way one quantity varied with respect to how the second quantity varied. This two-dimensional "landmark" was not just when the city was perpendicular to the road, it was significant because it was a location where there was a noteworthy change in the way the quantities covary.

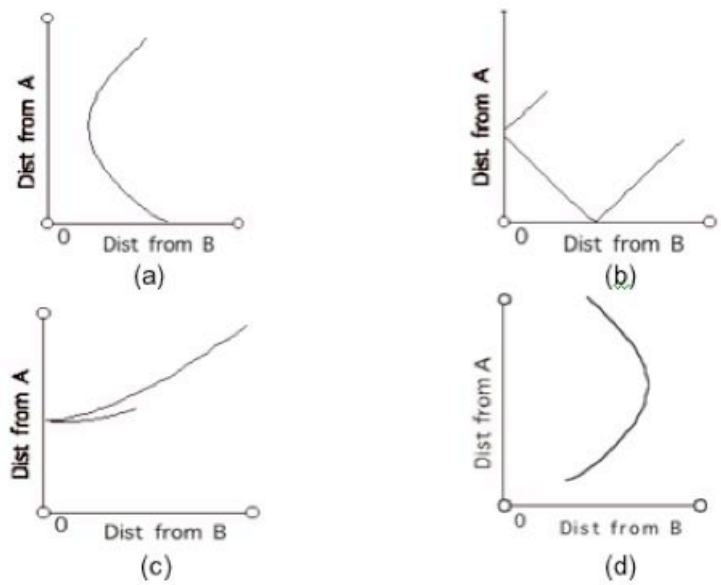
The reader will notice that this abstraction of the notion of a landmark was essential in order to make sense of the covariation. Without it, the two quantities appeared to relate in a somewhat random manner. Thus, in order to make sense of the covariation, the PSTs first needed to develop a way of organizing the covariation and one such way is through the abstraction of attributes of the behavior of two individual quantities into a two-dimensional landmark that, in addition to information about the behavior of each of the quantities individually, gives information about the relationship that exists between the two quantities.

Two additional comments must be made about the PSTs' developing knowledge. First, the reasoning described above was not limited to DH. As a homework assignment, the PSTs had to write-up solutions to the introduction to covariation homework assignment. In that write-up, all three PSTs – DH, SS, and KN – included the idea of two-dimensional landmarks in their discussion of how the two quantities co-vary. Once this way of thinking was presented by DH, it became the way of reasoning through the rest of the problems on this assignment. This is likely because the PSTs understood the behavior of each of the individual quantities and they simply lacked a means by which to organize the covariation. Once they were presented with a means, it was easily assimilated into the schemes that they brought to these problems. Second, this way of thinking made it possible for the PSTs to coordinate the two quantities and therefore physically coordinate their fingers representing the variable quantities. This, in turn, enabled the PSTs to visualize the behavior of the quantities as they covaried. It appears that their struggle earlier in the assignment was not in the physical coordination but, again, in developing a means to organize the relationship between the quantities.

Part 5: Imagining the Arrangement (Arrangement b)

Part 5 involved the reverse situation from that in parts 3 and 4. Rather than being given an arrangement and asked to explain what the graph would look like, the activity involved being given a graph depicting the covariation of the two quantities, and being asked to create an arrangement that would result in the given graph. For each of the graphs shown below in Figure 6-22, the PSTs were asked to (a) explore possible arrangements (locations of City A and City B that would result in such a graph) without a computer, (b) justify their prediction (out loud), (c) test their prediction by using GSP to locate City A and City B while the graph is showing, and (d) evaluate their prediction in light of their test (and analyze it if they were off). The activities in part 5 were envisioned to help the PSTs further develop their covariational reasoning skills. Specifically, they highlighted the importance and utility of the idea of a landmark and analyzing the graph on sub-intervals of the road on which there was some sort of predictable variation. Throughout the entire discussion, an image of Figure 6-22 was projected at the front of the room.

For each graph below: Locate, on paper (away from a computer), City A and City B relative to the road to produce that specific graph.



After making all your predictions:

- Test your predictions on a computer.
- If you were way off on a graph, explain how you were thinking.
- If you were accurate, explain as if to someone else how to answer questions like these.

Figure 6-22: Part 5 – Introduction to Covariation

Summary of Instruction

The PSTs were able to use covariational reasoning to describe a situation that would result in such a graph. The following excerpt is taken from the classroom discussion regarding the graph in the upper-right hand corner of Figure 6-22:

Excerpt 6-6 (Session 4, 09/09/04)

1. KN: B is going to be–

   [KN traces the graph using two fingers, as in Figure 6-23]

2. TI: –KN, show the others what you're doing.

3. KN: Well … I'm just tracing the graph with my finger.

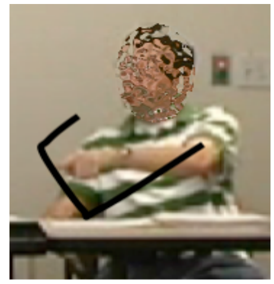


Figure 6-23: KN Tracing the Graph

4. TI: You're doing more than that.

5. KN: In, well … I'm seeing where the– (pause) Am I doing more than that?

6. TI: Because you have two fingers involved. You're not just tracing the graph with one finger–

7. KN: –right, I'm tracing the graph with both fingers.

8. SS: Wait, well…

[Long pause as each student begins to trace with both fingers, one horizontally along the table and the second moving vertically above the first]

9. DH: It means that they turn around at exactly the same point. To have those sharp points. Don't they have to?

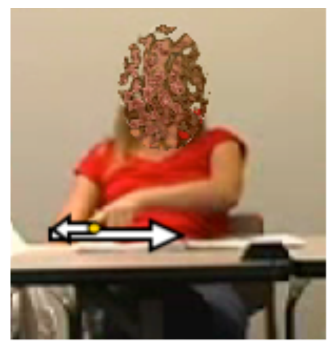


Figure 6-24: SS: B Gets Smaller then Larger

10. KN: No, the sharp point at the bottom – I'm starting from the right of the graph – I don't know why but... So you go down and B's still getting closer and closer, and then it gets a little farther away from B. So B has to be somewhere close to the end of the road. And, as B gets closer, A gets closer and then farther away.

11. SS: Oh! Because B's like this [moves horizontal finger left, then right – see Figure 6-24]. And at the same time–

12. DH: –so B's getting closer, closer, closer, farther and A's getting closer, closer, farther, farther, farther [Figure 6-25].

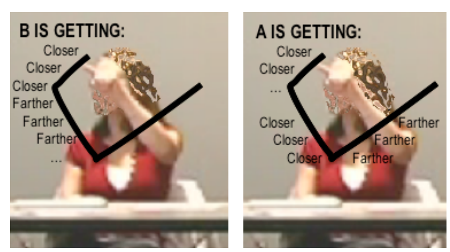


Figure 6-25: DH's Description of Individual Behavior

13. KN: A's closer to the middle, but not–

14. DH: –not exactly at the middle.

15. KN: A's close to the middle, but a bit to the left.

16. TI: So you said City B is far from the end?

17. DH: Well, no.

18. KN: City B is close to the mid–

19. SS: - and A is like close to, like three-quarters the way.

20. TI: All right. And …

21. KN: City A's between the midpoint and City B. Is that right?

22. TI: Like that [TI displays Figure 6-26 as the proposed arrangement]?

23. DH: Oh, it would have to be a little farther. Like… [TI shows result of proposed arrangement - Figure 6-27] … Yeah, like that.

24. KN: Cool.

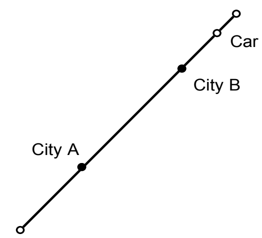


Figure 6-26: Proposed Arrangement

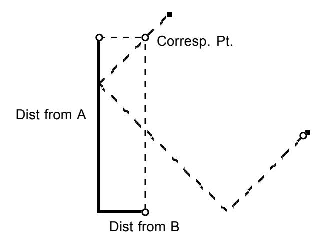


Figure 6-27: Result of Proposed Arrangement

In this excerpt, the PSTs began to call upon covariational reasoning in an effort to make sense of the graph. By tracing the graph with not one, but two fingers, the PSTs were relating the problem of determining the arrangement to their previous work in Parts 3 and 4. As in their previous attempts at explaining covariation, the PSTs describe the behavior of the individual quantities, however, in addition to talking about the behavior of individual quantities, the PSTs are beginning to reason in terms of how one quantity varies with respect to the other. For example, in line 10 we see KN referring to the relationship between the behavior of the distance from the car to City B with respect to the distance from the car to City A: "As B gets closer, A gets closer and then farther away." Further, we see indications of covariational reasoning in both SS and DH. In line 11, SS is obviously about to speak about how the distance from the car to City A varies as the distance from the car to City B gets smaller and then gets larger (see Figure

6-24). Finally, in line 12 and Figure 6-25, we see DH attempting to coordinate the variation of the two quantities. Once the PSTs' attention is focused on the covariation of the quantities, the conversation shifts to attempting to locate where on the road the cities might be[17] (lines 13-21).

Conversation about part 5 ended with the PSTs predicting that City B would have to be near the end of the road and City A would have to be near the middle (with both actually being on the road). TI then created an arrangement similar to the one that had been proposed (Figure 6-26). After confirmation that that was their conjectured arrangement, the resulting graph was displayed (Figure 6-27) and the PSTs "tinkered" with the location of the cities to result in a more accurate graph.

Discussion of Part 5

It is important to note that in the conversation shown in Excerpt 6-6, we see the PSTs attempting to figure out where key changes in the variation of the quantities occur. As the PSTs are discussing what appears to be one-dimensional landmarks (for example, see lines 11 and 12), they are in fact thinking about the covariation of quantities, as evidenced by their (at least attempt to) physically coordinate the two "distance" fingers. It may be helpful, as they speak about the location of B (or A), to imagine the PSTs moving their coordinated fingers back and forth over the graph in order to make sense of "the trip" and the location of the cities (as is shown in the embedded figures). It is this way of thinking that enabled them to present an acceptable arrangement that yielded an appropriate graph.

This activity also helped the PSTs generalize their work in parts 3 and 4: they were not just thinking of a particular graph. At the end of part 5, the PSTs, without any difficulty, began

[17] In actuality, they are speaking of where along the road would the segment from the car to the city be perpendicular to the road.

talking about what might happen to a graph as locations of either city were changed. The PSTs had naturally moved to a higher level of reflection involving imaging and predicting changes in graphs as a result of changing initial conditions.

PSTs' Write-Ups of Introduction to Covariation

As an assignment for session 5, the PSTs were asked to "write-up" their responses to the entire *Introduction to Covariation* Assignment. These write-ups were to contain fairly detailed solutions to the problems as well as too include discussions about how the PSTs were thinking about the problems.

An excerpt from SS's write-up of part 5 is shown below in Figure 6-28 (The reader should refer to Figure 6-22 for the graphs the PSTs were analyzing). In Figure 6-28, SS referred to locations along the road (for example "2/3 of the way"). There are indications in her discussions of arrangements that she had thought of these locations as two-dimensional landmarks. First, in (A) she referred to "while B's distance changes" (at the middle of the road) "A's distance was constantly going up." In (B), she noted that she should have made a better prediction of the location of B because "A changed directions before B." This comment pointed to the fact that she was aware of two specific locations on the road (2/3 of the way and 3/4 of the way). In addition, her rationale for rejecting her initial arrangement was that she knew that "A changed directions before B," indicating that the she was aware that between those two locations, A was decreasing while B was increasing.

Predictions

1. City B would be in the middle of the path and a little away from the path, and city A would be at one end or the other, past the end point.
2. City B would be about 2/3 of the way down the path and city A would be the same on the other side – both directly on the path of the car.
3. City B will be about 2/3 of the way directly down the path of the car and City A would be close, but not perpendicular to the end of the path.
4. City B would be off the path but in-between the end points, and City A would at one at the other end so that it would always be increasing.

Actual Findings

1. I was correct! I was able to trace the graph and see that it came closer to and farther away from City B about the same distance, so I knew that City B would be between the two endpoints, but not on the line. While City B's distance changed directions, City A's distance was constantly going up, so I knew it had to be in a place where the distance away from City A would be constantly growing.
2. City A was in the correct place, being 2/3 of the way between the endpoints and on the line. I found that the city had to be on the line because the graphed line came exactly to the edge of the graph. At the same time, City B was correct that it was on the graph, but it was about 3/4 of the way down the path instead of my prediction. I should have known this because it appears that A changed directions before B.
3. I was correct that City B would be 2/3 of the way directly down the path of the car, which I could see from the graph by the fact that the graph touched the edge where the (0) mark was about 2/3 of the way down the line that was graphed. City A ended up being almost perpendicular to City B and a good distance from the edge of the graph.
4. It is impossible to get the graph to face that direction!

Figure 6-28: Excerpt from SS's Write-Up

KN described his thought process for Activities 4c-4f in general:

The easiest way I found to solve these problems was using my fingers as visual aids. I put my finger of my right hand on the table to track the horizontal coordinates (distance from B). I put my finger from my left hand in the air directly above that finger to track the vertical coordinates (distance from A). I then traced the graphs in the air with my left finger, keeping my right finger directly beneath my left finger. By noticing when each finger changes direction, you can determine where the car passes each city. For instance, if your finger in the air goes down then up while tracing the graph, this indicates the car was getting closer to City A then passed it and started getting farther away. If that happened halfway through tracing the graph, the car passed City A when it got to the midpoint of the road. If it did it towards the end of drawing the graph, the car passed City A towards the end of the road. What was hard was to do the same for City B. It was important to think about the two varying separately and then think about what would happen if you moved them together. It was in that way, that you can make sense of when A

> is increasing and B is decreasing, when both are increasing, etc. (KN, *Introduction to Covariation* Write-up).

In his description of how to think about these problems, we see his image of the situation involving tracking the two quantities with his fingers. He claimed that it was necessary to think about the two fingers moving together to “make sense of when A is increasing and B is decreasing, when they both are increasing, etc.” In his comments, KN is focusing on where A changes direction in an attempt to determine the behavior of the quantities on the specified sub-intervals of the road.

Summary of Phase I

In the introduction to graphing activities, the PSTs initially struggled with the task of keeping track of quantities that were not visually perceivable. The PSTs were quickly able to track the behavior of the quantity in question by focusing their attention on non-visible attributes of the motion (as opposed to *the motion*). The PSTs did so by physically modeling the behavior of the quantities with their fingers. Once the focus of the classroom instruction shifted explicitly from the quantities to the graph, however, the PSTs’ conceptual attention was no longer on the quantities but rather on visualizing the shape of the graph. It appeared that at some point in the progression, the PSTs’ attention shifted from the covariation of quantities to the graph that was the result of the covariation. This shift in the PSTs’ attention came when the PSTs became conscious of the fact that their activity resulted in a “graph,” something that these undergraduate mathematics majors feel as if they understand thoroughly and completely. At that point, the PSTs returned to their understanding of a function as correspondence and the points as being traced by one “finger” (Silverman, 2004b). This in no way means that they are not aware that in tracing points along the curve, there are corresponding values of a variable quantity, rather it simply means that the focus of their attention has shifted from the quantities that are varying to points that are moving in the plane. After taking part in the *Introduction to Graphing* activities, which consisted of developing a sense of a variable quantity through physically modeling (a) one quantity, (b) a second quantity, and finally (c) the two together, and the discussion surrounding these activities, the PSTs did pay attention to the way quantities varied and attempted to make sense of the implications of this variation (and ultimately the covariation).

It was apparent in the beginning of the "*Introduction to Covariation*" activities that the PSTs were attempting to focus their attention on the varying quantities, however they lacked a means for organizing the covariation. Towards the end of the introduction to covariation activities, we saw the notion of a two-dimensional landmark emerge from the PSTs' activities trying to physically model the relationships in the City A & B assignment. At the end of Phase I, the PSTs in this class have begun to develop a technique of organizing the covariation of quantities, however this method of organization is not yet a way of thinking for them. In the next section, I focus on the next phase of the course, which involved a number of instructional activities designed to further reinforce the PSTs' developing understanding of functions as covariation of quantities and to help them come to see the utility in this "new" way of thinking.

CHAPTER VII

APPLICATIONS OF COVARIATIONAL REASONING

Phase II of this study took place during the 6 class sessions that followed Activities 1 and 2. The class sessions were primarily devoted to discussing two problem sets, both of which contained a number of what appeared to be standard school-mathematics problems.

The first problem set, titled “Models and Functions,” consisted of seven word problems from a number of mathematical topics (distance-rate-time, proportionality, right triangles, etc.). The second problem set, titled “Functions and Graphs,” was comprised six of problems (not word problems). Table 7-1 below gives an overview of Phase II.

Table 7-1: Overview of Phase II

Dates	Problem Sets	# of Class Sessions
9/14, 21, 23	Problem Set #1: Models and Functions	2.5
9/16, 28, 30	Problem Set #2: Functions and Graphs	2.5

Problem Set #1: Models and Functions

Table 7-2 shows a breakdown of the class activities for Problem Set #1, which is shown in Figure 7-1. For each problem in Problem Set #1, students were to

a. draw a labeled diagram of the situation;

b. list the functions that describe how quantities covary;

c. for each function definition, state what each variable and expression represents about the situation; and

d. explain what the behavior of the function's graph shows you about dynamic aspects of the situation.

Problem Set #1

1. A community group has 2000 perimeter-feet of prefabricated wall to build the superstructure for a single-story, rectangular convention hall. The hall needs to be further subdivided by two walls through its interior so that the hall is made of three huge rectangular rooms. What outer dimensions will give the convention hall as much floor space as is possible? (See a GSP sketch of this example and a Graphing Calculator example of an explained model.)

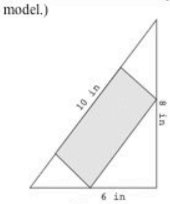


Figure 1

2. Bob drank 2/3L of water in 5/7 minute. In how many minutes will he drink 3/8L of water?

3. An A-frame barn is to be built so that it is 30 ft high, 40 ft wide and 60 ft long. A rectangular room is to be built inside the barn so that its ceiling abuts the roof. What dimensions will maximize the volume of the room?

4. You have a circle of radius 5 ft. Find the dimensions of the triangle circumscribed by this circle that has the largest possible area.

5. A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in (Figure 1). Find the dimensions of the rectangle with greatest area assuming the rectangle has two vertices on the triangle's hypotenuse and one vertex on each of the other sides, as in the figure to the right.

6. Jamie Johnson rides frequently with her father to Chicago. On one particular trip it took 2 hours for them to travel the 110 miles from home to Chicago. They made the trip in two parts. Jamie kept an eye on the speedometer and estimated that in the first part they averaged 40 miles per hour. She estimated that in the second part they averaged 60 miles per hour. About how long did they drive in each part of the trip?

7. Statistical data from trucking companies suggests that the operating cost of a certain truck (excluding driver's wages) is $12+x/6$ cents per mile when the truck travels at x miles per hour. If the driver earns $6.00 per hour, what is the most economical speed to operate the truck on a 400 mile turnpike where the minimum speed is 40 miles per hour and the maximum speed is 65 miles per hour?

Figure 7-1: Problem Set #1

Table 7-2: Overview of Problem Set #1(Activity 3)

Session	Date	Problem Set #1	Approx. Duration
5	9/14	Part 1: Introduction to Modeling – *The Lone Ranger Problem*	6 min
		Part 2: Example of Modeling – Problem 1: *The Community Building Problem*	16 min
7	9/21	Part 4: Student presentations – Problem 2: *The Drinking Problem*	43 min
8	9/23	Part 4: *The Community Building Problem* Revisited	14 min
		Part 5: *The Drinking Problem* Revisited	11 min
		Part 6: Student work on Problem 3: *The A-Frame Barn Problem*	10 min

Problem Set #1: Part 1 (The Lone Ranger Problem)

The 9/14 session began with a discussion of ongoing projects and administrative items. The rest of the time was spent on introducing *The Lone Ranger Problem* (Figure 7-2) and in a detailed discussion of the kind of reasoning behind the example solution contained in the handout (Figure 7-3). Students had not examined the sample solution prior to this discussion.

The Lone Ranger Problem

It is 1873 in Territorial New Mexico. The Lone Ranger is chasing a desperate bank robber over the desert. Both are on horseback, and the bank robber got an 11-mile head start. The bank robber's horse can run steadily at 16 miles per hour. The Lone Ranger's horse can run steadily at 19 miles per hour. The bank robber is heading for the Mexican border, for if he can cross the border he will be safe. The town that they started from is 49 miles from the border. Will the Lone Ranger catch the bank robber?

Figure 7-2: The Lone Ranger Problem

Summary of Instruction

TI began the conversation on *The Lone Ranger Problem* by focusing the students' attention on the quantities involved in the situation. An excerpt of this conversation is shown below.

Excerpt 7-1 (Session 5, 09/14/04)

1. TI: All right, so, here's an example problem [referring to the Lone Ranger problem]: Here's the setting [reads text of *The Lone Ranger Problem*]. Now it is this part here that I want you to focus on [directions (b) and (c)]. It is important to be very clear about what's varying and how the things that are varying are related to one another. … And there are some things that aren't varying. OK, what is one of the things that are not varying in the situation?

2. DH: The distance to the border.

3. TI: Well, the distance from where to the border?

4. KN: The town.

5. SS: Yeah.

6. TI: Right. The town to the border. … Right. Because their distance to the border does vary.

7. SS: Their speed doesn't differ.

8. TI: OK, they're rock solid horses.

9. SS: Well, apparently … since they're going 16 and 19 respectively.

10. TI: So we know what's not varying. What *is* varying?

11. SS: The distance from the robber to the border.

12. KN: Right, the distance to the border.

13. DH: There are two distances varying, the robber's and the Lone Ranger's.

14. TI: That's right. There *are* two distances varying.

    One of the themes of these problems is that in understanding the problem you will need to find one quantity that all of the other quantities can vary with respect to=

15. KN: =time.

16. TI: Time for what? … The amount of time it takes me to get home?

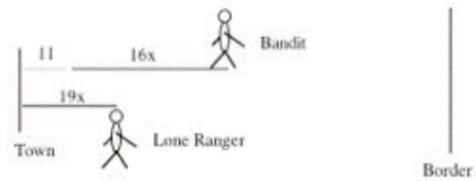
17. KN: The elapsed time.

18. TI: Well, there are actually two elapsed times. … The lone ranger started after the robber, right? …

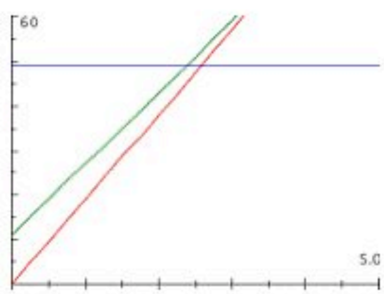
19. TI: Now, this [Figure 7-3] is written as if you wrote it. You see I have given [detailed descriptions for each of the solution requirements (a) through (d)]. So this is an expanded solution to a problem … where you're saying what everything stands for, and how to interpret … where the graph comes from and how to interpret it.

In lines 1-9, we see the students' attention on understanding the specifics of the situation – they were quick to mention not only the fixed distance between the town and the border, but also the unrealistic "rock solid" horses who can travel miles at a constant speed. In lines 10-14, TI asked the students about which quantities were varying and they correctly noted that there was more than one quantity varying: the elapsed time, the distance the robber has traveled, and the distance the Lone Ranger has traveled. With regard to elapsed time, TI pointed out that there is not a universal "elapsed time". Rather, each agent has an elapsed time. TI then oriented the discussion to the crux of approaching problems from a covariational perspective: determining which quantity to track in order to effectively produce a solution to the given problem.

A. Sketch



B. Functions that describe how the quantities vary	C. What each variable and expression represents
	x = number of hours that have passed since LR began chasing the bandit.
$f(x) = 19x$	This is the Lone Ranger's distance from town x hours after he started chasing the bank robber
$g(x) = 16x + 11$	This is the robber's distance from town x hours after the Lone Ranger started chasing him
$h(x) = 49$	This is the border's distance from town x hours after the Lone Ranger started chasing the bank robber.



D. Explanation of Graph's Behavior

- *The graphs show the robber's distance from town and the Lone Ranger's distance from town at each moment after the Lone Ranger started chasing the robber.*
- *Any vertical distance between ranger's and robber's graphs shows the distance between the two at that moment during the chase.*
- *The horizontal graph shows how far from town the border is at every moment during the chase.*

Figure 7-3: Sample Solution to Lone Ranger Problem

Discussion of Part 1 (The Lone Ranger Problem)

In his solution, TI described the independent variable x as the number of hours that have elapsed since the lone ranger began chasing the robber and **not** as an unknown. All of the other quantities being investigated were then written as functions of the independent variable. These functions represent two things simultaneously. First, they are a rule for calculating a particular value of a quantity (in this case, distances), at a particular value of the independent variable (Lone Ranger's elapsed time). Second, the function can represent a variable quantity (or a quantity which has already varied), when it is considered over the domain of all admissible values of the independent variable. It is in this way that the *Applications of Covariation* activities are related to the *Introduction to Covariation* activities: much like the "distance to A" and "distance to B" in *Cities A & B*, the covariation of the Lone Ranger's elapsed time (x) and his distance from town after chasing the robber for x hours ($f(x)$)[18] can be used to draw conclusions about the scenario and, further, to justify them.

This example extended the covariational theme, begun in *Cities A and B*, to settings that the students viewed as routine mathematics. This way of thinking included (a) determining one variable that each of the other quantities could be expressed in terms of, (b) tracking the covariation of the quantities of interest (i.e. graphing them), and (c) analyzing the resulting graph and making conclusions about the scenario.

Problem Set #1: Part 2 (The Community Building Problem)

The Community Building Problem (Figure 7-4) was discussed immediately following the Lone Ranger problem. The purpose of this discussion was to orient the students' towards identifying the variable and constant quantities and to focus on covariation.

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[18] One can also track the covariation of the elapsed time and the robber's distance from town after the lone ranger has been chasing the robber for x hours ($g(x)$) or the border's distance from town x hours after the lone ranger began chasing the robber ($h(x)$).

The Community Building Problem

A community group has 2000 perimeter-feet of prefabricated wall to build the superstructure for a single-story, rectangular convention hall. The hall needs to be further subdivided by two walls through its interior so that the hall is made of three huge rectangular rooms. What outer dimensions will give the convention hall as much floor space as is possible?

Figure 7-4: The Community Building Problem

Summary of Instruction

The discussion began with TI again asking the students about which quantities varied and which did not. The ensuing conversation is shown below in Excerpt 7-2.

Excerpt 7-2 (Session 5, 09/14/04)

1. TI: Let's try thinking about this one together [TI projects The Community Building problem at the front of the class]. OK? … So what's varying, what's staying the same? And how are things related?

2. DH: Well, the one thing that's not changing is the total amount of wall. They have 2000 pre-fabricated feet of wall.

3. TI: All right …

4. SS: It is, like, a "maximize" calculus problem.

5. DH: Umm. We need some function, like f(x) = 2000 … we have y equals, we have 2000 feet of wall, so y = 2000.

6. TI: Now we can do this using "f of x" notation, but for the moment let's just use y, all right? [TI Graphs the graph of y = 2000 (Figure 7-5)].

7. SS: And, we need some function to maximize … a function to represent the area … like length times width and solve for one of them.

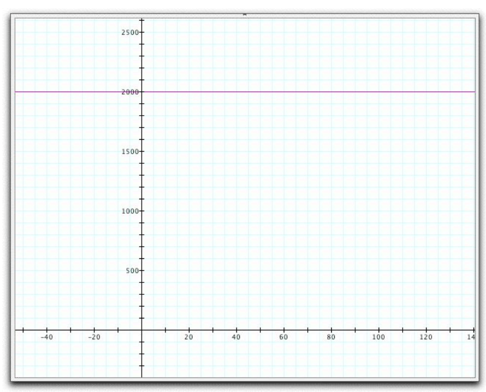

Figure 7-5: Graph of y = 2000

8. DH: So we could call the length of one side x and=

9. TI: =now let's slow down. Is there anything else that's staying the same?

Excerpt 7-2 provides insight into the students' envisioned solution method for *The Community Building Problem* as well as their situated understandings of function. In line 4, we see SS classify the problem as a "maximize" problem similar to those she encountered her calculus class. This is significant because in traditional calculus classes, maximization and minimization problems involve finding a value of x and are done without explicit attention to quantities or covariation. In line 8, her desire to "solve for one of the sides" affirms the fact that she is speaking about solving an equation and looking for an unknown as opposed to tracking covariation of quantities, the objective of the lesson. In line 5, we see DH introduce the idea of a function, though she does not appear to mean function in the spirit of the instruction in Phase I. She was using function notation to represent a particular value of the amount of pre-fabricated wall material – both y and $f(x)$ represented the constant amount of wall material available (2000 feet).

TI then moved the conversation on to the variables involved in the situation. At first, the conversation was centered on lengths of the sides and the perimeter of the community building. It was apparent that there were differing interpretations about the problem scenario: in one, DH assumed that the fixed amount of wall material implied that the perimeter of the community building would be constant; in the other, SS and DH were aware of the relationships between area and the exterior and interior walls of the community building. TI used this confusion to highlight the importance of understanding the problem and of a dynamic sketch and image of the situation. To emphasize the utility of a sketch, TI projected a Geometer's Sketchpad [GSP] sketch depicting the problem scenario on the screen at the front of the room (Figure 7-6). In the figure, each vertical and horizontal line represented a wall that needed to be created from the given amount of wall material. The dynamic GSP sketch allowed the user to drag the dot in the upper-left hand corner and adjusts the figure to meet the requirements of the problem (i.e. the community building is to remain a rectangle and the two interior walls remain parallel to the same set of the exterior walls).

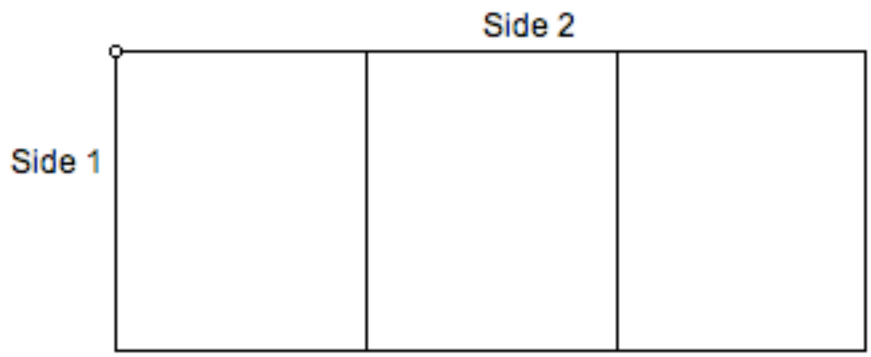

Figure 7-6: Diagram of Community Building Problem

Before dragging the dot in the corner, TI asked the students to think about what was varying and what stayed the same: "Think about changing the diagram. How could this diagram be different and still fit the problem?" The following conversation occurred in response to this question:

Excerpt 7-3 (Session 5, 09/14/04)

1. SS: You could partition it, like, horizontally, if you wanted, I guess. Well, wait, what do you mean?
2. TI: Well given that this is the way that the building is arranged, how could this diagram be different and still fit the conditions of the problem. Could it be taller?
3. All: Yeah.
4. DH: But you said given as this diagram?

TI's intent with this question was to focus the students' attention on the dynamic nature of the situation and to help them use this variability to decide a likely candidate for the independent variable. Despite the fact that throughout the previous activities (*Introduction to Graphing* and *Introduction to Covariation*) and the previous problem (*The Lone Ranger Problem*) the students were able and willing to discuss the variable quantities, in Excerpt 7-3 both DH and SS spoke as if the diagram was a static picture.

TI's goal for the discussion was to help the PSTs understand the problem as being fundamentally dynamic. This understanding would consist of recognizing (1) the lengths of the two sides of the community building, as well as the area of the community building, as variable quantities, (2) that all but one of the variables could be written in terms of an independent variable, and (3) that by understanding how the area of the building covaries with the independent variable, one can make conclusions about the solution of the problem. TI was able to guide the classroom discussion toward this goal by focusing on the variability of the variables.

Excerpt 7-4 (Session 5, 09/14/04)

1. TI: OK, so what's varying?
2. SS: The area.
3. TI: You know that the area is varying, but that's kind of like saying –
4. DH: –the length of Side 1 and the length of Side 2.
5. TI: OK, the length of Side 1 and the length of Side 2.
6. DH: Because the dividers have to be the same length as side 1.
7. TI: All right…
8. DH: So basically you have 4 Side 1's and 2 Side 2's.
9. TI: OK. … And where are those sides coming from?

10. SS: Where are they coming from? They're like the outer-thing of the wall.

11. DH: They're like coming from the 2000 perimeter.

12. TI: They're coming from the 2000 linear feet of material.

13. DH: So basically you'd have 2000 = 4x +2y.

14. TI: 4 times Side 1 plus 2 times Side 2.

15. TI: Now, as we do that, … What can you think of as the independent variable here. If you're going to model something with functions, then something's got to be an independent variable, and something's got to be a dependent variable.

16. DH: I always like to make Side 1 the independent variable.

17. TI: All right, and that's the way it works in this diagram. Because if I drag Side 1, the diagram adjusts (Figure 7-7) but if I move Side 2, it just moves the whole diagram, it moves everything [the building moves, but the walls remain the same length]. So, in this diagram, Side 1 is the independent variable. So if I change the length of Side 1, say I make it longer … What is going to happen?

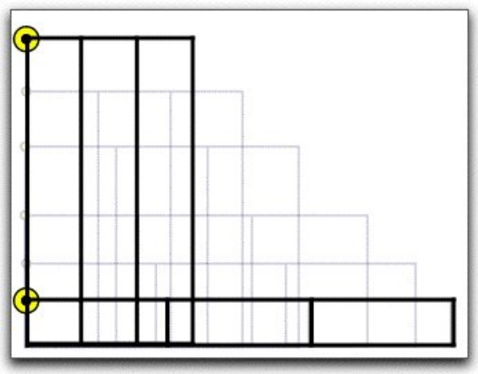

Figure 7-7: Varying One Side of the Community Building

18. SS: It is going to get taller and thinner.

19. KN: Side 2 is going to get smaller.

20. SS: Well, yeah, I mean, you're changing it [the length of Side 1], so Side 2 changes and then the figure can get taller and skinner, squat and fattier, like, that's what you're trying to figure out. Where is it going to give you the biggest area?"

In Excerpt 7-4, the class agreed on using Side 1 as the independent variable and noted that the two relationships involved were (a) four times the length of Side 1 [S1] plus two times the length of Side 2 [S2] must equal the 2000 feet of wall material available and (b) the area of the room will be equal to S1 times S2. These two relationships can be formalized to the following two functions: $y = \frac{1}{2}(2000 - 4x)$ and $A = xy$, or $A = x((2000 - 4x)/2)$ with x

representing the length of Side 1 and *y* representing the length of Side 2. In lines 18 and 19, SS and KN referred to the situation and described what would happen to the length of Side 2 as the length of Side 1 varied. In their comments, we see the students' reference to the "smooshing" of the room, indicating their awareness of the variability of the lengths of the walls. In addition, SS's comment (line 20) indicates that she had an awareness of the relationship between these lengths and the changing area of the room.

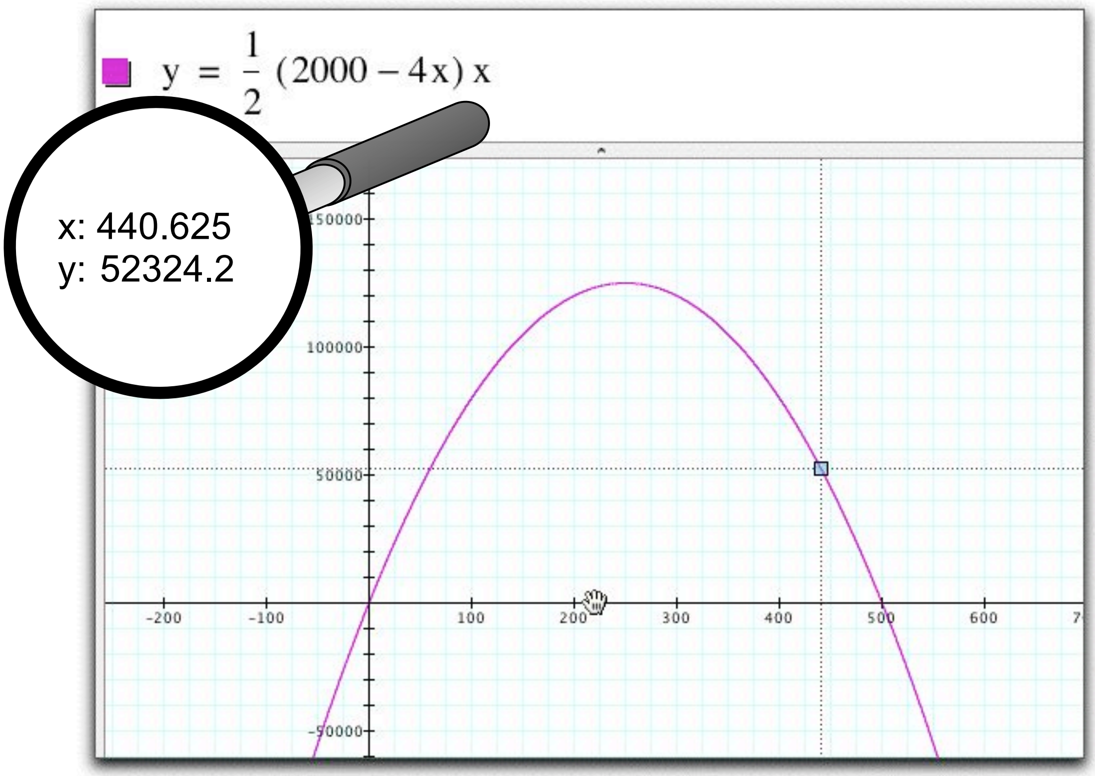

Figure 7-8: Graph of Covariation of Side 1 and Area of Community Building

TI graphed the equation $y = \frac{1}{2} \cdot (2000 - 4x) \cdot x$ on a computer graphing utility and projected the graph at the front of the room (Figure 7-8). Little time was spent on discussing the origin of this equation; it was assumed that the students understood that the two equations

$y = ½ (2000 – 4x)$ and $A = xy$ could be combined to get the equation whose graph is shown in Figure 7-8. TI envisioned that this graph would be a productive setting for the students to make connections between the previous instruction (*Introduction to Graphing*) and *The Community Building Problem*. To help make this connection, TI used the trace feature of the graphing utility (when one "clicks" on the graph, the program highlights the specified point with crosshairs and displays the *x*- and *y*- coordinates of a point that can be dragged along the curve – see crosshairs and magnifying glass in Figure 7-9). TI then orchestrated the following conversation:

Excerpt 7-5 (Session 5, 09/14/04)

1. TI: OK, let me ask you this. What does that point right there [TI clicks on a point on the graph. See Figure 7-9a] tell us? …. Can you read that? It says x is 437.5 and y is 54687.5 … What does that tell us?
2. DH: When the length of side 1 is that length, the total area is 54687.5
3. TI: All right… And now what does that point tell us. [TI clicks on a different point on the graph. See Figure 7-9b.]
4. SS: When the side is 360.625, the area has gotten a lot bigger, like all the way up to 100524.
5. TI: All right. … So and then [in each of the students' written solutions to the problems] you would say how to interpret the graph. … In terms of what the graph has to do with the problem of figuring out how to construct a building with maximum floor area.
6. TI: And again, you can just come up here [indicates solution to Lone Ranger Problem] for an example of the kind of detail that you are to go into.

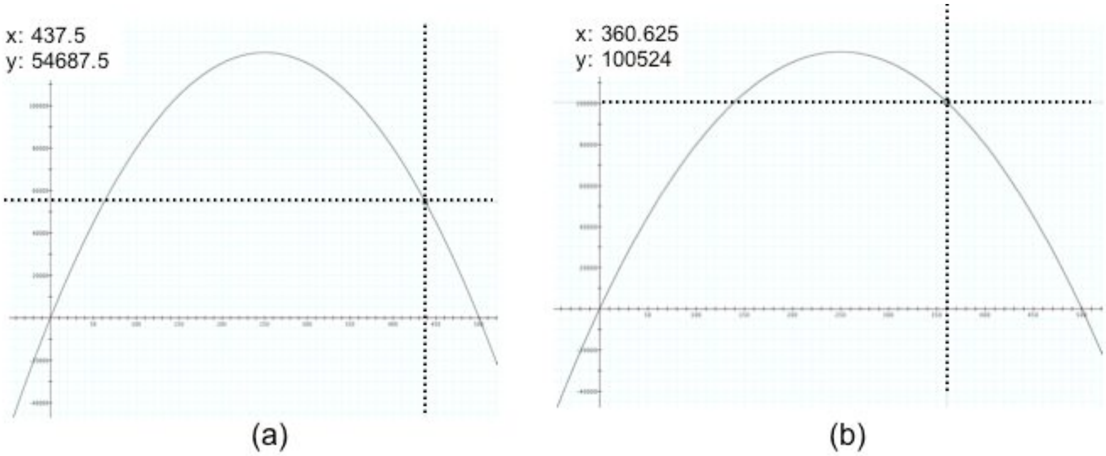

Figure 7-9 (a & b): Specific Points on Graph of Side 1 and Area

Excerpt 7-5 is of consequence because we see SS begin to speak of quantities varying: in line 4, she indicates that she is not just focused on the value of the side of the room and the area of the room, but that she is paying explicit attention to the fact that the area is changing. This observation, though trivial on the surface, indicates that a conceptual "moment of attention" on the variation of the quantities that had previously been lacking in the discussion of this problem.

TI closed this session by summarizing their work in the session and relating their work on the Community Building problem to their prior work in Activity 2 (*Introduction to Graphing*). He used the features of GSP to calculate the area of the community building Figure 7-10 and noted:

> So, here's the old finger and fairy dust. OK? Here's the length of side 1 [moves cursor over the heavy vertical line in Figure 7-10], and here's the area [indicates the area calculation in Figure 7-10]. As I change the length of side 1, the area changes. See this is the finger and fairy dust. All right. Now one of the things that you'll do once we get into geometer's sketchpad is you'll have an assignment where your assignment is to create illustrations like this to help your students understand a problem, not to solve it, but to understand it (TI, 09/14/04).

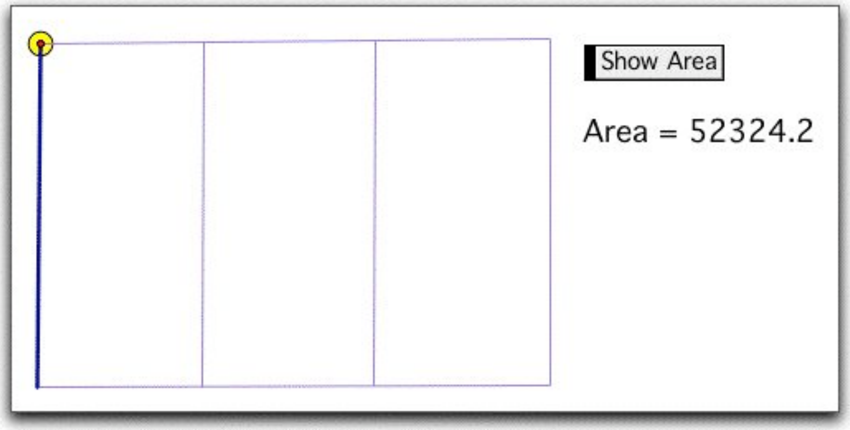

Figure 7-10: Introducing the Area in the GSP Sketch

TI concluded this class with a comment about an important theme running through the problems in Activity 3: helping the PSTs understand the problem in a way that attention to covariation can help them reason logically through a problem.

Discussion of Part 2 (The Community Building Problem).

Prior to this instruction, the PSTs had spent more than two class sessions focusing on variables and graphs as tracking the covariation of variables, however when presented with a problem to apply their understanding, they reverted back to their previous strategies. In Excerpt 7-2, despite TI's question about variables, the PSTs were quick to propose following the common procedure for "maximize" problems from algebra or calculus classes that includes identifying the function to be maximized and finding the extrema (using differentiation or a formula for the vertex). The students' comments indicate that they saw few connections (and saw little need to make connections) with the ideas of variables and covariation. This will be a recurring theme in this study.

In the same excerpt, we saw DH introduce the word function when attempting to represent the constant amount of wall material. With respect to her use of the word function, it is important to note that when functions are viewed as representing covariation of quantities, a constant function is pathological: in the case of $f(x) = 2000$, we can think of x as varying, but it is awkward to think of 2000 as varying. Thus, rather than thinking in terms of functions as a relationship between variables, it is more likely that she was thinking about representing the quantity 2000 graphically. To do so, she needed to assign some variable to be 2000 since that is how you graph a constant – you put "$y =$" or "$f(x) =$" in front of it.

Though the PSTs were quick to accept the diagram proposed by TI (Figure 7-6), Excerpt 7-3 indicates that their understanding of the diagram was different from TI's. In contrast to TI's image of the diagram as a dynamic depiction of the problem scenario, the PSTs viewed the diagram as a static drawing. In light of the PSTs' predisposition to solving equations to find an unknown, we can conjecture as to their use of the diagram: it helps them define the equations containing their unknown. Their thinking might go something like this: *First, the four vertical segments of wall are the same length and the two horizontal segments are the same length, and thus the entire amount of wall material (2000 feet) must be equal to* $4x + 2y$. *So our first equation is* $2000 = 4x + 2y$. *Second, the figure appears to be a rectangle, and since the area of a rectangle is length times width, the area of the community building (A) is given by a second equation,* $A = xy$. Given these two equations, one can then use any of the techniques of solving systems of equations (linear combination, substitution, graphing, or matrix algebra) to solve the problem. It is worthwhile to mention that this way of thinking is consistent with the way in which school mathematics texts typically present and discuss solutions to word problems. In a survey of a number of algebra texts, I found that diagrams were regularly employed as a tool to assist in the assigning of unknowns and the generating equations to be solved, not for reasoning about the problem scenario, understanding the variables at play, and formulating the relationships between the variables. Thus, the interpretation of the figure as a static drawing is further evidence that to the PSTs, *The Community Building Problem* was not yet about covariation of quantities.

Throughout the discussion, TI continually attempted to focus the PSTs' attention on the variability of the lengths of the sides. His efforts were successful midway through Excerpt 7-4 when the PSTs began to speak about how changes in the length of one side would affect the length of the other side. This is evident in the PSTs' comments that when you lengthen the vertical wall, the room "becomes tall and thin" and as you shorten the length of the vertical wall, the room "smooshes and becomes short and squat." TI used this attention to variability to relate their work on this problem with the activities in Part I. The PSTs' focus on solving equations and static diagrams brings into question if the PSTs saw the relationship between the dynamic sketch and the "fingers and fairy dust" from activity 2.

The final sentence in Excerpt 7-5 is worth discussing because it highlighted the way in which TI believed the class discussions relate to the teaching and learning of high school mathematics. A recurring theme throughout the course was that through focusing on understanding the problem (the variables and the way in which those variables covary), one could proceed through a problem and determine appropriate solution methods *in ways that make sense to them*. There is no evidence, at this point, that the PSTs had come to understand this big-picture idea.

Problem Set #1: Part 3 (The Drinking Problem)

As homework, TI assigned *The Drinking Problem* (Figure 7-11), the second problem from Problem Set #1. This problem is interesting in that it would not typically be thought of as a problem involving variable quantities – there are the "givens" and the unknown. TI's intent with this problem was to help the PSTs realize that, though they could *solve* the problem by executing a memorized procedure, they could also think of it from the perspective of covarying quantities.

The Drinking Problem

Bob drank 2/3L of water in 5/7 minute. In how many minutes will he drink 3/8L of water?

Figure 7-11: The Drinking Problem

SS's "Physics" Solution

SS prefaced her solution with the fact that it was more of a "physics" solution than a "mathematical" one. She also noted that her solution was not very clear and described it as "just multiplying it out to figure out how long it would take him to drink one liter" (SS, 9/21/04). Upon being pressed to explain what she did, she explained that her reasoning involved using a "known" to find the rate at which Bob was drinking water and then using this rate to find the unknown time it took him to drink the 3/8 L. She wrote the following solution on the board:

Drank = 2/3 L   Time = 5/7 m

Drank = 3/8 L   Time = ? m

How long to drink a L?

① $\frac{2\text{ L}}{3} \div \frac{5\text{ min}}{7} = \frac{2\text{ L}}{3} \times \frac{7}{5\text{ min}} = \frac{14}{15}\frac{\text{L}}{\text{min}}$ = liters per min

② If $\frac{3}{8}$L, $\frac{3\text{ L}}{8} \div \frac{14\text{ L}}{15\text{ min}} = \frac{3\ \cancel{\text{L}}}{8} \div \frac{15\text{ min}}{14\ \cancel{\text{L}}} = \frac{45}{112}$ min = # of minutes

Figure 7-12: SS's Solution to the Drinking Problem

Though her thought process of finding the rate at which Bob was drinking apeared insightful, her actions indicate that her understandings were about conversions of units. Rather than finding a rate and understanding its significance with regard to how one quantity changed in relation to the other, she was simply trying to find a number that had appropriate units. For example, in Figure 7-12 (calculation ①) we first see her calculation of the rate, which she described as “flipping and multiplying.” In calculation ②, she described her process as “wanting to show that liters were on the top in one and on the bottom in the other so that they would cancel.”

TI questioned SS about why she performed the calculations shown in Figure 7-12 and the following conversation ensued:

Excerpt 7-6 (Session 7, 09/21/04)

1. TI: Now, you’re trying to find a value for something [in your calculations in Figure 7-12]. So what … every number is a number of something … and every time you calculate, you’re finding a value of something. So what are you starting with, what are you doing with it, and what are you finding?

2. SS: I figured [inaudible] you wanted it down to one simple thing. Which is your rate, and so I tried to simplify it using this data here.

3. TI: And what were you going to do with that rate?

4. SS: And then apply it to the second one [calculation ②]. So that (pause) because … like … here [first line in Figure 7-12], I know all the information and here [second line in Figure 7-12] where there is an unknown … so I know they apply to each other. Because they have the same rate. So, if I can find the rate for the first one, then I can apply it to this one.

5. TI: So 2 Liters.

6. SS: 2/3 of a liter, but I wanted to show that liters was on the top, because that's what happens is like, here and here, you can cancel

7. DH: I have a good way to show that. Instead of doing the division signs, you basically make a table and then you have 2 on the top, thirds [on the bottom] and then you can cancel out on each table section.

8. SS: That would be a good way to do it.

9. TI: Well, I think these are actually small details.

10. SS: So that way you could show liters per minute… So that's like it is like very visual to show that, like, liters is on the bottom. And this is liters times minutes. Then when you figure it out, it will be liters per minute. Because its' really helpful. This is what they mean by visualization of the procedures they already know.

11. KN: Do you know that 45-112ths is the answer just because it is in terms of minutes. It seems like you're trying to find an answer, just trying to find some answer that's in minutes. Because you know that's what you're trying to find.

12. SS: Well, like, OK. I know that if I got it in minutes, then it would be the correct one. ONE SOLUTION.

We see that though SS could solve the problem, she articulated little understanding as to why she performed the calculations she did. In the excerpt, "2/3 L ÷ 5/7 min" was a calculation to be performed because she knew that the units of the rate had to be in liters per minute, not because of any significance of the constituent or resultant quantities. Similarly, she spoke of "applying" the rate because it has an unknown (line 4), again with emphasis on the units not on the meaning of the quantity. In response to KN's comment about her calculations being geared towards finding some number with the right units, not a number with a particular significance, she clarified "I know that if I got it in minutes, then it [was] correct" (line 12), but does not attempt to explain the significance of the solution in terms of what was known about the scenario. It appears that she was correct that she had approached this problem using dimensional analysis, a traditional physics technique in which which students perform operations on the given numbers in order to generate a quantity which has the desired units[19].

Discussion of SS's Solution

SS's solution was grounded in the manipulations of numbers and units in order to solve for an unknown with the correct units. Though SS could probably explain the meaning of 14/15 L/min in terms of the behavior of the quantities, this thought was not important in her solution of the problem. In line 7, we see DH's attention on manipulating units – rather than questioning the rationale for SS's calculations, she attempts to provide an additional way to keep track of the units. In line 10, SS provides a reason for her attention to dimensional analysis: "Then when you figure it out, it will be liters per minute. Because it is really helpful. This is what they mean by visualization of the procedures they already know." Thus, for SS, the important aspects of a solution method are that it helps students easily remember the steps in a procedure; that is what it meant *to understand*. To her, DH's table method was a way to help the students visualize the procedure. This stands in stark contrast with TI's intent, which was to help the PSTs visualize the

---

[19] Dimensional analysis, which involves performing the operations indicated in a formula used to solve a problem on the units (e.g. in $E = mc^2$, the units on the amount of energy in a particular amount of matter is equal to $kg \cdot (m / s)^2$, where the units of mass are in kg and the units of the speed of light are in m/s. Often, this technique is used by students as a method *to find solutions* by blindly performing operations on the given quantities, provided the operations on the units work out in to the correct unit.

variables and relationships and reason with them en route to making sense of what a solution might look like.

DH's Multiplicative Reasoning Solution

DH's approach involved using a double number line to figure out how long it would take Bob to drink 3/8 L of water. The double number line was discussed briefly in class sessions 2 and 3 (prior to Activity 1) in the context of discussing a rate schema (Thompson & Thompson, 1996; Thompson & Thompson, 1994).

Excerpt 7-7 (Session 7, 09/21/04)

1. DH: So I said, first of all, we need to find a common denominator for the number of liters we're talking about. So, we have 2/3 is the starting value and 3/8 liters is the, like, what we're looking for, basically the time it took. The common denominator for both of those is 24 and then [I found equivalent fractions with the common denominator 24] and then got 16/24 and 9/24.

From there, I was able to make a number line and broke it up into 24 parts [Figure 7-13]. So, each of these tick marks represents 1/24th of a liter and this liter represents 24-24ths.

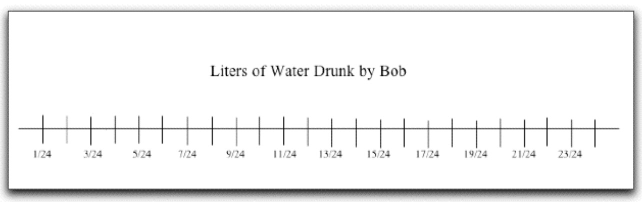

Figure 7-13: DH's First Number Line

2. TI: 24-24ths of what?

3. DH: Of, um, 1 Liter. So … I labeled this number line as liters of water drunk by Bob. And then I made another number line [Figure 7-14] right below that, and called it elapsed time and its measured in minutes.

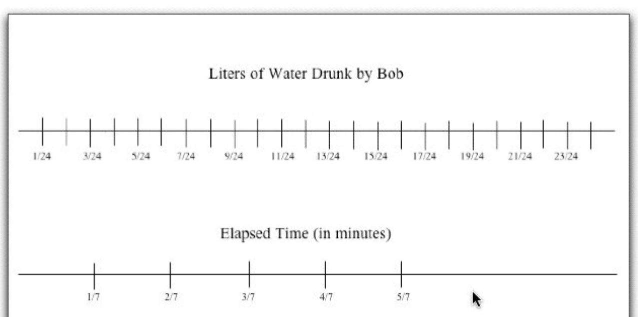

Figure 7-14: DH's Double Number Line

4. SS: So they correspond exactly to each other. Kind of like the-

5. DH: - yeah, well, we know that in 5/7 of a minute, he drank 2/3 of a liter of water and what I converted to 24ths I know that in 5/7 of a minute he drank 16-24ths. So right at this tick mark, is where 5/7 is going to be because we know that it is how much water bob drank in 5/7 of one minute (Figure 7-15).

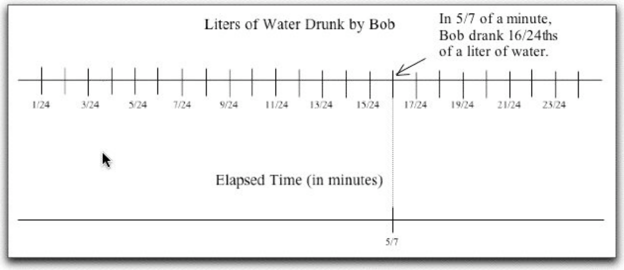

Figure 7-15: 5/7 min. on DH's Double Number Line

6. And so then, I broke [the 5/7 of a minute] up into 5 parts and I marked out each 1/7 of a minute. The next thing I said was OK, we want to find out how many liters drunk every 1/7 of a minute. And this is where I did something very similar to SS's. I said he drunk 2/3 of a liter in 5/7 of a minute, in order to find how much he drank in 1/7 of a minute, we're going to need to divide this by five - that's how many 1/7 of a liter there are in 5/7 of a liter. I got that in every 1/7th of a minute, Bob drinks 2/15 liters (Figure 7-16).

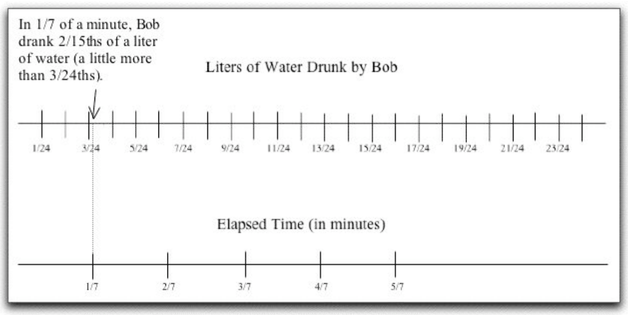

Figure 7-16: DH's Double Number Line (with 5/7 of a minute divided into 5 parts)

7. So then I was able to create an equation that says, OK, if 2/15 is drank every 1/7 of a minute, then multiplying that by 7 will get the number of liters drank in 1 minute. And then I created equations that said f(x) = 14/15 x, and this is how many liters of water bob drinks after x minutes. And our other equation, we know that f(x) = 3/8 and we know that this is how many liters of water Bob drinks. We want to find out how long it takes him to drink this much water in liters. So, I solved the two equations for each other. And I got the same answer as SS, 45/112 min, which is about equal to 0.4 minutes to drink 3/8 of a liter.

DH's solution involved finding the constant rate at which Bob drank. She set up the equation $f(x) = (14/15)x$ to represent the number of liters of water that Bob drinks after x minutes and $g(x) = 3/8$ to represent the number of liters that Bob is to drink (the specified amount we want to know how long it will take him to drink). Finally, she found the solution by solving the equation $f(x) = g(x)$ for x to yield an answer of $x = 45/112$.

Discussion of DH's Solution

DH included a diagram depicting a "double number line" in discussing her solution. While this resembled TI's earlier use of a double number line in describing a "speed schema" as a way of thinking about the concept of rate, DH did not specifically say that this was her motivation. The speed schema involved understanding that:

> speed is the quantification of motion
>
> completed motion involves two completed quantities – distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation);
>
> speed is a quantification of completed motion and is made by multiplicatively comparing distance traveled and amount of time required to go that distance;
>
> there is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go m distance units in s time units at a constant speed, then at this speed you will go $\frac{a}{b} \cdot m$ distance units in $\frac{a}{b} \cdot s$ time units. (Thompson & Thompson, 1994, p. 5).

In her solution, we see DH thinking in terms of "rates" as described in the speed schema. Of interest, however, is the method she employs to find a solution to the problem. In her approach, she solves the equation $(14/15)x = 3/8$. Her goal in solving this problem was to find an equation that contained an unknown which she could determine. DH drew her double number line along with her discussion of her solution, but her calculations could have been performed without it. She employed the double number line more as a means of recording her calculations than as a tool for reasoning about covariation "And so then, I broke [the 5/7 of a minute] up into 5 parts and I marked out each 1/7 of a minute" (line 7). Her calculation of the amount of water

Bob drank in 1/7 of a minute was grounded in the speed schema, not in reasoning with the quantities depicted by the double number line. She did not discuss how her double number line could be used to help one better understand why the equation (or more specifically, why $3/8 \div 14/15$) was an appropriate calculation. The imagery involved in understanding why this was an appropriate calculation involves a segment that intially connects 0 L and 0 minutes and that connects endpoints of segments on each so that the length of the liter segment is always 14/15 the length of the time segment. The final calculation would then involve determining how many 1/7ths of a minute there are in 3/8ths of a minute, which would tell how many 2/15ths of a liter he would drink in 3/8ths of a minute.

KN's "Graphical" Solution

KN approached the problem differently from the other two and was aware of it. Before discussing his solution, he commented on the others' solutions:

> The reason most people have problems drawing a graph is because most graphs you think of like minutes and liters [draws coordinate axis with elapsed time on the *x*-axis and amount drank on the *y*-axis]. When I did mine, I did liters and minutes [the opposite way, amount drank on the *x*-axis and elapsed time on the *y*-axis], because we know how many liters he drank and we're trying to find the time. So I had minutes be the y-coordinate. So he drank 2/3 of a liter in 5/7 of a minute. That's 14/15 liters per minute. So that means in one minute, he can drink 15/14 of a liter [writes "14/15 liters per minute = x" on the board]. So then I had my equation *f*(*x*) = 15/14 *x*. This equation told me the amount of time it would take someone to drink a given amount of water [KN draws the graph in Figure 7-17 as he explains the equation and his calculation], so then I plugged in at the liters 3/8, and got about 0.402 (KN, 9/21/04).

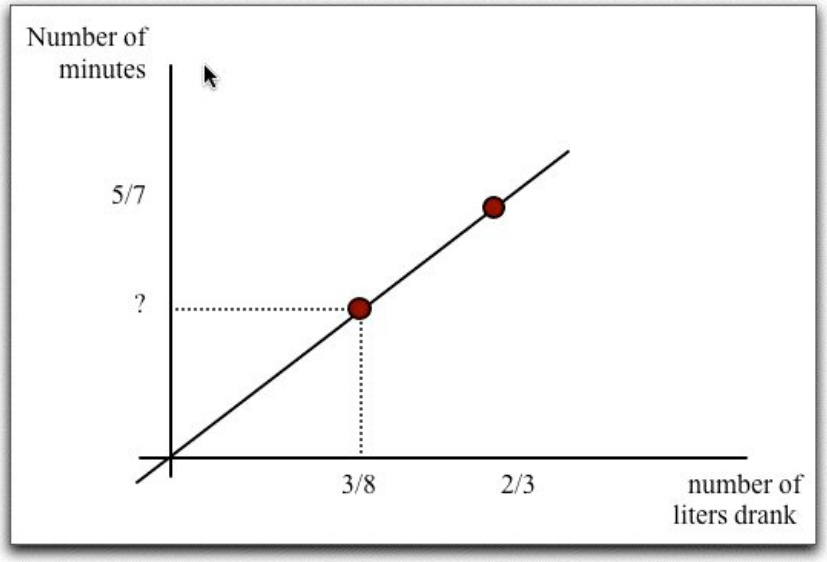

Figure 7-17: KN's Graph

The following conversation took place in response to KN's comment.

Excerpt 7-8 (Session 7, 09/21/04)

1. SS: Wait … wouldn't it be 14/15x?
2. KN: If it was liters per minute, but we are trying to … this is for any given amount of liters x.
3. SS: Yeah, but your original equation is 14/15 liters per minute. So why would it suddenly change.
4. DH: Because his x is liters, not minutes.
5. TI: Its minutes per liter.
6. KN: 14 L in 15 min. You also know it would take 15 minutes to drink 14 liters, so you kind of ask yourself, how many liters is he going to drink in 15 minutes.
7. SS: I just don't see the algebra. How'd you switch it around like that?
8. TI: Because his graph is going to be in minutes per liter.
9. SS: Yes, I understand that, but he changed, like that one equation says 14/15 of a liter per minute and then he writes 15/14 liter in a minute

10. KN: 14/15 liters in a minute so that's how many liters in one minute? So you cross multiply… [14/15 liters per min = x min /liter].

11. SS: Wait, can I please do something? [She writes 14/15 liters per minute = x min per liter → x = 15x = 14 → x = 14/15 on board]

12. TI: I don't think you meant x is 15/14ths, I think that you meant is=

13. KN: =Right, I just did that pretty quickly to explain=

14. TI: =No, not even that, just that the rate of change of time with respect to capacity is 15/14ths.

15. KN: OK to be honest, I didn't set up that ratio. I just

16. TI: You didn't?

17. SS: OK, because that was just really bothering me. Because it was like, visually thinking.

18. DH: It is 14/15ths liters per minute, so I know in one minute it has to be 15/14ths of a liter. I just kind of made that up as I was going along.

DH's solution involved treating the number of liters being drank as an independent variable and determining the amount of time it would take Bob to drink a given amount of water. He used this rate as the slope of a line that would tell him the amount of time it would take to drink some amount of water. While doing so, he drew and referred to the graph shown in Figure 7-17: he indicated the point which represented drinking 2/3 of a liter in 5/7 of a minute, the unknown amount of time it would take to drink 3/8 of a liter, and waved his hand along the line when referring to the amount of time it would take someone to drink a given amount of water.

Though a great deal of confusion arose about DH is specifying the rate of change of minutes with respect to the number of liters drunk, he was clear in his conception that they varied together and that the answer to the question amounted to answering the question, "At what moment in time will Bob have drunk 3/8 L of water when he drinks at the rate of 15/14 min/L?" Lines 13-18 indicate that when he solved this problem, he did not set up equations; he just reasoned about the quantities.

Discussion of KN's Solution.

KN exhibited a more advanced understanding of the situation than the other two students. His solution method also highlights the relationship between rates and covariation. First, his focus was determining which quantity to track in order to best think about the problem. His decision to make the amount of water drank the independent variable was quite sensible. In fact, a logical interpretation of his rationale *"because we know how many liters he drank and we're trying to find the time"* is that he was thinking in terms of covariation: he knew through what interval to vary the amount of water drunk and he needed to determine through what interval the elapsed time would vary. He then used his understanding of the way in which quantities accumulate to describe the rate at which the time changes with respect to a change in the amount of water drank. This is significant because reasoning in terms of accumulation is an indicator of one's attention to the variability in a quantity: when one thinks of a quantity accumulating, they are inherently thinking about the values of that quantity changing.

KN is correct that his interpretation of the problem, with the amount drank being recorded on the x-axis and the elapsed time on the y-axis, involved an uncommon use of time as a dependent variable. As a result, the other students are quick to suggest that the rate should be 14/15 not 15/14 and there was a long discussion about why the rate of 15/14 is acceptable. Ultimately, the problem was in the fact that SS was unable to think in terms of a rate of minutes per liter, regardless of the fact that the axes drawn on the whiteboard depicted a coordinate system with liters on the horizontal axis and time on the vertical axis. Her attention was on manipulating symbols, not on what the 15/14 min/L represented.

Follow-up Discussion on The Drinking Problem.

We have seen in the previous sections that though each of the students was able to "solve" the drinking problem, not one of the three was able to explain why it was that they were doing the calculations they were doing and what they had to do with the problem. Each of the students' initial inclinations were simply to answer the question, and it was as if they thought that putting all the details on the board for public consumption, their thinking about the problem would be self-evident.

TI closed the class session by describing what might guide his explanation of the problem and solution:

> I first would note that the students will not think about a dynamic situation. They'll just think "OK, we're told about 2/3 of a liter and 5/7 of a minute, 1 L OK, so we need to find out one thing, an amount of water. Now what do I do with those numbers to find it?" You were very clearly thinking about an amount of water varying with time, correct? … So, you were … so there's also the possibility of thinking about how much water he drank at 1/64th of a minute and 2/13ths of a minute. For him to drink 2/3 of a liter, he had to start drinking zero. So, that was one aspect that's different about presenting a solution as a student and presenting a solution as a teacher. That you have to make sure that the students are inside the big picture. What is that you imagine going on and how do your actions fit with what's going on (TI, 9/21/04).

Thus, TI's purpose in an explanation of the problem was to orient the PSTs (a) to think about the problem situation and understand how they might logically proceed and (b) to introduce how covariation can be employed to understand and solve the problem. It was believed by TI that with their background in reasoning about quantities, this example would help them as they engage with the rest of the problems in Problem Set #1.

Problem Set #1: Part 4 (The Community Building Problem Revisited)

Description of Part 4.

TI returned to *The Community Building Problem* one week after the initial discussion of *The Community Building Problem* and after the PSTs had spent time out of class thinking about *The Community Building Problem* and the rest of the problems in Problem Set #1. Though not instructed specifically to do so, they worked on the problems as a group.

TI prefaced the conversation with the following: "Now, I don't want you to, I'm going to go around and ask you to talk about what you did about the problems. I don't want you to give blow-by-blow details. Let's talk about how you thought about the problem so that it became sensible to take a particular approach" (TI, 9/23/04). In short, TI intended the discussion to be about how one might think about the problem so that the solution methods (i.e. in the case of the Community Building Problem, graphing the equation, finding the vertex of the parabola and recognizing the significance of both the *x*- and *y*-coordinate) would seem like a sensible thing for someone to want to do.

Summary of Instruction

As they had in their previous discussion of *The Community Building Problem,* the PSTs' first inclination was to draw a diagram of the situation on the board and to label the sides of the diagram (Figure 7-18). This time, though, they were focused on the variability of the sides of the building: "We decided to made side 1 and side 2 the variables. In this problem we said that they're the things that are going to be varying. So we know that they're not going to be, oh what's the word? Umm … static" (DH, 9/23/04). This comment indicates that the PSTs were conscious of the fact that the variables depicted in the diagram could vary. The conversation then shifted to tracking an independent variable and how the figure changed as the independent variable changed.

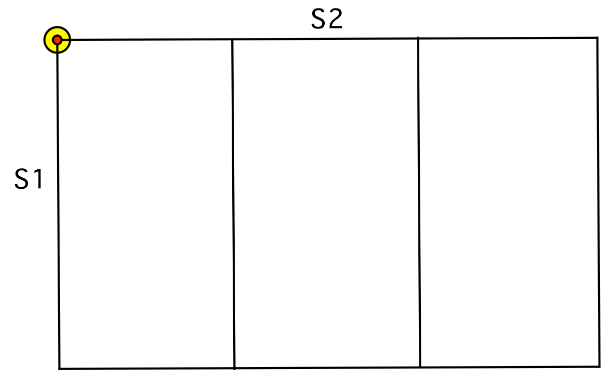

Figure 7-18: Students' Suggested Diagram

Excerpt 7-9 (Session 8, 09/23/04)

1. KN: We can think of one of the sides as the independent variable, so when you change that, it directly affects the length of side 2. … the dependent variable.
2. TI: And why?
3. KN: Because …
4. SS: [quietly] You can find all possible values? I don't know.
5. TI: No, I mean why must … Say what you said again, KN.
6. KN: Assign say side one to be the independent variable and then as you move, if you make side one longer, side 2 has to decrease… side 2 changes because the two vary together.
7. TI: All right, the two vary together, but why, if you make side 1 longer, why shouldn't side 2 get longer too?
8. KN: Because there's a fixed amount of wall that we're working with. As you move part of the wall from one side you're taking it away from the other.
9. TI: Yeah. So if you move part of the wall from this part [vertical walls], you have to take it away from that part [horizontal walls]. So that's the kind of talk I was looking for. Very qualitative. Not, "Here's what you have to do to solve it." But how can I think about this so that it makes sense with regard to the description of the setting.

In comparison to their prior reasoning about *The Community Building Problem* (Part 2), in the above excerpt, we see a shift in the way the PSTs thought about the problem. Rather than focusing on the diagram as a static source of formulas to be combined and solved, the solution entails viewing the diagram as a dynamic representation of the scenario (line 6). In addition, when questioned about *why* when one side gets larger the other side must get smaller, KN relates the behavior of the sides to the constraints of the problem (line 8). Thus, their dynamic representation embodies the way in which the lengths of the two sides of the community building covary.

TI then returned the conversation to the ultimate question regarding maximizing the area of the community building. Using the dynamic image of the room as a guide, DH proposed considering a set of values for S1 (and the corresponding set of values of S2) and thinking about the area of the room over that range. In the following excerpt, TI guides the PSTs through a qualitative description of why we know there will be a maximum area given the constraints in the amount of wall material available.

Excerpt 7-10 (Session 8, 09/23/04)

1. TI: We haven't talked about how the area varies. So how do we have to think about this to even … to even think that there might be a place where, in fact, you have the largest area?
2. DH: You have to think in the range from Side 1 being a length of zero to the range of Side 2 being a length of zero and everything in between. And as Side 1 increases, … Umm…
3. TI: Start off with Side one of length zero. What's the area?
4. DH: Zero.
5. TI: As you increase the length of Side 1, what happens to the area?
6. DH & KN: It increases?

7. TI: Will it just continue to increase forever?

8. ALL: No.

9. TI: Why not?

10. DH: Because eventually Side 2 will get close to zero.

11. TI: Right. So Side 2 will get close to zero. In fact, you can make it so that Side 2 is zero. So if its zero here, and zero there [TI indicates two points in the air] and its getting bigger at the beginning as you make Side 1 bigger, then what must it also do.

12. KN: Reach a maximum.

13. TI: It must get smaller at some point. Right? To get back to zero. So, if it gets bigger, getting larger than zero and then at some point it starts getting smaller and starts going back to zero, at some point it is as big as its going to get.

This excerpt is significant in that we see a marked change in the PSTs reasoning – we now see the PSTs reasoning in terms of the covariation of quantities over a given range. In lines 2-12, we see DH reasoning about the covariation of Side 1 and the area of the community building – focusing on Side 1 as the independent variable and on the interval from when Side 1 is zero to when Side 2 is zero. Ultimately, in line 12, KN notes that since the area will be zero initially (when Side 1 equals zero) and at some point in the future (when Side 2 equals zero), the area must reach a maximum value somewhere between when Side 1 equals zero and Side 2 equals zero (in essence, she is applying Rolle's Theorem). It is significant in that they are reasoning about the covariation of the quantities without the aide of the graph of Side 1 vs. the area of the room as in Part 2 (Figure 7-8). Rather, they are imagining what would happen as the quantities covary.

Discussion of Part 4 (The Community Building Problem Revisited)

Any discussion of this part must be prefaced by the fact that the PSTs knew how to solve this problem prior to instruction, and therefore, it is important to clarify in what ways the PSTs have developed. The PSTs' initial inclinations, as discussed early in Part 2, indicate that they did not understand how to think about this problem in a way that emphasized understanding the situation, the variables at play, and how covariation can be used as a tool for understanding the logic of standard solution methods. In short, they knew how to solve the problem, but they did not have an image of the situation that would guide them in helping others see a solution method as a sensible, logical approach to the problem. In Part 4, we saw the PSTs developing ability to visualize the covariation of quantities. It is believed that this way of thinking will enable the PSTs to have qualitatively different kinds of conversations about maximizing or minimizing with their students.

Problem Set #1: Part 5 (Bob's Drinking Problem Revisited)

Immediately following the discussion of *The Community Building Problem* in Part 4, TI revisited *The Drinking Problem*. He presented the students with the same task as he had in Part 4: "How can we think about the [*The Drinking Problem*] in a way that it will support modeling it with a function and thinking, in fact, that there is a solution?"

Summary of Instruction

TI introduced this segment of instruction with the following excerpt:

Excerpt 7-11 (Session 8, 09/23/04)

1. DH: At zero minutes, Bob has drunk zero liters of water. We know that he is drinking the water and after a period of 5/7 of a minute, he has drunk a total of 2/3 of a liter of water. So somewhere in there, at one of the points (time, amount of water drank), we know he drank 3/8 of a liter, we just don't know where along here [indicates the horizontal axis].

2. KN: And he drank the water at a constant rate.

3. TI: And he drank the water at a constant rate…

4. KN: Right.

5. TI: That's why its called a model. We're not saying he really did, but we're going to assume that he did. But what you said, though, was good. Somewhere along that, somewhere during the time that he was drinking water, in order for him to get to 2/3 of a liter, he had to have consumed 3/8 of a liter.

6. DH: Since he's drinking at a constant rate, we can use that kind of like rise and run and think how far over we'd have to run in order to rise in order to get a rise of 3/8 L and be on the line.

7. TI: Now suppose that the question was about 3/4 of a liter? Then he wouldn't have actually – if he drank 2/3 L, he wouldn't have actually gotten to 3/4 L. So then how would you think about it?

8. KN: He continues drinking at this rate, how long will it take him to-

9. TI: - right, yeah. So, if he were to continue drinking at that rate, so now you see you're making the connection very explicit between the situation, the idea of rate of change, the idea that he's accumulating water as time goes on.

In this excerpt, we see the PSTs' reasoning about relationships between variable quantities. In line 1, when DH notes that it happens "somewhere in there," she is envisioning a number of possible corresponding pairs of elapsed time and amount drunk and one of those possible pairs satisfying the given conditions. In line 2, KN notes "he drinks water at a constant rate," which was the final piece in thinking about this problem conceptually. DH follows this line of thinking by drawing an analogy between the constant rate and slope of the line - she notes that they just need to "think how far over we'd have to run in order to rise in order to get a rise of 3/8 L and be on the line" (line 6). It is also significant to note that the PSTs have begun to think about *The Drinking Problem* in a way that allows them to consider the problem of finding how long it would take for Bob to drink some amount greater than the given 2/3 of a liter. Without reasoning about quantities and proportionality, this problem tends to be significantly more difficult.

Discussion of The Drinking Problem

In Excerpt 7-11, we see a significant shift in the way in which the PSTs thinking about *The Drinking Problem*. In Part 3, none of the PSTs were able to reason through the problem. They were all were able to set up and solve equations, but their discussions in Part 3 indicated that they still struggled when trying to discuss a coherent, conceptual understanding of the solution and their method. In contrast, we now see the PSTs reasoning about the covariation of quantities, and accordingly speaking about how one might approach the problem in a way that makes sense. This way of thinking went something like this: At the start, Bob has drunk zero liters of water. Bob drinks water at a constant rate and can drink 2/3 of a liter in 5/7 of a minute. Since this is a constant rate, for any fractional part of 5/7 of a minute, he will drink the same fractional part of 2/3 of a liter. Therefore, if we know how much of a fraction of 2/3 L 3/8 L is, it will take him the same fractional part of 5/7 of a minute to drink 3/8 L of water. It is this way of thinking that both allows a solution method to emerge from understanding the problem, and, more importantly, gives meaning to the calculations that the PSTs were performing in Part 2.

It is also worthwhile to note that reasoning about quantities in this way also allows one to consider the problem of determining how much time it would take if Bob were to drink some amount greater than the given 2/3 of a liter. KN noted that the reasoning for this problem would be very similar: "if he continues drinking at that rate, how long would it take him to drink a specified amount of water." This realization indicates that the PSTs are beginning to develop a Key Developmental Understanding of function: through envisioning the quantities that are covarying according to a specific relationship and through understanding that relationship, KN found reasoning through a related problem fairly easy and straightforward.

Also in this excerpt, we see the first occurrence of the PSTs describing the relationships between the quantities at play and how those relationships give insight towards how and why one might solve the problem without significant intervention by TI.

Problem Set #1: Part 6 (The A-Frame Barn Problem)

Description of Part 5

The A-Frame Barn Problem was the final problem discussed in class. This problem, at its core, is very similar to *The Community Building Problem*, however there are additional intricacies that needed to be dealt with in order to specify a relationship between one variable quantity and the volume of the room. The text of the problem is shown in Figure 7-19.

The A-Frame Barn Problem

An A-frame barn is to be built so that it is 30 ft high, 40 ft wide and 60 ft long. A rectangular room is to be built inside the barn so that its ceiling abuts the roof. What dimensions will maximize the volume of the room?

Figure 7-19: The A-Frame Barn Problem

In this problem, students are presented with the dimensions of the structure (Figure 7-20) and they need to figure out how this information relates to the dimensions of the room.

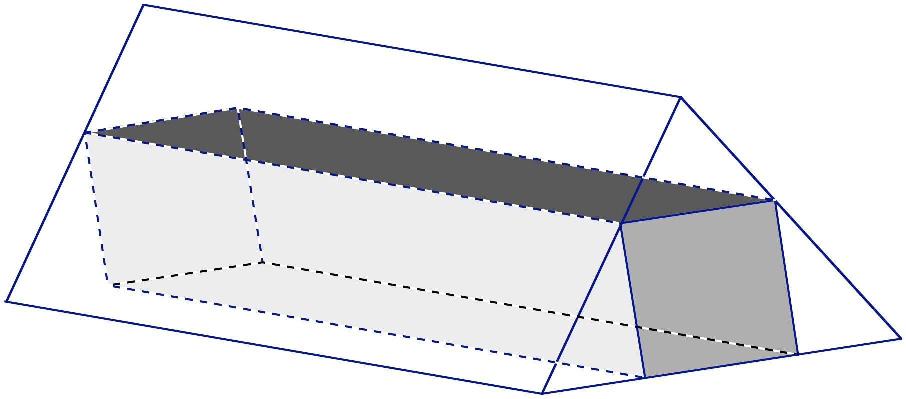

Figure 7-20: The A-Frame Barn

Summary of Instruction

The discussion of *The A-Frame Barn Problem* began with SS drawing a figure similar to Figure 7-20 on the board to be sure that she "understood where everything was." She then presented her method, which involved the reduction of the number of variables: "So I tried to reduce it to two variables. I was trying to look at one side, like a 2-D side. Like a triangle side and then maximizing the intersection" (SS, 9/23/04). Her reasoning was an attempt to reduce the number of variables in the problem, and ultimately construct a graph that would track the two varying quantities. Unfortunately, the "variable" she had removed from the scenario was a constant (60 feet) and not a variable (Figure 7-21). Despite this fact, her comment was, in essence, a suggestion to look at the face of the barn, which was an important step in the PSTs' discussion of the problem.

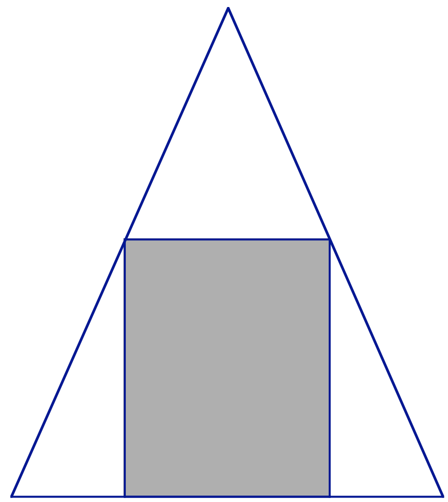

Figure 7-21: SS's "Reduction of Variables"

TI suggested using SS's diagram, but pushed the PSTs to think about the variables in the problem: "OK, remember, we're going to try to vary one thing, but what is it that we want to vary as a result of varying that one thing?" The following excerpt details the conversation that followed:

Excerpt 7-12 (Session 8, 09/23/04)

1. SS: Wait, is this just like that other problem where it is like … like the height could be zero and the area would be zero and then later, the width could be zero and the area would be zero again. And the problem is, like, realizing that there's a maximum somewhere in the middle [SS moves her fingers in air, tracing the area varying with one finger along the desk and one in the air – Figure 7-22]. That's where the solution is going to be [indicates the top of the parabola].

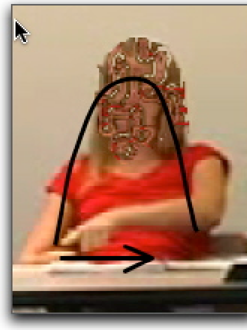

Figure 7-22: SS tracing her finger in the air.

2. TI: OK … Yeah so the height of the room could be zero, the width of the room could be zero. Increasing the width of the room from 0, the area will increase, but then for it to get back to zero, it will have to, at some point, start decreasing. Now, what's varying. What variables can we think of as varying that will, in fact, vary the area of that room's face.

3. SS: The width and the length of the room.

4. TI: The width and the length. OK, now can we vary one of those.

5. DH: By varying one of them, the other, by default varies.

6. TI: In what way?

7. DH: Well, if you make the width smaller, the height is going to have to get taller.

8. TI: Why?

9. SS: Because it has to abut the roof=

10. TI: =right, because of the constraints of the problem.

11. SS: You can't have the width stay at a length of 20 or something and the height go out the ceiling.

12. TI: Right. It is supposed to be a rectangular face and if they didn't vary together, it couldn't stay together. So, if I move this point then, in fact, I get the two varying together [PT moves "corner" of room – yellow point in Figure 7-23]. Now what is it … OK now in terms of developing a model, what about that point [the yellow point in Figure 7-23] are we going to … we have to quantify it. What about that point are we going to track?

In this excerpt, we see the first evidence of the PSTs relating their understanding of one problem to a different problem. SS was correct when she noted that this problem was very similar to *The Community Building Problem* (line 1). The similarity she noticed was not in the surface characteristics of the problems (that they were both about maximizing rooms) – she recognized the similarity in the way the variables covary: "like the height could be zero and the area would be zero and then later, the width could be zero and the area would be zero again. And the problem is, like, realizing that there's a maximum somewhere in the middle" (line 1). SS's explanation and gestures indicate that her reasoning about covariation gave insight into a solution and solution method for the problem – though she did not give an answer for the problem, her reasoning clearly indicated the direction one might go in order to find a solution.

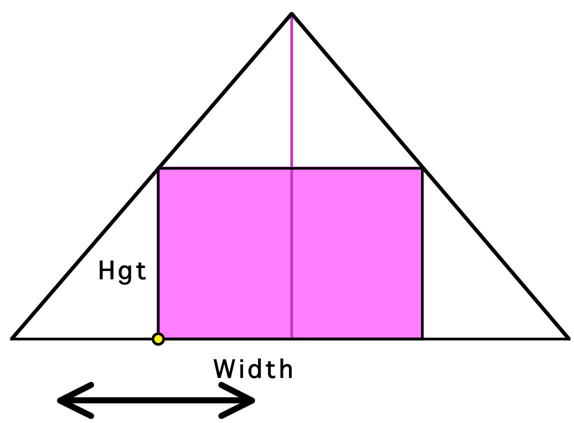

Figure 7-23: GSP Sketch of The A-Frame Barn

In the remainder of Excerpt 7-12, we see the PSTs reasoning about the variables and relationships between variables in the problem. With little guidance from TI, the PSTs realized that the relationships between the variables arise as a result of the constraints of the problem (lines 5-11). At the end of, TI initiated a conversation that served to transition from a discussion of qualitative features of the situation to a mathematical model. The PSTs chose to define that the height of the room would be a variable (y in Figure 7-24). They had trouble deciding whether to define the second variable as the width of the room or half of the width of room. From the video data, it is not clear why the PSTs chose to define the second variable (x in Figure 7-24) as half of the width of the room, but as this simplifies the next step, TI accepted this definition. Eventually, a diagram similar to Figure 7-24b was agreed upon as an acceptable diagram of the situation. Despite having this resource, the students were still unsure of how to proceed.

After a few minutes of silence, TI proposed using similar triangles, and almost in unison, the students responded "Oh!" With that missing piece of information, the PSTs were able to propose a way to model *The A-Frame Barn Problem* with functions:

Excerpt 7-13 (Session 8, 09/23/04)

DH: So we need to set up a ratio and then we can use that ratio to express one of the variables in terms of the other. And then the area, which is length times width could be expressed as a function of just one variable.

In the quote, DH is proposing using the property of similar triangles that the lengths of corresponding sides of similar triangles are always in the same proportion. Mathematically, this would be expressed as $\frac{30}{x} = \frac{y}{20 - x}$, where y represents the height of the room and x represents half of the width of the room. The height of the room could then be expressed as a function of the width: $y = \frac{60 - 30x}{20} = \frac{6 - 3x}{2}$, and the area of the room could then be expressed as a function of only one variable: $A(x) = 2x \cdot y = 2x \cdot \frac{6 - 3x}{2} = 6x - 3x^2$.

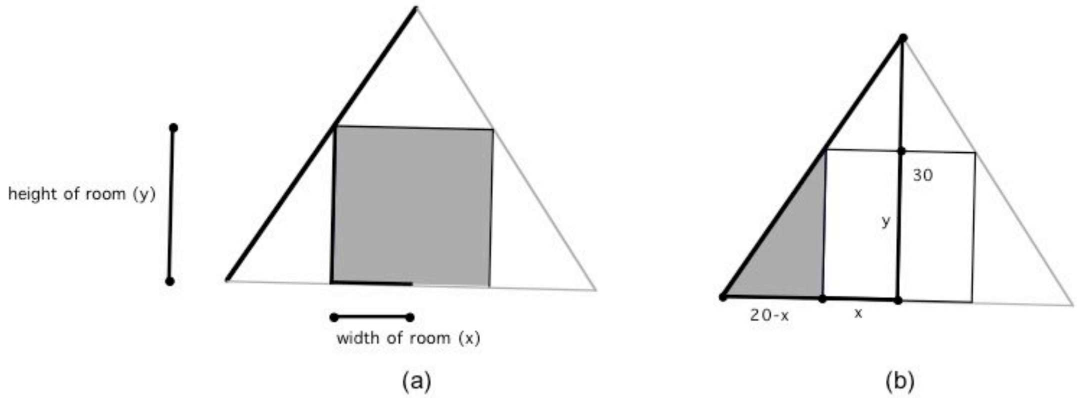

Figure 7-24: Variables in the A-Frame Barn Problem

TI then stopped the conversation before the PSTs generated an equation for area of the room in terms of the width of the room. The activities concluded with the following excerpt:

Excerpt 7-14 (Session 8, 09/23/04)

1. TI: Now, with this one, there were some technicalities … given that if we know half the width of the room, then we can calculate the height. So that took some … it was detail work. But what's the crux of solving this problem?

2. DH: Reducing it down to a number of variables that we can deal with and trying to think about how what you want varies as a function of something you know.

3. TI: Right, Imagining how its going to work, what varies, and then trying to think of how you can think of that varying as a function of one thing varying.

Discussion of The A-Frame Barn Problem

In their discussion of *The A-Frame Barn Problem*, we see the PSTs focusing on the variables in the scenario in a largely independent manner. With only "little nudges" by TI, we see the PSTs thinking about the variables at play, figuring out a way of expressing the relationships so that the quantity of interest can covary with a single quantity, and generating a function from which a solution could be derived (through minimal calculations like taking the derivative or averaging the roots).

The most significant comment that must be made about the PSTs' work on *The A-Frame Barn Problem*, was encapsulated by DH's comments in Excerpt 7-13 and Excerpt 7-14. Her comments indicate that she was developing an image of the utility of covariation in making sense of how to think about applied mathematics problems.

Additional Problem from Problem Set #1

There were four additional problems from Problem Set #1 that were not discussed in class. These problems consisted of two geometry problems, a distance-rate-time problem, and a cost-minimizing problem (see Figure 7-1 for the entire problem set). As a result of a number of problems regarding the PSTs' work as data, the student work on these problems will not be analyzed in depth. Regardless, the PSTs' work does shed light on how prevalent the ways of thinking discussed in the previous sections are (albeit, possibly anecdotally).

SS's problem write-ups can be described as defining unknowns, writing equations, and solving for unknowns. Her explanations consisted of trying to explain clearly each of the steps of her solution. In addition, for each of her solutions, she attempted to relate the answer she had found to the graph of the equations she had used. The explanation of this relationship consisted of comments like "graph the possibilities and see where the minimum value lies," and "Graphing the lengths of the sides in this problem showed that there was an obvious maximum value. At this maximum value, there were the values for y and x." Her solutions do not hinge on understanding quantities and the relationships between quantities, but rather solving for unknowns and relating the now known maximum or minimum value to the graph of the quantity being maximized or minimized.

DH generated figures and tables in order to answer the questions. Her graphs were not dynamic (they were generated using computer drawing tools) and as a result served as more of a means of justifying her solutions than as a tool for reasoning about the variables at play and the problem. She often noted the fact that she was attempting to focus on the covariation, but that she lacked a means of inscription for the covariation. As an example of this, she used tables to express function values for two different functions of the same independent variable. She then added the function values together and plotted (by drawing circles on a computer drawing application) different values of the sum of the two values. With regards to these tables, DH explained that she was aware of the covariation, but she just proceeded anyway:

> And when I was doing this problem, since that's just so natural, I didn't even think about it. So essentially I was just looking at two points, because it is just, because I just assume I know that there's going to be a connection between these two points. So I probably wasn't thinking of it as anything other than the two static points (DH, Problem Set #1 Write-Up).

Thus, though DH was not reasoning in terms of covariation of quantities, she understood that covariation was working in the background.

Though tables of values may, in some cases, indicate covariational reasoning, there is no evidence that DH was thinking in terms of variable quantities. Another interpretation is that she was looking for a value of the sum, which was the greatest, not imagining running through the independent variable and keeping track of the dependent variable, which was the sum of two functions. Much like SS, the majority of DH's text and inscriptions focused on justifying answers.

KN's covariational reasoning was by far the most advanced of the students. He was the only student to speak of the variability of the independent variable over an interval and the resulting variability of the dependent variable, though he only did so on one problem. He described his function as representing "The cost of the trip for any given *x*. … Since the speed limit is between 40 mph and 65 mph, we know we can only focus on that part of the graph. … Since between 40 mph and 65 mph, we see the cost decreasing and then increasing, we can know that the minimum cost will be around 65 mph."

Summary of Problem Set #1

There are a number of issues that emerged from the analysis of the data from problem set #1. First, in each of the problems, with the exception of the final problem attempted in class, the PSTs' initial inclination with respect to these problems was to generate equations to be solved. Despite the fact that they had success in understanding the relationships between varying quantities and answering fairly technical questions regarding the way the quantities in Chapter 6 (*Cities A & B)* co-varied, it was not until the final problem that they even considered the fact that one could think about the quantities at play in order to better understand the problem. Further, the notion of *how one might approach the problem* was not apparent in the conversations.

The PSTs' written solutions to the problems further highlight this issue. Though the PSTs were eventually able to reason through the problems discussed in class, when it came to their write-ups of the problems discussed in class (as well as those that were not discussed in class), the salient characteristics of their solutions remained "finding unknowns". It was as if on each problem, the PSTs needed to start from the beginning with labeling unknowns and solving equations. At this point, the data indicates that the PSTs needed an external "nudge" by TI to think about the situations presented in terms of covariation of quantities.

This is a theme that will recur throughout the discussion of the next set of activities: when engaging with each problem, the PSTs tended to begin with this approach. Once they were reminded that *variables vary* (What are the quantities that are varying in this situation? What happens as each of them vary?) and to *slow down* (variables vary a little at a time), they were able to reason through the problems. This is one of the interesting emergent themes in this study: We have seen evidence that the PSTs can reason about the situations in terms of simultaneous covariation of quantities; what is of interest is the fact that they tended not to.

Problem Set #2: Functions and their Graphs

The in-class activities devoted to Problem Set #2 were very similar in form to that of those in Problem Set #1. TI introduced the problems by discussing the "big ideas" of how one might think about problems of this sort and reminded the PSTs to remember that "variables vary" and to "slow down" and then the PSTs went to work on the problems. The problems in Problem Set #2, however, differed significantly from Problem Set #1. In Problem Set #2, the problems were more mathematically abstract and did not involve modeling of real world phenomena. Rather they involved problems where the mathematics itself was to be a context within which to investigate the covariation of quantities.

Table 7-3 describes the class sessions devoted to discussion of problems from this set. In addition to the in-class activities listed in the table, I will discuss the PSTs' written solutions to additional problems at the end of this section. The full text of Problem Set #2 is shown below in Figure 7-25.

Table 7-3: Breakdown of Activity 4: Problem Set #2

Session	Date	Activity	Approx. Duration
9	9/28	Part 1: Types of Explanations: Explaining the behavior of $f(x) = \sin x + 0.01\lvert\sin(100x)\rvert$	41 min
		Part 2: Problem # 1: Families of Polynomial Functions (The Quadratic).	18 min
		Part 2: Problem # 1(cont'd): Families of Polynomial Functions (The Cubic)	6 min
10	9/30	Part 3: Problem #2: Intro to "Mod" functions	14 min
		Part 4: Student work on Problem 2: $f(x) = x^2 \bmod 2$	13 min
		Part 5: Student work on Problem 2: $f(x) = x^3 \bmod 2$	6 min

Problem Set #2

In each of the following, recall that *a good explanation is one that provides a strong sense of* <u>*why*</u> *things work as they do*.

1. Explain the behavior of the families of functions in (a), (b), and (c) so that an explanation of why the functions in (a) behave as the do for varying values of *n* is the basis for why the functions in (b) and (c) behave as they do for varying values of *n*.

   a. $f(x) = x^2 + nx$
   b. $g(x) = x^3 + nx$
   c. $f(x) = \cos x + nx$

   It will be useful to think these functions as sums of functions when trying to describe their behavior and to explain why they behave as they do.

2. We normally think that *b* and *a* in "*a* mod *b*" stand for whole numbers. 27 mod 3 is 0, because 27 ÷ 3 has remainder 0. 27 mod 5 is 2, because 27 ÷ 5 has a remainder of 2. But we can think generalize this idea to fractions and irrational numbers, too. The definition of "*b* mod *a*" that does this is:

   (*a* mod *b*) is the <u>remainder obtained</u> when subtracting *mb* from *a*, where *m* is the largest integer less than or equal to $\frac{a}{b}$.

   By this definition, (6.5 mod 2.1) = 0.2, since 3 is the greatest integer less than or equal to $\frac{6.5}{2.1}$, and 6.5 – (3)(2.1) = 0.2. Similarly, (6.5 mod -2.1) = –1.9 because -4 is the largest integer less than or equal to $\frac{6.5}{-2.1}$ and 6.5 – (-4)(-2.1) = –1.9 (you should prove this for yourself).

   Given that *a* mod *b* is defined as above, for each function in the following list (a) describe its behavior and (b) *explain* its behavior.

$f_1(x) = x^2 \bmod 2$ [entered as $y = \mathrm{mod}(x^2, 2)$]
$f_2(x) = x + (x^2 \bmod 2)$ [entered as $y = x + \mathrm{mod}(x^2, 2)$]
$f_3(x) = x^3 \bmod 2$ [entered as $y = \mathrm{mod}(x^3, 2)$]
$f_4(x) = x^2 \bmod x$ [entered as $y = \mathrm{mod}(x^2, x)$]
$f_5(x) = x^2 \bmod \cos(x)$ [entered as $y = \mathrm{mod}(x^2, \cos(x))$]

Figure 7-25: Excerpt from Problem Set #2

Problem Set #2: Part 1 (Exploring the behavior of $f(x) = \sin x + 0.01|\sin(100x)|$ *)*

TI proposed the task of explaining the behavior of the function $f(x) = \sin x + 0.01|\sin(100x)|$ (Figure 7-26). This problem is traditionally difficult for students for two reasons. First, the graph (Figure 7-26) that appears, at first glance, to be a typical sine graph, is not (the callout in Figure 7-26 shows some of the complexity of this function). Second, the rules for dilations and contractions of periodic functions that permeate trigonometry instruction in traditional school mathematics[20] are not easily applicable to this problem. For these reasons, TI envisioned this problem as further problematizing the PSTs' understanding of functions. This problem also served to highlight the fact that understanding functions does not happen as a result of memorizing properties or formulas; more important is a way of thinking about functions from which the properties and formulas emerge.

The task was posed in the context of discussing the nature of good or poor explanations in mathematics. TI noted that there are two broad classifications of mathematical explanations[21]. A Type I explanation provides descriptions without discussing the reasoning behind what was done. It fails to give us a sense of *why* things work the way they do. A Type II explanation describes both how a function behaves and also why it behaves the way it does. This discussion was another attempt to shed light on the utility of covariation – explanations in terms of how one variable varies with respect to a second variable are inherently a Type II explanation.

Throughout the entire discussion, a graph of the function was projected at the front of the room (Figure 7-26).

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[20] For example, in $f(x) = \sin bx$, the (b) term will affect period. The period of the function is determined by the formula $P = 2\pi/b$.

[21] TI discusses the nature of Type I and Type II explanations at the following web-page: http://pat-thompson.net/MTED2800/Explanations/ExplanationAnalBad.htm

Summary of Instruction

In response to TI's request to explain the behavior of the function, the following interchange took place:

Excerpt 7-15 (Session 9, 09/28/04)

1. DH: Well, it is bumpy and not curvy because it is absolute value, so every value that's negative one in the negative graph is going to be one.
2. TI: It is reflected up, all right.
3. SS: Yeah, the absolute value has to do something with that.

KN: That makes sense.

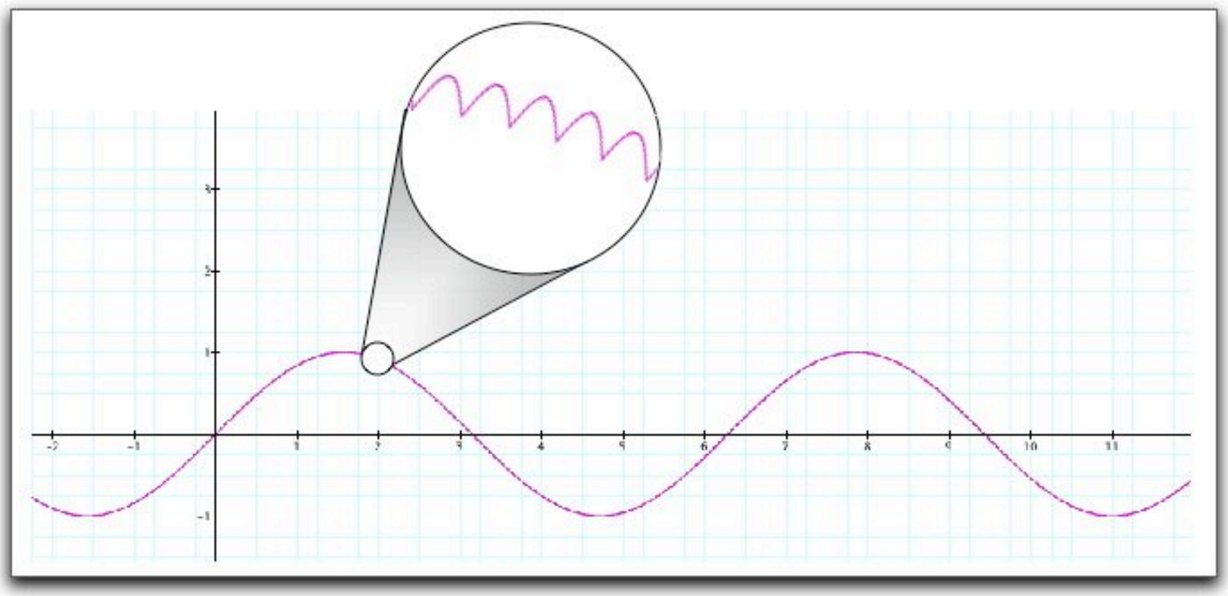


Figure 7-26: Graph of f(x) = sin(x) + 0.01|sin(100x)|

In this brief interchange, we see that the PSTs' initial inclination was to describe broad global characteristics of the function, as it had been in their initial attempts at explaining *The Community Building Problem* and *The Drinking Problem.* Initially, the PSTs did not see this problem as being about covariation.

In an effort to shift the conversation to the covariation of quantities, TI projected a sample Type II explanation at the front of the classroom (Figure 7-27). TI believed the figure and the explanation could serve as a didactic object for shifting the PSTs' attention to the way the quantities varied. TI then initiated the conversation shown in Excerpt 7-16.

The graph of 0.*01abs(sin(100x))* is as shown below (when blown up over a small interval). As (100*x*) varies through an interval of length pi, abs(sin(100*x*)) varies from 0 to 1 to 0. This means that abs(sin(100*x*)) varies from 0 to 1 to 0 as *x* varies through intervals of length pi/100.

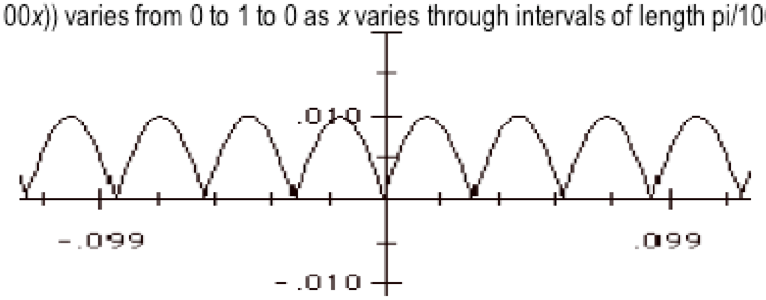


Figure 7-27: Part of a "Good" Explanation of f(x) = sin(x) + 0.01|sin(100x)|

Excerpt 7-16 (Session 9, 09/28/94)

1. TI: OK, now what about this business of starting with 100x being negative 2π? Why is it 100x instead of x?
2. KN: It seems like they already figured out the interval was length π/100. And then plugged 100x in instead of x.
3. SS: Does it have something to do with the amplitude and the period?
4. TI: Perhaps. What's the period of sine?
5. KN: 2π.
6. TI: And what's that with respect to? What has to vary by 2π for sine to repeat?
7. DH: What has to vary by 2π? The graph has to vary by 2π, right?

8. TI: Yeah. So that's the idea that it is periodic, isn't it? If you say the period is 2π then that means that–

9. DH: –the graph goes through one cycle.

10. TI: It begins repeating itself every interval- at the end of every interval of 2π. Now, what's varying? What is it that varies for the, for sine to have a period of 2π?

11. DH: x?

12. SS: Do you want to say x or theta or something like that?

In this excerpt we see the PSTs' struggling to make sense of periodic functions in terms of covariation of quantities. In line 2, we see KN relating TI's question to the formula for calculating the period described previously ($P = \frac{2\pi}{b}$), though KN either made a calculational error (the period is actually π/50) or he was considering a two-period length interval. SS's comment about the amplitude and period (line 4), the two common "buzz-words" in trigonometry further verifies the claim that the students initially referred to formulas for answers about the behavior of the function.

In the remainder of the excerpt, the students appear not to understand TI's questions about the specifics of periodic functions. Though they understood the definition of, and how to find, the period and the amplitude, in line 7, DH was unsure how to answer TI's question ("What has to vary by 2π in order for the function to be periodic?). While TI's focus was on the quantities that would result in the observed periodicity, DH was focused on periodicity as a characteristic of the graph (lines 6 and 8) – not of the covariation of quantities. At the end of the excerpt, we see both DH and SS guessing that it is an independent variable that varies by 2π. Both the intonation of DH's comment (line 10) and the wording of SS's comment (line 11) indicate that they did not necessarily understand why this was significant.

TI recognized that the PSTs were struggling and asked them to try to explain their difficulties.

Excerpt 7-17 (Session 9, 09/28/04)

1. TI: OK. What's hard about this?
2. DH: I'm not used to thinking about it this way.
3. TI: OK, that's true, but we've done a lot of stuff that you weren't used to.
4. DH: That wasn't easy either. [Smiles]. But with these problems, I thought I knew about trig functions like sine and cosine. But the way you're asking, I really don't know how to answer.
5. SS: Like I used to know, we used to have in other classes, we'd have an equation like you knew part of the equation meant what. Like … the period of the function and the number in front of x, we'd know how many cycles. And I guess I just can't totally compare that to what we're doing.

Excerpt 7-17 sheds some important light on the PSTs struggle to answer TI's questions. The excerpt indicates that TI had succeeded at problematizing trigonometry for DH and SS. In particular, we see SS noting an incongruity between her understanding of trigonometry and the way TI was pushing her to think. TI believed this to be a catalyst for developing deeper, more conceptual understandings of functions.

In an attempt to help the PSTs relate periodicity to their understanding of covariation, TI chose to focus the conversation on the relationship between the quantity being tracked on the coordinate axis (values of x) and the *argument* of the sine function ($100x$ in $f(x) = \sin x + 0.01|\sin(100x)|$). TI believed that the PSTs must understand two specific ideas about how the quantities vary in this situation to make sense of the graph. First, the PSTs needed to keep track of two quantities, x, which was recorded along the horizontal axis, and $100x$, the argument of the sine function. The crux of this situation to TI was that the PSTs come to realize that as x varies, $100x$ varies by 100 times that amount. Second, the PSTs needed to understand that, with respect to trigonometric functions, the function values repeat themselves whenever the *argument* of the trigonometric function varies by 2π.

Excerpt 7-18 (Session 8, 09/24/04)

1. TI: Let's say we have sine of something. [Writes sin ( ) on the board.] What about this? You know that's going to have a period of 2π, right. And what's going to vary by 2π?

2. KN: x?

3. TI: How about if we say whatever goes in there [indicates inside the parentheses with an arrow]? If whatever is in here … if this [points to parentheses] changes by 2π, then sine is going to start repeating itself, right?

   [DH shakes head, indicating she is not following].

4. TI: No, DH?

5. DH: I just don't understand what you're saying.

6. TI: Sine … OK, what kinds of things do you need to put in there so we can actually evaluate sine? … A number. (pause) That's the only thing you can put in there to actually evaluate sine. So, if I put, if I write u [places a U above the parentheses] what am I, isn't it taken for granted that u stands for a number? If I write, 20u [writes 20u], isn't it taken for granted that 20u stands for a number? So when this number varies by 2π … Here's a number [pointing to 20u]. How much does that have to vary for sine to repeat itself?

7. KN: Well, we just said it has to vary by 2π.

8. TI: That's right. But it is 20u that has to vary by 2π, not u. Suppose that I write 157y [inside the parentheses]. How much does 157y have to vary by in order for sine to repeat itself?

9. DH: 2π.

10. TI: 2π. How much does y have to vary so that sine repeats itself?

11. DH: 157 over 2π … or

12. KN: Oh, now I see where the π/100 comes from…it is the other way around.

13. DH: Oh, right … 2π/157.

14. TI: Yeah. So now, look back here [to the original problem]. … So you have 100x being -2π … that's like saying start at u being -2π, or 157y or something. It is just the thing you're going to evaluate sine at – make that -2π. OK, now, in principle, you call that the argument of the function, whatever you evaluate the function at, that's its argument. So, in that case, what is the argument of sine?

15. DH: 157y.

16. TI: And what's the argument for sine in this example?

17. DH: 100x.

18. TI: All right, so how much does 100x have to change by in order for [sin(100x)] to start repeating itself?

19. DH: 2π.

20. TI: So, then x has to change by how much?

21. DH: π/50.

This excerpt clearly shows the students making connections between the graph and the equation – in the end DH had reasoned through the covariation and deduced that the period of the function would be π/50. Rather than relying on formulas, she did so by focusing on understanding the way in which the quantities covaried.

TI continued this line of discussion, but shifted his purpose to help the PSTs develop an image of how a multiplier in the argument affects the period of the function. To accomplish this goal, he had the PSTs physically act out the three quantities at play in order to assist the PSTs in developing imagery that will assist them in making sense of why the graph behaves as it does. This activity relied on physical modeling of quantities (as was introduced in Chapter 6 and recurred throughout the activities of Chapter 7). In this case, the modeling consisted of one PST tracking x with his finger moving horizontally along the desk. A second PST tracked $100x$ with their fingers in a similar manner. Finally, the final PST was responsible for tracking $|\sin(100x)|$

vertically. As a final exercise, the teacher noted that the graph of $f(x) = |\sin(100x)|$ could be generated by keeping track of the values of x with the horizontal finger and sin $(100x)$ with the coordinated vertical finger (Figure 7-28). The following excerpt details how the PSTs came to think about the behavior of the function.

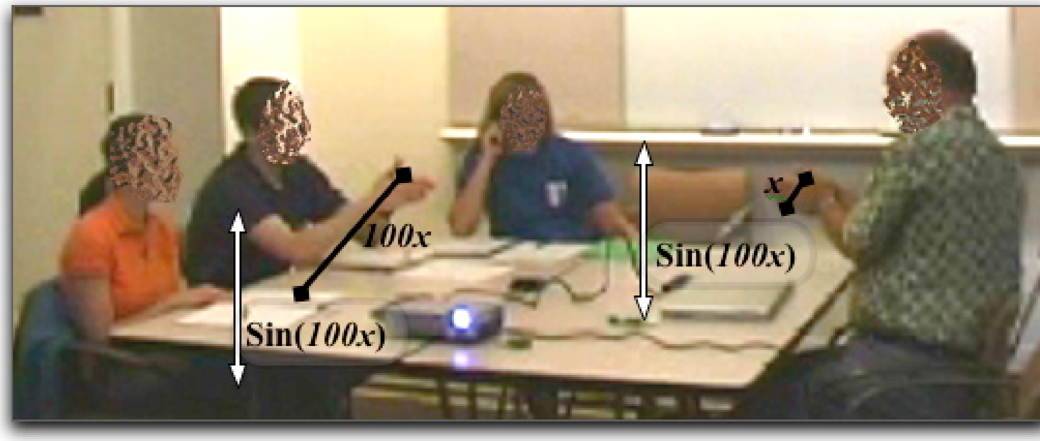


Figure 7-28: Modeling the Behavior in Part 2

Excerpt 7-19 (Session 9, 09/28/04)

1. TI: All right so let me ask this, suppose I move an inch. How much do you [KN, modeling 100x] have to move?
2. KN: 100 inches.
3. TI: All right, so as I move [my finger – representing x], you move yours [representing 100x] and you two, DH & SS, think about what sine is doing. So, if this distance [indicates a distance of approximately one foot] is π, then what's it going to do when KN moves that far? What's sine going to do?
4. DH: Do you mean when 100x moves that far?
5. TI: Yes, he's got 100x. So … when he goes from zero to π.
6. DH: It is going to go through half a cycle.
7. TI: It is going to go through half a cycle. Right. Now, DH and SS, remember, you're keeping track of sin 100x in relation to my value of x. All right? Now, I move a teeny tiny bit. OK, now what has sine done, with regards to his argument?

8. DH: It has gone up and down a lot of times … times.

9. TI: OK, so let's go back to our function $f(x) = \sin x + 0.01|\sin(100x)|$. You tell me. … What are you adding to sin x?

10. SS: Tiny, tiny numbers, between zero and 0.01 and back to zero.

11. TI: Tiny numbers.

12. DH: So for the large negative numbers, the tiny numbers are going to make the points on the graph slightly less negative – you're adding small positive numbers to the sine.

13. TI: Right.

14. DH: So you'll get a version of the sine curve with bumps … OK. And the bumps will bounce back [off of the sine curve] when … every π/50?

15. SS: Like every π/50, [the graph of sine] has little peaks added to it. So it is still doing the sine, but its got a little bit extra, like you're adding the absolute value of sine to it [Figure 7-29].

16. TI: Does this help you imagine what's going on? (pause) Remember, graphs are all about covariation.

17. DH: Finally, it clicked. … So as x varies every π/50, what's being added to the sine starts at zero, gets up to 0.01, and then back to zero.

Figure 7-29: SS Demonstrating the Absolute Value of Sine

In this excerpt, we see the PSTs' developing a sense of the relationship between the independent variable (x) to values of the argument of the sine function ($100x$). This sense was developed through the physical modeling of the quantities, which was helpful in that the PSTs began to develop an image of the sine function as being a function of an argument that is not necessarily the independent variable. The physical modeling of the quantity "the argument" allowed the PSTs to see both what the sine function was being evaluated at and how this quantity related to the independent variable.

In Excerpt 7-19, we also see DH presenting a conceptual understanding of why the graph behaves as it does. Her explanations rely on the notion of a landmark. In line 16, we see DH organizing the covariation by sub-dividing an interval of x-values into sub-intervals on which there is predictable behavior of $|0.01 \sin(100x)|$. She then describes the behavior of $f(x)$ on these sub-intervals.

Discussion of Activity 4, Part 1.

The PSTs' work on this problem highlights the fact that students with formal preparation in mathematics and who ostensibly understand particular mathematical content still may struggle to understand the content in a way that supports conceptual discussions. When the PSTs first encountered the problem, their initial inclination was to describe the shape of the graph (the "bumpyness") and to refer to the formulas that they believed described the salient features of the graph.

When pushed to reason in terms of the quantities, the PSTs experienced significant difficulty. One way of explaining their struggle, which is consistent with KN's comment regarding the period or "interval" (Excerpt 7-16, line 7), is that they envisioned a periodic function as a graph. They were aware of the fact that this graph has certain characteristics (for example, the distance from one "peak" to the next "peak" is the period, which can be calculated by the formula discussed previously). What is clear about this conceptualization of a periodic function is that there is nothing varying – one can run his or her finger along the graph and trace out the points on that graph, but it is the graph that has permanence. The graph is not generated by the covariation of quantities, it simply *is*. When one understands a periodic function in this way, there is no need to concern oneself with the quantities which vary and which are represented on the coordinate axes. This understanding is consistent with the focus of traditional trigonometry instruction, which involves calculating the period, amplitude, and phase-shift of the graph of a trigonometric function – not the period, amplitude, and phase-shift of the function that relates an independent variable to the dependent variable.

When TI pushed the PSTs to think about and explain the difficulties they had with this problem, they noted the differences between what they thought trigonometry was and what TI was asking them to think about (Excerpt 7-17). The excerpt indicates that the PSTs have two competing understandings of function – the school math (characterized by the memorization of both classes of graphs organized by physical characteristics and formulas that describe characteristics of the graphs) and the covariation – that they are yet to reconcile. The initial discussion (Excerpt 7-15 and Excerpt 7-16) demonstrates the fact that at this point, the school-math understanding is primary for the PSTs.

There are two possible reasons that the school-math understanding was primary. First, the PSTs have spent 4 or more years studying mathematics from this perspective and therefore it is only natural for them to try to explain the function describing broad characteristics of the graph and using formulas to specify what they thought to be the salient aspects of the function. Second, as was evidenced in the class discussions, thinking in terms of covariation requires one to imagine the variation of a third quantity ($100x$, in this example), which is not directly evident in the graph. Much like their work in *Cities A & B* (Activity 2), the PSTs needed to pay attention to quantities that were not visually perceptible. TI helped the PSTs develop an image of how the quantities related, having them model the variation of $100x$, and this image enabled the PSTs to begin to describe how the quantities covaried. It is important to mention that despite the fact that the students did not have an image that would support their explanations, their initial inclination was not to develop one. Rather, they returned to describing the shape of the graph. This result further indicates the fact that the PSTs' understandings of function are compartmentalized. They understood that they could use covariation in order to make sense of word-problems (Activity 3), however, that way of thinking did not permeate their initial work described in this section.

Again, the PSTs needed a "nudge" from TI before they considered turning to covariation as an explanatory mechanism.

Finally, at the end of this section, both SS and DH (Excerpt 7-19, lines 10-17) provided a Type II explanation of why the graph behaves as it does that is couched in the language of covariation and in understanding the behavior of each part of the function independently. Thus, this section highlights the fact that standard mathematics problems can serve as didactic objects that are conceived of as helping students develop an understanding that will support the ability to take part in conceptual conversation. Ultimately, the goal of this study is to understand if this knowledge can support PSTs' engagement in such a conversation.

Problem Set #2: Part 2 (Families of Polynomial Functions)

The first problem from Problem Set #2 that was discussed in class dealt with families of polynomial functions. TI anticipated that this problem would provide an occasion for the PSTs to further explore functions as covariation and learn to apply that understanding. In addition, TI anticipated that the discussions would provide an environment within which the PSTs could further develop their understanding of functions as a KDU by highlighting the implications of understanding functions as covariation. He anticipated that the PSTs would look at the family of graphs generated by $f(x) = x^2 + nx$ as if a single graph was physically pushed so that its vertex followed a path in the plane. He also anticipated that the PSTs would see the family of graphs generated by $f(x) = x^3 + nx$ as if a single graph is systematically bent and twisted. Neither explanation is mathematical, and both explanations are rooted in viewing graphs as primary objects of experience instead of as emerging from quantities' covariation. When viewed as primary objects, the graphs generated by the two families of functions (Figure 7-30) cannot be seen as behaving according to one underlying principle. When viewed covariationally, the two families' graphs can be seen as behaving identically.

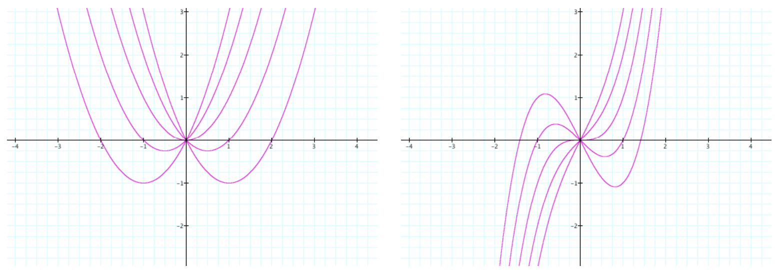


Figure 7-30: Families of graphs from f(x) = x^2+nx and from g(x) = x^3+nx

The text of *The Families of Polynomials Problem* is shown in Figure 7-31.

Families of Functions

Explain the behavior of the families of functions in (a) $f(x) = x^2 + nx$ and (b) $g(x) = x^3 + nx$ so that your explanation of why the functions in (a) behaves as they do for varying values of n is the basis for explaining why the functions in (b) behave as they do for varying values of n.

Figure 7-31: Problem #1 – Families of Polynomial Functions

In the previous session (class session 8), as an introduction to Problem Set #2, TI provided two "hints" to guide the PSTs as they considered *Families of Polynomial Functions*. First, he suggested that the PSTs think of the function $f(x) = x^2 + nx$ as the sum of two functions ($f(x) = x^2$ and $g(x) = nx$). Second, he provided dynamic animation, using Graphing Calculator[22] to assist the PSTs in exploring the behavior of the family of functions. Using the graphing software, students can create animations by specifying through what interval n is to vary and the increment

[22] Graphing Calculator is a powerful graphing package developed by Pacific Technology. It is available on the web at www.pacifict.com.

of variation in each frame. Figure 7-32 is meant to illustrate the dynamic animation of $f(x)$, with each of the graphs as frames of the animation (to imagine the animation, consider each of the frames displayed in rapid succession).

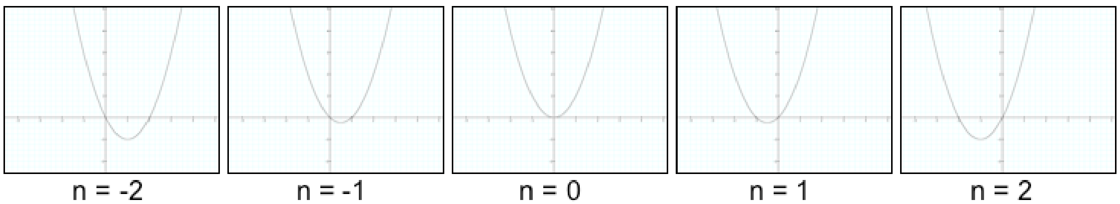


Figure 7-32: Graphs of f(x) = x^2 + nx for n = -2, -1, 0 , 1, 2

The Quadratic

Despite the fact that the PSTs had looked at this problem for homework, they were confused by the problem. DH felt the difficulty she was experiencing with the problem did not have to do with following TI's hint and considering the function as the sum of two smaller functions:

> Well, it wasn't that. I could think of it as sums of functions, but it was hard to *describe* it – there are so many things to describe in the graph. Say … say like where the vertex is in the different graphs, but then how does that explain it? … I just don't know (DH, 09/28/04).

SS experienced a similar frustration, indicating that she could describe how the vertex "jumped" from the fourth to the third quadrants, but thought that that was not the kind of explanation TI was looking for. As has been mentioned numerous times, the PSTs' initial inclination was to describe global characteristics of the graph (i.e. the jumping vertex) and not to turn to covariation as a means to explain the behavior which results in the graph.

TI's goal in the next segment was to shift the students' attention to the quantities varying. Before they could do so, they needed to be able to visualize $x^2 + nx$ as a variable quantity created by taking the sum of x^2 and nx for every real number x. TI's envisioned trajectory consisted of helping the PSTs develop an image of the $x^2 + nx$ as consisting of the sum of two "sub-functions," and that by understanding the behavior of the sub-functions and how the two are composed to get f, the students could explain the behavior of f in terms of covariation. As with the previous problem, we have two intermediate results that provide insight into the behavior of f, but do not show up directly in the graph of f.

TI began by generating graphs of the two sub-functions of f, $f(x) = x^2$ and $g(x) = nx$ (Figure 7-33). The graphing utility automatically defines n to be a parameter that can be varied (via a "slider" which can be seen at the bottom center of Figure 7-33) independently of the variables x and $f(x)$. The conversation began with the students explaining what was meant by TI's hint to think of the functions as a sum. The following excerpt describes the ensuing conversation.

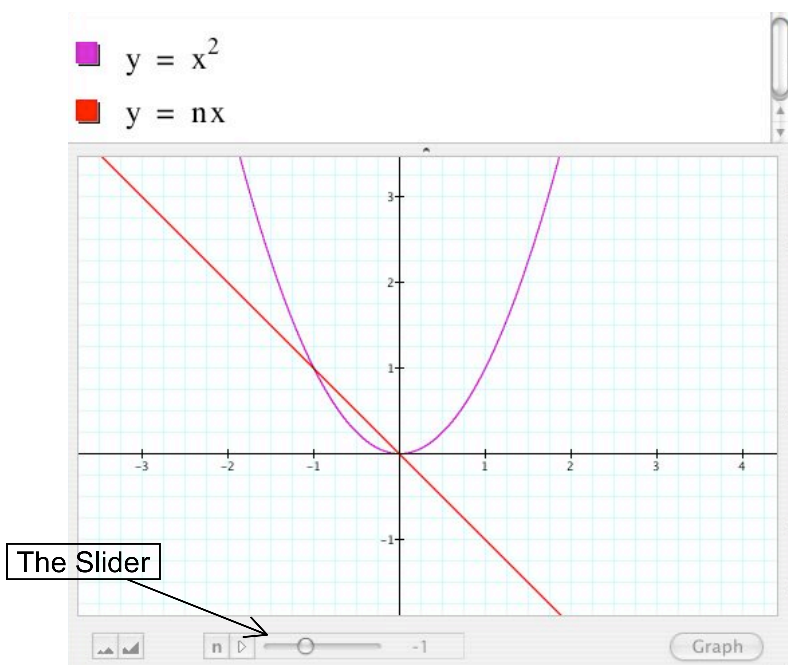


Figure 7-33: Graphs of f(x) = x^2 and g(x) = nx for n = –1

Excerpt 7-20 (Session 9, 09/28/04)

1. TI: How would we show … how would you see the sum of those two functions on that graph?

2. DH: You'd pick an x-value and find the y-value for both x^2 and nx. [pause] I mean in this case it would be y = -x [because n = -1]. Add those two values together and make a new point.

3. TI: Right, and then go up or down that far. OK, imagine that. Right here… [TI points at the x-axis at approximately x = -1.5; Figure 7-34] Where's the sum?

4. KN: It is going to be above both of them.

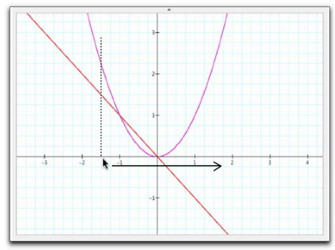


Figure 7-34: PSTs imagining the sum at x = -1.5 and as x varies

5. TI: Right, now tell me what's going to happen, kind of trace out the value of the sum as I move the pointer [TI moves pointer from x = -1.5; Figure 7-34].

6. KN: It's getting smaller and smaller.

7. DH: And there [she indicates between -1 and -0.5] it is going to be between the two of them.

8. TI: Is it going to be between the two?

9. KN: At -1, it is going to be double.

10. TI: At -1 its going to be at 2.

11. DH: Oh, oh, yeah, adding two positives.

12. KN: Right, they're both positive. So it is coming down, it is still above the blue line.

13. TI: And now where is it [TI points to a very small negative number]?

14. DH: It is almost at zero.

15. [DH moves her fingers in a coordinated manner as TI moves the pointer; Figure 7-35]

Figure 7-35: DH Imagining the Behavior of f(x)

16. TI: It is almost at zero [just to the left of x = 0]. Now [at x = 0]?

17. DH: It is at zero.

18. TI: Now [at approximately 0.25] where is it?

19. SS: It will be kind of around zero there too.

20. DH: But it will be more negative than it will be positive.

21. KN: Yeah, the line goes down more rapidly than the parabola goes up.

22. All: [Indicate affirmative].

23. TI: So, it is going to be below.

24. DH: So as soon as the distance between the x-axis and y=x and the y = nx are equal, that's where it will have a sum of zero. Instead of it being more negative than it is positive, it changes to being more positive than it is negative. And somewhere in there, in the middle there-

25. TI: - OK. So, and that happens right here, right [TI indicates x = 1]. So where will the vertex be?

26. KN: To the left of one. Between zero and 1. If you go to x = 1, y = x^2 = 1 and y = -x equals -1, so if you add those together, you'll get the point (1,0). So somewhere in-between, there must be a point where the sum of the two is at a minimum.

27. SS: Since the vertex will be between the two roots.

28. TI: So you get … so the vertex is somewhere between 0 and 1.

In the excerpt, we see the PSTs predict the location of the vertex of f for n = -1 [henceforth $f_{-1}(x)$]. They are able to do so with the assistance of TI who helps the PSTs focus their attention on understanding the behavior of the two sub-functions. He then instructed them to imagine the value of the sum for particular values of x (line 3) and moved the pointer along the *x-axis* and asked the PSTs to imagine the sum as x varied (lines 5-17). In line 15, we see DH spontaneously invoking "fingers and fairy dust" to help her imagine the result of the covariation. In lines 20 and 21, we see DH and KN conclude that the values of f_{-1} will be negative to the right of $x = 0$ because "it will be more negative than it will be positive" (line 20). Similarly, DH concluded that at $x = 1$, $f_{-1}(x)$ would equal zero because "that's where it [the two sub-functions] will have a sum of zero" (line 23). Finally, DH and SS use reasoning similar to Rolle's Theorem to conclude that the vertex will be between $x = 0$ and $x = 1$.

TI then moved the classroom discussion on to $f_{-0.5}(x)$, and walked the PSTs through a similar discussion. The salient aspects of the discussion included identifying the values of x for which $f_{-0.5}(x) = 0$ (when x^2 and $-0.5x$ are both positive and when $f_n(x) = -f_n(x)$[23]) and dividing x-axis into three regions with cut-points at the zeroes of $f_{-0.5}(x)$ (Figure 7-36). As was the case with $n = -1$, on the left-most interval, when both sub-functions are positive, $f_{-0.5}$ is also positive. On the middle interval, between the zeroes, $-0.5x$ is more negative than x^2 and therefore $f_{-0.5}$ is negative. Finally, on the right-most interval, x^2 is more positive than $-0.5x$ and therefore $f_{-0.5}$ is positive. The PSTs were able to appropriately identify the behavior of $f_{-0.5}$ as x varied through each of the three intervals.

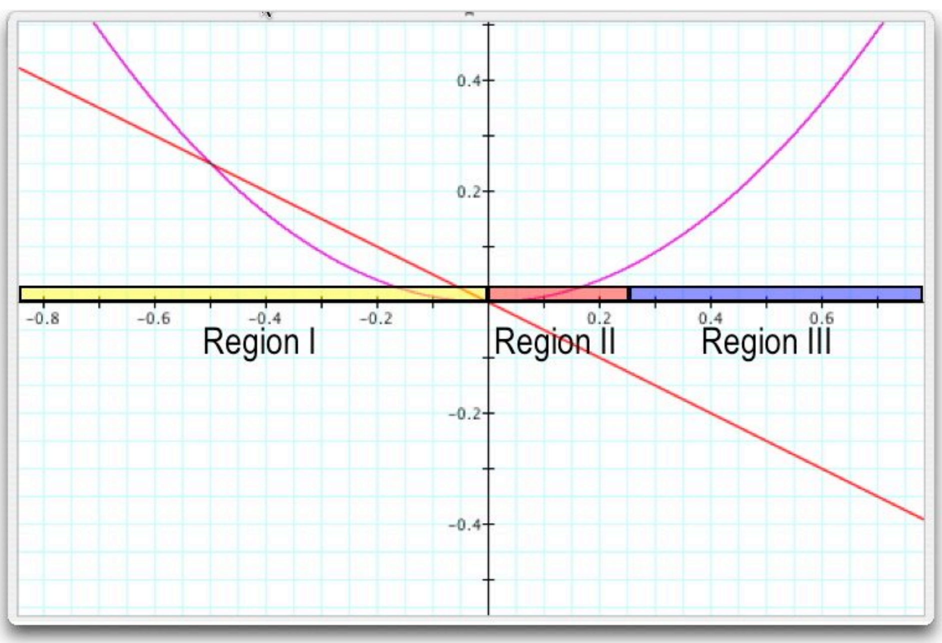


Figure 7-36: Graph of $y = x^2$ and $y = -0.5x$

[23] We would technically need four intervals, including when both sub-functions are negative. However, in this particular case, x^2 is never negative.

In the midst of the conversation about the behavior of $f_{-0.5}$, TI initiated a shift from a conversation about the zeroes and the vertex of $f_{-0.5}$ to a conversation about the location of the vertices of the family of functions f_n.

Excerpt 7-21 (Session 9, 09/28/04)

1. KN: Somewhere around there it looks like it is going to start coming back up.
2. TI: Right about there it looks like its going to be=
3. SS: =oh, close to the axis.
4. KN: So … somewhere between 0 and [x = -0.5, the second zero of f -0.5], is the vertex – its going to reach a minimum because as x gets bigger, the x^2 increases more while the -0.5x keeps decreasing the same. And then after here [x = -0.5], it starts going up more rapidly.
5. TI: So compare these two. Where was the vertex here [moves slider to n=-1.0]?
6. KN: Between zero and one.
7. TI: Between zero and one. Somewhere about here, right? What about the vertex here [moves slider to n=-0.5]?
8. SS: Farther to the left.
9. TI: Why?
10. DH: Well, we know that the second line is decreasing slower than the first line. So it'll take less time for the x^2 to overtake the line.
11. TI: Right. So if you compare the two, the vertex of this one [n = -0.5] is farther to the left.
12. DH: And higher.
13. TI: Right, it is still below the axis, but somewhere closer to the axis. It is somewhere close to the line, correct?
14. All: [Agreement].
15. TI: Down here [with n = 0.5], where is the vertex in this one in relation to the others?

16. KN: In the top one, the vertex is farther along the x axis

17. TI: So it is farther to the left.

18. KN: And it is also up – because the shorter line wins out, the farther up it'll go.

19. TI: So as we go from -1 to -1/2, the vertex goes from down here to up here [indicating locations with finger]. And as we go from -1/2 to 1/2, the vertex goes from here to here. OK. Are you guys starting to get a sense for why, how this family of functions behaves?

In line 1, we see DH locating the vertex by imagining the behavior of $f_{-0.5}$ by considering the sum of the two constituent sub-functions. Her explanation is guided by the idea of locating a landmark that will occur when the positive values of x^2 become larger than the negative values of $-0.5x$ (line 4). TI then shifts the conversation to comparing the behavior of the family of functions. It is in this context that we begin to see covariational reasoning emerge. For example, in line 10, DH notes that it will take less time for x^2 to overtake $-0.5x$ than $-x$.

Discussion of Activity 2, Part 2 – The Quadratic

At the beginning of the discussion of the *Families of Functions* discussion, we saw DH confused about what it meant to "explain" the behavior of the families of functions: "there are so many things to describe in the graph" (DH, 09/28/04). In both DH's and SS's comments, we saw them focusing on the graph and describing what they saw: when viewing the animation, they saw the vertex jumping and the rest of the points moving accordingly. It seems sensible to ask "aside from the jumping of the vertex, what else could I describe?" It should be noted that descriptions of this type are Type I explanations, explanations that do not give any reasoning beyond describing what happens. Thus, this section further verifies the claim describing functions in terms of the covariation of quantities is not the default way of thinking for these PSTs.

TI was able to "nudge" the PSTs towards reasoning about quantities as a means for explaining the behavior of the function. In Excerpt 7-20, we saw a transition from the PSTs answering TI's questions about particular points on f to them imagining the behavior of f by reasoning about the covariation in the constituent sub-functions. By doing so, the PSTs were able to identify the two zeroes of the f as x-values where f went from increasing to decreasing as a result of the values of the sub-functions attaining equal but opposite values. The students then used the information about the roots of the f to determine the approximate location of the vertex. This location of the vertex was grounded in pointwise analysis of the function.

In Excerpt 7-21, we see the beginning of the PSTs' shift to describing the behavior of f_n over an interval. In describing the behavior of the vertex as n varies from -1 to 0/5, we see the PSTs reasoning about how the values of f_n vary throughout the three regions shown in Figure 7-36 and how that can be used to better approximate the location of the vertex. Prior to the excerpt, the PSTs had decided that the landmarks occur when the two sub-functions are equal, but opposite (or zero), and that the behavior of the functions on the first and third regions was predictable. To fully explain the function, they focused their attention on the behavior of the function on Region II (Figure 7-36). DH's explanation of the relationship of the location of the vertices in f_{-1} and $f_{-0.5}$ is particularly insightful: "… we know that the second line is decreasing slower than the first line. So, it'll take less time for x^2 to overtake the line." I interpret this utterance as indicating her focus on the covariation of quantities: by taking less time, DH is imagining an interval of x and thinking about how much the values of x^2, $-0.5x$, and $-x$ will change on that interval. In line 19, TI believes that the PSTs are able to generalize from the situation: as the coefficient of x (the parameter) gets closer to zero, it will take less time for x^2 to overtake the linear function and therefore the vertex of f_n will be closer to the origin. A similar argument was used to explain why, as n approaches zero, the vertex of f_n approaches $y = 0$: "Because the shorter line wins out, the farther up it'll go" (line 18). In this case, since the landmark (the vertex) gets closer to $x = 0$, the linear function has decreased less on the interval from $x = 0$ to the x-coordinate of the vertex. Since it has decreased less before the quadratic wins, the vertex will be closer to the x-axis.

As a result of TI's nudging, the PSTs took part in conceptual conversations about the behavior of the families of polynomial functions. In particular, the students had developed important understandings of *why* quadratic polynomials – polynomials of the form $p(x) = ax^2 + bx + c$ – behave as they do. This is a topic that a traditional high school mathematics curriculum devotes a significant amount of time to, but one that is often reduced to the quadratic formula and finding points via the "vertex formula."

As with a number of instructional tasks in the course, the crux of this problem was that understanding the behavior of complicated functions can be aided by reasoning through sub-functions that do not show up in the final graph and by reasoning about the behavior of functions "microscopically"—examining the behaviors of functions over very small intervals and then larger intervals comprising them. With respect to the PSTs' final explanations, DH commented "Oh. Is that all? I was trying to describe *the whole graph*" (DH, 09/28/04). Taken against the background of her initial comment (page 171), I interpret this as her realization that the goal of an explanation was not to describe the shape of the graph, but why, in terms of the behavior of quantities, did the graph have the significant features it did. This comment, as well as a number of the PSTs' utterances in Excerpt 7-21 indicate that they had the capacity to reason about functions in terms of the covariation of quantities. But, again, their initial work on this problem indicates that they were not inclined to reason in terms of covariation spontaneously.

The Cubic

The ultimate purpose of this activity was that it provide a setting within which the PSTs could begin to develop an appreciation for both the conceptual and pedagogical power of a covariational understanding of functions. When viewing the two families' behaviors without covariation, one sees them as unrelated and therefore requiring unrelated principles to explain them. When viewing the two families' behaviors within a covariational perspective, the two families exhibit the same underlying principle—we are adding a linear function to another function, and the linear function has the same effect in both cases. As such, the final 6 minutes of class session 9 was spent discussing Part (b) of the Families of Functions Problem, where the PSTs were to use their explanation of the quadratic to understand the behavior of the cubic function $g(x) = x^3 + nx$. Excerpt 7-22 and Excerpt 7-23 give the reader a sense of the short discussion.

Excerpt 7-22 (Session 9, 09/28/04)

[The discussion is about g(x) with n = 1. Figure 7-37]

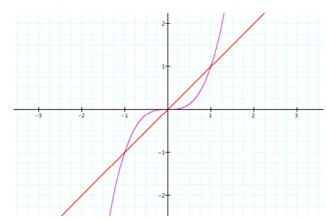


Figure 7-37: Graph of g(x) for n = 1

1. TI: Just run through the covariation and tell me what the sum is going to look like.
2. DH: Well starting all the way at the left, the cubic is more negative than the line -
3. KN: - they're both negative, so it'll be below both of the graphs.
4. DH: And it is moving … Oh, none of them are positive.
5. SS: So it will be negative until -1 since when it hits x = -1, it is -2, we know that much, and it crosses through zero.
6. KN: So as it goes from -1 to 0, it is increasing at a slower rate than it was before that.
7. SS: Yeah, until it passes zero and then it does the same thing, but positively.
8. TI: So … but go ahead and describe what we're going to get and then tell me why it is going to look that way.

9. DH: We're going to end up having a graph that is increasing at a slower rate.

10. TI: But why?

11. KN: Because the cubic is increasing fast then slowly and we're adding negative numbers to it that are increasing.

[TI displays graph; Figure 7-38]

12. TI: Does it agree?

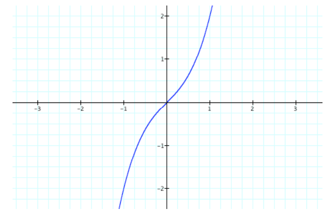


Figure 7-38: Graph of x^3

Excerpt 7-23 (Session 9, 09/28/04)

1. TI: So now if I make n = -1, what's the resulting graph going to look like [the graph of x^3 and –x are shown in Figure 7-39]?

2. SS: It is still going to be negative starting on the left. Because it's so, because both x^3 and –x both are very negative.

3. TI: Well that doesn't describe very much. Just saying that it is going to be very negative…

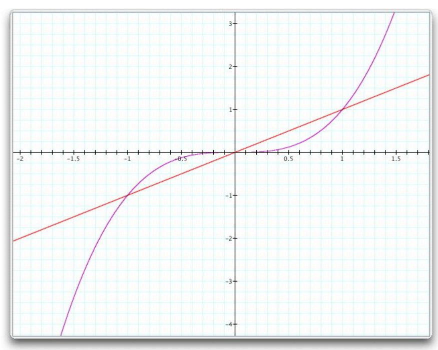


Figure 7-39: Graph of g(x) for n = -1

4. KN: We know it has to be above the x^3 graph, because we're talking x approaching zero from the negative side.

5. SS: You mean "above" like as in more positive than x^3?

6. KN: Yeah for each x point, it will be greater.

7. TI: So where out there is it going to balance, so that –

8. SS: – at -1. That's Where they both will cross 0 so the sum is zero.

9. TI: So it looks like the sum will be zero at -1. What about farther to the left of -1?

10. SS: It is going to be negative.

…

11. TI: All right, so it is going to come up until -1, and then what's it going to do?

12. DH: It's more positive – so it will go up.

13. DH: But then it is approaching zero.

14. TI: But then it is zero at zero. Right. And then what?

15. DH: Then it is more negative than it is positive.

16. TI: It is more negative than it is positive.

17. DH: It will be negative until it crosses at 1.

18. KN: And then it will keep going [up]. So it will kind of do that [traces increasing "cubic shape" with hand].

In these excerpts, we see the PSTs' explanations of the cubic using the same reasoning as they had for the quadratic. For both n = 1 (Excerpt 7-22) and n = -1 (Excerpt 7-23) we see the students determining the zeroes of the polynomial (where the values of the two constituent functions are equal, but with opposite signs so that the sum is zero). They use these zeroes as landmarks to organize their analysis of the covariation. For example, in Excerpt 7-22, line 5, we see SS note that for $x < -1$, g will be negative and below both of the sub-functions. In line 6, KN notes that there will be a vertex between $x = -1$ and $x = 0$. In both the excerpts, we see evidence of the PSTs explaining behavior of the function by analyzing the rate of change in g in terms of the rates of change in the sub-functions. This type of reasoning enabled the PSTs to understand why when $n = 1$ (in fact, when n is positive) the graph will be always increasing and when $n = -1$ (when n is negative) the graph will have a local maxima and a local minima.

Though there was not time to push the generalization in class, TI believed that the PSTs could imagine this behavior. He projected an image of $g(x)$ and for different values of n and moved through each n in succession ($-5 < x < 3$), asking the students to imagine each as a frame of a movie (Figure 7-40). He ended the class with the following comment:

So there's n = -5, n= -4, -3, -2, -1, 0, 1, 2, 3, … Is that what you saw when you played the movie before. In principle, is this any different from x^2 + nx? So it is by looking at them in terms of covariation that they become the same. When you look at them in terms of "gee, somebody's picking that one up and moving it around, and they're taking this one and bending it and stretching it, they look very different in terms of their behavior. But when you look at it in terms of covariation, it is the same thing. So that's the total explanation (TI, 09/28/04).

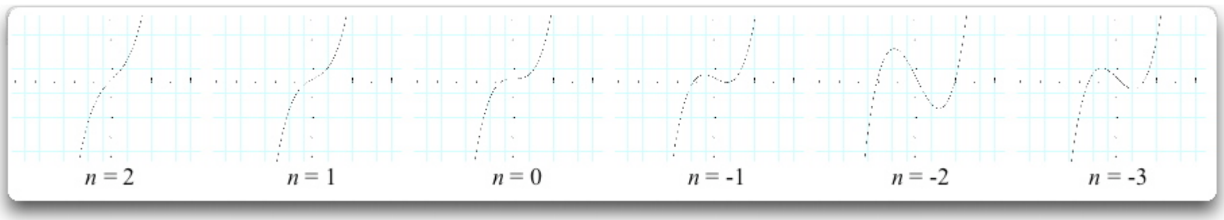


Figure 7-40: The Cubic as Frames of a Movie
(n = -3,-2,-1,0,1, 2)

Discussion of Activity 2, Part 2 – The Cubic

The class discussion of the family of cubic polynomials presents further evidence that the PSTs have developed the ability to reason about functions via covariation. Their technique was to determine landmarks for the given function and then to use covariation to analyze the behavior of the function between the landmarks. Thus, we are presented with further evidence that the PSTs were developing a KDU of functions – a particular understanding of function that enables them to find similar, yet conceptually more difficult problems accessible. In this case, their understanding of function enabled them to explain the behavior of both quadratic and cubic polynomials.

Student Work on Activity 2, Part 2 (Families of Polynomial Function)

As homework, the PSTs wrote-up their solutions to the *Families of Polynomial Functions* problem. Their write-ups provide insight into the ways in which the PSTs were thinking about functions. In each PST's work, he or she (1) determined what the characteristics of a landmark might be; (2) found the landmark(s); and (3) described how the quantities co-varied between the landmarks. This approach to explaining the behavior of functions was very similar to the means for explaining the behavior of the correspondence point in *Cities A & B* (Activity 2) and in the applied problems of Activity 3, however, with this activity, we see the PSTs applying their understanding of function as covariation in a more abstract setting. Once TI had helped them determine the characteristics of the landmark for the *Families of Functions* problem (i.e. where the sum of the two sub-functions was zero), they were able to use their knowledge to explain the behavior of each of the functions and the family of functions.

Though each of the PSTs' work showed this kind of reasoning about the behavior of the polynomial functions, the write-ups were qualitatively different: one treated n as a parameter while another treated n as another variable. The difference between the two is that a parameter is a quantity that is fixed before the independent and dependent variables can be tracked. When n is a parameter, there are a number of different graphs, or cases, which correspond to each value of n. When n is thought of as another variable, the complexity of the explanations necessarily increases significantly – what is really being described is a function of two variables: $f(x,n) = x^2 + nx$.

In the following, I give brief examples of the PSTs' written explanations of the behavior of the quadratic family. The PSTs' explanations of the cubic were very similar to their explanations of the quadratic.

KN's Explanation of the Quadratic: Slides of a movie. KN divided his explanation into three segments n = -1, -0.5, and 1. For each n, he then discussed how the quantities covaried. Below is his explanation for n = -1:

n = -1:

Both h(x) and k(x) lie above the x-axis when x is less than 0. So the sum of the two equations is bigger than either equation and therefore f(x) lies above both graphs when x is less than zero. As x approaches 0 from the left, f(x) begins to flatten out because h(x) flattens out. At x = 0, both functions equal 0, so there sum is also 0 and f(x) crosses the x-axis at x = 0. As x becomes positive, the line k(x) is farther below the x-axis than the curve h(x) is above the x-axis, so f(x) is negative. When x is approximately .5, the graph of h(x) starts increasing faster than k(x) decreases. This is the vertex of f(x). When x = 1, h(x) is as far above the x-axis as k(x) is below the x-axis, so their sum is 0 and f(x) = 0. When x is greater than 1, h(x) is always farther above the x-axis than k(x) is below the x-axis, so f(x) is above the x-axis. The slope of f(x) starts increasing as x increases because the slope of h(x) increases while the slope of k(x) stays constant.

Figure 7-41: KN's Written Explanation

KN located the landmarks and then described the way in which the quantities covary between the landmarks. In his explanation, he approximates the location of the vertex as a result of the covariation. He notes that since "The graph of $h(x)$ starts increasing faster than $k(x)$ decreases" (line 9), the vertex has to be at approximately 0.5. He concludes his discussion by noting "In sum, as n increases from –infinity to 0, the vertex of $f(x)$ moves in the positive y direction and in the negative x direction until n = 0. At that point, $f(x) = x^2$. When n increases from 0 to infinity, the vertex of f(x) moves into the third quadrant and gets farther away from both axes" (KN, PS2 Write-up).

DH's Explanation of the Quadratic: A Function of Two Variables

Rather than consider values of n separately, DH chose to divide the explanation up into a number of cases: when n was very negative, when n was near -1, when $n = 0$, and when n approaches infinity. As an example of her explanation, Figure 7-42 shows her explanation for the case when n is near -1.

For n near -1, the y values of the graph of $y = x^2 + nx$ are still positive until we reach $x = 0$. It is at this point that the distance between the $y = x^2$ graph and the $y = nx$ graph are equidistant from the x-axis. (This is where the values of the two equations are both zero.) Thus, we know that the graph of $y = x^2 + nx$ is zero and the graph crosses the x-axis. Using the same logic as above, the two graphs are, again, equidistant from the x-axis when x = -n. (This is the point where the graph of $y = x^2 + nx$ crosses over the x-axis again.) Thus, we know that there is a vertex somewhere between $x = 0$ and $x = -n$.

Figure 7-42: DH's Written Explanation

In her explanation, DH attempted to describe the behavior of the quadratic polynomial in terms of two variables. In her explanation, we see that she has noted that the vertex of f_n will occur sometime between $x = 0$ and $x = n$ (lines 6-7). Rather than imagining snapshots for particular values of n, she is imagining both n and x varying at the same time, and therefore she is imagining movie "clips." To understand the clips, however, one needs to analyze them frame-by-frame, and DH does not do so. As a result, her explanations are still a bit vague.

In addition to her lack of explaining clearly why the graph behaved as it did for n defined on different regions, DH often appeared to lose track of what she was explaining, often erroneously speaking of n when she meant x, and vice versa. This fact also indicates the complexity of analyzing functions of more than one variable.

Discussion of Student Work on Activity 2, Part 2.

The student work shown in this section shows two things. First, it further verifies the claim that the three PSTs had each developed the ability to reason about the behavior of functions via covariation. This in no way implies that all three PSTs had the same level of sophistication and coherence in their reasoning. Rather, they all were focused on identifying landmarks and analyzing the behavior of the function in the regions bound by those landmarks. Despite the fact that each PSTs concluded his or her explanation with a line similar to DH's:

"When we put it all together, we get a picture of a family of graphs that appear to "bounce" across the coordinate axes" (DH, PS2 Write-Up), the classroom conversations and written explanations provide evidence that they were not simply observing the vertex bouncing, but had developed some sense of how analyzing the covariation can help explain the vertex's location for any *n*.

There were differences in their explanations, both in the organization and detail. KN was the most organized and detailed. As he explained, he described why the function behaved as it did for a number of "frames" and then generalized the behavior of the family for all *n*. In essence, his explanation was an argument justifying his observations. DH's explanations could be described as a description of the images that she had in mind when she tried to explain the behavior of the family of functions. SS's contribution was significantly less detailed and less organized than the other two, and the reason for this is not clear.

Post-Instruction Interview

Each of the PSTs took part in a post-instruction interview within 5 days of the end of the *Families of Polynomial Functions* discussion. The five interview questions that will be discussed in this section were chosen because they were asked of each PST (some of the latter questions from the interview protocol were not discussed with all PSTs). In this section, I discuss themes that emerged in the PSTs' responses to the interview questions shown in Figure 7-43.

1. On the first day of class, you were asked to give your personal definition. We all know that a function is defined mathematically as “a mapping which assigns each element in the domain to at most one element in the range,” but how would you adjust your personal definition of function to take account what you have learned the past few weeks? Can you see developing an understanding of function such as this as being a worthwhile instructional goal? What insight might provide that the traditional understanding/definition might not?

2. We have talked about variables varying. Using that logic, how can you explain the fact that in the equation $2x - 1 = 0$, x does not appear to vary—it is 1/2?

3. How would you characterize the instructional sequence thus far through the course. What was the instructional purpose of the introduction to graphing (fingers and fairy dust) activity? What about the families of functions problem?

Figure 7-43: Selection of Post-Instruction Interview Questions

Definitions and Images of Functions

The first interview question involved the PSTs reviewing their personal definitions of function from the initial assessment and commenting on both their prior definition and how, if at all, that definition had changed. A theme that emerged throughout the conversations about their personal definition of function was that though their definition of function had not changed, they felt that the meaning that they ascribe to a function had been deepened significantly. Each of the students felt that the salient characteristic of a function was the unambiguousness of the mapping from one set to a second set. In Excerpt 7-24 KN described his “deeper understanding” of functions[24].

[24] The “JS” in the transcript refers to the author, who was also the interviewer.

Excerpt 7-24 (KN Post-Instruction Interview, 09/28/04)

1. JS: And the first question is the idea of a function, which was on the first day of class, you were asked a question, give your personal definition of a function.

2. KN: Um, I don't think it is changed as much as now I really understand it more. As before, I knew that was the definition, because that's what I was told was the definition, but now I know why that's the definition.

3. JS: So why?

4. KN: So why is that the definition?

5. JS: Yeah.

6. KN: Ok, maybe, let me rephrase what I just said. Maybe I don't know exactly why that's the definition of a function, but, um...

7. JS: It is not an easy question.

8. KN: It is not an easy question, that's for sure.

9. JS: No.

10. KN: I Just think, ok, the definition is about the same, you know, with an independent variable and a dependent variable and the vertical line test, but I guess I just have a better sense of functions and how they relate to other math topics. This would allow me to improve my explanations of functions, I think.

In this excerpt, we see that KN believed that functions were still about independent variables, dependent variables, and the vertical line test (line 10), and that the significant difference in his understanding of function was that he now knew *why* it was the definition (line 2) and he would now be able to explain it better (line 10). When pushed to explain what it was he knew better, he was not able to specify. He stated that he had a better "sense" of functions.

DH spoke of a similar extension of her definition of function:

> Yeah, I think that my general definition of a function has not changed. I still think of a function as a mapping, but now I see that there may be more to it. What I said the first day of class is probably more conservative than what we have been talking about. Everything we've been talking about, about using different coordinate systems, paying attention to the way the fact that each point represents two values, and how r cosine two theta really *is* a function, it is just not

> a function of x and y...I *knew* all that, but it wasn't the important stuff. Now my understanding of functions is more inclusive. It really is more than the vertical line test. It has more to do with what we see when we see functions. Now rather than a graph being like a picture, I see where it comes from – like the bars on the Cities A&B diagram (DH, 09/28/04).

In addition, we see her mention something similar to KN's "sense of functions." I interpret her comment to indicate that rather than seeing functions as a picture – or, using TI's analogy from Activity 1: *Introduction to Graphing*, a piece of wire – she envisioned the graph in conjunction with the quantities which covary to result in the graph. Though there is no data available to address the question of whether she believed the graph to be pre-eminent or the result of tracking the covariation, both DH and KN's developing image of covariation is a first step towards it.

SS was not able to give as detailed of an explanation as the other two PSTs, but she did feel that her definition of functions had changed:

> From this class I believe it is more than just the vertical line test. It is broader, yet more defined. Functions are covariation of variables, like a relationship between variables. I feel like looking at how the variables vary is important in my understanding (SS, 09/29/04).

From SS's statement, it is evident that she has identified variation and relationships between variables as a significant aspect of understanding functions. This is a significant shift from her responses on the initial questionnaire, which described functions as equations that can be used to produce a new set of data.

The interview data strengthens our conclusion that there had been a significant change in the PSTs' personal definition of function. Whereas on the initial assessment, the PSTs' focus was on the idea of a mapping and the unambiguousness of that mapping, in DH and KN's responses, we see a shift to the importance of understanding functions as more than an abstract definition or a mapping from a class of equations to a picture. We also see emphasis on variability, developing an image of the relationship between the quantities, and understanding functions as a unifying idea in mathematics.

Functions and the Purpose of Instruction

The reasons for the PSTs' desire to move beyond an abstract definition of function varied. For example, KN mentioned that if a student were to only understand functions in terms of the set-theoretic, mapping definition, they would not really know what a function was. Below, he claims that "the point of all this" is to understand physical phenomena, like walking, and from understanding the physical phenomena, ideas like the unambiguous mapping emerge as logical.

> KN: Right. The way I would think about it is if I tried to explain to someone else before what a function was using that definition, I don't think they'd know what it was.
>
> JS: So what wouldn't they understand?
>
> KN: Like so what? What's the point of all this? It is a nice little picture, but if you're talking in terms of someone walking, and they can't be in two places at once, that's something that everybody's familiar with and that will help you remember. The little, you know, the figurative, nothing can be mapped to two different points, is kind of abstract. If you're just trying to explain to someone a function just using the definition, just explaining the definition, isn't enough. Because, a definition is just a definition. It doesn't have to do with what a function really is (KN, 09/28/04).

His comments indicate that he believed that when ideas emerge in context, they have deeper meaning for persons having them. It is significant that he mentioned that when someone walks "they can't be in two places at once." This statement is an analog for the injective requirement for functions, however it has significant instructional implications. When one sees the injective requirement to emerge in explanations of observable, real world phenomena, the way in which they would teach that topic is likely to be different.

KN also noted that it is important for the students to understand what a function really is. DH and SS also discussed the importance of *understanding* functions:

> Well, it is just, as a math student, it just gets me away from the idea that certain equations represent certain pictures. Like y equals x squared is some u-shaped figure, instead of y equals x squared to, ok, as x increases, y increases, so...by so much. I mean, it is important to be able to answer the question "Why does a parabola look like that?" with something more than "it just does, it is a quadratic" (DH, 09/28/04).

> Yeah, the point is to understand why the graph looks like it does. Like in the families of functions, I could describe it, but it took a while to explain why it looked the way it did. That kind of understanding of functions is really important (SS, 09/29/04).

By understanding, the PSTs are referring to being able to give explanations of the sort that they had been asked to give in the class. When asked by the interviewer to specify what was important about the explanations, they gave two types of responses: (i) broad and vague descriptions of understanding as important and (ii) descriptions of understanding as being generative. The predominant description was type (i), which was typified by SS's comment "I mean like with being able to grasp a lot deeper what's going on, I mean what's *really* going on, not just like "ooh, I can draw the graph," but truly understanding the graph" (SS, 09/29/04). Two examples of a type (ii) description are given below.[25]

> Uh...but, I think that it gives you a much better conceptual idea of what's going on, so...you...I mean, it goes back to the understanding *why*. Like, it does help me like, the whole … tracing your finger like distance from something or the … um, I thought that was *very* helpful … like … Understanding. You know when something just *clicks*, and you're like I've been told this and I've memorized it, and I could spit it out to you (pause) but it doesn't mean I'm gonna have it launch a memory, doesn't mean I can apply it to other things or build *upon* that. But when it is represented this way, you, it is just, it builds a deeper understanding and better understanding, and I feel like, I feel like I could go into deeper problems based off of *that* way rather than [the other way] (SS, 09/29/04).

[25] Ellipses (…) indicate hesitations, not omitted text

> The important thing was to see graphs instead of as pictures as relationships, 'cause if you're shown a graph, you're like oh it is some curve thing, but if you're doing it with your fingers, you're like if x increases, then the y value increases. So you just see it as more of a relationship than as a picture. This kind of understanding allows you to work on other problems in a similar way – like in the families of functions problem (KN, 09/28/04).

Though there was a mix of type (i) and type (ii) comments, the comments that referred to the idea of an understanding of function as being generative (KN and SS above), indicate that the PSTs' understanding of functions were developing into key pedagogical understandings of function as covariation of quantities.

Continuing Confusions

Though there are indications that the PSTs conception of function had shifted more towards understanding functions as covariation, and there are indications that the PSTs were at least partially aware of the utility of their developing understanding, the PSTs' understandings of function were by no means stable. In discussing her developing understanding of function, SS noted the fact that the word itself is confusing and commonly used with a number of different meanings.

> Functions, *function*, *FUNCtion.* The problem here is that people use the word *function* for multiple things, not just– (pause) Ok, I'm like this is the true problem is that we were all raised with the word *function* [uses hands to make quotation marks] is used for lots of different things, not the true- (pause) So then I'm like, what truly is a true function. I guess I've just had no time to process all of this (SS, 09/29/04).

Though she is correct, that "function" is commonly used as a synonym for "equation," this quote raises questions about her current conception of function. Though previously in the interview, she had mentioned that variables varying and covariation were important, we see that she believes that she does not understand what is and is not important about functions.

In her comment (page 189), DH explained that her understanding of functions now includes a number of mathematical ideas that she previously had thought of as separate as related ideas. Interestingly, she also noted that her definition of function had not changed. She later clarified this apparent contradiction by discussing how these new aspects of function would affect her teaching:

> Though it was important for me to learn about why functions look the way they do, I'm not sure how important it is for my students. I mean, I need to be able to explain it, but the students are tested on things like the "Which relation is a function?" or "What graph is a function?" I'm not sure how this would help them with those questions (DH, 09/28/04).

Thus, her understanding of function had changed, but she was unsure of the salient issues with regards to functions. More specifically, she was unsure of the pedagogical implications of her developing understandings. In both DH's and SS's comments, the conflict between the two understandings of function (school-math and covariation), which had been hinted at throughout this analysis, is made evident. It is worth noting that DH seemed to envision herself teaching a curriculum as presented instead of as something that she could modify. The assessments that she envisioned being given to her students were unaligned with her emerging understanding of the role of covariation in providing a foundation for students' understandings of function.

The PSTs confusions about relationships among the ideas of variable, function, and equation was evident in their responses to Question #2 from Figure 7-43: *We have talked about variables varying. Using that logic, how can you explain the fact that in the equation $2x - 1 = 0$, x does not appear to vary—it is 1/2?* The PSTs' responses to the question are shown below:

> For this static situation, x is equal to one half. I mean, the reason they're variables, is that it just stands for … it stands for a constant. And what we're talking about with the *change* in variable in a function is that at every infinite point along a graph, *at that point*, the variables stand for a particular constant, but throughout the entire graph, they represent a *number* of constants (DH, 09/28/04).

> Well, it is not variable in that situation. In this case, it is an unknown. In this case "variable" is really a misnomer. If you plotted the function and got the graph, there would be variables. Maybe it has to do with if there are two variables, like x and y (KN, 09/28/04).

> This is kind of confusing. X is a variable, but in this case it is just a number. Maybe this isn't a function. [JS: Does it pass the vertical line test?]
> Oh. Yeah it does. I don't know. Is it possible to be both a variable and a number? I don't know. This highlights the problem I've been having in this class. I don't understand the point of math. I mean, in all my years of like learning math, it has come very easily to me, and like, so I never had any trouble, so I was like sure whatever. I'm just really confused. (SS, 09/29/04).

The significance of these PST statements is twofold. First, it shows that they had yet to develop ideas of variable, function, and equation as a coherent scheme. It seems they were untroubled by the understandings they had developed through *doing* mathematics because in *doing* mathematics they could compartmentalize their understandings around patterns of activity in response to different types of performance-requests—*Find x, Show* the graph, *Solve* for *x*, etc. Second, SS's comment reveals a principle source of disequilibrium for PSTs in this course: They were being asked to develop understandings of variable, function, and equation that cut across their existing compartments.

CHAPTER VIII

ANALYSIS OF PSTS' PLANNING FOR, AND TEACHING, A LESSON

The final phase of the study was designed to analyze the relationships between PSTs' understandings of mathematics and their developing pedagogical thinking. PSTs planned and taught a lesson on functions in polar coordinates. The project's objective was that they would use their understanding of the concept of function to help their high school student (HSS) understand and reason through the assessment questions shown in Figure 8-1. In the guidelines for the project, and in his explanation of the project, TI focused on helping the HSSs *understand* mathematics, not memorizing procedures or rules. The written instructions for the project included the following passage:

> Design your lesson so that it will not only help your student answer the questions about the graph and function, but also so that your student will *understand* the questions and their solutions. Focus your instruction so that your student sees his or her solutions to the questions *as making sense* as opposed to remembering what he or she should say (Project Assignment Sheet).

The course project consisted of five parts: interviewing the HSS, conceptualizing the lesson, planning the lesson, teaching the lesson, and writing a reflective essay. An overview of the project with the planned dates is shown below in Table 8-1.

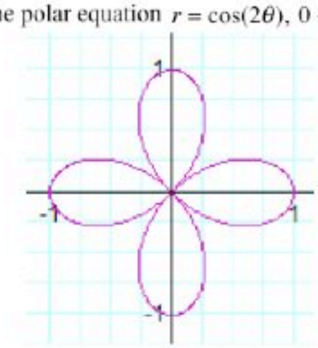
Table 8-1: Overview of Project

Dates	Description of Project
Before 9/14	Interview HSS
9/16	Conceptualizing Paper Due/Discussed in Class
9/26—9/27	Meetings with Instructor to Formalize Plan
9/30	Plan for Instruction Due
Before 10/5	Teach Lesson
10/19	Project Reflective Due

Assessment Questions: Polar Equations and Graphs

Please answer the following questions to the best of your ability on a separate sheet of paper.

Given the following graph of the polar equation $r = \cos(2\theta),\ 0 < \theta < 2\pi$,



a. Does the graph represent a function? Explain your answer.
b. What is varying that the function produces this graph?
c. Explain why the graph looks as it does.
d. What would the graph look like if, instead of $r = \cos(2\theta)$, you graphed
   i. $r = \cos(\theta),\ 0 < \theta < 2\pi$?
   ii. $r = \cos(4\theta),\ 0 < \theta < 2\pi$?
e. What would the graphs of $r = \cos n\theta$, $n = 3, 5, 6, 7, 8, 9$, and 10, look like? How do you know?

Figure 8-1: Assessment Questions for HSSs

The PSTs were assigned a high school sophomore at a comprehensive, independent college-preparatory school as their HSS. The HSSs were compensated for their participation in this study. In this chapter, the PSTs' HSSs will be referred to as follows[26]: DH's HSS was Joe, KN's HSS was Jamie, and SS's HSS was Nick.

[26] HSSs are referred to by pseudonyms.

Activities Prior to Lesson on Functions of Polar Coordinates

PSTs' Interviews with their High School Student

Each PST interviewed his or her HSS. The purpose of the interview was twofold: to help the PSTs develop an understanding of their HSS's mathematical background and to help develop a positive rapport with the HSS. In the discussion that follows, I use the PSTs' write-ups as the primary source of data. I viewed the interviews in their entirety and will comment on the accuracy (and at times, the inaccuracy) of the PSTs' comments. It is important to remember that of interest to this study is what the PSTs thought to be important, and ultimately *not* what the HSSs were able to do or say.

This interview was conducted at the beginning of Phase II of instruction and a summary of the interview was submitted to the course instructor on 09/16/2004.

Highlights of the Interview and PST Write-ups

The PSTs were given an interview protocol designed to support productive conversations with their HSS. The interview protocol consisted of three questions addressing the HSS's beliefs about what a function was and its importance to mathematics in general. The final four questions dealt with the HSS's understandings of functions – they were asked for a conventional definition of a function, to compare and evaluate a set of seven possible definitions of function (none of which were entirely incorrect), and to discuss an applied functional situation involving linear motion. The entire interview protocol can be found in Appendix A.

The main theme that emerged in the interviews and the PST write-ups was *what counts as an acceptable definition of function?* Each of the PSTs noted that their HSS had some understanding of functions. Both Jamie's (KN) and Joe's (DH) initial understandings of function were as "a thing you plug a number in to to get another number out." Both KN and DH noted that this kind of an understanding of function, though not technically incorrect, was insufficient. KN explained that this understanding of function was not adequate because, though it did provide insight into particular points on the graph, it does not highlight the fact that functions are any kind of relationship between variable quantities.

Each of the PSTs noted his or her HSS's knowledge of the vocabulary of functions: the HSSs regularly referred to terms like domain, range, independent variable, and dependent variable as all "having to do with the functions." SS was the only PST who thought that her HSS was able to correctly and confidently discuss the significance of each—KN and DH felt that their HSSs knew the correct words but did not understand their meaning or their importance. Both KN and DH felt that their HSSs had rudimentary understandings of functions and used the rest of the interview to understand the specifics of their HSS's understandings of function.

Eventually, each of the HSSs mentioned what the PSTs thought to be the more formal and specific definition of a function: "for every input, there can be no more than one output." Both DH and KN questioned whether their HSSs really understood the importance of this uniqueness clause in the traditional definition of function. Their main concern was that both Jamie and Joe recalled that a function must be a unique mapping in the context of discussing the graph of a function. DH noted that when she presented Joe with a velocity-time graph "it was like a light-bulb went off in his head and he remembered the vertical line test as a way to determine [whether a graph was a function or not]". A similar event occurred when KN presented Jamie the graph of a circle and asked whether it was a function. She responded that it was not and drew two U-shaped graphs to explain why not: she noted that only the horizontal one passed the vertical line test, "because [in the horizontal one], for every x-value there would be more than one y-value."

Though Joe and Jamie both were able to *say* “for every input, there can be no more than one output,” DH and KN noted that when asked to explain further, their HSSs turned to a graphical inscription as an aide. They believed that their HSSs’ understandings of functions were limited to the idea of the vertical line test and a graph—DH even commented that “Joe seemed more confident about questions where he was given the actual graph—he felt that questions about relationships were too abstract.”

In DH’s write-up, she believed that Joe’s understanding of function was largely grounded in graphs and pictures:

Excerpt 8-1 (DH: Interview Write-up, 09/14/04)

> I can see that Joe has an idea of what a function looks like if he is given a graph (because he can use devices like the vertical line test to test for a function) but he seems to understand little about the real-world applications. … I can see that the way Joe leaned about functions was probably looking at examples of graphs of functions, but [he] never really got a good working definition or explanation of what a function is and how to define it in a way that would be useful.

By working definition, I interpret DH to mean an image of functions that would enable Joe to engage with a wide array of functional situations, including applied situations. KN noted that Jamie had created her own definition of function: “a formula, algebraic expression, or equation that expresses a certain relation between two quantities such that the quantities in the first set correspond to exactly one element in the second set.” With regard to this definition, KN believed that her definition might have helped Jamie describe the technicalities of the definition of a function, but that it would likely not help her understand how functions relate to applied situations, like using a function to track the distance walked as a function of elapsed time. In his interview with Jamie, KN commented that if thought about distance as a function of time: “We know that the you can’t be in more than one place at the same time … that’s all that functions are about. They say that if I walk for a given amount of time, I can't have walked two different distances" (KN, Interview with Jamie, 09/12/04). In these examples, we see KN and DH had come to value a more imagistic working definition of function.

Both KN and DH noted that their HSSs had a rudimentary definition and understanding of function, however as evidenced above, they felt that that their HSSs' understandings of function needed to be improved in a number of ways. First, they believed that the HSSs needed to understand functions as more than output generators. Second, they believed that the HSSs needed to understand that functions were not just graphs that pass the vertical line test. They needed to understand what the vertical line test means, especially with respect to real-world applications of functions. Finally, Both KN and DH hinted at the idea of developing a "working definition" or image of functions. It is important to mention that neither KN nor DH specified what this working definition might be. I suspect that they did not specify one because, at that time, they were unable to clearly give one themselves.

In contrast, to KN and DH, SS felt her HSS had a solid understanding of function and the related vocabulary. She felt he had answered the questions with confidence and was able to give good examples and justifications for why each was or was not a function. Though it was apparent, through a careful viewing of the interview video, that Nick's understanding of functions as output generators and as manifestations of the vertical line test was slightly more advanced than that of Jamie and Joe, SS did not analyze his responses to the interview questions critically—she simply described his answers. Thus, we are not provided with significant insight as to what she believes constitutes a sufficient understanding of function—possibly with the exception of the obvious fact that she values his being able to say "for every x there is exactly one y."

Writing and Discussing Conceptual Narratives

The second part of the course project involved each of the PSTs writing two narratives describing (1) how they imagined their respective HSS would reason about the assessment questions (Figure 8-1) before taking part in instruction (which was to be designed in part 3 of the course project) and (2) how they imagined their respective HSS would reason about the assessment questions, were he or she to understand the ideas perfectly. The assignment was clarified by TI on the assignment sheet: "By *narrative* I mean a semi-transcript of what you imagine you'd hear if your HSS were to think out loud while reasoning about the questions." These narratives provide us with insight into the aspects of understanding of functions that are particularly salient to the PSTs. The narratives were completed out of class concurrently with Problem Set #1 and were discussed in class on 9/16/04.

Uniqueness of the Mapping

In response to assessment question (a), each of the PSTs believed that before instruction, their HSS would respond that the graph in Figure 8-1 did not represent a function. For KN and DH, this belief was grounded in the assumption that the HSSs would rely solely on the vertical line test as the means to determine whether a graph was a function or not. SS believed her HSS would reason similarly, though rather than use the term *vertical line test*, she believed Nick would note that "it is making too many values for each x."

DH believed that, after instruction, her HSS would be more comfortable with explaining the definition of a function. She presented the following as an acceptable answer to question (a):

Excerpt 8-2 (DH: Written Narrative, 09/16/04)

Yes, this graph does represent a function. Even though my previous instinct to use the "vertical line test" to test for functionality, I realize that that method only

> works for functions graphed in rectangular coordinates. As this graph represents a function in polar coordinates, the "vertical line test" does not apply. However, I now understand that what the vertical line test is essentially testing for is to see if there is more than one value for *y* for every [any] given *x*. In this case, we want to check to make sure that there is only one value of *r* for every θ. When examining every value for θ ($0 \leq \theta \leq 2\pi$) it becomes clear that for every given θ there is, in fact, only one value of *r*. Therefore, this graph does, in fact, represent a function.

SS's envisioned response was very similar. KN discussed an acceptable answer a bit differently—he presented it in terms of a conversation. He envisioned presenting his HSS with three questions: (1) Does the graph represent a function? (2) How can you say it is a function since it doesn't pass the vertical line test? and (3) Could you develop a test comparable to the vertical line test? His anticipated responses to (1) and (2) were very similar to DH's (above). His response to (3) is of special interest:

Excerpt 8-3 (KN: Written Narrative, 09/16/04)

> KN: So could you develop a test comparable to the vertical line test for polar functions?
>
> Jamie: Yes, instead of drawing vertical lines, draw lines coming out of the origin in every direction. If it passes through the graph at more than one point, then there is not exactly one length r for every angle measure theta. The polar equation would therefore not be a function.

The "angle line test" is a bit misleading because multiple points that appear to lie along a ray whose base is at the origin do not necessarily mean that those points all are located at the same angle—two points that appear to be on the same ray may actually be located by different angle measures ($\theta_1 = d$ radians and $\theta_2 = d + 2\pi n$ radians). However, KN's attempt does show evidence of his thinking critically about the mathematics involved in the problem.

Variables and Variability

Assessment question (b) dealt with the HSSs identifying the variables involved in the situation. Each of the PSTs anticipated that, before instruction, their HSS would have difficulty with the letters *r* and *θ*. They believed that their HSSs would be "used to" *x*'s and *y*'s. In addition, each PST believed that his or her HSS would note that both *r* and *θ* are variables because the output values for *r* change when you change the input values for *θ*. DH envisioned being able to orchestrate the following conversation with her HSS (after her lesson with him):

Excerpt 8-4 (DH: Written Narrative, 09/16/04)

11. DH: Can you tell me what variables are?

12. Joe: A variable is the thing you are looking for in an equation.

13. DH: Right. Now, does the variable always stay the same? Does it always represent the same number?

14. Joe: No.

15. DH: Good. So, essentially, we can say that variables vary, right? They change?

16. Joe: Yeah

17. DH: OK, so this question is basically just asking for the variables in this problem.

18. DH: Oh, ok. Well, you have the x-axis here (points to horizontal axis) and the y-axis here (points to vertical axis), and I've really only seen graphs that use x and y as the coordinates, so I guess x and y are the variables. Therefore, x and y are what's varying to make this graph.

In her simulated transcript, we see that two understandings of variable are at odds. When Joe spoke of a variable as "the thing you are looking for in an equation" (line 2), he referred to the common practice of solving for an *unknown*. It appears that the shift from variable as unknown to variable as truly variable was transparent to DH.

After instruction, DH and SS believed their HSSs' responses to question (b) would be unproblematic. In their narratives, they simply wrote that they would be able to specify *r* and *θ* as the variables. KN simply omitted discussing possible HSS responses to the question.

The Graphs' Appearances

Questions (c) - (e) deal with explaining why the graph of $r = \cos(2\theta)$ and the graphs of $r = \cos(n\theta)$ look the way they do. The common belief among PSTs was that the HSSs would not understand the question, but would nonetheless be able to answer it. Each PST believed that his or her HSS would comment about the shape ("it looks like a flower") and would guess as to the relationships between the coefficient of θ and the graphs ("it will get bigger" or "there will be more petals").

In contrast, the PSTs believed that, after instruction, their HSSs would be able to describe why the graph looks as it does. The PST responses and explanations of what this description would consist of, however, varied significantly. SS simply stated that Nick would be able to explain that with even numbers in front, "the whole graphed rose will have 2 times as many petals as the number in front" and described a similar rule for odd numbers. DH envisioned that Joe would comment that rather than thinking of the graph as being traced out from left to right, you need to imagine "a particle moving along the path of the graph. [Then] we would see its exact distance from the center (r) at every given value of θ." With respect to questions (d) and (e), she believed that she would have to remind him "if you were looking at a regular cosine graph, what does it mean to have that number in front of θ?" before he was able to explain how the graph changes as n is varied. KN imagined his HSS as thinking about plotting a number of points "for every angle between 0 and 2π." With regards to the behavior of $r = \cos(n\theta)$, he anticipated the following response:

Excerpt 8-5 (KN: Written Narrative, 09/16/04)

> Cos(2θ), 0 < θ < 2π, has values ranging between cos(0) and cos(4π). Those values correspond to the length of r. Therefore, r equals 1 at 4 different points (0, π, 2π, and 3π). That is why the graph has four separate "petals". Cos(θ) 0 < θ < 2π would have values ranging between cos(0) and cos(2π). In that range, r equals 1 only once. So there would only be one "petal".

Three different interpretations of what it means to understand the family of functions appeared in the PSTs' narratives. SS believed that understanding this family of functions is about establishing a pattern between the coefficient of θ and the number of petals on the graph. DH believed the salient issues were developing imagery that would support understanding graphing functions in polar coordinates. KN believed his HSS would identify critical points on the graph (i.e. when $r = 1$) and use that to understand why the graph looks as it does.

Initial Lesson Plans and Individual Meetings with TI

The PSTs were to draft a lesson plan and then discuss it individually with TI. In the meetings, TI planned to assist the PSTs in thinking about how the big ideas of variability and covariation could be used as the basis for teaching polar coordinates. The PSTs also met with TI to ensure that they had planned adequately before interacting with the HSSs. Only DH came to the meeting having thought about the lesson[27]; KN and SS had simply expected to "talk about functions in general" and possibly to "use a calculator to show [the HSSs] stuff." As a result, rather than giving insight into the PSTs thinking about functions, this discussion of what

[27] It is worthwhile to mention that since the individual meetings ended up not being an appropriate setting to approve the students' lessons, they each had to send an electronic copy of the lesson plan to TI. TI required that they improve their lesson plans prior to teaching their HSS.

transpired between the PSTs and TI in the individual meetings serves as a base line against which we will analyze the PSTs' final instructional plan and instruction.

Each of the conversations began with TI asking the PST where they would like to begin, to which each of the PSTs proposed introducing polar coordinates by highlighting their relationship to rectangular coordinates. TI then asked each PST what polar coordinates, and more generally what any coordinate system, was for. Only KN, after significant thought, was able to propose that a coordinate system was for locating points (TI had to remind DH and SS of this). The following excerpt indicates SS's confusion about the nature of coordinate systems:

Excerpt 8-6 (SS Meeting with TI, 09/21/04)

1. TI: So what are polar coordinates?
2. SS: It's the relationship between an angle and a radius.
3. TI: Uh-huh.
4. SS: It's different than like x and y.
5. TI: What are they for?
6. SS: What do you mean? Like, why graph in polar coordinates?
7. TI: What are polar coordinates for?
8. SS: Um, graphing...Graphing in a different dimension, a motion...It's a much simpler way to graph things that involve cosine and sine and theta...I mean, it's obviously a little confusing at first, but once you get used to visualizing polar coordinates, you can see the motion of something distance-wise.

The conversation shown in Excerpt 8-6 was typical of the PSTs' initial comments regarding the nature and purpose of polar coordinates and graphs of functions in polar coordinates. Though the PSTs were likely able to work with polar coordinates and generate graphs of polar coordinates via memorized procedures, formulas, and algorithms, their comments indicate that there was little understanding of polar coordinates (and coordinate systems in general) – especially understanding in a way that would support conceptual teaching of polar coordinates.

TI led each of the PSTs through a similar conversation that he anticipated would help them come to see how covariation could be used to make sense of functions of polar coordinates and polar graphs. He believed that this understanding would support the development of lesson plans (and ultimately instruction) that focused on the idea that understanding function as covariation could support understanding functions of polar coordinates and ultimately understanding why polar graphs behave as they do.

Instructor's Agenda for Lesson Planning Meetings

TI appeared to have an agenda for the individual meetings. Responses from one who understands polar coordinates in the way TI envisions might respond are given in italics. Each of the PSTs ultimately produced responses similar to those in italics after being unable to answer them initially.

What are polar coordinates for? *Polar coordinates, like in any coordinate system, are simply a way of locating points.*

How are locations described using polar coordinates? *Polar coordinates are similar to rectangular coordinates, but rather than locating points by distances from the x- and y-axes, they are located by an angle from the positive x-axis and a distance from the origin.*

How are measuring an x-coordinate and measuring a θ-coordinate similar? *In rectangular coordinates, we imagine x varying along a number line. A similar image with polar coordinates would be a ray, with initial point at located at the origin, whose direction varies counter-clockwise from the positive x-axis. As the angle varies, the corresponding distance* *from the origin can be located. The difference is like the difference between how taxi-drivers and air-traffic controllers map locations.*

Emergent Issue: Implicit Assumptions of Covariation

As noted in their conceptual narratives, each PST felt that his or her HSS would have difficulty with the variables r and θ. As a result, they felt that their first objective for instruction would be to help their HSSs understand how r and θ are defined. For example, in her meeting with TI, DH commented: "I know I will have to show him that θ is the angle between the x-axis and the r is the distance from the origin. I'll probably give him a number of points in polar coordinates and ask him to plot them." Though the different PSTs had slightly different ways to explain what r and θ were, they all arrived at the meeting believing once they did, that they could then move to graphing polar functions.

When TI attempted to focus their attention on covariation, DH and KN both proposed the idea of creating a table to help organize the corresponding values of r and θ. They felt that, once they had a table, their HSSs could graph the function. Consequently, they did not see a need for their HSSs to focus on the argument of the cosine function; for each n they would simply generate a table of values of the polar coordinates, graph them, and connect the points with a smooth curve. I showed in previous chapters that the PSTs had the ability to reason about covariation of quantities, and therefore it seems likely that *to them* this table was a way of organizing their covariation. For example, in Excerpt 8-5, we saw that KN was aware of the argument of the cosine's impact on the covariation: he notes that as θ varies from 0 to 2π, the argument of cosine varies from 0 to 4π, and thus as θ varies from 0 to 2π, the values of the function will vary from 1 to 0 to -1 to 0 to 1 twice, resulting in 4 petals. In contrast to then, when discussing their planned instruction, KN and DH move directly from calculating coordinates to plotting points.

It seems that KN and DH had implicitly assumed that the HSSs would be aware of the covariation that KN and DH understood implicitly. TI noted this, commenting that, traditionally, HSSs look at a table of values and see the values in each column varying one at a time (i.e., they will find a pattern in the "*x*" column and then find a pattern in the "*y*" column). He intended to push the PSTs to think about how they could help their HSSs to come to understand how the quantities vary together. To do so, he proposed the following analog for the "fingers and fairy dust" from *Introduction to Graphing and Covariation* (Chapter 6): *Use a pencil, ruler or other rigid, straight object to represent the varying direction determined by the angle θ. Then for each θ, use your finger to model r, the distance from the origin. Finally, use the same fairy dust to create the graph as a record of the covariation.*

Each PST came to the following broad instructional goals for his or her lesson plan[28]: (1) Help the HSSs understand the importance of a coordinate system, how the polar coordinate system locates points, and the fact that a point can be located using both rectangular and polar coordinates; (2) Help the HSSs understand functions and graphs as records of covariation, beginning in rectangular coordinates and moving to polar coordinates; (3) Help the HSSs understand sine and cosine functions and how changing the argument affects the covariation. TI believed that if the above three points were accomplished, and if the HSSs could reason through

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[28] The similarity of the students' goals and the relative congruence of the goals to the TI's agenda suggests that it was TI who brought them to that point and that they may not have arrived their spontaneously.

why $f(x) = \cos(\mathrm{n}x)$ behaves as it does in rectangular coordinates, then they could explain why $f(\theta) = \cos(\mathrm{n}\theta)$ behaves as it does in polar coordinates.

The PSTs were required to get their lesson plans approved by TI prior to working with their HSS. DH and KN each wrote a plan in semi-narrative form. They included what they (as teacher) would say or do, the kinds of responses they expected their HSSs to give, and how they would build on the HSS's thinking. SS's lesson plan was significantly shorter than the others and focused solely on what the teacher would do.

Lessons on Functions Graphed in Polar Coordinates

In this section, I will describe each PST's lesson with only occasional commentary on it. I will provide commentary and analysis on them individually and collectively after describing all three.

As would be expected from the relative uniformity of the conversations with TI, each of the PSTs had the following as the goals of instruction: (i) understand covariation and understand functions in terms of covariation; (ii) understand the purpose of coordinate systems in general and polar coordinates in particular; (iii) review sine and cosine functions and understand the periodicity of trigonometric functions; (iv) physically modeling the quantities and tracking the correspondence point within a polar coordinate system. Though the PSTs' plans shared the big ideas, the ways in which they planned to help their HSSs achieve the goals and the order in which they addressed (i) – (iv) varied greatly.

Highlights of SS's Lesson

Covariation

SS planned to use the *Introduction to Graphing and Covariation* activities (Chapter 6) as the first activity with Annie[29]. She began instruction by asking Annie if she had ever heard of covariation and Annie said that she had not. The following conversation then took place.

Excerpt 8-7 (SS's Lesson, 11/22/04)

1. SS: So what we're going to do is we're going to … well, my first "mini-goal" is to help you understand covariation. So, think about this situation … about like somebody walking. They're going from point A to point B … so they're spanning a certain distance, correct? And they're spanning that distance over a certain amount of time. OK, so what happens is … the idea of covariation is that its not just an equation, where there are variables, but it is like something that's actually a concept going on. What are the variables in this situation?

2. A: Time and distance.

3. SS: Right. So you have the guy, and he's walking along from point A to point B. What would his distance look like, the total distance he's traveled, if you were to move your finger along. Like, from this point on the desk to this point on the desk. Like what would it look like … you can just draw it in the air.

4. A: A straight line?

5. SS: Well, how would it look? Let's look at just the distance? Would it speed up or would it be constant?

6. A: It would be constant.

7. SS: And what about the time, like, let's say he does it in x minutes. Like how would you track the time? Like physically do it with your finger. Like this [SS moves her finger vertically in the air]. Like here's me [SS stands against wall]. Track me as I walk from this wall to the door.

   [Annie tracks the distance vertically.]

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[29] After many attempts to schedule a time with Nick, SS chose to work with a different student, Annie, who was a student at an after-school program that SS worked in. Though she did not have an opportunity to assess Annie's understanding of function, she was familiar enough with her that the initial conversation was comfortable.

8. SS: Good. So you see how the two variables are varying? That's covariation—both variables are varying.

In Excerpt 8-7, we are provided with insight into SS's reconstructed understanding of the *Introduction to Graphing and Covariation* activities. In line 3, we see SS asking Annie what the walker's distance would look like if she were to move her finger along the desk. In contrast to TI's support during activities that SS had taken part in[30], SS was not focused on Annie understanding what a particular location of the distance finger or a particular location of the time finger represented. Rather than having the goal of developing an image of distance as a varying quantity that was represented by a line segment between the starting point and her finger, for SS, the salient aspects of this activity is the physical activity. In addition, in line 9, we see the salient aspects of the instruction to SS were the variability of the variables, not how the variables varied with respect to each other. This led to a problem when she attempted to have Annie model the covariation. Finally, in line 19 we see SS's primary instructional technique: SS felt as if once her HSS "saw" the variables varying, she understood it.

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[30] This instruction was detailed in Chapter 6.

Excerpt 8-8 (SS's Lesson, 11/22/04)

1. SS: OK, so that's your total time. Now, we're going to track distance and time together. So try to move them together as I walk.

2. A: So like this [Figure 8-2]?

3. SS: OK. But remember to track the distance as you're doing the time. So your [distance] hand will move [horizontally]. So this finger will constantly stay on top of the other finger. Like this [SS demonstrates moving fingers]

Figure 8-2: Annie tracking distance and time together

[Annie moves fingers, one horizontally and the other directly above, without SS walking]

4. SS: Good. Now do them together.

…

5. SS: So do you know what you've created? You made a graph.

6. A: Oh, yeah.

7. SS: Does that make sense? Like how something would vary, it creates a graph because these things vary. Like if there was no time movement, it's the time movement that keeps this [the distance finger] moving along the graph.

8. A: Right.

9. SS: So, isn't that neat? You can use covariation to get a sense of what the graph looks like.

In Excerpt 8-8, we see that SS's goal for instruction was to have the HSS see that the graph is created because things vary. While this is true, the fact that the quantities vary does not explain why a graph looks the way it does or how one graph might differ from another graph. The variability of quantities is necessary for understanding functions as covariation, but it is not sufficient. In line 11, we also get a glimpse into SS's intent for the activity: to get a better sense of what the graph looks like. Thus, for SS, the activity was grounded in the fact that *the graph already exists* and that covariation can be used to discover characteristics of the graph.

Polar Coordinates

SS's instructional segment on polar coordinates began with SS trying to explain how polar coordinates are used:

Excerpt 8-9 (SS's Lesson, 11/22/04)

> Polar coordinates are … another way of graphing things varying. Instead of linearly, like, for instance, an air traffic controller. You know that radar screen—the circle and the thing that goes "beep, beep, beep." That's polar coordinates. The rings are distance from the center and over here [she points to their previous distance-time graph] we were looking at distance in the *x*-direction. The line that was circling was the angle, called theta. You see?

SS then proceeded to define r and θ in a standard way and mentioned that polar coordinates are just another way to locate points. She then took Annie through an activity that involved graphing a point given its polar coordinates. It is worth noting that SS's descriptions in Excerpt 8-7, of the resemblance between polar coordinates and an air traffic controller's radar screen, are quite vague, disconnected, and unoperationalized. It is hard to see how they could help someone who doesn't already understand polar coordinates.

Understanding the Cosine Function

SS then moved on to discussing the cosine function, which Annie did not remember having learned. SS began by drawing two 30-60-90 right triangles, one oriented with the 30^o angle at the lower left and one with the 60^o angle at the lower left. She then reminded Annie of the rules for sine and cosine of an angle in a right triangle: sine is opposite over hypotenuse and cosine is adjacent over hypotenuse. Using right triangle trigonometry, they generated the table of values shown in Figure 8-2.

θ	$\sin\theta$	$\cos\theta$
0	0	1
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2} \approx 0.866$
60°	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{2}$
90°	1	0

Figure 8-3: Table of Values Generated by SS and Annie

SS then moved on to graphing sin(θ) in polar coordinates. SS had Annie move her forearm to represent θ (the angle) and her finger along her forearm to represent *r* (the distance from the origin). She began by asking Annie to locate the first two polar points (0,0) and (30^{o}, ½) from their table for sin (θ). They then modeled what happens between the two points (Figure 8-4) by "connecting the dots" while moving their forearms. This discussion continued as they made their way through one full revolution by 30^{o} increments.

Figure 8-4: SS and Annie "Graphing" Sin θ

SS continued by having Annie do the same activity for sinθ (the second column in the table they had generated), sin(2θ), and sin(3θ). Each time, they made a table, modeled the points using their forearm and finger, and discussed what the graph would look like. At this point, Annie did appear to understand SS's lesson as designed: she appeared comfortable generating points and connecting the dots.

It was in the context of polar coordinates that SS brought up the notion of a function. In her interview with SS, Annie showed familiarity with the vertical line test. She was able to explain that the vertical line test meant that to be a function, the graph could not have more than one *y*-value for any *x*-value. Annie displayed the same familiarity during this lesson. In an attempt to explain functions of polar coordinates, SS noted that "it is able to appear to have two *y*-values for a particular *x*-value, but it can still be a function because the object's radii changes as θ changes. So it really is the same, but in this case it can't have more than one *r* for every θ." She then showed Annie why each of the sine functions they had graphed was actually a function, "because as you move your arm around, your finger is only at one point."

Families of Functions

SS spent very little time discussing cos($n\theta$) as a family of functions graphed in polar coordinates. Rather than using the explanation of one graph to provide insight to how she might explain another, SS began each graph anew. Once they had the graphs of the cosθ, cos(2θ), and cos(3θ) drawn in polar coordinates (Figure 8-5), SS asked Annie what she thought the graph of cos(4θ) would look like. When Annie was unable to predict, they quickly went over what the graphs of cos(4θ) and cos(5θ) would look like. SS drew Annie's attention to the "petals" on the graph and suggested that she look for a pattern in the number of petals and the coefficient. There was no evidence that this technique for predicting the shape of the graph of cos($n\theta$) involved the analysis of quantities covarying.

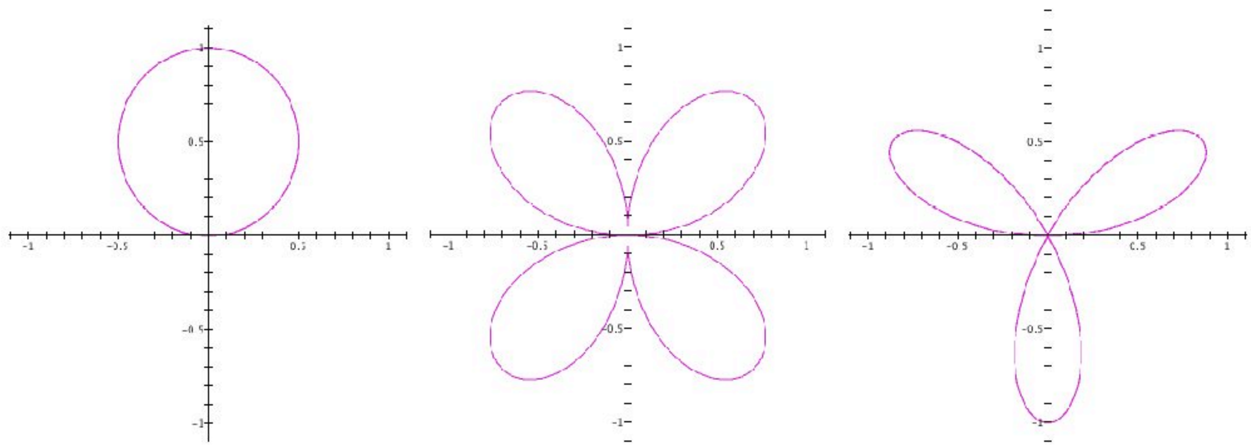

Figure 8-5: Graphs of sinθ, sin(2θ), and sin(3θ)

Summary of Annie's Performance on Assessment Task

In her write-up, SS claimed that Annie had answered all of the assessment items correctly, but her "correct" answers were not necessarily indicative of Annie's *understanding* of functions of polar coordinates as an instance of covariation of quantities – in fact in light of SS's instruction, this seems fairly unlikely. On assessment item (a), Annie answered that the graph did represent a function but did not explain why she thought it so. For item (b), Annie indicated that the variables, r and θ were varying, but again gave no explanation as to why she thought this was so. Annie omitted item (c) and used item (d) to answer item (e): she generated a table of values, plotted the corresponding points, and connected the points for cos(θ), for item (d), part (i), and cos(4θ), for item (d), part (ii), and then generalized a pattern from the number of petals in the graphs. There was no evidence that Annie used what she knew about cos(2θ) (or sin($n\theta$), for that matter) to inform her work on cos (θ) and cos (4θ). These responses indicate both that

Annie's answers to parts (a) and (b) were likely not grounded in covariation of quantities and Annie understood polar graphs and equations to be about generalizing patterns from visible characteristics in the graphs as opposed to understanding functional relationships between varying quantities.

Highlights of DH's Lesson

Due to technical problems, the audio recording of DH's work with Joe was extremely poor. As a result, my discussion of her lesson is grounded in both the video-recordings and an audio recording of an ad hoc, hastily-arranged conversation with DH about her lesson with Joe. The conversation took place approximately 9 hours after her lesson with Joe. My purposes in having this conversation was to ensure I understood both what she had planned to do and how the lesson unfolded. The emphasis was on her plan and her actions, and *not* on what Joe had learned.

Polar Coordinates

DH's first goal for instruction was to help Joe generalize his understanding of the rectangular coordinate system to the polar coordinate system. She began by helping Joe unpack his understanding of the rectangular coordinate system by asking him to plot several points and explain how he knew their location. She then asked Joe to think about a radar screen and described how it is used to determine the location of a plane:

> Have you ever seen a radar screen that has this "arm"/radius extending from a central point on the screen? Notice how the "arm" moves in a circle. Here we have an example of a polar coordinate system. Rather than measuring the plane's location as position in the x direction and position in the y direction, this system can measure the distance the plane is from a fixed point (r) and the angle (θ) the plane is from a fixed horizontal axis (what you are used to calling the "x-axis"), and this is measured in (r, θ).

DH then gave Joe a number of polar points to graph. After each point was plotted, DH asked Joe to explain why he located it where he did.

It was in the context of discussing the plotting of these points that DH planned to help Joe understand the fact that there can be more than one point in polar coordinates that gets plotted at the same place in the plane. This was also the setting that she planned to use to help Joe understand what it means for a function of polar coordinates to be a unique mapping. Two of the points she had Joe graph were (π,4) and (3π,4). DH questioned Joe as to why she had plotted (3π,4) on top of (π,4), and after a brief pause and a confused look, she continued: "OK, what was the *r* and what was the angle for each of the points you plotted?" Joe noticed that they had the same *r* but not the same angle. DH then responded:

> So, you can have two *different* points that look the *same* in polar coordinates. It seems a bit confusing, but we can look at it this way: if these two points were on a graph and we wanted to know whether or not they fit the description for a function, we could say that for every value of θ there is only one value of *r*. In this case, we have only one value of *r* for each θ. Yes, we have the same *r* value, but that is okay because there are not multiple values of *r* for one particular value of θ. If we had $\theta = \pi/2$, and we had both $r = 4$ and 5, then it wouldn't be a function.

After asking Joe if he understood, she moved on to the cosine function.

Cosine Functions

The next phase of DH's lesson involved helping Joe understand the cosine function. Though Joe had seen trigonometric functions in class, he was not able to recall anything particular about them, so DH needed to start from the beginning. She used a progression that was very similar to SS's, though she chose to use angles in radians[31] and she used three right triangles (two 30-60-90 triangles and a 45-45-90) to help him see the cosine of π/6, π/4, and π/3. Using

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[31] SS did not *explain* the significance of radians, she simply explained that an angle of π/2 was the same as a 90^{o} angle.

the triangles as a reference, Joe and DH generated a unit circle (Figure 8-6) and discussed the significance of each of the points on the unit circle (the first coordinate is the cosine of the angle and the second coordinate is the sine of the angle).

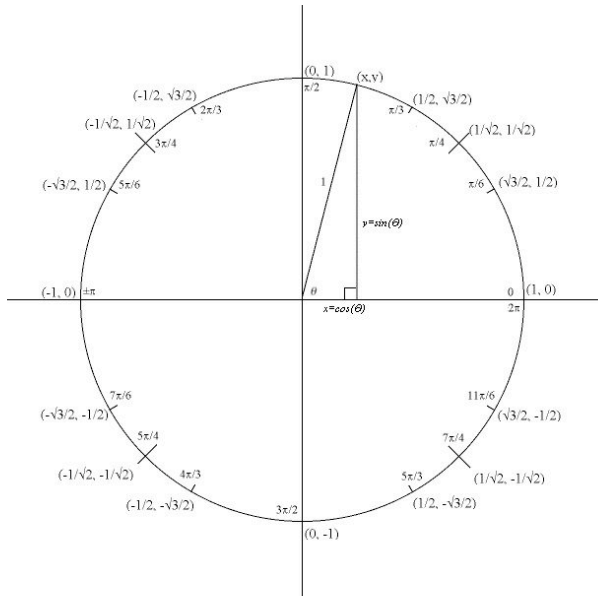

Figure 8-6: Unit Circle Generated by Joe and DH

Together, they generated the following table of values of θ and r =2cosθ (from 0 to π), using the unit circle. This table, which is shown in Figure 8-7, was used in her instruction on covariation.

θ	$r = 2\cos\theta$
0	2
π/6	$\sqrt{3}$
π/4	$\sqrt{2}$
π/3	1
π/2	0
2π/3	-1
3π/4	$-\sqrt{2}$
5π/6	$-\sqrt{3}$
π	-2

Figure 8-7: DH's Table of Values of r = 2cosθ

Covariation

The segment on covariation was steeped in DH's developing image of covariation. Throughout the course, DH had focused on tables of values as an important tool in understanding the way quantities covary. Though, she had initially focused on tables as a list of ordered pairs (as she had done in *The Drinking Problem*, Chapter 7), DH had come to use the table as a means of organizing instances of covariation. Her technique with Joe involved recording points of interest (where the graph appears to stop increasing and start decreasing or where the graph appears to cross the *x*-axis) in a table and then helping Joe reason about how the quantities covaried between those points. She began by referring to Figure 8-7 and noted "it is not just that we see θ varying from 0 to π/6. We can use the table to help us see that as θ varies from 0 to π/6, r varies from 2 to $\sqrt{3}$ … They both happen at the same time." She then plotted the points (0, 2) and (π/6, $\sqrt{3}$) and connecting the two points with a smooth line while repeating her statement "as θ varies from 0 to π/6, r varies from 2 to $\sqrt{3}$." Joe was able to graph the $r = \cos(2\theta)$ for $0 \leq \theta \leq \pi$ using the same technique. In her lesson plan, DH referred to this method as *using tables effectively* to understand covariation and graph functions.

DH had Joe "graph" $r = 2\cos\theta$ using the data from Figure 8-7. She did so by drawing a ray at a specified angle and asking Joe to plot the corresponding *r* value. Every time she plotted an additional point, she had Joe explain that *as* θ *varied from* u *to* v*, r varied from* a *to* b. She believed this approach would help him understand covariation as well as the fact that with this polar function, there is only one *r*-value for every θ, despite the fact that it did not pass the vertical line test.

Families of Functions

In order to understand the behavior of the family of cosine functions, DH had Joe generate tables of values and graph $y = \cos\theta$ and $y = \cos(2\theta)$ on the same Cartesian plane. Her plan was to help Joe see that "the first graph goes through one complete cycle in the allotted time frame whereas the graph of $\cos(2\theta)$ goes through two complete cycles in the same allotted time frame." Joe did note that there are twice as many "humps" in $\cos(2\theta)$ as there are in $\cos\theta$, but was not comfortable predicting what the graph of $\cos 3\theta$ would look like. Joe did predict that there would be *n*-times as many humps for $y = \cos(n\theta)$ once they had graphed $y = \cos(3\theta)$. DH closed the lesson by restating Joe's comments in more formal language: "So what you're saying is that $\cos(n\theta)$ will go through *n* times as many complete cycles as $y = \cos\theta$." Joe's generalization appeared to be an empirical abstraction. He looked for patterns across cases and not for an underlying principle by which they all behaved.

Summary of Joe's Performance on Assessment Task

Joe was able to answer the first three assessment tasks correctly, however DH mentioned in her reflective essay that she was unsure if he was remembering what they had done as part of instruction or if he was making sense of the problem and reasoning in terms of covariation. For example, he struggled to describe why the graph of $r = \cos(2\theta)$ behaves as it does (item (c)). Once DH hinted that he might want to "break it up into smaller pieces," Joe was able to use the language of covariation to describe how the values of *r* varied over intervals of θ. For the final two items, DH noted "Joe was at quite a disadvantage, as we had not really had sufficient time to go over the idea of periodic functions" (DH, Reflective Essay, 10/18/04). Joe's answers to items (e) and (f) consisted of an attempt to conjecture about the relationship between the coefficient and the shape of the graph: he proposed that $\cos(4\theta)$ might be "two times as big" as $\cos(2\theta)$[32]. Though his answer was incorrect, it is more interesting to note that DH had accepted his answer to item (c), while believing that he was unprepared to answer items (d) and (e). If Joe was unprepared to discuss the behavior of the periodic functions $\cos(\theta)$ and $\cos(4\theta)$, it seems unlikely that his discussion of $\cos(2\theta)$ could have been anything more than repeating what DH had said about $\cos(2\theta)$ a few minutes before. Thus, we see that DH has not yet made sense of the fact that a graph behaves the way it does because of the specific way in which the quantities in question covary – she thinks it is possible to understand how a function behaves by observing characteristics of graphs without first making sense of the covariation.

[32] What he was envisioning was a graph that had for each value of θ an *r* that was twice that of the *r* for $\cos(2\theta)$. It is important to note that his image of this consisted of a picture, not the underlying covariation of quantities.

Highlights of KN's Lesson

Covariation

KN began his lesson by focusing on covariation. He presented Jamie with the following problem: "The tuition at Vanderbilt doubles every year for two years, then stays the same for two years, then decreases by half for the next two years." In Excerpt 8-10, he has Jamie physically model the function. After Jamie had modeled the tuition function in the air, KN had her do the same thing at the whiteboard, with her right hand moving along the *x*-axis (which he had drawn) and a marker in her left hand. He then commented "So what you basically just did … is you showed how two different variables varied together. You showed as time goes on, the cost does something according to time. So the cost is dependent on the time."

Excerpt 8-10 (KN's Lesson, 10/4/04)

10. KN: OK, I want you to do this … you're going to put your right finger on the desk and this is going to trace time as time goes on. [KN moves right finger along the desk.]

11. J: OK.

12. KN: And put your left finger on the table and this is going to trace the cost. [KN moves his left finger vertically above the desk.] So as time goes, then trace the cost with your left hand. And keep your left hand above your right hand.

[Jamie moves fingers as in Figure 8-8]

Figure 8-8: Jamie Modeling Tuition Function

KN then asked Jamie to use her work on the tuition function as a model to explain how *y* varies as *x* varies, where $y = x^2$. She responded: "OK, when *x* is negative and the negative number gets smaller, *y* decreases, but as *x* turns to positive, the graph increases." He accepted her response and explained the point of the exercise: "The point is that when someone says 'draw $y = x^2$', you just draw a U-shaped thing, right? The point is to get you to notice how the two variables, how *y* and *x* vary together. It isn't just a picture … it's … as *x* increases, when *x* is negative, *y* decreases until it gets to zero. And you can do the same thing with your fingers."

Sine and Cosine Functions

KN found Jamie to be much less familiar with trigonometric functions than he anticipated and drew a sine graph (in rectangular form) as an introduction to sine and cosine functions. Since Jamie was not familiar with the graphs, he also labeled (0, 0), (0, π), (0, 2π) as the important points on the graph and asked her to label the remaining important points[33] (the labeled graph is shown in Figure 8-9). He then helped Jamie label the important points on the graph and then

[33] It is important to mention that KN was having Jamie label the points using the established pattern on the graph, not her understanding of how $\sin\theta$ behaves.

asked her to describe the covariation in the same way as she had for the college tuition function and for $y = x^2$. Jamie gave an acceptable explanation by observing the points labeled on the graph and noting "as *it* varies from 0 to π/2, *the graph* varies from 0 to 1." KN consistently corrected her use of "it" and "the graph" to the more proper *x* and *y*. After they had described the behavior of the sine, he drew the cosine graph and asked her to label the important points and describe the behavior. He concluded his instruction on sine and cosine graphs by formalizing the ideas of the dependent and independent variables, the domain and the range, and periodicity in a fairly traditional manner.

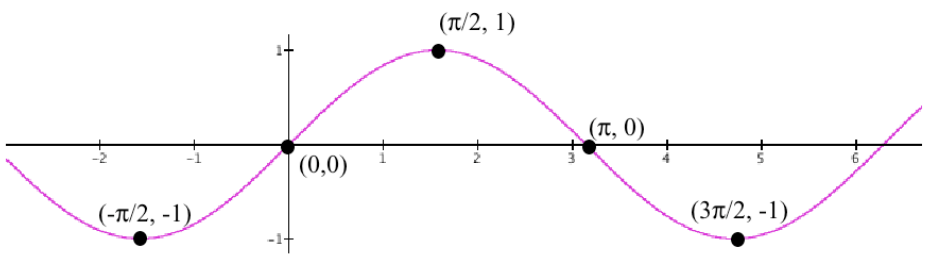

Figure 8-9: Jamie and KN's Sine Graph

Families of Functions

Once they had the graphs of sin *x* and cos *x*, KN asked Jamie to explain what the graph of $y = \sin(2x)$ might look like. In order to do so Jamie created a table of values and graphed the function. Jamie noted that the graph of $\sin(2x)$ hits the *x*-axis twice as many times as $\sin(x)$ and KN revoiced her comment: "So what you're saying is that the frequency, the number of times the graph goes up and down in a given amount of time, doubles." She was able to correctly predict that the graph of $\sin(nx)$ would have a frequency *n* times as great as $\sin(x)$, however this prediction was based on empirical abstraction across the graphs as opposed to an understanding of the way in which *x* and *nx* covary – i.e. that *nx* varies *n*-times as rapidly as *x* does.

Polar Coordinates

KN asked Jamie to plot five points and focused her attention on how she knew to locate each point where she did. He noted that she was using “a system with rules” to determine where to locate the points. He then commented that polar coordinates were another system that could be used to locate points, described how to locate points using distance from a fixed point and direction from that point, and asked her to give the polar coordinates for the five points she had already plotted using rectangular coordinates. Once he believed that Jamie understood how to use polar coordinates to locate points, he had her calculate and plot a number of points that satisfy the equation $r = \theta$. He then explained that she could use this graph to show the covariation: “as θ increases from 0, r increases from 0. And as θ continues to increase, so does r.” Once he had this graph, he explained that though it appears that this graph was not a function, in polar coordinates, a function cannot have more than one r for any θ. Jamie then explained why this graph was actually a function in polar coordinates, referring back to her table to convince herself that there was only one r for any particular θ.

KN’s final objective was to help Jamie understand how to graph trigonometric functions in polar coordinates. He did so by generating a table of values for $r = \sin\theta$, constantly referring back to the previously generated table of values for $y = \sin x$. and having Jamie plot the points. Excerpt 8-11 gives part of the conversation.

Excerpt 8-11 (KN's Lesson, 10/4/04)

1. KN: OK, Jamie, where would we plot the first point?
2. J: When $\theta = 0$, it’s zero. So I’d plot it at the origin.
3. KN: Right. When $\theta = 0$, $r = \sin 0$, which is 0.
4. J: OK, (pause) and when $\theta = \pi/4$, the radius equals $-\sqrt{2}/2$.

5. KN: And I'll figure out what that is … Because I don't know what $\sqrt{2}/2$ is. Do you?

6. J: No.

7. KN: OK, it's like 0.7.

8. J: OK.

9. KN: So what does 0.7 mean?

10. J: It's the length of the radius.

11. KN: So at that angle, there's a point that is 0.7 away from the origin. Plot that point.

12. J: So, like over here [Jamie indicates a location close to (45°, $\sqrt{2}/2$)]?

13. KN: Good.

14. [They work their way from $\theta = 0$ to $\theta = 2\pi$ in a similar manner.]

15. KN: So, what would the graph look like?

16. J: Like this? [Jamie connects the dots.]

17. KN: OK, good. Now how do you know that there will be all these points in here [KN indicates the portion of the graph between (0,0) and (π/4, $\sqrt{2}/2$); Region A (red) in Figure 8-10]?

18. J: Well, I just connected the two points.

19. KN: But how do you know that there will be points in there? How do you know that there is a point on the graph in between the two points? [Long pause; no answer from Jamie.] Remember in all these graphs back here, I've asked you to explain how y varies as x varies, so in this case, we'd say that as θ varies from 0 to $\pi/4$, the r varies from 0 to $\sqrt{2}/2$ or about 0.7.

[…]

20. So how do you know what that these points will be in here? [KN indicates the portion of the graph between (π/5, $\sqrt{2}/2$) and (π/2, 1), which is Region B (green) in Figure 8-10]? You've said it several times today.

21. J: Because as θ varies from 45° to 90°, the r varies from 0.7 to 1.

22. KN: Good.

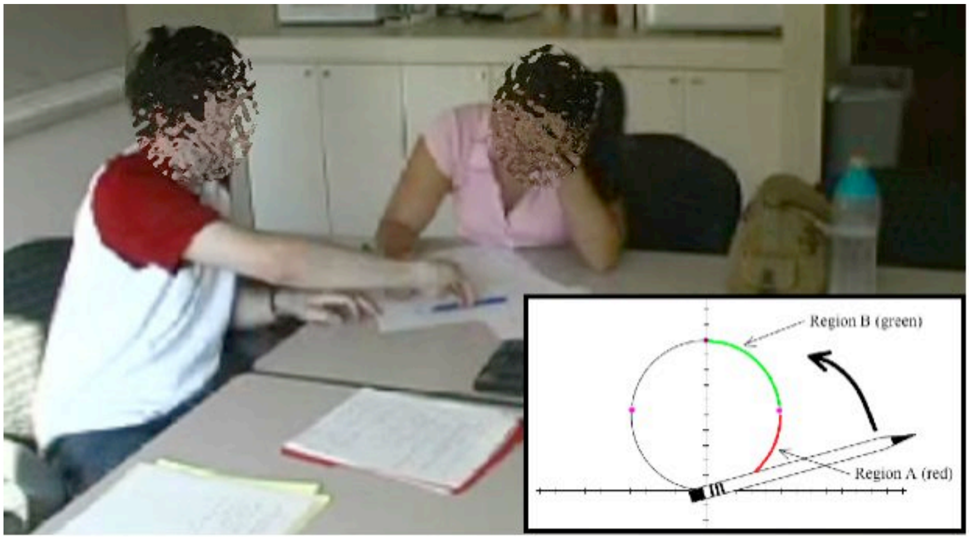

Figure 8-10: KN and Jamie Graphing r = cosθ

Summary of Jamie's Performance on Assessment Task

Jamie responded correctly to the first three assessment items, however much like Joe and Annie, her responses lacked enough explanation to make conclusions about the understandings behind her answers. She was unable to answer items (d) and (e) correctly, but her response to item (d) is particularly interesting. She began by creating a table of values, using the same values of the independent variable as she had used for cos(2θ) (Figure 1-11), and plotted the points. Next to the table, she noted that she did not know how to graph the function because she did not know what happened between the points. Thus, Jamie had developed an understanding that it was important to understand what happened between landmarks, but she had yet to make sense of where to look for this information – had she been reasoning in terms of covariation, she would have turned to the functional relationship for this information. As it stands, it appears that, to her, the values in this table were *points* and not simultaneous values of covarying quantities.

θ	cos(4θ)
0	1
π/4	-1
π/2	1
3π/4	-1
. . .	. . .

Figure 8-11: Jamie's Table of Values for Cos(4θ)

Emergent Issues in the Instruction.

The similarities in DH, KN, and SS's instruction goes beyond the main topics that each had discussed in their meeting with TI—there was a striking similarity in the understanding of functions and covariation that the PSTs drew upon while creating and teaching their lessons. Each PST wanted their HSS to physically model the covariation of quantities much as they had done in the *Introduction to Graphing and Covariation* activities. They did deviate from the *Introduction to Graphing and Covariation* instructional sequence in one significant way: Rather than understanding how the quantities varied and using the graph to record and keep track of the variation (and ultimately covariation) of the quantities, their attention was on understanding the graph by focusing on how a point on the graph might vary along the graph (much like a wooden bead moves along sculpted metal wires in children's toys). For example, in Excerpt 8-7, we see SS asking Annie to use her fingers to model time and distance without attention being paid to the significance of each quantity or the importance of the landmarks. Further, her explanation of the importance of covariation was "Like how something would vary, it creates a graph because these things vary. Like if there was no time movement … it's the time movement that keeps this [the distance finger] moving along the graph." (Excerpt 8-8, line 8). In this lesson segment, SS's actions suggested an unawareness of the need for tight coherence in Annie's thinking of what she (Annie) was modeling by coordinating her fingers. Annie seemed not to understand what her fingers' locations represented, nor how moving her fingers was supposed to be constrained by the behaviors of the represented quantities. I suspect that Annie would have ended with a similar understanding of SS's point had SS gone straight to having Annie move one finger through the air, then proclaiming, "See, you've traced a graph."

In DH and KN's instruction, we saw them helping their HSSs calculate important points and then, for each region between these important points, asking their HSSs to physically model (using fingers and fairy dust or an analog) and reason about how one quantity varied as the other varied. Both the activities that they orchestrated and the way in which they (and their HSSs) described why the graph appeared as it did (DH on Page 222, KN and Jamie in Excerpt 8-11) were very similar to the activities and descriptions detailed in Chapter 6. There was, however, one significant difference: In their work with the HSSs, their descriptions of the graph were grounded in points that lie on the graph. Thus, though they did appear to be reasoning about quantities, they were really reasoning about how the graph looked between particular points on it. For example, consider Figure 8-12. The PSTs helped the HSSs calculate and plot points of interest (purple in Figure 8-12) and then reasoned about how black point behaved between the purple points. Their reasoning could be described as follows: *since we know the black point is moving between the points (0,0) and (π/2, 1), we know that as x varies between 0 and π/2, the y has to vary between 0 and 1.* The instructional conversations that the PSTs orchestrated were grounded in helping their HSSs reason about graphs and to speak about how the x- and y-coordinates (or r- and θ-coordinates) change between specific points on the graph. But their attention, nevertheless, was on the "space" in a quadrant that lay between two (purple) points.

Put another way, their image of the covariation of *x* and *y* (or of *r* and *θ*) was *driven* by their anticipation of the graph's shape. The graph's shape did not emerge from the covariation of two quantities. This claim is supported by the HSSs' inability to reason through assessment item (d) –without the graph the HSSs could not construct the covariation.

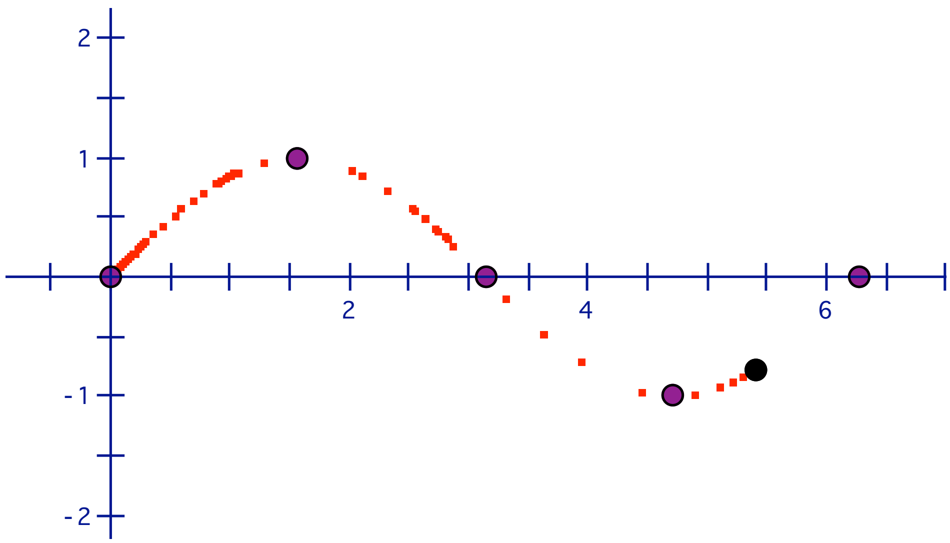

Figure 8-12: Keeping Track of the Variation of the Black Point

In Excerpt 8-11, we saw KN helping Jamie understand why there will be points between (0,0) and ($\pi/4$, $\sqrt{2}/2$) and between ($\pi/4$, $\sqrt{2}/2$) and ($\pi/2$, 1) (the red and green regions in Figure 8-10). He helped her see that we are assured of having the "in-between" points because of covariation, namely the fact that they know that "as θ varies from 45° to 90°, the *r* varies from 0.7 to 1" (Excerpt 8-11, line 21). It is important to mention that it is unlikely that KN believes that it is simply because "as θ varies from 45° to 90°, the *r* varies from 0.7 to 1" that he is guaranteed the in-between points; his explanation is grounded in his own understanding of covariation. In chapter 6, I showed that he had developed the imagery that would scaffold his statement. When KN spoke of the variables varying, he was likely envisioning two quantities covarying and the corresponding movement of the correspondence point. However, in his instruction it was *a point on the graph*, not the variable quantities and the correspondence point, that was the salient issue.

Post-Instruction Assignments

The final part of the project involved the PSTs writing reflective essays for TI about their work with their HSS. Each PST had a video of their instruction as a reference for their essay. This reflective essay was due approximately three weeks after their lesson. Follow-up interviews, which focused on their instruction and reflective essay, were held approximately one week after they turned in their essay. The class revisited trigonometry and polar coordinates during the time in which they wrote their essays, and then moved on to a unit on rates of change. In addition to the content covered in the sequel to their project, TI spent a considerable amount of class time making the logic of his lessons explicit. He orchestrated class discussions about the particular mathematical understandings that served as a background for the instructional activities in which the PSTs were taking part. He also focused PSTs attention on "sub-goals" of instruction and the purposeful sequencing of these activities.

There were two crosscutting ideas in PSTs' reflective essays and their follow-up interviews—the importance of covariation and the importance of deep understanding of mathematics.

The Importance of Covariation

In their reflective essays, each of the PSTs indicated that they felt that they should have spent more time helping their HSSs understand covariation. This is despite the fact that they had specifically mentioned covariation in their initial lesson plans as one of the few key ideas that they wanted their HSSs to understand. PSTs' comments provide us with insight as to their reasons for the perceived inadequacy.

> I wrote in my project report that I needed to spend more time on building up to covariation. When I tried to have Jamie track the two variables with her fingers. I started by asking her to track the two variables together, with one on top of the other. What I realize now was that I moved too fast through this … it would have been really important for her to track the variables independently first. I think I rushed through this because I was so comfortable with tracking the two different things at once—I could almost do it in my head. I thought she would be able to catch on very quickly (KN, Follow-Up Interview, 11/3/04).

> I realize now that I could have slowed down when explaining about the relationship between independent and dependent variables and their covariation in relation to functions. As this was a really core concept that I needed to solidify with Joe in order for him to have the possibility of giving the kind of answers I wanted him to for the assessment question (DH, Reflective Essay, 10/18/04).

> What I learned was that I took a lot for granted. First, there was the simple idea that Joe didn't know what the symbols, like theta, were. More importantly, though, I think I really rushed through covariation when it was probably the most important thing we could have talked about. In order for Joe to have answered the assessment questions—even for him to understand polar functions—he needed to understand that there were variables that were varying and the graph was tracking how those variables covaried. My problem was that I rushed through helping him understand what was going on—that there was an angle, or an arc-length, that was varying and as it varied, $\sin(\theta)$ varied in some way related to the variation of θ. I understood most of this—well maybe not the angles and arc-lengths—but it didn't come across in my lesson. I think I had just spent so much time thinking and doing covariation that it was, kind of, understood. I guess I didn't think I needed to spend that much time on it. (DH, Follow-Up Interview, 11/3/04).

Thus, for DH and KN, the idea of covariation had become transparent to them. KN later described it as a "mindset" that he used to approach problems, but not something he was conscious of. SS was also aware that she skimmed over ideas that she now realized were fundamental to Annie having a chance to understand what SS was trying to teach. It seems that all three realized that many of their students' difficulties stemmed from an inadequate understanding of covariation, and the PSTs realized that they should have provided this in their teaching.

The Importance of Deep Understanding of Mathematics

Both DH and KN noted that as a result of their work with their HSS, they recognized the importance of focusing on mathematical understandings in their planning. Though they had written their lesson plans in narrative form, they felt that the narrative they described was more of a discussion with someone with similar understandings to their own as opposed to a HSS's understandings. In his essay, KN noted that

> I realize now that when I taught the lesson, I did not even have a good understanding of [how you measure the argument of a trigonometric function]. I did not anticipate her having this problem … this proves to me how important it is for a teacher to have a deep understanding of the subject matter being taught (KN, Reflective Essay, 10/18/04).

When questioned about this comment in the follow-up interview, he clarified his thoughts:

> I did understand everything in my lesson plan, but in my plan I kind of glossed over the idea of what a radian was. I now understand what a radian is [conceptual foundations of radian measure had been covered in class] and why it is important when graphing trigonometric functions. But more important, I think, is the idea that when preparing for lessons, I realize now that it is important to plan what you want students to learn and what they need to know in order to learn it, rather than what you are going to ask them to do or how you are going to teach them. Had I thought about this before teaching Jamie, I think some of my weaknesses would have been evident and I could have worked on them (KN, Follow-Up Interview, 11/3/04)

DH made a similar observation about her instruction on families of trigonometric functions:

> I think he saw that y = cos(2x) goes through 2 complete cycles in the same period that it takes for y = cos(x) to complete only one cycle, but I know he does not understand that this happened because the argument varies by 2π. For all he is concerned the graph does that because I told him it does (DH, Reflective Essay, 10/18/04).

> I really thought I understood polar coordinates and functions before my lesson. I thought I planned a good lesson. I think I tried to do too much, though. I spent all this time teaching him about plotting points and graphs and stuff, but like, I think the important thing was understanding the covariation. And understanding trig functions. If he could understand what the independent variable is and what sine represents … and trace out how they vary … those were the big ideas (DH, Follow-Up Interview, 11/30/04)

> In thinking about my plan and my lesson, I think that the … I really needed to understand the logic of the lesson better. I mean, I wasn't prepared for him not to know how to graph cosine functions. I wasn't prepared to teach him about cosine functions and how to graph them—I was just going to have him compare the graphs and find the relationship. Now I see that understanding the cosine function is really important because the big ideas are periodic functions and the argument varying by 2π. Now, after the past few classes, my lesson logic would include understanding the cosine function in terms of an angle, which is an arc length, and the *x*- or *y*-coordinate of a point on a circle centered at the origin (DH, Follow-Up Interview, 11/3/04).

In these excerpts, we see that, upon reflection, the PSTs have recognized that some of their instructional difficulties stem from their lack of understanding of the mathematical ideas they were teaching. They also indicate that they now understand that their understanding needs to be structured much like a "logic of the lesson" that they had been discussing in class. Later in his interview, KN described this lesson logic as consisting of nested understandings of mathematics: "Two things are important. First there is what you want the HSS to understand—the big idea—and second there is the order in which they need to learn the things that are necessary for them to understand it" (KN, Follow-Up Interview, 11/3/04).

Discussion of Instruction

In this chapter, I documented the PSTs' participation in an instructional activity that took place over approximately five weeks. Their conceptual narratives, which were completed towards the middle of Phase I of instruction, provided insight into what the PSTs understood to be the key aspects of instruction. These key aspects, which included the uniqueness of the mapping, variables and variability, and the appearance of the graphs, are consistent with the PSTs' initial assessment and their experiences in the course. These three themes also appeared in their instructional plans. However, in their plans they included an understanding of covariation as an additional goal of instruction. Though they included covariation in their plans, the majority of the emphasis throughout their instruction was on the first three: teaching an analog for the vertical line test, defining r and θ, and plotting points to get a better sense of what the graph looks like.

The PSTs' pedagogical conceptualization of mathematical ideas can be characterized through analyses of their lesson plans and instruction:

- *Coordinate systems* are used to locate points, and the important thing for HSSs to know about coordinate systems are the variables that are used to represent the coordinates of a location and a rule for determining the specific coordinates. Once they know the rule, they need to practice locating the points until they get comfortable with it.
- *The sine and cosine functions* are periodic functions, which means that the graph repeats itself. The important things for HSSs to know are values of sine and cosine of 30-60-90 and 45-45-90 right triangles.
- *Families of functions* are groups of functions that have the same sort of shape. The "shape" is determined by finding and plotting points.
- *Covariation* is everything that goes on between the points that you plot.

These pedagogical understandings are not consistent with the PSTs' understandings of functions and graphs discussed in detail in Chapters 6 and 7. It is important to note that their pedagogical conceptualization of the mathematical concept can only be understood when it is examined through the lens of their current understandings (mathematical, pedagogical, and otherwise). In other words, it is not necessarily the case that the PSTs think that the conceptualizations discussed in the previous paragraph are the only important issues in understanding the ideas. Rather, the pedagogical conceptualizations are the result of the assimilation of the course content and assignments onto their current schemes *for understanding* mathematics. For example, creating a table of values was the way that the PSTs had organized the coordinated values of the variable quantities. They *learned* to use the table as a tool to reason about the way in which the variables covary. It, therefore, makes sense for them to think that if they were to help their HSSs understand tables of values, they would see functions and graphs as they do.

I argued in previous chapters that the PSTs had developed a key developmental understanding of functions as covariation of quantities. In this chapter, we see evidence of the fact that a key developmental understanding of function is not sufficient for developing instruction focused on understanding functions as covariation of quantities. This result was not unexpected. In Chapter 2, I suggested that PSTs may need to develop a *key pedagogical understanding*. I said,

> A *key developmental understanding* is a particular understanding of a mathematical idea that facilitates understanding a variety of additional mathematical topics. A *key pedagogical understanding* involves an individual's awareness of the pedagogical implications of those key developmental understandings of important mathematical ideas.

A key pedagogical idea is one way of thinking about particular understandings of mathematics content that can support the development of instructional environments where HSSs can engage in productive mathematical conversations.

Though there was no evidence in PSTs' lesson plans or instruction indicating that they had developed key pedagogical understandings, their reflective essays and the follow-up interviews showed some indications of progress. Specifically, the PSTs came to understand covariation in a different way. Rather than focusing on covariation as something to do, both KN and DH noted that it was something that they needed to build up through focusing on the variation of the individual quantities and then moving to the coordination of the variation. They both indicated that approaching the lesson with an emphasis on covariation of quantities would have enabled them both to help their HSSs be more successful on the assessment and also reduce the amount of material their HSS needed to know. Their understanding of covariation indicates a shift in their pedagogical conceptualization of covariation. Rather than *the thing you use to make sense of what goes on between points you've plotted*, they needed to come to understand covariation as a way of thinking about the situation and had begun to realize that that way of thinking needs to be developed carefully (first with the notion of variables varying independently, then in a coordinated manner, then as producing a graph as a record of the covariation). The paradoxical nature of this finding should be noted: *It took significant effort, both on TI's and the PSTs' part, for the PSTs to become conscious of the instructional development that they, themselves, had experienced first hand.*

Follow-Up with PSTs

As already mentioned, TI placed a great deal of emphasis on understanding the logic of a lesson. What he meant by this was that in designing instruction, a teacher's focus needs to be on two things: what the teacher wants the HSS to understand and how the teacher envisions positioning the HSS to develop this understanding. Thus, planning for instruction is about the continual interplay between what the teacher asks the HSSs to do and the pedagogical purpose of those activities.

In service of having the PSTs realize that objective, TI assigned the following task: *Design a lesson logic, which includes the activities the HSSs will take part in and the rationale for those activities, to teach the point-slope formula and point-point formula* (TI, November 11, 2004). Their completed lesson logic is shown below in Figure 8-13.

Step	Action	Reason
1.	Give the students a point in the xy-plane and a rate of change. *[Note: Here it may be helpful to give them this rate of change in a context they are familiar with. If they have encountered rates of change when talking about the distance a car travels over some amount of time, refer back to such an example so students can relate this to their prior knowledge about the subject of rate of change.]* Now, ask the students to find a second point where the rate of change between these two points is the given rate of change.	This forces students to think about the relationship between the *x* and *y* coordinates. They need to think "If I am given a rate of change, this means that when *x* increases by some amount, *y* increases by the rate of change times however much *x* increased by."
2.	Ask different students in the class the point that they chose and have them explain how they found it. Most likely, different students will find different points that satisfy the above instructions.	This emphasizes the fact that many different pairs of points can share the same rate of change between them.
3.	Assume that at least one of the students found his or her point by finding the value of *y* when *x* increased by 1. Ask students what would happen if *x* changed by 2, 4, ½ or by $^1/_{100}$ instead of by 1.	This helps students to see that there are many points that satisfy the scenario.
4.	Ask the students if they have found all the points that satisfy the above scenario that the rate of change between their new point and the initial point is the given rate of change. If not, ask them where the other points are that do satisfy the scenario.	Students should see that any point on the line determined by their two points would also satisfy the relationship. The students should see that since a line is made up of an infinite number of points, then an infinite number of points satisfies the scenario.
5.	We now want the students to talk in more general terms. Ask them what would happen if, given some rate of change, you increased x by an arbitrary amount. Have them formalize this in an equation.	By getting away from thinking only in terms of whole number intervals, they should see that to find the change in the *y*-value, you can multiply the change in the *x*-value by the rate of change. The students would come up with the equation Δy = (rate of change)$\cdot\Delta x$.
6.	Ask students to think about what they would do if somebody else did exactly what they just did (started with an initial point and a rate of change and found a series of points that all share the same rate of change), and told the students two of their points, and the students	This gets the students thinking about their method. It moves them away from thinking in terms of "there is a set formula for finding the rate of change or a line" and, instead, gets them to think of the relationship between change in *x* and change in *y* and the rate of

	were asked how they could use this information to figure out what rate of change the other person started with.	change of the associated line.
7.	Ask the students how they found the rate of change between those two points. The students should be able to tell you that to find the rate of change, you simply divide the change in y by the change in x.	By seeing that the (rate of change) = Δy / Δx, the students will be able to see why both the point-point and point-slope formulas work.
	[…]	[…]
12.	Ask the students what the change in *x* is and what the change in *y* is between these two points. They should see that the change in x is (x-2) while the change in y is (y-3).	This step is to get them to see even though *x* and *y* do not have particular values, you can still discuss how much their values change over any interval.
13.	Using the equation they came up with earlier—Δy = (rate of change)·Δx—have them plug in their values of Δx and Δy that they just found. They will get the equation (y-3) = 3·(x-2).	Now the students will see that the line drawn on the board can be symbolized using an equation.
14.	Ask the students what you would need then, to write an equation for any line. They should see that all is needed is the slope of that line and a point on that line.	This gets the students to understand the point-slope formula for a line.
15.	Ask the students how you would write an equation for a line if you were instead given just two points that were on the line. Since the students already know how to find the slope from two points on a line (Steps 6&7), they should see how to do this.	This gets the students to understand the point-point formula for a line.

Figure 8-13: PSTs' Lesson Logic for Equations of Lines

In this lesson logic, we see a number of significant shifts in the PSTs' mathematical and pedagogical understandings. In contrast to their lesson plans on polar functions that were content driven, this lesson logic was centered around HSS's understanding. Each of the reasons describe the particular way that they desire their HSSs to be thinking. For example in step 1, we see that the desired understanding of a rate of change is in terms of proportionality: "They need to think *If I am given a rate of change, this means that when x increases by some amount, y increases by the rate of change times however much x increased by*." Their logic also is focused on generalizing: in step 4, the action involves asking about all the points that satisfy a given relationship and their rationale is that they desire their HSSs to come to think about a line emerging from considering all the points that satisfy the given relationship.

The striking differences between the PSTs work with their HSS and in the lesson logic raises the question of what might have transpired in the weeks between the two – the lesson logic task was assigned almost a month after my interviews with DH and KN. Unfortunately, data from the instruction that took place during this time was not collected and any claims about the mechanisms for their development are largely speculative. During this period the class continued to stress ideas rooted in covariation, which included average rate of change, Riemann sums, and the fundamental theorem of calculus from a covariational perspective. It seems that the students needed more time to fully understand the content and to explore additional related mathematical topics before they could fully make sense of the implications of functions as simultaneous covariation of quantities. Additionally, throughout the remainder of the course, the instructor explicitly focused classroom discussions on the logic of the lesson and incorporated that focus into their daily work – in addition to completing the assignments, classroom time was devoted to unpacking what one needed to know and understand in order to answer the questions in the desired way and possible instruction which might position students to develop such understandings.

TI envisioned these classroom discussions involving the de-construction of mathematical ideas as a vehicle for developing content knowledge that would support conceptual instruction guided by the PSTs' own mathematical understandings. The creation of this lesson logic required the PSTs to understand mathematical ideas, but also required them to break down those understandings into smaller cognitive objectives that, were HSSs to develop them, would result in a high likelihood of their coming to understand the mathematics in the desired ways. For example, rather than showing or telling the HSSs that given a constant rate of change that an arbitrary change in x, the change in y could always be determined, they constructed a sequence of activities (steps 1-4 of Figure 8-13) that would help the HSSs to understand that this idea as a consequence of the understandings they abstracted from their activity.

In Chapter 2, I discussed a conceptual analysis of understanding function as covariation of quantities. This included (i) A variable is a measurable quantity of variable magnitude; (ii) A function is a relationship between the two variables; and (iii) The graph of a function results by keeping track of the simultaneous covariation of quantities. In Chapters 6 and 7, I presented evidence that the PSTs had developed covariational conceptions of function. In particular, the evidence indicated that they had come to reason in terms of (i) and (ii) and were developing a sense of (iii). The lesson logic presented in Figure 8-13 indicates that the PSTs had come to understand (iii) more deeply during the 4 weeks between the end of this study and the generation of the lesson logic. Moreover, the lesson logic provides us with evidence that the PSTs have developed an awareness of a similar conceptual analysis of functions as covariation of quantities. Their conceptual development, which focused on the variability of the quantities (lines 3, 12, 13), the relationship between two variables (lines 1, 2), and the graph as resulting from tracking the simultaneous covariation (lines 4, 6), is evidence of this awareness and indicative of a developing key pedagogical understanding.

CHAPTER IX

SUMMARY AND CONCLUSIONS

This final chapter highlights the study's key findings as well as its contributions, implications and limitations. I will also provide an overview of the three phases of the study as the findings, contributions and implications are grounded in an examination of the PSTs across phases.

Summary of Findings

Research Question #1: Pre-Service Teachers Understandings of Function

The instructional phase of this study had as its goal to understand three pre-service teachers' developing understanding of function as simultaneous covariation of quantities. The setting for the study was an instructional sequence that employed simultaneous covariation of quantities as a vehicle to assist the PSTs in developing an understanding of function that enabled them to engage with a wide array of problems dealing with applied and abstract functional situations. This instructional sequence was envisioned as positioning the PSTs to develop a Key Developmental Understanding [KDU] of function. In Chapter II, I discussed Simon's (2002) idea of a KDU as a way to think about the mathematical understandings that a mathematics course for pre-service teachers [PSTs] would ideally engender. These KDUs include a (i) conceptual advance that enables one to find different, yet conceptually related ideas and problems understandable, solvable and sometimes even trivial and (ii) a fundamental transformation of one's understanding that necessitates a shift in the way the individual interprets the world around them.

The instructional phases of the study (Chapters VI and VII) were designed to study the PSTs' understandings of function and better understand both if and how KDUs of function might be developed. Activity 1 was intended to help the PSTs develop imagery that would support their development of a covariational conception of function. The PSTs progressed through activities in which they physically modeled quantities, first separately, then together, and finally in a coordinated manner. The PSTs were ultimately able to model the variable quantities but struggled to model the covariation. Analysis of Activity 1 indicated that though they were able to imagine the variation of the two quantities, they lacked a means of coordinating the variation. In addition, analysis indicated that one reason for their difficulty in coordinating the quantities was an incongruity between their existing understanding of functions and graphs (as solid wires that points move along) and the desired understanding of functions and graphs (graphs as resulting from tracking the values of covarying quantities). Once the PSTs realized that the activity of modeling the quantities resulted in a traditional distance-time graph, their understanding of the focus of the activity shifted from coordinating the variation of the quantities to making their fingers vary in a particular way.

Activity 2 provided the PSTs with situations within which they needed to represent covariation of two quantities in a graph and interpret graphs in terms of the constituent quantities. Though for the majority of the problems the PSTs' initial inclination was to speak of global characteristics of the graph (bigger/smaller, fatter/skinnier), towards the end of the activity we saw a significant shift in the PSTs' thinking. Once KN introduced the idea of reasoning about the covariation of quantities in terms of two-dimensional landmarks (a way of reasoning about the covariation that, interestingly, the instructor had introduced on at least two previous occasions), this means of organizing the covariation of quantities became part of the public discourse and was regularly called upon in class and as part of their written work by the each of PSTs.

In Activity 3, we saw a stable pattern emerge in the way in which the PSTs engaged with the word problems. For all but the last problem discussed in class, the PSTs began by generating and graphing equations and attempting to solve for the unknowns. Analysis of their solution methods indicated that they used graphs to check the internal validity of their calculation of the unknown and the graphs provided little support for their understanding of the situation or explaining the significance of the calculations and the solution. Only after nudges by the instructor did they turn to analyzing the covariation. Throughout Activity 3, the PSTs consistently showed that they had developed understandings of functions and covariation that enabled them to reason through the applied problems, but their initial inclination was not to do so.

Activity 4 engaged PSTs in activities designed to further focus their attention on the utility of analyzing situations in terms of covariation of quantities. In the problems of Activity 4, traditional understandings of trigonometric functions and polynomials were insufficient for answering the questions asked. Though again, the PSTs' initial inclination was to reason in terms of equations, formulas, and static graphs, the PSTs were able to apply covariational reasoning to each of the scenarios and discuss why the graphs behaved as they did. In Chapter VII, I claimed that this was evidence of the possibility that standard mathematics problems could serve as didactic objects that are conceived of as helping HSSs develop understandings that will support the ability to take part in conceptual conversation.

In returning to the question of whether the PSTs had developed a key developmental understanding of function as covariation of quantities, the preceding analysis of the instructional phase of this study provides contradictory results. First, a KDU involves a conceptual advance. Throughout Chapter VII, we saw evidence of such a conceptual advance: rather than functions and graphs being ancillary accoutrements to memorized procedures for finding an unknown, the PSTs came to use functions and graphs as tools for reasoning about how the quantities in question covary, about what solutions might be, and ultimately about why particular solution techniques were appropriate for achieving the desired outcome. This way of reasoning allowed the PSTs to reason through a variety of problems dealing with applied functional situations.

As to the question of whether the PSTs have experienced "a fundamental transformation" of their understanding that necessitated a shift in the way the individual interprets the world around them, the results are not as simple. They have clearly transformed *the way they can* interpret the situations in the world around them, however, even towards the end of the instructional phase, their tendency was not to interpret situations in terms of covariation of quantities. With regards to this, I conclude that they have transformed the way in which they understand functions, but at this point the transformation consists of an incomplete accommodation of their new understanding of function on to their existing conceptual structures. At the conclusion of Chapter VII, I claimed that the PSTs' new understanding highlights the fact that standard mathematics problems can serve as didactic objects that are conceived of as helping HSSs develop an understanding that will support the ability to take part in conceptual conversation. In other words, the PSTs developed the capacity to speak conceptually about the covariation that arises in a variety of functional relationships. However, the incomplete accommodation indicates that that this understanding is likely not yet a KDU.

Research Question #2: Understanding the Relationships Between PSTs' Understanding of Covariation and Teaching for Understanding of Functions

This study is grounded in the belief that how a teacher understands a mathematical concept will influence the instructional conversations that a teacher envisions as assisting his or her students in developing a consistent, and ideally similar, understanding. Though it could not be concluded that the PSTs developed a KDU of function, it would seem likely that the PSTs' new mathematical understandings of functions would exhibit themselves in their work with the HSSs. Chapter VIII affirms this notion, despite the fact that the PSTs' instruction on functions and covariation is significantly different from the course instruction.

With regards to the idea of a key pedagogical understanding, we saw that the PSTs' goals for their teaching revolved around showing their HSSs the mathematical ideas that they felt to be particularly important as opposed to helping their HSSs come to understand the mathematical ideas in a particularly powerful way. As a result, we do not have sufficient evidence to make claims about the development of KPUs. Instead, I use the term pedagogical conceptualizations of mathematical ideas to refer to mathematical understandings that have pedagogical implications yet do not meet the criteria of a KPU.

Analysis of the PSTs' lesson plans and instruction shed light on their pedagogical conceptualizations of the mathematical ideas: *Coordinate systems* are used to locate points, and the important thing for HSSs to know about coordinate systems are the variables that are used to represent the coordinates of a location and a rule for determining the specific coordinates. Once they know the rule, they need to practice locating the points until they get comfortable with it. *The sine and cosine functions* are periodic functions, which means that the graph repeats itself. The important things for HSSs to know are values of sine and cosine of 30-60-90 and 45-45-90 right triangles. *Families of functions* are groups of functions that have the same sort of shape.

The "shape" is be determined by finding and plotting points. *Covariation* is everything that goes on between the points that you plot. These understandings served as a basis for the PSTs' work with their HSSs.

What should be noted about these understandings is that when viewed from the perspective of the PSTs, these understandings make sense. Their instruction was grounded in showing the HSSs what they had learned to do as opposed to what they had come to understand. The inclination to show HSSs the seemingly salient aspects of a mathematics understanding is a common practice among developing teachers – Simon, Tzur and their colleagues (1998, 2000) have dubbed this the teacher possessing a "perception-based perspective" of teaching and learning mathematics. This perception- based perspective has as one of its key tenets that "Students learn mathematics by direct perception of mathematical objects, principles, and the relationships between them" (Heinz, Kinzel, Simon, & Tzur, 2000, p. 102).

This tendency to focus on demonstrating objects, principles and relationships does not shed light on the incongruity between the mathematical understandings PSTs discussed in Chapters VII and VIII and those described above. As noted in the previous section, at the time of the lesson plan and their instruction, the PSTs were still in disequilibrium with respect to their understanding of functions: they were still trying to accommodate their current existing conceptual structures to incorporate the new understanding of functions. What we see in their instruction is what they have come to see as the salient aspects of using covariation to make sense of functional situations. These salient aspects are remnants of their ongoing accommodation. The PSTs' goals for instruction, the instructional tasks chosen, and the instructional conversations they orchestrated were clearly grounded in the sense they had made of the mathematical content.

Contributions and Implications

In Chapter II, I began by proposing the construct of a key developmental understanding as a way to think about goals for the mathematical development of PSTs and key pedagogical understandings as abstractions of a KDU that entails (a) a teacher being aware of the pedagogical utility of the KDU and (b) an understanding of how the KDU might develop from PSTs' existing understandings. The study was then designed to understand the complexities of the development of a KDU and the relationships between the PSTs understandings of functions and their planning and enacting of instruction.

Despite the fact that the PSTs did not fully develop a KDU of functions as covariation, they did develop an understanding of function that supported their ability to speak conceptually about functional relationships in terms of simultaneous covariation of quantities. The first significant finding of this study is that this understanding, though most definitely necessary, is not sufficient for a teacher to have the ability to conceive of instruction that would enable their HSSs to develop an understanding consistent with the teacher's own understanding. A common interpretation of this inability would be that the PSTs had not developed sufficient pedagogical knowledge (knowledge of teaching that would support a conception-based perspective of teaching (Simon, Tzur, Heinz, & Kinzel, 2000; Simon et al., 1998)) or pedagogical content knowledge (instructional methods particular to teaching the content). However, this study documents the fact that the ways in which one understands content, particularly what one understands to be the salient aspects of the content, has an impact on the kinds of instruction and instructional conversations a teacher has the possibility of employing. Had the PSTs developed a more robust understandings involving particular instructional strategies for teaching functions of polar coordinates, their instruction would likely not be significantly different unless this new understanding influences their understandings of what it means to understand polar coordinates.

In short, this study affirms the belief that rather than teaching the way one was taught, one teaches what they know—an individual's understandings of mathematical content and their pedagogical conceptualizations of the content are the lens through which all instructional activities are conceptualized. When viewed in this light, the two enduring debates in teacher education, (i) how much content knowledge and (ii) what other kinds of knowledge are necessary for teaching for understanding, seem a bit misguided. First, when (i) and (ii) are investigated, they are done so using measures inconsistent with the kinds of mathematical understanding detailed in this study. For example, the numerous studies that note that mathematics teachers often do not understand fractions (or any other sophisticated mathematical idea) and demonstrate that there is a shift in the practices of teachers who take part in in-service development centered on developing fractional content knowledge or pedagogical content knowledge.

The results of this study indicate that unless new mathematical understandings and pedagogical conceptualizations of those understandings are developed, the underlying goals of instruction (what the teachers want their students to learn) will likely remain the same. Few researchers, particularly Peg Smith and her colleagues at the University of Pittsburgh and Pat Thompson and Kay McClain at Vanderbilt University and Arizona State University, take the time to better understand the question of *how might we want our teachers to come to understand a mathematical topic that will support teaching with understanding.* More work is needed to better understand and develop particularly powerful pedagogical conceptualizations of mathematics.

A second significant finding of this study deals with the tremendous difficulty that the PSTs had in developing key developmental understandings of function and related pedagogical conceptualizations of that understanding. Under the best of possible scenarios, including the teacher being an experienced mathematics teacher educator who began investigating students' understandings of functions more than a decade and a small class of significantly above average students, the fact that the students struggled to develop a coherent understanding highlights the fact that the development of these understandings is a truly difficult task. The difficulty of the task emphasizes that one significant impediment to the quality of teacher education is the fact that it is a vicious cycle: it is an extremely difficult task for teachers to develop an understanding of functions (or any mathematical concept) that is inconsistent with their prior understandings. This points to the need for teachers to develop, while students themselves, understandings of mathematics that can develop into key developmental and key pedagogical understandings. Thus, the study is a call for the need for the introduction of key developmental understandings of mathematics content as goals for mathematics instruction in elementary school and high school.

Finally, this study contributes to the development of a conceptual framework for thinking about ways to support the development of content knowledge for conceptual teaching. In Chapter II, I proposed a developmental notion of content knowledge for teaching, which was grounded in research in educational psychology and mathematics education, as such a framework. This study concluded that the PSTs had not developed key developmental or key pedagogical understandings, but that fact does not necessarily diminish the importance of the framework. This study verifies the notion that, first and foremost, a good teacher of mathematics must not only understand mathematics, but they must understand it in a particular way. This study also indicates that these mathematical understandings are not enough, for the pedagogical conceptualization of the mathematics and the salient aspects of both the mathematical ideas and mathematics instruction tend to be transparent to those who have learned them. Thus, awareness of the salient aspects of their own development needs to be incorporated into a refined developmental trajectory of content knowledge for conceptual teaching.

The usefulness of a developmental notion of content knowledge for teaching as a framework for thinking about and investigating phenomena is worthwhile, but the study also sheds some light on the more critical question for teacher educators: *What kinds of instruction can support the development of key pedagogical understandings?* Though the ultimate purpose of this study was to investigate this question, the data gathered did not align itself well with answering it. At the end of Chapter VIII, I discussed two post-instruction assignments that showed the students had developed an awareness of the pedagogical power of their understanding and had developed small insights into its developmental nature. Classroom data is not available to further investigate the mechanisms through which these more advanced pedagogical conceptualizations developed, but there are two possibilities. First, between their work with their HSS and the two post-instruction assignments, the students had taken part in almost four weeks of classroom instruction dealing with trigonometry and rates of change from a covariational perspective. Thus, the possibility that the PSTs needed (a) more time to fully understand both the content and the implications of the content and (b) that the students needed to explore additional related mathematical ideas in order for the pedagogical power of functions as simultaneous covariation of quantities to become part of their developing understandings. Second, throughout the remainder of the course, the instructor explicitly focused classroom discussions on the logic of the lesson and incorporated that focus into their daily work. This study indicates that these three instructional moves (prolonged engagement, focusing on related ideas that emphasize the utility of a mathematical understanding, and a focus on unpacking mathematical understandings) are likely candidates for effective instructional techniques for the development of content knowledge for conceptual teaching. Further investigation into the specific nature of the instruction is needed.

Limitations of the Study

The first obvious limitation of the study was the sample size. Had the research questions being investigated in this project involved understanding if the instruction was effective or not, the small number of HSSs and the resulting lack of reliability of the results would have overshadowed any attempts by the researcher to insure the internal and external validity of the data. The effectiveness of the instruction, however, was not the ultimate purpose of this investigation. The goal of this study was to understand the intricacies and ultimately refine the proposed developmental model of content knowledge for teaching. The study did so by documenting the difficulties three PSTs experienced while taking part in instruction specifically designed for this purpose. Though a larger group of PSTs would have likely increased the likelihood of replicating the specific results documented in this study, that fact does not dismiss the difficulties that this group of three atypical students experienced: DH, KN, and SS, who each graduated in the top 10% of their high school classes, gained admittance to a highly selective private university, and are mid-way through undergraduate programs in mathematics and mathematics education, experienced significant difficulty developing coherent mathematical understandings and pedagogical conceptualizations of mathematics. Under the best of possible circumstances, the difficulties these students faced highlights the notion that developing understandings of the sort described in this study is a difficult endeavor. The work with these three students has served to illuminate possible improvements to the developmental trajectory of content knowledge for teaching – follow up research on this is needed.

A second limitation of the study was the nature and availability of the data. The three major sources of data analyzed in this study included whole-class discussions, students written work, and individual interviews with the PSTs. By its very nature, the data from whole-class discussions both extensive and incomplete – much was said in the classroom discussions, but students often did not articulate their ideas and ways of thinking. Triangulation of whole class discussions and the students written work was therefore essential to ensure the credibility of the claims. PSTs' work often did not provide specific discussion of the way in which the PSTs conceptualized and approached the issues at hand. As a result, the majority of the claims made in this dissertation are grounded in interpretations of conversations and written work. The large-scale findings discussed in this chapter are themes that emerged across the entire study, however following Saldanha (2003), the individual analyses detailed in Chapters VI through VIII "should be taken as *viable* rather than *hard* claims about [the PSTs'] understandings and underlying images and conceptual operations."

Finally, the emergent nature of the study resulted in limitations for the results reported in this document. As planned, to gain insight into the relationships between PSTs' mathematical understandings and their developing pedagogy, the PSTs were interviewed twice, once at the end of instruction and once after their work with their HSS. Retrospective analysis indicated that additional data regarding the development of the PSTs' mathematical understandings would have likely been a valuable resource – more frequent interviews would have allowed for making substantive claims about individual PSTs' development. In addition, as tends to happen in investigations involving human interactions, retrospective analysis indicated that there was a significant shift in the ways in which the students came to think about teaching functions and related ideas. Unfortunately, this significant shift happened after the collection of data from class discussions ended. As previously noted, little data was available to make sense of what, specifically, facilitated this development. Additional work is needed to better understand the significant aspects of the instructional sequence the students took part in and the impact of unpacking students' developing mathematical understandings via focusing on the logic of a lesson.

APPENDIX A

Understandings of Function

PST Interview with HSS

Name of Vanderbilt Student: ____
Name of USN Student: ____
Date of Interview: __________________________

1) In what grade or math class did you first hear the word *function* used mathematically? Did the idea of function make sense? Please explain.
2) Explain the idea of *function* as if to a person who is unfamiliar with the word and its meaning.
3) Is the idea of function important in mathematics? Please explain.
4) State the conventional mathematical definition of *function*. Why do you think it is stated this way?
5) Which of the following six definitions of function do you believe are acceptable? For each, state briefly why you think it is acceptable or why you think it is unacceptable.
   a) A function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set.
   b) A function is a computational process that produces some value of one variable (e.g., y) from any given value of another variable (e.g., x).
   c) A function is a dependence relation between two variables (e.g., y depends on x).
   d) A function is a rule that connects the value of x with the value of y.
   e) A function is a computational process that produces some value of one variable (e.g. y) from any given value of another variable (e.g., x).
   f) A function is a formula, algebraic expression, or equation that expresses a certain relation between quantities.
   g) A function is a collection of numbers in a certain order that can be expressed as a graph.
6) Which of the above definitions in 5 (a – g) would you classify as “best?” Why? Which of these definitions are confusing? How would you describe your confusion?
7) Consider the following statement: “Each morning it takes me 18 minutes to walk 1.2 miles from my house to school.”
   a) What variables are in the description of this situation?
   b) Does the following scenario depicted in the above sentence entail a function? Why or why not?

APPENDIX B

PST Awareness of Pedagogical Purpose
Interview with PST

1. What is "covariation" all about?
What was TI's purpose for you studying it? Talk about the conceptual development thus far. What were the important ideas? Did you learn them? TO what do you attribute your developing knowledge? What might the "logic of the lesson" for this part of the course look like? I.e. what is TI trying to help you understand and how is he accomplishing it.

2. Cross-Cutting Ideas
In class, you all have mentioned that there was some sort of "cross-cutting" features of the problems you encountered in this course? What, if anything, do they all have in common?
Have available copies of the problem sets for their reference.

3. Unpacking the Activities
What was the purpose of the Fingers Activity? Did it work? It was hard in the beginning, is it still hard? Why was it hard? Explain the behavior of x^2 mod x (on desktop). How are the fingers and fairy dust related to understanding the "mod" function? What was key to thinking about the "mod" function? What about with the idea of "smoothness?" Let's say a student was able to draw the graph of y = sin x using their fingers. Is that a conceptual advance over traditional instruction? Why or why not?

4. Assessing Extension and Application of Covariational Reasoning
Explain the behavior of $y = \sin(e^x)$ and $y = e^{\sin x}$. What would be helpful? [Expect reliance on graphs. When unsure, revert to tracing graph, not focusing on quantities – have graphs and table of values ready.] Focus on interpretations of inscriptions (are they tracking quantities or tracing graph). Does this activity have anything to do with Cities A&B?

5. Revisiting Parametric Equations
Given graph of the parametrically defined function $x = t \cos t$ and $y = 2t$, with $0 < t < 3\pi$, examine function questions from miniproject.

Design a 75-minute lesson that will enable a high school student to make sense of functions graphed in parametric equations and graphs of parametric equations. An assessment item that you will give your student is given above. Design your lesson so that it will not only help your student answer the questions about the graph and function, but also so that your student will understand the questions and their solutions. Focus your instruction so that your student sees his or her solutions to the questions as making sense as opposed to remembering what he or she should say. What would they need to know…

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