THEORY OF PARABOLIC DIFFERENTIAL EQUATIONS ON SINGULAR MANIFOLDS AND ITS APPLICATIONS TO GEOMETRIC ANALYSIS

By

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Chapter I

Introduction

1. Background and main results

In this thesis, we conduct a systematic study of the theory of parabolic differential equations on manifolds with singularities. The majority of the material in this work is based on the author's recent manuscripts [**80, 81**]. The particular class of manifolds considered here is called *singular manifolds*. Roughly speaking, a manifold (M, g) is singular if it is conformal to a *uniformly regular Riemannian manifold* $(\mathsf{M}, g/\rho^2)$, by which we mean a manifold whose local patches are of comparable sizes, and all transit maps and curvatures are uniformly bounded. The conformal factor ρ is called a singularity function for (M, g). The concept of *singular manifolds* used in this paper and the concept of *uniformly regular Riemannian manifolds* were first introduced by H. Amann in [**4, 5**].

Perhaps one of the most important motivations for such a theory stems from geometric analysis. An interesting problem in geometry is whether one can find a "standard model" in each class of metrics on some manifold, thus reducing topological questions to differential geometry ones. One of the well-known representatives of this kind is the Yamabe problem. On a compact manifold (M, g), this problem aims at finding a metric with constant scalar curvature, which is conformal to g.

If we already have a metric, as a natural question, one might want to ask how we can "drive" the given metric into a "standard" one, or at least improve it. For example, the famous Ricci flow deforms the metric tensor g of an m-dimensional compact closed manifold by the law:

$$\begin{cases} \partial_t g = -2\mathrm{Ric}(g) & \text{on} \quad \mathsf{M}_T; \\ g(0) = g_0 & \text{on} \quad \mathsf{M}. \end{cases}$$

Here $\mathrm{Ric}(g)$ is the Ricci tensor of the metric g, and $\mathsf{M}_T := (0, T) \times \mathsf{M}$ for $T \in (0, \infty]$. The Ricci flow tends to "flatten out", or "round out", a manifold, depending on its initial "shape". Another example of evolution of metrics is the Yamabe flow. As an alternative approach to the Yamabe problem, R. Hamilton introduced the normalized Yamabe flow, which asks whether a metric, driven by this flow, converges conformally to one with constant scalar curvature. The normalized Yamabe flow reads as follows.

$$\begin{cases} \partial_t g = (s_g - R_g) g & \text{on} \quad \mathsf{M}_T; \\ g(0) = g_0 & \text{on} \quad \mathsf{M}, \end{cases} \tag{I.1}$$

where R_g is the scalar curvature with respect to the evolving metric g, and s_g is the average of R_g. On an m-dimensional closed compact manifold (M, g_0), an easy computation shows that the normalized Yamabe flow is equivalent to the (unnormalized) Yamabe flow.

$$\begin{cases} \partial_t g = -R_g g & \text{on} \quad \mathsf{M}_T; \\ g(0) = g_0 & \text{on} \quad \mathsf{M}. \end{cases} \tag{I.2}$$

This parabolic equation has a history of its own. The concept of closed manifolds in this article refers to manifolds without boundary, not necessarily compact. On closed compact manifolds, global existence and regularity of solutions to equation (I.2) has been well studied. R.G. Ye [**91**] proved that the unique solution to (I.2) exists globally and smoothly for any smooth initial metric. Fix a background metric $\tilde{g}$ such that $(\mathsf{M}, \tilde{g})$ is compact. In particular, we can take $\tilde{g} = g_0$. Let $g = u^{\frac{4}{m-2}} \tilde{g}$ for some $u > 0$. By rescaling the time variable, equation (I.2) is equivalent to

$$\begin{cases} \partial_t u = u^{-\frac{4}{m-2}} L_{\tilde{g}} u & \text{on} \quad \mathsf{M}_T; \\ u(0) = u_0 & \text{on} \quad \mathsf{M} \end{cases} \tag{I.3}$$

with a positive function u_0. In addition, $L_{\tilde{g}} u := \Delta_{\tilde{g}} u - \frac{m-2}{4(m-1)} R_{\tilde{g}} u$ is the conformal Laplacian operator with respect to the metric $\tilde{g}$, and $\Delta_{\tilde{g}}$ denotes the Laplace-Beltrami operator with respect to $\tilde{g}$. By the compactness of $(\mathsf{M}, \tilde{g})$, uniform ellipticity of the operator $u_0^{-\frac{4}{m-2}} L_{\tilde{g}}$ is guaranteed. That is why well-posedness is relatively easy in the compact case.

It was conjectured by R. Hamilton that the solution of the Yamabe flow (I.2) converges to a metric with constant scalar curvature as $t \to \infty$. B. Chow [**20**] commenced the study of Hamilton's conjecture and proved convergence in the case when (M, g_0) is locally conformally flat and has positive Ricci curvature. Later, this result was improved by R.G. Ye [**91**], wherein the author removed the restriction on the positivity of the Ricci curvature by lifting the flow to a sphere, and deriving a Harnack inequality. In the case that $3 \le m \le 5$, H. Schwetlick and M. Struwe [**78**] showed that the normalized Yamabe flow evolves any initial metric to one with constant scalar curvature as long as the initial Yamabe energy is small. In [**13**], S. Brendle was able to remove the smallness assumption on the initial Yamabe energy. A convergence result is stated in [**14**] by the same author for higher dimension cases.

Evolution equations, such as the Ricci flow and the Yamabe flow, driving metrics by their curvatures, comprise an important class of so-called geometric flows, or geometric evolution equations. Geometric flows have been studied in depth on compact manifolds. However, many well established analytic tools and strategies are limited, or even fail, in the case of non-compact manifolds, and in particular, manifolds with singularities. In general, it is not known how to start some geometric flows, even just for a short time, without imposing further conditions. For instance, the theory for the Yamabe flow on non-compact manifolds is far from being settled. Even local well-posedness is only established for restricted situations. Its difficulty can be observed from the fact that, losing the compactness of (M, g_0), equation (I.3) can exhibit degenerate and singular behaviors simultaneously. The investigation of the Yamabe flow on non-compact manifolds was initiated by L. Ma and Y. An in [**58**]. Later, conditions on extending local in time solutions were explored in [**59**]. In [**58**], the authors showed that for a complete closed non-compact Riemannian manifold (M, g_0) with Ricci curvature bounded from below and with a uniform bound on the scalar curvature in the sense that:

$$\mathrm{Ric}_{g_0} \geq -K g_0, \quad |R_{g_0}| \leq C,$$

equation (I.2) has a short time solution on $\mathsf{M} \times [0, T(g_0)]$ for some $T(g_0) > 0$. If in addition $R_{g_0} \leq 0$, then this solution is global. The proof is based on the widely used technique consisting of exhausting M with a sequence of compact manifolds with boundary and studying the solutions to a sequence of initial boundary value problems. Then uniform estimates for these solutions and their gradients are obtained by means of the maximum principle on manifolds with Ricci curvature bounded from below in the sense given above. The uniform boundedness of the scalar curvature plays an indispensable role in the proof for local well-posedness, although this has not been pointed out explicitly in [**58**]. Hence one approach to the investigation of geometric flows on non-compact manifolds is trying to find curvature conditions, at least for local existence.

Another way to understand geometric flows, mainly in the case of manifolds with singularities, is to generalize the ideas for (degenerate or singular) elliptic differential operators. One way to view manifolds with singularities is to consider the so-called ringed spaces, that is, to view a manifold M as a pair consisting of a punctured manifold $\mathsf{M} \setminus \{\mathsf{p}\}$ (by removing the singularity p) and a subalgebra of differential operators degenerating at a specified rate near the singularity, rather than just M itself, see [**64**]. These considerations lead to the study of degenerate or singular differential operators on manifolds with singularities.

The study of differential operators on manifolds with singularities is motivated by a variety of applications from applied mathematics, geometry and topology. All of it is related to the seminal paper by V.A. Kondrat'ev [**54**]. There is a tremendous amount of literature on pseudo-differential calculus of differential operators of Fuchs type, which has been introduced independently by R.B. Melrose [**60, 61**] and B.-W. Schulze [**64, 75, 76, 77**]. One branch of these lines of research is connected with the so-called b-calculus and its generalizations on manifolds with cylindrical ends. See [**60, 61**]. Many authors have been very active in this direction. Research along another line, known as conical differential operators, was also initiated a long time ago. Operators in this line of research are modelled on conical manifolds. The investigation of conical singularities was initiated by J. Cheeger in [**17, 18, 19**], and then continued by many other authors. A comparison between the b-calculus and the cone algebra can be found in [**55**]. The amount of research on pseudo-differential calculus of differential operators of Fuchs type is enormous, and thus it is literally impossible to list all the work.

To distinguish these two lines of research, for instance, we look at a compact Riemannian manifold (M, g) with boundary $(\partial\mathsf{M}, g_{\partial\mathsf{M}})$. R.B. Melrose deforms a collar neighborhood of $\partial\mathsf{M}$ into a cylindrical end by equipping $\partial\mathsf{M}$ with the singular metric $(dt/t)^2 + g_{\partial\mathsf{M}}$ for $t \in (0,1]$. Setting $t = e^s$ with $s \in (-\infty, 0]$, it is easy to see that this metric is indeed asymptotically cylindrical. On the other hand, B.-W. Schulze uses the degenerate metric $dt^2 + t^2 g_{\partial\mathsf{M}}$. The associated Laplace-Beltrami operators with respect to these two metrics are

$$((t\partial_t)^2 + \Delta_{g_{\partial\mathsf{M}}}), \quad \text{and} \quad t^{-2}((t\partial_t)^2 + \Delta_{g_{\partial\mathsf{M}}}), \tag{I.4}$$

respectively. Many efforts have been made to generalize research in these directions to more complicated types of singularities, e.g., edge and corner pseudo-differential calculus. However, for higher order singularities, the corresponding algebra becomes significantly more complicated, although many ideas and structures can be extracted, e.g., from the calculus of boundary value problems, c.f. [**56, 75, 77**]. Therefore, a natural question to ask is whether we can find a general approach, which is less sensitive to the geometric structure near the singularities, to analyzing differential equations on manifolds with singularities.

In 2012, H. Amann [**4, 5**] introduced the geometric framework of *singular manifolds* and *uniformly regular Riemannian manifolds* to study differential equations on non-compact manifolds and on manifolds with singularities. To illustrate the geometric generality of these two concepts, we show in Example III.6 that manifolds with cone, edge and corner ends are *singular manifolds*. Meanwhile, manifolds with asymptotically cylindrical ends are examples of *uniformly regular Riemannian manifolds*. It is proved in [**30**] that the class of all closed *uniformly regular Riemannian manifolds* coincides with the family of complete manifolds with bounded geometry, which reveals that the concept of *uniformly regular Riemannian manifolds* extends one of the perhaps most extensively studied class of non-compact manifolds.

If two real-valued functions f and g are equivalent in the sense that $f/c \leq g \leq cf$ for some $c \geq 1$, then we write $f \sim g$. From our description in the first paragraph, a *singular manifold* (M, g) with singularity function $\rho \sim \mathbf{1}_{\mathsf{M}}$ is *uniformly regular*. Therefore, the family of *uniformly regular Riemannian manifolds* is a subclass of *singular manifolds*. We will focus on the latter class in this thesis.

The framework of *singular manifolds*, as we can observe from its definition, does not depend on the order of the singularities or the specific geometric structure near the singularities. Moreover, function space theory, for instance, interpolation, embedding and trace theorems, is well established for *singular manifolds* in [**4, 5**]. It is known that although most function spaces, like Sobolev-Slobodeckii spaces, can be defined invariantly on non-compact manifolds, the corresponding function space theory does not hold in this generality. These preparations and the uniform structure of *singular manifolds* enable us to build up the theory of parabolic differential equations on *singular manifolds*, with singularities of arbitrarily high dimensions and of various structures.

To clarify the role of the differential operators in this article, we look at

(I.5)
$$\mathscr{A} = \rho^{-\lambda}\mathcal{A},$$

on a *singular manifold* (M, g). Here $\rho \in C^\infty(\mathsf{M}, (0,1))$ is a singularity function and $\lambda > 0$, or $\rho \in C^\infty(\mathsf{M}, (1,\infty))$ and $\lambda < 0$. In (I.5), $\mathcal{A}$ is a *uniformly ρ-elliptic operator*. We consider a second order differential operator

$$\mathcal{A}u := -\mathrm{div}(\vec{a} \cdot \mathrm{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u$$

as an example. By *uniformly strongly ρ-ellipticity*, we mean that the principal symbol of $\mathcal{A}$ fulfils

$$\hat{\sigma}\mathcal{A}(x,\xi) := (\vec{a}(x) \cdot \xi | \xi)_{g^*} \sim \rho^2 |\xi|^2_{g^*}.$$

Here g^* is the cotangent metric induced by g, $\vec{a}$ is a symmetric $(1,1)$-tensor field on (M, g), and the operation $[u \mapsto \vec{a} \cdot \mathrm{grad} u]$ denotes center contraction. See Section 5.1 for the precise definition. These operators, as we can immediately observe from the above relationship, can exhibit degenerate or singular behaviors while approaching the singular ends. However, if we look at the localization of the operator $\mathcal{A}$ in an atlas satisfying that the size of each local patch is proportional to the value of ρ in that patch, then the local expressions of $\mathcal{A}$ have uniform ellipticity constants in all local coordinates. That is why we call $\mathcal{A}$ "uniformly strongly elliptic."

In [**6**], H. Amann built up the L_p-maximal regularity for second order *uniformly strongly ρ-elliptic* operators. Given a densely embedded Banach couple $E_1 \overset{d}{\hookrightarrow} E_0$, an operator A is said to have L_p-maximal regularity if for any $J = [0,T]$ and each $f \in L_p(J; E_0)$, the equation

$$\begin{cases} u_t + Au = f, \quad J \in T; \\ \qquad\quad u = 0, \end{cases}$$

has a unique solution $u \in H^1_p(J; E_0) \cap L_p(J; E_1)$.

We generalize the concept of *uniformly strong ρ-ellipticity* to elliptic operators of arbitrary even order acting on tensor bundles in Chapter 4. A linear operator

$$\mathcal{A} := \sum_{r=0}^{2l} \mathsf{C}(a_r, \nabla^r \cdot) \tag{I.6}$$

of order $2l$, where a_r is a $(\sigma + \tau + r, \tau + \sigma)$-tensor field, is said to be *uniformly strongly ρ-elliptic* if its principal symbol satisfies that for each cotangent field ξ and every (σ, τ)-tensor field η, it holds

$$(\mathsf{C}(a_{2l}(x), \eta \otimes (-i\xi)^{\otimes 2l}) | \eta)_g \sim \rho^{2l} |\eta|^2_g |\xi|^{2l}_{g^*}. \tag{I.7}$$

Moreover, in Chapter 4, we show that this ellipticity condition can actually be replaced by a weaker one, called *normal ρ-ellipticity*. But for the sake of simplicity, we still stay with *uniformly* *strong ρ-ellipticity* stated above. In Chapter 4, by imposing some mild regularity condition on the coefficients a_r of $\mathcal{A}$, called *s-regularity*, we are able to prove the following result.

Theorem I.1. *Let $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\vartheta \in \mathbb{R}$. Suppose that $\mathcal{A}$ is a $2l$-th order linear differential operator, which is uniformly strongly ρ-elliptic and s-regular. Then*

$$\mathcal{A} \in \mathcal{H}(bc^{s+2l,\vartheta}(\mathsf{M}, V_\tau^\sigma), bc^{s,\vartheta}(\mathsf{M}, V_\tau^\sigma)).$$

Here $u \in bc^{s,\vartheta}(\mathsf{M}, V_\tau^\sigma)$ if $\rho^\vartheta u$ is a (σ, τ)-tensor field with *little Hölder* continuity of order s. The precise definition of weighted *little Hölder* spaces will be presented in Section 2.2. An operator A is said to belong to the class $\mathcal{H}(E_1, E_0)$ for some densely embedded Banach couple $E_1 \overset{d}{\hookrightarrow} E_0$, if $-A$ generates a strongly continuous analytic semigroup on E_0 with $dom(-A) = E_1$. By means of a well-known result of G. Da Prato, P. Grisvard [**65**] and S. Angenent [**9**], this theorem implies that the operator $\mathcal{A}$ enjoys the maximal regularity property in a Hölder framework, called continuous maximal regularity. We refer the reader to Chapter 4 for the precise definition of continuous maximal regularity.

In virtue of a theorem by G. Da Prato and P. Grisvard [**65**], S.B. Angenent [**9**], and P. Clément, G. Simonett [**22**], continuous maximal regularity theory gives rise to local existence and uniqueness for quasilinear or even fully nonlinear equations. To briefly illustrate the scope of the work in Chapter 4, for example, we take a look at a model quasilinear bilaplacian problem on a *singular manifold* (M, g)

$$\begin{cases} \partial_t u + u^n \Delta_g^2 u = 0 & \text{on} \quad \mathsf{M}_T; \\ u(0) = u_0 & \text{on} \quad \mathsf{M}. \end{cases}$$

Here Δ_g is the Laplace-Beltrami operator with respect to g. This model is closely related to the thin film equation. Suppose that U is a properly chosen open subset of some *little Hölder* space $bc^{s,\vartheta}(\mathsf{M})$. If for every $u \in U$ the principal symbol of the operator $u^n \Delta_g^2$ satisfies

$$u^n (g^*(-i\xi, -i\xi))^2 \sim \rho^4 |\xi|_{g^*}^4, \quad \xi \in T^*\mathsf{M},$$

then the above equation has a unique classical solution in weighted *little Hölder* spaces for any $u_0 \in U$. Similar results also hold true for differential operators acting on tensor fields. In addition, it is worthwhile mentioning that L_p and continuous maximal regularity theories on *uniformly regular Riemannian manifolds* are established by H. Amann, G. Simonett and the author, see [**6, 82**].

Maximal regularity theory has proven itself a powerful tool in the study of linear and nonlinear evolution equations in recent decades, especially for higher order differential equations, see [**1**], [**8**]-[**10**], [**49**], [**50**], [**31**]-[**35**], [**57**], [**70**], [**72**], [**83**] for example. In addition to proving well-posedness results, in several examples of Chapter 7 and 8, we will present how to establish regularity of solutions to parabolic equations on *singular manifolds* by means of maximal regularity theory. A detailed discussion of this technique can be found in [**79**], where the author proved regularity of solutions to the Ricci flow, the surface diffusion flow, and the mean curvature flow.

To line up with the aforementioned work on pseudo-differential calculus on manifolds with singularities, the author would like to point out that on an asymptotically cylindrical manifold, the Laplace-Beltrami operator

$$((t\partial_t)^2 + \Delta_{g_{\partial \mathsf{M}}})$$

is *uniformly strongly ρ-elliptic*. Hence, the theory established in [**6**] and Chapter 4 can to some extend be considered as a generalization of the work on b-calculus by R.B. Melrose [**60, 61**] and his collaborators.

In contrast to the "uniformly strongly elliptic" operator $\mathcal{A}$, by our choice of ρ and λ in (I.5), the ellipticity constants of the localizations for the operator $\mathscr{A}$ in local coordinates blow up while approaching the singular ends of the manifold (M, g). The rate of the blow-up of the ellipticity constant is characterized by the power λ. For this reason, we will call such an $\mathscr{A}$ a (ρ, λ)-singular elliptic operator. More precisely, a second order elliptic differential operator $\mathscr{A}$ is (ρ, λ)-singular if its principal symbol satisfies

$$\hat{\sigma}\mathscr{A}(x, \xi) \sim \rho^{2-\lambda} |\xi|^2_{g^*},$$

for any cotangent field ξ. To illustrate the behavior of the operator $\mathscr{A}$, we consider the Euclidean space $\mathbb{R}^N$ as a *singular manifold* with ∞ as a singular end, and take $\mathcal{A}$ to be the Laplacian in (I.5). Then the operator $\mathscr{A}$, in some sense, looks like one with unbounded coefficients at infinity on $\mathbb{R}^m$.

Our approach to (ρ, λ)-singular elliptic operators in Chapter 5 is based on the traditional strategy of associating differential operators with densely defined, closed and sectorial forms. This method, being utilized by many authors, has displayed its power in establishing L_p-semigroup theory for second order differential operators on domains in $\mathbb{R}^m$. See, for example, [**25, 63, 67, 68**] and the references therein. To the best of the author's knowledge, there are only very few papers on the generation of analytic semigroups for differential operators with unbounded diffusion coefficients in $\mathbb{R}^m$ or in an exterior domain with regular boundary, among them [**39, 44, 46, 62, 63**]. In all these articles, the drift coefficients have to be controlled by the diffusion and potential terms. In [**63**], the authors use a form operator method to prove a semigroup result for operators with unbounded coefficients in a weighted Sobolev space. The drawback of the method used in [**63**] is reflected by the difficulty to precisely determine the domains of the differential operators. This is, in fact, one of the most challenging tasks in the form operator approach. One of the most important features of the work in Chapter 5 is that with the assistance of the theory of function spaces and differential operators on *singular manifolds* established in [**4, 5, 6**], we can find a precise characterization for the domains of the second order (ρ, λ)-singular elliptic operators.

A conventional method to render the associated sesquilinear form of an elliptic operator $\mathscr{A}$ densely defined, closed and sectorial is to perturb $\mathscr{A}$ by a spectral parameter $\omega > 0$. See [**63, 68**] for instance. Then $\mathscr{A}$ generates a quasi-contractive semigroup. However, for a (ρ, λ)-singular elliptic operator, e.g., the operator $\mathscr{A}$ in (I.5), because of the existence of the multiplier $\rho^{-\lambda}$, we need to perturb $\mathscr{A}$ by a weight function of the form $\omega\rho^{-\lambda}$. This feature arising from our approach creates an essential difficulty for a parabolic theory of differential equations on manifolds with singularities. We take the conical Laplace-Beltrami

$$t^{-2}((t\partial_t)^2 + \Delta_{g_{\partial \mathsf{M}}})$$

in (I.5) as an example. This operator is $(\rho, 2)$-singular elliptic. In order to prove that this operator generates a contractive semigroup, we need to perturb it not by a constant ω, but actually by a weight function ωt^{-2}. More generally, in Section 5.1, by imposing some weak regularity assumption on the coefficients of $\mathscr{A}$, called (ρ, λ)-regularity, we shall prove the following result.

Theorem I.2. *Suppose that the differential operator*

$$\mathscr{A}u := -\mathrm{div}(\vec{a} \cdot \mathrm{grad}u) + \mathsf{C}(\nabla u, a_1) + a_0 u,$$

is (ρ, λ)-regular and (ρ, λ)-singular. For ω so large that the conditions (A3)-(A5) in Section 5.1 are satisfied, define $\mathscr{A}_\omega := \mathscr{A} + \omega\rho^{-\lambda}$. Then

$$\mathscr{A}_\omega \in \mathcal{H}(W_p^{2,\lambda'-\lambda}(\mathsf{M}, V_\tau^\sigma), L_p^{\lambda'}(\mathsf{M}, V_\tau^\sigma)), \quad 1 < p < \infty,$$

and the semigroup $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ *is contractive. Here* $\sigma=\tau=0$*, when* $p\neq 2$*.*

Here $L_p^{\lambda'}(\mathsf{M},V_\tau^\sigma)$ and $W_p^{2,\lambda'-2}(\mathsf{M},V_\tau^\sigma)$ are some weighted Sobolev spaces whose definition will be given in Section 2.2. The extra condition on the perturbation term $\omega\rho^{-\lambda}$ is equivalent to a largeness condition on the potential term a_0 of the operator $\mathscr{A}$ in Theorem I.2. The commutator of weight functions of the form $\omega\rho^{-\lambda}$ and differential operators is usually not a perturbation in the sense of [**41, 73**]. Thus the extra term $\omega\rho^{-\lambda}$, in general, cannot be removed by a "soft" method, like the perturbation theory of semigroups. In some cases, e.g., the conic Laplace-Beltrami operator, we find it more practical to put a control on the diffusion or drift term. This is a quite natural condition which has been used in [**39, 44, 46, 62, 63**]. In all these articles, the growth of the drift coefficients have to be controlled by the diffusion and potential terms.

In Section 5.2, we are able to remove the compensation condition $\omega\rho^{-\lambda}$ for a class of *singular manifolds* called *singular manifolds* with $\mathscr{H}_\lambda$-ends. To the best of the author's knowledge, this concept is introduced here for the first time. We will show that it is possible to create *singular manifolds* with $\mathscr{H}_\lambda$-ends with singularities of arbitrarily high dimension. To illustrate how to construct such manifolds, we look at the following example of manifolds with "holes". First, we start with an m-dimensional complete closed manifold $(\mathscr{M},g)$ with bounded geometry. Then we remove finitely many $\Sigma_j\subset\mathscr{M}$. Each Σ_j is an m-dimensional compact manifold with boundary. Let

$$\mathsf{M}:=\mathscr{M}\setminus\cup_j\Sigma_j.$$

Since the boundary $\partial\Sigma_j$ is not contained in M, the manifold (M,g) is incomplete. The resulting manifold with "holes" is a *singular manifold* with $\mathscr{H}_\lambda$-ends. On such a manifold (M,g) with "holes", as an example of the work in Chapter 5, we can show that the Laplace-Beltrami operator satisfies

$$\Delta_g:=\mathrm{div}_g\circ\mathrm{grad}_g\in\mathcal{H}(W_p^{2,\lambda'-2}(\mathsf{M}),L_p^{\lambda'}(\mathsf{M})),\quad 1<p<\infty.$$

For general (ρ,λ)-singular elliptic operators, we have the following result.

Theorem I.3. *Suppose that* $(\mathsf{M},g;\rho)$ *is a singular manifold satisfying* $\rho\leq 1$*,*

$$|\nabla\rho|_g\sim\mathbf{1},\quad \|\Delta\rho\|_\infty<\infty$$

on $\mathsf{M}_r := \{\mathsf{p} \in \mathsf{M} : \rho(\mathsf{p}) < r\}$ *for some* $r \in (0,1]$*. Moreover, assume that the set*

$$S_{r_0} := \{\mathsf{p} \in \mathsf{M} : \rho(\mathsf{p}) = r_0\}$$

is compact for $r_0 \in (0,r)$*. Let* $\lambda' \in \mathbb{R}$*, and* $\lambda \in [0,1) \cup (1,\infty)$

(a) *Then* $(\mathsf{M}, g; \rho)$ *is a singular manifold with* $\mathscr{H}_\lambda$*-ends.*

(b) *Furthermore, assume that the differential operator*

$$\mathscr{A} u := -\mathrm{div}(\rho^{2-\lambda} \mathrm{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u$$

is (ρ,λ)*-regular. Then*

$$\mathscr{A} \in \mathcal{H}(W_p^{2,\lambda'-\lambda}(\mathsf{M}, V_\tau^\sigma), L_p^{\lambda'}(\mathsf{M}, V_\tau^\sigma)), \quad 1 < p < \infty.$$

Here $\sigma = \tau = 0$ *when* $p \neq 2$*.*

2. Introduction to applications

We will apply Theorem I.1-I.3 to several well-known evolution equations in Chapter 6-8.

As an example in geometric analysis, we prove the following well-posedness result for the Yamabe flow (I.2) on an m-dimensional *singular manifold* $(\mathsf{M}, \tilde{g})$.

Theorem I.4. *Suppose that* $u_0 \in U_\vartheta^s := \{u \in bc^{s,\vartheta}(\mathsf{M}) : \inf \rho^\vartheta u > 0\}$ *with* $s \in (0,1)$*, and* $\vartheta = (m-2)/2$*. Then equation* (I.3) *has a unique smooth local positive solution* $\hat{u}$ *with*

$$\hat{u} \in C(J(u_0), U_\vartheta^s)$$

existing on $J(u_0) := [0, T(u_0))$ *for some* $T(u_0) > 0$*.*

Under the conformal change $g_0 = u_0^{\frac{4}{m-2}} \tilde{g}$, we have

$$R_{g_0} = -\frac{4(m-1)}{m-2} u_0^{-\frac{m+2}{m-2}} L_{\tilde{g}} u. \tag{I.8}$$

To show that in Theorem I.4 we may start with a metric with unbounded scalar curvature, we take $\rho = \mathbf{1}_\mathsf{M}$ for computational brevity, i.e., $(\mathsf{M}, \tilde{g})$ to be *uniformly regular*. Then there is some $C > 1$

such that for any $u_0 \in U^s_\vartheta$

$$1/C \leq \|u_0^{-\frac{m+2}{m-2}}\|_\infty \leq C, \quad \|R_{\tilde{g}}\|_\infty \leq C.$$

But at the same time, there are ample examples of $u_0 \in U^s_\vartheta$ with unbounded derivatives. In view of formula (I.8), it is not hard to create g_0 with unbounded scalar curvature. Therefore, the Yamabe flow can admit a unique smooth solution while starting at a metric with unbounded curvature, and these solutions evolve into metrics with bounded curvatures instantaneously.

As mentioned in Section 1.1, all previous well-posedness results concerning the Yamabe flow require uniform boundedness of the scalar curvature. Evolving geometric flows, like the Yamabe flow, along the class of metrics with unbounded curvatures is far more challenging and leads to many unexpected phenomena. Indeed, its difficulty can be illustrated by an example in [**43**].

Example I.5 (G. Giesen, P.M. Topping [**43**])**.** *Suppose that* T^2 *is a torus equipped with an arbitrary conformal structure and* $\mathsf{p} \in \mathsf{T}^2$*. Let* h *be the unique complete, conformal, hyperbolic metric on* $\mathsf{T}^2 \setminus \{\mathsf{p}\}$*. Then there exists a smooth Ricci flow* $g(t)$ *on* T^2 *for* $t > 0$ *such that* $g(t) \to h$ *smoothly locally on* $\mathsf{T}^2 \setminus \{\mathsf{p}\}$ *as* $t \to 0$*.*

Hence a Ricci flow with unbounded curvature may pull "points at infinity" to within finite distance in finite time. This phenomenon contrasts the classic situation, which means that while starting a Ricci flow with an initial metric with bounded curvature and evolving it with bounded curvature, any curve heading to infinity retains its infinite length. It is well known that in dimension two, the Yamabe flow agrees with the Ricci flow. The above observation points out part of the difficulty in evolving the Yamabe flow with a metric of unbounded curvature. This is why the result in Theorem I.4 is quite unexpected.

Finally, based on [**47**], in every conformal class we can find a metric $\tilde{g}$ making $(\mathsf{M}, \tilde{g})$ *uniformly regular.* Hence, Theorem I.4 applies to every conformal class.

In addition to its application to geometric evolution equations, we also apply Theorems I.1-I.3 to the heat equation and its two well-known relatives, namely, the porous medium equation and the parabolic p-Laplacian equation, on *singular manifolds.* A detailed discussion of these equations will be presented in Chapter 7.

Another application of Theorems I.1-I.3 concerns parabolic degenerate boundary value problems and boundary blow-up problems on domains with compact boundary. The order of the degeneracy or singularity is measured by the rate of decay and blow-up in the ellipticity condition while approaching the boundary. See Chapter 6 for a precise description. This result extends the work in [**40, 89**]. Rooting from this theory, Theorem I.1-I.3 can thus be applied to parabolic equations in Euclidean spaces. In Section 8.1, we prove a local existence and uniqueness theorem for a generalized multidimensional thin film equation

$$\begin{cases} \partial_t u + \operatorname{div}(u^n D\Delta u + \alpha_1 u^{n-1}\Delta u Du + \alpha_2 u^{n-2}|Du|^2 Du) = f & \text{on} \quad \Omega_T; \\ u(0) = u_0 & \text{on} \quad \Omega \end{cases} \tag{I.9}$$

if the initial data u_0 decays sufficiently fast to the boundary of its support. Here α_1, α_2 are two constants, $n > 0$, and $\Omega \subset \mathbb{R}^m$ is a sufficiently smooth domain. This generalized model was first investigated by J.R. King in [**52**] in the one dimensional case. Later, a multidimensional counterpart has been studied with periodic boundary conditions on cubes in [**12**]. An interesting waiting-time phenomenon can be observed from our approach. The mathematical investigation of the thin film equation was initiated by F. Bernis and A. Friedman in [**11**]. An intriguing feature of free boundary problems associated with degenerate parabolic equations is the waiting-time phenomenon of the support of the solutions. This phenomenon has been widely observed and studied by many mathematicians. See [**23, 27, 42, 48, 84**] for example. The waiting-time phenomenon for the case $\alpha_1, \alpha_2 = 0$, the original thin film equation, has been explored in several of the papers listed above. Our result extends the results in the above literature for the generalized system (I.9). In Section 8.2, a generalization of the parabolic Heston equation, an extensively studied model in financial mathematics, is investigated.

3. Assumptions and notations

3.1. Geometric assumptions. Following H. Amann [**4, 5**], let (M, g) be a C^∞-Riemannian manifold of dimension m with or without boundary endowed with g as its Riemannian metric such that its underlying topological space is separable. An atlas $\mathfrak{A} := (\mathsf{O}_\kappa, \varphi_\kappa)_{\kappa \in \mathfrak{K}}$ for M is said to be normalized if

$$\varphi_\kappa(\mathsf{O}_\kappa) = \begin{cases} \mathsf{Q}^m, & \mathsf{O}_\kappa \subset \mathring{\mathsf{M}}, \\ \mathsf{Q}^m \cap \mathbb{H}^m, & \mathsf{O}_\kappa \cap \partial\mathsf{M} \neq \emptyset, \end{cases}$$

where $\mathbb{H}^m$ is the closed half space $\bar{\mathbb{R}}^+ \times \mathbb{R}^{m-1}$ and Q^m is the unit cube at the origin in $\mathbb{R}^m$. We put $\mathsf{Q}^m_\kappa := \varphi_\kappa(\mathsf{O}_\kappa)$ and $\psi_\kappa := \varphi_\kappa^{-1}$.

The atlas $\mathfrak{A}$ is said to have *finite multiplicity* if there exists $K \in \mathbb{N}$ such that any intersection of more than K coordinate patches is empty. Put

$$\mathfrak{N}(\kappa) := \{\tilde{\kappa} \in \mathfrak{K} : \mathsf{O}_{\tilde{\kappa}} \cap \mathsf{O}_\kappa \neq \emptyset\}.$$

The finite multiplicity of $\mathfrak{A}$ and the separability of M imply that $\mathfrak{A}$ is countable.

An atlas $\mathfrak{A}$ is said to fulfil the *uniformly shrinkable* condition, if it is normalized and there exists $r \in (0,1)$ such that $\{\psi_\kappa(r\mathsf{Q}^m_\kappa) : \kappa \in \mathfrak{K}\}$ is a cover for M.

Following H. Amann [**4, 5**], we say that (M, g) is a **uniformly regular Riemannian manifold** if it admits an atlas $\mathfrak{A}$ such that

(R1) $\mathfrak{A}$ is uniformly shrinkable and has finite multiplicity. If M is oriented, then $\mathfrak{A}$ is orientation preserving.

(R2) $\|\varphi_\eta \circ \psi_\kappa\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$, $\eta \in \mathfrak{N}(\kappa)$, and $k \in \mathbb{N}_0$.

(R3) $\psi^*_\kappa g \sim g_m$, $\kappa \in \mathfrak{K}$. Here g_m denotes the Euclidean metric on $\mathbb{R}^m$ and $\psi^*_\kappa g$ denotes the pull-back metric of g by ψ_κ.

(R4) $\|\psi^*_\kappa g\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$ and $k \in \mathbb{N}_0$.

Here $\|u\|_{k,\infty} := \max_{|\alpha| \le k} \|\partial^\alpha u\|_\infty$, and it is understood that a constant $c(k)$, like in (R2), depends only on k. An atlas $\mathfrak{A}$ satisfying (R1) and (R2) is called a *uniformly regular atlas*. (R3) reads as

$$|\xi|^2/c \le \psi^*_\kappa g(x)(\xi,\xi) \le c|\xi|^2, \quad \text{for any } x \in \mathsf{Q}^m_\kappa, \xi \in \mathbb{R}^m, \kappa \in \mathfrak{K} \text{ and some } c \ge 1.$$

In [**30**], it is shown that the class of *uniformly regular Riemannian manifolds* coincides with the family of complete Riemannian manifolds with bounded geometry, when $\partial\mathsf{M} = \emptyset$.

Assume that $\rho \in C^\infty(\mathsf{M}, (0,\infty))$. Then $(\rho, \mathfrak{K})$ is a *singularity datum* for M if

(S1) $(\mathsf{M}, g/\rho^2)$ is a *uniformly regular Riemannian manifold.*

(S2) $\mathfrak{A}$ is a uniformly regular atlas.

(S3) $\|\psi_\kappa^*\rho\|_{k,\infty} \leq c(k)\rho_\kappa$, $\kappa \in \mathfrak{K}$ and $k \in \mathbb{N}_0$, where $\rho_\kappa := \rho(\psi_\kappa(0))$.

(S4) $\rho_\kappa/c \leq \rho(\mathsf{p}) \leq c\rho_\kappa$, $\mathsf{p} \in \mathsf{O}_\kappa$ and $\kappa \in \mathfrak{K}$ for some $c \geq 1$ independent of κ.

Two *singularity data* $(\rho, \mathfrak{K})$ and $(\tilde{\rho}, \tilde{\mathfrak{K}})$ are equivalent, if

(E1) $\rho \sim \tilde{\rho}$.

(E2) $\mathrm{card}\{\tilde{\kappa} \in \tilde{\mathfrak{K}} : \mathsf{O}_{\tilde{\kappa}} \cap \mathsf{O}_\kappa \neq \emptyset\} \leq c$, $\kappa \in \mathfrak{K}$.

(E3) $\|\varphi_{\tilde{\kappa}} \circ \psi_\kappa\|_{k,\infty} \leq c(k)$, $\kappa \in \mathfrak{K}$, $\tilde{\kappa} \in \tilde{\mathfrak{K}}$ and $k \in \mathbb{N}_0$

We write the equivalence relationship as $(\rho, \mathfrak{K}) \sim (\tilde{\rho}, \tilde{\mathfrak{K}})$. (S1) and (E1) imply that

$$1/c \leq \rho_\kappa/\tilde{\rho}_{\tilde{\kappa}} \leq c, \quad \kappa \in \mathfrak{K}, \quad \tilde{\kappa} \in \tilde{\mathfrak{K}} \text{ and } \mathsf{O}_{\tilde{\kappa}} \cap \mathsf{O}_\kappa \neq \emptyset. \tag{I.10}$$

A singularity structure, $\mathfrak{S}(\mathsf{M})$, for M is a maximal family of equivalent *singularity data*. A *singularity function* for $\mathfrak{S}(\mathsf{M})$ is a function $\rho \in C^\infty(\mathsf{M}, (0, \infty))$ such that there exists an atlas $\mathfrak{A}$ with $(\rho, \mathfrak{A}) \in \mathfrak{S}(\mathsf{M})$. The set of all *singularity functions* for $\mathfrak{S}(\mathsf{M})$ is the *singular type*, $\mathfrak{T}(\mathsf{M})$, for $\mathfrak{S}(\mathsf{M})$. By a **singular manifold** we mean a Riemannian manifold M endowed with a singularity structure $\mathfrak{S}(\mathsf{M})$. Then M is said to be *singular of type* $\mathfrak{T}(\mathsf{M})$. If $\rho \in \mathfrak{T}(\mathsf{M})$, then it is convenient to set $[\![\rho]\!] := \mathfrak{T}(\mathsf{M})$ and to say that $(\mathsf{M}, g; \rho)$ is a *singular manifold.* A *singular manifold* is a *uniformly regular Riemannian manifold* iff $\rho \sim \mathbf{1}_{\mathsf{M}}$.

We refer to [**6, 7**] for examples of *uniformly regular Riemannian manifolds* and *singular manifolds.*

A *singular manifold* M with a uniformly regular atlas $\mathfrak{A}$ admits a *localization system subordinate to* $\mathfrak{A}$, by which we mean a family $(\pi_\kappa, \zeta_\kappa)_{\kappa \in \mathfrak{K}}$ satisfying:

(L1) $\pi_\kappa \in \mathcal{D}(\mathsf{O}_\kappa, [0,1])$ and $(\pi_\kappa^2)_{\kappa \in \mathfrak{K}}$ is a partition of unity subordinate to $\mathfrak{A}$.

(L2) $\zeta_\kappa := \varphi_\kappa^* \zeta$ with $\zeta \in \mathcal{D}(\mathsf{Q}^m, [0,1])$ satisfying $\zeta|_{\mathrm{supp}(\psi_\kappa^* \pi_\kappa)} \equiv 1$, $\kappa \in \mathfrak{K}$.

(L3) $\|\psi_\kappa^* \pi_\kappa\|_{k,\infty} \leq c(k)$, for $\kappa \in \mathfrak{K}$, $k \in \mathbb{N}_0$.

The reader may refer to [**4**, Lemma 3.2] for a proof.

Lastly, for each $k \in \mathbb{N}$, the concept of C^k**-uniformly regular Riemannian manifold** is defined by modifying (R2), (R4) and (L1)-(L3) in an obvious way. Similarly, C^k**-singular manifolds** are defined by replacing the smoothness of ρ by $\rho \in C^k(\mathsf{M}, (0, \infty))$ and altering (S1)-(S3) accordingly.

3.2. Notations. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. $\mathbb{N}_0$ is the set of all natural numbers including 0.

For any interval I containing 0, $\dot{I} := I \setminus \{0\}$. Given any topological set U, $\mathring{U}$ denotes the interior of U.

For any two Banach spaces X, Y, $X \doteq Y$ means that they are equal in the sense of equivalent norms. $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from X to Y, and $\mathcal{L}\mathrm{is}(X, Y)$ stands for the set of all bounded linear isomorphisms from X to Y.

For any Banach space E, we abbreviate $\mathfrak{F}(\mathbb{R}^m, E)$ to $\mathfrak{F}(E)$, where $\mathfrak{F}$ stands for any function space defined in this article. The precise definitions for these function spaces will be presented in Section 2.

Given any Banach space X and manifold $\mathscr{M}$, let $\|\cdot\|_\infty$, $\|\cdot\|_{s,\infty}$, $\|\cdot\|_p$ and $\|\cdot\|_{s,p}$ denote the usual norm of the Banach spaces $BC(\mathscr{M}, X)(L_\infty(\mathscr{M}, X))$, $BC^s(\mathscr{M}, X)$, $L_p(\mathscr{M}, X)$ and $W^s_p(\mathscr{M}, X)$, respectively.

We denote $\mathbb{K}$-valued function spaces with domain $U \in \{\mathsf{M}, \Omega\}$ by $\mathfrak{F}(U)$ if $\Omega \subset \mathbb{R}^m$.

Chapter II

Weighted function spaces on singular manifolds

In this section, we define the weighted function spaces on *singular manifolds*, following the work of H. Amann in [**4, 5**].

Let $\mathbb{A}$ be a countable index set. Suppose E_α is for each $\alpha \in \mathbb{A}$ a locally convex space. We endow $\prod_\alpha E_\alpha$ with the product topology, that is, the coarsest topology for which all projections $pr_\beta : \prod_\alpha E_\alpha \to E_\beta$, $(e_\alpha)_\alpha \mapsto e_\beta$ are continuous. By $\bigoplus_\alpha E_\alpha$ we mean the vector subspace of $\prod_\alpha E_\alpha$ consisting of all finitely supported elements, equipped with the inductive limit topology, that is, the finest locally convex topology for which all injections $E_\beta \to \bigoplus_\alpha E_\alpha$ are continuous.

1. Tensor bundles

Suppose $(\mathsf{M}, g; \rho)$ is a *singular manifold.* Given $\sigma, \tau \in \mathbb{N}_0$,

$$T^\sigma_\tau \mathsf{M} := T\mathsf{M}^{\otimes\sigma} \otimes T^*\mathsf{M}^{\otimes\tau}$$

is the (σ,τ)-tensor bundle of M, where $T\mathsf{M}$ and $T^*\mathsf{M}$ are the tangent and the cotangent bundle of M, respectively. We write $\mathcal{T}^\sigma_\tau\mathsf{M}$ for the $C^\infty(\mathsf{M})$-module of all smooth sections of $T^\sigma_\tau\mathsf{M}$, and $\Gamma(\mathsf{M}, T^\sigma_\tau\mathsf{M})$ for the set of all sections.

For abbreviation, we set $\mathbb{J}^\sigma := \{1, 2, \ldots, m\}^\sigma$, and $\mathbb{J}^\tau$ is defined alike. Given local coordinates $\varphi = \{x^1, \ldots, x^m\}$, $(i) := (i_1, \ldots, i_\sigma) \in \mathbb{J}^\sigma$ and $(j) := (j_1, \ldots, j_\tau) \in \mathbb{J}^\tau$, we set

$$\frac{\partial}{\partial x^{(i)}} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\sigma}}, \quad \partial_{(i)} := \partial_{i_1} \circ \cdots \circ \partial_{i_\sigma} \quad dx^{(j)} := dx^{j_1} \otimes \cdots \otimes dx^{j_\tau}$$

with $\partial_i = \frac{\partial}{\partial x^i}$. The local representation of $a \in \Gamma(\mathsf{M}, T^\sigma_\tau \mathsf{M})$ with respect to these coordinates is given by

$$a = a^{(i)}_{(j)} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)} \tag{II.1}$$

with coefficients $a^{(i)}_{(j)}$ defined on O_κ

We denote by $\nabla = \nabla_g$ the Levi-Civita connection on $T\mathsf{M}$. It has a unique extension over $\mathcal{T}^\sigma_\tau \mathsf{M}$ satisfying, for $X \in \mathcal{T}^1_0 \mathsf{M}$,

(i) $\nabla_X f = \langle df, X \rangle, \quad f \in C^\infty(\mathsf{M})$,

(ii) $\nabla_X (a \otimes b) = \nabla_X a \otimes b + a \otimes \nabla_X b, \quad a \in \mathcal{T}^{\sigma_1}_{\tau_1} \mathsf{M},\ b \in \mathcal{T}^{\sigma_2}_{\tau_2} \mathsf{M}$,

(iii) $\nabla_X \langle a, b \rangle = \langle \nabla_X a, b \rangle + \langle a, \nabla_X b \rangle, \quad a \in \mathcal{T}^\sigma_\tau \mathsf{M},\ b \in \mathcal{T}^\tau_\sigma \mathsf{M}$,

where $\langle \cdot, \cdot \rangle : \mathcal{T}^\sigma_\tau \mathsf{M} \times \mathcal{T}^\tau_\sigma \mathsf{M} \to C^\infty(\mathsf{M})$ is the extension of the fiber-wise defined duality pairing on M, cf. [**4**, Section 3]. Then the covariant (Levi-Civita) derivative is the linear map

$$\nabla : \mathcal{T}^\sigma_\tau \mathsf{M} \to \mathcal{T}^\sigma_{\tau+1} \mathsf{M}, \quad a \mapsto \nabla a$$

defined by

$$\langle \nabla a, b \otimes X \rangle := \langle \nabla_X a, b \rangle, \quad b \in \mathcal{T}^\tau_\sigma \mathsf{M}, \quad X \in \mathcal{T}^1_0 \mathsf{M}.$$

For $k \in \mathbb{N}_0$, we define

$$\nabla^k : \mathcal{T}^\sigma_\tau \mathsf{M} \to \mathcal{T}^\sigma_{\tau+k} \mathsf{M}, \quad a \mapsto \nabla^k a$$

by letting $\nabla^0 a := a$ and $\nabla^{k+1} a := \nabla \circ \nabla^k a$. We can also extend the Riemannian metric $(\cdot|\cdot)_g$ from the tangent bundle to any (σ, τ)-tensor bundle $T^\sigma_\tau \mathsf{M}$ such that $(\cdot|\cdot)_g := (\cdot|\cdot)_{g^\tau_\sigma} : T^\sigma_\tau \mathsf{M} \times T^\sigma_\tau \mathsf{M} \to \mathbb{C}$ by

$$(a|b)_g = g_{(i)(\tilde{i})} g^{(j)(\tilde{j})} a^{(i)}_{(j)} \bar{b}^{(\tilde{i})}_{(\tilde{j})}$$

in every coordinate with $(i), (\tilde{i}) \in \mathbb{J}^\sigma$, $(j), (\tilde{j}) \in \mathbb{J}^\tau$ and

$$g_{(i)(\tilde{i})} := g_{i_1, \tilde{i}_1} \cdots g_{i_\sigma, \tilde{i}_\sigma}, \quad g^{(j)(\tilde{j})} := g^{j_1, \tilde{j}_1} \cdots g^{j_\tau, \tilde{j}_\tau}.$$

In addition,

$$|\cdot|_g := |\cdot|_{g^\tau_\sigma} : \mathcal{T}^\sigma_\tau \mathsf{M} \to C^\infty(\mathsf{M}), \quad a \mapsto \sqrt{(a|a)_g}$$

is called the (vector bundle) *norm* induced by g.

We assume that V is a $\mathbb{C}$-valued tensor bundle on M and E is a $\mathbb{C}$-valued vector space, i.e.,

$$V = V^\sigma_\tau := \{T^\sigma_\tau \mathsf{M}, (\cdot|\cdot)_g\}, \quad \text{and} \quad E = E^\sigma_\tau := \{\mathbb{C}^{m^\sigma \times m^\tau}, (\cdot|\cdot)\},$$

for some $\sigma, \tau \in \mathbb{N}_0$. Here $(a|b) :=$trace$(b^* a)$ with b^* being the conjugate matrix of b. By setting $N = m^{\sigma+\tau}$, we can identify $\mathfrak{F}^s(\mathsf{M}, E)$ with $\mathfrak{F}^s(\mathsf{M})^N$.

Recall that for any $a \in V^\sigma_{\tau+1}$

$$(a^\sharp)^{(i;k)}_{(j)} := g^{kl} a^{(i)}_{(j;l)}, \quad (i) \in \mathbb{J}^\sigma, \quad (j) \in \mathbb{J}^\tau, \quad k, l \in \mathbb{J}^1.$$

We have $|a^\sharp|_{g^\tau_{\sigma+1}} = |a|_{g^{\tau+1}_\sigma}$. For any $(i_1) \in \mathbb{J}^{\sigma_1}$ and $(i_2) \in \mathbb{J}^{\sigma_2}$, the index $(i_1; i_2)$ is defined by

$$(i_1; i_2) = (i_{1,1}, \cdots, i_{1,\sigma_1}; i_{2,1}, \cdots, i_{2,\sigma_2}).$$

Given any $a \in V^{\sigma+1}_\tau$,

$$(a_\flat)^{(i)}_{(j;k)} := g_{kl} a^{(i;l)}_{(j)}.$$

Similarly, we have $|a_\flat|_{g^{\tau+1}_\sigma} = |a|^\tau_{g_{\sigma+1}}$.

Suppose that $\sigma + \tau \geq 1$. We put for $a \in V$ and $\alpha_i \in T^*\mathsf{M}$, $\beta^j \in T\mathsf{M}$

$$(G^\tau_\sigma a)(\alpha_1, \cdots, \alpha_\tau; \beta^1, \cdots, \beta^\sigma) := a((\beta^1)_\flat, \cdots, (\beta^\sigma)_\flat; (\alpha_1)^\sharp, \cdots, (\alpha_\tau)^\sharp).$$

Then it induces a conjugate linear bijection

$$G^\tau_\sigma : V \to V', \quad (G^\tau_\sigma)^{-1} = G^\sigma_\tau.$$

Consequently, for $a, b \in V$

$$(a|b)_g = \langle a, G^\tau_\sigma \bar{b} \rangle.$$

From this, it is easy to show

$$|G^\tau_\sigma a|_{g^\sigma_\tau} = |a|_{g^\tau_\sigma}. \tag{II.2}$$

Throughout the rest of this paper, unless stated otherwise, we always assume that

- $(\mathsf{M}, g; \rho)$ is a *singular manifold.*
- $\rho \in \mathfrak{T}(\mathsf{M})$, $s \geq 0$, $1 < p < \infty$ and $\vartheta \in \mathbb{R}$.
- $(\pi_\kappa, \zeta_\kappa)_{\kappa \in \mathfrak{K}}$ is a localization system subordinate to $\mathfrak{A}$.
- $\sigma, \tau \in \mathbb{N}_0$, $V = V^\sigma_\tau := \{T^\sigma_\tau \mathsf{M}, (\cdot|\cdot)_g\}$, $E = E^\sigma_\tau := \{\mathbb{C}^{m^\sigma \times m^\tau}, (\cdot|\cdot)\}$.

In [**4**, Lemma 3.1], it is shown that M satisfies the following properties:

(P1) $\psi^*_\kappa g \sim \rho^2_\kappa g_m$ and $\psi^*_\kappa g^* \sim \rho^{-2}_\kappa g_m$, where g^* is the induced contravariant metric.

(P2) $\rho^{-2}_\kappa \|\psi^*_\kappa g\|_{k,\infty} + \rho^2_\kappa \|\psi^*_\kappa g^*\|_{k,\infty} \leq c(k)$, $k \in \mathbb{N}_0$ and $\kappa \in \mathfrak{K}$.

(P3) For $\sigma, \tau \in \mathbb{N}_0$ given, then

$$\psi^*_\kappa(|a|_g) \sim \rho^{\sigma-\tau}_\kappa |\psi^*_\kappa a|_{g_m}, \quad a \in \mathcal{T}^\sigma_\tau \mathsf{M},$$

and

$$|\varphi^*_\kappa b|_g \sim \rho^{\sigma-\tau}_\kappa \varphi^*_\kappa(|b|_{g_m}), \quad b \in \mathcal{T}^\sigma_\tau \mathsf{Q}^m_\kappa.$$

For $K \subset \mathsf{M}$, we put $\mathfrak{K}_K := \{\kappa \in \mathfrak{K} : \mathsf{O}_\kappa \cap K \neq \emptyset\}$. Then, given $\kappa \in \mathfrak{K}$,

$$\mathbb{X}_\kappa := \begin{cases} \mathbb{R}^m & \text{if } \kappa \in \mathfrak{K} \setminus \mathfrak{K}_{\partial \mathsf{M}}, \\ \mathbb{H}^m & \text{otherwise}, \end{cases}$$

endowed with the Euclidean metric g_m.

Given $a \in \Gamma(\mathsf{M}, V)$ with local representation (II.1) we define $\psi^*_\kappa a \in E$ by means of $\psi^*_\kappa a = [a^{(i)}_{(j)}]$, where $[a^{(i)}_{(j)}]$ stands for the $(m^\sigma \times m^\tau)$-matrix with entries $a^{(i)}_{(j)}$ in the $((i),(j))$ position, with (i), (j) arranged lexicographically.

2. Definitions of weighted function spaces

For the sake of brevity, we set $\boldsymbol{L}_{1,loc}(\mathbb{X}, E) := \prod_\kappa L_{1,loc}(\mathbb{X}_\kappa, E)$. Then we introduce two linear maps for $\kappa \in \mathfrak{K}$:

$$\mathcal{R}^c_\kappa : L_{1,loc}(\mathsf{M}, V) \to L_{1,loc}(\mathbb{X}_\kappa, E),\ u \mapsto \psi^*_\kappa(\pi_\kappa u),$$

and

$$\mathcal{R}_\kappa : L_{1,loc}(\mathbb{X}_\kappa, E) \to L_{1,loc}(\mathsf{M}, V),\ v_\kappa \mapsto \pi_\kappa \varphi_\kappa^* v_\kappa.$$

Here and in the following it is understood that a partially defined and compactly supported tensor field is automatically extended over the whole base manifold by identifying it to be zero outside its original domain. We define

$$\mathcal{R}^c : L_{1,loc}(\mathsf{M}, V) \to \boldsymbol{L}_{1,loc}(\mathbb{R}^m), \quad u \mapsto (\mathcal{R}^c_\kappa u)_\kappa,$$

and

$$\mathcal{R} : \boldsymbol{L}_{1,loc}(\mathbb{R}^m) \to L_{1,loc}(\mathsf{M}, V), \quad (v_\kappa)_\kappa \mapsto \sum_\kappa \mathcal{R}_\kappa v_\kappa.$$

In the rest of this section we assume that $k \in \mathbb{N}_0$. In the first place, we list some prerequisites for the *Hölder* and *little Hölder* spaces on $\mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$ from [**5**, Section 11]. Given any Banach space F, the Banach space $BC^k(\mathbb{X}, F)$ is defined by

$$BC^k(\mathbb{X}, F) := (\{u \in C^k(\mathbb{X}, F) : \|u\|_{k,\infty} < \infty\}, \|\cdot\|_{k,\infty}).$$

The closed linear subspace $BUC^k(\mathbb{X}, F)$ of $BC^k(\mathbb{X}, F)$ consists of all functions $u \in BC^k(\mathbb{X}, F)$ such that $\partial^\alpha u$ is uniformly continuous for all $|\alpha| \leq k$. Moreover,

$$BC^\infty(\mathbb{X}, F) := \bigcap_k BC^k(\mathbb{X}, F) = \bigcap_k BUC^k(\mathbb{X}, F).$$

It is a Fréchet space when equipped with the natural projective topology.

For $0 < s < 1$, $0 < \delta \leq \infty$ and $u \in F^{\mathbb{X}}$, the seminorm $[\cdot]^\delta_{s,\infty}$ is defined by

$$[u]^\delta_{s,\infty} := \sup_{h \in (0,\delta)^m} \frac{\|u(\cdot + h) - u(\cdot)\|_\infty}{|h|^s}, \quad [\cdot]_{s,\infty} := [\cdot]^\infty_{s,\infty}.$$

Let $k < s < k+1$. The *Hölder* space $BC^s(\mathbb{X}, F)$ is defined as

$$BC^s(\mathbb{X}, F) := (\{u \in BC^k(\mathbb{X}, F) : \|u\|_{s,\infty} < \infty\}, \|\cdot\|_{s,\infty}),$$

where $\|u\|_{s,\infty} := \|u\|_{k,\infty} + \max_{|\alpha|=k} [\partial^\alpha u]_{s-k,\infty}$.

The *little Hölder* space of order $s \geq 0$ is defined by

$$bc^s(\mathbb{X}, F) := \text{the closure of } BC^\infty(\mathbb{X}, F) \text{ in } BC^s(\mathbb{X}, F).$$

By [**5**, formula (11.13), Corollary 11.2, Theorem 11.3], we have

$$bc^k(\mathbb{X}, F) = BUC^k(\mathbb{X}, F),$$

and for $k < s < k+1$

$$u \in BC^s(\mathbb{X}, F) \text{ belongs to } bc^s(\mathbb{X}, F) \text{ iff } \lim_{\delta\to 0}[\partial^\alpha u]^\delta_{s-[s],\infty} = 0, \qquad |\alpha| = [s].$$

Now we are ready to introduce the weighted *Hölder* and *little Hölder* spaces on *singular manifolds*. Define

$$BC^{k,\vartheta}(\mathsf{M}, V) := (\{u \in C^k(\mathsf{M}, V) : \|u\|_{k,\infty;\vartheta} < \infty\}, \|\cdot\|_{k,\infty;\vartheta}),$$

where $\|u\|_{k,\infty;\vartheta} := \max_{0\le i\le k}\|\rho^{\vartheta+i+\tau-\sigma}|\nabla^i u|_g\|_\infty$. We also set

$$BC^{\infty,\vartheta}(\mathsf{M}, V) := \bigcap_k BC^{k,\vartheta}(\mathsf{M}, V)$$

endowed with the conventional projective topology. Then

$$bc^{k,\vartheta}(\mathsf{M}, V) := \text{ the closure of } BC^{\infty,\vartheta} \text{ in } BC^{k,\vartheta}(\mathsf{M}, V).$$

Let $k < s < k+1$. Now the *Hölder* space $BC^{s,\vartheta}(\mathsf{M}, V)$ is defined by

$$BC^{s,\vartheta}(\mathsf{M}, V) := (bc^{k,\vartheta}(\mathsf{M}, V), bc^{k+1,\vartheta}(\mathsf{M}, V))_{s-k,\infty}. \tag{II.3}$$

Here $(\cdot,\cdot)_{\theta,\infty}$ is the real interpolation method, see [**1**, Example I.2.4.1] and [**57**, Definition 1.2.2]. $BC^{s,\vartheta}(\mathsf{M}, V)$ equipped with the norm $\|\cdot\|_{s,\infty;\vartheta}$ is a Banach space by interpolation theory, where $\|\cdot\|_{s,\infty;\vartheta}$ is the norm of the interpolation space in definition (II.3). For $s \ge 0$, we define the weighted *little Hölder* spaces by

$$bc^{s,\vartheta}(\mathsf{M}, V) := \text{ the closure of } BC^{\infty,\vartheta}(\mathsf{M}, V) \text{ in } BC^{s,\vartheta}(\mathsf{M}, V). \tag{II.4}$$

We denote by $\mathcal{D}(\mathsf{M}, V)$ the space of smooth sections of V that is compactly supported in M. The weighted Sobolev space $W^{k,\vartheta}_p(\mathsf{M}, V)$ is defined as the completion of $\mathcal{D}(\mathsf{M}, V)$ in $L_{1,loc}(\mathsf{M}, V)$ with respect to the norm

$$\|\cdot\|_{k,p;\vartheta} : u \mapsto (\textstyle\sum_{i=0}^k \|\rho^{\vartheta+i+\tau-\sigma}|\nabla^i u|_g\|_p^p)^{\frac{1}{p}}.$$

Note that $W_p^{0,\vartheta}(\mathsf{M},V) = L_p^\vartheta(\mathsf{M},V)$ with equal norms. In particular, we can define the weighted spaces $L_q^\vartheta(\mathsf{M},V)$ for $q \in \{1,\infty\}$ in a similar manner.

Analogously, the weighted Besov spaces are defined for $k \in \mathbb{N}$ by

$$B_p^{k,\vartheta}(\mathsf{M},V) := (W_p^{k-1,\vartheta}(\mathsf{M},V), W_p^{k+1,\vartheta}(\mathsf{M},V))_{1/2,p}. \tag{II.5}$$

The weighted Sobolev-Slobodeckii spaces are defined as

$$W_p^{s,\vartheta}(\mathsf{M},V) := (W_p^{k,\vartheta}(\mathsf{M},V), W_p^{k+1,\vartheta}(\mathsf{M},V))_{s-k,p}, \tag{II.6}$$

for $k < s < k+1$, where $(\cdot,\cdot)_{\theta,p}$ is the real interpolation method [**85**, Section 1.3].

Whenever $\partial\mathsf{M} \neq \emptyset$, we denote by $\mathring{\mathfrak{F}}_p^{s,\vartheta}(\mathsf{M},V)$ the closure of $\mathcal{D}(\mathring{\mathsf{M}},V)$ in $\mathfrak{F}_p^{s,\vartheta}(\mathsf{M},V)$ for $\mathfrak{F} \in \{B,W\}$. In particular,

$$\mathring{W}_p^{s,\vartheta}(\mathsf{M},V) = W_p^{s,\vartheta}(\mathsf{M},V), \quad 0 \le s < 1/p.$$

See [**4**, Theorem 8.3(ii)].

In the special case that (M,g) is uniformly regular, since $\rho \sim \mathbf{1}_{\mathsf{M}}$, the definition of any weighted space $\mathfrak{F}^{s,\vartheta}(\mathsf{M},V)$ is actually independent of the weight ϑ. In this case, all spaces are indeed unweighted. We thus denote these spaces simply by $\mathfrak{F}^s(\mathsf{M},V)$.

3. Basic properties

In the following context, assume that E_κ is a sequence of Banach spaces for $\kappa \in \mathfrak{K}$. Then $\mathbf{E} := \prod_\kappa E_\kappa$. For $1 \le q \le \infty$, we denote by $l_q^\vartheta(\mathbf{E}) := l_q^\vartheta(\mathbf{E};\rho)$ the linear subspace of $\mathbf{E}$ consisting of all $\boldsymbol{x} = (x_\kappa)$ such that

$$\|\boldsymbol{x}\|_{l_q^\vartheta(\mathbf{E})} := \begin{cases} (\sum_\kappa \|\rho_\kappa^{\vartheta+m/q} x_\kappa\|_{E_\kappa}^q)^{1/q}, & 1 \le q < \infty, \\ \sup_\kappa \|\rho_\kappa^\vartheta x_\kappa\|_{E_\kappa}, & q = \infty \end{cases}$$

is finite. Then $l_q^\vartheta(\mathbf{E})$ is a Banach space with norm $\|\cdot\|_{l_q^\vartheta(\mathbf{E})}$.

For $\mathfrak{F} \in \{bc, BC, W_p, \mathring{W}_p\}$, we put $\boldsymbol{\mathfrak{F}}^s := \prod_\kappa \mathfrak{F}_\kappa^s$, where $\mathfrak{F}_\kappa^s := \mathfrak{F}^s(\mathbb{X}_\kappa, E)$. Denote by

$$l_{\infty,\mathrm{unif}}^\vartheta(\boldsymbol{bc}^k)$$

the linear subspace of $l_\infty^\vartheta(\boldsymbol{BC}^k)$ of all $\boldsymbol{u}=(u_\kappa)_\kappa$ such that $\rho_\kappa^\vartheta \partial^\alpha u_\kappa$ is uniformly continuous on $\mathbb{X}_\kappa$ for $|\alpha|\leq k$, uniformly with respect to $\kappa \in \mathfrak{K}$. Similarly, for any $k<s<k+1$, we denote by

$$l_{\infty,\mathrm{unif}}^\vartheta(\boldsymbol{bc}^s)$$

the linear subspace of $l_{\infty,\mathrm{unif}}^\vartheta(\boldsymbol{bc}^k)$ of all $\boldsymbol{u}=(u_\kappa)_\kappa$ such that

$$\lim_{\delta\to 0}\max_{|\alpha|=k}\rho_\kappa^\vartheta[\partial^\alpha u_\kappa]_{s-k,\infty}^\delta=0,$$

uniformly with respect to $\kappa\in\mathfrak{K}$.

In the sequel, we always assume $\mathfrak{F}\in\{bc, BC, W_p, \mathring{W}_p\}$, unless stated otherwise. Define

$$L_\vartheta : l_q^{\vartheta'+\vartheta}(\boldsymbol{\mathfrak{F}}^s)\to l_q^{\vartheta'}(\boldsymbol{\mathfrak{F}}^s)) : \quad (u_\kappa)_\kappa \mapsto (\rho_\kappa^\vartheta u_\kappa)_\kappa,$$

where $q=$ "∞, unif" for $\mathfrak{F}=bc$, or $q=\infty$ for $\mathfrak{F}=BC$, or $q=p$ for $\mathfrak{F}\in\{W_p,\mathring{W}_p\}$. Then the following result follows immediately from the definition of weighted l_q spaces.

$$L_\vartheta\in\mathcal{L}\mathrm{is}(l_q^{\vartheta'+\vartheta}(\boldsymbol{\mathfrak{F}}^s), l_q^{\vartheta'}(\boldsymbol{\mathfrak{F}}^s)), \quad \text{with} \quad (L_\vartheta)^{-1}=L_{-\vartheta}. \tag{II.7}$$

Proposition II.1. *$\mathcal{R}$ is a retraction from $l_q^\vartheta(\boldsymbol{\mathfrak{F}}^s)$ onto $\mathfrak{F}^{s,\vartheta}(\mathsf{M},V)$ with $\mathcal{R}^c$ as a coretraction. Here $q=$ "∞, unif" for $\mathfrak{F}=bc$, or $q=\infty$ for $\mathfrak{F}=BC$, or $q=p$ for $\mathfrak{F}\in\{W_p,\mathring{W}_p\}$.*

PROOF. In [**4, 5**], a different retraction and coretraction system between the spaces $\mathfrak{F}^{s,\vartheta}(\mathsf{M},V)$ and $l_b(\boldsymbol{\mathfrak{F}}^s)$ is defined as follows.

$$\mathcal{R}_{q,\kappa}^{\vartheta;c}:=\rho_\kappa^{\vartheta+m/q}\mathcal{R}_\kappa^c, \quad \text{and} \quad \mathcal{R}_{q,\kappa}^\vartheta:=\rho_\kappa^{-\vartheta-m/q}\mathcal{R}_\kappa;$$

and

$$\mathcal{R}_q^{\vartheta;c}: L_{1,loc}(\mathsf{M},V)\to \boldsymbol{L}_{1,loc}(\mathbb{R}^m), \quad u\mapsto(\mathcal{R}_{q,\kappa}^{\vartheta;c}u)_\kappa,$$

$$\mathcal{R}_\infty^\vartheta: \boldsymbol{L}_{1,loc}(\mathbb{R}^m)\to L_{1,loc}(\mathsf{M},V), \quad (v_\kappa)_\kappa\mapsto\sum_\kappa \mathcal{R}_{q,\kappa}^\vartheta v_\kappa.$$

We have the following relationship between these two retraction and coretraction systems:

$$\mathcal{R}_q^{\vartheta;c}=L_{\vartheta+m/q}\circ\mathcal{R}^c, \quad \mathcal{R}_q^\vartheta=\mathcal{R}\circ L_{-\vartheta-m/q}.$$

Now the assertion follows straight away from (II.7), [**4**, Theorems 6.1, 6.3, 7.1, 11.1] and [**5**, Theorems 12.1, 12.3, formula (12.7)]. □

In the sequel, $(\cdot,\cdot)^0_{\theta,\infty}$ and $[\cdot,\cdot]_\theta$ denote the continuous interpolation method and the complex interpolation method, respectively. See [**1**, Example I.2.4.2, I.2.4.4] for definitions.

Proposition II.2. *Suppose that* $0 < s_0 < s_1 < \infty$, $0 < \theta < 1$ *and* $\vartheta \in \mathbb{R}$. *Then*

$$(l^\vartheta_q(\mathfrak{F}^{s_0}), l^\vartheta_q(\mathfrak{F}^{s_1}))_\theta \doteq l^\vartheta_q(\mathfrak{F}^{s_\theta}) \doteq [l^\vartheta_q(\mathfrak{F}^{s_0}), l^\vartheta_q(\mathfrak{F}^{s_1})]_\theta$$

holds for $s_0, s_1, s_\theta \notin \mathbb{N}$ *for* $\mathfrak{F} \in \{bc, BC, W_p\}$, *or* $s_0, s_1, s_\theta \notin \mathbb{N} + 1/p$ *for* $\mathfrak{F} = \mathring{W}_p$. *When* $\mathfrak{F} = bc$, $(\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty}$, *when* $\mathfrak{F} = BC$, $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,\infty}$, *or when* $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$, $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,p}$. *Here* $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$ *for any* $\xi_0, \xi_1 \in \mathbb{R}$.

PROOF. When $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$, the assertion with weight $\vartheta = 0$ follows from [**5**, Lemmas 11.10, 11.11] and [**1**, Proposition I.2.3.2]. The rest of the statement is a consequence of (II.7) and [**1**, Proposition I.2.3.2].

When $\mathfrak{F} \in \{bc, BC\}$, the assertion with weight $\vartheta = -1/p$ follows from [**3**, formula (3.4.1), Theorem 3.7.1, Corollary 4.9.2] and [**85**, Theorem 1.18.1]. (II.7) implies the remaining assertions. □

Proposition II.3. *Suppose that* $0 < s_0 < s_1 < \infty$, $0 < \theta < 1$ *and* $\vartheta \in \mathbb{R}$. *Then*

$$(\mathfrak{F}^{s_0,\vartheta}(\mathsf{M}, V), \mathfrak{F}^{s_1,\vartheta}(\mathsf{M}, V))_\theta \doteq \mathfrak{F}^{s_\theta,\vartheta}(\mathsf{M}, V) \doteq [\mathfrak{F}^{s_0,\vartheta}(\mathsf{M}, V), \mathfrak{F}^{s_1,\vartheta}(\mathsf{M}, V)]_\theta$$

holds for $s_0, s_1, s_\theta \notin \mathbb{N}$ *for* $\mathfrak{F} \in \{bc, BC, W_p\}$, *or* $s_0, s_1, s_\theta \notin \mathbb{N} + 1/p$ *for* $\mathfrak{F} = \mathring{W}_p$. *When* $\mathfrak{F} = bc$, $(\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty}$, *when* $\mathfrak{F} = BC$, $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,\infty}$, *or when* $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$, $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,p}$.

PROOF. The assertion is a direct consequence of Propositions II.1 and II.2. □

Let $V_j = V^{\sigma_j}_{\tau_j} := \{T^{\sigma_j}_{\tau_j}\mathsf{M}, (\cdot|\cdot)_g\}$ with $j = 1, 2, 3$ be $\mathbb{K}$-valued tensor bundles on M. Let $\oplus$ be the Whitney sum. By bundle multiplication from $V_1 \times V_2$ into V_3, denoted by

$$\mathsf{m} : V_1 \oplus V_2 \to V_3, \quad (v_1, v_2) \mapsto \mathsf{m}(v_1, v_2),$$

we mean a smooth bounded section $\mathfrak{m}$ of $\mathrm{Hom}(V_1 \otimes V_2, V_3)$, i.e.,

$$\mathfrak{m} \in BC^\infty(\mathsf{M}, \mathrm{Hom}(V_1 \otimes V_2, V_3)), \tag{II.8}$$

such that $\mathsf{m}(v_1, v_2) := \mathfrak{m}(v_1 \otimes v_2)$. (II.8) implies that for some $c > 0$

$$|\mathsf{m}(v_1, v_2)|_g \le c|v_1|_g|v_2|_g, \quad v_i \in \Gamma(\mathsf{M}, V_i) \text{ with } i = 1, 2.$$

Its point-wise extension from $\Gamma(\mathsf{M}, V_1 \oplus V_2)$ onto $\Gamma(\mathsf{M}, V_3)$ is defined by:

$$\mathsf{m}(v_1, v_2)(p) := \mathsf{m}(p)(v_1(p), v_2(p))$$

for $v_i \in \Gamma(\mathsf{M}, V_i)$ and $p \in \mathsf{M}$. We still denote it by m. We can prove the following point-wise multiplier theorems for function spaces over *singular manifolds*.

Proposition II.4. *Let $k \in \mathbb{N}_0$. Assume that the tensor bundles $V_j = V_{\tau_j}^{\sigma_j} := \{T_{\tau_j}^{\sigma_j}\mathsf{M}, (\cdot|\cdot)_g\}$ with $j = 1, 2, 3$ satisfy*

$$\sigma_3 - \tau_3 = \sigma_1 + \sigma_2 - \tau_1 - \tau_2. \tag{II.9}$$

Suppose that $\mathsf{m} : V_1 \oplus V_2 \to V_3$ is a bundle multiplication, and $\vartheta_3 = \vartheta_1 + \vartheta_2$. Then $[(v_1, v_2) \mapsto \mathsf{m}(v_1, v_2)]$ is a bilinear and continuous map for the following spaces.

(a) $BC^{t,\vartheta_1}(\mathsf{M}, V_1) \times \mathfrak{F}^{s,\vartheta_2}(\mathsf{M}, V_2) \to \mathfrak{F}^{s,\vartheta_3}(\mathsf{M}, V_3)$, *where $t > s \ge 0$ for $\mathfrak{F} \in \{bc, W_p, \mathring{W}_p\}$.*

(b) $BC^{k,\vartheta_1}(\mathsf{M}, V_1) \times \mathfrak{F}^{k,\vartheta_2}(\mathsf{M}, V_2) \to \mathfrak{F}^{k,\vartheta_3}(\mathsf{M}, V_3)$, *where $k \in \mathbb{N}_0$ for $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$.*

(c) $\mathfrak{F}^{s,\vartheta_1}(\mathsf{M}, V_1) \times \mathfrak{F}^{s,\vartheta_2}(\mathsf{M}, V_2) \to \mathfrak{F}^{s,\vartheta_3}(\mathsf{M}, V_3)$ *for $\mathfrak{F} \in \{bc, BC\}$.*

PROOF. The statement follows from [**5**, Theorem 13.5]. □

Proposition II.5. *For $\mathfrak{F} \in \{bc, BC, W_p, \mathring{W}_p\}$, we have*

$$f_\vartheta : [u \mapsto \rho^\vartheta u] \in \mathcal{L}\mathrm{is}(\mathfrak{F}^{s,\vartheta'+\vartheta}(\mathsf{M}, V), \mathfrak{F}^{s,\vartheta'}(\mathsf{M}, V)), \quad (f_\vartheta)^{-1} = f_{-\vartheta}.$$

PROOF. By (S3) and (S4), we infer that $\boldsymbol{\rho} := (\zeta \frac{\psi_\kappa^* \rho^\vartheta}{\rho_\kappa^\vartheta})_\kappa \in \bigcap_k l_\infty(\boldsymbol{BC}^k)$, where ζ is defined in (L2). Then it follows from the point-wise multiplication results in [**2**, Appendix A2] and [**86**, Corollary 2.8.2] that for $\boldsymbol{u} = (u_\kappa)_\kappa$ and any $s \ge 0$

$$[\boldsymbol{u} \mapsto (\zeta \frac{\psi_\kappa^* \rho^\vartheta}{\rho_\kappa^\vartheta} u_\kappa)_\kappa] \in \mathcal{L}(l_q^{\vartheta'}(\boldsymbol{\mathfrak{F}}^s)),$$

where $q=\infty$ for $\mathfrak{F}=BC$, or $q=p$ for $\mathfrak{F}\in\{W_p,\mathring{W}_p\}$. Given $u\in\mathfrak{F}^{s,\vartheta'}(\mathsf{M},V)$,

$$\begin{aligned}\|\rho^\vartheta u\|_{\mathfrak{F}^{s,\vartheta'}} &= \|\mathcal{R}\mathcal{R}^c\rho^\vartheta u\|_{\mathfrak{F}^{s,\vartheta'}}\le C\|\mathcal{R}^c\rho^\vartheta u\|_{l_q^{\vartheta'}(\boldsymbol{\mathfrak{F}}^s)}\\ &=\|\boldsymbol{\rho}L_\vartheta\mathcal{R}^c u\|_{l_q^{\vartheta'}(\boldsymbol{\mathfrak{F}}^s)}\le C\|\boldsymbol{\rho}\|_{l_\infty(\boldsymbol{BC}^k)}\|\mathcal{R}^c u\|_{l_q^{\vartheta'+\vartheta}(\boldsymbol{\mathfrak{F}}^s)}\\ &\le C(\rho,\vartheta,s)\|u\|_{\mathfrak{F}^{s,\vartheta'+\vartheta}}.\end{aligned}$$

Now the open mapping theorem implies that the asserted result holds for $\mathfrak{F}\in\{BC,W_p,\mathring{W}_p\}$.

Given any $u\in bc^{s,\vartheta'+\vartheta}(\mathsf{M},V)$, there exists $(u_n)_n\in BC^{\infty,\vartheta'+\vartheta}(\mathsf{M},V)$ such that

$$u_n\to u,\quad \text{in}\quad BC^{s,\vartheta'+\vartheta}(\mathsf{M},V).$$

We already have

$$\|\rho^\vartheta u\|_{s,\infty;\vartheta'}\le C\|u\|_{s,\infty;\vartheta'+\vartheta},$$

and $(\rho^\vartheta u_n)_n\in BC^{\infty,\vartheta'}(\mathsf{M},V)$. By the conclusion for $\mathfrak{F}^s=BC^s$, we infer that as $n\to\infty$

$$\|\rho^\vartheta(u-u_n)\|_{s,\infty;\vartheta'}\le C\|u-u_n\|_{s,\infty;\vartheta'+\vartheta}\to 0.$$

We have established the asserted result for weighted *little Hölder* spaces in view of the definition (II.4). □

Proposition II.6. *For $\mathfrak{F}\in\{bc,BC,W_p,\mathring{W}_p\}$, $\sigma,\tau\in\mathbb{N}_0$ and $\vartheta\in\mathbb{R}$,*

$$\nabla\in\mathcal{L}(\mathfrak{F}^{s,\vartheta}(\mathsf{M},V^\sigma_\tau),\mathfrak{F}^{s-1,\vartheta}(\mathsf{M},V^\sigma_{\tau+1})).$$

Proof. The case $s\in\mathbb{N}$ and $\mathfrak{F}\in\{BC,W_p,\mathring{W}_p\}$ is immediate from the definition of the weighted function spaces. When $\mathfrak{F}=bc$, the integer case can be proven by a density argument as in the proof for Proposition II.5. The non-integer case follows from [**5**, Theorem 16.1]. □

Let $\hat{g}=g/\rho^2$. Then $(\mathsf{M},\hat{g})$ is a *uniformly regular Riemannian manifold.* We denote the (σ,τ)-tensor fields with respect to $\hat{g}$ by $\hat{V}=\hat{V}^\sigma_\tau$. The definitions of the corresponding weighted function spaces $\mathfrak{F}^{s',\vartheta'}(\mathsf{M},\hat{V})$ do not depend on the choice of ϑ' in this case. We denote the unweighted spaces by $\mathfrak{F}^{s'}(\mathsf{M},\hat{V})$. The reader may refer to [**79**] for the precise definitions for these unweighted spaces on uniformly regular Rimannian manifolds.

Proposition II.7. *For* $\mathfrak{F} \in \{bc, BC, W_p, \mathring{W}_p\}$*, it holds that*

$$\mathfrak{F}^s(\mathsf{M}, \hat{V}) \doteq \mathfrak{F}^{s,-1/p}(\mathsf{M}, V)$$

Proof. The assertion follows from Proposition II.1 and [**79**, Propositions 2.1, 2.2]. □

4. Surface divergence

Proposition II.8. *For* $\mathfrak{F} \in \{bc, BC, W_p, \mathring{W}_p\}$*, we have*

$$[a \mapsto a^\sharp] \in \mathcal{L}(\mathfrak{F}^{s,\vartheta}(\mathsf{M}, V^\sigma_{\tau+1}), \mathfrak{F}^{s,\vartheta+2}(\mathsf{M}, V^{\sigma+1}_\tau)).$$

$$[a \mapsto a_\flat] \in \mathcal{L}(\mathfrak{F}^{s,\vartheta}(\mathsf{M}, V^{\sigma+1}_\tau), \mathfrak{F}^{s,\vartheta-2}(\mathsf{M}, V^\sigma_{\tau+1})).$$

Proof. We only prove the second assertion. The first one follows in an analogous manner. For any $X \in T\mathsf{M}$,

$$\nabla_X a_\flat = \nabla_X \langle g, a\rangle = \langle \nabla_X g, a\rangle + \langle g, \nabla_X a\rangle = \langle g, \nabla_X a\rangle = (\nabla_X a)_\flat.$$

The third equality follows from the metric preservation of the Levi-Civita connection. This implies

$$\nabla(a_\flat) = (\nabla a)_\flat.$$

By induction, we have

$$\nabla^k(a_\flat) = (\nabla^k a)_\flat.$$

Then the statement for the case $s \in \mathbb{N}_0$ is an immediate consequence of the definitions of the corresponding function spaces. The non-integer case follows by interpolation theory, Proposition II.3 and Definitions (II.3), formula (II.6) when $\mathfrak{F} \in \{BC, W_p, \mathring{W}_p\}$. When $\mathfrak{F} = bc$, the assertion follows from a density argument and Definition (II.4) as in the proof for Proposition II.5. □

We denote by $\mathsf{C}^{\sigma+1}_{\tau+1} : V^{\sigma+1}_{\tau+1} \to V^\sigma_\tau$ the contraction with respect to position $\sigma+1$ and $\tau+1$, that is for any $(i) \in \mathbb{J}^\sigma$, $(j) \in \mathbb{J}^\tau$ and $k, l \in \mathbb{J}^1$ and $\mathsf{p} \in \mathsf{M}$

$$\mathsf{C}^{\sigma+1}_{\tau+1} a := \mathsf{C}^{\sigma+1}_{\tau+1} a^{(i;k)}_{(j;l)} \frac{\partial}{\partial x^{(i)}} \otimes \frac{\partial}{\partial x^k} \otimes dx^{(j)} \otimes dx^l := a^{(i;k)}_{(j;k)} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)}$$

in every local chart. Recall that the surface divergence of tensor fields with respect to g is the map

(II.10) $$\mathrm{div} = \mathrm{div}_g : C^1(\mathsf{M}, V_\tau^{\sigma+1}) \to C(\mathsf{M}, V_\tau^\sigma), \quad a \mapsto \mathsf{C}_{\tau+1}^{\sigma+1}(\nabla a).$$

Suppose that $\partial\mathsf{M} \neq \emptyset$. Since $T(\partial\mathsf{M})$ is a subbundle of codimension 1 of the vector bundle $(T\mathsf{M})_{\partial\mathsf{M}}$ over $\partial\mathsf{M}$, there exists a unique vector field $\boldsymbol{n}$ in $(T\mathsf{M})_{\partial\mathsf{M}}$ of length 1 orthogonal to $T(\partial\mathsf{M})$, and inward pointing. In every local coordinate, $\varphi_\kappa = \{x_1, \cdots, x_m\}$

$$\boldsymbol{n} = \frac{1}{\sqrt{g_{11}|\partial\mathsf{O}_\kappa}} \frac{\partial}{\partial x^1}.$$

Put $V' := V_\sigma^\tau$. Let $\mathsf{C} : V_{\tau+\tau_1}^{\sigma+\sigma_1} \times V' \to V_{\tau_1}^{\sigma_1}$ denote the complete contraction. For any $a \in V_{\tau+\tau_1}^{\sigma+\sigma_1}$ and $b \in V'$, the complete contraction (on the right) is defined by

$$\mathsf{C}(a,b) = a_{(j;j_1)}^{(i;i_1)} b_{(i)}^{(j)} \frac{\partial}{\partial x^{(i_1)}} \otimes dx^{(j_1)},$$

with $(i) \in \mathbb{J}^\sigma$,$(i_1) \in \mathbb{J}^{\sigma_1}$, $(j) \in \mathbb{J}^\tau$,$(j_1) \in \mathbb{J}^{\tau_1}$, in local coordinates. The complete contraction (on the left) is defined in an analogous manner. Note that the complete contraction is a bundle multiplication.

Theorem II.9. *For any* $a \in W_2^{1,-\vartheta}(\mathsf{M}, V')$ *and* $b \in \mathring{W}_2^{1,\vartheta}(\mathsf{M}, V_\tau^{\sigma+1})$

$$-\int_{\mathsf{M}} \langle \mathrm{div} b, a\rangle \, dV_g = \int_{\mathsf{M}} \langle b, \nabla a\rangle \, dV_g.$$

Proof. By the divergence theorem and the density of the spaces $\mathcal{D}(\mathsf{M}, V')$ and $\mathcal{D}(\mathring{\mathsf{M}}, V_\tau^{\sigma+1})$ in $W_2^{1,-\vartheta}(\mathsf{M}, V')$ and $\mathring{W}_2^{1,\vartheta}(\mathsf{M}, V_\tau^{\sigma+1})$, it suffices to show that

(II.11) $$\mathrm{div}(\mathsf{C}(b,a)) = \langle \mathrm{div} b, a\rangle + \langle b, \nabla a\rangle,$$

for any $a \in \mathcal{D}(\mathsf{M}, V')$ and $b \in \mathcal{D}(\mathring{\mathsf{M}}, V_\tau^{\sigma+1})$. Definition (II.10) yields

$$\mathrm{div}(\mathsf{C}(b,a)) = \mathrm{div}(a_{(i)}^{(j)} b_{(j)}^{(i;k)} \frac{\partial}{\partial x^k}) = \partial_k (a_{(i)}^{(j)} b_{(j)}^{(i;k)}) + \Gamma_{lk}^l a_{(i)}^{(j)} b_{(j)}^{(i;k)},$$

for $(i) \in \mathbb{J}^\sigma$, $(j) \in \mathbb{J}^\tau$. By [**4**, formula (3.17)] and (II.10)

$$\begin{aligned}&\langle \mathrm{div} b, a\rangle \\ &= a^{(j)}_{(i)}\partial_k(b^{(i;k)}_{(j)}) + (\sum_{s=1}^{\sigma}\Gamma^{i_s}_{kh}b^{(i_1,\cdots,h,\cdots,i_\sigma;k)}_{(j)} - \sum_{t=1}^{\tau}\Gamma^{h}_{kj_t}b^{(i;k)}_{(j_1,\cdots,h,\cdots,j_\tau)} + \Gamma^k_{kh}b^{(i;h)}_{(j)})a^{(j)}_{(i)},\end{aligned}$$

and

$$\langle b, \nabla a\rangle = (\partial_k a^{(j)}_{(i)})b^{(i;k)}_{(j)} + (\sum_{t=1}^{\tau}\Gamma^{j_t}_{kh}a^{(j_1,\cdots,h,\cdots,j_\tau)}_{(i)} - \sum_{s=1}^{\sigma}\Gamma^{h}_{ki_s}a^{(j)}_{(i_1,\cdots,h,\cdots,i_\sigma)})b^{(i;k)}_{(j)}.$$

This proves (II.11). □

Corollary II.10. *For any* $a \in W^{1,-\vartheta+2\sigma-2\tau}_2(\mathsf{M}, V)$ *and* $b \in \mathring{W}^{1,\vartheta}_2(\mathsf{M}, V^{\sigma+1}_\tau)$

$$-\int_{\mathsf{M}}(\mathrm{div} b|a)_g\, dV_g = \int_{\mathsf{M}}(b|\mathrm{grad} a)_g\, dV_g.$$

Proof. In [**5**, p. 10], it is shown that for any $X \in T\mathsf{M}$ and $a \in V$

$$\nabla_X(G^\tau_\sigma a) = G^\tau_\sigma(\nabla_X a).$$

Therefore,

$$\nabla G^\tau_\sigma a = (G^{\tau+1}_\sigma(\nabla u))_\flat.$$

Now it is an easy task to check

$$\nabla G^\tau_\sigma u = (G^{\tau+1}_\sigma(\mathrm{grad} u)_\flat)_\flat = G^\tau_{\sigma+1}(\mathrm{grad} u). \tag{II.12}$$

This implies the asserted result. □

Proposition II.11. $\mathrm{div} \in \mathcal{L}(\mathfrak{F}^{s+1,\vartheta}(\mathsf{M}, V^{\sigma+1}_\tau), \mathfrak{F}^{s,\vartheta}(\mathsf{M}, V^\sigma_\tau))$ *for* $\mathfrak{F} \in \{BC, W_p, \mathring{W}_p\}$.

Proof. Given any $a \in \mathfrak{F}^{s,\vartheta}(\mathsf{M}, V^{\sigma+1}_\tau)$, it is easy to see that

$$\|\mathcal{R}^c \mathsf{C}^{\sigma+1}_{\tau+1} a\|_{l^\vartheta_q(\mathfrak{F}^s)} \le C\|\mathcal{R}^c a\|_{l^\vartheta_q(\mathfrak{F}^s(E^{\sigma+1}_{\tau+1}))}$$

with $\psi^*_\kappa(\pi_\kappa a^{(i;k)}_{(j;k)})$ in the $((i),(j))$ position. Here $q = \infty$ for $\mathfrak{F} = BC$, or $q = p$ for $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$. Combining with Proposition II.1, it implies that

$$\mathsf{C}^{\sigma+1}_{\tau+1} \in \mathcal{L}(\mathfrak{F}^{s,\vartheta}(\mathsf{M}, V^{\sigma+1}_{\tau+1}), \mathfrak{F}^{s,\vartheta}(\mathsf{M}, V)).$$

Using Proposition II.6, we can now prove the asserted result. □

5. Spaces of negative order

For any $u \in \mathcal{D}(\mathsf{M}, V)$ and $v \in \mathcal{D}(\mathsf{M}, V')$, we put

$$\langle u, v \rangle_{\mathsf{M}} := \int_{\mathsf{M}} \langle u, \bar{v} \rangle \, dV_g.$$

Then we define

$$W_p^{-s,\vartheta}(\mathsf{M}, V) := (\mathring{W}_{p'}^{s,-\vartheta}(\mathsf{M}, V'))' \tag{II.13}$$

by mean of the duality pairing $\langle \cdot, \cdot \rangle_{\mathsf{M}}$. It is convenient to denote by $\mathring{W}_p^{-s,\vartheta}(\mathsf{M}, V)$ the closure of $\mathcal{D}(\mathring{\mathsf{M}}, V)$ in $W_p^{-s,\vartheta}(\mathsf{M}, V)$. Then

$$\mathring{W}_p^{t,\vartheta}(\mathsf{M}, V) = W_p^{t,\vartheta}(\mathsf{M}, V), \quad t < 1/p. \tag{II.14}$$

We refer the reader to [**4**, Section 12] for more details. Given $u \in \mathfrak{F}^{-s,\vartheta}(\mathsf{M}, V)$ with $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$ and $v \in \mathcal{D}(\mathring{\mathsf{M}}, V_\sigma^{\tau+1})$

$$\langle \nabla u, v \rangle_{\mathsf{M}} := -\int_{\mathsf{M}} \langle u, \mathrm{div}(\bar{v}) \rangle \, dV_g. \tag{II.15}$$

Theorem II.9 shows for $u \in \mathfrak{F}^{-s,\vartheta}(\mathsf{M}, V_\tau^{\sigma+1})$ and $v \in \mathcal{D}(\mathring{\mathsf{M}}, V')$

$$\langle \mathrm{div} u, v \rangle_{\mathsf{M}} = -\int_{\mathsf{M}} \langle u, \nabla \bar{v} \rangle \, dV_g.$$

By means of Proposition II.6 and II.11, it is not hard to prove the following proposition.

Proposition II.12. *Suppose that* $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$. *Then*

$$\nabla \in \mathcal{L}(\mathfrak{F}^{-s,\vartheta}(\mathsf{M}, V), \mathfrak{F}^{-s-1,\vartheta}(\mathsf{M}, V_{\tau+1}^{\sigma})),$$

and

$$\mathrm{div} \in \mathcal{L}(\mathfrak{F}^{-s,\vartheta}(\mathsf{M}, V_\tau^{\sigma+1}), \mathfrak{F}^{-s-1,\vartheta}(\mathsf{M}, V)).$$

Let $\langle\cdot|\cdot\rangle_{2,\vartheta'}$ be the inner product in $L_2^{\vartheta'}(\mathsf{M},V)$, that is,

$$\langle u|v\rangle_{2,\vartheta'} := \int_{\mathsf{M}} \rho^{2\vartheta'+2\tau-2\sigma}(u|v)_g \, dV_g. \tag{II.16}$$

Proposition II.13. *Suppose that* $\mathfrak{F} \in \{W_p, \mathring{W}_p\}$ *and* $s \in \mathbb{R}$. *Then*

$$[u \mapsto \rho^{2\tau-2\sigma} G_\sigma^\tau u] \in \mathcal{L}\mathrm{is}(\mathfrak{F}^{s,\vartheta}(\mathsf{M},V), \mathfrak{F}^{s,\vartheta}(\mathsf{M},V')).$$

PROOF. The statement follows from Proposition II.5, an analogue of the proof for Proposition (II.8) and the open mapping theorem. □

In virtue of Proposition II.5 and II.13, now one readily checks that

$$(\mathring{W}_p^{s,\vartheta}(\mathsf{M},V))'_{\vartheta'} \doteq W_{p'}^{-s,2\vartheta'-\vartheta}(\mathsf{M},V), \tag{II.17}$$

where $(\mathring{W}_p^{s,\vartheta}(\mathsf{M},V))'_{\vartheta'}$ is the dual space of $W_p^{s,\vartheta}(\mathsf{M},V)$ with respect to $\langle\cdot|\cdot\rangle_{2,\vartheta'}$.

Chapter III

Examples of singular manifolds

1. Singular manifolds of pipe and wedge type

As was shown by the examples in [**7**], we can find manifolds with singularities of arbitrarily high dimension. Among them, a very important family is the *singular manifolds* of pipe and wedge type.

Following [**7**], throughout we write $J_0 := (0,1]$ and $J_\infty := [1,\infty)$, and assume $J \in \{J_0, J_\infty\}$.

We denote by $\mathscr{R}(J)$ the set of all $R \in C^\infty(J,(0,\infty))$ with $R(1) = 1$ such that $R(\alpha) := \lim_{t\to\alpha} R(t)$ exists in $[0,\infty]$ if $J = J_\alpha$ with $\alpha \in \{0,\infty\}$. We write $R \in \mathscr{C}(J)$ if

$$\text{(III.1)} \qquad \begin{cases} \text{(i)} & R \in \mathscr{R}(J), \text{ and } R(\infty) = 0 \text{ if } J = J_\infty; \\ \text{(ii)} & \int_J dt/R(t) = \infty; \\ \text{(iii)} & \|\partial_t^k R\|_\infty < \infty, \quad k \geq 1. \end{cases}$$

The elements in $\mathscr{C}(J)$ are called *cusp characteristics* on J.

The following results from [**7**] are the cornerstones of the construction of *singular manifolds* of pipe and wedge type.

Lemma III.1. [**7**, Theorem 3.1] *Suppose that ρ is a bounded singularity function on (M, g), and $\tilde\rho$ is one for $(\tilde{\mathsf{M}}, \tilde g)$. Then $\rho \otimes \tilde\rho$ is a singularity function for $(\mathsf{M} \times \tilde{\mathsf{M}}, g + \tilde g)$.*

Lemma III.2. [**7**, Lemma 3.4] *Let $f : \tilde{\mathsf{M}} \to \mathsf{M}$ be a diffeomorphism of manifolds. Suppose that $(\mathsf{M}, g; \rho)$ is a singular manifold. Then so is $(\tilde{\mathsf{M}}, f^*g; f^*\rho)$.*

Lemma III.3. [**7**, Lemma 5.2] *Suppose that $R \in \mathscr{C}(J)$. Then R is a singularity function for (J, dt^2).*

Assume that $(B, g_B; b)$ is a d-dimensional singular submanifold of $\mathbb{R}^{\bar d}$ with singularity function b, and $R \in \mathscr{C}(J)$. The (model) (R,B)-pipe $P(R,B)$ on J, also called R-pipe over B on J, is defined by

$$P(R,B) = P(R,B;J) := \{(t, R(t)y) : t \in J,\, y \in B\} \subset \mathbb{R}^{1+\bar{d}}.$$

It is a $(1+d)$-dimensional submanifold of $\mathbb{R}^{1+\bar{d}}$. An R-pipe is an R-cusp if $R(\alpha) = 0$ with $\alpha \in \{0, \infty\}$. The map

$$\phi_P = \phi_P(R) : P \to J \times B : \quad (t, R(t)y) \to (t, y)$$

is a diffeomorphism, the *canonical stretching diffeomorphism* of P.

We assume that

$$\text{(III.2)} \qquad b \text{ is bounded.}$$

Then the above three lemmas imply the following result.

Lemma III.4. *Assume that* (III.2) *is fulfilled. Then*

$$(P(R,B), \phi_P^*(dt^2 + g_B); \phi_P^*(R \otimes b))$$

is a singular manifold.

Assume that (Γ, g_Γ) is a connected *uniformly regular Riemannian manifold* without boundary. Then the (model) Γ-wedge over the (R,B)-pipe, $P(R,B)$, is defined by

$$W = W(R,B,\Gamma) := P(R,B) \times \Gamma.$$

If Γ is a one-point space, then W is naturally identified with P. Thus every pipe is also a wedge.

Lemma III.5. *Assume that* (III.2) *is fulfilled. Then*

$$(W(R,B,\Gamma), \phi_P^*(dt^2 + g_B) + g_\Gamma; \phi_P^*(R \otimes b) \otimes \mathbf{1}_\Gamma)$$

is a singular manifold.

In the following examples, (B, g_B) always denotes a compact closed C^∞-Riemannian manifold.

Example III.6. (a) *Suppose* M *is a cone, i.e.,* $\mathsf{M} = ([0,1] \times B)/(\{0\} \times B)$. *We equip* M *with the conventional metric* $g = dt^2 + t^2 g_B$. *Then* $(\mathsf{M}, g; \rho)$ *is a singular manifold of wedge type.*

(b) *Suppose* M *is an edge manifold, that is,* $(\mathsf{M}, g; \rho) = (P(t, B) \times \mathbb{R}^d, dt^2 + t^2 g_B + g_d; \phi_P^*(t \otimes \mathbf{1}_B) \otimes \mathbf{1}_{\mathbb{R}^d})$*. Then* $(\mathsf{M}, g; \rho)$ *is a singular manifold of wedge type.*

(c) *Let* $[t \mapsto T(t)], [r \mapsto R(r)] \in \mathscr{C}(J_0)$ *be two cusp characteristics on* J_0*, i.e.,* $(J_0, dt^2; T)$ *and* $(J_0, dr^2; R)$ *are two singular manifolds. In order to avoid possible confusion, we denote the* canonical stretching diffeomorphisms *of the pipes generated by* T *and* R *by* ϕ_{P_T} *and* ϕ_{P_R}*, respectively.*

Suppose M *is a* (T, R)*-corner manifold. More precisely,* $(\mathsf{M}, g; \rho) =$

$$(P(T, P(R, B)), \phi_{P_T}^*(dt^2 + \phi_{P_R}^*(dr^2 + g_B)); \phi_{P_T}^*(T \otimes \phi_{P_R}^*(R \otimes \mathbf{1}_B))).$$

Then $(\mathsf{M}, g; \rho)$ *is a singular manifold of wedge type.*

In some references, the authors equip an edge with the metric $g = dt^2/t^2 + g_B + g_\Gamma/t^2$, which makes (M, g) uniformly regular. This case has been studied in depth in [**6**].

2. Manifolds with holes

In this section, we construct another interesting class of manifolds, that is, manifolds with holes.

Given any compact submanifold $\Sigma \subset (\mathsf{M}, g)$, the distance function is a well-defined smooth function in a collar neighborhood $\mathscr{U}_\Sigma$ of Σ. The distance ball at Σ with radius r is defined by

$$\mathbb{B}_{\mathsf{M}}(\Sigma, r) := \{\mathsf{p} \in \mathsf{M} : \mathrm{dist}_{\mathsf{M}}(\mathsf{p}, \Sigma) < r\}.$$

Lemma III.7. *Suppose that* $(\mathscr{M}, g)$ *is a uniformly regular Riemannian manifold, and*

$$\mathbf{\Sigma} = \{\Sigma_1, \cdots, \Sigma_k\}$$

is a finite set of disjoint m*-dimensional compact manifolds with boundary such that* $\Sigma_j \subset \mathring{\mathscr{M}}$*. Put*

$$\mathsf{M} := \mathscr{M} \setminus \cup_{j=1}^k \Sigma_j$$

and

$$\mathscr{B}_{j,r} := \bar{\mathbb{B}}_{\mathscr{M}}(\partial\Sigma_j, r) \cap \mathsf{M}, \quad j = 1, \cdots, k.$$

Then we can find a singularity function ρ satisfying

$$\rho|_{\mathscr{B}_{j,r}} \sim \mathrm{dist}_{\mathscr{M}}(\cdot, \partial\Sigma_j),$$

for some $r \in [0, \delta)$, where $\delta < \mathrm{diam}(\mathscr{M})$ fulfils that $\mathscr{B}_{i,\delta} \cap \mathscr{B}_{j,\delta} = \emptyset$ for $i \neq j$, and

$$\rho \sim \mathbf{1}, \quad \textit{elsewhere on } \mathsf{M}.$$

Then (M, g) is a singular manifold.

PROOF. This lemma immediately follows from [**7**, Theorem 1.6]. □

Manifolds satisfying the conditions in Lemma III.7 are called *singular manifolds with holes.* We will show in Proposition V.22 below that the singularity function can actually be chosen to satisfy $\rho|_{\mathscr{B}_{j,r}} = \mathrm{dist}_{\mathscr{M}}(\cdot, \partial\Sigma_j)$.

More generally, by [**7**, Theorem 1.6], we indeed have the following result.

Lemma III.8. *Suppose that $(\mathscr{M}, g)$ is a uniformly regular Riemannian manifold, and*

$$\mathbf{\Sigma} = \{\Sigma_1, \cdots, \Sigma_k\}$$

is a finite set of disjoint compact closed submanifolds of codimension at least 1 such that $\Sigma_j \subset \partial\mathscr{M}$ if $\Sigma_j \cap \partial\mathscr{M} \neq \emptyset$. Put $\mathsf{M} := \mathscr{M} \setminus \cup_{j=1}^{k}\Sigma_j$ and

$$\mathscr{B}_{j,r} := \bar{\mathbb{B}}_{\mathscr{M}}(\Sigma_j, r) \cap \mathsf{M}, \quad j = 1, \cdots, k.$$

Then we can find a singularity function ρ satisfying

$$\rho|_{\mathscr{B}_{j,r}} \sim \mathrm{dist}_{\mathscr{M}}(\cdot, \Sigma_j),$$

for some $r \in [0, \delta)$, where $\delta < \mathrm{diam}(\mathscr{M})$ fulfils that $\mathscr{B}_{i,\delta} \cap \mathscr{B}_{j,\delta} = \emptyset$ for $i \neq j$, and

$$\rho \sim \mathbf{1}, \quad \textit{elsewhere on } \mathsf{M}.$$

Then (M, g) is a singular manifold.

Chapter IV

Continuous maximal regularity for normally ρ-elliptic operators

Throughout this chapter, we always assume that $(\mathsf{M}, g; \rho)$ is a *singular manifold* without boundary.

Following [**82**], letting $l \in \mathbb{N}_0$, $\mathcal{A} : C^\infty(\mathsf{M}, V) \to \Gamma(\mathsf{M}, V)$ is called a linear differential operator of order l on M if we can find $\mathfrak{a} = (a_r)_r \in \prod_{r=0}^{l} \Gamma(\mathsf{M}, V^{\sigma+\tau+r}_{\tau+\sigma})$ such that

$$\mathcal{A} = \mathcal{A}(\mathfrak{a}) := \sum_{r=0}^{l} \mathsf{C}(a_r, \nabla^r \cdot). \tag{IV.1}$$

Recall $\mathsf{C}(\cdot,\cdot)$ denotes complete contraction. Making use of [**4**, formula (3.18)], one can check that for any l-th order linear differential operator so defined, in every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$ there exists some linear differential operator

$$\mathcal{A}_\kappa(x, \partial) := \sum_{|\alpha| \leq l} a^\kappa_\alpha(x) \partial^\alpha, \quad \text{with } a^\kappa_\alpha \in \mathcal{L}(E)^{\mathsf{Q}^m_\kappa}, \tag{IV.2}$$

called the local representation of $\mathcal{A}$ with respect to $(\mathsf{O}_\kappa, \varphi_\kappa)$, such that for any $u \in C^\infty(\mathsf{M}, V)$

$$\psi^*_\kappa(\mathcal{A}u) = \mathcal{A}_\kappa(\psi^*_\kappa u).$$

Proposition IV.1. *Let $t > s \geq 0$, $\vartheta \in \mathbb{R}$ and $\mathfrak{F} \in \{bc, BC, W_p\}$. Suppose that $\mathcal{A} = \mathcal{A}(\mathfrak{a})$ with $\mathfrak{a} = (a_r)_r \in \prod_{r=0}^{l} BC^t(\mathsf{M}, V^{\sigma+\tau+r}_{\tau+\sigma})$. Then*

$$\mathcal{A} \in \mathcal{L}(\mathfrak{F}^{s+l,\vartheta}(\mathsf{M}, V), \mathfrak{F}^{s,\vartheta}(\mathsf{M}, V)).$$

Proof. The assertion is a direct consequence of Propositions II.4 and II.6. $\square$

Corollary IV.2. *Let $s \geq 0$, $\vartheta \in \mathbb{R}$ and $\mathfrak{F} \in \{bc, BC\}$. Suppose that $\mathcal{A} = \mathcal{A}(\mathfrak{a})$ with $\mathfrak{a} = (a_r)_r \in \prod_{r=0}^{l} bc^s(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r})$. Then*

$$\mathcal{A} \in \mathcal{L}(\mathfrak{F}^{s+l,\vartheta}(\mathsf{M}, V), \mathfrak{F}^{s,\vartheta}(\mathsf{M}, V)).$$

Given any angle $\phi \in [0, \pi]$, set

$$\Sigma_\phi := \{z \in \mathbb{C} : |\arg z| \leq \phi\} \cup \{0\}.$$

A linear operator $\mathcal{A} := \mathcal{A}(\mathfrak{a})$ of order l is said to be *normally ρ-elliptic* if there exists some constant $\mathcal{C}_\mathfrak{e} > 0$ such that for every pair $(\mathsf{p}, \xi) \in \mathsf{M} \times \Gamma(\mathsf{M}, T^*M)$ with $|\xi(\mathsf{p})|_{g^*(\mathsf{p})} \neq 0$ for all $\mathsf{p} \in \mathsf{M}$, the principal symbol

$$\hat{\sigma}\mathcal{A}^\pi(\mathsf{p}, \xi(\mathsf{p})) := \mathsf{C}(a_l, (-i\xi)^{\otimes l})(\mathsf{p}) \in \mathcal{L}(T_\mathsf{p}\mathsf{M}^{\otimes\sigma} \otimes T_\mathsf{p}^*\mathsf{M}^{\otimes\tau})$$

satisfies

$$S := \Sigma_{\pi/2} \subset \rho(-\hat{\sigma}\mathcal{A}^\pi(\mathsf{p}, \xi(\mathsf{p}))), \tag{IV.3}$$

and

$$(\rho^l(\mathsf{p})|\xi(\mathsf{p})|_{g^*(\mathsf{p})}^l + |\mu|)\|(\mu + \hat{\sigma}\mathcal{A}^\pi(\mathsf{p}, \xi(\mathsf{p})))^{-1}\|_{\mathcal{L}(T_p\mathsf{M}^{\otimes\sigma}\otimes T_p^*\mathsf{M}^{\otimes\tau})} \leq \mathcal{C}_\mathfrak{e}, \quad \mu \in S. \tag{IV.4}$$

The constant $\mathcal{C}_\mathfrak{e}$ is called the *ρ-ellipticity constant* of $\mathcal{A}$. To the best of the author's knowledge, this ellipticity condition is the first one formulated for elliptic operators acting on tensor fields on manifolds with singularities.

We can also introduce a stronger version of the ellipticity condition for $\mathcal{A}$. $\mathcal{A}$ is called *uniformly strongly ρ-elliptic* if there exists some constant $C_e > 0$ such that for all $(\mathsf{p}, \xi, \eta) \in \mathsf{M} \times \Gamma(\mathsf{M}, T^*M) \times \Gamma(\mathsf{M}, T_\tau^\sigma\mathsf{M})$ the principal symbol satisfies

$$\hat{\sigma}\mathcal{A}^\pi(\mathsf{p}, \xi(\mathsf{p}))(\eta(\mathsf{p})) \geq C_e\rho^l(\mathsf{p})|\eta(\mathsf{p})|_{g(\mathsf{p})}^2|\xi(\mathsf{p})|_{g^*(\mathsf{p})}^l.$$

Here $\hat{\sigma}\mathcal{A}^\pi(\mathsf{p}, \xi(\mathsf{p}))(\eta(\mathsf{p})) := (\mathsf{C}(a_l, \eta \otimes (-i\xi)^{\otimes l})(\mathsf{p})|\eta(\mathsf{p}))_{g(\mathsf{p})}$. In [**6**], H. Amann has used the *uniformly strong ρ-ellipticity* condition to establish an L_p-maximal regularity theory for second order differential operators acting on scalar functions.

We can readily check that a *uniformly strongly ρ-elliptic* operator $\mathcal{A}$ must be *normally ρ-elliptic*. If $\mathcal{A}$ is of odd order, then by replacing ξ with $-\xi$ in (IV.3), it is easy to see that $\rho(\hat{\sigma}\mathcal{A}^{\pi}(\mathsf{p},\xi(\mathsf{p}))) = \mathbb{C}$. This is a contradiction. Therefore, every *normally ρ-elliptic* operator is of even order.

We call a linear operator $\mathcal{A} := \mathcal{A}(\mathfrak{a})$ *s-regular* if

(IV.5) $$a_r \in bc^s(\mathsf{M}, V^{\sigma+\tau+r}_{\tau+\sigma}), \quad r = 0, 1, \cdots, l.$$

This reveals the existence of some constant $\mathcal{C}_{\mathfrak{a}}$ such that

(IV.6) $$\|a_r\|_{s,\infty} \leq \mathcal{C}_{\mathfrak{a}}, \quad r = 0, 1, \cdots, l.$$

We consider how (IV.5) affects the behavior of the localizations $\mathcal{A}_{\kappa}$. Given any linear differential operator $\mathcal{A}$ of order $2l$, by an analogy of Proposition II.1, we infer that

$$(\psi^*_{\kappa} a_r)_{\kappa} \in l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s(\mathsf{Q}^m_{\kappa}, E^{\sigma+\tau+r}_{\tau+\sigma})), \quad r = 0, 1, \cdots, 2l,$$

or equivalently

$$(\psi^*_{\kappa}(a_r)^{(i)}_{(j)})_{\kappa} \in l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s(\mathsf{Q}^m_{\kappa})), \quad (i) \in \mathbb{J}^{\sigma+\tau+r}, \quad (j) \in \mathbb{J}^{\tau+\sigma}, \quad r = 0, 1, \cdots, 2l.$$

By [**4**, formula (3.18)], the coefficients of $\mathcal{A}_{\kappa}$, i.e., a^{κ}_{α}, are linear combinations of the products of $(a_r)^{(i)}_{(j)}$ and possibly the derivatives of the Christoffel symbols of the metric g. Thus [**4**, formula (3.19)] shows that

(IV.7) $$(a^{\kappa}_{\alpha})_{\kappa} \in l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s(\mathsf{Q}^m_{\kappa}, \mathcal{L}(E))), \quad |\alpha| \leq 2l.$$

Given any Banach space X, a linear differential operator of order l

$$\mathcal{A} := \mathcal{A}(x,\partial) := \sum_{|\alpha|\leq l} a_{\alpha}(x)\partial^{\alpha}$$

defined on an open subset $U \subset \mathbb{R}^m$ with $a_{\alpha} : U \to \mathcal{L}(X)$ is said to be *normally elliptic* if its principal symbol $\hat{\sigma}\mathcal{A}^{\pi}(x,\xi) := \sum_{|\alpha|=l} a_{\alpha}(x)(-i\xi)^{\alpha}$ satisfies

$$S := \Sigma_{\pi/2} \subset \rho(-\hat{\sigma}\mathcal{A}^{\pi}(x,\xi))$$

and there exists some $\mathcal{C}_\mathfrak{e} > 0$ such that

$$(|\xi|^l + |\mu|)\|(\mu + \hat{\sigma}\mathcal{A}^\pi(x,\xi))^{-1}\|_{\mathcal{L}(X)} \leq \mathcal{C}_\mathfrak{e}, \quad \mu \in S, \tag{IV.8}$$

for all $(x,\xi) \in U \times \dot{\mathbb{R}}^m$, where $\dot{\mathbb{R}}^m := \mathbb{R}^m \setminus \{0\}$. The constant $\mathcal{C}_\mathfrak{e}$ is called the *ellipticity constant* of $\mathcal{A}$. As above, one can check that $\mathcal{A}$ must be of even order.

Proposition IV.3. *A linear differential operator $\mathcal{A} := \mathcal{A}(\mathfrak{a})$ of order $2l$ is normally ρ-elliptic iff all its local realizations*

$$\mathcal{A}_\kappa(x,\partial) = \sum_{|\alpha|\leq 2l} a^\kappa_\alpha(x)\partial^\alpha$$

are normally elliptic on Q^m_κ with a uniform ellipticity constant $\mathcal{C}_\mathfrak{e}$ *in condition* (IV.8).

Proof. We first assume that $\mathcal{A} := \mathcal{A}(\mathfrak{a})$ is *normally ρ-elliptic.* In every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$, by definition we have

$$\hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi) = \sum_{|\alpha|=2l} a^\kappa_\alpha(x)(-i\xi)^\alpha = \psi^*_\kappa \mathsf{C}(a_{2l}, (-i\xi^\mathsf{M})^{\otimes 2l})(\mathsf{p})$$

with $(x,\xi) \in \mathsf{Q}^m_\kappa \times \dot{\mathbb{R}}^m$ and $\mathsf{p} = \psi_\kappa(x)$. Here ξ^M is a 1-form satisfying $\xi^\mathsf{M}|_{\mathsf{O}_\kappa} = \xi_j dx^j$. By [**82**, formula (3.2)] and (IV.3), we conclude $S := \Sigma_{\pi/2} \subset \rho(-\hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi))$. For every $\mu \in S$, $\eta, \varsigma \in E^\sigma_\tau$ with $\varsigma = (\mu + \hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi))\eta$, and $\xi \in \dot{\mathbb{R}}^m$, one computes

$$\begin{aligned}
&(|\xi|^{2l}_{g_m} + |\mu|)|(\mu + \hat{\sigma}\mathcal{A}^\pi(x,\xi))^{-1}\varsigma|_{g_m} = (|\xi|^{2l}_{g_m} + |\mu|)|\eta|_{g_m} \\
&\leq C\rho_\kappa^{\tau-\sigma}(C'\rho^{2l}(\mathsf{p})|\xi^\mathsf{M}(\mathsf{p})|^{2l}_{g^*(\mathsf{p})} + |\mu|)|d\psi_\kappa(x)\eta|_{g(\mathsf{p})} && \text{(IV.9)} \\
&\leq M\rho_\kappa^{\tau-\sigma}(\rho^{2l}(\mathsf{p})|\xi^\mathsf{M}(\mathsf{p})|^{2l}_{g^*(\mathsf{p})} + |\mu|)|d\psi_\kappa(x)\eta|_{g(\mathsf{p})} && \text{(IV.10)} \\
&\leq M\mathcal{C}_\mathfrak{e}\rho_\kappa^{\tau-\sigma}|(\mu + \mathsf{C}(a_{2l}, (-i\xi^\mathsf{M})^{\otimes 2l})(\mathsf{p}))d\psi_\kappa(x)\eta|_{g(\mathsf{p})} && \text{(IV.11)} \\
&\leq M'\mathcal{C}_\mathfrak{e}\rho_\kappa^{\tau-\sigma}\rho_\kappa^{\sigma-\tau}|\psi^*_\kappa(\mu + \mathsf{C}(a_{2l}, (-i\xi^\mathsf{M})^{\otimes 2l})(\mathsf{p}))d\psi_\kappa(x)\eta|_{g_m} && \text{(IV.12)} \\
&= M'\mathcal{C}_\mathfrak{e}|(\mu + \hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi))\eta|_{g_m} = M'\mathcal{C}_\mathfrak{e}|\varsigma|_{g_m}.
\end{aligned}$$

In (IV.9), we have adopted (S4) and (P3). In (IV.10), the constant $M = C\max\{C', 1\}$ is independent of the choices of κ and x. (IV.11) follows from (IV.4), and (IV.12) is a direct consequence of (P3).

The "if" part follows by a similar argument. □

Proposition IV.4. *Let* $\mathfrak{F} \in \{bc, BC\}$, $s \in \mathbb{R}_+ \setminus \mathbb{N}$ *and* $\vartheta \in \mathbb{R}$. *Suppose that* $\mathcal{A} = \mathcal{A}(\mathfrak{a})$ *is a* $2l$*-th order linear differential operator, which is normally* ρ*-elliptic and* s*-regular with bounds* $\mathcal{C}_{\mathfrak{e}}$ *and* $\mathcal{C}_{\mathfrak{a}}$ *defined in* (IV.4) *and* (IV.6). *Then there exist* $\omega = \omega(\mathcal{C}_{\mathfrak{e}}, \mathcal{C}_{\mathfrak{a}})$, $\phi = \phi(\mathcal{C}_{\mathfrak{e}}, \mathcal{C}_{\mathfrak{a}}) > \pi/2$ *and* $\mathcal{E} = \mathcal{E}(\mathcal{C}_{\mathfrak{e}}, \mathcal{C}_{\mathfrak{a}})$ *such that* $S = \omega + \Sigma_\phi \subset \rho(-\mathcal{A})$ *and*

$$|\mu|^{1-i} \|(\mu + \mathcal{A})^{-1}\|_{\mathcal{L}(\mathfrak{F}^{s,\vartheta}(\mathsf{M},V), \mathfrak{F}^{s+2li,\vartheta}(\mathsf{M},V))} \leq \mathcal{E}, \quad \mu \in S, \quad i = 0, 1.$$

Proof. With the convention $b =$ "∞, unif" for $\mathfrak{F} = bc$, and $b = \infty$ for $\mathfrak{F} = BC$, we set

$$E_0 := \mathfrak{F}^{s,\vartheta}, \quad E_\theta := \mathfrak{F}^{s+2l-1,\vartheta}, \quad E_1 := \mathfrak{F}^{s+2l,\vartheta},$$

and

$$l_b^\vartheta(\boldsymbol{E}_0) := l_b^\vartheta(\boldsymbol{\mathfrak{F}}^s), \quad l_b^\vartheta(\boldsymbol{E}_\theta) := l_b^\vartheta(\boldsymbol{\mathfrak{F}}^{s+2l-1}), \quad l_b^\vartheta(\boldsymbol{E}_1) := l_b^\vartheta(\boldsymbol{\mathfrak{F}}^{s+2l}).$$

(i) Define $h : \mathbb{R}^m \to \mathsf{Q}^m$: $x \mapsto \zeta(x)x$. Here ζ is defined in (L2). It is easy to see that $h \in BC^\infty(\mathbb{R}^m, \mathsf{Q}^m)$. Let

$$\bar{\mathcal{A}}_\kappa(x, \partial) := \sum_{|\alpha| \leq 2l} \bar{a}_\alpha^\kappa(x) \partial^\alpha := \sum_{|\alpha| \leq 2l} (a_\alpha^\kappa \circ h)(x) \partial^\alpha.$$

It is not hard to check with the assistance of (IV.7) that the coefficients $(\bar{a}_\alpha^\kappa)_\kappa$ satisfy

$$(\bar{a}_\alpha^\kappa)_\kappa \in l_{\infty,\text{unif}}(\boldsymbol{bc}^s(\mathcal{L}(E))), \quad |\alpha| \leq 2l,$$

and by Proposition IV.3 that $\bar{\mathcal{A}}_\kappa$ are all *normally elliptic* with a uniform *ellipticity constant* for all $\kappa \in \mathfrak{K}$. In virtue of [**2**, Theorems 4.1, 4.2 and Remark 4.6], these two conditions imply the existence of some constants $\omega_0 = \omega_0(\mathcal{C}_{\mathfrak{e}}, \mathcal{C}_{\mathfrak{a}})$, $\phi = \phi(\mathcal{C}_{\mathfrak{e}}, \mathcal{C}_{\mathfrak{a}}) > \pi/2$ and $\mathcal{E} = \mathcal{E}(\mathcal{C}_{\mathfrak{e}}, \mathcal{C}_{\mathfrak{a}})$ such that

(IV.13) $$S_0 := \omega_0 + \Sigma_\phi \subset \rho(-\bar{\mathcal{A}}_\kappa), \quad \kappa \in \mathfrak{K},$$

and

(IV.14) $$|\mu|^{1-i} \|(\mu + \bar{\mathcal{A}}_\kappa)^{-1}\|_{\mathcal{L}(\mathfrak{F}^s(E), \mathfrak{F}^{s+2li}(E))} \leq \mathcal{E}, \quad \mu \in S_0, \quad i = 0, 1, \quad \kappa \in \mathfrak{K}.$$

Let $\bar{\mathcal{A}} : l_b^\vartheta(\boldsymbol{E}_1) \to \boldsymbol{E}$: $[(u_\kappa)_\kappa \mapsto (\bar{\mathcal{A}}_\kappa u_\kappa)_\kappa]$. First, it is not hard to verify by means of the point-wise multiplication results in [**2**, Appendix A2] that

(IV.15) $$\bar{\mathcal{A}} \in \mathcal{L}(l_b^\vartheta(\boldsymbol{E}_1), l_\infty^\vartheta(\boldsymbol{E}_0)).$$

By Proposition II.2 and the well-known interpolation theory, for any $s < t \notin \mathbb{N}$,

$$l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{t+2l}) \overset{d}{\hookrightarrow} l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{s+2l}).$$

Hence for any $\boldsymbol{u} \in l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{s+2l})$, we can choose

$$(\boldsymbol{u}_n)_n := ((u_{n,\kappa})_\kappa)_n \subset l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{t+2l})$$

converging to $\boldsymbol{u}$ in $l^{\vartheta}_{\infty}(\boldsymbol{bc}^{s+2l})$. Since s is arbitrary, we see that the estimate (IV.15) still holds when s is replaced by t, i.e.,

$$\bar{\mathcal{A}}\boldsymbol{u}_n \in l^{\vartheta}_{\infty}(\boldsymbol{bc}^{t}) \hookrightarrow l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{s}).$$

What is more, $\bar{\mathcal{A}}\boldsymbol{u}_n =: \boldsymbol{v}_n \to \bar{\mathcal{A}}\boldsymbol{u}$ in the $l^{\vartheta}_{\infty}(\boldsymbol{BC}^{s})$-norm. Since $l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{s+2l})$ is a Banach space, it yields $\bar{\mathcal{A}}\boldsymbol{u} \in l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{s})$. Therefore

$$\bar{\mathcal{A}} \in \mathcal{L}(l^{\vartheta}_{b}(\boldsymbol{E}_1), l^{\vartheta}_{b}(\boldsymbol{E}_0)). \tag{IV.16}$$

For any $\mu \in S_0$, it is easy to see that $\mu + \bar{\mathcal{A}} : \boldsymbol{\mathfrak{F}}^{s+2l} \to l^{\vartheta}_{\infty}(\boldsymbol{E}_0)$ is a bijective map. We write the inverse of $\mu + \bar{\mathcal{A}}$ as $(\mu + \bar{\mathcal{A}})^{-1}$ and compute for $\boldsymbol{u} := (u_\kappa)_\kappa \in l^{\vartheta}_{b}(\boldsymbol{E}_0)$

$$\begin{aligned}\|(\mu + \bar{\mathcal{A}})^{-1}\boldsymbol{u}\|_{l^{\vartheta}_{\infty}(\boldsymbol{BC}^{s+2l})} &= \sup_{\kappa\in\mathfrak{K}} \rho^{\vartheta}_{\kappa}\|(\mu + \bar{\mathcal{A}}_\kappa)^{-1}u_\kappa\|_{s+2l,\infty} = \sup_{\kappa\in\mathfrak{K}} \|(\mu + \bar{\mathcal{A}}_\kappa)^{-1}\rho^{\vartheta}_{\kappa}u_\kappa\|_{s+2l,\infty} \\ &\leq \mathcal{E} \sup_{\kappa\in\mathfrak{K}} \|\rho^{\vartheta}_{\kappa}u_\kappa\|_{\mathfrak{F}^s(E)} = \mathcal{E}\|\boldsymbol{u}\|_{l^{\vartheta}_{b}(\boldsymbol{E}_0)}.\end{aligned} \tag{IV.17}$$

In the case $\mathfrak{F} = bc$, (IV.17) only shows that for each $\boldsymbol{u} \in l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{s})$ and $\mu \in S$ $(\mu + \bar{\mathcal{A}})^{-1}\boldsymbol{u} \in l^{\vartheta}_{\infty}(\boldsymbol{BC}^{s+2l})$. It remains to prove $(\mu + \bar{\mathcal{A}})^{-1}\boldsymbol{u} \in l^{\vartheta}_{\infty,\mathrm{unif}}(\boldsymbol{E}_1)$. This can be answered by a density argument as in the proof for (IV.16).

Hence $S_0 \subset \rho(-\bar{\mathcal{A}})$. Similarly, one checks

$$|\mu|\|(\mu + \bar{\mathcal{A}})^{-1}\|_{\mathcal{L}(l^{\vartheta}_{b}(\boldsymbol{E}_0))} \leq \mathcal{E}, \quad \mu \in S_0.$$

(ii) Given any $u \in E_1(\mathsf{M}, V)$ and $\mu \in S$, one computes

$$\begin{aligned}
[\mathcal{R}^c_\kappa(\mu + \mathcal{A}) - (\mu + \bar{\mathcal{A}}_\kappa)\mathcal{R}^c_\kappa]u &= \psi^*_\kappa(\pi_\kappa(\mu + \mathcal{A})u) - (\mu + \bar{\mathcal{A}}_\kappa)\psi^*_\kappa(\pi_\kappa u) \\
&= \psi^*_\kappa \pi_\kappa(\mu + \bar{\mathcal{A}}_\kappa)\psi^*_\kappa u - (\mu + \bar{\mathcal{A}}_\kappa)\psi^*_\kappa(\pi_\kappa u) \\
&= \psi^*_\kappa \pi_\kappa \bar{\mathcal{A}}_\kappa \psi^*_\kappa u - \bar{\mathcal{A}}_\kappa \psi^*_\kappa(\pi_\kappa u) \\
&= -\sum_{|\alpha| \le 2l} \sum_{0 < \beta \le \alpha} \binom{\alpha}{\beta} \bar{a}^\kappa_\alpha \partial^{\alpha - \beta}(\zeta \psi^*_\kappa u)\partial^\beta(\psi^*_\kappa \pi_\kappa) =: \mathcal{B}_\kappa u.
\end{aligned}$$

Note that $\zeta \equiv 1$ on $\mathrm{supp}(\psi^*_\kappa \pi_\kappa)$ for all $\kappa \in \mathfrak{K}$. Define for any $u \in C^\infty(\mathsf{M}, V)$

$$\mathcal{B}u := (\mathcal{B}_\kappa u)_\kappa.$$

Similar to the computation for (IV.16), we can easily check

$$\mathcal{B}\mathcal{R} \in \mathcal{L}(l^\vartheta_b(\boldsymbol{E}_\theta), l^\vartheta_b(\boldsymbol{E}_0)).$$

By Proposition II.2, we have

$$l^\vartheta_b(\boldsymbol{E}_\theta) \doteq (l^\vartheta_b(\boldsymbol{E}_0), l^\vartheta_b(\boldsymbol{E}_1))_\theta,$$

where either $(\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty}$ for $\mathfrak{F} = bc$, or $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,\infty}$ for $\mathfrak{F} = BC$, and $\theta = 1 - 1/(2l)$.

It follows from interpolation theory and (II.7) that for every $\varepsilon > 0$ there exists some positive constant $C(\varepsilon)$ such that for all $\boldsymbol{u} \in l^\vartheta_b(\boldsymbol{E}_1)$

$$\|\mathcal{B}\mathcal{R}\boldsymbol{u}\|_{l^\vartheta_b(\boldsymbol{E}_0)} \le \varepsilon \|\boldsymbol{u}\|_{l^\vartheta_b(\boldsymbol{E}_1)} + C(\varepsilon)\|\boldsymbol{u}\|_{l^\vartheta_b(\boldsymbol{E}_0)}$$

Given any $\boldsymbol{u} \in l^\vartheta_b(\boldsymbol{E}_0)$ and $\mu \in S_0$,

$$\begin{aligned}
\|\mathcal{B}\mathcal{R}(\mu + \bar{\mathcal{A}})^{-1}\boldsymbol{u}\|_{l^\vartheta_b(\boldsymbol{E}_0)} \le& \varepsilon \|(\mu + \bar{\mathcal{A}})^{-1}\boldsymbol{u}\|_{l^\vartheta_b(\boldsymbol{E}_1)} + C(\varepsilon)\|(\mu + \bar{\mathcal{A}})^{-1}\boldsymbol{u}\|_{l^\vartheta_b(\boldsymbol{E}_0)} \\
\le& \mathcal{E}(\varepsilon + \frac{C(\varepsilon)}{|\mu|})\|\boldsymbol{u}\|_{l^\vartheta_b(\boldsymbol{E}_0)}.
\end{aligned}$$

Hence we can find some $\omega_1 = \omega_1(\mathcal{C}_\mathfrak{e}, \mathcal{C}_\mathfrak{a}) \ge \omega_0$ such that for all $\mu \in S_1 := \omega_1 + \Sigma_\phi$

$$\|\mathcal{B}\mathcal{R}(\mu + \bar{\mathcal{A}})^{-1}\|_{\mathcal{L}(l^\vartheta_b(\boldsymbol{E}_0))} \le 1/2,$$

which implies that $S_1 \subset \rho(-\bar{\mathcal{A}} - \mathcal{B}\mathcal{R})$ and

$$\|(I + \mathcal{B}\mathcal{R}(\mu + \bar{\mathcal{A}})^{-1})^{-1}\|_{l^\vartheta_b(\boldsymbol{E}_0)} \le 2.$$

Now we compute for any $\boldsymbol{u} \in l_b^\vartheta(\boldsymbol{E}_0)$ and $\mu \in S_1$

$$\begin{aligned}|\mu|\|(\mu+\bar{\mathcal{A}}+\mathcal{B}\mathcal{R})^{-1}\boldsymbol{u}\|_{l_b^\vartheta(\boldsymbol{E}_0)} =&|\mu|\|(\mu+\bar{\mathcal{A}})^{-1}(I+\mathcal{B}\mathcal{R}(\mu+\bar{\mathcal{A}})^{-1})^{-1}\boldsymbol{u}\|_{l_b^\vartheta(\boldsymbol{E}_0)}\\ \leq&\mathcal{E}\|(I+\mathcal{B}\mathcal{R}(\mu+\bar{\mathcal{A}})^{-1})^{-1}\boldsymbol{u}\|_{l_b^\vartheta(\boldsymbol{E}_0)}\\ \leq&2\mathcal{E}\|\boldsymbol{u}\|_{l_b^\vartheta(\boldsymbol{E}_0)},\end{aligned}$$

where $I = \mathrm{id}_{l_b^\vartheta(\boldsymbol{E}_0)}$, and a similar computation yields

$$\|(\mu+\bar{\mathcal{A}}+\mathcal{B}\mathcal{R})^{-1}\boldsymbol{u}\|_{l_b^\vartheta(\boldsymbol{E}_1)} \leq 2\mathcal{E}\|\boldsymbol{u}\|_{l_b^\vartheta(\boldsymbol{E}_0)}.$$

One readily checks

$$\mathcal{R}^c(\mu+\mathcal{A})u = (\mu+\bar{\mathcal{A}})\mathcal{R}^c u + B\mathcal{R}\mathcal{R}^c u = (\mu+\bar{\mathcal{A}}+\mathcal{B}\mathcal{R})\mathcal{R}^c u.$$

For $\mu \in S_1$, we immediately have

$$\mathcal{R}(\mu+\bar{\mathcal{A}}+\mathcal{B}\mathcal{R})^{-1}\mathcal{R}^c(\mu+\mathcal{A}) = \mathcal{R}(\mu+\bar{\mathcal{A}}+\mathcal{B}\mathcal{R})^{-1}(\mu+\bar{\mathcal{A}}+\mathcal{B}\mathcal{R})\mathcal{R}^c = \mathrm{id}_{E_1(\mathsf{M},V)}.$$

Therefore, $\mu+\mathcal{A}$ is injective for $\mu \in S_1$.

(iii) Given $u \in C^\infty(E) := C^\infty(\mathbb{R}^m, E)$, we define

$$\mathcal{C}_\kappa u := [(\mu+\mathcal{A})\mathcal{R}_\kappa - \mathcal{R}_\kappa(\mu+\bar{\mathcal{A}}_\kappa)]u.$$

An easy computation shows that for each $u \in C^\infty(E)$

$$\psi_\kappa^*\mathcal{C}_\kappa u = \sum_{|\alpha|\leq 2l}\bar{a}_\alpha^\kappa\partial^\alpha(\psi_\kappa^*\pi_\kappa u) - \psi_\kappa^*\pi_\kappa(\sum_{|\alpha|\leq 2l}\bar{a}_\alpha^\kappa\partial^\alpha u) = \sum_{|\alpha|\leq 2l}\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\bar{a}_\alpha^\kappa\partial^{\alpha-\beta}(\zeta u)\partial^\beta(\psi_\kappa^*\pi_\kappa).$$

It is obvious that $\mathcal{C}_\kappa \in \mathcal{L}(\mathfrak{F}^{s+2l-1}(E), \mathfrak{F}^s(\mathsf{M},V))$. Moreover, with $\boldsymbol{u} = (u_\kappa)_\kappa$, it is a simple matter to verify as for (IV.16) that

$$[\boldsymbol{u} \mapsto (\psi_\kappa^*\mathcal{C}_\kappa u_\kappa)_\kappa] \in \mathcal{L}(l_b^\vartheta(\boldsymbol{E}_\theta), l_b^\vartheta(\boldsymbol{E}_0)).$$

Define $\mathcal{C} : l_b^\vartheta(\boldsymbol{E}_\theta) \to E_1(\mathsf{M},V)$: $[\boldsymbol{u} \mapsto \sum_\kappa \mathcal{C}_\kappa u_\kappa]$. Then given any $\boldsymbol{u} \in l_b^\vartheta(\boldsymbol{E}_1)$

$$(\mu+\mathcal{A})\mathcal{R}\boldsymbol{u} = \mathcal{R}(\mu+\bar{\mathcal{A}})\boldsymbol{u} + \mathcal{R}\mathcal{R}^c\mathcal{C}\boldsymbol{u} = \mathcal{R}(\mu+\bar{\mathcal{A}}+\mathcal{R}^c\mathcal{C})\boldsymbol{u}.$$

It follows in an analogous way to the proof for Proposition II.1 that

$$[\boldsymbol{u} \mapsto \sum_{\kappa} \varphi_{\kappa}^{*}(\zeta u_{\kappa})] \in \mathcal{L}(l_b^{\vartheta}(\boldsymbol{E}_0), E_0(\mathsf{M}, V)).$$

In view of $\mathcal{C}\boldsymbol{u} = \sum_{\kappa} \varphi_{\kappa}^{*}(\zeta \psi_{\kappa}^{*} \mathcal{C}_{\kappa} u_{\kappa})$, we obtain

$$\mathcal{C} \in \mathcal{L}(l_b^{\vartheta}(\boldsymbol{E}_{\theta}), E_0(\mathsf{M}, V))$$

and thus

$$\mathcal{R}^c \mathcal{C} \in \mathcal{L}(l_b^{\vartheta}(\boldsymbol{E}_{\theta}), l_b^{\vartheta}(\boldsymbol{E}_0)).$$

Now it is not hard to verify via an analogous computation as in (ii) that there exists some $\omega_2 = \omega_2(\mathcal{C}_{\mathfrak{c}}, \mathcal{C}_{\mathfrak{a}}) \geq \omega_1$ such that $S_2 := \omega_2 + \Sigma_{\phi} \subset \rho(-\bar{\mathcal{A}} - \mathcal{R}^c \mathcal{C})$ and

$$|\mu|^{1-i} \|(\mu + \bar{\mathcal{A}} + \mathcal{R}^c \mathcal{C})^{-1}\|_{\mathcal{L}(l_b^{\vartheta}(\boldsymbol{E}_0), l_b^{\vartheta}(\boldsymbol{E}_i))} \leq 2\mathcal{E}, \quad \mu \in S_2, \quad i = 0, 1.$$

Then we have

$$(\mu + \mathcal{A})\mathcal{R}(\mu + \bar{\mathcal{A}} + \mathcal{R}^c \mathcal{C})^{-1} \mathcal{R}^c = \mathcal{R}(\mu + \bar{\mathcal{A}} + \mathcal{R}^c \mathcal{C})(\mu + \bar{\mathcal{A}} + \mathcal{R}^c \mathcal{C})^{-1} \mathcal{R}^c = \mathrm{id}_{E_0(\mathsf{M}, V)}.$$

Thus, $\mu + \mathcal{A}$ is surjective for $\mu \in S_1$, and $\mathcal{R}(\mu + \bar{\mathcal{A}} + \mathcal{R}^c \mathcal{C})^{-1} \mathcal{R}^c$ is a right inverse of $(\mu + \mathcal{A})$. Furthermore,

$$\begin{aligned} &|\mu|^{1-i} \|(\mu + \mathcal{A})^{-1}\|_{\mathcal{L}(E_0(\mathsf{M}, V), E_i(\mathsf{M}, V))} \\ =&|\mu|^{1-i} \|\mathcal{R}(\mu + \bar{\mathcal{A}} + \mathcal{R}^c \mathcal{C})^{-1} \mathcal{R}^c\|_{\mathcal{L}(E_0(\mathsf{M}, V), E_i(\mathsf{M}, V))} \leq C\mathcal{E}, \quad \mu \in S_1, \quad i = 0, 1. \end{aligned}$$

This completes the proof □

Recall that an operator A is said to belong to the class $\mathcal{H}(E_1, E_0)$ for some densely embedded Banach couple $E_1 \overset{d}{\hookrightarrow} E_0$, if $-A$ generates a strongly continuous analytic semigroup on E_0 with $dom(-A) = E_1$. By the well-known semigroup theory, Proposition IV.4 immediately implies

Theorem IV.5. *Let $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\vartheta \in \mathbb{R}$. Suppose $\mathcal{A}$ satisfies the conditions in Proposition IV.4. Then*

$$\mathcal{A} \in \mathcal{H}(bc^{s+2l,\vartheta}(\mathsf{M}, V), bc^{s,\vartheta}(\mathsf{M}, V)).$$

Remark IV.6. Note that the embedding $BC^{s+2l,\vartheta}(\mathsf{M},V) \overset{d}{\hookrightarrow} BC^{s,\vartheta}(\mathsf{M},V)$, in general, does not hold. So we cannot formulate a similar statement to Theorem IV.5 for weighted *Hölder* spaces.

For some fixed interval $I = [0,T]$, $\gamma \in (0,1)$, and some Banach space X, we define

$$BUC_{1-\gamma}(I,X) := \{u \in C(\dot{I},X); [t \mapsto t^{1-\gamma}u] \in C(\dot{I},X), \lim_{t\to 0^+} t^{1-\gamma}\|u(t)\|_X = 0\},$$

$$\|u\|_{C_{1-\gamma}} := \sup_{t\in\dot{I}} t^{1-\gamma}\|u(t)\|_X,$$

and

$$BUC^1_{1-\gamma}(I,X) := \{u \in C^1(\dot{I},X) : u, \dot{u} \in BUC_{1-\gamma}(I,X)\}.$$

Recall that in the above definition $\dot{I} = I \setminus \{0\}$. Moreover, we put

$$BUC_0(I,X) := BUC(I,X) \quad \text{and} \quad BUC^1_0(I,X) := BUC^1(I,X).$$

In addition, if $I = [0,T)$ is a half open interval, then

$$C_{1-\gamma}(I,X) := \{v \in C(\dot{I},X) : v \in BUC_{1-\gamma}([0,t],X), \ \ t < T\},$$

$$C^1_{1-\gamma}(I,X) := \{v \in C^1(\dot{I},X) : v, \dot{v} \in C_{1-\gamma}(I,X)\}.$$

We equip these two spaces with the natural Fréchet topology induced by the topology of the spaces $BUC_{1-\gamma}([0,t],X)$ and $BUC^1_{1-\gamma}([0,t],X)$, respectively.

Assume that $E_1 \overset{d}{\hookrightarrow} E_0$ is a densely embedded Banach couple. Define

$$\mathbb{E}_0(I) := BUC_{1-\gamma}(I,E_0), \quad \mathbb{E}_1(I) := BUC_{1-\gamma}(I,E_1) \cap BUC^1_{1-\gamma}(I,E_0). \tag{IV.18}$$

For $A \in \mathcal{H}(E_1,E_0)$, we say $(\mathbb{E}_0(I),\mathbb{E}_1(I))$ is a pair of *maximal regularity* of A, if

$$(\tfrac{d}{dt} + A, \gamma_0) \in \mathcal{L}\mathrm{is}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\gamma),$$

where γ_0 is the evaluation map at 0, i.e., $\gamma_0(u) = u(0)$, and $E_\gamma := (E_0,E_1)^0_{\gamma,\infty}$. Symbolically, we denote this property by

$$A \in \mathcal{M}_\gamma(E_1,E_0).$$

Now following a well-known theorem by G. Da Prato and P. Grisvard [**65**] and S. Angenent [**9**] and the proof of [**82**, Theorem 3.7], we have

Theorem IV.7. *Let $\gamma \in (0,1]$, $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\vartheta \in \mathbb{R}$. Suppose that $\mathcal{A}$ satisfies the conditions in Proposition IV.4. Then*

$$\mathcal{A} \in \mathcal{M}_\gamma(bc^{s+2l,\vartheta}(\mathsf{M}, V), bc^{s,\vartheta}(\mathsf{M}, V)).$$

Remark IV.8. In order to prove the statement in Theorem IV.7, it suffices to require $(\mathsf{M}, g; \rho)$ to be a $C^{2l+[s]+1}$-*singular manifold*.

Chapter V

Singular equations of second order

Throughout this chapter, we always assume that $(\mathsf{M}, g; \rho)$ is a *singular manifold*, possibly with boundary. Since $W_p(\mathsf{M}, V) = \mathring{W}_p(\mathsf{M}, V)$ when $\partial\mathsf{M} = \emptyset$, in this chapter, we always focus on the space $\mathring{W}_p(\mathsf{M}, V)$.

1. Singular elliptic operators with large potential terms

Let $\sigma, \tau \in \mathbb{N}_0$ and $\lambda' \in \mathbb{R}$. Suppose that $\mathscr{A} : C^\infty(\mathsf{M}, V) \to \Gamma(\mathsf{M}, V)$ is a second order differential operator defined as follows.

$$\mathscr{A}u := -\mathrm{div}(\vec{a} \cdot \mathrm{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u, \tag{V.1}$$

with $\vec{a} \in C^1(\mathsf{M}, T_1^1\mathsf{M})$, $a_1 \in \Gamma(\mathsf{M}, T\mathsf{M})$ and $a_0 \in \mathbb{C}^{\mathsf{M}}$, for any $u \in C^\infty(\mathsf{M}, V)$ and some $\lambda \in \mathbb{R}$. Here $\mathrm{grad} = \mathrm{grad}_g$, $\nabla = \nabla_g$, and $\mathrm{div} = \mathrm{div}_g$. We put for all $\omega \geq 0$

$$\mathscr{A}_\omega u := \mathscr{A}u + \omega\rho^{-\lambda}u.$$

Center contraction $[u \mapsto \vec{a} \cdot \mathrm{grad} u]$ is defined by the relationship

$$\cdot : V_1^1 \times V_\tau^{\sigma+1} \to V_\tau^{\sigma+1} : (a, b) \mapsto a \cdot b,$$

and in every local chart for $\mathsf{p} \in \mathsf{M}$, we have

$$\begin{aligned}(a \cdot b)(\mathsf{p}) &:= \{a_k^l \frac{\partial}{\partial x^l} \otimes dx^k(\mathsf{p})\} \cdot \{b_{(j)}^{(i;h)} \frac{\partial}{\partial x^{(i)}} \otimes \frac{\partial}{\partial x^h} \otimes dx^{(j)}(\mathsf{p})\} \\ &:= a_k^l b_{(j)}^{(i;k)} \frac{\partial}{\partial x^{(i)}} \otimes \frac{\partial}{\partial x^l} \otimes dx^{(j)}(\mathsf{p})\end{aligned}$$

with $(i) \in \mathbb{J}^\sigma$, $(j) \in \mathbb{J}^\tau$ and $l, k, h \in \mathbb{J}^1$. Here we write a differential operator in divergence form, which will benefit us in giving a precise bound for the constant ω.

1.1. L_2-theory. We impose the following assumptions on the coefficients of $\mathscr{A}$ and the compensation term $\omega\rho^{-\lambda}$.

(A1) $\mathscr{A}$ is (ρ,λ)-regular, by which we means that $\vec{a} \in BC^{1,\lambda-2}(\mathsf{M}, T^1_1\mathsf{M})$ is symmetric and

$$a_1 \in L^\lambda_\infty(\mathsf{M}, T\mathsf{M}), \quad a_0 \in L^\lambda_\infty(\mathsf{M}).$$

(A2) $\mathscr{A}$ is (ρ,λ)-singular elliptic. More precisely, there exists some $C_{\hat{\sigma}} > 0$ such that

$$(\vec{a}\cdot\xi|\xi)_{g(\mathsf{p})} \geq C_{\hat{\sigma}}\rho^{2-\lambda}|\xi|^2_g(\mathsf{p}), \quad \xi \in V^{\sigma+1}_\tau, \quad \mathsf{p}\in\mathsf{M}.$$

(A3) $\omega > \omega_{\mathscr{A}}$, where $\omega_{\mathscr{A}} \in \mathbb{R}$ satisfies for some $C_1 < 2$

$$\operatorname{essinf}(\operatorname{Re}(\rho^\lambda a_0) + \omega_{\mathscr{A}}) > 0; \tag{V.2}$$

$$\rho^{\lambda-1}|(2\lambda' + 2\tau - 2\sigma)\vec{a}\cdot\operatorname{grad}\log\rho + a_1|_g \leq C_1\sqrt{C_{\hat{\sigma}}(\operatorname{Re}(\rho^\lambda a_0) + \omega_{\mathscr{A}})}; \tag{V.3}$$

$$\rho^{\lambda-1}|(2\lambda' - \lambda + 2\tau - 2\sigma)\vec{a}\cdot\operatorname{grad}\log\rho + a_1|_g \leq C_1\sqrt{C_{\hat{\sigma}}(\operatorname{Re}(\rho^\lambda a_0) + \omega_{\mathscr{A}})}. \tag{V.4}$$

We may replace the compensation term $\omega\rho^{-\lambda}$ by a largeness condition for the potential term a_0, which can be stated as follows.

(A3') $\operatorname{Re}(\rho^\lambda a_0)$ is so large that there exists some $C_1 < 2$ and $\omega_{\mathscr{A}} < 0$ such that

$$\operatorname{essinf}(\operatorname{Re}(\rho^\lambda a_0) + \omega_{\mathscr{A}}) > 0;$$

$$\rho^{\lambda-1}|(2\lambda' + 2\tau - 2\sigma)\vec{a}\cdot\operatorname{grad}\log\rho + a_1|_g \leq C_1\sqrt{C_{\hat{\sigma}}(\operatorname{Re}(\rho^\lambda a_0) + \omega_{\mathscr{A}})};$$

$$\rho^{\lambda-1}|(2\lambda' - \lambda + 2\tau - 2\sigma)\vec{a}\cdot\operatorname{grad}\log\rho + a_1|_g \leq C_1\sqrt{C_{\hat{\sigma}}(\operatorname{Re}(\rho^\lambda a_0) + \omega_{\mathscr{A}})}.$$

Note that in (A3') only negative values $\omega_{\mathscr{A}}$ are admissible, which is different from (A3).

Throughout, we assume that the singular data $[\![\rho]\!]$ and the constant λ satisfy

$$\begin{cases} \|\rho\|_\infty \leq 1, \quad \lambda \geq 0, \quad \text{or} \\ \|\rho\|_\infty \geq 1, \quad \lambda \leq 0. \end{cases} \tag{V.5}$$

Note that the case $\lambda = 0$ has been studied in [**6**]. The results in [**6**] are similar to those established in Chapter 4. In this case, no restriction for $\|\rho\|_\infty$ is required.

Given $\lambda' \in \mathbb{R}$, let $X := \mathring{W}_2^{1,\lambda'-\lambda/2}(\mathsf{M}, V)$. Then we can associate with $\mathscr{A}_\omega$ a form operator $\mathfrak{a}_\omega$ with $D(\mathfrak{a}_\omega) = X$, defined by

$$\mathfrak{a}_\omega(u,v) = \langle \vec{a} \cdot \mathrm{grad} u | \mathrm{grad} v \rangle_{2,\lambda'} + \langle \mathsf{C}(\nabla u, (2\lambda' + 2\tau - 2\sigma)\vec{a} \cdot \mathrm{grad} \log \rho + a_1) | v \rangle_{2,\lambda'}$$
$$+ \langle (a_0 + \omega \rho^{-\lambda}) u | v \rangle_{2,\lambda'}$$

for all $u, v \in X$. Recall that $\langle \cdot | \cdot \rangle_{2,\lambda'}$ is the inner product in $L_2^{\lambda'}(\mathsf{M}, V)$, see (II.16).

Lemma V.1. *For any $\sigma, \tau, \sigma', \tau' \in \mathbb{N}_0$, it holds that*

(a) $|\vec{a} \cdot \xi|_{g^\tau_{\sigma+1}} \le |\vec{a}|_{g^1_1} |\xi|_{g^\tau_{\sigma+1}}, \quad \xi \in V^{\sigma+1}_\tau.$

(b) $(a|b)_{g^\tau_\sigma} \le |a|_{g^\tau_\sigma} |b|_{g^\tau_\sigma}, \quad a, b \in V^\sigma_\tau.$

(c) $|\mathsf{C}(a,b)|_{g^\tau_\sigma} \le |a|_{g^{\sigma'}_{\tau'}} |b|_{g^{\tau+\tau'}_{\sigma+\sigma'}}, \quad a \in V^{\tau'}_{\sigma'}, b \in V^{\sigma+\sigma'}_{\tau+\tau'}.$

Proof. Statement (a) can be verified via direct computation. Statements (b) and (c) follow from identity (II.2) and [**6**, formula (A5)]. ☐

Proposition V.2. $\mathfrak{a}_\omega$ *is continuous and X-coercive. More precisely,*

(Continuity) there exists some constant M such that for all $u, v \in X$

$$|\mathfrak{a}_\omega(u,v)| \le M \|u\|_X \|v\|_X;$$

(X-Coercivity) for ω large enough, there is some M such that for any $u \in X$

$$\mathrm{Re}(\mathfrak{a}_\omega(u,u)) \ge M \|u\|^2_X.$$

Proof. (i) By [**6**, formula (5.8)], we have

$$\mathrm{grad} \log \rho \in BC^{1,2}(\mathsf{M}, T\mathsf{M}). \qquad \text{(V.6)}$$

Proposition II.4, (A1) and Lemma V.1 then imply that

$$\vec{a} \cdot \mathrm{grad} \log \rho = \mathsf{C}(\vec{a}, \mathrm{grad} \log \rho) \in BC^{1,\lambda}(\mathsf{M}, T\mathsf{M}).$$

For any $u, v \in X$,

$$
\begin{aligned}
& |\mathfrak{a}_\omega(u,v)| \\
\leq & \int\limits_{\mathsf{M}} \rho^{2\lambda'+2\tau-2\sigma} |\vec{a} \cdot \mathrm{grad} u|_g |\mathrm{grad} v|_g \, dV_g \\
& + \int\limits_{\mathsf{M}} \rho^{2\lambda'+2\tau-2\sigma} |(2\lambda' + 2\tau - 2\sigma)\vec{a} \cdot \mathrm{grad}\log\rho + a_1|_g |\nabla u|_g |v|_g \, dV_g \\
& + \int\limits_{\mathsf{M}} \rho^{2\lambda'+2\tau-2\sigma} (\rho^\lambda a_0 + \omega) |\rho^{-\lambda/2} u|_g |\rho^{-\lambda/2} v|_g \, dV_g \\
\leq & \|\rho^{\lambda-2}\vec{a}\|_\infty \Big(\int\limits_{\mathsf{M}} |\rho^{\lambda'+1-\lambda/2+\tau-\sigma} |\nabla u|_g|^2 \, dV_g\Big)^{1/2} \Big(\int\limits_{\mathsf{M}} |\rho^{\lambda'+1-\lambda/2+\tau-\sigma} |\nabla v|_g|^2 \, dV_g\Big)^{1/2} \\
& + \|\rho^{\lambda-1} |(2\lambda' + 2\tau - 2\sigma)\vec{a} \cdot \mathrm{grad}\log\rho + a_1|_g\|_\infty \\
& \Big(\int\limits_{\mathsf{M}} |\rho^{\lambda'+1-\lambda/2+\tau-\sigma} |\nabla u|_g|^2 \, dV_g\Big)^{1/2} \Big(\int\limits_{\mathsf{M}} |\rho^{\lambda'-\lambda/2+\tau-\sigma} |v|_g|^2 \, dV_g\Big)^{1/2} \\
& + \|\rho^\lambda a_0 + \omega\|_\infty \Big(\int\limits_{\mathsf{M}} |\rho^{\lambda'-\lambda/2+\tau-\sigma} u|_g^2 \, dV_g\Big)^{1/2} \Big(\int\limits_{\mathsf{M}} |\rho^{\lambda'-\lambda/2+\tau-\sigma} v|_g^2 \, dV_g\Big)^{1/2} \\
\leq & M(\omega) \|u\|_X \|v\|_X.
\end{aligned}
$$

This proves the continuity of $\mathfrak{a}_\omega$.

(ii) Given any $u \in X$, we have

$$
\begin{aligned}
& \mathrm{Re}(\mathfrak{a}_\omega(u,u)) \\
\geq & C_{\hat{\sigma}} \int\limits_{\mathsf{M}} |\rho^{\lambda'+1-\lambda/2+\tau-\sigma} |\mathrm{grad} u|_g|^2 \, dV_g \\
& - C_1 \int\limits_{\mathsf{M}} \sqrt{C_{\hat{\sigma}} (\mathrm{Re}(\rho^\lambda a_0) + \omega_{\mathscr{A}})} |\rho^{\lambda'-\lambda/2+\tau-\sigma} u|_g |\rho^{\lambda'+1-\lambda/2+\tau-\sigma} \nabla u|_g \, dV_g \\
& + \int\limits_{\mathsf{M}} (\mathrm{Re}(\rho^\lambda a_0) + \omega) |\rho^{\lambda'-\lambda/2+\tau-\sigma} u|_g^2 \, dV_g \\
\geq & (1 - \frac{C_1^2}{4}) C_{\hat{\sigma}} \int\limits_{\mathsf{M}} |\rho^{\lambda'+1-\lambda/2+\tau-\sigma} |\nabla u|_g|^2 \, dV_g + (\omega - \omega_{\mathscr{A}}) \int\limits_{\mathsf{M}} |\rho^{\lambda'-\lambda/2+\tau-\sigma} u|^2 \, dV_g \\
\geq & M(\omega) \|u\|_X^2
\end{aligned}
$$

for all $\omega > \omega_{\mathscr{A}}$ and some $M(\omega) > 0$. In the second line, we have adopted Lemma V.1 and (V.3). $\square$

Proposition V.2 shows that $\mathfrak{a}_\omega$ with $D(\mathfrak{a}_\omega) = X$ is densely defined, sectorial and closed on $L_2^{\lambda'}(\mathsf{M}, V)$. By [**51**, Theorems VI.2.1, IX.1.24], there exists an associated operator T such that $-T$ generates a contractive strongly continuous analytic semigroup on $L_2^{\lambda'}(\mathsf{M}, V)$, i.e., $\|e^{-tT}\|_{\mathcal{L}(L_2^{\lambda'}(\mathsf{M},V))} \leq 1$ for all $t \geq 0$, with domain

$$D(T) := \{u \in X, \exists! v \in L_2^{\lambda'}(\mathsf{M}, V) : \mathfrak{a}_\omega(u, \phi) = \langle v|\phi\rangle_{2,\lambda'}, \forall \phi \in X\}, \quad Tu = v,$$

which is a core of $\mathfrak{a}_\omega$. T is unique in the sense that there exists only one operator satisfying

$$\mathfrak{a}_\omega(u, v) = \langle Tu, v\rangle_{2,\lambda'}, \quad u \in D(T), \, v \in X.$$

On the other hand, by (II.12) and definition (II.15), we can get

$$\langle \mathscr{A}_\omega u|v\rangle_{2,\lambda'} = \mathfrak{a}_\omega(u, v), \quad u, v \in X.$$

So by the uniqueness of T, we have

$$\mathscr{A}_\omega|_{D(T)} = T.$$

Therefore, $-\mathscr{A}_\omega$ generates a contractive strongly continuous analytic semigroup on $L_2^{\lambda'}(\mathsf{M}, V)$ with domain $D(\mathscr{A}_\omega)$:

$$D(\mathscr{A}_\omega) := \{u \in X, \exists! v \in L_2^{\lambda'}(\mathsf{M}, V) : \mathfrak{a}_\omega(u, \phi) = \langle v|\phi\rangle_{2,\lambda'}, \forall \phi \in X\}, \quad \mathscr{A}_\omega u = v.$$

In the rest of this subsection, our aim is to show that $D(\mathscr{A}_\omega) \doteq \mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V)$. Define

$$\mathscr{B}_\omega u := -\mathrm{div}(\rho^\lambda \vec{a} \cdot \mathrm{grad} u) + \mathsf{C}(\nabla u, \rho^\lambda a_1) + (\rho^\lambda a_0 + \omega)u.$$

(A1)-(A2) imply that

$$\rho^\lambda \vec{a} \in BC^{1,-2}(\mathsf{M}, T_1^1\mathsf{M}), \quad \rho^\lambda a_1 \in L_\infty(\mathsf{M}, T\mathsf{M}), \quad \rho^\lambda a_0 \in L_\infty(\mathsf{M}),$$

and

$$(\rho^\lambda \vec{a} \cdot \xi|\xi)_{g(\mathsf{p})} \geq C_{\hat{\sigma}} \rho^2(\mathsf{p}) |\xi|^2_{g(\mathsf{p})}, \quad \xi \in V_\tau^{\sigma+1}, \quad \mathsf{p} \in \mathsf{M}.$$

By [**6**, Theorem 5.2], we obtain

$$\mathscr{B}_\omega \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'-\lambda}(\mathsf{M}, V)). \tag{V.7}$$

Note that although [**6**, Theorem 5.2] is only formulated for scalar functions, this theorem can be easily generalized to arbitrary tensor fields.

For any $u \in \mathcal{D}(\mathsf{M}, V)$, one checks that

$$\rho^{-\lambda}\mathscr{B}_\omega u = \mathscr{A}_\omega u - \lambda \mathsf{C}(\nabla u, \vec{a} \cdot \operatorname{grad} \log \rho) =: \mathscr{A}_\omega u + \mathcal{P}_\lambda u. \tag{V.8}$$

It follows from Propositions II.4, II.6 and (V.6) that

$$\mathcal{P}_\lambda \in \mathcal{L}(\mathring{W}_2^{1,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)).$$

Combining with Proposition II.5, we have

$$\rho^\lambda \mathcal{P}_\lambda \in \mathcal{L}(\mathring{W}_2^{1,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'-\lambda}(\mathsf{M}, V)).$$

Let $\mathfrak{B}_\omega := \mathscr{B}_\omega - \rho^\lambda \mathcal{P}_\lambda$. By well-known perturbation results of analytic semigroups and Definition (II.6), we infer that

$$\mathfrak{B}_\omega \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda}(V), L_2^{\lambda'-\lambda}(V)). \tag{V.9}$$

Then for $\omega > \omega_{\mathscr{A}}$, the previous discussion on $\mathscr{A}_\omega$ and (V.4) show that $-\mathfrak{B}_\omega$ generates a contractive strongly continuous analytic semigroup on $L_2^{\lambda'-\lambda}(V)$. Then, together with (V.7), this implies that for ω sufficiently large,

$$\mathfrak{B}_\omega \in \mathcal{L}\mathrm{is}(D(\mathfrak{B}_\omega), L_2^{\lambda'-\lambda}(V)) \cap \mathcal{L}\mathrm{is}(\mathring{W}_2^{2,\lambda'-\lambda}(V), L_2^{\lambda'-\lambda}(V)).$$

Now we infer that $D(\mathfrak{B}_\omega) \doteq \mathring{W}_2^{2,\lambda'-\lambda}(V)$. Observe that $D(\mathfrak{B}_\omega)$ is invariant for $\omega > \omega_{\mathscr{A}}$. Thus for all $\omega > \omega_{\mathscr{A}}$, the operator $-\mathfrak{B}_\omega$ generates a contractive strongly continuous analytic semigroup on $L_2^{\lambda'-\lambda}(V)$ with domain $\mathring{W}_2^{2,\lambda'-\lambda}(V)$.

Theorem V.3. *Suppose that the differential operator*

$$\mathscr{A} u := -\mathrm{div}(\vec{a} \cdot \mathrm{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u,$$

is (ρ, λ)-regular and (ρ, λ)-singular elliptic, and the constant ω satisfies (A3). Define $\mathscr{A}_\omega := \mathscr{A} + \omega\rho^{-\lambda}$. Then

$$\mathscr{A}_\omega \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)) \cap \mathcal{L}\mathrm{is}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)),$$

and the semigroup $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ *is contractive.*

Proof. By Propositions II.4, II.6, II.11 and Lemma V.1, we obtain

$$\mathscr{A}_\omega \in \mathcal{L}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V), L_2^{\lambda'}(\mathsf{M},V)).$$

This implies together with the definition of $D(\mathscr{A}_\omega)$ that

$$\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V) \hookrightarrow D(\mathscr{A}_\omega).$$

We have shown that for $\omega > \omega_{\mathscr{A}}$,

$$\mathscr{A}_\omega = \rho^{-\lambda}\mathfrak{B}_\omega \in \mathcal{L}\mathrm{is}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V), L_2^{\lambda'}(\mathsf{M},V)). \tag{V.10}$$

Now by (V.10), we can establish

$$D(\mathscr{A}_\omega) \doteq \mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V).$$

The asserted statement thus follows. □

Corollary V.4. *Suppose that* $\mathscr{A}$ *is* (ρ,λ)*-regular and* (ρ,λ)*-singular elliptic, and satisfies (A3'). Then*

$$\mathscr{A} \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V), L_2^{\lambda'}(\mathsf{M},V)) \cap \mathcal{L}\mathrm{is}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V), L_2^{\lambda'}(\mathsf{M},V)),$$

and the semigroup $\{e^{-t\mathscr{A}}\}_{t\geq 0}$ *is contractive.*

1.2. L_p-theory for scalar functions. In this subsection, we assume that $V = \mathbb{C}$. The aim of this subsection is to prove that the differential operator $\mathscr{A}_\omega$ generates a contractive strongly continuous analytic semigroup on $L_p^{\lambda'}(\mathsf{M})$ with $1 < p < \infty$ for large ω.

We first show the following Riesz-Thorin interpolation theorem for the weighted L_p-spaces with $1 \leq p \leq \infty$.

Lemma V.5. *Let* $1 \leq p_0 < p_1 \leq \infty$, $\theta \in (0,1)$, *and* $\vartheta \in \mathbb{R}$. *Define* $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then for every* $f \in L_{p_0}^{\vartheta}(\mathsf{M}) \cap L_{p_1}^{\vartheta}(\mathsf{M})$,

$$\|f\|_{L_{p_\theta}^{\vartheta}} \leq \|f\|_{L_{p_0}^{\vartheta}}^{1-\theta} \|f\|_{L_{p_1}^{\vartheta}}^{\theta}.$$

PROOF. Observe that the operator f_ϑ defined in Proposition II.5 is indeed an isometry from $L_p^\vartheta(\mathsf{M})$ to $L_p(\mathsf{M})$ for $1 \leq p \leq \infty$. Then we have

$$\|f\|_{L^\vartheta_{p_\theta}} = \|\rho^\vartheta f\|_{L_{\mathsf{p}_\theta}} \leq \|\rho^\vartheta f\|^{1-\theta}_{L_{p_0}} \|\rho^\vartheta f\|^\theta_{L_{p_1}} = \|f\|^{1-\theta}_{L^\vartheta_{p_0}} \|f\|^\theta_{L^\vartheta_{p_1}}.$$

□

The adjoint, $\mathscr{A}^*_\omega(\lambda')$, of $\mathscr{A}_\omega$ with respect to $L_2^{\lambda'/2}(\mathsf{M})$ can be easily computed as follows.

$$\mathscr{A}^*_\omega(\lambda')u = -\mathrm{div}(\vec{a}\cdot \mathrm{grad} u) - \mathsf{C}(\nabla u, 2\lambda'\vec{a}\cdot \mathrm{grad}\log\rho + \bar{a}_1) + (b(\lambda', \vec{\boldsymbol{a}}) + \omega\rho^{-\lambda})u,$$

where with $\vec{\boldsymbol{a}} := (\vec{a}, a_1, a_0)$

$$b(\lambda', \vec{\boldsymbol{a}}) := \bar{a}_0 - \mathrm{div}(\lambda'\vec{a}\cdot \mathrm{grad}\log\rho + \bar{a}_1) - \lambda'(\lambda'\vec{a}\cdot \mathrm{grad}\log\rho + \bar{a}_1|\mathrm{grad}\log\rho)_g.$$

Here $\bar{a}_0, \bar{a}_1$ are the complex conjugate of a_0, a_1, and we have used the equality

$$\langle \mathsf{C}(\nabla u, a)|v\rangle_{2,\lambda'/2} = -\langle u|\mathsf{C}(\nabla v, \bar{a})\rangle_{2,\lambda'/2} - \langle u|(\mathrm{div}\bar{a} + \lambda'(\bar{a}|\mathrm{grad}\log\rho)_g)v\rangle_{2,\lambda'/2}$$

for $a \in C^1(\mathsf{M}, T\mathsf{M})$ and $u, v \in \mathcal{D}(\mathring{\mathsf{M}})$.

The adjoint, $\mathscr{A}_\omega(\lambda') := (\mathscr{A}^*_\omega(\lambda'))^*$, of $\mathscr{A}^*_\omega(\lambda')$ with respect to $L_2(\mathsf{M})$ is

$$\mathscr{A}_\omega(\lambda')u = -\mathrm{div}(\vec{a}\cdot \mathrm{grad} u) + \mathsf{C}(\nabla u, 2\lambda'\vec{a}\cdot \mathrm{grad}\log\rho + a_1) + (\tilde{b}(\lambda', \vec{\boldsymbol{a}}) + \omega\rho^{-\lambda})u,$$

where

$$\tilde{b}(\lambda', \vec{\boldsymbol{a}}) := a_0 + \mathrm{div}(\lambda'\vec{a}\cdot \mathrm{grad}\log\rho) - \lambda'(\lambda'\vec{a}\cdot \mathrm{grad}\log\rho + a_1|\mathrm{grad}\log\rho)_g.$$

We impose the following conditions on the compensation term $\omega\rho^{-\lambda}$.

(A4) $\omega > \omega_{\mathscr{A}}$, where $\omega_{\mathscr{A}} \in \mathbb{R}$ satisfies for some $C_1 < 2$

$$\mathrm{essinf}(\mathrm{Re}(\rho^\lambda b(\lambda', \vec{\boldsymbol{a}}) + \omega_{\mathscr{A}}) > 0;$$

$$\rho^{\lambda-1}|2\lambda'\vec{a}\cdot \mathrm{grad}\log\rho + a_1|_g \leq C_1\sqrt{C_{\hat{\sigma}}(\mathrm{Re}(\rho^\lambda b(\lambda', \vec{\boldsymbol{a}})) + \omega_{\mathscr{A}})};$$

$$\mathrm{essinf}(\mathrm{Re}(\rho^\lambda \tilde{b}(\lambda', \vec{\boldsymbol{a}}) + \omega_{\mathscr{A}}) > 0;$$

$$\rho^{\lambda-1}|2\lambda'\vec{a}\cdot \mathrm{grad}\log\rho + a_1|_g \leq C_1\sqrt{C_{\hat{\sigma}}(\mathrm{Re}(\rho^\lambda \tilde{b}(\lambda', \vec{\boldsymbol{a}})) + \omega_{\mathscr{A}})},$$

and

(A5) $\omega > \omega_{\mathscr{A}}$, where $\omega_{\mathscr{A}} \in \mathbb{R}$ satisfies for some $C_1 < 2$

$$\operatorname{essinf}(\operatorname{Re}(\rho^{\lambda}\tilde{b}(\lambda' - \lambda, \vec{\boldsymbol{a}}) + \omega_{\mathscr{A}}) > 0;$$
$$\rho^{\lambda-1}|(2\lambda' - \lambda)\vec{a} \cdot \operatorname{grad} \log \rho + a_1|_g \leq C_1 \sqrt{C_{\hat{\sigma}}(\operatorname{Re}(\rho^{\lambda} b(\lambda', \vec{\boldsymbol{a}})) + \omega_{\mathscr{A}})};$$
$$\rho^{\lambda-1}|(2\lambda' - \lambda)\vec{a} \cdot \operatorname{grad} \log \rho + a_1|_g \leq C_1 \sqrt{C_{\hat{\sigma}}(\operatorname{Re}(\rho^{\lambda}\tilde{b}(\lambda' - \lambda, \vec{\boldsymbol{a}})) + \omega_{\mathscr{A}})}.$$

We can also formulate an analogue of (A3') for the largeness of the potential term a_0 to replace the compensation condition (A4) and (A5).

Then the discussion in Section 5.1.1 and (A4) imply that $-\mathscr{A}_{\omega}^{*}(\lambda')$ and $-\mathscr{A}_{\omega}(\lambda')$ generate contractive strongly continuous analytic semigroups on $L_2(\mathsf{M})$ for all ω satisfying (A4).

Definition V.6. *Let $q \in [1, \infty]$ and $\vartheta \in \mathbb{R}$. A strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on $L_2^{\vartheta}(\mathsf{M})$ is said to be L_q^{ϑ}-contractive if*

$$\|T(t)u\|_{0,q;\vartheta} \leq \|u\|_{0,q;\vartheta}, \quad t \geq 0, \quad u \in L_2^{\vartheta}(\mathsf{M}) \cap L_q^{\vartheta}(\mathsf{M}).$$

Theorem V.7. *Suppose that the differential operator*

$$\mathscr{A} u := -\operatorname{div}(\vec{a} \cdot \operatorname{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u,$$

is (ρ, λ)-regular and (ρ, λ)-singular elliptic. For ω satisfying (A3)-(A5), define $\mathscr{A}_{\omega} := \mathscr{A} + \omega\rho^{-\lambda}$. Then

$$\mathscr{A}_{\omega} \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'}(\mathsf{M})) \cap \mathcal{L}\mathrm{is}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'}(\mathsf{M})), \quad 1 < p < \infty,$$

and the semigroup $\{e^{-t\mathscr{A}_{\omega}}\}_{t \geq 0}$ is contractive.

Proof. (i) By Proposition II.1, it is not hard to verify that $u \in X$ implies $(|u| - 1)^{+}\operatorname{sign} u \in X$ and

$$\nabla((|u| - 1)^{+}\operatorname{sign} u) = \begin{cases} \nabla u, & |u| > 1; \\ 0, & |u| \leq 1. \end{cases} \tag{V.11}$$

Here it is understood that

$$\mathrm{sign} u := \begin{cases} u/|u|, & u \neq 0; \\ 0, & u = 0. \end{cases}$$

Now following a similar proof to step (ii) of Proposition V.2, we get

$$\mathrm{Re}(\mathfrak{a}_\omega(u, (|u|-1)^+ \mathrm{sign} u)) \geq 0, \quad \omega > \omega_{\mathscr{A}}.$$

By [**67**, Theorem 2.7], the semigroup $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ is L_∞-contractive.

Similarly, based on (A4), we can show that $\{e^{-t\mathscr{A}^*_\omega(\lambda')}\}_{t\geq 0}$ and $\{e^{-t\mathscr{A}_\omega(\lambda')}\}_{t\geq 0}$ are L_∞-contractive as well. By a well-known argument, see [**25**, Chapter 1.4], this implies that for each $1 < p < \infty$, $\{e^{-t\mathscr{A}^*_\omega(\lambda')}\}_{t\geq 0}$ and $\{e^{-t\mathscr{A}_\omega(\lambda')}\}_{t\geq 0}$ can be extended to contractive strongly continuous analytic semigroups on $L_p(\mathsf{M})$ with angle

$$\theta_p \geq \theta(1 - |2/p - 1|),$$

where θ is the smaller one of the angles of the semigroups on $L_2(\mathsf{M})$ generated by $\{e^{-t\mathscr{A}^*_\omega(\lambda')}\}_{t\geq 0}$ and $\{e^{-t\mathscr{A}_\omega(\lambda')}\}_{t\geq 0}$.

(ii) Pick $v \in L_2^{\lambda'}(\mathsf{M}) \cap L_1^{\lambda'}(\mathsf{M})$ and $u \in L_2(\mathsf{M}) \cap L_\infty(\mathsf{M})$. We then have

$$\begin{aligned} |\langle e^{-t\mathscr{A}_\omega} v | u \rangle_{2,\lambda'/2}| &= |\langle v | e^{-t\mathscr{A}^*_\omega(\lambda')} u \rangle_{2,\lambda'/2}| = |\langle \rho^{\lambda'} v | e^{-t\mathscr{A}^*_\omega(\lambda')} u \rangle_{2,0}| \\ &\leq \|v\|_{L_1^{\lambda'}} \|e^{-t\mathscr{A}^*_\omega(\lambda')} u\|_{L_\infty} \\ &\leq \|v\|_{L_1^{\lambda'}} \|u\|_{L_\infty}. \end{aligned} \tag{V.12}$$

We have thus established the $L_1^{\lambda'}$-contractivity of the semigroup $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$. It is then an immediate consequence of Lemma V.5 that $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ is $L_p^{\lambda'}$-contractive for $1 \leq p \leq 2$.

(iii) Now we modify a widely used argument, see [**25**, Chapter 1.4], for weighted L_p-spaces. Choose $u \in L_2^{\lambda'}(\mathsf{M})$ with $\mathrm{supp}(u) \subset K$ with $K \subset \mathsf{M}$ satisfying $V_g(K) < \infty$. Then

$$\begin{aligned} \lim_{t\to 0^+} \|\chi_K e^{-t\mathscr{A}_\omega} u\|_{L_1^{\lambda'}} &= \lim_{t\to 0^+} \langle \rho^{-\lambda'} \chi_K | |e^{-t\mathscr{A}_\omega} u| \rangle_{2,\lambda'} \\ &= \langle \rho^{-\lambda'} \chi_K | |u| \rangle_{2,\lambda'} = \|u\|_{L_1^{\lambda'}} \end{aligned} \tag{V.13}$$

by the strong $L_2^{\lambda'}$-continuity of $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$. On the other hand, we also have

$$\|e^{-t\mathscr{A}_\omega}u\|_{L_1^{\lambda'}} \leq \|u\|_{L_1^{\lambda'}}.$$

This together with (V.13) implies that

$$\lim_{t\to 0^+} \|\chi_{\mathsf{M}\setminus K} e^{-t\mathscr{A}_\omega}u\|_{L_1^{\lambda'}} = 0.$$

Now one can compute that

$$\begin{aligned}\lim_{t\to 0^+} \|e^{-t\mathscr{A}_\omega}u - u\|_{L_1^{\lambda'}} &\leq \lim_{t\to 0^+} \|\chi_K(e^{-t\mathscr{A}_\omega}u - u)\|_{L_1^{\lambda'}} \\ &\leq \lim_{t\to 0^+} \|e^{-t\mathscr{A}_\omega}u - u\|_{L_2^{\lambda'}} \mu(K)^{1/2} = 0.\end{aligned}$$

The set of such u contains $\mathcal{D}(\mathsf{M})$ and thus is dense in $L_1^{\lambda'}(\mathsf{M})$. This establishes the strong continuity of $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ on $L_2^{\lambda'}(\mathsf{M}) \cap L_1^{\lambda'}(\mathsf{M})$. Lemma V.5 then implies the strong continuity of $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ on $L_2^{\lambda'}(\mathsf{M}) \cap L_p^{\lambda'}(\mathsf{M})$ for $1 \leq p \leq 2$.

By (II.17), $L_p^{\lambda'}(\mathsf{M})$ is reflexive for $1 < p < \infty$. The strong continuity of $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ on $L_2^{\lambda'}(\mathsf{M}) \cap L_p^{\lambda'}(\mathsf{M})$ for $2 < p < \infty$ now follows from [**45**, Theorem 1.4.9] and the strong continuity of $\{e^{-t\mathscr{A}_\omega^*(\lambda')}\}_{t\geq 0}$ on $L_q(\mathsf{M})$ with $1 < q < 2$.

(iv) Assume that $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ is analytic on $L_2^{\lambda'}(\mathsf{M})$ with angle ϕ. We define

$$H_z := \rho^{\lambda'} e^{-\mathscr{A}_\omega h(z)} \rho^{-\lambda'}, \quad \text{on} \quad S := \{z \in \mathbb{C} : 0 \leq \mathrm{Re} z \leq 1\},$$

where $h(z) := re^{i\theta z}$ with $r > 0$ and $|\theta| < \phi$. Then given any $u \in L_2(\mathsf{M}) \cap L_1(\mathsf{M})$ and $v \in L_2(\mathsf{M}) \cap L_\infty(\mathsf{M})$, we have

$$\begin{aligned}|\langle H_z u|v\rangle_{2,0}| &\leq \|H_z u\|_{L_2}\|v\|_{L_2} \leq \|e^{\mathscr{A}_\omega h(z)}\rho^{-\lambda'}u\|_{L_2^{\lambda'}}\|v\|_{L_2} \\ &\leq \|\rho^{-\lambda'}u\|_{L_2^{\lambda'}}\|v\|_{L_2} \leq \|u\|_{L_2}\|v\|_{L_2}\end{aligned} \tag{V.14}$$

for $z \in S$. Similarly, one can verify that $\langle H_z u|v\rangle_{2,0}$ is continuous on S and analytic inside $\mathring{S}$. Moreover,

$$|\langle H_z u|v\rangle_{2,0}| \leq \|u\|_{L_1}\|v\|_{L_\infty}, \quad \text{if } \mathrm{Re} z = 0.$$

By the Stein interpolation theorem, see [**25**, Section 1.1.6], we conclude that for all $0 < t < 1$, $u \in L_2^{\lambda'}(\mathsf{M}) \cap L_1^{\lambda'}(\mathsf{M})$ and $\frac{1}{p} = 1 - t + \frac{t}{2}$,

$$\|\rho^{\lambda'} u\|_{L_p} \geq \|H_t \rho^{\lambda'} u\|_{L_p} = \|\rho^{\lambda'} e^{-\mathscr{A}_\omega h(t)} u\|_{L_p} = \|e^{-\mathscr{A}_\omega h(t)} u\|_{L_p^{\lambda'}}.$$

Therefore, $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ can be extended to a contractive strongly continuous analytic semigroup on $L_p^{\lambda'}(\mathsf{M})$ with angle $\phi(2-2/p)$ for $1 < p < 2$.

When $2 < p < \infty$, the analytic extension of $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ follows from a duality argument as in (V.12).

(v) In order to determine the domain of $\mathscr{A}_\omega$, we apply a similar discussion to the proof for Theorem V.3. We consider the adjoint, $\mathfrak{B}^*_\omega(\lambda' - \lambda)$, of $\mathfrak{B}_\omega = \mathscr{B}_\omega - \rho^\lambda \mathcal{P}_\lambda$ with respect to $L_2^{(\lambda'-\lambda)/2}(\mathsf{M})$, i.e.,

$$\mathfrak{B}^*_\omega(\lambda' - \lambda)u = -\mathrm{div}(\rho^\lambda \vec{a} \cdot \mathrm{grad} u) - \rho^\lambda \mathsf{C}(\nabla u, (2\lambda' - \lambda)\vec{a} \cdot \mathrm{grad}\log\rho + \bar{a}_1) + \rho^\lambda b(\lambda', \vec{\boldsymbol{a}})u,$$

and the adjoint, $\mathfrak{B}_\omega(\lambda' - \lambda)$, of $\mathfrak{B}^*_\omega(\lambda' - \lambda)$ with respect to $L_2(\mathsf{M})$, i.e.,

$$\begin{aligned}\mathfrak{B}_\omega(\lambda' - \lambda)u = &-\mathrm{div}(\rho^\lambda \vec{a} \cdot \mathrm{grad} u) + \rho^\lambda \mathsf{C}(\nabla u, (2\lambda' - \lambda)\vec{a} \cdot \mathrm{grad}\log\rho + a_1)\\ &+ \rho^\lambda \tilde{b}(\lambda' - \lambda, \vec{\boldsymbol{a}})u.\end{aligned}$$

Following Step (i)-(iv), under Assumptions (A3) and (A5), we can show that $\mathfrak{B}_\omega$ generates a contractive strongly continuous analytic semigroup on $L_p^{\lambda'-\lambda}(\mathsf{M})$ for any $1 < p < \infty$. By [**6**, Theorem 5.2], for ω large enough,

$$\mathfrak{B}_\omega \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'-\lambda}(\mathsf{M})) \cap \mathcal{L}\mathrm{is}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'-\lambda}(\mathsf{M})).$$

An analogous argument to the proof for Theorem V.3 and the discussion prior to this proof yields that

$$\mathscr{A}_\omega = \rho^{-\lambda}\mathfrak{B}_\omega \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'}(\mathsf{M})) \cap \mathcal{L}\mathrm{is}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'}(\mathsf{M})).$$

□

Remark V.8. The proof of L_∞-contractivity for unweighted L_p-spaces in [**67**, Theorem 2.7] suggests that there seems to be a more straightforward way to prove $L_\infty^{\lambda'}$-contractivity.

In fact, we can show that $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$ is $L_\infty^{\lambda'}$-contractive if for any $u \in X$

(i) $(|u| - \rho^{-\lambda'})^+ \mathrm{sign} u \in X$, and

(ii) $\mathrm{Re}\mathfrak{a}_\omega(u, (|u| - \rho^{-\lambda'})^+ \mathrm{sign} u) \geq 0$.

However, Condition (ii), in general, does not hold for all $u \in X$.

Remark V.9. When the tensor field $V \neq \mathbb{C}$, it requires much more effort to establish the L_p-semigroup theory for the differential operator

$$\mathscr{A} u := -\mathrm{div}(\vec{a} \cdot \mathrm{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u.$$

The author is not aware of how to obtain the $L_\infty^{\lambda'}$-contractivity of the semigroup $\{e^{-t\mathscr{A}_\omega}\}_{t\geq 0}$. Instead, one needs to go through the local expressions of $\mathscr{A}_\omega$ and establish a similar contractivity property for these local expressions, and then prove generation of analytic semigroups of the local expressions. However, the drawback of this technique is reflected by the fact that it is hard to determine the precise bound for the constant ω. Indeed, we only know that for ω sufficiently large,

$$\mathscr{A}_\omega \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}, V), L_p^{\lambda'}(\mathsf{M}, V)) \cap \mathcal{L}\mathrm{is}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}, V), L_p^{\lambda'}(\mathsf{M}, V)),$$

for $1 < p < \infty$, and the semigroup $\{e^{-\mathscr{A}_\omega}\}_{t\geq 0}$ is bounded on $L_p^{\lambda'}(\mathsf{M}, V)$. Because it is hard to apply this result, a rigorous proof for this assertion will not be stated in this article.

2. Singular elliptic operators on singular manifolds with $\mathscr{H}_\lambda$-ends

2.1. Differential operators on singular manifolds with *property* $\mathscr{H}_\lambda$. In the first subsection, we will exhibit a technique to remove the "largeness" assumption on the potential term or the compensation term $\omega\rho^{-\lambda}$.

Suppose that $(\mathsf{M}, g; \rho)$ is a *singular manifold*, possibly with boundary. Without loss of generality, we assume that M is connected. Before beginning the discussion of any particular model, we first consider a variant of the operator $\mathscr{A}$ defined in (V.1), i.e.,

$$\mathscr{A} u := -\mathrm{div}(\vec{a} \cdot \mathrm{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u.$$

Put $v := e^{-z\mathfrak{h}}u$ for some $z = a + ib \in \mathbb{C}$ with $|z| = 1$, and $\mathfrak{h} \in C^2(\mathsf{M}, \mathbb{R})$. Then

$$\begin{aligned}\mathscr{A}u =& -\operatorname{div}(\vec{a}\cdot \operatorname{grad}(e^{z\mathfrak{h}}v)) + \mathsf{C}(\nabla e^{z\mathfrak{h}}v, a_1) + e^{z\mathfrak{h}}a_0 v\\ =& -\operatorname{div}(e^{z\mathfrak{h}}\vec{a}\cdot \operatorname{grad}v) - z\operatorname{div}(\vec{a}\cdot(e^{z\mathfrak{h}}v\otimes \operatorname{grad}\mathfrak{h})) + e^{z\mathfrak{h}}\mathsf{C}(\nabla v, a_1)\\ &+ ze^{z\mathfrak{h}}\mathsf{C}(\nabla\mathfrak{h}, a_1)v + e^{z\mathfrak{h}}a_0 v\\ =& e^{z\mathfrak{h}}\{\mathscr{A}v - 2z\mathsf{C}(\nabla v, \vec{a}\cdot \operatorname{grad}\mathfrak{h})\\ &- [z\operatorname{div}(\vec{a}\cdot \operatorname{grad}\mathfrak{h}) + z^2(\vec{a}\cdot \operatorname{grad}\mathfrak{h}|\operatorname{grad}\mathfrak{h})_g - z\mathsf{C}(\nabla\mathfrak{h}, a_1)]v\}.\end{aligned} \tag{V.15}$$

In the sequel, we let $\vec{a} := \rho^{2-\lambda}g_\flat$, which means that we will consider differential operators of the following form

$$\mathscr{A}u := -\operatorname{div}(\rho^{2-\lambda}\operatorname{grad}u) + \mathsf{C}(\nabla u, a_1) + a_0 u$$

with ρ and λ satisfying (V.5). Assume that $\mathscr{A}$ is (ρ,λ)-regular.

Define

$$\mathscr{A}_\mathfrak{h}v := \mathscr{A}v - 2z\rho^{2-\lambda}\mathsf{C}(\nabla v, \operatorname{grad}\mathfrak{h}) - [z\operatorname{div}(\rho^{2-\lambda}\operatorname{grad}\mathfrak{h}) + z^2\rho^{2-\lambda}|\operatorname{grad}\mathfrak{h}|_g^2 - z\mathsf{C}(\nabla\mathfrak{h}, a_1)]v. \tag{V.16}$$

By (V.15), we thus have $\mathscr{A}_\mathfrak{h} = e^{-z\mathfrak{h}} \circ \mathscr{A} \circ e^{z\mathfrak{h}}$.

A function $h \in C^2(\mathsf{M}, \mathbb{R})$ is said to belong to the class $\mathscr{H}_\lambda(\mathsf{M}, g; \rho)$ with parameters (c, M), if

($\mathscr{H}_\lambda 1$) $M/c \le \rho|\operatorname{grad}h|_g \le Mc$;

($\mathscr{H}_\lambda 2$) $M/c \le \rho^\lambda \operatorname{div}(\rho^{2-\lambda}\operatorname{grad}h) \le Mc$.

Observe that if $h \in \mathscr{H}_\lambda(\mathsf{M}, g; \rho)$ with parameters $(c, 1)$, then $Mh \in \mathscr{H}_\lambda(\mathsf{M}, g; \rho)$ with parameters (c, M).

Definition V.10. *A singular manifold* $(\mathsf{M}, g; \rho)$ *is said to enjoy* property $\mathscr{H}_\lambda$*, if there exists some* $h \in \mathscr{H}_\lambda(\mathsf{M}, g; \rho)$.

We impose the following additional assumptions on $(\mathsf{M}, g; \rho)$, and on the constant $z = a + ib$.

(H1) $(\mathsf{M}, g; \rho)$ satisfies *property* $\mathscr{H}_\lambda$. Hence there is an $\mathfrak{h} \in \mathscr{H}_\lambda(\mathsf{M}, g; \rho)$ with parameters (c, M).

(H2) $a = \operatorname{Re}z \in (-\frac{1}{2Mc^3}, 0)$, and $|z| = 1$.

Let $A_z := -z\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}\mathfrak{h}) - z^2\rho^{2-\lambda}|\mathrm{grad}\mathfrak{h}|_g^2 + z\mathsf{C}(a_1, \nabla\mathfrak{h}) + a_0$. By (H1), one can check that the operator $\mathscr{A}_\mathfrak{h}$ is (ρ,λ)-regular and (ρ,λ)-singular elliptic with $C_{\hat{\sigma}} = 1$. Moreover, (H1) implies

$$\rho^2|\mathrm{grad}\mathfrak{h}|_g^2/(Mc^3) \le \rho^\lambda|\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}\mathfrak{h})|.$$

Lemma V.1(c) yields

$$\rho^\lambda|\mathsf{C}(\nabla\mathfrak{h}, a_1)| \le \rho|\mathrm{grad}\mathfrak{h}|_g\|a_1\|_{\infty;\lambda}.$$

Note that (H2) gives $b^2 - a^2 - \frac{a}{Mc^3} > 1$. We then have

$$\begin{aligned} \rho^\lambda\mathrm{Re}(A_z) &= \rho^2(b^2-a^2)|\mathrm{grad}\mathfrak{h}|_g^2 - a\rho^\lambda\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}\mathfrak{h}) + \rho^\lambda(a\mathsf{C}(\nabla\mathfrak{h}, a_1) + \mathrm{Re}(a_0)) \\ &\ge \rho^2(b^2 - a^2 - \frac{a}{Mc^3} - \frac{c}{M}\|a_1\|_{\infty;\lambda})|\mathrm{grad}\mathfrak{h}|_g^2 + \rho^\lambda\mathrm{Re}(a_0) \\ &> C_0\rho^2|\mathrm{grad}\mathfrak{h}|_g^2 - \omega_\mathscr{A} \end{aligned} \tag{V.17}$$

for some $C_0 > 1$ and $\omega_\mathscr{A} < 0$ by choosing M sufficiently large and the real part of z, i.e., a, satisfying (H2) accordingly. This shows that

$$\rho|2z\mathrm{grad}\mathfrak{h}|_g = 2\rho|\mathrm{grad}\mathfrak{h}|_g < \frac{2}{\sqrt{C_0}}\sqrt{\rho^\lambda\mathrm{Re}(A_z) + \omega_\mathscr{A}}.$$

For any $\lambda' \in \mathbb{R}$, let

$$I(\lambda', \lambda, \tau, \sigma) := \{2\lambda' + 2\tau - 2\sigma, 2\lambda' - \lambda + 2\tau - 2\sigma\}.$$

By choosing M large enough and making $z = a + ib$ satisfying (H2), it holds that

$$\begin{aligned} &\rho^{\lambda-1}|-2z\rho^{2-\lambda}\mathrm{grad}\mathfrak{h} + t\rho^{2-\lambda}\mathrm{grad}\log\rho + a_1|_g \\ &< \frac{2}{\sqrt{C_1}}\sqrt{\rho^\lambda\mathrm{Re}(A_z) + \omega_\mathscr{A}} \end{aligned} \tag{V.18}$$

for all $t \in I(\lambda', \lambda, \tau, \sigma)$ and some $\omega_\mathscr{A} < 0$, $C_1 \in (1, C_0)$. Therefore, $\omega_\mathscr{A} < 0$ satisfies (V.2)-(V.4).

We consider the following condition.

(H3) M is sufficiently large such that (V.17) and (V.18) hold.

Summarizing the above discussions, for $z = a + ib$ and M satisfying (H2) and (H3), we conclude from Theorem V.3 with $\omega = 0$ that

$$\mathscr{A}_\mathfrak{h} \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)) \cap \mathcal{L}\mathrm{is}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)), \tag{V.19}$$

and the semigroup $\{e^{-t\mathscr{A}_\mathfrak{h}}\}_{t\geq 0}$ is contractive.

For any function space $\mathfrak{F}^{s,\vartheta}(\mathsf{M},V)$ defined in Section 2.2, the space

$$e^{z\mathfrak{h}}\mathfrak{F}^{s,\vartheta}(\mathsf{M},V):=\{u\in L_{1,loc}(\mathsf{M},V):\ e^{-z\mathfrak{h}}u\in\mathfrak{F}^{s,\vartheta}(\mathsf{M},V)\}$$

is a Banach space equipped with the norm $\|\cdot\|_{e^{z\mathfrak{h}}\mathfrak{F}^{s,\vartheta}}$, where

$$\|u\|_{e^{z\mathfrak{h}}\mathfrak{F}^{s,\vartheta}}:=\|e^{-z\mathfrak{h}}u\|_{\mathfrak{F}^{s,\vartheta}}.$$

It is easy to see that

$$e^{z\mathfrak{h}}\in\mathcal{L}\mathrm{is}(e^{-z\mathfrak{h}}\mathfrak{F}^{s,\vartheta}(\mathsf{M},V),\mathfrak{F}^{s,\vartheta}(\mathsf{M},V)). \tag{V.20}$$

Theorem V.11. *Suppose that* $(\mathsf{M},g;\rho)$ *is a singular manifold with property* $\mathscr{H}_\lambda$, *and we choose* $\mathfrak{h}\in\mathscr{H}_\lambda(\mathsf{M},g;\rho)$ *with parameters* (c,M). *Let* $\lambda'\in\mathbb{R}$, ρ *and* λ *satisfy* (V.5). *Furthermore, assume that the differential operator*

$$\mathscr{A}u:=-\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}u)+\mathsf{C}(\nabla u,a_1)+a_0u$$

is (ρ,λ)*-regular. Then, for any constant* $z=a+ib$ *and* M *fulfilling (H2) and (H3), we have*

$$\mathscr{A}\in\mathcal{H}(e^{z\mathfrak{h}}\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V),e^{z\mathfrak{h}}L_2^{\lambda'}(\mathsf{M},V))\cap\mathcal{L}\mathrm{is}(e^{z\mathfrak{h}}\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V),e^{z\mathfrak{h}}L_2^{\lambda'}(\mathsf{M},V)),$$

and the semigroup $\{e^{-t\mathscr{A}}\}_{t\geq 0}$ *is contractive.*

Proof. (V.19) implies that $S:=\Sigma_\theta\subset\rho(-\mathscr{A}_\mathfrak{h})$ so that

$$|\mu|^{1-k}\|(\mu+\mathscr{A}_\mathfrak{h})^{-1}\|_{\mathcal{L}(L_2^{\lambda'}(\mathsf{M},V),\mathring{W}_2^{2k,\lambda'-k\lambda}(\mathsf{M},V))}\leq\mathcal{E},\quad \mu\in S,\quad k=0,1,$$

for some $\theta\in[\pi/2,\pi)$ and $\mathcal{E}>0$. By (V.20) and $\mathscr{A}=e^{z\mathfrak{h}}\circ\mathscr{A}_\mathfrak{h}\circ e^{-z\mathfrak{h}}$, it holds that $S\subset\rho(-\mathscr{A})$ and for all $\mu\in S$ and $k=0,1$

$$\begin{aligned}&|\mu|^{1-k}\|(\mu+\mathscr{A})^{-1}\|_{e^{z\mathfrak{h}}\mathcal{L}(L_2^{\lambda'}(\mathsf{M},V),e^{z\mathfrak{h}}\mathring{W}_2^{2k,\lambda'-k\lambda}(\mathsf{M},V))}\\&=|\mu|^{1-k}\|(\mu+e^{z\mathfrak{h}}\circ\mathscr{A}_\mathfrak{h}\circ e^{-z\mathfrak{h}})^{-1}\|_{\mathcal{L}(e^{z\mathfrak{h}}L_2^{\lambda'}(\mathsf{M},V),e^{z\mathfrak{h}}\mathring{W}_2^{2k,\lambda'-k\lambda}(\mathsf{M},V))}\leq\mathcal{E}'.\end{aligned}$$

Then the assertion follows from the well-known semigroup theory. ☐

Remark V.12. Because the choice of the constant z and M is not unique, it seems that the assertion in Theorem V.11 is not well formulated.

However, as is shown in Section 5.2.3 below, this is indeed not a problem. In Theorem V.26, we will generalize the result in Theorem V.11 to *singular manifolds* with $\mathscr{H}_\lambda$-ends, which roughly speaking, means that a manifold satisfies *property* $\mathscr{H}_\lambda$ close to the singularities and is uniformly regular elsewhere.

As we will see in Theorem V.27 and Corollary V.29 below, for most of the practical examples, once an $\mathfrak{h} \in \mathscr{H}_\lambda(\mathsf{M}, g; \rho)$ with parameters $(c, 1)$ is fixed, we will see that the space $e^{zM\mathfrak{h}} \mathring{W}_p^{s,\vartheta}(\mathsf{M}, V)$ actually coincides with the weighted Sobolev-Slobodeckii space $\mathring{W}_p^{s,\vartheta+aM}(\mathsf{M}, V)$, for any $z = a + ib$ and M fulfilling (H3) and (H4).

Note that $aM \in (-\frac{1}{2c^3}, 0)$ in fact only depends on the constant c. Since the weight λ' is arbitrary, in Theorem V.11, we actually have that for any λ',

$$\mathscr{A} \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)) \cap \mathcal{L}\mathrm{is}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)).$$

The result in Theorem V.11 thus parallels to those in Section 5.1.

2.2. Singular manifolds with $\mathscr{H}_\lambda$-ends.

Definition V.13. *An m-dimensional singular manifold $(\mathsf{M}, g; \rho)$ is called a singular manifold with $\mathscr{H}_\lambda$-ends if it satisfies the following conditions.*

(i) $\mathbf{G} = \{G_1, \cdots, G_n\}$ *is a finite set of disjoint closed subsets of* M*. Each* $(G_i, g; \rho_i)$ *is an* m*-dimensional singular manifold satisfying property* $\mathscr{H}_\lambda$*.*

(ii) G_0 *is closed in* M*, and* (G_0, g) *is an* m*-dimensional uniformly regular Riemannian manifold.*

(iii) $\{G_0\} \cup \mathbf{G}$ *forms a covering for* M*.* $\partial_0 G_i := G_0 \cap G_i \subset \partial G_0 \cap \partial G_i$*.*

(vi) *Let* $\rho_i := \rho|_{G_i}$*. Either of the following conditions holds true*

$$\rho_i \leq 1, \quad i = 1, \cdots, n; \qquad or \qquad \rho_i \geq 1, \quad i = 1, \cdots, n.$$

G_i *are called the* $\mathscr{H}_\lambda$*-ends of* M*.*

In the following, we will present several examples of *singular manifolds* with $\mathscr{H}_\lambda$-ends, and show how to construct such manifolds in a systematic way.

The proof for the following lemma is straightforward.

Lemma V.14. *Suppose that* $(\mathsf{M}, g; \rho)$ *has* property $\mathscr{H}_\lambda$*,* $\mathfrak{h} \in \mathscr{H}(\mathsf{M}, g; \rho)$ *with parameter* (c, M)*, and* (B, g_B) *is a uniformly regular Riemannian manifold. Then* $(\mathsf{M} \times B, g + g_B; \rho \otimes \mathbf{1}_B)$ *also has* property $\mathscr{H}_\lambda$*, and*

$$\mathfrak{h} \otimes \mathbf{1}_B \in \mathscr{H}_\lambda(\mathsf{M} \times B, g + g_B; \rho \otimes \mathbf{1}_B)$$

with parameter (c, M)*.*

Lemma V.15. *Let* $f : \tilde{\mathsf{M}} \to \mathsf{M}$ *be a diffeomorphism of manifolds. Suppose that* $(\mathsf{M}, g; \rho)$ *has* property $\mathscr{H}_\lambda$*, and* $\mathfrak{h} \in \mathscr{H}(\mathsf{M}, g; \rho)$ *with parameters* (c, M)*.*

Then so does $(\tilde{\mathsf{M}}, f^*g; f^*\rho)$*, and* $f^*\mathfrak{h} \in \mathscr{H}(\tilde{\mathsf{M}}, f^*g; f^*\rho)$ *with parameters* (c, M)*.*

Proof. It is a simple matter to check that $(f^{-1}(\mathsf{O}_\kappa), f^*\varphi_\kappa)_{\kappa \in \mathfrak{K}}$ forms a uniformly regular atlas for $\tilde{\mathsf{M}}$ and

$$(f^*\varphi_\kappa)_* f^*\mathfrak{h} = \psi_\kappa^* \mathfrak{h}, \quad (f^*\varphi_\kappa)_*(f^*g) = \psi_\kappa^* g.$$

As a direct consequence, we have the identities

$$(f^*\varphi_\kappa)_* \mathrm{grad}_{f^*g} f^*\mathfrak{h} = \psi_\kappa^* \mathrm{grad}_g \mathfrak{h},$$

and

$$(f^*\varphi_\kappa)_* \mathrm{div}_{f^*g}((f^*\rho)^{2-\lambda} \mathrm{grad}_{f^*g} f^*\mathfrak{h}) = \mathrm{div}_g(\rho^{2-\lambda} \mathrm{grad}_g \mathfrak{h}).$$

$\square$

The following examples show that we can construct a family of *singular manifolds* with $\mathscr{H}_\lambda$-ends in a great variety of geometric constellations. In particular, we can find manifolds with $\mathscr{H}_\lambda$-type singularities of arbitrarily high dimension.

Let $J_0 := (0, 1]$ as in Chapter 3. We will introduce some subsets of the class $\mathscr{C}(J_0)$, which is very useful for constructing examples of *singular manifolds* with $\mathscr{H}_\lambda$-ends. We call a *cusp characteristic*

$R \in \mathscr{C}(J_0)$ a *mild cusp characteristic* if R satisfies (III.1) and (V.21) below.

$$\dot{R} \sim \mathbf{1}_{J_0}. \tag{V.21}$$

If R further satisfies

$$|\ddot{R}| < \infty, \tag{V.22}$$

then we call it a *uniformly mild cusp characteristic.* We write $R \in \mathscr{C}_{\mathscr{U}}(J_0)$.

Example V.16. *$R(t) = t$, $R(t) = \frac{4}{\pi} \arctan t$, $R(t) = \log(1 + (e-1)t)$, $R(t) = 2t/3 + \sin(\frac{\pi}{2}t)/3$ are examples of uniformly mild cusp characteristics.*

Lemma V.17. *Suppose that $R \in \mathscr{C}_{\mathscr{U}}(J_0)$ and $\lambda \in [0,1) \cup (1,\infty)$. Then $(J_0, dt^2; R)$ is a singular manifold with $\mathscr{H}_\lambda$-end.*

Proof. First, by [**7**, Lemma 5.2], $(J_0, dt^2; R)$ is a *singular manifold.* We set

$$\mathfrak{h}(t) = \operatorname{sign}(1-\lambda) \log R(t). \tag{V.23}$$

Then $R(t)|\dot{\mathfrak{h}}(t)| = \dot{R}(t) \sim \mathbf{1}_{J_0}$ on J_0, and

$$R^\lambda(t) \frac{d}{dt}(R^{2-\lambda}(t)\dot{\mathfrak{h}}(t)) = |1-\lambda||\dot{R}(t)|^2 + \operatorname{sign}(1-\lambda)R(t)\ddot{R}(t) \sim \mathbf{1}_{I_c},$$

where $I_c := (0,c]$ for c small enough. Then the assertion follows from the fact that $([c,1], dt^2)$ is uniformly regular for any $c > 0$. □

Remark V.18. We can actually show that $(J_0, dt^2; R)$ is a *singular manifold* with *property $\mathscr{H}_\lambda$* with

$$\mathfrak{h}(t) := \operatorname{sign}(\lambda - 1) \int_t^1 ds/R(s) \in \mathscr{H}_\lambda(J_0, dt^2; R),$$

as long as R is a *mild cusp characteristic.* But for the sake of practical usage, we will see in Section 5.2.3 below that (V.23) benefits us more in establishing the correspondence of the space $e^{z\mathfrak{h}}\mathring{W}_p^{s,\vartheta}(\mathsf{M}, V)$ with weighted Sobolev-Slobodeckii spaces.

Suppose that $R \in \mathscr{C}_{\mathscr{U}}(J_0)$, (B, g_B) is a uniformly regular Riemannian submanifold of $\mathbb{R}^{d-1}$, and (Γ, g_Γ) is a compact connected Riemannian manifold without boundary. We call (M, g) a *uniformly* *mild* Γ*-wedge over* $P(R,B)$, if there is a diffeomorphism $f : \mathsf{M} \to W(R,B,\Gamma)$ such that $g = f^*(\phi_P^*(dt^2 + g_B) + g_\Gamma)$.

Proposition V.19. *Let* $\lambda \in [0,1) \cup (1,\infty)$*. Assume that* (M, g) *is a uniformly mild* Γ*-wedge over* $P(R,B)$*. Then* (M, g) *is a singular manifold with* $\mathscr{H}_\lambda$*-end.*

Proof. Lemma III.5 implies that

$$(\mathsf{M}, g; f^*(\phi_P^*(R \otimes \mathbf{1}_B) \otimes \mathbf{1}_\Gamma))$$

is a *singular manifold*. We define

$$\mathfrak{h}(t) := \mathrm{sign}(1-\lambda) \log R(t).$$

Put $I_c := (0,c]$ and $\mathsf{M}_c := f^{-1}(P(R|_{I_c}, B) \times \Gamma)$. It follows from Lemmas III.1, V.14, V.15, and V.17 that for $c > 0$ sufficiently small, (M_c, g) has *property* $\mathscr{H}_\lambda$ with

$$f^*(\phi_P^*(\mathfrak{h} \otimes \mathbf{1}_B) \otimes \mathbf{1}_\Gamma) \in \mathscr{H}_\lambda(\mathsf{M}_c, g; f^*(\phi_P^*(R|_{I_c} \otimes \mathbf{1}_B) \otimes \mathbf{1}_\Gamma)).$$

□

Remark V.20. As before, in fact, we only need to require R to be a *mild cusp characteristic*. Let $\mathfrak{h}(t) := \mathrm{sign}(\lambda - 1) \int_t^1 ds/R(s)$. Then (M, g) has *property* $\mathscr{H}_\lambda$ with

$$f^*(\phi_P^*(\mathfrak{h} \otimes \mathbf{1}_B) \otimes \mathbf{1}_\Gamma) \in \mathscr{H}_\lambda(\mathsf{M}, g; f^*(\phi_P^*(R \otimes \mathbf{1}_B) \otimes \mathbf{1}_\Gamma)).$$

Example V.21. *Proposition V.19 implies that the following manifolds enjoy property* $\mathscr{H}_\lambda$.

(a) *A cone manifold* $(\mathsf{M}, g; \rho) = (P(t,B), dt^2 + t^2 g_B; \phi_P^*(t \otimes \mathbf{1}_B))$ *enjoys property* $\mathscr{H}_\lambda$ *for* $\lambda \in [0,1) \cup (1,\infty)$.

(b) *An edge manifold* $(\mathsf{M}, g; \rho) = (P(t,B) \times \mathbb{R}^d, dt^2 + t^2 g_B + g_d; \phi_P^*(t \otimes \mathbf{1}_B) \otimes \mathbf{1}_{\mathbb{R}^d})$ *enjoys property* $\mathscr{H}_\lambda$ *for* $\lambda \in [0,1) \cup (1,\infty)$.

Proposition V.22. *Suppose that* (M, g) *is a singular manifold with holes. More precisely,* $(\mathscr{M}, g)$ *is a uniformly regular Riemannian manifold.* $\mathbf{\Sigma} = \{\Sigma_1, \cdots, \Sigma_k\}$ *is a finite set of disjoint m-dimensional compact manifolds with boundary such that* $\Sigma_j \subset \mathring{\mathscr{M}}$*. Put* $\mathsf{M} := \mathscr{M} \setminus \cup_{j=1}^k \Sigma_j$ *and*

$$\mathscr{B}_{j,r} := \bar{\mathbb{B}}_{\mathscr{M}}(\partial\Sigma_j, r) \cap \mathsf{M}, \quad j = 1, \cdots, k.$$

Then we can find a singularity function ρ satisfying

$$\rho|_{\mathscr{B}_{j,r}} =: \rho_j = \mathrm{dist}_{\mathscr{M}}(\cdot, \partial\Sigma_j),$$

for some $r \in [0, \delta)$, where $\delta < \mathrm{diam}(\mathscr{M})$ fulfils that $\mathscr{B}_{i,\delta} \cap \mathscr{B}_{j,\delta} = \emptyset$ for $i \neq j$, and

$$\rho \sim \mathbf{1}, \quad \textit{elsewhere on } \mathsf{M}.$$

Moreover, $(\mathsf{M}, g; \rho)$ is a singular manifold with $\mathscr{H}_\lambda$-ends for $\lambda \in [0,1) \cup (1, \infty)$.

Proof. By Lemma III.7, (M, g) is a *singular manifold*. We will show that $\rho_j := \mathsf{dist}_{\mathscr{M}}(\cdot, \partial\Sigma_j)$ is a singularity function for $\mathscr{B}_{j,r}$ and

$$\mathfrak{h}_j := \mathrm{sign}(1-\lambda) \log \rho_j \in \mathscr{H}_\lambda(\mathscr{B}_{j,r}, g; \rho_j)$$

for sufficiently small r. By the collar neighborhood theorem, there exists an open neighborhood $\mathscr{V}_{j,\varepsilon}$ of $\partial\Sigma_j$ in the closure of M in $\mathscr{M}$, i.e., $\bar{\mathsf{M}}$, and a diffeomorphism f_j such that

$$f_j : \mathscr{V}_{j,\varepsilon} \to \partial\Sigma_j \times [0, \varepsilon), \quad f_j^* g|_{\mathscr{V}_{j,\varepsilon}} = g|_{\partial\Sigma_j} + dt^2,$$

for some $\varepsilon > 0$. Note that ρ_j is a well defined smooth function in $\mathscr{V}_{j,\varepsilon}$ for ε sufficiently small. Let $T^\perp\partial\Sigma_j$ denote the normal bundle of $\partial\Sigma_j$ in $\bar{\mathsf{M}}$. At every point $\mathsf{p} \in \partial\Sigma_j$, there exists a unique $\nu_{\mathsf{p}} \in T_{\mathsf{p}}^\perp\partial\Sigma_j$ such that

$$T_{\mathsf{p}} f_j \nu_{\mathsf{p}} = e_1 \in T_0\mathbb{R}.$$

Then, $f_j^{-1}(\mathsf{p}, t) = exp_{\mathsf{p}}(t\nu_{\mathsf{p}})$, where exp_{p} is the exponential map at p. Therefore,

$$f_j^* \rho_j(\mathsf{p}, t) := t\beta_j(\mathsf{p}), \quad \text{in } \partial\Sigma_j \times [0, \varepsilon),$$

for some $\beta_j \in C^\infty(\partial\Sigma_j)$ and $\beta_j \sim \mathbf{1}_{\partial\Sigma_j}$. Because of the compactness of $\partial\Sigma_j$, by choosing ε small enough, we can easily show that

$$|\nabla\rho_j|_g \sim \mathbf{1}_{\mathscr{V}_{j,\varepsilon}}, \quad |\Delta\rho_j| < \infty, \quad \text{in } \mathscr{V}_{j,\varepsilon}.$$

Here Δ is the Laplace-Beltrami operator with respect to the metric g defined by $\Delta = \Delta_g :=$ div $\circ$ grad. Since $\mathscr{B}_{j,r} \subset \mathscr{V}_{j,\varepsilon}$ for r small enough, in view of

$$\rho_j \mathrm{grad}\mathfrak{h}_j = \mathrm{sign}(1-\lambda)\mathrm{grad}\rho_j,$$

and

$$\rho_j^{\lambda}\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}\mathfrak{h}_j) = \mathrm{sign}(1-\lambda)\rho_j\Delta\rho_j + |1-\lambda||\mathrm{grad}\rho_j|_g^2,$$

we immediately conclude that $\mathfrak{h}_j$ satisfies $(\mathscr{H}_\lambda 1)$ and $(\mathscr{H}_\lambda 2)$ in $\mathscr{B}_{j,r}$ for r small enough.

Because $|\nabla\rho_j|_g \sim \mathbf{1}$ in $\mathscr{B}_{j,r}$ for r small enough, we can infer from the implicit function theorem that

$$S_{j,r_0} := \{\mathsf{p} \in \mathsf{M} : \mathsf{dist}_{\mathscr{M}}(\mathsf{p}, \partial\Sigma_j) = r_0\} \cap \mathsf{M}$$

is a compact submanifold for some $r_0 \in (0, r)$. By the tubular neighborhood theorem, we can easily show that $(\mathscr{B}_{j,r_0}, g)$ and $(\mathsf{M} \setminus \cup_{j=1}^k \mathring{\mathscr{B}}_{j,r_0}, g)$ are all manifolds with boundary.

By [**7**, Corollary 4.3], $(\partial\Sigma_j, g|_{\partial\Sigma_j})$ is uniformly regular. In particular, taking β_j as a singularity function, $(\partial\Sigma_j, g|_{\partial\Sigma_j}; \beta_j)$ can be considered as a *singular manifold*. By Lemmas V.14 and V.15, we conclude that for r sufficiently small $(\mathscr{B}_{j,r_0}, g; \rho_j)$ is a *singular manifold* with boundary S_{j,r_0}.

Based on the collar neighborhood theorem, we can find an open neighborhood $\mathscr{U}_{j,\varepsilon} \subset \mathscr{B}_{j,r}$ of S_{j,r_0} in $\mathsf{M} \setminus \cup_{j=1}^k \mathring{\mathscr{B}}_{j,r_0}$ such that there is a diffeomorphism

$$\phi_j : \mathscr{U}_{j,\varepsilon} \to S_{j,r_0} \times [0,\varepsilon), \quad \phi_j^* g|_{\mathscr{U}_{j,\varepsilon}} = g|_{S_{j,r_0}} + dt^2,$$

with $\phi_j(S_{j,r_0}) = S_{j,r_0} \times \{0\}$. We choose a function $\xi \in BC^\infty([0,\varepsilon), [0,1])$ such that

$$\xi|_{[0,\varepsilon/4]} \equiv 0, \quad \xi|_{[\varepsilon/2,\varepsilon)} \equiv 1.$$

Put $\xi_{j,0} := \phi_j^*(\mathbf{1}_{S_{j,r_0}} \otimes \xi)$. Similarly, we can find $\xi_{j,j} \in BC^\infty(\mathscr{U}_{j,\varepsilon}, [0,1])$ such that

$$\xi_{j,j}|_{\phi_j^{-1}(S_{j,r_0}\times[0,\varepsilon/2])} \equiv 1, \quad \xi_{j,j}|_{\phi_j^{-1}(S_{j,r_0}\times[3\varepsilon/4,\varepsilon))} \equiv 0.$$

We define $\xi_i \in C^\infty(\mathsf{M}, [0,1])$ with $i = 0, \cdots, k$ as follows. For $j = 1, \cdots, k$,

$$\xi_j(\mathsf{p}) = \begin{cases} 1, & \mathsf{p} \in \mathscr{B}_{j,r_0}, \\ \xi_{j,j}, & \mathsf{p} \in \mathscr{U}_{j,\varepsilon}, \\ 0, & \text{elsewhere}, \end{cases} \quad \text{and} \quad \xi_0(\mathsf{p}) = \begin{cases} 0, & \mathsf{p} \in \mathscr{B}_{j,r_0}, \\ \xi_{j,0}, & \mathsf{p} \in \mathscr{U}_{j,\varepsilon}, \\ 1, & \text{elsewhere}. \end{cases}$$

Put $\rho := \xi_0 \mathbf{1}_{\mathsf{M}} + \sum_{j=1}^k \xi_j \rho_j$. Then it is not hard to see that ρ is a singularity function for (M, g) such that $\rho \sim \mathbf{1}$ on $\mathsf{M} \setminus \cup_{j=1}^k \mathring{\mathscr{B}}_{j,r_0}$ and $\rho|_{\mathscr{B}_{j,r_0}} = \rho_j$. Therefore, $(\mathsf{M} \setminus \cup_{j=1}^k \mathring{\mathscr{B}}_{j,r_0}, g)$ is a *uniformly regular Riemannian manifold.*

Summarizing the above discussions, we have proved that $(\mathsf{M}, g; \rho)$ is a *singular manifold* with $\mathscr{H}_\lambda$-ends. ☐

From the above proof, it is easy to see that the following corollary holds.

Corollary V.23. *Suppose that $(\mathscr{M}, g)$ is a uniformly regular Riemannian manifold with compact boundary. Let $\lambda \in [0,1) \cup (1,\infty)$. Put $\mathsf{M} := \mathring{\mathscr{M}}$ and*

$$\mathscr{B}_r := \bar{\mathbb{B}}_{\mathscr{M}}(\partial\mathscr{M}, r) \cap \mathsf{M}, \quad j = 1, \cdots, k.$$

Then there exists a singularity function ρ satisfying

$$\rho|_{\mathscr{B}_r} =: \rho_j = \mathrm{dist}_{\mathscr{M}}(\cdot, \partial\mathscr{M}),$$

for some $r > 0$, and

$$\rho \sim \mathbf{1}, \quad \textit{elsewhere on } \mathsf{M}.$$

Moreover, $(\mathsf{M}, g; \rho)$ is a singular manifold with $\mathscr{H}_\lambda$-ends.

Remark V.24. More generally, we can take $\mathbf{\Sigma} = \{\Sigma_1, \cdots, \Sigma_k\}$ to be a finite set of disjoint compact closed submanifolds of codimension at least 1 such that $\Sigma_j \subset \partial\mathscr{M}$ if $\Sigma_j \cap \partial\mathscr{M} \neq \emptyset$. In [**7**, Theorem 1.6], it is shown that $\mathsf{M} := \mathscr{M} \setminus \cup_{j=1}^k \Sigma_j$ is a *singular manifold*. Indeed, we can prove that this is a *singular manifold* with $\mathscr{H}_\lambda$-ends. The proof is quite similar to that for Proposition V.22, but more technical. To keep this thesis at a reasonable length, we will not present a proof herein.

Remark V.25. In Proposition V.22, we can also allow $\mathbf{\Sigma} = \{\mathsf{p}_1, \cdots, \mathsf{p}_k\}$ to be a finite set of discrete points in $\mathring{\mathscr{M}}$. Then

$$(\mathsf{M}, g; \rho) := (\mathscr{M} \setminus \cup_{i=1}^k \bar{\mathbb{B}}_{\mathscr{M}}(\mathsf{p}_i, r), g; \rho)$$

is still a *singular manifold*. Here ρ is defined in the same way as in Proposition V.22.

An estimate for $\Delta\rho_j$ can be obtained from the fact that for r sufficiently small

$$\Delta\rho_j(\mathsf{p}) = \frac{m-1}{\rho_j(\mathsf{p})} + O(\rho_j(\mathsf{p})), \quad \text{in } \mathscr{B}_{j,r}.$$

See [**21**, formulas (1.134), (1,159)]. Taking $\mathfrak{h}_j = \log \rho_j$, we have

$$\rho_j^\lambda \mathrm{div}(\rho_j^{2-\lambda} \mathrm{grad}\mathfrak{h}_j) = \rho_j \Delta\rho_j + (1-\lambda)|\mathrm{grad}\rho_j|_g^2 = m - \lambda + O(\rho_j^2),$$

since $|\mathrm{grad}\rho_j|_g = 1$. We immediately have

$$\mathrm{sign}(m-\lambda)\mathfrak{h}_j \in \mathscr{H}_\lambda(\mathscr{B}_{j,r}, g; \rho)$$

for sufficiently small r and $\lambda \geq 0$ with $\lambda \neq m$.

Therefore, $(\mathsf{M}, g; \rho)$ is indeed a *singular manifold* with $\mathscr{H}_\lambda$-ends.

2.3. L_p-theory on singular manifolds with $\mathscr{H}_\lambda$-ends.

Theorem V.26. *Suppose that $(\mathsf{M}, g; \rho)$ is a singular manifold with $\mathscr{H}_\lambda$-ends. Let $\lambda' \in \mathbb{R}$, ρ and λ satisfy* (V.5). *Furthermore, assume that the differential operator*

$$\mathscr{A}u := -\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}u) + \mathsf{C}(\nabla u, a_1) + a_0 u$$

is (ρ, λ)-regular. Then, for any constant $z = a + ib$ and M satisfying (H2) and (H3) on all the $\mathscr{H}_\lambda$-ends G_i with $i = 1, \cdots, n$, we have

$$\mathscr{A} \in \mathcal{H}(e^{z\mathfrak{h}}\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), e^{z\mathfrak{h}}L_2^{\lambda'}(\mathsf{M}, V)).$$

Proof. Without loss of generality, we may assume that $\partial_0 G_i \neq \emptyset$ for $i = 1, \cdots, n$. It is not hard to see that $\partial_0 G_i$ is a component of ∂G_i.

(i) Based on the collar neighborhood theorem, we can find an open neighborhood U_i of $\partial_0 G_i$ in G_i such that there is a diffeomorphism

$$\phi_i : U_i \to \partial_0 G_i \times [0,1), \quad \phi_i^* g|_{U_i} = g|_{\partial_0 G_i} + dt^2,$$

with $\phi_i(\partial_0 G_i) = \partial_0 G_i \times \{0\}$, and

$$\rho_i|_{U_i} \sim \mathbf{1}_{U_i}, \quad i = 1, \cdots, n.$$

We choose functions $\xi, \tilde{\xi} \in BC^\infty([0,1),[0,1])$ such that

$$\xi|_{[0,1/2]} \equiv 1, \quad \xi|_{[3/4,1)} \equiv 0; \quad \tilde{\xi}|_{[0,1/4]} \equiv 0, \quad \tilde{\xi}|_{[1/2,1)} \equiv 1.$$

Set $\hat{\pi}_{i,0} := \phi_i^*(\mathbf{1}_{\partial_0 G_i} \otimes \xi)$ and $\hat{\pi}_{i,i} := \phi_i^*(\mathbf{1}_{\partial_0 G_i} \otimes \tilde{\xi})$. We define $\tilde{\pi}_j \in C^\infty(\mathsf{M}, [0,1])$ with $j = 0, \cdots, n$ as follows. For $i = 1, \cdots, n$,

$$\tilde{\pi}_i(\mathsf{p}) = \begin{cases} 1, & \mathsf{p} \in G_i \setminus U_i, \\ \hat{\pi}_{i,i}, & \mathsf{p} \in U_i, \\ 0, & \text{elsewhere}, \end{cases} \quad \text{and} \quad \tilde{\pi}_0(\mathsf{p}) = \begin{cases} 1, & \mathsf{p} \in G_0, \\ \hat{\pi}_{i,0}, & \mathsf{p} \in U_i, \\ 0, & \text{elsewhere}. \end{cases}$$

For $j = 0, \cdots, n$, we set

$$\pi_j = \frac{\tilde{\pi}_j}{\sqrt{\sum_{i=0}^n \tilde{\pi}_i^2}}.$$

Then $(\pi_j^2)_{j=0}^n$ forms a partition of unity on M, and $\pi_j \in BC^{\infty,0}(\mathsf{M})$.

Put $\hat{G}_0 := G_0 \cup \bigcup_{i=1}^n \bar{U}_i$, which is uniformly regular. Define

$$\mathring{\boldsymbol{W}}_2^{s,\vartheta}(\mathsf{M}, V) := \prod_{j=0}^n \mathring{W}_2^{s,\vartheta}(X_j, V),$$

where $X_j := G_j$ for $j = 1, \cdots, n$, and $X_0 := \hat{G}_0$. It is understood that on X_0, the singularity function can be taken as $\mathbf{1}_{X_0}$, and thus the definition of weighted function spaces on X_0 is independent of the choice of the weight ϑ. We further introduce two maps:

$$\Lambda^c : \mathring{W}_2^{s,\vartheta}(\mathsf{M}, V) \to \mathring{\boldsymbol{W}}_2^{s,\vartheta}(\mathsf{M}, V) : \quad u \mapsto (\pi_j u)_{j=0}^n,$$

and

$$\Lambda : \mathring{\boldsymbol{W}}_2^{s,\vartheta}(\mathsf{M}, V) \to \mathring{W}_2^{s,\vartheta}(\mathsf{M}, V) : \quad (u_j)_{j=0}^n \mapsto \sum_{j=0}^n \pi_j u_j.$$

By Proposition II.4, we immediately conclude that Λ is a retraction from the space $\mathring{\boldsymbol{W}}_2^{s,\vartheta}(\mathsf{M}, V)$ to $\mathring{W}_2^{s,\vartheta}(\mathsf{M}, V)$ with Λ^c as a coretraction.

(ii) We show that there exists some $\mathfrak{h} \in C^2(\mathsf{M})$ such that $\mathfrak{h}_i := \mathfrak{h}|_{G_i} \in \mathscr{H}_\lambda(G_i, g; \rho_i)$ with uniform parameters (c, M) for $i = 1, \cdots, n$, and $\mathfrak{h}_0 := \mathfrak{h}|_{G_0} \in BC^2(G_0)$.

Since G_i has *property* $\mathscr{H}_\lambda$, we can find $\mathfrak{h}_i \in \mathscr{H}_\lambda(G_i, g; \rho_i)$ with uniform parameters (c, M) on all $\mathscr{H}_\lambda$-ends G_i for $i = 1, \cdots, n$. Note that for $u \in C^2(\mathsf{M})$, it follows from [**6**, formula A.9] and (II.12) that

$$|\Delta u| = |C_{\tau+1}^{\sigma+1} \nabla \text{grad} u| = |\nabla \text{grad} u|_g = |\text{grad}^2 u|_g = |\nabla^2 u|_g.$$

Therefore, $(\mathscr{H}_\lambda 1)$ and $(\mathscr{H}_\lambda 2)$ actually imply that $\mathfrak{h}_i \in BC^{2,0}(G_i)$.

Since $\partial_0 G_i$ is a compact submanifold of M, by the tubular neighborhood theorem, we can find an closed neighborhood $\tilde{U}_i$ of $\partial_0 G_i$ in M such that $\tilde{U}_i \cap G_j = \emptyset$ for $j \neq 0, i$, and there is a diffeomorphism

$$\tilde{\phi}_i : \tilde{U}_i \to \partial_0 G_i \times [-1,1], \quad \tilde{\phi}_i^* g|_{\tilde{U}_i} = g|_{\partial_0 G_i} + dt^2,$$

with the convention $\tilde{\phi}_i : \tilde{U}_i \cap G_i \to \partial_0 G_i \times [0,1)$, and $\rho_i|_{\tilde{U}_i} \sim \mathbf{1}_{\tilde{U}_i}$ for $i = 1, \cdots, n$. By a similar construction as in Step (i), we can find $\xi, \tilde{\xi} \in BC^\infty([-1,1],[0,1])$ with

$$\xi|_{[-1,-1/2]} \equiv 1, \quad \xi|_{[-1/4,1]} \equiv 0; \quad \tilde{\xi}|_{[-1,-3/4]} \equiv 0, \quad \tilde{\xi}|_{[-1/2,1]} \equiv 1.$$

Set $\xi_{i,0} := \tilde{\phi}_i^*(\mathbf{1}_{\partial_0 G_i} \otimes \xi)$ and $\xi_{i,i} := \tilde{\phi}_i^*(\mathbf{1}_{\partial_0 G_i} \otimes \tilde{\xi})$. Then we define

$$\xi_i := \begin{cases} \xi_{i,i}, & \text{on } \tilde{U}_i; \\ 1, & \text{on } G_i \setminus \tilde{U}_i; \\ 0, & \text{elsewhere,} \end{cases} \quad \text{and} \quad \xi_0 := \begin{cases} \xi_{i,0}, & \text{on } \tilde{U}_i; \\ 0, & \text{on } G_i \setminus \tilde{U}_i; \\ 1, & \text{elsewhere.} \end{cases}$$

The compactness of $\partial_0 G_i$ and [**7**, Corollary 4.3] imply that $\partial_0 G_i$ is uniformly regular. Therefore, we find for $\partial_0 G_i$ a uniformly regular atlas $\hat{\mathfrak{A}}_i := (\hat{\mathsf{O}}_{\kappa,i}, \hat{\varphi}_{\kappa,i})_{\kappa \in \mathfrak{K}_i}$, and a localization system $(\hat{\pi}_{\kappa,i})_{\kappa \in \mathfrak{K}_i}$. We set

$$\mathsf{O}_{\kappa,i} = \tilde{\phi}_i^{-1}(\hat{\mathsf{O}}_{\kappa,i} \times [-1,1]), \quad \varphi_{\kappa,i} = (\hat{\varphi}_{\kappa,i}, \mathrm{id}) \circ \tilde{\phi}_i,$$

and $\pi_{\kappa,i} := \tilde{\phi}_i^*(\hat{\pi}_{\kappa,i} \otimes \mathbf{1}_{[-1,1]})$. Then $(\pi_{\kappa,i}^2)_{\kappa \in \mathfrak{K}_i}$ forms a partition of unity on $\tilde{U}_i$.

Let $\psi_{\kappa,i} = [\varphi_{\kappa,i}]^{-1}$. We define

$$\mathcal{R}_i^c : BC^k(\mathsf{M}_i) \to \boldsymbol{BC}^k(\mathbb{U}), \quad u \mapsto (\psi_{\kappa,i}^*(\pi_{\kappa,i} u))_{\kappa \in \mathfrak{K}_i},$$

and

$$\mathcal{R}_i : \boldsymbol{BC}^k(\mathbb{U}) \to BC^k(\mathsf{M}_i), \quad (u_\kappa)_{\kappa \in \mathfrak{K}_i} \mapsto \sum_{\kappa \in \mathfrak{K}_i} \psi_{\kappa,i}^*(\pi_{\kappa,i} u_\kappa).$$

Here $\boldsymbol{BC}^k(\mathbb{U}) := \prod_{\kappa \in \mathfrak{K}_i} BC^k(\mathbb{U}_\kappa)$ and

$$\mathbb{U}_\kappa = \begin{cases} \mathbb{R}^{m-1} \times [-1,1], & \text{if } \mathsf{M}_i = \tilde{U}_i; \\ \mathbb{R}^{m-1} \times [0,1], & \text{if } \mathsf{M}_i = \tilde{U}_i \cap G_i. \end{cases}$$

Then alike to Proposition II.1, we can show that $\mathcal{R}_i$ is a retraction from $\boldsymbol{BC}^k(\mathbb{U})$ to $BC^k(\mathsf{M}_i)$ with $\mathcal{R}_i^c$ as a coretraction.

By a well-known extension theorem, there exists a universal extension operator

$$\mathfrak{E} \in \mathcal{L}(BC^k(\mathbb{R}^{m-1} \times [0,1]), BC^k(\mathbb{R}^{m-1} \times [-1,1])).$$

Set $\boldsymbol{\mathfrak{E}} \in \mathcal{L}(\boldsymbol{BC}^k(\mathbb{R}^{m-1} \times [0,1]), \boldsymbol{BC}^k(\mathbb{R}^{m-1} \times [-1,1]))$ and

$$\mathfrak{E}_i := \mathcal{R}_i \circ \boldsymbol{\mathfrak{E}} \circ \mathcal{R}_i^c, \quad i = 1, \cdots, n.$$

Note that $(G_i \cup \tilde{U}_i, g; \rho)$ is a *singular manifold*. Then

$$\mathfrak{E}_i \in \mathcal{L}(BC^{k,0}(G_i), BC^{k,0}(G_i \cup \tilde{U}_i)), \quad i = 1, \cdots, n, \quad k \in \mathbb{N}_0.$$

Here we adopt the convention that $\mathfrak{E}_i u(\mathsf{p}) = u(\mathsf{p})$ for any point $\mathsf{p} \in G_i \setminus \tilde{U}_i$. Put $\tilde{\mathfrak{h}}_i := \mathfrak{E}_i \mathfrak{h}_i$. We thus have $\tilde{\mathfrak{h}}_i \in BC^{2,0}(G_i \cup \tilde{U}_i)$. Now we define

$$\mathfrak{h} = \xi_0 \mathbf{1}_{\mathsf{M}} + \sum_{i=1}^n \xi_i \tilde{\mathfrak{h}}_i.$$

Then $\mathfrak{h} \in C^2(\mathsf{M})$ satisfies the desired properties.

(iii) One can verify that for $j = 0, \cdots, n$ and any $u \in \mathcal{D}(\mathsf{M}, V)$

$$\begin{aligned} \pi_j \mathscr{A}_{\mathfrak{h}} v =& \mathscr{A}_{\mathfrak{h}}(\pi_j v) + 2\rho^{2-\lambda} \mathsf{C}(\nabla v, \mathrm{grad}\pi_j) \\ &+ [\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}\pi_j) - \mathsf{C}(\nabla \pi_j, a_1) + 2z\rho^{2-\lambda}\mathsf{C}(\nabla\pi_j, \mathrm{grad}\mathfrak{h})]v \\ =& : \mathscr{A}_{\mathfrak{h}}(\pi_j v) + \mathcal{B}_j v, \end{aligned} \tag{V.24}$$

where the operator $\mathscr{A}_{\mathfrak{h}}$ is defined in (V.16). Note that $\rho|_{\cup_{j=0}^n \mathrm{supp}(|\nabla \pi_j|_g)} \sim \mathbf{1}$, and thus

$$\mathrm{grad}\pi_j \in BC^{\infty,\vartheta}(\mathsf{M}, T\mathsf{M})$$

for any $\vartheta \in \mathbb{R}$. Based on these observations and Propositions II.4, II.6, II.11, and [**4**, Corollaries 7.2, 12.2], we infer that

$$\mathcal{B}_j \in \mathcal{L}(\mathring{B}_2^{1,\lambda'-\lambda/2}(\mathsf{M}, V), L_2^{\lambda'}(X_j, V)), \quad j = 0, \cdots, n. \tag{V.25}$$

Set $\mathscr{A}_{\mathfrak{h}_j} := \mathscr{A}_{\mathfrak{h}}|_{X_j}$. (V.19) and [**6**, Theorem 5.2] yield

$$\mathscr{A}_{\mathfrak{h}_j} \in \mathcal{H}(\mathring{W}_2^{\lambda'-\lambda}(X_j, V), L_2^{\lambda'}(X_j, V)), \quad j = 0, \cdots, n.$$

Put $\bar{\mathscr{A}_{\mathfrak{h}}} := (\mathscr{A}_{\mathfrak{h}_j})_{j=0}^n$ and

$$E_1 := \mathring{\boldsymbol{W}}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), \quad E_0 := \boldsymbol{L}_2^{\lambda'}(\mathsf{M}, V).$$

Then there exist some $\theta \in [\pi/2, \pi)$, $\omega_0 \geq 0$ and $\mathcal{E} > 0$ such that $S_0 := \omega_0 + \Sigma_\theta \subset \rho(-\bar{\mathscr{A}_{\mathfrak{h}}})$ and

$$|\mu|^{1-k} \|(\mu + \bar{\mathscr{A}_{\mathfrak{h}}})^{-1}\|_{\mathcal{L}(E_0, E_k)} \leq \mathcal{E}, \quad k = 0, 1, \quad \mu \in S_0.$$

Put

$$\mathcal{B} := (\mathcal{B}_j)_{j=0}^n \in \mathcal{L}(\mathring{B}_2^{1,\lambda'-\lambda/2}(\mathsf{M}, V), E_0).$$

From Definition (II.5), it is not hard to show that

$$\mathring{\boldsymbol{B}}_2^{1,\lambda'-\lambda/2}(\mathsf{M}, V) \doteq (E_1, E_0)_{1/2,2}.$$

Then by (V.25), we have

$$\mathcal{B}\Lambda \in \mathcal{L}(\mathring{\boldsymbol{B}}_2^{1,\lambda'-\lambda/2}(\mathsf{M}, V), E_0).$$

Combining with interpolation theory, we infer that for every $\varepsilon > 0$ there exists some positive constant $C(\varepsilon)$ such that for all $\boldsymbol{u} = (u_j)_{j=0}^n \in E_1$

$$\|\mathcal{B}\Lambda \boldsymbol{u}\|_{E_0} \leq \varepsilon \|\boldsymbol{u}\|_{E_1} + C(\varepsilon)\|\boldsymbol{u}\|_{E_0}.$$

Given any $\boldsymbol{u} \in E_0$ and $\mu \in S_0$,

$$\begin{aligned}\|\mathcal{B}\Lambda(\mu + \bar{\mathscr{A}_{\mathfrak{h}}})^{-1}\boldsymbol{u}\|_{E_0} \leq& \varepsilon\|(\mu + \bar{\mathscr{A}_{\mathfrak{h}}})^{-1}\boldsymbol{u}\|_{E_1} + C(\varepsilon)\|(\mu + \bar{\mathscr{A}_{\mathfrak{h}}})^{-1}\boldsymbol{u}\|_{E_0} \\ \leq& \mathcal{E}(\varepsilon + \frac{C(\varepsilon)}{|\mu|})\|\boldsymbol{u}\|_{E_0}.\end{aligned}$$

Hence we can find some $\omega_1 \geq \omega_0$ such that for all $\mu \in S_1 := \omega_1 + \Sigma_\theta$

$$\|\mathcal{B}\Lambda(\mu + \bar{\mathscr{A}_{\mathfrak{h}}})^{-1}\|_{\mathcal{L}(E_0)} \leq 1/2,$$

which implies that $S_1 \subset \rho(-\bar{\mathscr{A}_{\mathfrak{h}}} - \mathcal{B}\Lambda)$ and

$$\|(I + \mathcal{B}\Lambda(\mu + \bar{\mathscr{A}_{\mathfrak{h}}})^{-1})^{-1}\|_{\mathcal{L}(E_0)} \leq 2.$$

Now one can easily verify that

$$|\mu|^{1-k}\|(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}+\mathcal{B}\Lambda)^{-1}\|_{\mathcal{L}(E_0,E_k)}\leq 2\mathcal{E},\quad k=0,1,\quad \mu\in S_1.$$

(iv) (V.24) shows that

$$\Lambda^c(\mu+\mathscr{A}_{\mathfrak{h}})u=(\mu+\bar{\mathscr{A}_{\mathfrak{h}}})\Lambda^c u+\mathcal{B}\Lambda\Lambda^c u=(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}+\mathcal{B}\Lambda)\Lambda^c u.$$

For any $\mu\in S_1$, this yields

$$\Lambda(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}+\mathcal{B}\Lambda)^{-1}\Lambda^c(\mu+\mathscr{A}_{\mathfrak{h}})=\Lambda(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}+\mathcal{B}\Lambda)^{-1}(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}+\mathcal{B}\Lambda)\Lambda^c=\mathrm{id}_{\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M},V)}.$$

This proves the injectivity of $\mu+\mathscr{A}$ for $\mu\in S_1$.

(v) On the other hand, one can also view $\mathcal{B}_j$ as an operator from $\mathring{B}_2^{1,\lambda'-\lambda/2}(X_j,V)$ to $L_2^{\lambda'}(\mathsf{M},V)$. Then

$$\mathcal{B}_j\in\mathcal{L}(\mathring{B}_2^{1,\lambda'-\lambda/2}(X_j,V),L_2^{\lambda'}(\mathsf{M},V)).$$

Let $\mathfrak{B}\boldsymbol{u}:=\sum_{j=0}^n\mathcal{B}_j u_j$ for $\boldsymbol{u}=(u_j)_{j=0}^n$. Following an analogous argument as in (iii), we infer that there exists some $\omega_2\geq\omega_1$ such that $S_2:=\omega_2+\Sigma_\theta\subset\rho(-\bar{\mathscr{A}_{\mathfrak{h}}}+\Lambda^c\mathfrak{B})$ and

$$|\mu|^{1-k}\|(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}-\Lambda^c\mathfrak{B})^{-1}\|_{\mathcal{L}(E_0,E_k)}\leq 2\mathcal{E},\quad k=0,1,\quad \mu\in S_2. \tag{V.26}$$

We further have

$$(\mu+\mathscr{A}_{\mathfrak{h}})\Lambda(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}-\Lambda^c\mathfrak{B})^{-1}\Lambda^c=\Lambda(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}-\Lambda^c\mathfrak{B})(\mu+\bar{\mathscr{A}_{\mathfrak{h}}}-\Lambda^c\mathfrak{B})^{-1}\Lambda^c=\mathrm{id}_{L_2^{\lambda'}(\mathsf{M},V)}.$$

Thus, $\mu+\mathscr{A}$ is surjective for $\mu\in S_2$. Moreover, together with (V.26), we have

$$|\mu|^{1-k}\|(\mu+\mathscr{A}_{\mathfrak{h}})^{-1}\|_{\mathcal{L}(L_2^{\lambda'}(\mathsf{M},V),\mathring{W}_2^{2k,\lambda'-\lambda}(\mathsf{M},V))}\leq\mathcal{E}',\quad k=0,1,\quad \mu\in S_2$$

for some $\mathcal{E}'>0$. Now the asserted statement follows from the well-known semigroup theory and a similar argument to the proof for Theorem V.11. □

The following theorem is the main result of this thesis.

Theorem V.27. *Suppose that* $(\mathsf{M},g;\rho)$ *is a singular manifold satisfying* $\rho\leq 1$,

$$|\nabla\rho|_g\sim\mathbf{1},\quad \|\Delta\rho\|_\infty<\infty$$

on $\mathsf{M}_r := \{\mathsf{p} \in \mathsf{M} : \rho(\mathsf{p}) < r\}$ *for some* $r \in (0,1]$*. Moreover, assume that the set*

$$S_{r_0} := \{\mathsf{p} \in \mathsf{M} : \rho(\mathsf{p}) = r_0\}$$

is compact for $r_0 \in (0, r)$*. Let* $\lambda' \in \mathbb{R}$*, and* $\lambda \in [0,1) \cup (1, \infty)$*.*

(a) *Then* $(\mathsf{M}, g; \rho)$ *is a singular manifold with* $\mathscr{H}_\lambda$*-ends.*

(b) *Furthermore, assume that the differential operator*

$$\mathscr{A} u := -\mathrm{div}(\rho^{2-\lambda} \mathrm{grad} u) + \mathsf{C}(\nabla u, a_1) + a_0 u$$

is (ρ, λ)*-regular. Then*

$$\mathscr{A} \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}, V), L_p^{\lambda'}(\mathsf{M}, V)), \quad 1 < p < \infty.$$

Here $V = \mathbb{C}$ *if* $p \neq 2$*, or* $V = V_\tau^\sigma$ *with* $\sigma, \tau \in \mathbb{N}_0$ *if* $p = 2$*.*

Proof. (i) For $M > 0$, we set

$$\mathfrak{h}(\mathsf{p}) = M \mathrm{sign}(1-\lambda) \log \rho(\mathsf{p}), \quad \mathsf{p} \in \mathsf{M}. \tag{V.27}$$

A direct computation shows that

$$\rho \mathrm{grad} \mathfrak{h} = M \mathrm{sign}(1-\lambda) \mathrm{grad} \rho,$$

and

$$\rho^\lambda \mathrm{div}(r^{2-\lambda} \mathrm{grad} \mathfrak{h}) = M \mathrm{sign}(1-\lambda) \rho \Delta \rho + M|1-\lambda| |\mathrm{grad} \rho|_g^2.$$

Together with (S3) and (S4), one can then easily show that $\mathfrak{h} \in BC^{2,0}(\mathsf{M})$, and

$$\mathfrak{h} \in \mathscr{H}_\lambda(\mathsf{M}_{r_1}, g; \rho) \tag{V.28}$$

with parameters (c, M) for some $r_1 \leq r$ sufficiently small.

By the implicit function function theorem, S_{r_0} is a compact submanifold. Then the assertion that $(\mathsf{M}, g; \rho)$ is a *singular manifold* with $\mathscr{H}_\lambda$-ends is simply a consequence of the tubular neighborhood theorem.

(ii) The retraction-coretraction system defined in the proof for Theorem V.26 allows us to decompose the problem into generation of analytic semigroup on every $\mathscr{H}_\lambda$-end, and then to glue the complete operator together by the perturbation argument used therein.

We thus can reduce the assumptions on the manifold $(\mathsf{M}, g; \rho)$ to only assuming $(\mathsf{M}, g; \rho)$ to be a *singular manifold* with *property* $\mathscr{H}_\lambda$, and *property* $\mathscr{H}_0$ if $\lambda \neq 0$. Moreover,

$$\mathfrak{h} = M \mathrm{sign}(1-\lambda) \log \rho \in \mathscr{H}_\lambda(\mathsf{M}, g; \rho), \quad \mathfrak{h}_0 = M \log \rho \in \mathscr{H}_0(\mathsf{M}, g; \rho),$$

both with parameter (c, M).

The reason to include the extra assumption that $(\mathsf{M}, g; \rho)$ has *property* $\mathscr{H}_0$ will be self-explanatory in Step (v) below, while we determine the domain of the $L_p^{\lambda'}(\mathsf{M})$-realization of the operator $\mathscr{A}$.

(iii) Take $\mathfrak{h}$ as in (V.27) and $z = a + ib$, M satisfying (H2) and (H3) in Section 5.2.1. In Theorem V.26, we have shown that

$$\mathscr{A} \in \mathcal{H}(e^{z\mathfrak{h}} \mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), e^{z\mathfrak{h}} L_2^{\lambda'}(\mathsf{M}, V)).$$

We have $e^{z\mathfrak{h}} = \rho^{\mathrm{sign}(1-\lambda) zM} = \rho^{\mathrm{sign}(1-\lambda) aM} \rho^{\mathrm{sign}(1-\lambda) bMi}$. By (V.6) and Proposition II.8, we infer that

$$\nabla \log \rho \in BC^{1,0}(\mathsf{M}, T^*M),$$

which implies

$$\nabla \rho^{\mathrm{sign}(1-\lambda) bMi} = \mathrm{sign}(1-\lambda) bMi \rho^{\mathrm{sign}(1-\lambda) bMi} \nabla \log \rho \in BC^{1,0}(\mathsf{M}, T^*M).$$

Combining with $|\rho^{\mathrm{sign}(1-\lambda) bMi}| \equiv 1$, we thus have

$$\rho^{\mathrm{sign}(1-\lambda) bMi} \in BC^{2,0}(\mathsf{M}).$$

By Propositions II.4, II.5 and the fact that $e^{z\mathfrak{h}} e^{-z\mathfrak{h}} = e^{-z\mathfrak{h}} e^{z\mathfrak{h}} = \mathbf{1}_{\mathsf{M}}$, we infer that

$$e^{z\mathfrak{h}} \in \mathcal{L}\mathrm{is}(\mathring{W}_p^{s,\vartheta}(\mathsf{M}, V), \mathring{W}_p^{s,\vartheta + \mathrm{sign}(\lambda-1) aM}(\mathsf{M}, V)), \quad 1 < p < \infty, \quad 0 \leq s \leq 2.$$

A similar argument to Theorem V.11 yields

$$\mathscr{A} \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda+\mathrm{sign}(\lambda-1) aM}(\mathsf{M}, V), L_2^{\lambda'+\mathrm{sign}(\lambda-1) aM}(\mathsf{M}, V)).$$

Since λ' is arbitrary and $\text{sign}(\lambda-1)aM \in (-1/2c^3, 1/2c^3)$, it implies that

$$\mathscr{A} \in \mathcal{H}(\mathring{W}_2^{2,\lambda'-\lambda}(\mathsf{M}, V), L_2^{\lambda'}(\mathsf{M}, V)), \quad \lambda' \in \mathbb{R}.$$

(iv) Now we look at the general case $1 < p < \infty$ and suppose that $V = \mathbb{C}$. Recall that the adjoint, $\mathscr{A}^*(\vartheta)$, of $\mathscr{A}$ with respect to $L_2^{\vartheta/2}(\mathsf{M})$ is

$$\mathscr{A}^*(\vartheta)u = -\text{div}(\rho^{2-\lambda}\text{grad}u) - \mathsf{C}(\nabla u, 2\vartheta\rho^{2-\lambda}\text{grad}\log\rho + a_1) + b(\vartheta, \vec{\boldsymbol{a}})u,$$

where with $\vec{\boldsymbol{a}} = (\vec{a}, a_1, a_0)$ and

$$b(\vartheta, \vec{\boldsymbol{a}}) := \bar{a}_0 - \text{div}(\vartheta\rho^{2-\lambda}\text{grad}\log\rho + a_1) - \vartheta(\vartheta\rho^{2-\lambda}\text{grad}\log\rho + a_1|\text{grad}\log\rho)_g.$$

To simplify our usage of notation in the following computations, we first focus on the case $\lambda > 1$. The remaining case follows easily by symmetry. Recall that when $\lambda > 1$, we can set

$$\mathfrak{h}(\mathsf{p}) = -M\log\rho(\mathsf{p}), \quad \mathsf{p} \in \mathsf{M}, \quad M > 0.$$

Let $\mathscr{A}_{\mathfrak{h}}^*(\vartheta) := e^{-z\mathfrak{h}} \circ \mathscr{A}^*(\vartheta) \circ e^{z\mathfrak{h}}$. Since $\mathscr{A}^*(\vartheta)$ is (ρ,λ)-regular and (ρ,λ)-singular elliptic, by choosing $z = z(\vartheta) = a + ib$ and $M = M(\vartheta)$ satisfying (H2) and (H3), we have

$$\mathscr{A}_{\mathfrak{h}}^*(\vartheta) \in \mathcal{H}(\mathring{W}_2^{2,-\lambda}(\mathsf{M}), L_2(\mathsf{M})) \cap \mathcal{L}\text{is}(\mathring{W}_2^{2,-\lambda}(\mathsf{M}), L_2(\mathsf{M})).$$

We have thus established

$$\mathscr{A}^*(\vartheta) \in \mathcal{H}(\mathring{W}_2^{2,aM-\lambda}(\mathsf{M}), L_2^{aM}(\mathsf{M})) \cap \mathcal{L}\text{is}(\mathring{W}_2^{2,aM-\lambda}(\mathsf{M}), L_2^{aM}(\mathsf{M})),$$

and the semigroup $\{e^{-t\mathscr{A}^*(\vartheta)}\}_{t\geq 0}$ is contractive. Note that

$$aM \in (-1/2c^3, 0) \subset (-1, 0)$$

only depends on c. Henceforth, we always take $\alpha := aM = -1/4c^3$.

For the adjoint, $\mathscr{A}(\vartheta; 2\alpha)$, of $\mathscr{A}^*(\vartheta)$ with respect to $L_2^{\alpha}(\mathsf{M})$, we can show similarly that

$$\mathscr{A}(\vartheta; 2\alpha) \in \mathcal{H}(\mathring{W}_2^{2,\alpha-\lambda}(\mathsf{M}), L_2^{\alpha}(\mathsf{M})) \cap \mathcal{L}\text{is}(\mathring{W}_2^{2,\alpha-\lambda}(\mathsf{M}), L_2^{\alpha}(\mathsf{M})),$$

and $\{e^{-t\mathscr{A}(\vartheta;2\alpha)}\}_{t\geq 0}$ is contractive. Let $\mathscr{A}_{\mathfrak{h}}(\vartheta; 2\alpha) = e^{-z\mathfrak{h}} \circ \mathscr{A}(\vartheta; 2\alpha) \circ e^{z\mathfrak{h}}$.

The L_∞-contractivity of $\{e^{-t\mathscr{A}_\mathfrak{h}}\}_{t\geq 0}$, $\{e^{-t\mathscr{A}_\mathfrak{h}^*(\vartheta)}\}_{t\geq 0}$ and $\{e^{-t\mathscr{A}_\mathfrak{h}(\vartheta;2\alpha)}\}_{t\geq 0}$ can be built up by a similar argument to Section 5.1.2. It yields for any $u\in L_2^\alpha(\mathsf{M})\cap L_\infty^\alpha(\mathsf{M})$

$$\begin{aligned}\|e^{-t\mathscr{A}}u\|_{L_\infty^\alpha} &= \|e^{-t\mathscr{A}}e^{z\mathfrak{h}}e^{-z\mathfrak{h}}u\|_{L_\infty^\alpha}\\ &= \|e^{z\mathfrak{h}}e^{-t\mathscr{A}_\mathfrak{h}}e^{-z\mathfrak{h}}u\|_{L_\infty^\alpha}\\ &\leq \|e^{-t\mathscr{A}_\mathfrak{h}}e^{-z\mathfrak{h}}u\|_{L_\infty}\leq \|e^{-z\mathfrak{h}}u\|_{L_\infty}\leq \|u\|_{L_\infty^\alpha}.\end{aligned}\tag{V.29}$$

(V.29) follows from $\mathscr{A}_\mathfrak{h}=e^{-z\mathfrak{h}}\circ\mathscr{A}\circ e^{z\mathfrak{h}}$ and

$$\begin{aligned}e^{-z\mathfrak{h}}e^{-t\mathscr{A}}e^{z\mathfrak{h}}v &= e^{-z\mathfrak{h}}\lim_{n\to\infty}[\frac{n}{t}(\frac{n}{t}+\mathscr{A})^{-1}]^n e^{z\mathfrak{h}}v=\lim_{n\to\infty}e^{-z\mathfrak{h}}e^{z\mathfrak{h}}[\frac{n}{t}e^{-z\mathfrak{h}}(\frac{n}{t}+\mathscr{A})^{-1}e^{z\mathfrak{h}}]^n v\\ &=\lim_{n\to\infty}[\frac{n}{t}(\frac{n}{t}+e^{-z\mathfrak{h}}\circ\mathscr{A}\circ e^{z\mathfrak{h}})^{-1}]^n v=\lim_{n\to\infty}[\frac{n}{t}(\frac{n}{t}+\mathscr{A}_\mathfrak{h})^{-1}]^n u=e^{-t\mathscr{A}_\mathfrak{h}}v.\end{aligned}$$

A similar argument applies to $\{e^{-t\mathscr{A}^*(\vartheta)}\}_{t\geq 0}$ and $\{e^{-t\mathscr{A}(\vartheta;2\alpha)}\}_{t\geq 0}$ as well. Thus we have established the L_∞^α-contractivity of the semigroups $\{e^{-t\mathscr{A}}\}_{t\geq 0}$, $\{e^{-t\mathscr{A}^*(\vartheta)}\}_{t\geq 0}$, and $\{e^{-t\mathscr{A}(\vartheta;2\alpha)}\}_{t\geq 0}$.

Now we make use of the duality argument in Step (ii) of the proof for Theorem V.7 again. For any $u\in L_2^{\vartheta-\alpha}(\mathsf{M})\cap L_1^{\vartheta-\alpha}(\mathsf{M})$ and $v\in L_2^\alpha(\mathsf{M})\cap L_\infty^\alpha(\mathsf{M})$, it holds

$$\begin{aligned}|\langle e^{-t\mathscr{A}}v|u\rangle_{2,\vartheta/2}| &= |\langle v|e^{-t\mathscr{A}^*}u\rangle_{2,\vartheta/2}|=|\langle\rho^\vartheta v|e^{-t\mathscr{A}^*}u\rangle_{2,0}|\\ &\leq \|\rho^\vartheta v\|_{L_1^{-\alpha}}\|e^{-t\mathscr{A}^*}u\|_{L_\infty^\alpha}\\ &\leq \|v\|_{L_1^{\vartheta-\alpha}}\|u\|_{L_\infty^\alpha}.\end{aligned}$$

Taking $\vartheta=\lambda'+\alpha$, the above inequality proves that $\{e^{-t\mathscr{A}}\}_{t\geq 0}$ is indeed $L_1^{\lambda'}$-contractive. Applying this duality argument to $\{e^{-t\mathscr{A}^*(\vartheta)}\}_{t\geq 0}$ and $\{e^{-t\mathscr{A}(\vartheta;2\alpha)}\}_{t\geq 0}$ repeatedly with respect to $\langle\cdot|\cdot\rangle_{2,\alpha}$, we can then obtain the L_1^α-contractivity of these two semigroups. Similarly, we have

$$|\langle e^{-t\mathscr{A}}v|u\rangle_{2,\vartheta/2}|\leq \|v\|_{L_\infty^{\vartheta-\alpha}}\|u\|_{L_1^\alpha}.$$

Hence, by Lemma V.5 $\{e^{-t\mathscr{A}}\}_{t\geq 0}$ is indeed $L_p^{\lambda'}$-contractive for all $1\leq p\leq\infty$. After carefully following the proof for Theorem V.7, one can show that $\{e^{-t\mathscr{A}}\}_{t\geq 0}$ can be extended to a contractive strongly continuous analytic semigroup on $L_p^{\lambda'}(\mathsf{M})$ for $1<p<\infty$.

(v) To determine the domain of the realization of $\mathscr{A}$ on $L_p^{\lambda'}(\mathsf{M})$, we look at the operator

$$\mathfrak{B}u:=-\mathrm{div}(\rho^\lambda\vec{a}\cdot\mathrm{grad}u)+\mathsf{C}(\nabla u,\rho^\lambda a_1)+\rho^\lambda a_0 u+\lambda\mathsf{C}(\nabla u,\rho^\lambda\vec{a}\cdot\mathrm{grad}\log\rho).$$

We have computed in Section 5.1.1 that $\mathscr{A} = \rho^{-\lambda}\mathfrak{B}$. Since in Step (ii), we assume that $(\mathsf{M}, g; \rho)$ has *property* $\mathscr{H}_0$, following an analogous discussion to Step (iii)-(iv), we can show that $-\mathfrak{B}_\omega := -\mathfrak{B} - \omega$ generates a contractive strongly continuous analytic semigroup on $L_p^{\lambda'-\lambda}(\mathsf{M})$ with domain $D(\mathfrak{B}_\omega)$ for $1 < p < \infty$ and any $\omega \geq 0$. In particular, $D(\mathfrak{B}_\omega)$ is independent of ω. On the other hand, by [**6**, Theorem 5.2], for ω sufficiently large and $\lambda' \in \mathbb{R}$, $1 < p < \infty$

$$\mathfrak{B}_\omega \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'-\lambda}(\mathsf{M})) \cap \mathcal{L}\mathrm{is}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'-\lambda}(\mathsf{M})).$$

Therefore, we indeed have $D(\mathfrak{B}) \doteq \mathring{W}_p^{\lambda'-\lambda}(\mathsf{M})$ and

$$\mathfrak{B} \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'-\lambda}(\mathsf{M})) \cap \mathcal{L}\mathrm{is}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'-\lambda}(\mathsf{M})).$$

Now it follows from a similar argument to the proof for Theorem V.3 that

$$\mathscr{A} \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'}(\mathsf{M})) \cap \mathcal{L}\mathrm{is}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}), L_p^{\lambda'}(\mathsf{M})), \quad \lambda' \in \mathbb{R}, \quad 1 < p < \infty.$$

□

We say $u, v \in \mathbb{R}^{\mathsf{M}}$ are C^k-equivalent, which is denoted by $u \sim_k v$, if

$$u \sim v, \quad |\nabla^i u|_g \sim |\nabla^i v|_g, \quad i = 1, \cdots, k.$$

Definition V.28. *An m-dimensional singular manifold $(\mathsf{M}, g; \rho)$ is called a singular manifold with holes and uniformly mild wedge ends if it fulfils the following conditions.*

(i) $(\mathscr{M}, g)$ *is an m-dimensional uniformly regular Riemannian manifold, and*

$$\mathbf{\Sigma} = \{\Sigma_1, \cdots, \Sigma_k\}$$

is a finite set of disjoint m-dimensional compact manifolds with boundary such that $\Sigma_j \subset \mathring{\mathscr{M}}$. Put $G_0 := \mathscr{M} \setminus \cup_{j=1}^k \Sigma_j$ and

$$\mathscr{B}_{j,r} := \bar{\mathbb{B}}_{\mathscr{M}}(\partial\Sigma_j, r) \cap G_0, \quad j = 1, \cdots, k.$$

Furthermore, the singularity function ρ satisfies

$$\rho \sim_2 \mathrm{dist}_{\mathscr{M}}(\cdot, \partial\Sigma_j) \quad \text{in } \mathscr{B}_{j,r} \tag{V.30}$$

for some $r \in (0,\delta)$*, where* $\delta < \mathrm{diam}(\mathscr{M})$ *and* $\mathscr{B}_{i,\delta} \cap \mathscr{B}_{j,\delta} = \emptyset$ *for* $i \neq j$*, and*

$$\rho \sim \mathbf{1}, \quad \textit{elsewhere on } G_0.$$

(ii) $\mathbf{G} = \{G_1, \cdots, G_n\}$ *is a finite set of disjoint* m*-dimensional uniformly mild wedges. More precisely, there is a diffeomorphism* $f_i : G_i \to W(R_i, B_i, \Gamma_i)$ *with* $R_i \in \mathscr{C}_{\mathscr{U}}(J_0)$*. Let* $I_r := (0, r]$ *and*

$$\mathscr{G}_{i,r} := f_i^{-1}(\phi_P(I_r \times B_i) \times \Gamma_i), \quad i = 1, \cdots, n.$$

Moreover, the singularity function ρ *satisfies*

$$\rho \sim_2 f_i^*(\phi_P^*(R_i|_{I_r} \otimes \mathbf{1}_{B_i}) \otimes \mathbf{1}_{\Gamma_i}) \quad \textit{in } \mathscr{G}_{j,r} \tag{V.31}$$

for some $r \in (0, 1]$*, and*

$$\rho \sim \mathbf{1}, \quad \textit{elsewhere on } G_i.$$

(iii) $\{G_0\} \cup \mathbf{G}$ *forms a covering for* M*.* $\partial_0 G_i := G_0 \cap G_i \subset \partial G_0 \cap \partial G_i$*.*

One can easily see that (V.30) and (V.31) imply that

$$|\Delta\rho| < \infty \quad \text{in } \mathscr{B}_{j,r} \text{ and } \mathscr{G}_{j,r}. \tag{V.32}$$

The following corollary does not directly stem from Theorems V.26 and V.27. But using the ideas in their proofs, we can prove this corollary without difficulty.

Corollary V.29. *Suppose that* $(\mathsf{M}, g; \rho)$ *is a singular manifold with holes and uniformly mild wedge ends. Let* $\lambda' \in \mathbb{R}$*, and* $\lambda \in [0,1) \cup (1,\infty)$*. Furthermore, assume that the differential operator*

$$\mathscr{A} u := -\mathrm{div}(\rho^{2-\lambda}\mathrm{grad}u) + \mathsf{C}(\nabla u, a_1) + a_0 u$$

is (ρ, λ)*-regular. Then*

$$\mathscr{A} \in \mathcal{H}(\mathring{W}_p^{2,\lambda'-\lambda}(\mathsf{M}, V), L_p^{\lambda'}(\mathsf{M}, V)), \quad 1 < p < \infty.$$

Here $V = \mathbb{C}$ *if* $p \neq 2$*, or* $V = V_\tau^\sigma$ *with* $\sigma, \tau \in \mathbb{N}_0$ *if* $p = 2$*.*

Proof. If $S_{i,r} := \{\mathsf{p} \in G_i : \rho(\mathsf{p}) = r\}$ is compact for small r and all $i = 1, \cdots, n$, then by Theorem V.27 the asserted result will be true. However, in general, $S_{i,r}$ might not be compact. Nevertheless, looking into the proofs for Theorem V.26 and Theorem V.27, the compactness of $S_{i,r}$ will only be responsible for Step (i) and (ii) in the proof for Theorem V.26.

Firstly, we take $\mathfrak{h} := \mathrm{sign}(1-\lambda)\log\rho$. Then $\mathfrak{h} \in C^2(\mathsf{M})$ satisfies

$$\mathfrak{h} \in \mathscr{H}_\lambda(\mathscr{B}_{j,r}, g; \rho) \text{ and } \mathfrak{h} \in \mathscr{H}_\lambda(\mathscr{G}_{i,r}, g; \rho)$$

with parameters $(c, 1)$ for some $r > 0$, following from (V.32) and a similar argument to the proofs for Propositions V.19 and V.22. Furthermore,

$$\mathfrak{h} \in BC^2(\mathsf{M} \setminus (\bigcup_{j=1}^{k} \mathring{\mathscr{B}}_{j,r} \cup \bigcup_{i=1}^{n} \mathring{\mathscr{G}}_{i,r})).$$

Thus the properties of $\mathfrak{h}$ listed in Step (ii) of the proof for Theorem V.26 are all satisfied.

Next, we prove the existence of the retraction-coretraction system defined in Step (i) of the proof for Theorem V.26. For any $r \in (0, 1)$, picking $(r_0, r]$ with $r_0 > 0$, we can construct a collar neighborhood of $S_{i,r}$ on G_i by

$$U_i := f_i^{-1}(\phi_P^{-1}(B_i \times (r_0, r]) \times \Gamma_i).$$

Moreover, we choose $\xi, \tilde{\xi} \in BC^\infty((r_0, r], [0, 1])$ such that

$$\xi|_{[\frac{r+r_0}{2}, r]} \equiv 1, \quad \xi|_{(r_0, \frac{r+3r_0}{4}]} \equiv 0; \quad \tilde{\xi}|_{(r_0, \frac{r+r_0}{2}]} \equiv 1, \quad \tilde{\xi}|_{[\frac{3r+r_0}{4}, r]} \equiv 0.$$

Now we can define $\hat{\pi}_{i,0} := f_i^*(\phi_P^*(\mathbf{1}_{B_i} \otimes \xi) \otimes \mathbf{1}_{\Gamma_i})$, and $\hat{\pi}_{i,i} := f_i^*(\phi_P^*(\mathbf{1}_{B_i} \otimes \tilde{\xi}) \otimes \mathbf{1}_{\Gamma_i})$. The rest of the proof just follows from a similar argument to Step (i) of the proof for Theorem V.26. ☐

Remark V.30. In view of Remarks V.24 and V.25, the assertion in Corollary V.29 remains true if we replace the condition of *singular manifolds with holes* by removing a finite set of disjoint compact submanifolds $\{\Sigma_1, \cdots, \Sigma_k\}$ or discrete points $\{\mathsf{p}_1, \cdots, \mathsf{p}_k\}$ from a *uniformly regular Riemannian manifold* $(\mathscr{M}, g)$. Here $\Sigma_i \subset \partial\mathscr{M}$ if $\Sigma_i \cap \partial\mathscr{M} \neq \emptyset$, or $\mathsf{p}_i \in \mathring{\mathscr{M}}$.

Remark V.31. From our proofs in Section 5.1 and 5.2, it is a simple matter to check that we do not require the *singular manifold* $(\mathsf{M}, g; \rho)$ to enjoy smoothness up to C^∞. Indeed, in order to prove all the results in Chapter 5, it suffices to require $(\mathsf{M}, g; \rho)$ to be a C^2-*singular manifold.*

Chapter VI

Domains with compact boundary as singular manifolds

Suppose that $\Omega \subset \mathbb{R}^m$ is a C^k-domain with compact boundary for $k > 2$. Then Ω satisfies a *uniform exterior and interior ball condition*, i.e., there is some $r > 0$ such that for every $x \in \partial\Omega$ there are balls $\mathbb{B}(x_i, r) \subset \Omega$ and $\mathbb{B}(x_e, r) \subset \mathbb{R}^m \setminus \Omega$ such that

$$\partial\Omega \cap \bar{\mathbb{B}}(x_i, r) = \partial\Omega \cap \bar{\mathbb{B}}(x_e, r) = x.$$

For $a \leq r$, we denote the a-tubular neighborhood of $\partial\Omega$ by T_a. Let

$$d_{\partial\Omega}(x) := \mathsf{dist}(x, \partial\Omega), \quad x \in \Omega,$$

i.e., the distance function to the boundary. We define $\mathsf{d} : \Omega \to \mathbb{R}^+$ by

$$\text{(VI.1)} \qquad \mathsf{d} = d_{\partial\Omega} \quad \text{if } \Omega \text{ is bounded,} \quad \text{or} \quad \begin{cases} \mathsf{d} = d_{\partial\Omega} & \text{in } \Omega \cap \mathsf{T}_a, \\ \mathsf{d} \sim \mathbf{1} & \text{in } \Omega \setminus \mathsf{T}_a \end{cases} \quad \text{otherwise.}$$

Then we have the following proposition.

Proposition VI.1. *Let $\beta \geq 1$. Suppose that $\Omega \subset \mathbb{R}^m$ is a C^k-domain with compact boundary and $k > 2$. Then $(\Omega, g_m; \mathsf{d}^\beta)$ is a C^{k-1}-singular manifold.*

Proof. The case of $k = \infty$ is a direct consequence of [**7**, Theorem 1.6]. When $k < \infty$, one notices that, to parameterize T_a, we need to use the outward pointing unit normal of $\partial\Omega$, which is C^{k-1}-continuous. By a similar argument to [**7**, Theorem 1.6], we can then prove the asserted statement. □

Given any finite dimensional Banach space X, by defining the *singular manifold* $(\mathsf{M}, g; \rho)$ by $(\Omega, g_m; \mathsf{d}^\beta)$, we denote the weighted function spaces defined on Ω by $\mathfrak{F}^{s,\vartheta}_\beta(\Omega, X)$, i.e.,

$$\mathfrak{F}^{s,\vartheta}_\beta(\Omega, X) = bc^{s,\vartheta}(\mathsf{M}, X), \quad \mathfrak{F} \in \{bc, BC, W_p\}.$$

In particular, we set $\mathfrak{F}^{s,\vartheta}(\Omega, X) := \mathfrak{F}^{s,\vartheta}_1(\Omega, X)$.

In view of Remark IV.8, we have the following continuous maximal regularity theorem for elliptic operators with higher order degeneracy on domains.

Theorem VI.2. *Let $\gamma \in (0, 1]$, $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $\vartheta \in \mathbb{R}$, $\beta \geq 1$ and $k = 2l + [s] + 2$. Suppose that $\Omega \subset \mathbb{R}^m$ is a C^k-domain and the differential operator*

$$\mathcal{A} := \sum_{|\alpha| \leq 2l} a_\alpha \partial^\alpha$$

satisfies

(a) *for any $\xi \in \mathbb{S}^{m-1}$*

$$S := \Sigma_{\pi/2} \subset \rho(-\hat{\sigma}\mathcal{A}^\pi(x, \xi)),$$

and for some $\mathcal{C}_\mathfrak{e} > 0$

$$(\mathsf{d}^{2l\beta}(x) + |\mu|)\|(\mu + \hat{\sigma}\mathcal{A}^\pi(x, \xi))^{-1}\|_{\mathcal{L}(X)} \leq \mathcal{C}_\mathfrak{e}, \quad \mu \in S;$$

(b) $a_\alpha \in bc^{s,-|\alpha|}_\beta(\Omega, \mathcal{L}(X))$.

Then

$$\mathcal{A} \in \mathcal{M}_\gamma(bc^{s+2l,\vartheta}_\beta(\Omega, X), bc^{s,\vartheta}_\beta(\Omega, X)).$$

Remarks VI.3.

(a) Condition (a) in Theorem VI.2 can be replaced by the following condition. For any $\xi \in \mathbb{S}^{m-1}$ and $\eta \in X$,

$$\langle \hat{\sigma}\mathcal{A}^\pi(x, \xi)\eta, \eta \rangle_X \sim \mathsf{d}^{2l\beta}|\eta|^2_X.$$

Here $\langle \cdot, \cdot \rangle$ is the inner product in X. So the result in Theorem VI.2 corresponds to the case of degenerate boundary value problems with strong degeneration. This generalizes the results of [**40, 89**] to unbounded domains and elliptic operators with order higher than two.

(b) In Theorem VI.2, taking X to be any infinite dimensional Banach space is also admissible.

Next, we make use of the theory established in Chapter 5 to prove an existence and uniqueness result for second order degenerate boundary value problems with weak degeneration, or second order boundary blow-up problems.

Given any Banach space X, $s \in (0,1)$, and any perfect interval J, we denote by

$$C^s(J,X)$$

the set of all $u \in C(J,X)$ such that u is Hölder continuous of order s.

Assume that $\Omega \subset \mathbb{R}^m$ is a C^3-domain with compact boundary. Take $\rho = \mathsf{d}$. Let $J = [0,T]$, $\Omega_T := (0,T] \times \mathsf{M}$, and $\Omega_0 := \{0\} \times \mathsf{M}$. We consider the following initial value problem.

$$\text{(VI.2)} \qquad \begin{cases} u_t + \mathscr{A}u = f & \text{on} \quad \Omega_T; \\ u = u_0 & \text{on} \quad \Omega_0. \end{cases}$$

Here

$$\mathscr{A}u = -a\Delta u + a_1 \cdot \nabla u + a_0 u,$$

and the coefficients (a, a_1, a_0) satisfy for some $s \in (0,1)$ and $\lambda \in (0,1) \cup (1,\infty)$

$$\text{(VI.3)} \qquad a_1 \in C^s(J; BC^{0,\lambda}(\Omega, T_0\mathbb{R}^m)), \quad a_0 \in C^s(J; L_\infty^\lambda(\Omega));$$

and if $\lambda = 2$

$$\text{(VI.4)} \qquad a \in C^s(J; \mathbb{R}_+);$$

or if $\lambda \neq 2$

$$\text{(VI.5)} \qquad a \in C^s(J, BC^{2,\lambda-2}(\Omega)), \quad \text{for every} \quad t \in J, \quad a(t)^{\frac{1}{2-\lambda}} \sim_2 \mathsf{d}.$$

Observe that (VI.3) can be equivalently stated as

$$\mathsf{d}^{\lambda-1} a_1 \in C^s(J; BC(\Omega, \mathbb{R}^m)), \quad \mathsf{d}^\lambda a_0 \in C^s(J; L_\infty(\Omega)).$$

By (VI.4) and (VI.5), we can verify that when $\lambda \neq 2$, $(\Omega, g_m; a^{\frac{1}{2-\lambda}})$ is a C^2-*singular manifold* with $\mathscr{H}_\lambda$-ends. When $\lambda = 2$, we take the *singular manifold* to be $(\Omega, g_m, \mathsf{d})$. In both cases, the conditions in Theorem V.27 are satisfied.

Now we conclude from Remark V.31 and [**1**, Theorem II.1.2.1] that

Theorem VI.4. *Suppose that $\Omega \subset \mathbb{R}^m$ is a C^3-domain with compact boundary. Let $s \in (0,1)$, $\lambda \in (0,1) \cup (1,\infty)$, $\lambda' \in \mathbb{R}$ and $1 < p < \infty$. Assume that the coefficients (a, a_1, a_0) of the differential operator*

$$\mathscr{A}u = -a\Delta u + a_1 \cdot \nabla u + a_0 u$$

satisfy (VI.3)-(VI.5). *Then given any*

$$(f, u_0) \in C^s(J; L_p^{\lambda'}(\Omega)) \times L_p^{\lambda'}(\Omega),$$

the initial value problem (VI.2) *has a unique solution*

$$u \in C^{1+s}(J \setminus \{0\}; L_p^{\lambda'}(\Omega)) \cap C^s(J \setminus \{0\}; W_p^{\lambda'-\lambda}(\Omega)).$$

The case $\lambda = 0$ corresponds to *normally ρ-ellipticity* and thus is covered by Theorem VI.2.

Remark VI.5. Based on (VI.5), we can readily observe that the principle symbol of $\mathscr{A}$ satisfies

$$\hat{\sigma}\mathscr{A}(x,\xi) = a(t)|\xi|^2 \sim \mathsf{d}^{2-\lambda}|\xi|^2, \quad \lambda \neq 2.$$

Therefore, (VI.2) can either be a degenerate boundary value problem or be a boundary blow-up problem. This supplements to the results in [**40, 89**] with weak degeneration case, i.e., $\lambda \in (0,1) \cup (1,2)$, or with boundary singularity case, i.e., $\lambda > 2$.

Chapter VII

Applications to geometric analysis

1. The Laplace-Beltrami operator

Suppose that $(\mathsf{M}, g; \rho)$ is a *singular manifold*. Recall that the Laplace-Beltrami operator with respect to g is defined by

$$\Delta = \Delta_g := \mathrm{div}_g \circ \mathrm{grad}_g = \mathrm{div} \circ \mathrm{grad}.$$

One readily checks that Δ is (ρ, λ)-regular and (ρ, λ)-singular elliptic with $C_{\hat{\sigma}} = 1$, $\lambda = 2$.

Let $\mathsf{M}_T := (0, T] \times \mathsf{M}$, and $\mathsf{M}_0 := \{0\} \times \mathsf{M}$. Then Theorem V.27, Corollary V.29 and [**1**, Theorem II.1.2.1] imply the following existence and uniqueness theorem for the heat equation.

Theorem VII.1. *Suppose that either* $(\mathsf{M}, g; \rho)$ *is a singular manifold with holes and uniformly mild wedge ends, or* $(\mathsf{M}, g; \rho)$ *satisfies the conditions in Theorem V.27. Let* $\lambda' \in \mathbb{R}$ *and* $J = [0, T]$. *Then for any*

$$(f, u_0) \in C^s(J; L_p^{\lambda'}(\mathsf{M})) \times L_p^{\lambda'}(\mathsf{M}),$$

with some $s \in (0, 1)$, *the boundary value problem*

$$\text{(VII.1)} \qquad \begin{cases} u_t - \Delta u = f & \text{on} \quad \mathsf{M}_T \\ u = 0 & \text{on} \quad \partial \mathsf{M}_T \\ u = u_0 & \text{on} \quad \mathsf{M}_0 \end{cases}$$

has a unique solution

$$u \in C^{1+s}(J \setminus \{0\}; L_p^{\lambda'}(\mathsf{M})) \cap C^s(J \setminus \{0\}; \mathring{W}_p^{\lambda' - 2}(\mathsf{M})).$$

In the case of L_2-spaces, making use of [**71**, Theorem 1.6], we have the following corollary.

Corollary VII.2. *Under the conditions in Theorem VII.1, let* $V = V^{\sigma}_{\tau}$ *be a tensor field on* M *and* $1 < p < \infty$. *Then for any*

$$(f, u_0) \in L_p([0,T]; L^{\lambda'}_2(\mathsf{M}, V)) \times \mathring{B}^{1,\lambda'-1}_2(\mathsf{M}, V),$$

the boundary value problem (VII.1) *has a unique solution*

$$u \in L_p([0,T]; \mathring{W}^{2,\lambda'-2}_2(\mathsf{M}, V)) \times W^1_p([0,T]; L^{\lambda'}_2(\mathsf{M}, V)).$$

Remark VII.3. A similar result can also be formulated for the wave equation on *singular manifolds with holes and uniformly mild wedge ends*, or *singular manifolds* satisfying the conditions in Theorem V.27. We refer the reader to [**69**] for the corresponding semigroup theory for hyperbolic equations.

2. The porous medium equation

We consider the porous medium equation on a *singular manifold* $(\mathsf{M}, g; \rho)$ without boundary, which reads as follows.

$$\text{(VII.2)} \qquad \begin{cases} \partial_t u - \Delta u^n = f; \\ u(0) = u_0 \end{cases}$$

for $n > 1$. On Euclidean spaces, J.L. Vázquez [**87, 88**] proved existence and uniqueness of non-negative weak solutions of Dirichlet problems for the porous medium equation. In a landmark article [**24**], P. Daskalopoulos and R. Hamilton showed existence and uniqueness of smooth solutions for the porous medium equation, and the smoothness of the free boundary, namely, the boundary of the support of the solution, under mild assumptions on the initial data. In the past decade, there has been rising interest in investigating the porous medium equation on Riemannian manifolds. See [**16, 26, 53, 66, 90, 92**] for example. To the best of the author's knowledge, research in this direction is all restricted to the case of complete, or even compact, manifolds. The result that we state in this section is the first one concerning existence and uniqueness of solutions to the porous medium equation on manifolds with singularities.

Let

$$P(u) := -nu^{n-1}\Delta, \quad Q(u) := n(n-1)|\mathrm{grad} u|^2_g u^{n-2}.$$

A direct computation shows that equation (VII.2) is equivalent to

$$\begin{cases} \partial_t u + P(u)u = Q(u) + f; \\ \\ u(0) = u_0. \end{cases}$$

Given any $0 < s < 1$, put $\vartheta = -2/(n-1)$. In the current context, $V = \mathbb{R}$, thus we abbreviate the notation $bc^{s',\vartheta}(M, V)$ to $bc^{s',\vartheta}(\mathsf{M})$ for any $s' \geq 0$. Let

$$E_0 := bc^{s,\vartheta}(\mathsf{M}), \quad E_1 := bc^{2+s,\vartheta}(\mathsf{M}), \quad E_{1/2} := (E_0, E_1)^0_{1/2,\infty}.$$

Then by Proposition II.3, $E_{1/2} \doteq bc^{1+s,\vartheta}(\mathsf{M})$. Let

$$U^{1+s}_{\vartheta} := \{u \in E_{1/2} : \inf \rho^{\vartheta} u > 0\},$$

which is open in $E_{1/2}$.

For any $\beta \in \mathbb{R}$, define $\mathsf{P}_\beta : U^{1+s}_{\vartheta} \to L_{1,loc}(\mathsf{M}) : u \mapsto u^{\beta}$. One readily checks that [**82**, Proposition 6.3] still holds true for *singular manifolds*. Hence by [**82**, Proposition 6.3] and Proposition II.5, we obtain

$$[u \mapsto u^{\beta}] = [u \mapsto \rho^{-\beta\vartheta} \mathsf{P}_\beta(\rho^{\vartheta} u)] \in C^{\omega}(U^{1+s}_{\vartheta}, bc^{1+s,\beta\vartheta}(\mathsf{M})). \tag{VII.3}$$

In view of (P2), we infer that $\mathcal{R}^c g^* \in l^2_\infty(\boldsymbol{BC}^k(E^2_0))$ for any $k \in \mathbb{N}_0$. Then Proposition II.1 yields

$$g^* \in BC^{\infty,2}(\mathsf{M}, V^2_0). \tag{VII.4}$$

One may check via Proposition II.4, (VII.3) and (VII.4) that

$$u^{n-1} g^* \in bc^{1+s}(\mathsf{M}, V^2_0), \quad u \in U^{1+s}_{\vartheta}.$$

On account of the expression $\Delta_g v = \mathsf{C}(g^*, \nabla^2 v)$, it is then a direct consequence of Corollary IV.2 and [**15**, Proposition 1] that

$$P \in C^{\omega}(U^{1+s}_{\vartheta}, \mathcal{L}(E_1, E_0)). \tag{VII.5}$$

In the above, $\nabla := \nabla_g$, where ∇_g is Levi-Civita connection of g. Given any $\vartheta' \in \mathbb{R}$, by Propositions II.6 and II.8, one obtains

$$\operatorname{grad} \in \mathcal{L}(BC^{k+1,\vartheta'}(\mathsf{M}, V^{\sigma}_{\tau}), BC^{k,\vartheta'+2}(\mathsf{M}, V^{\sigma+1}_{\tau})). \tag{VII.6}$$

A density argument as in the proof for Proposition II.5 yields

$$\mathrm{grad} \in \mathcal{L}(bc^{k+1,\vartheta'}(\mathsf{M}, V_\tau^\sigma), bc^{k,\vartheta'+2}(\mathsf{M}, V_\tau^{\sigma+1})).$$

Interpolation theory and definition (II.3) implies that (VII.6) also holds for *Hölder* spaces of non-integer order. Applying the density argument as in the proof for Proposition II.5 once more, we establish the assertion for weighted *little Hölder* spaces of non-integer order, that is, for any $s' \geq 0$

$$\mathrm{grad} \in \mathcal{L}(bc^{s'+1,\vartheta'}(\mathsf{M}, V_\tau^\sigma), bc^{s',\vartheta'+2}(\mathsf{M}, V_\tau^{\sigma+1})). \tag{VII.7}$$

We have the expression $|\mathrm{grad} u|_g^2 = \mathsf{C}(\nabla u, \mathrm{grad} u)$. Since complete contraction is a bundle multiplication, we infer from Propositions II.4, II.6 and (VII.7) that

$$[u \mapsto |\mathrm{grad} u|_g^2] \in C^\omega(U_\vartheta^{1+s}, bc^{s,2\vartheta+2}(\mathsf{M})). \tag{VII.8}$$

Proposition II.4, (VII.3) and (VII.8) immediately imply

$$Q \in C^\omega(U_\vartheta^{1+s}, E_0). \tag{VII.9}$$

Given any $u \in U_\vartheta^{1+s}$, one verifies that the principal symbol of $P(u)$ fulfils

$$-n\mathsf{C}(u^{n-1}g^*, (-i\xi)^{\otimes 2}) = n\rho^2(\rho^\vartheta u)^{n-1}|\xi|_{g^*}^2 \geq n(\inf \rho^\vartheta u)^{n-1}\rho^2|\xi|_{g^*}^2,$$

for any cotangent field ξ. Hence for any $u \in U_\vartheta^{1+s}$, $P(u)$ is *normally ρ-elliptic*. It follows from Theorem IV.7 that

$$P(u) \in \mathcal{M}_\gamma(E_1, E_0), \quad u \in U_\vartheta^{1+s}. \tag{VII.10}$$

Theorem VII.4. *Suppose that* $u_0 \in U_\vartheta^{1+s} := \{u \in bc^{1+s,\vartheta}(\mathsf{M}) : \inf \rho^\vartheta u > 0\}$ *with* $0 < s < 1$, *and* $\vartheta = -2/(n-1)$. *Then given any*

$$f \in bc^{s,\vartheta}(\mathsf{M}),$$

equation (VII.2) *has a unique local positive solution*

$$\hat{u} \in C_{1/2}^1(J(u_0), bc^{s,\vartheta}(\mathsf{M})) \cap C_{1/2}(J(u_0), bc^{2+s,\vartheta}(\mathsf{M})) \cap C(J(u_0), U_\vartheta^{1+s})$$

existing on $J(u_0) := [0, T(u_0))$ *for some* $T(u_0) > 0$. *Moreover,*

$$\hat{u} \in C^\infty(\dot{J}(u_0) \times \mathsf{M}).$$

Proof. In virtue of (VII.5), (VII.9) and (VII.10), [**22**, Theorem 4.1] immediately establishes the local existence and uniqueness part. The short term positivity of the solution follows straightaway from the continuity of the solution. To argue for the asserted regularity property of the solution $\hat{u}$, we look at $v := \rho^\vartheta \hat{u}$. By multiplying both sides of equation VII.2 with ρ^ϑ, we have

$$\begin{cases} \partial_t v - \rho^\vartheta \Delta \rho^{2-\vartheta} v^n = \rho^\vartheta f; \\ \qquad\qquad v(0) = \rho^\vartheta u_0. \end{cases}$$

One checks

$$\begin{aligned} \rho^\vartheta \Delta \rho^{2-\vartheta} v^n = & n\rho^2 v^{n-1} \Delta v + n(n-1)\rho^2 |\mathrm{grad} v|_g^2 v^{n-2} \\ & + 2n(2-\vartheta)\rho^2 (\mathrm{grad} \log \rho | \mathrm{grad} v)_g v^{n-1} \\ & + (2-\vartheta)[\rho\Delta\rho + (1-\vartheta)|\mathrm{grad}\rho|_g^2] v^n. \end{aligned}$$

Let $\hat{g} = g/\rho^2$. Recall that $(\mathsf{M}, \hat{g})$ is a *uniformly regular Riemannian manifold*. Put $U^{1+s} := \{v \in bc^{1+s}(\mathsf{M}) : \inf v > 0\}$. By [**6**, formula (5.15)],

$$\rho^2 |\mathrm{grad} v|_g^2 = |\mathrm{grad}_{\hat{g}} v|_{\hat{g}}^2.$$

We have

$$(\mathrm{grad} \log \rho | \mathrm{grad} v)_g = (\mathrm{grad} \log \rho | \mathrm{grad}_{\hat{g}} v)_{\hat{g}}.$$

It follows from [**6**, formula (5.8)] that $\rho^2 \mathrm{grad} \log \rho \in BC^{1,0}(\mathsf{M}, T\mathsf{M})$. By Proposition II.7,

$$\rho^2 \mathrm{grad} \log \rho \in BC^1(\mathsf{M}, \widehat{T\mathsf{M}}).$$

[**82**, formula (5.6)] implies $v^{n-1} \in bc^{1+s}(\mathsf{M})$ for all $v \in U^{1+s}$. It is immediate from (S3) that $\rho \in BC^{\infty,-1}(\mathsf{M})$. By Propositions II.4-II.8 and II.11, we can show that

$$\rho\Delta\rho + (1-\vartheta)|\mathrm{grad}\rho|_g^2 \in BC^1(\mathsf{M}).$$

Put

$$P(v) := -n\rho^2 v^{n-1} \Delta,$$

and

$$\begin{aligned} Q(v) :=& \rho^{\vartheta}\Delta\rho^{2-\vartheta}v^n + P(v)v \\ =& n(n-1)\rho^2|\mathrm{grad} v|_g^2 v^{n-2} + 2n(2-\vartheta)\rho^2(\mathrm{grad}\log\rho|\mathrm{grad} v)_g v^{n-1} \\ &+ (2-\vartheta)[\rho\Delta\rho + (1-\vartheta)|\mathrm{grad}\rho|_g^2]v^n. \end{aligned}$$

Then by the above discussion, we infer that

$$P \in C^{\omega}(U^{1+s}, \mathcal{L}(bc^{2+s}(\mathsf{M}), bc^s(\mathsf{M}))), \quad Q \in C^{\omega}(U^{1+s}, bc^s(\mathsf{M})).$$

For each $v \in U^{1+s}$, we can check that $P(v)$ is *normally elliptic* in the sense of [**82**, Section 3]. Applying the parameter-dependent diffeomorphism technique in [**79**], we can establish

$$v \in C^{\infty}(\dot{J}(u_0) \times \mathsf{M}),$$

which in turn implies

$$\hat{u} \in C^{\infty}(\dot{J}(u_0) \times \mathsf{M}).$$

□

Remark VII.5. It is clear Theorem VII.4 still holds true for the fast diffusion case of the porous medium equation (the plasma equation).

Before concluding this section, we comment on the Cauchy problem for the porous medium equation and its waiting-time phenomenon. Since our conclusion for the porous medium equation, to some extend, can be viewed as a simpler version of the corresponding theory of the thin film equation in Section 8.1, we will only state our results without providing proofs. More details can be found in Section 8.1.

Remark VII.6. Suppose that $\mathrm{supp}(u_0) =: \Omega \subset \mathbb{R}^m$ is a C^4-domain with compact boundary, and $u_0 \in U_{\vartheta}^{1+s} := \{u \in bc^{1+s,\vartheta}(\Omega) : \inf \mathsf{d}^{\vartheta}u > 0\}$ with $0 < s < 1$, $\vartheta = -2/(n-1)$. We know from Proposition VI.1 that $(\Omega, g_m; \mathsf{d})$ is a C^3-*singular manifold*, where d is defined in (VI.1). Then by

Theorems VI.2 and VII.4, for every $f \in bc^{s,\vartheta}(\Omega)$, the equation

$$\begin{cases} \partial_t u + \Delta u^n = f & \text{on} \quad \Omega; \\ u(0) = u_0 & \text{on} \quad \Omega, \end{cases}$$

has a unique solution on $J(u_0) = [0, T(u_0))$

$$\hat{u} \in C^1_{1/2}(J(u_0), bc^{s,\vartheta}_1(\Omega)) \cap C_{1/2}(J(u_0), bc^{2+s,\vartheta}_1(\Omega)) \cap C(J(u_0), U^{1+s}_\vartheta). \tag{VII.11}$$

Furthermore, by identifying $\hat{u}, f, u_0 \equiv 0$ in $\mathbb{R}^m \setminus \Omega$, $\hat{u}$ is indeed a strong L_1-solution of the Cauchy problem

$$\begin{cases} \partial_t u + \Delta u^n = f & \text{on} \quad \mathbb{R}^m; \\ u(0) = u_0 & \text{on} \quad \mathbb{R}^m \end{cases}$$

in the sense of [**88**, Definition 9.1], except that the interval of existence $[0, \infty)$ in [**88**, Definition 9.1] is replaced by $J(u_0)$. This solution is unique by [**88**, Theorem 9.2]. Another observation from (VII.11) is that $\hat{u}$ enjoys the so-called waiting-time property, that is,

$$\operatorname{supp}[\hat{u}(t,\cdot)] = \operatorname{supp}[\hat{u}(0,\cdot)], \quad t \in (0, T(u_0)).$$

3. The Yamabe flow

Suppose that $(\mathsf{M}, g_0; \rho)$ is a *singular manifold* without boundary of dimension m for $m \geq 3$. The Yamabe flow reads as

$$\begin{cases} \partial_t g = -R_g g; \\ g(0) = g^0, \end{cases} \tag{VII.12}$$

where R_g is the scalar curvature with respect to the metric g. g^0 is in the conformal class of the background metric g_0 of M, i.e., $[g_0]$.

We seek solutions to the Yamabe flow (VII.12) in $[g_0]$. Let $c(m) := \frac{m-2}{4(m-1)}$, and define the conformal Laplacian operator L_g with respect to the metric g as:

$$L_g u := \Delta_g u - c(m) R_g u.$$

Let $g = u^{\frac{4}{m-2}} g_0$ for some $u > 0$. It is well known that by rescaling the time variable equation (VII.12) is equivalent to

$$\begin{cases} \partial_t u^{\frac{m+2}{m-2}} = \dfrac{m+2}{m-2} L_0 u; \\ \\ u(0) = u_0, \end{cases}$$

where $L_0 := L_{g_0}$ and u_0 is a positive function. See [**59**, formula (7)]. It is equivalent to solving the following equation:

$$\text{(VII.13)} \qquad \begin{cases} \partial_t u = u^{-\frac{4}{m-2}} L_0 u; \\ \\ u(0) = u_0. \end{cases}$$

A well-known formula of scalar curvature in local coordinates yields

$$R_g = \frac{1}{2} g^{ki} g^{lj} (g_{jk,li} + g_{il,kj} - g_{jl,ki} - g_{ik,lj}).$$

(P2) implies that

$$\mathcal{R}^c R_{g_0} \in l_\infty^2(\boldsymbol{BC}^k(\mathbb{R})),$$

for any $k \in \mathbb{N}_0$. By Proposition II.1, we infer that

$$\text{(VII.14)} \qquad R_{g_0} \in BC^{\infty,2}(\mathsf{M}).$$

Put

$$P(u)h := -u^{-\frac{4}{m-2}} \Delta_{g_0} h, \quad Q(u) := -c(m) u^{\frac{m-6}{m-2}} R_{g_0}.$$

Given any $0 < s < 1$, we choose $0 < \alpha < s$, $\gamma = (s - \alpha)/2$. Let $\vartheta = (m-2)/2$ and

$$E_0 := bc^{\alpha,\vartheta}(\mathsf{M}), \quad E_1 := bc^{2+\alpha,\vartheta}(\mathsf{M}), \quad E_\gamma := (E_0, E_1)^0_{\gamma,\infty}.$$

Then by Proposition II.3, $E_\gamma \doteq bc^{s,\vartheta}(\mathsf{M})$. Put

$$U^s_\vartheta = \{u \in E_\gamma : \inf \rho^\vartheta u > 0\}.$$

In view of (VII.14), it follows from an analogous discussion as in (VII.5) and (VII.9) that

$$\text{(VII.15)} \qquad P \in C^\omega(U^s_\vartheta, \mathcal{L}(E_1, E_0)), \quad Q \in C^\omega(U^s_\vartheta, E_0).$$

A similar computation as in (VII.10) yields

$$P(u) \in \mathcal{M}_\gamma(E_1, E_0), \quad u \in U^s_\vartheta. \tag{VII.16}$$

Theorem VII.7. *Suppose that* $u_0 \in U^s_\vartheta := \{u \in bc^{s,\vartheta}(\mathsf{M}) : \inf \rho^\vartheta u > 0\}$ *with* $0 < s < 1$, *and* $\vartheta = (m-2)/2$. *Then for every fixed* $\alpha \in (0, s)$, *equation* (VII.13) *has a unique local positive solution*

$$\hat{u} \in C^1_{1-\gamma}(J(u_0), bc^{\alpha,\vartheta}(\mathsf{M})) \cap C_{1-\gamma}(J(u_0), bc^{2+\alpha,\vartheta}(\mathsf{M})) \cap C(J(u_0), U^s_\vartheta)$$

existing on $J(u_0) := [0, T(u_0))$ *for some* $T(u_0) > 0$ *with* $\gamma = (s-\alpha)/2$. *Moreover,*

$$\hat{g} \in C^\infty(\dot{J}(u_0) \times \mathsf{M}, V^0_2).$$

In particular, if the metric g_0/ρ^2 *is real analytic, then*

$$\hat{g} \in C^\omega(\dot{J}(u_0) \times \mathsf{M}, V^0_2).$$

Proof. Local existence and uniqueness follows directly from (VII.15), (VII.16), and [**22**, Theorem 4.1]. The regularity part follows by a similar way to the proof of Theorem VII.4. □

Remark VII.8. The initial metric $g^0 = u_0^{\frac{4}{m-2}} g_0$ in the above theorem can have unbounded scalar curvature. To make this already long paper not any longer, we will give more details on this observation elsewhere.

4. The evolutionary p-Laplacian equation

In this section, we investigate the well-posedness of the following evolutionary p-Laplacian equation on a *singular manifold* $(\mathsf{M}, g; \rho)$.

$$\begin{cases} \partial_t u - \mathrm{div}(|\mathrm{grad} u|_g^{p-2} \mathrm{grad} u) = f; \\ u(0) = u_0. \end{cases} \tag{VII.17}$$

Here $1 < p < \infty$ with $p \neq 2$, and $\mathrm{grad} = \mathrm{grad}_g$, $\mathrm{div} = \mathrm{div}_g$. This problem has been studied extensively on Euclidean spaces. The two books [**28, 29**] contain a detailed analysis and a historical account of this problem. There are several generalizations of the elliptic p-Laplacian equation on

Riemannian manifolds. But fewer have been achieved for its parabolic version above. See [**26**] for instance.

One computes

$$\begin{aligned}\operatorname{div}(|\mathrm{grad}u|_g^{p-2}\nabla u) &= |\mathrm{grad}u|_g^{p-2}\Delta u + (p-2)|\mathrm{grad}u|_g^{p-4}\mathsf{C}((\mathrm{grad}u)^{\otimes 2}, \nabla^2 u)\\ &= |\mathrm{grad}u|_g^{p-4}\mathsf{C}(|\mathrm{grad}u|^2 g^* + (p-2)(\mathrm{grad}u)^{\otimes 2}, \nabla^2 u).\end{aligned}$$

Let

$$\vec{a}(u) := -|\mathrm{grad}u|_g^{p-4}(|\mathrm{grad}u|^2 g^* + (p-2)(\mathrm{grad}u)^{\otimes 2}).$$

For any $0 < s < 1$, we put $\vartheta = p/(2-p)$ and

$$E_0 := bc^{s,\vartheta}(\mathsf{M}), \quad E_1 := bc^{2+s,\vartheta}(\mathsf{M}), \quad E_{1/2} := (E_0, E_1)^0_{1/2,\infty}.$$

Proposition II.3 implies $E_{1/2} \doteq bc^{1+s,\vartheta}(\mathsf{M})$. Let

$$U_\vartheta^{1+s} := \{u \in E_{1/2} : \inf \rho^{\vartheta+1}|\mathrm{grad}u|_g > 0\}.$$

This is an open subset of $E_{1/2}$.

We infer from (VII.3) and (VII.8) that

$$[u \mapsto |\mathrm{grad}u|_g^{p-2}] \in C^\omega(U_\vartheta^{1+s}, bc^{s,-2}(\mathsf{M})),$$

and from [**5**, Example 13.4(b)], Propositions II.4 and II.8 that

$$[u \mapsto |\mathrm{grad}u|_g^{p-4}(\mathrm{grad}u)^{\otimes 2}] \in C^\omega(U_\vartheta^{1+s}, bc^{s,0}(\mathsf{M}, V_0^2)).$$

In virtue of (VII.4) and Proposition II.4, we have

$$[u \mapsto \vec{a}(u)] \in C^\omega(U_\vartheta^{1+s}, bc^{s,0}(\mathsf{M}, V_0^2)). \tag{VII.18}$$

The principal symbol can be computed as in Section 7.2.

$$\begin{aligned}\mathsf{C}(\vec{a}(u), (-i\xi)^{\otimes 2})(\mathsf{p}) =& |\mathrm{grad}u(\mathsf{p})|_{g(\mathsf{p})}^{p-2}|\xi(\mathsf{p})|^2_{g^*(\mathsf{p})} + (p-2)|\mathrm{grad}u(\mathsf{p})|_{g(\mathsf{p})}^{p-4}[\mathsf{C}(\mathrm{grad}u, \xi)(\mathsf{p})]^2\\ =& |\mathrm{grad}u(\mathsf{p})|_{g(\mathsf{p})}^{p-2}|\xi(\mathsf{p})|^2_{g^*(\mathsf{p})} + (p-2)|\mathrm{grad}u(\mathsf{p})|_{g(\mathsf{p})}^{p-4}(\nabla u(\mathsf{p})|\xi(\mathsf{p}))^2_{g^*(\mathsf{p})}.\end{aligned}$$

For $p > 2$, one checks for any $\xi \in \Gamma(\mathsf{M}, T^*\mathsf{M})$

$$\mathsf{C}(\vec{a}(u), (-i\xi)^{\otimes 2})(\mathsf{p}) \geq |\mathrm{grad}u(\mathsf{p})|^{p-2}_{g(\mathsf{p})} |\xi(\mathsf{p})|^2_{g^*(\mathsf{p})} \geq (\inf \rho^{\vartheta+1}|\mathrm{grad}u|_g)^{p-2} \rho^2(\mathsf{p})|\xi(\mathsf{p})|^2_{g^*(\mathsf{p})},$$

and for $1 < p < 2$

$$\begin{aligned}\mathsf{C}(\vec{a}(u), (-i\xi)^{\otimes 2})(\mathsf{p}) &\geq |\mathrm{grad}u(\mathsf{p})|^{p-2}_{g(\mathsf{p})} |\xi(\mathsf{p})|^2_{g^*(\mathsf{p})} + (p-2)|\mathrm{grad}u(\mathsf{p})|^{p-2}_{g(\mathsf{p})} |\xi(\mathsf{p})|^2_{g^*(\mathsf{p})} \\ &= (p-1)|\mathrm{grad}u(\mathsf{p})|^{p-2}_{g(\mathsf{p})} |\xi(\mathsf{p})|^2_{g^*(\mathsf{p})} \\ &\geq (p-1)(\sup \rho^{\vartheta+1}|\mathrm{grad}u|_g)^{p-2} \rho^2(\mathsf{p})|\xi(\mathsf{p})|^2_{g^*(\mathsf{p})},\end{aligned}$$

holds for all $u \in U^{1+s}_\vartheta$. In the second step, we have used the Cauchy-Schwarz inequality. Therefore, $\mathsf{C}(\vec{a}(u), \nabla^2 \cdot)$ is *normally ρ-elliptic* for every $u \in U^{1+s}_\vartheta$.

Theorem VII.9. *Suppose that* $u_0 \in U^{1+s}_\vartheta := \{u \in bc^{1+s,\vartheta}(\mathsf{M}) : \inf \rho^{\vartheta+1}|\mathrm{grad}u|_g > 0\}$ *with* $0 < s < 1$*, and* $\vartheta = p/(2-p)$*. Then given any*

$$f \in bc^{s,\vartheta}(\mathsf{M}),$$

equation (VII.17) *has a unique local solution*

$$\hat{u} \in C^1_{1/2}(J(u_0), bc^{s,\vartheta}(\mathsf{M})) \cap C_{1/2}(J(u_0), bc^{2+s,\vartheta}(\mathsf{M})) \cap C(J(u_0), U^{1+s}_\vartheta)$$

existing on $J(u_0) := [0, T(u_0))$ *for some* $T(u_0) > 0$*. Moreover,*

$$\hat{u} \in C^\infty(\dot{J}(u_0) \times \mathsf{M}).$$

PROOF. The assertion follows in a similar way to the proof of Theorem VII.4. □

Chapter VIII

Further applications to degenerate boundary value problems or boundary blow-up problems

1. The thin film equation on domains

Suppose that $\Omega \subset \mathbb{R}^m$ is a C^6-domain with compact boundary. Then by the discussion in Chapter 6, $(\Omega, g_m; \mathsf{d}^\beta)$ with $\beta \geq 1$ is a *singular manifold*, where d is defined in (VI.1). We consider the following thin film equation with $n > 0$ and degenerate boundary condition. Physically, the power exponent is determined by the flow condition at the liquid-solid interface, and is usually constrained to $n \in (0, 3]$. Since the other choices of n make no difference in our theory, $n \in [3, \infty)$ is also included herein.

(VIII.1)
$$\begin{cases} \partial_t u + \operatorname{div}(u^n D\Delta u + \alpha_1 u^{n-1}\Delta u Du + \alpha_2 u^{n-2}|Du|^2 Du) = f & \text{on} \quad \Omega_T; \\ u(0) = u_0 & \text{on} \quad \Omega. \end{cases}$$

Here α_1, α_2 are two constants, and D denotes the gradient in $\mathbb{R}^m$. An easy computation shows that

$$\begin{aligned} &\operatorname{div}(u^n D\Delta u + \alpha_1 u^{n-1}\Delta u Du + \alpha_2 u^{n-2}|Du|^2 Du) \\ &= u^n \Delta^2 u + (n+\alpha_1)u^{n-1}(Du|D\Delta u)_{g_m} + \alpha_1 u^{n-1}(\Delta u)^2 \\ &\quad + [\alpha_1(n-1)+\alpha_2]u^{n-2}|Du|^2\Delta u + \alpha_2(n-2)u^{n-3}|Du|^4 \\ &\quad + 2\alpha_2 u^{n-2}(\nabla^2 u Du|Du)_{g_m}. \end{aligned}$$

For any $0 < s < 1$, take $\vartheta = -4/n$

$$E_0 := bc^{s,\vartheta}_\beta(\Omega), \quad E_1 := bc^{4+s,\vartheta}_\beta(\Omega), \quad E_{1/2} = (E_0, E_1)^0_{1/2,\infty}.$$

Then $E_{1/2} \doteq bc_\beta^{2+s,\vartheta}(\Omega)$. Let $U_\vartheta^{2+s} := \{u \in E_{1/2} : \inf \mathsf{d}^{\beta\vartheta} u > 0\}$. For any $u \in U_\vartheta^{2+s}$ and $v \in E_1$, we define

$$\begin{aligned} P(u)v := & u^n \Delta^2 v + (n+\alpha_1)u^{n-1}(Du|D\Delta v)_{g_m} + \alpha_1 u^{n-1}\Delta u \Delta v \\ & + [\alpha_1(n-1)+\alpha_2]u^{n-2}|Du|^2\Delta v + \alpha_2(n-2)u^{n-4}|Du|^4 v \\ & + 2\alpha_2 u^{n-2}(\nabla^2 v Du|Du)_{g_m}. \end{aligned}$$

It follows from a similar argument as in Section 7.2 that

$$P \in C^\omega(U_\vartheta^{2+s}, \mathcal{L}(E_1, E_0))$$

and for every $u \in U_\vartheta^{2+s}$, the principal symbol of $P(u)$ can be computed as

$$\begin{aligned} \hat{\sigma} P(u)(x,\xi) &= u^n(x)(g_m((-i\xi),(-i\xi)))^2 \\ &= \mathsf{d}^{4\beta}(x)(\mathsf{d}^\vartheta u)^n(x)|\xi|^4 \geq (\inf \mathsf{d}^\vartheta u)^n \mathsf{d}^{4\beta}(x)|\xi|^4. \end{aligned}$$

Thus $P(u)$ is *normally ρ-elliptic.*

Theorem VIII.1. *Given any $\beta \geq 1$, suppose that $u_0 \in U_\vartheta^{2+s} := \{u \in bc_\beta^{2+s,\vartheta}(\Omega) : \inf \mathsf{d}^{\beta\vartheta} u > 0\}$ with $0 < s < 1$, $\vartheta = -4/n$. Then for every $f \in bc_\beta^{s,\vartheta}(\Omega)$, equation* (VIII.1) *has a unique local solution*

$$\hat{u} \in C^1_{1/2}(J(u_0), bc_\beta^{s,\vartheta}(\Omega)) \cap C_{1/2}(J(u_0), bc_\beta^{4+s,\vartheta}(\Omega)) \cap C(J(u_0), U_\vartheta^{2+s})$$

existing on $J(u_0) := [0, T(u_0))$ for some $T(u_0) > 0$. Moreover,

$$\hat{u} \in C^\infty(\dot{J}(u_0) \times \Omega).$$

Proof. The proof is essentially the same as that for Theorem VII.4 except that we use Theorem VI.2 instead of Theorem IV.7. □

In the case $\alpha_1 = 0$, we can admit lower regularity for the initial data.

Corollary VIII.2. *Given any $\beta \geq 1$, suppose that $u_0 \in U_\vartheta^{1+s} = \{u \in bc_\beta^{1+s,\vartheta}(\Omega) : \inf \mathsf{d}^{\beta\vartheta} u > 0\}$ with $0 < s < 1$, $\vartheta = -4/n$. Then for every $f \in bc_\beta^{s,\vartheta}(\Omega)$, equation* (VIII.1) *has a unique local* *solution*

$$\hat{u} \in C^1_{3/4}(J(u_0), bc^{s,\vartheta}_\beta(\Omega)) \cap C_{3/4}(J(u_0), bc^{4+s,\vartheta}_\beta(\Omega)) \cap C(J(u_0), U^{1+s}_\vartheta)$$

existing on $J(u_0) := [0, T(u_0))$. *Moreover,*

$$\hat{u} \in C^\infty(\dot{J}(u_0) \times \Omega).$$

In some publications, a more general form of the thin film equation is considered with u^n replaced by $\Psi(u) = u^n + \delta u^3$ with $\delta \geq 0$ and $n \in (0, 3]$. The term δu^3 is sometimes omitted because it is relatively small compared to u^n for $n < 3$ near the free boundary $\text{supp}[u(t, \cdot)]$.

(VIII.2)
$$\begin{cases} \partial_t u + \text{div}(\Psi(u) D\Delta u + \alpha_1 u^{n-1} \Delta u Du + \alpha_2 u^{n-2} |Du|^2 Du) = f & \text{on} \quad \Omega_T; \\ u(0) = u_0 & \text{on} \quad \Omega. \end{cases}$$

For any $u \in U^{2+s}_\vartheta$, it is easy to check that $u^3 \in bc^{2+s,3\vartheta}_\beta(\Omega) \hookrightarrow bc^{2+s,n\vartheta}_\beta(\Omega)$. Now the computations shown above for equation (VIII.1) still hold for the new system undoubtedly.

Corollary VIII.3. *Suppose that the conditions in Theorem VIII.1 are satisfied. Then equation* (VIII.2) *has a unique local solution*

$$\hat{u} \in C^1_{1/2}(J(u_0), bc^{s,\vartheta}_\beta(\Omega)) \cap C_{1/2}(J(u_0), bc^{4+s,\vartheta}_\beta(\Omega)) \cap C(J(u_0), U^{2+s}_\vartheta)$$

existing on $J(u_0) := [0, T(u_0))$ *for some* $T(u_0) > 0$. *Moreover,*

$$\hat{u} \in C^\infty(\dot{J}(u_0) \times \Omega).$$

Remark VIII.4. We may observe that the solution $\hat{u}$ obtained in Theorem VIII.1 is actually a solution to the following initial value problem with conditions on the free boundary $\partial[\text{supp}(u)]$. Indeed, assume that $\text{supp}(u_0) = \bar{\Omega}$ and Ω is a C^6-domain with compact boundary. Let $\Omega(t) :=$

$\mathrm{supp}[u(t,\cdot)]$. If the initial data u_0 satisfies the conditions in Theorem VIII.1, then

$$\text{(VIII.3)}\qquad \begin{cases} \partial_t u + \mathrm{div}(u^n D\Delta u + \alpha_1 u^{n-1}\Delta u Du + \alpha_2 u^{n-2}|Du|^2 Du) = f & \text{in} \quad \Omega(t); \\ u = 0 & \text{on} \quad \partial\Omega(t); \\ u^n \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega(t); \\ u(0) = u_0 & \text{in} \quad \Omega \end{cases}$$

has at least one classical solution. The third condition reflects conservation of mass. This is a generalization of the problem studied in [**23, 48**]. The existence of a solution can be observed from the fact that the solution $\hat{u}$ to the first and fourth lines satisfies

$$\hat{u}(t,\cdot) \in bc_\beta^{4+s,\vartheta}(\Omega) \cap U_\vartheta^{2+s}, \quad t \in \dot{J}.$$

Hence, for $t \in \dot{J}$ there are two continuous positive functions $c(t) < C(t)$ such that

$$\text{(VIII.4)}\qquad c(t) \le \mathsf{d}^{\beta\vartheta}(x)\hat{u}(t,x) \le C(t), \quad x \in \Omega,$$

and

$$\mathsf{d}^{(\vartheta-1)\beta}(x)\hat{u}^n(t,x)|D\Delta\hat{u}(t,x)|_{g_m} \le C(t), \quad x \in \Omega.$$

The second inequality follows from (VII.3), (VII.7) and that $\Delta \in \mathcal{L}(bc_\beta^{4+s,\vartheta}(\Omega), bc_\beta^{2+s,2+\vartheta}(\Omega))$. The above two inequalities imply that for every $t \in \dot{J}$, as $x \to \partial\Omega$

$$|\hat{u}(t,x)| \le C(t)\mathsf{d}^{-\beta\vartheta}(x) \to 0, \quad \hat{u}^n(t,x)|D\Delta\hat{u}(t,x)|_{g_m} \le C(t)\mathsf{d}^{(1-\vartheta)\beta}(x) \to 0.$$

The fact that $\hat{u}(t,\cdot) > 0$ on Ω is a consequence of (VIII.4). Therefore,

$$\text{(VIII.5)}\qquad \mathrm{supp}[\hat{u}(t,\cdot)] = \Omega(t) = \Omega, \quad t \in J,$$

and $\hat{u}$ is indeed a solution to equation (VIII.3). If we seek solutions in the class

$$C^1_{1/2}(J(u_0), bc_\beta^{s,\vartheta}(\Omega)) \cap C_{1/2}(J(u_0), bc_\beta^{4+s,\vartheta}(\Omega)),$$

then $\hat{u}$ is actually the unique solution. Note that the solution to equation (VIII.3) is, in general, not unique unless a third condition is prescribed on the free boundary $\partial[\mathrm{supp}(u)]$. A conventional supplementary condition is to set the contact angle to be zero.

By identifying $\hat{u}, f, u_0 \equiv 0$ on $\mathbb{R}^m \setminus \Omega$, $\hat{u}$ is nothing but a weak solution to the Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{div}(u^n D\Delta u + \alpha_1 u^{n-1}\Delta u Du + \alpha_2 u^{n-2}|Du|^2 Du) = f & \text{on} \quad \mathbb{R}^m_T; \\ u(0) = u_0 & \text{on} \quad \mathbb{R}^m \end{cases}$$

belonging to the class $C_{1/2}(J; W^2_1(\mathbb{R}^m))$ for $\beta \in [1, n/(2n-4)]$ when $n \in (2,3]$, or for all $\beta \geq 1$ while $n \in (0,2]$ in the sense that for all $\phi \in C_0(\mathring{J}; W^2_\infty(\mathbb{R}^m))$

$$\int_{J\times\mathbb{R}^m} \{u\partial_t\phi - \Delta u \mathrm{div}(u^n D\phi) + \alpha_1 u^{n-1}\Delta u(D\phi|Du)_{g_m} + \alpha_2 u^{n-2}|Du|^2(D\phi|Du)_{g_m}\} = -\int_{J\times\mathbb{R}^m} f\phi.$$

To prove this statement, one first observes that, by the *uniform exterior and interior ball condition*, for some sufficiently small $a > 0$ there is some a-tubular neighborhood of $\partial\Omega$, denoted by T_a, such that T_a can be parameterized by

$$\Lambda : (-a, a) \times \partial\Omega \to \mathsf{T}_a : \quad (r, \mathsf{p}) \mapsto \mathsf{p} + r\nu_{\mathsf{p}},$$

where ν_{p} is the inward pointing unit normal of $\partial\Omega$ at p. By the implicit function theorem, there exists some C^5-function Θ such that

$$\Lambda^{-1} : \mathsf{T}_a \to (-a, a) \times \partial\Omega, \quad \Lambda^{-1}(x) = (\mathsf{d}(x), \Theta(x)),$$

where d is defined in (VI.1), and $\Theta(x)$ is the closest point on $\partial\Omega$ to x.

To verify that $\hat{u} \in C_{1/2}(J; W^2_1(\mathbb{R}^m))$, it suffices to check the integrability of $\hat{u}$ near $\partial\Omega$. Since $u \in C_{1/2}(J, bc^{4+s,\vartheta}_\beta(\Omega))$, there exists a positive function $P \in C_{1/2}(J)$ such that

$$\mathsf{d}^{(2+\vartheta)\beta}(x)|\nabla^2\hat{u}(t,x)| \leq P(t), \quad x \in \Omega, \quad t \in \dot{J}.$$

Then

$$\int_{\mathsf{T}_a} |\nabla^2\hat{u}(t,x)|\, dx \leq P(t) \int_{\mathsf{T}_a\cap\Omega} \mathsf{d}^{-(2+\vartheta)\beta}(x)\, dx \leq MP(t)\int_0^a\int_{\partial\Omega} r^{-(2+\vartheta)\beta}\, d\mu\, dr,$$

which is finite iff $n \in (0,2]$, or $\beta \in [1, n/(2n-4)]$ and $n \in (2,3]$. The last line follows from the compactness of $\partial\Omega$ and [**74**, formula (25)]. The argument for lower order derivatives of $\hat{u}$ is similar.

What is more, (VIII.5) states that the support of $\hat{u}$ has the global small term waiting-time property for all dimensions, that is, there exists some $T^* > 0$ such that

$$\text{supp}[u(t,\cdot)] = \text{supp}[u(0,\cdot)], \quad t \in (0, T^*). \tag{VIII.6}$$

To the best of the author's knowledge, this is the first known result for the generalized thin film equation (VIII.1). This result also supplements those in [**23, 48, 84**] for the case dimension $m \geq 4$ with $n \in (0,3]$ and to domains without the *external cone property* with $n \in [2,3]$. For any $y \in \partial\Omega$, Ω is said to satisfy *the external cone property* at y if for some $\theta \in (0, \pi/4)$ there is an infinite cone $\mathcal{C}(y,\theta)$ with vertex y and opening angle θ such that

$$\text{supp}[u_0] \cap \mathcal{C}(y,\theta) = \emptyset.$$

See [**48**, Theorem 4.1] for more details. A domain Ω is said to enjoy *the external cone property* if it satisfies this property at every $y \in \partial\Omega$. Note that any $u_0 \in U_\vartheta^{2+s}$ fulfils the flatness condition of the initial data in [**48**, Theorem 4.1].

2. Generalized Heston operator

Let $\Omega = \mathbb{R} \times \mathbb{R}_+$. One can readily check that

$$(\mathsf{M}, g; \rho) := (\Omega, g_2; y), \quad g_2 = dx^2 + dy^2,$$

is a *singular manifold* with uniformly mild wedge end.

Let $J := [0,T]$. Consider the following initial value problem.

$$\begin{cases} u_t + \mathscr{A} u = f & \text{on} \quad \Omega_T \\ u(0) = u_0 & \text{on} \quad \Omega_0. \end{cases} \tag{VIII.7}$$

Here with $\alpha < 2$ and $z = (x,y)$

$$\begin{aligned} &\mathscr{A}(t,z)u(t,z) \\ :&= -\partial_i(y^\alpha a^{ij}\partial_j u(t,z)) + y^{\alpha-1}b^j(t,z)\partial_j u(t,z) + y^{\alpha-2}c(t,z)u(t,z), \end{aligned}$$

where $b^j(t,z) := b_0^j(t,z) + yb_1^j(t,z)$, and $c(t,z) := c_0(t,z) + yc_1(t,x) + y^2c_2(t,z)$. We impose the following assumptions on the coefficients.

(GH1)

$$(a^{ij}) = \frac{1}{2}\begin{pmatrix} 1 & \varrho\sigma \\ \varrho\sigma & \sigma^2 \end{pmatrix}, \quad \sigma > 0, \quad -1 < \varrho < 1,$$

(GH2) $b_i^j, c_i \in C^s(J; L_\infty(\Omega))$ for some $s \in (0,1)$.

This problem corresponds to the case $\lambda = 2 - \alpha$ in (V.1).

While $\alpha = 1$, $b_i^j \equiv$ const, $c_0 \equiv 0$, $c_1, c_2 \equiv$ const, $\mathscr{A}$ is called the Heston operator. (VIII.7) generalizes the Heston model in the following sense. It does not only exhibit degeneracy along the boundary, but boundary singularities may also appear. When $\alpha > 0$, the diffusion term is degenerate. Whereas $\alpha < 0$ corresponds to the situation that boundary singularities show for the highest order term.

The Heston operator has been studied in [**36, 37, 38**] and the references therein. In this section, we focus on the case $\alpha \neq 1$. The study of this kind of problem is new since the Schauder approach in the aforementioned articles relies on the particular choice the degeneracy factor y.

One can check by direct computations that after a change of spatial variables and rescaling of the temporal variable. Equation (VIII.7) can be transformed into

$$\begin{cases} u_t + \hat{\mathscr{A}} u = f & \text{on} \quad \Omega_T \\ u(0) = u_0 & \text{on} \quad \Omega_0. \end{cases}$$

Here

$$\begin{aligned} \hat{\mathscr{A}}(t,z)u(t,z) := &- \partial_j(y^\alpha \partial_j u(t,z)) + y^{\alpha-1}(\hat{b}_0^j(t,z) + y\hat{b}_1^j(t,z))\partial_j u(t,z) \\ &+ y^{\alpha-2}(\hat{c}_0(t,z) + y\hat{c}_1(t,x) + y^2\hat{c}_2(t,z))u(t,z), \end{aligned}$$

where $\hat{b}_i^j, \hat{c}_i \in C^s(J; L_\infty(\Omega))$. By Corollary V.29,

Theorem VIII.5. *Suppose that $\Omega = \mathbb{R} \times \mathbb{R}_+$. Let $s \in (0,1)$, $\alpha \in (-\infty, 1) \cup (1,2)$, $\lambda' \in \mathbb{R}$ and $1 < p < \infty$. Assume that (GH1) and (GH2) are satisfied. Then given any*

$$(f, u_0) \in C^s(J; L_p^{\lambda'}(\Omega)) \times L_p^{\lambda'}(\Omega),$$

the equation (VIII.7) *has a unique solution*

$$u \in C^{1+s}(J \setminus \{0\}; L_p^{\lambda'}(\Omega)) \cap C^s(J \setminus \{0\}; W_p^{\lambda'+\alpha-2}(\Omega)).$$

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