AMENABLE EXTENSIONS IN II_1 FACTORS

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OVERVIEW

The notion of amenability lies in the heart of the study of von Neumann algebras. In this thesis we consider a question about amenable extensions inside certain II_1 factors.

The paper is consist of five chapters. The first chapter gives a brief introduction on the basics of von Neumann algebras. The second chapter introduces amenability, starting from amenable groups then to amenable von Neumann algebras. We also discuss amenable extensions and maximal amenable subalgebras. The third chapter states the main theorems of this paper and discuss the strategy. The last two chapters are devoted to the proofs of the main theorems.

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Chapter 1

Introduction to von Neumann algebras

In this chapter, we will collect the basics of von Neumann algebras. von Neumann algebras were introduced by John von Neumann in order to establish the mathematical foundations for quantum mechanics. As we shall see, the theory has close connections with many branches of mathematics such as measure theory, group representations, ergodic theory, etc. The treatment is brief and most of proofs in this chapter are omitted, as they can be found in many standard textbooks in this field, [Dix81, KR97a, KR97b, Tak03].

1.1 Bounded operators on Hilbert spaces

Throughout this paper, we always assume a linear space is over the complex numbers $\mathbb{C}$, unless explicitly stated otherwise.

An *inner product space* is a linear vector space V equipped with a map $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ satisfying the following

- $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ if and only if $x=0$;
- $\langle ax+by,z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$;
- $\overline{\langle x,y\rangle} = \langle y,x\rangle$,

for any $x,y,z\in V$ and $a,b\in\mathbb{C}$. Such a map $\langle\cdot,\cdot\rangle$ is called an *inner product* on V.

An inner product space automatically becomes a normed space with the norm given by

$$\|x\| := \sqrt{\langle x,x\rangle}.$$

Definition 1.1.1. A *Hilbert space* is an inner product space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ such that with the norm induced by the inner product as above, $\mathscr{H}$ is a complete norm space (i.e. a Banach space).

A map between two vector spaces $T: V_1\to V_2$ is called a *linear operator* if $T(ax+by) = aT(x)+bT(y)$, for any $x,y\in V_1$ and $a,b\in\mathbb{C}$.

Definition 1.1.2. A linear map $T:\mathscr{H}_1\to\mathscr{H}_2$ between two Hilbert spaces is *bounded* if

$$\|T\| := \sup_{x\in\mathscr{H}_1:\|x\|=1}\|T(x)\| < \infty.$$

The quantity defined above is called the *operator norm* or the *uniform norm* for T. We denote by $B(\mathscr{H}_1,\mathscr{H}_2)$ the space of all bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$. In the case of $\mathscr{H}_1=\mathscr{H}_2=\mathscr{H}$, we simply write $B(\mathscr{H})$.

One also defines a conjugate linear map called the *adjoint*, $^* : B(\mathscr{H}) \to B(\mathscr{H})$ by

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \, , \forall x, y \in \mathscr{H}.$$

It is easy to see that $B(\mathscr{H})$ is closed under taking composition, addition, scalar multiplication and adjoint. Moreover, $(B(\mathscr{H}), \|\cdot\|)$ is a Banach space with $\|TS\| \leq \|T\|\|S\|$ and $\|T^*T\| = \|T\|^2 = \|T^*\|^2$ for all $T, S \in B(\mathscr{H})$. We call such an algebra with a norm and a $*$ operation satisfying all the above properties an abstract *C^*-algebra*.

1.2 von Neumann algebras

The operator norm gives $B(\mathscr{H})$ a nice topology which is locally convex. However in order to define von Neumann algebras, we need more topologies on $B(\mathscr{H})$.

Definition 1.2.1. Let $\mathscr{H}$ be a Hilbert space, $\{T_i\}_{i\in I} \subset B(\mathscr{H})$ a net of bounded operators and $T \in B(\mathscr{H})$.

We say that $T_i \to T$ in the *weak operator topology* (WOT), if

$$\langle T_i(x), y\rangle \to \langle T(x), y\rangle, \forall x, y \in \mathscr{H}.$$

We say that $T_i \to T$ in the *strong operator topology* (SOT), if

$$\|T_i(x) - T(x)\| \to 0, \forall x \in \mathscr{H}.$$

Remark 1.2.2. The closed unit ball $(B(\mathscr{H}))_1$ is compact under the weak operator topology.

Remark 1.2.3. When $\dim \mathscr{H} = \infty$, the weak operator topology is strictly weaker than the strong operator topology and the latter is strictly weaker than the uniform norm topology. When $\dim \mathscr{H} < \infty$, all three topologies coincide.

Definition 1.2.4. A self-adjoint subalgebra M of $B(\mathscr{H})$ is called a *von Neumann algebra* if M is closed under the weak operator topology and it contains the identity operator.

Given a subset $A \subset B(\mathscr{H})$, we define the *commutant* of A by $A' := \{B \in B(\mathscr{H}) : AB = BA\}$. The *bicommutant* A'' is given by $A'' := (A')'$.

Being WOT-closed allows one to carry out spectral calculus, polar decompositions and taking the least upper bound *within the algebra itself.* This is the key different feature compared to C*-algebras. For example, projections are abundant in a von Neumann algebra while there are unital C*-algebras which only contains trivial projections (e.g. the reduced C*-algebra associated with a non-abelian free group). Why we prefer the WOT over the uniform topology can be further explained by the following theorem, the first fundamental result in the development of the theory. It was shown, of course, by John von Neumann:

Theorem 1.2.5 (Bicommutant Theorem). *Let M be a unital self-adjoint subalgebra of $B(\mathscr{H})$, for some Hilbert space $\mathscr{H}$. Then the following statements are equivalent:*

1. *M is SOT-closed;*

2. *M is WOT-closed;*

3. *$M = M''$.*

Another fundamental theorem is due to Kaplansky:

Theorem 1.2.6 (Kaplansky Density Theorem). *Let $M \subset B(\mathscr{H})$ be a self-adjoint algebra which contains the identity. Then any self-adjoint element in the closed unit ball of the SOT (equivalently, WOT) closure of M, is in the SOT closure of the self-adjoint elements in the closed unit ball of M.*

Example 1.2.7. Here we give some examples of von Neumann algebras.

- $B(\mathscr{H})$ is a von Neumann algebra. In particular, when $\mathscr{H} = \mathbb{C}^n$, $B(\mathscr{H}) = M_{n\times n}(\mathbb{C})$.

- Let (X,μ) be a standard Borel probability space and let $\mathscr{H} = L^2(X,\mu)$. Each $a \in L^\infty(X,\mu)$ can be viewed as an element in $B(\mathscr{H})$ by point-wise multiplication: $L_a(f)(x) = a(x)f(x)$, for all $f \in \mathscr{H}$ and $x \in X$. $L^\infty(X,\mu)$ is an abelian von Neumann algebra.

- If $A_1 \subset B(\mathscr{H}_1), A_2 \subset B(\mathscr{H}_2)$ are two von Neumann algebras. One can form the direct sum $A_1 \oplus A_2 \subset B(\mathscr{H}_1 \oplus \mathscr{H}_2)$ in the obvious way.

- The tensor product $A_1 \overline{\otimes} A_2$ is the von Neumann algebra on $\mathscr{H}_1 \overline{\otimes} \mathscr{H}_2$ generated by $A_1 \otimes 1$ and $1 \otimes A_2$.

- Let $M \subset B(\mathscr{H})$ be a von Neumann algebra and let $p \in M$, $p' \in M'$ be projections. Then the *reduced (resp. induced) von Neumann algebra pMp* (resp. Mp') is the von

Neumann algebra on $p\mathscr{H}$ (resp. $p'\mathscr{H}$) generated by pMp (resp. Mp'). One has the nice relation that $(pMp)' = M'p$.

Example 1.2.8 (The group-measure-space construction). Let (X,μ) be a standard measure space and $\Gamma \curvearrowright (X,\mu)$ be a measure-class preserving action of a countable discrete group Γ. Then this induces an automorphism α of the von Neumann algebra $L^\infty(X,\mu)$ by

$$\alpha_g(a)(x) = a(g^{-1}\cdot x).$$

Define $\mathscr{H} := L^2(X,\mu)\overline{\otimes}\ell^2(\Gamma)$ and consider the representations $\pi : L^\infty(X,\mu) \to B(\mathscr{H})$ and the unitary representation $u : \Gamma \to U(\mathscr{H})$ given by

$$\begin{aligned} \pi(a)(f\otimes\delta_h) &= \alpha_h^{-1}(a)f\otimes\delta_h, \\ u_g(f\otimes\delta_h) &= f\otimes\delta_{g^{-1}h}, \end{aligned} \tag{1.2.1}$$

where $\{\delta_g : g\in\Gamma\}$ is the canonical orthonormal basis of $\ell^2(\Gamma)$.

We denote by $L^\infty(X,\mu)\rtimes\Gamma$ the von Neumann algebra generated by $\pi(L^\infty(X,\mu))$ and $u(\Gamma)$. This is the *group-measure space construction* due to Murray and von Neumann.

Example 1.2.9 (Group von Neumann algebras). As a special case of the above example, if we take X to be a one-point space, then the resulting von Neumann algebra is called the *(left) group von Neumann algebra* associated with Γ and we denote it by $L(\Gamma)$.

1.3 Type decompositions

The spectral theorem implies that the set $P(M)$ of a von Neumann algebra generates M. In fact, $P(M)$ is a complete lattice. One of the main achievement in the early stage of the theory is the type decomposition of von Neumann algebras, which is obtained by the comparison theory of projections.

Definition 1.3.1. Two projections p_1, p_2 of a von Neumann algebra M are said to be *equivalent* in M, if there is a *partial isometry* $v \in M$, such that $v^*v = p_1$ and $vv^* = p_2$. In this case, p_1 is called the *initial projection* of v, and p_2 is called the *final projection* of v. When p_1, p_2 are equivalent, we denote it as $p_1 \sim_M p_2$ or simply $p_1 \sim p_2$, if there is no confusion on which von Neumann we are talking about.

If there is a projection $p_3 \in M$ such that $p_3 \leq p_2$ and $p_1 \sim p_3$, then we say that p_2 *majorizes* p_1 and we write $p_1 \precsim p_2$. We write $p_1 \prec p_2$ if $p_1 \precsim p_2$ but $p_1 \nsim p_2$.

For a von Neumann algebra M, we denote by $Z(M) := M \cap M'$ the center of M. M is called a *factor* if $Z(M) = \mathbb{C}$. By a result of von Neumann, each separable von Neumann algebra can be written as a direct integral of factors.

Definition 1.3.2. A projection $p \in M$ is said to be *finite*, if $p \sim q \leq p$ implies that $q = p$. Otherwise, it is said to be *infinite*. p is said to be *abelian*, if pMp is abelian.

Remark 1.3.3. It is easy to see that abelian projections are finite.

Definition 1.3.4. A von Neumann algebra M is said to be *of type I*, if any nonzero central projection in M majorizes a nonzero abelian projection in M. If M has no nonzero abelian projections and if every nonzero central projection majorizes a nonzero finite projection in M, it is said to be *of type II*. If M is of type II and 1 is finite, then M is said to be *of type II_1*. If M is of type II with no nonzero finite central projections, then M is said to be *of type II_∞*. If M does not have finite projections, then it is *of type III*.

Now we are ready to state the main theorem.

Theorem 1.3.5. *Each von Neumann algebra can be uniquely decomposed into the direct sum of von Neumann algebras of type I, type II_1, type II_∞ and type III.*

Example 1.3.6. Now we exhibit examples of von Neumann algebras of each type.

- $B(\mathscr{H})$ is of type I since it is generated by minimal projections;
- $L^\infty(X,\mu)$ is of type I, since it is obviously abelian;
- All other examples of type II_1, II_∞ and III can be obtained via the group-measure-space construction. However we omit the details here.

1.4 Type II_1 factors

Our primary interest lies in the study of type II_1 factors. One thing that makes them particularly nice is the existence of a finite faithful normal trace.

Theorem 1.4.1. *A factor M is of type II_1 if and only if $\dim M = \infty$ and there exists a linear functional $\tau : M \to \mathbb{C}$ satisfying the following properties:*

- ***Finite**: $\tau(1) < \infty$;*
- ***Tracial**: $\tau(xy) = \tau(yx), \forall x, y \in M$;*
- ***Positive**: $\tau(x^*x) \geq 0, \forall x \in M$;*
- ***Faithful**: $\tau(x^*x) = 0$ implies that $x = 0$;*
- ***Normal**: if $\{a_i\} \subset M$ is an increasing net of positive elements such that $a_i \to a \in M$ in SOT, then $\tau(a_i) \to \tau(a)$.*

Such a τ is called a trace on M.

Remark 1.4.2.

- M is finite is equivalent to the existence of a trace on M.
- The trace τ on a factor is unique up to a multiplicative scalar. It is said to be *normalized* if $\tau(1) = 1$. In this paper we always assume that the trace is normalized.
- A trace τ on M gives rise to an inner product on M by

$$\langle x, y \rangle := \tau(y^*x).$$

  We let $L^2(M, \tau)$ to be the Hilbert completion of M with this inner product. Then, M can be faithfully represented on $L^2(M)$ by left multiplication. Thus, under this representation, elements of M can be both treated as vectors in the Hilbert space $L^2(M)$ and as bounded operators acting on $L^2(M)$. Given $x \in M$, we will write $\|x\|$ for the operator norm and $\|x\|_2 = \sqrt{\tau(x^*x)}$ for the norm in $L^2(M)$.

Example 1.4.3 (II$_1$ factors coming from groups). In previous sections we introduced the construction due to Murray and von Neumann called the group von Neumann algebras. Note that for any countable discrete group Γ, there is a natural trace on $L(\Gamma)$ given by

$$\tau(x) := \langle x\delta_e, \delta_e \rangle ,$$

where $e \in \Gamma$ is the identity element.

Moreover, each element $x \in L(\Gamma)$ can be write formally as an infinite linear combination of the canonical unitary elements $u_g, g \in \Gamma$: $x = \sum_{g\in\Gamma} a_g u_g$ and we have that

$$\tau(x^*x) = \sqrt{\sum_g |a_g|^2}.$$

A group Γ is ICC if any non-trivial element has infinite conjugacy class. It is not hard to see that for an infinite group Γ, Γ is ICC if and only if $L(\Gamma)$ is a II$_1$ factor.

Now we are ready to give the two examples of II$_1$ factors that are of particular interest:

- **The hyperfinite II$_1$ factor**: Let S_n be the group of permutations on $\{1, \cdots, n\}$ and let S_∞ to be the inductive limit of all S_n's. It is easy to see that S_∞ is an ICC group thus $L(S_\infty)$ is a II$_1$ factor.

  Moreover, notice that $L(S_n) \subset L(S_\infty)$ is isomorphic to $M_{n\times n}(\mathbb{C})$ and $\bigcup_{n\geq 1} L(S_n)$ is dense in $L(S_\infty)$ in the WOT. In other words, $L(S_\infty)$ is the WOT closure of an increasing sequence of finite dimensional algebras. We denote it by R and call it the *hyperfinite II$_1$ factor*.

- **The free group factor**: Let $\mathbb{F}_n$ be the free group with n generators. If $2 \leq n \leq \infty$, then $\mathbb{F}_n$ is ICC. The corresponding II$_1$ factor $L(\mathbb{F}_n)$ is called the *free group factor.*

The fundamental question of the theory of II$_1$ factors is to decide the isomorphism problem of factors. As the first two examples introduced by the co-founders of the theory, it is known that the hyperfinite II$_1$ factor and free group factors are not isomorphic.

1.5 Ultraproducts of II_1 factors

In the last section of the chapter, we recall the construction of ultraproducts of II_1 factors.

Definition 1.5.1. Let $\mathbb{N}$ be the set of positive integers. An *ultrafilter* on $\mathbb{N}$ is a set ω consisting of subsets of $\mathbb{N}$, such that

- $\emptyset \notin \omega$;
- If $A, B \subset \mathbb{N}$ and $B \in \omega$, $B \subset A$, then $A \in \omega$;
- If both A and B are in ω, then $A \cap B \in \omega$;
- For any $A \subset \mathbb{N}$, either A or $\mathbb{N} \backslash A$ is in ω.

ω is said to be *non-principal* or *free* if ω contains all the subset of the form $\{n \in \mathbb{N} : n \geq n_0\}$ for some n_0.

Remark 1.5.2. In fact, the set of ultrafilters on $\mathbb{N}$ can be identified with the Stone-Čech compactification $\beta(\mathbb{N})$ of $\mathbb{N}$. Free ultrafilters correspond to the points in $\beta(\mathbb{N}) \backslash \mathbb{N}$.

Definition 1.5.3. Let X be a topological space. A sequence $(x_i)_{i \geq 1}$ in X is said to *converge along* ω *to* $x \in X$, if for any open neighbourhood U of x, the set $\{i : x_i \in U\}$ is in ω. We usually write it as

$$\lim_{\omega} x_i = x.$$

Note that if X is Hausdorff, the such an x is unique.

Ultraproducts of general von Neumann algebras can be tricky to define. Fortunately in the case of II_1 factors, things are nice.

Definition 1.5.4. Let (M, τ) be a II_1 factor and let $\omega \in \beta(\mathbb{N}) \backslash \mathbb{N}$ be a free ultrafilter. Let $\prod_{n \in \mathbb{N}} M$ be the set $\{(x_n)_n : \sup_n \|x_n\| < \infty\}$. Let I_ω be the norm-closed ideal of $\prod_n M$ defined by

$$I_\omega = \{(x_n)_n \in \prod_n M : \lim_\omega \|x_n\|_2 = 0\},$$

where $\|x_n\|_2 = \sqrt{\tau(x^*x)}$. The *ultraproduct of* (M, τ) is defined to be $M^\omega := \prod_n M / I_\omega$.

There is a natural trace τ_ω on M^ω given by

$$\tau_\omega((x_n)_n) = \lim_\omega \tau(x_n).$$

Here is the basic results that we will use

Theorem 1.5.5. *Let (M, τ) be a II_1 factor and let $\omega \in \beta(\mathbb{N}) \backslash \mathbb{N}$ be a free ultrafilter. As above we define the ultraproduct M^ω. Then*

- *M^ω is a type II_1 factor;*
- *Each projection $p \in M^\omega$ lifts to a sequence of projections $(p_n)_n \in \prod_n M$. Similar result also holds for every unitary in M^ω.*

Chapter 2

Amenable subalgebras and extensions

Amenable groups were first introduced by von Neumann in his attempt to understand the Banach-Tarski paradox. There are many equivalent definitions for amenability of groups, either geometric, combinatorial or analytic. Roughly speaking, an amenable group is a "small" group which is similar to the group of integers. The first examples of non-amenable groups are groups which contains non-abelian free groups as subgroups. However, the question that whether every non-amenable group contains a free subgroup, took mathematicians many years to answer.

As we mentioned in Chapter 1, the hyperfinite II_1 factor is one of the first examples introduced by Murray and von Neumann. Gradually experts realize that it is closely related to other notions such semi-discreteness, Schwartz's property P and injectivity. Indeed, Connes' fundamental work on the classification of injective von Neumann algebras [Con76] shows that they are equivalent. Moreover, Connes' result implies that for an ICC group Γ, $L(\Gamma) \cong R$ if and only if Γ is amenable. Thus, he suggests the name amenability for those von Neumann algebras. Thanks to Connes, amenable von Neumann algebras are well understood. Thus, in order to study non-amenable von Neumann algebras, it is natural to consider their amenable subalgebras.

In this chapter we discuss questions related to amenable extensions. In particular, we give a brief review on the history of maximal amenable subalgebras.

Definition 2.1.1. (**Amenable groups**) Let Γ be a countable discrete group. Γ is *amenable* if one of the following equivalent conditions are satisfied:

- There is a mean φ on X (that is, a finitely additive probability measure on 2^X) which is invariant under left multiplication;
- The left regular representation $\lambda : \Gamma \to U(\ell^2(\Gamma))$ admits non-trivial almost-invariant vectors;
- There is a net $\{F_i\}_{i\in I}$ of finite subsets of Γ, such that

$$\lim_i \frac{|F_i \Delta g F_i|}{|F_i|} = 0, \forall g \in \Gamma,$$

  where Δ here means taking symmetric difference in set theory. Such a net is called a **Føner net**.

- Any continuous action $\Gamma \curvearrowright X$ on some compact Hausdorff space X admits an invariant Radon probability measure.

Example 2.1.2. (**Examples of amenable groups**) From the definition it is easy to see that finite groups and abelian groups are amenable. Moreover, amenability is closed under taking subgroups, quotients and inductive limits. In particular, S_∞ is amenable. One can also show that amenability is closed under extension, thus all solvable groups are amenable.

Example 2.1.3. (**Paradoxical decomposition and non-amenable groups**) A group Γ is said to admit a *paradoxical decomposition*, if there exists group elements $g_1, \cdots, g_n$, $h_1, \cdots h_m$ in Γ, for some $n, m \in \mathbb{N}$, and mutually disjoint subsets $A_1, \cdots, A_n, B_1, \cdots, B_m$ of Γ, such that $\bigcup_{1\le i\le n} g_i A_i = \bigcup_{1\le j\le m} h_j B_j = \Gamma$.

It is straightforward to see that existence of paradoxical decompositions is an obstruction for amenability. Tarski showed that being non-amenable is equivalent to the existence of paradoxical decompositions.

Let's now show that the free group with two generators is non-amenable. Suppose that a, b are the generators for $\mathbb{F}_2$. Then let A_+ be the set of elements which starts with an a on the left, in its reduced form. Similarly we can define A_-, B_+, B_-. Clearly these four sets are mutually disjoint. Note that

$$\mathbb{F}_2 = A_+ \cup aA_- = B_+ \cup bB_-.$$

Therefore, $\mathbb{F}_2$ is non-amenable.

Definition 2.2.1. Let $B \subset A$ be an inclusion of unital C*-algebras. *A conditional expectation from A onto B* is a complete positive map $E : A \to B$ such that $E(b_1 x b_2) = b_1 E(x) b_2$, for all $x \in A, b_i \in B, i \in \{1,2\}$.

Let M be a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$. M is called *hyperfinite*, or *approximately finitely dimentional* (AFD), if M is the WOT closure of an increasing net of finite dimentional von Neumann subalgebras. M is said to be *injective*, if there is a conditional expectation from $B(\mathscr{H})$ onto M. A *hypertrace* for M is a state φ on $B(\mathscr{H})$ such that $\varphi(mx) = \varphi(xm)$, for all $x \in B(\mathscr{H})$ and $m \in M$.

Theorem 2.2.2 (Connes, [Con76])**.** *AFD, injectivity and the existence of a hypertrace (and many other conditions) are equivalent for von Neumann algebras.*

We follow Connes' suggestion and call those von Neumann algebras *amenable*.

Remark 2.2.3. Type I von Neumann algebras are amenable. *R* is the unique separable amenable II_1 factor, up to isomorphism. In particular, all ICC amenable groups give rise to the same II_1 factor *R*.

2.3 Amenable extensions

Now we consider the following question: given an amenable subalgebra $A \subset M$ inside a II_1 factor, how can we amenably extend A within M? The following partial answer is well-known:

Proposition 2.3.1. *If M is a finite von Neumann algebra and $A \subset M$ is an amenable subalgebra. If u is an element from the normalizer $\mathcal{N}_M(A) = \{u \in U(M) : uAu^* = A\}$ of A inside M, then the von Neumann algebra generated by u and A is amenable.*

Proof. Since A is injective, there is a conditional expectation $E : B(L^2(M,\tau)) \Rightarrow A$ which extends the τ-preserving conditional expectation from M onto A. Define a state φ on $B(L^2(M))$ by

$$\varphi(x) = \mathrm{Lim}_n \frac{1}{n} \sum_{0 \leq i \leq n-1} \tau(E(u^i x u^{*i})), \forall x \in B(L^2(M)).$$

Then one checks that φ is a hypertrace for A and $\varphi \circ Ad(u) = \varphi$. Thus φ gives a hypertrace for the von Neumann algebra generated by A and u. □

Remark 2.3.2. Note that in the above amenable extension, one has to add normalizing unitaries *one at a time*. However, that are cases when we can add the entire normalizer *all at once.* Indeed, Ozawa and Popa [OP10] showed that the free group factors are *strongly solid*, meaning that for any diffuse amenable subalgebra of a free group factor, its normalizer again generators an amenable subalgebra. Many more examples are shown to be strongly solid [Sin11, Hou10, HS10, Avs11].

2.4 Maximal amenable subalgebras

Since amenable algebras are closed under inductive limits, Zorn's lemma implies that there always exists maximal amenable subalgebras. Fuglede and Kadison [FK51] showed that for any II_1 factor, there always exists a maximal hyperfinite subfactor, thus answered a question of Murray and von Neumann about the double relative commutant. Later on, during a conference at Baton Rouge in 1967, Kadison asked a series of famous questions about von Neumann algebras (see for example [Ge03]). Among them is the following:

Question. Is every self-adjoint element in a II_1 factor contained in a hyperfinite subfactor?

Popa answered this question in the negative, by showing that the generator masa in the free group factor is maximal amenable, [Pop83a].

If (M,τ) is a finite von Neumann algebra with a faithful normal tracial state τ and ω is a free ultrafilter, we'll write M^ω as the ultraproduct of (M,τ). The key insight of Popa [Pop83a] is that the inclusion $A \subset M$, where $M = L(\mathbb{F}_n)$ with $n \geq 2$ and A the generator masa, satisfies the *asymptotic orthogonality property*, which we define below:

Since Popa, there are many results considering maximal amenable subalgebras. Ge [Ge96, Theorem 4.5] showed that any diffuse amenable finite von Neumann algebra can be realized as a maximal amenable subalgebra of the free group factor. Shen [She06] showed that the $\bigotimes_{n\in\mathbb{N}} A$ is maximal amenable inside $\bigotimes_{n\in\mathbb{N}} M$, where A is the generator masa in the free group factor M, thus gave an example of a maximal masa in a McDuff-II_1 factor. Cameron, Fang, Ravichandran and White [CFRW10] proved that the radial masa in the free group factor is maximal amenable. Brothier [Bro14] gave an example in the setting of planar algebras. Boutonnet and Carderi [BC13] showed that the subalgebra coming from a maximal amenable subgroup in a hyperbolic group, is maximal amenable. Houdayer [Hou14a] showed that the factors coming from free Bogoljubov actions contains concrete maximal amenable masa's, see also [Hou15]. All these results use Popa's AOP approach.

Very recently a new method via the study of centralizers of states, is developed by

Boutonnet and Carderi [BC15]. In particular, they are able to show that the subalgebra coming from the upper-triangular matrix subgroup of $SL(3,\mathbb{Z})$, is maximal amenable inside $L(SL(3,\mathbb{Z}))$. See Ozawa's remark [Oza15] for an application of this new approach.

Chapter 3

Unique maximal amenable extensions

Given an amenable subalgebra inside a diffuse non-amenable II_1 factor, how many ways one can amenably extend it? In this direction, Jesse Peterson conjectures that there is a unique maximal amenable extension for any diffuse amenable subalgebra in a free group factor. At present this conjecture seems very far-fetched however we do get some partial answers. In this chapter, we will state the main results in this thesis, that for the radial masa in a free group factor and for the cup subalgebra in a planar II_1 factor, unique amenable extension results can be obtained. The main technique of the proofs will also be explained and the proofs will occupy the last two chapters.

3.1 Main results

One central theme in the theory is the study of the free group factors ([MvN43], [Voi96], [Ge98], [Oza04], [OP10]). One of the motivating questions of this paper is a conjecture by J. Peterson (see the end of [PT11]):

Conjecture. For the free group factor, any diffuse amenable subalgebra is contained in a *unique* maximal amenable subalgebra.

Houdayer's result on Gamma stability of free products [Hou15, Theorem 4.1] implies that the generator masa satisfies Peterson's conjecture. The proof again is relying on the AOP. See also Ozawa's proof [Oza15] via the centralizer approach.

One subalgebra of the free group factor under intense study is the radial masa. So let $M = L(\mathbb{F}_N)$ with $2 \leq N < \infty$ be the free group factor with finitely many generators and denote by C the von Neumann subalgebra of M generated by $\omega_1 := \sum_{g \in \mathbb{F}_N, |g|=1} u_g$. Note that ω_1 is only well-defined for free groups with finitely many generators. It was proved by Pytlik [Pyt81, Theorem 4.3] that C is a masa in M, called the *radial masa* or the *Laplacian masa*. Moreover, Rădulescu [Răd91, Theorem 7] showed that C is singular and Cameron, Fang, Ravichandra and White [CFRW10, Corollary 6.3] proved that it is maximal amenable in M.

Recall that a result of Popa [Pop83b, Corollary 4.3] shows that generator masa's coming from different generators cannot be unitarily conjugate inside M. This implies that the radial masa C cannot be unitarily conjugate with the generator masa A inside M. However, whether they are conjugate via some automorphism, is still unknown.

One of the main results of this paper is the following:

Theorem A. *[Wen16] Let $M = L(\mathbb{F}_N)$ with $2 \leq N < \infty$ and let $C \subset M$ be the radial masa. Then every amenable subalgebra of M having diffuse intersection with C, must be contained in C.*

This is the first example of such disjointness for an maximal amenable subalgebra which is not known to be in a free position.

Another new class of examples is constructed with Jones planar algebras [Jon99]. If $\mathscr{P}$ is a subfactor planar algebra, then we can associate to it a II_1 factor M [GJS10]. This II_1 factor is isomorphic to an interpolated free group factor $L(\mathbb{F}_t)$ where t is a linear combination of the index and the global index of $\mathscr{P}$ [Dyk94, Rad94, GJS11, Har13]. This factor admits a generic abelian subalgebra $A \subset M$ that we call the cup subalgebra. Brothier previously proved that the cup subalgebra is maximal amenable [Bro14].

Theorem B. *[BW16] The cup subalgebrais the unique maximal amenable extension for any diffuse subalgebra of itself.*

3.2 Strong AOP and maximal amenable extensions

The approach taken in this paper is to show a stronger version of Popa's AOP (see [Pop83a]).

Definition 3.2.1 (s-AOP). Let $A \subset M$ be an inclusion of finite von Neumann algebras. We say that the inclusion satisfies the *strong asymptotic orthogonality property* (s-AOP for short), if for any free ultrafilter ω on $\mathbb{N}$ and for any diffuse subalgebra $B \subset A$, $(x_n)_n \in B' \cap M^{\omega} \ominus A^{\omega}$ and $y_1, y_2 \in M \ominus A$, we have that $y_1(x_n)_n \perp (x_n)_n y_2$.

From s-AOP one can easily conclude unique maximal extension results, by the following theorem. The proof is inspired by [Pop83a, Lemma 3.1, Theorem 3.2] and [CFRW10, Lemma 2.2, Corollary 2.3]:

Theorem 3.2.2. *Let M be a strongly solid II_1 factor and $A \subset M$ a singular masa in M. Assume in addition that for any diffuse von Neumann subalgebra $B \subset A$ and any free ulltrafilter ω, the following holds:*

for any $(x_k)_k \in B' \bigcap M^{\omega} \ominus A^{\omega}$ and for any $y_1, y_2 \in M \ominus A$, we have that $y_1(x_k)_k \perp (x_k)_k y_2$.

Then any amenable subalegbra of M containing B, must be contained in A.

Proof. As shown by [CFRW10, Lemma 2.2, Corollary 2.3], AOP and singularity imply that A is maximal amenable in M.

Let $B \subset Q \subset M$ be an amenable subalgebra. By solidity of M, $A \subset B' \bigcap M$ is amenable. Since A is maximal amenable, we conclude $Q' \bigcap Q \subset B' \bigcap M \subset A$.

Let z be the maximal central projection of Q such that Qz is type II_1. Now suppose that $z \neq 0$.

Since Qz is amenable and of type II_1, Popa's intertwining theorem ([Pop06, Theorem A.1]) easily implies that there is a unitary $u \in (Qz)' \bigcap (Qz)^{\omega}$, such that $E_{A^{\omega}}(u) = 0$. For a proof, see [CFRW10, Lemma 2.2].

Now let C be a masa in Qz which contains Bz. Again by solidity and maximal injectivity, $C \subset Az$. Since Qz is of type II_1, there exists two non-zero projections $p_1, p_2 \in C$ and a partial isometry $v \in Qz$, such that $vv^* = p_1, v^*v = p_2, p_1p_2 = 0$. Then we have $E_A(v) = E_A(p_1vp_2) = p_1E_A(v)p_2 = 0$ so that $vu \perp uv$. However we also know that $vu = uv$, hence $v = 0$. This contradicts that $p_1, p_2 \neq 0$.

Thus, Q has to be of type I. Let C be a masa in Q containing B. Again $C \subset A$. By Kadison's result [Kad84], C is regular in Q. Both A and Q lie in the normalizer of C, so they together generate an amenable algebra containing A. By maximal amenability of A, it follows that $Q \subset A$. □

Ozawa and Popa [OP10] showed that free group factors are strongly solid and singularity for the subalgerbas are known. Therefore, our main task is to prove s-AOP for those examples we mentioned in the last section. This will be achieved in the last two chapters.

Chapter 4

Unique maximal extension for the radial masa

4.1 Preliminaries

This chapter is based on the paper [Wen16].

Let $\Gamma = \mathbb{F}_{N+1}, N \in \mathbb{N}$. Write $K := 2N+1$ for later use. Denote by $\omega_n = \sum_{g \in G, |g|=n} u_g$, for $n = 1, 2, 3, \cdots$ and let $\omega_0 = u_e$. Let $M = L(\Gamma)$ be the free group factor and let $C = \{\omega_1\}'' \subset M$ be the radial masa. $\{\omega_n\}_{n \geq 0}$ forms an orthogonal basis for $L^2(C)$.

Let $\mathscr{K}_i$ be the finite-dimensional subspace of $\mathscr{H} := L^2(M)$ spanned by all words of length i and we denote by Q_i the orthogonal projection from $\mathscr{H}$ onto $\mathscr{K}_i$. For $\xi \in \mathscr{K}_i$ and $n, m \in \mathbb{N} \bigcup \{0\}$, we define the following

$$\xi_{n,m} := \frac{Q_{i+m+n}(\omega_n \xi \omega_m)}{K^{(n+m)/2}}.$$

Rădulescu [Răd91] discovered that there is a nice decomposition of $\mathscr{H} \ominus L^2(C) = \bigoplus_{i \geq 1} \mathscr{H}_i$ into a direct sum of C-C-bimodules, each $\mathscr{H}_i$ has a distinguished unit vector ξ^i, which is from $\mathscr{K}_{l(i)}$, for some $l(i) \in \mathbb{N}$, such that $\mathscr{H}_i$ is generated by ξ^i as a C-C-bimodule.

Moreover, by [Răd91, Lemma 3, Lemma 6], for those i with $l(i) \geq 2$, we have that $\{\xi^i_{n,m}\}_{n,m \geq 0}$ forms an orthonormal basis for $\mathscr{H}_i$. For those i with $l(i) = 1$ (there are finitely many such i's), $\{\xi^i_{n,m}\}_{n,m \geq 0}$ is no longer an orthonormal basis for $\mathscr{H}_i$, however for any $i, j \geq 1$, the linear mapping $T_{i,j} : \mathscr{H}_i \to \mathscr{H}_j$, given by

$$T_{i,j}(\xi^i_{n,m}) = \xi^j_{n,m},$$

extends uniquely to an invertible bounded operator. Furthermore, there is a universal constant $C_1 > 0$ such that

$$\|T^{\pm 1}_{i,j}\| \leq C_1, \forall i, j \geq 1.$$

Remark 4.1.1. Recall that in a separable Hilbert space, a sequence of vectors $\{v_n\}$ forms a *Riesz basis* (for the basics of Riesz basis, see, e.g. [Chr01]), if $\{v_i\}$ is the image of some orthonormal basis under some bounded invertible operator. It is also equivalent to the fact that there exists some $A,B>0$ such that for any $(c_n)\in\ell^2$, $A\sum|c_n|^2\le\|\sum c_nv_n\|\le B\sum|c_n|^2$. In this case, every vector x in the Hilbert space has a *unique* decomposition $x=\sum c_nv_n$, for some $(c_n)\in\ell^2$. It follows that $\left\{\xi^i_{n,m}\right\}_{i\ge1,n,m\ge0}$ forms a Riesz basis for $L^2(M)\ominus L^2(C)$. Consequently, for any $x\in L^2(M)\ominus L^2(C)$, there is a *unique* decomposition $x=\sum_{i\ge1,n,m\ge0}a^i_{n,m}\xi^i_{n,m}$ for some $\left(a^i_{n,m}\right)_{n,m,i}\in\ell^2$. We call $\{\xi^i_{n,m}\}_{i\ge1,n,m\ge0}$ the *Rădulescu basis* for $L^2(M)\ominus L^2(C)$.

Sometimes it will be convenient to use the following convention: we write $\xi^i_{n,m}$ for all $n,m\in\mathbb{Z}$, where we define $\xi^i_{n,m}=0$ whenever $n<0$ or $m<0$.

The key computation in [CFRW10] is that when considering the AOP in the case of the radial masa, the Rădulescu basis plays the same role as the canonical basis for the generator masa case. However, in our approach, the Rădulescu basis suffers from a lack of right modularity. Instead, $\{\omega_n\xi^i\omega_m\}$, after proper normalization, is the more natural basis to work with.

We collect some relations between $\omega_n\xi^i\omega_m$ and $\xi^i_{n,m}$'s, due to Rădulescu, in the following lemma:

Lemma 4.1.2 (Lemma 2, 6 in [Răd91]). *The following statements hold for all $n,m\ge0$:*
(1) If $l(i)\ge2$, then $\omega_n\xi^i\omega_m=K^{\frac{n+m}{2}}\xi^i_{n,m}-K^{\frac{n+m-2}{2}}\left(\xi^i_{n,m-2}+\xi^i_{n-2,m}\right)+K^{\frac{n+m-4}{2}}\xi^i_{n-2,m-2}$;
(2) If $l(i)=1$, then there is some $\sigma=\sigma(i)\in\{-1,1\}$ such that

$$\begin{aligned}\omega_n\xi^i\omega_m=&K^{\frac{n+m}{2}}\xi^i_{n,m}-K^{\frac{n+m-2}{2}}\left(\xi^i_{n,m-2}+\xi^i_{n-2,m}+\sigma\xi^i_{n-1,m-1}\right)\\&+\sum_{k\ge2}(-\sigma)^kK^{\frac{n+m-2k}{2}}\left(\sigma\xi^i_{n-k-1,m-k+1}+\sigma\xi^i_{n-k+1,m-k-1}+2\xi^i_{n-k,m-k}\right);\end{aligned}$$

(3) For all $i, j \geq 1$, the linear mapping $S_{i,j} : \mathscr{H}_i \to \mathscr{H}_j$ given by

$$S_{i,j}\left(\omega_n \xi^i \omega_m\right) = \omega_n \xi^j \omega_m, \forall n, m \geq 0,$$

is well-defined and extends to an invertible bounded operator between the two subspaces, with $\sup_{i,j} \|S_{i,j}^{\pm 1}\| \leq C_2$, for some uniform constants $0 < C_2 < \infty$.

Lemma 4.1.3. $\left\{\eta_{n,m}^i := \dfrac{\omega_n \xi^i \omega_m}{K^{(n+m)/2}}\right\}_{i\geq 1, n,m\geq 0}$ *forms a Riesz basis for $L^2(M) \ominus L^2(C)$.*

Therefore, for any $x \in L^2(M) \ominus L^2(C)$, there is a unique decomposition $x = \sum_{i\geq 1, n,m\geq 0} b_{n,m}^i \eta_{n,m}^i$ where $\left(b_{n,m}^i\right)_{i\geq 1, n,m\geq 0} \in \ell^2$.

Proof. By (3) of the previous lemma, it suffices to prove the conclusion for some fixed $i \geq 1$ with $l(i) \geq 2$.

Fix $i \geq 1$ with $l(i) \geq 2$ and $(a_{n,m})_{n,m} \in \ell^2$. We will omit the superscript i, since no confusion will appear. Using part (1) of the previous lemma, we have

$$\begin{aligned}
\sum_{n,m\geq 0} a_{n,m}\eta_{n,m} &= \sum_{n,m\geq 0} a_{n,m}\left(\xi_{n,m} - \frac{\xi_{n,m-2}}{K} - \frac{\xi_{n-2,m}}{K} + \frac{\xi_{n-2,m-2}}{K^2}\right) \\
&= \sum_{n,m}\left(a_{n,m} - \frac{a_{n,m+2}}{K} - \frac{a_{n+2,m}}{K} + \frac{a_{n+2,m+2}}{K^2}\right)\xi_{n,m} \\
&= \sum_{n,m}\left(\left(a_{n,m} - \frac{a_{n,m+2}}{K}\right) - \frac{1}{K}\left(a_{n+2,m} - \frac{a_{n+2,m+2}}{K}\right)\right)\xi_{n,m},
\end{aligned}$$

hence by repeated use of the triangle inequality, we have

$$\begin{aligned}
\left\|\sum a_{n,m}\eta_{n,m}\right\|_2 &= \left(\sum_{n,m\geq 0}\left|\left(\left(a_{n,m} - \frac{a_{n,m+2}}{K}\right) - \frac{1}{K}\left(a_{n+2,m} - \frac{a_{n+2,m+2}}{K}\right)\right)\right|^2\right)^{1/2} \\
&\geq \left(\sum_{n,m\geq 0}\left|a_{n,m} - \frac{a_{n,m+2}}{K}\right|^2\right)^{1/2} - \frac{1}{K}\left(\sum_{n,m\geq 0}\left|a_{n+2,m} - \frac{a_{n+2,m+2}}{K}\right|^2\right)^{1/2} \\
&\geq \left(1 - \frac{1}{K}\right)\left(\sum_{n,m\geq 0}\left|a_{n,m} - \frac{a_{n,m+2}}{K}\right|^2\right)^{1/2} \\
&\geq \left(1 - \frac{1}{K}\right)^2\left(\sum_{n,m\geq 0}|a_{n,m}|^2\right)^{1/2}.
\end{aligned}$$

The other side of the inequality is easy, since each $a_{n,m}$ only appears at most four times. Thus there is a $B > 0$, such that

$$\left\|\sum a_{n,m}\eta_{n,m}\right\|_2^2 \leq B\sum |a_{n,m}|^2,$$

So we are done. $\square$

Remark 4.1.4. Because both $\|T_{i,j}^{\pm 1}\|$ and $\|S_{i,j}^{\pm 1}\|$ are uniformly bounded, there is a $C_0 > 0$ such that $\|T_{i,j}^{\pm 1}\| \leq C_0, \|S_{i,j}^{\pm 1}\| \leq C_0$, and for any $\left(c_{n,m}^i\right) \in \ell^2$,

$$\frac{1}{C_0}\sum_{i,n,m}\left|c_{n,m}^i\right|^2 \leq \|\sum_{n,m\geq 0, i\geq 1} c_{n,m}^i \xi_{n,m}^i\|_2^2 \leq C_0 \sum_{i,n,m}\left|c_{n,m}^i\right|^2,$$
$$\frac{1}{C_0}\sum_{i,n,m}\left|c_{n,m}^i\right|^2 \leq \|\sum_{n,m\geq 0, i\geq 1} c_{n,m}^i \eta_{n,m}^i\|_2^2 \leq C_0 \sum_{i,n,m}\left|c_{n,m}^i\right|^2$$

For each $k \in \mathbb{N}$, define $L_k, L_k' : L^2(M) \ominus L^2(C) \to L^2(M) \ominus L^2(C)$ as follows

$$L_k\left(\sum_{i\geq 1, n,m\geq 0} a_{n,m}^i \xi_{n,m}^i\right) := \sum_{i\geq 1, n\leq k, m\geq 0} a_{n,m}^i \xi_{n,m}^i,$$
$$L_k'\left(\sum_{i\geq 1, n,m\geq 0} b_{n,m}^i \eta_{n,m}^i\right) := \sum_{i\geq 1, n\leq k, m\geq 0} b_{n,m}^i \eta_{n,m}^i,$$

i.e. L_k (resp. L_k') is the left "projection" onto the span of $\left\{\xi_{n,m}^i\right\}_{i,n,m}$ (resp. $\left\{\eta_{n,m}^i\right\}_{i,n,m}$) with the first subscript no larger than k. However one should be warned that they are merely idempotents, instead of projections, due to the presence of those i's with $l(i) = 1$. We can also define R_k, R_k' for the right "projections" in the similar fashion. All these idempotents are bounded operators. Let $L_k \vee R_k := L_k + R_k - L_kR_k, L_k' \vee R_k' := L_k' + R_k' - L_k'R_k'$.

Lemma 4.1.5. *L_k' is right C-modular, $\forall k \geq 0$.*

Proof. Since $\{\omega_n\}_{n\geq 0}$ forms an orthogonal basis for $L^2(C)$ and $\{\eta_{n,m}^i\}_{i\geq 1, n,m\geq 0}$ is a Riesz basis for $L^2(M) \ominus L^2(C)$, it is sufficient to show that $L_k'(\eta_{n,m}^i \omega_l) = L_k'(\eta_{n,m}^i)\omega_l$.

The definition of the $\eta^i_{n,m}$'s clearly implies that $\eta^i_{n,m}\omega_l \in \operatorname{span}\{\eta^i_{n,k}\}_{k\geq 0}$, that is, multiplying ω_l on the right does not change neither the upper nor left index of $\eta^i_{n,m}$, thus $L'_k(\eta^i_{n,m}\omega_l) = L'_k(\eta^i_{n,m})\omega_l$ and the proof is complete. $\square$

We will need the following result from [CFRW10]:

Lemma 4.1.6 (Lemma 4.3, Theorem 6.2 in [CFRW10])**.** *Given $(x_k)_k \in M^\omega \ominus C^\omega$, if for every $k_0 \in \mathbb{N}$, we have that $\lim_{k\to\omega} \left\|(L_{k_0} \vee R_{k_0})(x_k)\right\|_2 = 0$, then for any $y_1, y_2 \in L^2(M) \ominus L^2(C)$, $y_1(x_k)_k \perp (x_k)_k y_2$.*

4.2 Proof of Theorem A

Now we state the key technical result in this chapter.

Proposition 4.2.1. ***[s-AOP for the radial masa]***

Let $\Gamma = \mathbb{F}_{N+1}$ be a non-abelian free group with finitely many generators and $M = L(\Gamma)$ the corresponding group von Neumann algebra. Denote by C the radial masa of M and suppose that $B \subset C$ is a diffuse von Neumann subalgebra. Then for any $(x_k)_k \in B' \bigcap M^\omega \ominus C^\omega$ and $y_1, y_2 \in M \ominus C$, where ω is a free ultrafilter, we have that $y_1(x_k)_k \perp (x_k)_k y_2$ in $L^2(M^\omega)$.

We will break the proof into several lemmas.

Let $(x_k)_k \in B' \bigcap M^\omega \ominus C^\omega$ and $y_1, y_2 \in M \ominus C$ be given. For each k, we can assume $x_k \in M \ominus C \subset L^2(M) \ominus L^2(C), ||x_k|| \leq 1$ and write its decompositions with respect to $\left\{\xi^i_{n,m}\right\}_{i\geq 1, n,m\geq 0}$ and $\left\{\eta^i_{n,m}\right\}_{i\geq 1, n,m\geq 0}$, respectively:

$$x_k = \sum_{i\geq 1, n,m\geq 0} a^{i,k}_{n,m} \xi^i_{n,m} = \sum_{i\geq 1, n,m\geq 0} b^{i,k}_{n,m} \eta^i_{n,m},$$

where both $\left(a^{i,k}_{n,m}\right)_{i\geq 1, n,m\geq 0}$ and $\left(b^{i,k}_{n,m}\right)_{i\geq 1, n,m\geq 0}$ are from ℓ^2.

Since B is diffuse, we can choose a sequence $\{u_k\}_k$ in the unitary group of B, which converges to 0 weakly. Recall that $\left\{\frac{\omega_i}{||\omega_i||_2}\right\}_{i\geq 0}$ is an orthonormal basis for $L^2(C)$. Moreover, for any fixed $N_0 \geq 0$, $\omega_n \omega_m$ will be supported on those ω_i's with $i > N_0$, provided that $|m-n| > N_0$. We first need to approximate each u_k using finite linear combinations of ω_i's.

Lemma 4.2.2. *For each fixed N_0, there exists a sequence $\{S_k\}_{k\geq 1}$ of non-empty, disjoint, finite subsets of $\mathbb{N} \cup \{0\}$ and a sequence of strictly increasing natural numbers $\{n_k\}_{k\geq 1}$, such that in the decomposition with respect to $\{\omega_i\}_{i\geq 0}$, the supports of elements from $\{\omega_m \omega_n : m \in S_i, n \leq N_0\}$ and the supports of elements from $\{\omega_m \omega_n : m \in S_j, n \leq N_0\}$ are disjoint, whenever $i, j \geq 1, i \neq j$. Moreover, there exists a sequence $\{v_k\}_k$ in C, with $v_k \in span\{\omega_i : i \in S_k\}$ such that $||v_k|| \leq 2$ and $||v_k - u_{n_k}||_2 \leq \frac{1}{2^k}$.*

Moreover, one can construct $\{v_k\}, \{S_k\}$ such that there is a sequence $\{F_k\}$ of strictly increasing natural numbers such that $L'_{N_0}(v_i x) = L'_{N_0}\left(v_i\left(L'_{F_{i+1}} - L'_{F_i}\right)(x)\right)$, for all $x \in L^2(M) \ominus L^2(C)$.

Proof. Throughout this lemma, for any $x \in C$, we always consider the Fourier expansion of x with respect to $\{\omega_i\}_{i\geq 0}$. Moreover, if $x = \sum_{i\geq 0} a_i\omega_i$ and $F \subset \mathbb{N}\cup\{0\}$, we will use the notation $P_F(x) := \sum_{i\in F} a_i\omega_i$.

We construct $\{S_k\}$, $\{n_k\}$ and $\{v_k\}$ inductively. Since $span\{\omega_n\}_{n\geq 0}$ is a weakly dense *-subalgebra of C, Kaplansky Density Theorem implies that there exists a sequence $\{z_k\}_k$ of elements in C, whose Fourier expansions are finitely supported, such that $||z_k|| \leq 3/2$ and $||u_k - z_k||_2 \leq \frac{1}{4^k}$. For each k, suppose that z_k is supported on $\{\omega_i\}_{i\in T_k}$, where $T_k \subset \mathbb{N}\cup\{0\}$ is some finite subset. Let $n_1 = 1$, $v_1 = z_1$ and $S_1 = T_1$. Then $||v_1|| \leq 2$ and $||v_1 - u_{n_1}||_2 \leq 1/2$ and $v_1 \in span\{\omega_i : i \in S_1\}$.

Now suppose that $S_1, \cdots, S_k$ and $n_1, \cdots, n_k$ have already been chosen. Then there exists a finite subset $F_{k+1} \subset \mathbb{N}\cup\{0\}$, such that $\bigcup_{1\leq i\leq k} S_i \subset F_{k+1}$ and for any $S \subset \mathbb{N}\cup\{0\}\backslash F_{k+1}$, we always have that in the decomposition with respect to $\{\omega_i\}_{i\geq 0}$, the supports of elements from $\{\omega_m\omega_n : m \in \cup_{1\leq i\leq k} S_i, n \leq N_0\}$ and the supports of elements from $\{\omega_m\omega_n : m \in S, n \leq N_0\}$ are disjoint (for example, one can let $F_{k+1} = \{0, 1, \cdots, max\cup_{1\leq i\leq k} S_i + 3N_0\}$). Now since $u_k \to 0$ weakly, there is a natural number $n_{k+1} > n_k$, such that with respect to the basis $\{\omega_i\}_{i\geq 0}$, the Fourier coefficient of $z_{n_{k+1}}$ has absolute value less than $\frac{1}{4^k|F_{k+1}|||\omega_i||}$, for each $0 \leq i \leq F_{k+1}$. Let $S_{k+1} := T_{n_{k+1}}\backslash F_{k+1}, v_{k+1} := P_{S_{k+1}}(z_{n_{k+1}})$. Then

$$\begin{aligned}
||v_{k+1} - u_{n_{k+1}}||_2 &\leq \left\|v_{k+1} - z_{n_{k+1}}\right\|_2 + \left\|z_{n_{k+1}} - u_{n_{k+1}}\right\|_2 \\
&= ||P_{(T_{n_{k+1}}\backslash S_{k+1})}(z_{n_{k+1}})||_2 + \left\|z_{n_{k+1}} - u_{n_{k+1}}\right\|_2 \\
&= \left\|P_{F_{k+1}}(z_{n_{k+1}})\right\|_2 + \left\|z_{n_{k+1}} - u_{n_{k+1}}\right\|_2 \\
&\leq \left(\frac{|F_{k+1}|}{2^{2k}|F_{k+1}|}\right)^{1/2} + \frac{1}{4^{k+1}} \\
&\leq \frac{1}{2^{k+1}},
\end{aligned}$$

and an easy estimate of the ℓ^1-norm gives us

$$\|v_{k+1}\| \leq \|z_{k+1}\| + \frac{|F_{k+1}|}{2^k|F_{k+1}|} \leq 2.$$

The last statement can be achieved by letting the supports of $\{v_k\}_k$ mutually far away. For example, choose the gap between S_i and S_j to be greater than $3N_0$ and let F_k be the collection of elements of $\mathbb{N}\cup\{0\}$ between $\min_{n\in S_k}|n| - N_0$ and $\max_{n\in S_k}|n| + N_0$. $\square$

Thus by taking a subsequence if necessary, we may assume that $\{v_k\}$ is a sequence in C, such that $v_k \in span\{\omega_i : i \in S_k\}$ for some finite subset $S_k \subset \mathbb{N}$, $||v_k|| \leq 2$, $||v_k - u_k||_2 \leq \frac{1}{2^k}$ and $v_i\omega_k \perp v_j\omega_k$, for all $i, j \geq 1, i \neq j$ and all $0 \leq k \leq N_0$ and there is a sequence $\{F_k\}$ of strictly increasing natural numbers such that $L'_{N_0}(v_i x) = L'_{N_0}\left(v_i\left(L'_{F_{i+1}} - L'_{F_i}\right)(x)\right)$, for all $x \in L^2(M) \ominus L^2(C)$.

Lemma 4.2.3. $\lim_{k\to\omega}\left\|L'_{N_0}(x_k)\right\|_2 = 0$.

Proof. Fix a small $\varepsilon > 0$. First choose some large $N_1 < N_2$ such that $2\sum_{i=N_1}^{N_2}\|v_i - u_i\|_2^2 \leq \varepsilon$ and $\dfrac{4\left\|L'_{N_0}\right\|^2 C_0^2 + 1}{N_2 - N_1} \leq \varepsilon$. Then we have

$$\begin{aligned}
\lim_{k\to\omega}\sum_{i=N_1}^{N_2}\left\langle L'_{N_0}(v_i x_k), L'_{N_0}(v_i x_k)\right\rangle &\geq \lim_{k\to\omega}\sum_{i=N_1}^{N_2}\left\langle L'_{N_0}(u_i x_k), L'_{N_0}(u_i x_k)\right\rangle - \varepsilon \\
&= \lim_{k\to\omega}\sum_{i=N_1}^{N_2}\left\langle L'_{N_0}(x_k u_i), L'_{N_0}(x_k u_i)\right\rangle - \varepsilon \\
&= \lim_{k\to\omega}\sum_{i=N_1}^{N_2}\left\langle L'_{N_0}(x_k)u_i, L'_{N_0}(x_k)u_i\right\rangle - \varepsilon \\
&= (N_2 - N_1)\lim_{k\to\omega}\left\|L'_{N_0}(x_k)\right\|_2^2 - \varepsilon.
\end{aligned}$$

The second line uses the assumption that $(x_k)_k \in B' \cap M^\omega$ and the third line uses the fact that L'_{N_0} is a right-C modular map, i.e.

$$L'_{N_0}(xa) = L'_{N_0}(x)a, \forall x \in L^2(M) \ominus L^2(C), \forall a \in C.$$

Meanwhile,

$$\begin{aligned}
\sum_{i=N_1}^{N_2} \left\langle L'_{N_0}(v_i x_k), L'_{N_0}(v_i x_k) \right\rangle &= \sum_{i=N_1}^{N_2} \left\langle L'_{N_0}\left(v_i \left(L'_{F_{i+1}} - L'_{F_i}\right)(x_k)\right), L'_{N_0}\left(v_i \left(L'_{F_{i+1}} - L'_{F_i}\right)(x_k)\right) \right\rangle \\
&\leq \left\|L'_{N_0}\right\|^2 \sum_{i=N_1}^{N_2} \left\langle v_i \left(L'_{F_{i+1}} - L'_{F_i}\right)(x_k), v_i \left(L'_{F_{i+1}} - L'_{F_i}\right)(x_k) \right\rangle \\
&\leq \sum_{i=N_1}^{N_2} \left\|L'_{N_0}\right\|^2 \|v_i\|^2 \left\| \left(L'_{F_{i+1}} - L'_{F_i}\right)(x_k) \right\|_2^2 \\
&\leq 4 \left\|L'_{N_0}\right\|^2 \sum_{i=N_1}^{N_2} C_0 \sum_{j \geq 1, F_i+1 \leq n \leq F_{i+1}, m \geq 0} \left| b^{j,k}_{n,m} \right|^2 \\
&\leq 4 \left\|L'_{N_0}\right\|^2 C_0 \sum_{j \geq 1, 0 \leq n \leq F_{N_2}, m \geq 0} \left| b^{j,k}_{n,m} \right|^2 \\
&\leq 4 \left\|L'_{N_0}\right\|^2 C_0^2 \|x_k\|_2^2 \leq 4 \left\|L'_{N_0}\right\|^2 C_0^2.
\end{aligned}$$

Therefore, we conclude that $\lim_{k\to\omega} \left\|L'_{N_0}(x_k)\right\|_2^2 \leq \dfrac{4||L'_{N_0}||^2 C_0^2 + 1}{N_2 - N_1} \leq \varepsilon$ can be made arbitrarily small. Thus the proof for Lemma 4.2.3 is complete. □

Lemma 4.2.4. $\lim_{k\to\omega} \|L_{N_0}(x_k)\|_2 = 0$.

Proof. We use the relations between $\eta^i_{n,m}$ and $\xi^i_{n,m}$, as stated above in Lemma 4.1.2 and Lemma 4.1.3.

First, since $x_k = \sum_{i\geq 1, l(i)\geq 2, n,m\geq 0} a^{i,k}_{n,m} \xi^i_{n,m} \oplus \sum_{i\geq 1, l(i)=1, n,m\geq 0} a^{i,k}_{n,m} \xi^i_{n,m}$, it suffices to consider separately the part with i's such that $l(i) \geq 2$ and the part with i's such that $l(i) = 1$.

For the i's with $l(i) \geq 2$, recall that $\eta^i_{n,m} = \xi^i_{n,m} - K^{-1}\left(\xi^i_{n,m-2} + \xi^i_{n-2,m}\right) + K^{-2}\xi^i_{n-2,m-2}$ so that $a^i_{n,m} = b^i_{n,m} - \dfrac{b^i_{n,m+2}}{K} - \dfrac{b^i_{n+2,m}}{K} + \dfrac{b^i_{n+2,m+2}}{K^2}$. Therefore

$$
\begin{aligned}
\left\| L_{N_0}\left(\sum_{i\geq 1, l(i)\geq 2, n,m\geq 0} a_{n,m}^{i,k}\xi_{n,m}^{i,k}\right)\right\|_2^2 &= \sum_{i\geq 1, l(i)\geq 2, N_0\geq n\geq 0, m\geq 0} \left|a_{n,m}^{i,k}\right|^2 \\
&= \sum_{i\geq 1, l(i)\geq 2, N_0\geq n\geq 0, m\geq 0} \left| b_{n,m}^{i,k} - \frac{b_{n,m+2}^{i,k}}{K} - \frac{b_{n+2,m}^{i,k}}{K} + \frac{b_{n+2,m+2}^{i,k}}{K^2}\right|^2 \\
&\leq 16 \sum_{i\geq 1, l(i)\geq 2, N_0+2\geq n\geq 0, m\geq 0} \left|b_{n,m}^{i,k}\right|^2 \\
&\leq 16C_0 \left\| L'_{N_0+2}\left(\sum_{i\geq 1, l(i)\geq 2, n,m\geq 0} b_{n,m}^{i,k}\eta_{n,m}^{i,k}\right)\right\|_2^2,
\end{aligned}
$$

and the last term goes to 0 as $k\to\omega$, by the previous lemma.

Now consider the i's with $l(i)=1$. As there are only finitely many such i's, we may restrict our attention to a single fixed i.

For some $\sigma\in\{1,-1\}$, we have that

$$
\begin{aligned}
\sum_{n,m\geq 0} b_{n,m}^{i,k}\eta_{n,m}^{i,k} = \sum_{n,m} b_{n,m}^{i,k}\Bigg(&\xi_{n,m}^{i,k} - \frac{\xi_{n-2,m}^{i,k}}{K} - \frac{\xi_{n,m-2}^{i,k}}{K} + \sigma\frac{\xi_{n-1,m-1}^{i,k}}{K} \\
&+ \sum_{l\geq 2}\frac{(-\sigma)^l}{K^l}\left(\sigma\xi_{n-l-1,m-l+1}^{i,k} + \sigma\xi_{n-l+1,m-l-1}^{i,k} + 2\xi_{n-l,m-l}^{i,k}\right)\Bigg) \\
= \sum_{n,m}\Bigg(&b_{n,m}^{i,k} - \frac{b_{n+2,m}^{i,k}}{K} - \frac{b_{n,m+2}^{i,k}}{K} + \frac{\sigma b_{n+1,m+1}^{i,k}}{K} \\
&+ \sum_{l\geq 2}\frac{(-\sigma)^l}{K^l}\left(\sigma b_{n+l+1,m+l-1}^{i,k} + \sigma b_{n+l-1,m+l+1}^{i,k} + 2b_{n+l,m+l}^{i,k}\right)\Bigg)\xi_{n,m}^{i,k}.
\end{aligned}
$$

Therefore, for any fixed $\varepsilon>0, N_0\geq 0$, we find a large integer $N_1\gg N_0$, to be specified later, and we let $K_0 = N_1 - N_0$. By the triangle inequality,

$$\left(\sum_{n\le N_0, m\ge 0} |a^{i,k}_{n,m}|^2\right)^{1/2} \le \left(\sum_{n\le N_0, m\ge 0}\left|b^{i,k}_{n,m} - \frac{b^{i,k}_{n+2,m}}{K} - \frac{b^{i,k}_{n,m+2}}{K} + \frac{\sigma b^{i,k}_{n+1,m+1}}{K}\right|^2\right)^{1/2}$$
$$+ \sum_{2\le l\le K_0} \frac{1}{K^l}\left(\sum_{n\le N_0, m\ge 0}\left|\sigma b^{i,k}_{n+l+1,m+l-1} + \sigma b^{i,k}_{n+k-1,m+k+1} + 2b^{i,k}_{n+k,m+k}\right|^2\right)^{1/2}$$
$$+ \sum_{l\ge K_0+1} \frac{1}{K^l}\left(\sum_{n\le N_0, m\ge 0}\left|\sigma b^{i,k}_{n+l+1,m+l-1} + \sigma b^{i,k}_{n+l-1,m+l+1} + 2b^{i,k}_{n+l,m+l}\right|^2\right)^{1/2}.$$

We estimate the third term in the above inequality first:

$$\sum_{l\ge K_0+1} \frac{1}{K^l}\left(\sum_{n\le N_0, m\ge 0}\left|\sigma b^{i,k}_{n+l+1,m+l-1} + \sigma b^{i,k}_{n+l-1,m+l+1} + 2b^{i,k}_{n+l,m+l}\right|^2\right)^{1/2}$$
$$\le \sum_{l\ge K_0+1} \frac{1}{K^l}\left(\sum_{n,m\ge 0}\left|\sigma b^{i,k}_{n+l+1,m+l-1} + \sigma b^{i,k}_{n+l-1,m+l+1} + 2b^{i,k}_{n+l,m+l}\right|^2\right)^{1/2}$$
$$\le \sum_{l\ge K_0+1} \frac{1}{K^l} 4\left(\sum_{n,m\ge 0}\left|b^{i,k}_{n,m}\right|^2\right)^{1/2}$$
$$\le \sum_{l\ge K_0+1} \frac{1}{K^l} 4C_0 \|x_k\|_2 \le \frac{4C_0}{K^{K_0}(K-1)},$$

hence we can choose N_1 large enough so that K_0 is large, such that the third term is less than $\varepsilon/3$, for any k.

Now we estimate the first and the second terms. To this end, we choose a large $k_0 = k_0(N_1, \varepsilon)$, such that for any $k \ge k_0$, we have that $4C_0K_0\left(\sum_{m\ge 0, n\le N_1+1}\left|b^{i,k}_{n,m}\right|^2\right)^{1/2}$ is less than $\varepsilon/3$. Thus both the first and the second term can be bounded above by $\varepsilon/3$. Combine all these pieces together, we conclude that

$$\left\|L_{N_0}\left(\sum_{n,m\ge 0} a^{i,k}_{n,m}\xi^{i,k}_{n,m}\right)\right\|_2 \le C_0\left(\sum_{n\le N_0, m\ge 0}|a^{i,k}_{n,m}|^2\right)^{1/2} \le C_0\varepsilon,$$

when k is close enough to ω. Since $\varepsilon > 0$ is arbitrary, we are done. $\square$

Proof of Proposition 4.2.1. The same proof for Lemma 4.2.4 shows that $\lim_{k\to\omega}\|R_{N_0}(x_k)\|_2 = 0$. So Lemma 4.1.6 applies. $\square$

Remark 4.2.5. In fact, the same conclusion as in Proposition 4.2.1 holds, if we replace the assumption "$B \subset C$ diffuse" by "$B \subset C^\omega$ diffuse".

Theorem 4.2.6. *The radial masa satisfies Peterson's conjecture.*

Proof. It is shown in [OP10] that $L(\mathbb{F}_N), N \geq 2$ is strongly solid, and the fact that the radial masa is singular is shown in [Răd91](another proof can be found in [SS03]). Therefore, Theorem 3.2.2 and Proposition 4.2.1 imply the result. $\square$

Remark 4.2.7. One can also use [Hou14b, Theorem 8.1] and Proposition 4.2.1 to conclude Theorem 4.2.6.

4.3 Some remarks

In fact, one can state a more general structural result for the inclusion $C \subset L(\mathbb{F}_{N+1})$.

Theorem 4.3.1. *Let $M = L(\mathbb{F}_{N+1})$ be a free group factor with $1 \leq N < \infty$ and let $C \subset M$ be the radial masa. If $Q \subset M$ is a von Neumann subalgebra that has a diffuse intersection with C, then there exists a sequence of central projections $e_n \in Z(Q), n \geq 0$ such that*

- *$e_0 Q \subset C$;*
- *For all $n \geq 1$, $e_n Q$ is a non-amenable II_1 factor such that $e_n(Q' \cap M^\omega) = e_n(Q' \cap M)$ is discrete and abelian (even contained in C).*

Proof. Let $e_0 \in Z(Q)$ be the maximal projection such that $e_0 Q$ is amenable. Then $Qe_0 \oplus C(1-e_0)$ is amenable and has a diffuse intersection with C so it is contained in C by Theorem 4.2.6. Moreover, $Q(1-e_0)$ has a discrete center, by solidity of M. This gives a sequence of central projections $\{e_n\}_{n \geq 1}$ such that for all $n \geq 1$, $e_n Q$ is a non-amenable II_1 factor.

Now fix $n \geq 1$. By [Ioa15, Lemma 2.7], one can find a central projection $e \in Z((e_n Q)' \cap e_n M e_n)$ such that

- $e((e_n Q)' \cap e_n M e_n) = e((e_n Q)' \cap (e_n M e_n)^\omega)$ is discrete;
- $(e_n - e)((e_n Q)' \cap (e_n M e_n)^\omega)$ is diffuse.

By [Pet09, Proof of Theorem 4.3], the fact that $(e_n - e)((e_n Q)' \cap (e_n M e_n)^\omega)$ is diffuse implies that $(e_n - e)Q$ is amenable. Since $e_n Q$ has no direct summand, this forces $e = e_n$.

Finally, $(Q \cap C)' \cap M$ is amenable, again by solidity. As it contains C, it has to be equal to C. In particular $Q' \cap M \subset (Q \cap C)' \cap M \cap C$. So the last part of the theorem is true. □

Remark 4.3.2. In [Hou15, Theorem 3.1], Houdayer showed the general situation for free products of σ-finite von Neumann algebras, which contains the strong-AOP for the generator masa in a free group factor as a special case. Also, the strong-AOP as in Proposition 4.2.1 means that for any diffuse subalgebra B of the radial masa C, the inclusion $C \subset M$

has the *AOP relative to B*, in the sense of [Hou14b, Definition 5.1]. The unique maximal injective extension for any diffuse subalgebra of the generator masa is first shown by Houdayer [Hou15, Theorem 4.1]. A proof via the study of centralizers is obtained by Ozawa [Oza15].

Remark 4.3.3. Note that the disjointness result as in Theorem 4.2.6 is not true for arbitrary maximal amenable masa of a II_1 factor. For instance, if the inclusion $A \subset M$ has some nice decomposition, then A does not have the uniqueness property as the generator masa in the above corollary. We give some such examples:

- Let $M = A_1 *_{A_0} A_2$ be the amalgamated free product with A_i amenable, and A_0 diffuse, $A_0 \neq A_i$, $i = 1,2$, then A_0 can be contained in different maximal amenable subalgebras.

- Let M_1, M_2 both be the free group factor and $A_i \subset M_i$ the corresponding generator masa, $i = 1,2$. Then $A = A_1 \overline{\otimes} A_2$ is a maximal injective subalgebra inside $M = M_1 \overline{\otimes} M_2$. However, many other injective subalgebras could contain the diffuse subalgebra $A_1 \otimes 1$.

- Let $\Lambda < \Gamma$ be a singular subgroup in the sense of Boutonnet and Carderi ([BC15, Definition 1.2]) and suppose Γ acts on a finite diffuse amenable von Neumann algebra Q. Then $Q \rtimes \Lambda$ is maximal injective inside $Q \rtimes \Gamma$, by [BC15, Theorem 1.3]. However again there are lots of different injective subalgebras containing Q but are not contained in $Q \rtimes \Lambda$.

Remark 4.3.4. We would like to mention an example in the ultra-product setting. Let $A \subset M$ be a singular masa inside a separable II_1 factor. Then for any free ultrafilter ω, $\mathscr{A} := A^\omega$ is a maximal injective masa in $\mathscr{M} := M^\omega$, a result due to Popa ([Pop14, Theorem 5.2.1]). However, it is well known that any two separable abelian subalegebras in a ultraproduct of II_1 factors are unitarily conjugate ([Pop83b, Lemma 7.1]). In particular, A is both contained in a maximal injective masa and a maximal hyperfinite subfactor of $\mathscr{M}$.

Chapter 5

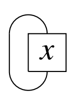
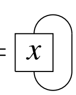
Unique maximal extension for the cup

5.1 Preliminaries

This chapter is based on the joint work with Arnaud Brothier [BW16].

5.1.1 Planar algebras

A planar algebra is a collection of complex $*$-algebras $\mathscr{P} = (\mathscr{P}_n^{\pm} : n \geqslant 0)$ on which the set of shaded planar tangles acts. See [Jon99, Jon12] for more details. We follow similar conventions that was used in [CJS14] for drawing a shaded planar tangle. We decorate strings with natural numbers to indicate that they represent a given number of parallel strings. The distinguished interval of a box is decorated by a dollar sign if it is not at the top left corner. We do not draw the outside box and will omit unnecessary decorations. The left and right traces of a planar algebra are the maps $\tau_l : \mathscr{P}_n^{\pm} \longrightarrow \mathscr{P}_0^{\mp}$ and $\tau_r : \mathscr{P}_n^{\pm} \longrightarrow \mathscr{P}_0^{\pm}$ defined for any $n \geqslant 0$ such that

$\tau_l(x) =$  and $\tau_r(x) =$  for any $x \in \mathscr{P}_n^{\pm}$.

Suppose that $\mathscr{P}_0^{\pm} = \mathbb{C}$. The planar algebra is called spherical if the two traces agree on each element of $\mathscr{P}$. We say that $\mathscr{P}$ is non-degenerate if the sesquilinear forms $(x,y) \mapsto \tau_l(xy^*)$ and $(x,y) \mapsto \tau_r(xy^*)$ are positive definite. A subfactor planar algebra is a planar algebra such that each space $\mathscr{P}_n^{\pm}$ is finite dimensional, $\mathscr{P}_0^{\pm} = \mathbb{C}$, $\mathscr{P}$ is spherical and non-degenerate. The modulus of a subfactor planar algebra is the value of a closed loop.

5.1.2 Construction of a II_1 factor

We recall a construction due to Jones et al. [JSW10]. Consider the direct sum $Gr\mathscr{P} = \bigoplus_{n\geqslant 0} \mathscr{P}_n^+$ that we equipped with the following Bacher product and involution:

$$xy = \sum_{a=0}^{\min(2n,2m)} \boxed{x}\overset{a}{\frown}\boxed{y}, \text{ and } x^\dagger = -\boxed{x^*}-^{\$}, \text{where } x \in \mathscr{P}_n^+ \text{ and } y \in \mathscr{P}_m^+.$$

Consider the linear form $\tau : Gr\mathscr{P} \longrightarrow \mathbb{C}$ that sends $x \in \mathscr{P}_0^+$ to itself and 0 to any element in $\mathscr{P}_n^+$ if $n \neq 0$. The vector space $Gr\mathscr{P}$ endowed with those operation is an associative $*$-algebra with a faithful tracial state. Let H be the completion of $Gr\mathscr{P}$ for the inner product $(x,y) \mapsto \tau(xy^*)$. The left multiplication of $Gr\mathscr{P}$ on H is bounded and defines a $*$-representation [GJS10, JSW10]. Let M be the von Neumann algebra generated by $Gr\mathscr{P}$ inside $B(H)$. It is an interpolated free group factor [GJS11, Har13]. We define another multiplication on $Gr\mathscr{P}$ by requiring that if $x \in \mathscr{P}_n$ and $y \in \mathscr{P}_m$, then

$$x \bullet y = \boxed{x}\ \boxed{y} \in \mathscr{P}_{n+m}^+.$$

Denote by $x^{\bullet n}$ the n-th power of x for this multiplication. Remark, $\|a \bullet b\|_2 = \|a\|_2 \|b\|_2$, for all $a \in \mathscr{P}_n$ and $b \in \mathscr{P}_m$. Therefore, this multiplication is a continuous bilinear form for the L^2-norm $\|\cdot\|_2$ of M. We extend this operation on $L^2(M) \times L^2(M)$ and still denote it by $\bullet$.

5.1.3 The cup subalgebra

Let $\cup$ be the unity of the $*$-algebra $\mathscr{P}_1^+$, viewed as an element of M [GJS10]. Let $A \subset M$ be the von Neumann subalgebra generated by $\cup$. We call it the cup subalgebra.

5.2 Proof of Theorem B

Proposition 5.2.1. *Let (A,τ) be a tracial von Neumann algebra and $B\subset A$ a diffuse subalgebra. Denote by $L^2(A)$ the Gelfand-Naimark-Segal completion of A for the trace τ. Consider a sequence $\xi=(\xi_n : n\geqslant 0)$ of unit vectors of the coarse bimodule $L^2(A)\otimes L^2(A)$. Suppose that for any $b\in B$ we have $\lim_{n\to\infty}\|b\cdot\xi_n-\xi_n\cdot b\|_2=0$. Then, if $p\in B(L^2(A))$ is a finite rank projection, then $\lim_{n\to\infty}\|(p\otimes 1)\xi_n\|_2=\lim_{n\to\infty}\|(1\otimes p)\xi_n\|_2=0$.*

Proof. Let A,B,ξ, and p as above. It is sufficient to prove the proposition when p is a rank one projection. Let $\eta\in L^2(A)$ be a unit vector such that $p=p_\eta$ is the rank one projection onto $\mathbb{C}\eta$. Consider $0<\varepsilon<1$ and a natural number I such that $16/(I+1)<\varepsilon$. Denote by ε_n the quantity $\frac{\varepsilon}{2^{n+4}(2n+1)}$ for any $n\geqslant 0$. Since B is diffuse, there exists a sequence of unitaries $(u_n)_n$ in B such that $\lim_{n\to\infty}\langle u_n\cdot\zeta_1,\zeta_2\rangle=0$ for any $\zeta_1,\zeta_2\in L^2(A)$. Hence, there exists a subsequence $(v_n)_n$ such that $|\langle v_n\cdot\eta, v_m\cdot\eta\rangle|\leqslant\varepsilon_{\max(n,m)}$ for any natural numbers $n\neq m$. By [**?** , Proposition 2.3] and by taking a subsequence if necessary we have,

$$2=2\|\xi_n\|^2\geqslant\sum_{i=0}^{I}\|(p_{v_i\cdot\eta}\otimes 1)\xi_n\|^2-2\sum_{i,j=0}^{I}|\langle v_i\cdot\eta, v_j\cdot\eta\rangle|,\ \text{for any } n\geqslant 0.$$

Hence,

$$\sum_{i=0}^{I}\|(p_{v_i\cdot\eta}\otimes 1)\xi_n\|^2\leqslant 2+2\sum_{i,j=0}^{I}\varepsilon_{\max(i,j)}\leqslant 2+2\sum_{i=0}^{I}(2i+1)\varepsilon_i\leqslant 2+\varepsilon/4,\ \text{for any } n\geqslant 0.$$

Let $\lambda : B\longrightarrow B(L^2(A)\otimes L^2(A))$ be the left action of B on the coarse bimodule $L^2(A)\otimes L^2(A)$. Observe, $p_{v_i\cdot\eta}\otimes 1=\lambda(v_i)\circ(p_\eta\otimes 1)\circ\lambda(v_i)^*$ and v_i is a unitary, for any $i\geqslant 0$. Therefore, $\|(p_{v_i\cdot\eta}\otimes 1)\xi_n\|=\|(p_\eta\otimes 1)v_i^*\cdot\xi_n\|$ for any $n,i\geqslant 0$. By assumption, there exists

$N > 0$ such that for any $n \geqslant N$ and $i \leqslant I$ we have $\|v_i^* \cdot \xi_n - \xi_n \cdot v_i^*\| < \varepsilon/4$. Therefore,

$$\begin{aligned}\|(p_\eta \otimes 1)\xi_n\| &= \|(p_\eta \otimes 1)(\xi_n \cdot v_i^*)\| \\ &\leqslant \|(p_\eta \otimes 1)(v_i^* \cdot \xi_n - \xi_n \cdot v_i^*)\| + \|(p_\eta \otimes 1)(v_i^* \cdot \xi_n)\| \\ &\leqslant \varepsilon/4 + \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| \text{ for any } n \geqslant N, i \leqslant I.\end{aligned}$$

We obtain

$$\begin{aligned}\sum_{i=0}^{I} \|(p_\eta \otimes 1)\xi_n\|^2 &\leqslant \sum_{i=0}^{I} (\varepsilon^2/16 + \varepsilon/2\|(p_{v_i \cdot \eta} \otimes 1)\xi_n\| + \|(p_{v_i \cdot \eta} \otimes 1)\xi_n\|^2) \\ &\leqslant (I+1)(\varepsilon/16 + \varepsilon/2) + (2 + \varepsilon/4) \text{ for any } n \geqslant N.\end{aligned}$$

Therefore, $\|(p_\eta \otimes 1)\xi_n\|^2 \leqslant \varepsilon/16 + \varepsilon/2 + (2+\varepsilon/2)\varepsilon/16 \leqslant \varepsilon$ for any $n \geqslant N$. The same proof shows that there exists $M > 0$ such that for any $n \geqslant M$ we have $\|(1 \otimes p_\eta)\xi_n\|^2 \leqslant \varepsilon$. This proves the proposition. □

Fix a subfactor planar algebra $\mathscr{P}$ with modulus $\delta > 1$ and denote by $A \subset M$ its associated cup subalgebra. Consider the subspace $V_n \subset \mathscr{P}_n^+, n \geqslant 0$ of elements that vanishes when they are capped off on the top left corner and vanished when they are capped off on the top right corner. Let $V \subset L^2(M)$ be their orthogonal direct sum. By [JSW10, Theorem 4.9], the following map is an isomorphism of A-bimodules:

$$\phi : L^2(A) \oplus (L^2(A) \otimes V \otimes L^2(A)) \longrightarrow L^2(M), a + b \otimes v \otimes c \longmapsto a + b \bullet v \bullet c.$$

This implies that the A-bimodule $L^2(M) \ominus L^2(A)$ is isomorphic to an infinite direct sum of the coarse bimodule. We identify $L^2(M)$ with $\phi^{-1}(L^2(M))$.

Consider the finite dimensional subspace $L_m = \mathrm{Span}(\cup^{\bullet k} : k \leqslant m) \subset A$ for $m \geqslant 0$, where $\cup^{\bullet 0} = 1 \in \mathscr{P}_0^+$. Denote by $L_m^\perp$ the orthogonal complement of L_m inside $L^2(A)$ for any $m \geqslant 0$.

Lemma 5.2.2. *Let $m \geqslant 0$ and $x \in M \cap L_m^\perp \otimes V \otimes L_m^\perp$, $y \in M \cap L_m \otimes V \otimes L_m$. Then $xy \in$*

$L_m^{\perp} \otimes V \otimes L_m$ *and* $yx \in L_m \otimes V \otimes L_m^{\perp}$. *In particular,* $xy \perp yx$.

Proof. Consider $x = \cup^{\bullet k} \bullet v \bullet \cup^{\bullet l}$ and $y = \cup^{\bullet s} \bullet w \bullet \cup^{\bullet t}$, where $s,t < m+1 \leqslant k,l$ and $v,w \in V \cap Gr\mathscr{P}$. We have that

$$xy = \sum_{i=0}^{s+1} \delta^{[i/2]} \cup^{\bullet k} \bullet v \bullet \cup^{\bullet(l+s-i)} \bullet w \bullet \cup^{\bullet t},$$

where $[i/2] = i/2$ if i is even and $i/2 - 1/2$ if i is odd. Observe, $L_m^{\perp}$ is equal to the closure of $\mathrm{Span}(\cup^{\bullet k} : k \geqslant m+1)$. Therefore, $xy \in L_m^{\perp} \otimes V \otimes L_m$ and similarly $yx \in L_m \otimes V \otimes L_m^{\perp}$. The space $M \cap L_m^{\perp} \otimes V \otimes L_m$ (resp. $M \cap L_m \otimes V \otimes L_m^{\perp}$) is the weak closure of $\mathrm{Span}(\cup^{\bullet k} \bullet v \bullet \cup^{\bullet l} : k,l \geqslant m+1, v \in V \cap Gr\mathscr{P})$ (resp. $\mathrm{Span}(\cup^{\bullet s} \bullet w \bullet \cup^{\bullet t} : s,t \leqslant m, w \in V \cap Gr\mathscr{P})$). This concludes the proof by a density argument. $\square$

We are ready to prove the s-AOP of the inclusion of the cup subalgebra.

Proposition 5.2.3. *The inclusion of the cup subalgebra $A \subset M$ has s-AOP.*

Proof. Let $\mathscr{P}$ be a subfactor planar algebra, $A \subset M$ its associated cup subalgebra, and $B \subset A$ a diffuse subalgebra. Consider $x \in M^{\omega} \ominus A^{\omega}$ in the relative commutant of B and $y \in M \ominus A$, where ω is a free ultrafilter on $\mathbb{N}$. Let us show that $xy \perp yx$. Observe, $Gr\mathscr{P}$ is a weakly dense $*$-subalgebra of M. Therefore, we can assume that $y \in Gr\mathscr{P}$ by Kaplansky density theorem. This implies that there exists $m \geqslant 0$ such that $y \in Gr\mathscr{P} \cap L_m \otimes V \otimes L_m$. Let $(x_n)_n$ be a representative of x in the ultrapower M^{ω}. We can assume that for any $n \geqslant 0$ we have $x_n \in L^2(M) \ominus L^2(A)$. Let $p \in B(L^2(A))$ be the orthogonal projection onto L_m. It is a finite rank projection. Therefore, by Proposition 5.2.1, $(p \otimes 1)x = (1 \otimes p)x = 0$. Hence, we can assume that $x_n \in L_m^{\perp} \otimes V \otimes L_m^{\perp}$ for any $n \geqslant 0$. Lemma 5.2.2 implies that $x_n y \perp y x_n$ for any $n \geqslant 0$. This implies that $xy \perp xy$. $\square$

Thus, by Theorem 3.2.2, we complete the proof of Theorem B.

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