ASYMPTOTICS FOR FABER POLYNOMIALS AND POLYNOMIALS ORTHOGONAL OVER REGIONS IN THE COMPLEX PLANE

By

Erwin Miña Díaz

Dissertation
Submitted to the Faculty of the
Graduate School of Vanderbilt University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY
in
Mathematics

August, 2006
Nashville, Tennessee

Approved:
Professor Edward B. Saff
Professor Akram Aldroubi
Professor Douglas P. Hardin
Professor Dechao Zheng
Professor Prodyot K. Basu

To my beloved family, specially to my parents, Magaly and Tomás, whose teachings and uncountable sacrifices are the source of my achievements

ACKNOWLEDGEMENTS

I would like to extend my appreciation to my advisor, Professor Edward B. Saff, for his guidance and support during my doctoral studies. He has suggested most of the problems investigated in the present work and contributed with very insightful ideas.

I would also like to thank the other members of my committee, Professors Akram Aldroubi, Prodyot Basu, Douglas Harding and Dechao Zheng, for their general help and support.

I am very indebted to Professors Laurent Baratchart, Guillermo López Lagomasino, Andrei Martínez Finkelshtein and Nikos Stylianopoulos, for sharing their ideas and suggesting possible ways to follow when discussing several of the problems considered in this dissertation.

I am particularly thankful to Professor Baratchart and the APICS project members for their kindness and warm hospitality toward me during the semester I spent at INRIA, Sophia-Antipolis, France. That research stay was possible thanks to the support provided by the NSF Vanderbilt-INRIA collaborative research grant for graduate students.

Finally, I would like to express my deepest gratitude to my friends and teachers, Manuel Bello Hernández and Guillermo López Lagomasino for their constant encouragement and invaluable help along my career path.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In this dissertation we study the asymptotic behavior and zero distribution of certain polynomials that are associated with a Jordan curve L in the complex plane. We first consider in Chapter II the so-called Faber polynomials for L, and in the subsequent chapters we deal with polynomials that are orthogonal over the interior of L. A further description of these polynomials and the results obtained for them is given in what follows.

I.1 Faber polynomials

Let $\phi(z)$ be a function with a Laurent expansion at infinity of the form

$$\phi(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \cdots, \quad b_1 > 0\,. \tag{1}$$

The nth Faber polynomial $F_n(z)$, $n = 0, 1, \ldots$, associated with ϕ is the polynomial part of the Laurent expansion at infinity of the function $[\phi(z)]^n$. Thus,

$$[\phi(z)]^n = F_n(z) + \mathcal{O}\left(z^{-1}\right) \quad \text{as } z \to \infty. \tag{2}$$

These polynomials were introduced in 1903 by G. Faber in connection with the problem of generalizing the classical Taylor series from the unit disk to a simply connected domain. The monograph by Suetin [30] and the paper by Curtiss [2] are excellent reference sources for the many applications that Faber polynomials have found in approximation theory, orthogonal polynomials and complex function theory.

Faber polynomials can alternatively be introduced by means of their generating function. If ψ denotes the inverse map of ϕ, then for fixed $z \in \mathbb{C}$,

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \tag{3}$$

where the series on the right-hand side of (3) converges uniformly on any closed region $\{w : |w| \geq r\}$ on which the generating function on the left-hand side of (3) is analytic as a function of the variable w.

In particular, it follows that if $|z| < r$, then

$$F_n(z) = \frac{1}{2\pi i} \oint_{|w|=r} \frac{w^n \psi'(w) dw}{\psi(w) - z}. \tag{4}$$

Assume now that 1 is the smallest number among all those $r \geq 0$ for which ψ has an analytic and univalent continuation from a neighborhood of ∞ (where it is originally defined) to the exterior $\{w : |w| > r\}$ of the circle of radius r. Indeed, we can always normalize ϕ (multiplying it by a suitable number) to have our assumption satisfied. Denote by Δ_1 the exterior of the unit circle, and let Ω be the image by ψ of Δ_1. Then, it is a rather straightforward consequence of the definition of F_n that

$$F_n(z) = [\phi(z)]^n (1 + o(1)) \tag{5}$$

uniformly on closed subsets of Ω as $n \to \infty$ (notice that ϕ has an analytic and univalent continuation to Ω).

Our interest is in the behavior of $F_n(z)$ for points z in the complement of Ω, assuming that $L := \partial\Omega$ is a Jordan curve such that ψ has a singularity on the unit circle. In other words, we assume that ψ has a one-to-one continuous extension to $\overline{\Delta_1}$, but cannot be analytically continued across the unit circle.

Ullman [32] proved a general result implying that in this case, all points of L

attract zeros of the Faber polynomials. However, this does not discard the possibility that many other points in the interior domain G of L (perhaps all of them!) could attract zeros of the F_n's as well. The fact is that the single assumption that L is a Jordan curve is too weak to predict the behavior of F_n on both L and its interior G. Thus, several works have study this question under additional assumptions on the smoothness of L and/or the boundary behavior of ψ near the unit circle. In particular, under some of these conditions, the domain of validity of (5) has been extended to include L, or portions of L (cf. [30], [23]). However, the results for $z \in G$ are restricted to estimates on the rate of decay of the Faber polynomials on G ([30, p. 61], [6]).

In Chapter II of this dissertation we study the behavior of the Faber polynomials for the specific case when the Jordan curve L is piecewise analytic with no inner cusps (see Figure 3 in page 16). We also impose an extra condition to the smooth corners and outer cusps of L (see Section II.2 for details).

We prove that (5) also holds *uniformly* on any closed subset of $\overline{\Omega}$ separated from the nonsmooth corners of L (see Theorem II.2.1). In addition, we also give the *exact rate of decay* of the $o(1)$ term in (5). We emphasize the word "uniformly" because a previous theorem in [23] had already established the validity of (5) for points z in L (nonsmooth corners excluded) but in the pointwise sense. Most interesting of all is the asymptotic representation that we obtain for $F_n(z)$ for values of $z \in G$ (see Theorem II.2.4). This representation shows that the behavior of $F_n(z)$ in G is ruled by only some of the corners of L. For instance, if L has neither smooth corners nor cusps, then the "dominant" corners are those with smaller exterior angles. In particular, we derive from this representation fine statements on the location and distribution of the zeros of the F_n's (see Section II.3 for details).

To derive our asymptotic formulas, we make use of the relationship that exists between the singularities of the generating function in (3) and the behavior of the coefficients $F_n(z)$ of its Laurent expansion at ∞. When L is piecewise analytic, the singularities of ψ on its circle of convergence are finitely many, and the behavior of ψ near each singularity is known from the work of Lehman and others (see [13] and the references therein). The way this specific type of singularities affects the coefficients $F_n(z)$ is discerned by analyzing the behavior of the integral representation (4).

I.2 Polynomials orthogonal over regions

Let G be the interior of a closed Jordan curve $L = \partial G$ in the complex plane, and let m denote the two-dimensional Lebesgue measure. Let $\{P_n(z)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to area measure over G. This is the sequence of polynomials

$$P_n(z) = \kappa_n z^n + \cdots \,, \quad \kappa_n > 0, \quad n \geq 0,$$

satisfying the orthonormality conditions

$$\int_G P_n(z)\overline{P_m(z)}\, dm(z) = \delta_{n,m}\,, \quad n, m \geq 0.$$

In our investigation, we are concerned with the general question of describing the asymptotic behavior and limiting distribution of the zeros of the polynomials P_n.

This question has been studied to some extent in previous works (see e.g., [17], [14]). In [14], Levin, Saff and Stylianopoulos (L.S.S.) found that the zero distribution of the polynomials P_n is related to the analytic continuation properties of a conformal map φ of G onto the unit disk. For example, a simplified version of their fundamental result (Thm. 2.1) is the following: *if the mapping φ has a singularity on the boundary L of G, then every point of L attracts zeros of the P_n's.*

If the map φ can be analytically continued across the Jordan curve L, then either L is analytic or L is a finite union of analytic arcs joining at corners having interior

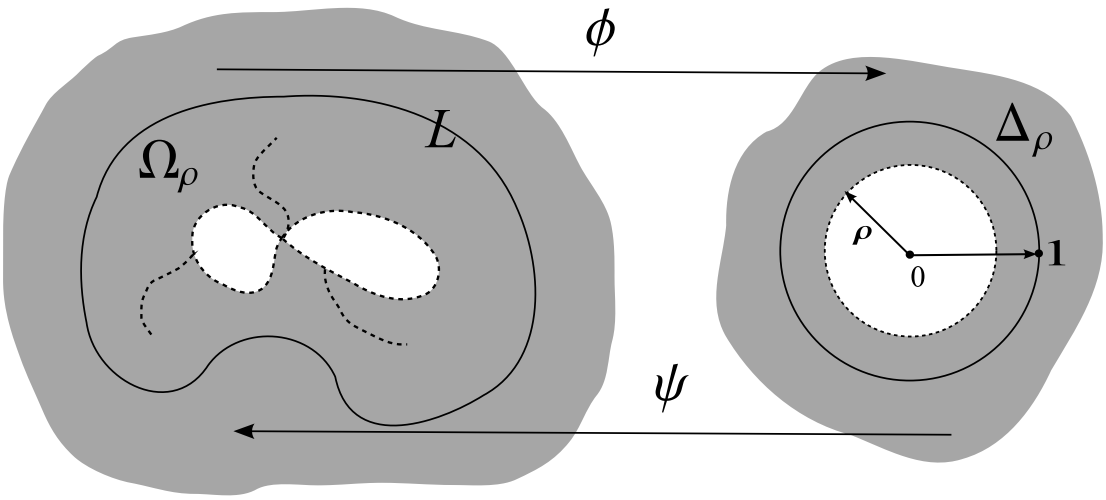


Figure 1: Sets Ω_ρ and Δ_ρ. $\partial\Omega_\rho$ = dotted lines.

angles of the form π/N, $N \geq 2$ an integer.

The existing results for the second case (i.e., when L is not analytic) are limited to the analysis of some particular cases. We do not explicitly investigate this case, although some results will follow as a particular case of Theorem V.2.7 of Chapter V for a curve L formed by the union of two circular arcs.

In Chapter III of this dissertation we investigate the first situation, that is, when the curve L is analytic. Let ψ be the conformal map of the exterior Δ_1 of the unit circle onto the exterior Ω of L, normalized so that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. Let ρ be the smallest number among all those $r \geq 0$ for which ψ has an analytic and univalent continuation from Δ_1 onto the exterior of the circle of radius r. Because of the analyticity of L, $\rho < 1$. Denote by Δ_ρ the exterior of the circle of radius ρ , and let $\Omega_\rho = \psi(\Delta_\rho)$ be the image of Δ_ρ by ψ. Then, $\Omega_\rho \supset \Omega$ and ϕ (the inverse map of ψ) is analytic and univalent on $\Omega_\rho \setminus \{\infty\}$ (see Figure 1 above).

Carleman [1] proved that

$$P_n(z) = \sqrt{\frac{n+1}{\pi}}\, \phi'(z)[\phi(z)]^n \left(1 + o(1)\right), \tag{6}$$

uniformly on $\overline{\Omega}$ as $n \to \infty$. This result was later improved by Korovkin [11] (it also

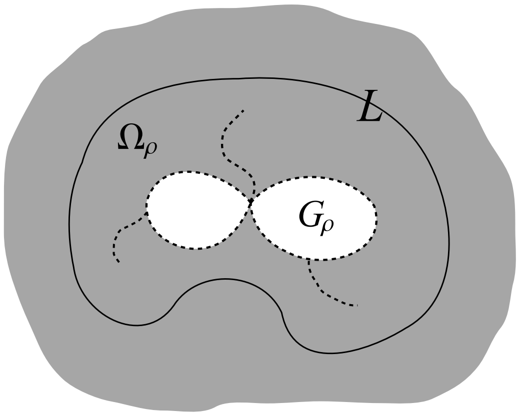


Figure 2: Two possible scenarios: $G_\rho \neq \emptyset$ (left, white region), $G_\rho = \emptyset$ (right).

appears in Johnston [10]), who proved that the same formula holds uniformly on any closed set of Ω_ρ (see Section III.1 for more details).

Carleman's formula (6) implies that the zeros of the P_n's accumulate (as n gets large) on the compact set $\mathbb{C} \setminus \Omega_\rho$. In particular, if the interior G_ρ of $\mathbb{C} \setminus \Omega_\rho$ is empty, then every point of $\partial\Omega_\rho$ attracts zeros of the P_n, and these distribute, in the limit, as the so-called *equilibrium measure* of $\partial\Omega_\rho$ (see [19, Prop. 2.4]). However, no results have been given for the most interesting case when $G_\rho \neq \emptyset$ (see Figure 2 above).

Given the simplicity of the curve L, it is rather surprising that (6) is the only known result on the asymptotic behavior of the P_n's.

Our main result in Chapter III is, in fact, an improvement of Carleman's formula (see Thm. III.2.1). We prove the following integral representation for P_n: *if φ is a conformal map of G onto the unit disk, then*

$$P_n(z) = \frac{\varphi'(z)}{\sqrt{\pi(n+1)}} \cdot \frac{1}{2\pi i} \oint_L \frac{\varphi'(t)[\phi(t)]^{n+1}dt}{[\varphi(t)-\varphi(z)]^2} + o(1), \tag{7}$$

uniformly on compact subsets of G as $n \to \infty$.

It is an easy exercise to check that (7) implies (6). The natural question that arises is whether the interior mapping φ plays a role in the behavior of P_n if G_ρ is not empty. The answer is positive.

To show this, we investigate the case when the boundary L_ρ of Ω_ρ is a piecewise analytic Jordan curve without smooth corners or cusps. Employing the same method of proof for our results on Faber polynomials, we use the integral representation (7) to prove that (6) holds uniformly on any closed set of $\overline{\Omega}_\rho$ that does not contain corners of L_ρ. The asymptotic behavior of P_n at each corner of L_ρ is also given (see Theorem III.2.2).

Furthermore, we obtain an asymptotic representation for the polynomials P_n in G_ρ from which fine statements on the location and distribution of the zeros of these polynomials can be derived (see Section III.3). This asymptotic representation (Theorem III.2.4) shows that the asymptotic behavior of P_n inside G_ρ is ruled by only some of the corners of L_ρ, that is, those with smaller exterior angles.

In Chapter IV we study the particular case in which the analytic curve L is the lemniscate

$$L := \left\{ z : |z^K - 1| = r^K \right\},$$

where $K \in \mathbb{N} \setminus \{1\}$ and $r > 1$ are two (fixed) given numbers. For this curve,

$$G_\rho = \left\{ z : |z^K - 1| < 1 \right\}$$

is a "rose of K petals" (see Figure 14 in page 79) consisting of K open components.

This case is not covered by the above results because although

$$L_\rho = \left\{ z : |z^K - 1| = 1 \right\}$$

is a piecewise analytic curve, it is not, however, a Jordan curve. In fact, we will prove that this time the zeros of the corresponding orthonormal polynomials behave differently (see Section IV.1 for details). With this example we show that the behavior of P_n is also affected by the connectivity of the open set G_ρ.

It is worth noticing that the analysis of this case is carried out by reducing the problem to a similar one for orthogonal polynomials on the unit circle. To treat the new problem we prove in Section IV.2 a theorem of independent interest about the asymptotic behavior of polynomials orthogonal on the unit circle with respect to positive analytic weights.

In Chapter V, we study the asymptotic behavior of the sequences of polynomials that are orthogonal with respect to "analytic weights" over the interior G of a Jordan curve L. We consider a not identically zero analytic function w defined on G and satisfying the integrability condition

$$\int_G |w(z)|^2 \, dm(z) < \infty \,,$$

together with its corresponding sequence of orthonormal polynomial $\{P_n(z;w)\}_{n=0}^{\infty}$ defined by the relations

$$P_n(z;w) = \kappa_n^w z^n + \cdots \,, \qquad \kappa_n^w > 0, \qquad n \geq 0,$$

$$\int_G P_n(z;w)\overline{P_m(z;w)}|w(z)|^2 dm(z) = \delta_{n,m}, \quad n,\, m \geq 0$$

(recall m is the area measure).

Here, an essential role is played by the reproducing kernel $K_w(z,\zeta)$ of the Hilbert space $\mathcal{B}_w^2(G)$ consisting of all analytic functions on G that are square integrable with respect to $|w|^2 dm|_G$. For fixed $\zeta \in G$, $K_w(\cdot,\zeta)$ is the unique function in $\mathcal{B}_w^2(G)$ having the *reproducing property*

$$f(\zeta) = \int_G f(z)\overline{K_w(z,\zeta)}|w(z)|^2 dm(z) \quad \text{for all } f \in \mathcal{B}_w^2(G) \,.$$

While the zeros of the polynomials that are orthogonal with respect to area measure over G are only affected by the geometry of L, the zero distribution of the polynomials

$P_n(z;w)$ depends both on the geometry of L and on the analytic properties of the weight w. This is so because the zeros of these polynomials are influenced by the relative position of the singularities of the functions $K_w(\cdot,\zeta)$, $\zeta \in G$, and $K_w(z,\zeta)$ is, in turn, constructed from w and a conformal map φ of G onto the unit disk.

For instance, we prove Theorem V.2.1, which is a generalization of the fundamental theorem (L.S.S.) mentioned above and characterizes, in terms of the behavior of the zeros of the $P_n(z;w)$'s, the case when there is at least one $\zeta \in G$ for which $K_w(\cdot,\zeta)$ has a singularity on $L = \partial G$. In this case, every point of L attracts zeros, and for the values of n belonging to some subsequence, the zeros of $P_n(z;w)$ follow the equilibrium distribution of the curve L.

When w has finitely many zeros, the kernel $K_w(z,\zeta)$ can be constructed from the zeros of w and the map φ by using an iterative procedure (Proposition V.3.4) that essentially goes back to Nehari [20]. Using this procedure we obtain formulas for $K_w(z,\zeta)$ of crucial importance for locating its singularities (see Lemma V.3.6). We also give a determinant representation for it valid when w has simple zeros (see Proposition V.3.5).

To investigate the situation in which **no** $K_w(\cdot,\zeta)$ has a singularity on L, we study two specific cases in detail.

Firstly, L is taken to be the unit circle (the nicest analytic curve) and w a meromorphic function (a rather complicated weight). We prove that for n varying through some subsequence of the natural numbers, the zeros of $P_n(z;w)$ distribute *uniformly* over a critical circle lying inside the unit disk and determined by the relative position of the zeros and poles of w (see Theorem V.2.3).

Secondly, L is taken to be a piecewise analytic curve bounded by two circular arcs forming a π/N-angle, and w is an entire function. In this case, the zeros of $P_n(z;w)$ distribute according to a measure that is supported on a "bubble-shaped" set lying inside L (see Figure 17 in page 99 and Theorem V.2.7).

The results contained in Chapter V have been published in [19] and were obtained in collaboration with Dr. Edward B. Saff and Dr. Nikos S. Stylianopoulos.

CHAPTER II

FABER POLYNOMIALS FOR PIECEWISE ANALYTIC JORDAN CURVES

II.1 Introduction

Let L be a closed Jordan curve in the complex plane $\mathbb{C}$ and let $\Omega = \mathrm{ext}(L)$ be the unbounded component of $\overline{\mathbb{C}} \setminus L$. By the Riemann mapping theorem, there is a unique conformal map $\phi(z)$ of Ω onto the exterior of the unit circle $\{w : |w| > 1\}$ with a Laurent expansion at infinity of the form

$$\phi(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \cdots , \quad b_1 > 0 \, .$$

The nth Faber polynomial $F_n(z)$, $n = 0, 1, \ldots$, associated with the curve L is the polynomial part of the Laurent expansion at infinity of the function $[\phi(z)]^n$. Thus,

$$[\phi(z)]^n = F_n(z) + \mathcal{O}\left(z^{-1}\right) \quad \text{as } z \to \infty. \tag{8}$$

Since their introduction by G. Faber in 1903, Faber polynomials have constituted an active subject of research with important applications in approximation theory and complex function theory (see, e.g., [2], [30], [37], and the references therein). In particular, several works have been devoted to the study of the asymptotic behavior of the Faber polynomials and their zeros, and it is precisely the purpose of this chapter to investigate this question for the case when the curve L is piecewise analytic with no inner cusps.

Before mentioning those results in the existing literature that are relevant to our investigation, let us state some basic, but very useful well-known facts that are a direct consequence of the definition of Faber polynomials.

Because L is a Jordan curve, the mapping ϕ has a one-to-one continuous extension to $\overline{\Omega}$, which we also denote by ϕ. Let ψ be the inverse of ϕ. For every $r \geq 1$, let

$$L_r := \{z : |\phi(z)| = r\}, \quad \Omega_r := \operatorname{ext}(L_r), \quad G_r := \operatorname{int}(L_r),$$

so that $L_1 = L$, $\Omega_1 = \Omega$ is the exterior of L, and G_1 is the interior of L. Then, from the Cauchy integral formula and Cauchy theorem one immediately gets from (8) that for all $R > 1$,

$$F_n(z) = \frac{1}{2\pi i} \oint_{L_R} \frac{[\phi(\zeta)]^n d\zeta}{\zeta - z}, \quad z \in G_R\,, \tag{9}$$

$$F_n(z) = [\phi(z)]^n + \frac{1}{2\pi i} \oint_{L_R} \frac{[\phi(\zeta)]^n d\zeta}{\zeta - z}, \quad z \in \Omega_R\,. \tag{10}$$

It follows from (10) and the maximum modulus principle for analytic functions that for every $1 < \sigma < \rho$,

$$F_n(z) = [\phi(z)]^n \left[1 + \mathcal{O}\left(\frac{\sigma^n}{\rho^n}\right)\right], \tag{11}$$

uniformly on $\overline{\Omega}_\rho$ as $n \to \infty$. Moreover, if L is rectifiable, then (10) is also valid for $R = 1$, and σ can be replaced by 1 in (11).

The asymptotic representation (11) readily implies that for any $\rho > 1$, there is a natural number $N(\rho)$ such that $F_n(z) \neq 0$ for all $z \in \overline{\Omega}_\rho$ whenever $n > N(\rho)$. Hence, all possible accumulation points of the zeros of the Faber polynomials lie on $L \cup G_1$.

However, for discerning the behavior of $F_n(z)$ in the remaining points of the complex plane, formula (9) does not seem to be as appropriate as the one that results from it through the change of variables $z = \psi(w)$:

$$F_n(z) = \frac{1}{2\pi i} \oint_{\mathbb{T}_R} \frac{t^n \psi'(t) dt}{\psi(t) - z}, \quad z \in G_R\,. \tag{12}$$

The right-hand side of (12) is the Cauchy integral formula for the coefficient of $1/w^{n+1}$ in the Laurent expansion of the function $\psi'(\cdot)/(\psi(\cdot)-z)$ at ∞; in other words,

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{n=0}^{\infty}\frac{F_n(z)}{w^{n+1}}. \tag{13}$$

This is the generating function of the Faber polynomials. From the discussion in Chapter I, it is now clear that the asymptotic behavior of the F_n's (and by consequence, that of their zeros) is influenced by the nature of the singularities of ψ via the generating function.

One way in which one can investigate such an influence is precisely by studying the behavior as $n \to \infty$ of the complex integrals in (12). This idea was successfully exploited by J. L. Ullman in his fundamental paper [32] (see also [33]). Using (12) as the key ingredient, Ullman was able to characterize the set of possible accumulation points of the zeros of the Faber polynomials defined by (13) for an arbitrary map ψ with a simple pole at ∞ (regardless of the geometric significance of ψ). A later complement to his theorems was produced by A. B. J. Kuijlaars and E. B. Saff in [12]. Using methods of potential theory, they obtained results on the weak*-convergence of the sequence of normalized counting measures of the zeros of the F_n's (see Section II.3 for definitions).

Ullman's results are of great generality, but do not say much in more specific cases like the one we are interested in, namely, when the curve L is piecewise analytic with no inner cusps (some additional conditions will be imposed on the smooth corners and outer cusps of L). Loosely speaking, we shall deal with the case when ψ has a finite number of singularities $w_1, ..., w_s$ on the unit circle, each w_k causing ψ to transform a small circular arc centered at w_k onto two analytic arcs that meet at $z_k = \psi(w_k)$ at a specified angle. Ullman's contribution to this case limits itself to ensure that every point of the curve L is an accumulation point of the zeros of the F_n's, while

Kuijlaars and Saff's improvement establishes the existence of a subsequence of the sequence of normalized counting measures of the zeros of the F_n's that converges in the weak*-sense to the equilibrium measure of L.

We will be able to say more. For example, a consequence of the asymptotic formulas that we derive for F_n is that any smooth portion of L will be eventually free of zeros of F_n, and that every compact set lying on the interior of L contains at most a finite (independent of n) number of zeros of F_n. This already implies that, indeed, the full sequence of normalized counting measures of the zeros of the F_n's converges to the equilibrium measure of L.

As for the asymptotic behavior of the Faber polynomials, the estimate (11) can be improved for a piecewise analytic curve L . Theorem 1.1 in I. E. Pritsker's paper [23] implies that if $z \in L$ is not a corner, then

$$F_n(z) = [\phi(z)]^n(1 + o(1)), \quad (n \to \infty) \tag{14}$$

while if z_0 is a corner of L, then

$$F_n(z_0) = \alpha[\phi(z_0)]^n(1 + o(1)), \quad (n \to \infty) \tag{15}$$

where α is the exterior angle formed by the two analytic arcs of L that meet at z_0.

Thus, at least in the pointwise sense (recall (11)), (14) is true for every z that lies in the exterior or the smooth portions of L. We will show that this is also true in the *uniform* sense and will give the exact rate of decay of the $o(1)$ term in (14) (see Theorem II.2.1).

As for the behavior of F_n in the interior of L, the most recent results are due to D. Gaier, who gave in [6] uniform estimates on the decay of F_n of the form

$$n^\lambda F_n(z) = \mathcal{O}\,(1)\,, \quad (n \to \infty) \tag{16}$$

for z lying on an arbitrary compact set of G_1. The constant $\lambda > 0$ depends on the exterior angles at the corners of L (see Section II.2 below).

We substantially improve this result by deriving an asymptotic representation for F_n inside L that, roughly speaking, gives us the relevant part of the $\mathcal{O}(1)$ term in (16) (see Theorem II.2.4).

Summarizing, in this chapter we shall employ the same integral representation (12) to obtain finer results on the asymptotic behavior of the Faber polynomials corresponding to a piecewise analytic Jordan curve with no inner cusps. These are Theorems II.2.1 and II.2.4 of Section II.2. Their proof is postponed to Section II.4. In Section II.3 we state and prove a series of corollaries to these theorems about the zero distribution of the Faber polynomials.

II.2 Asymptotic behavior of F_n

Let L be a piecewise analytic Jordan curve with corners at the points $z_1, z_2, \ldots, z_s$, $s \geq 1$, such that for all $1 \leq k \leq s$, the two analytic arcs that meet at z_k form an *exterior* angle $\lambda_k \pi$, $0 < \lambda_k \leq 2$. Thus, inner cusps are excluded. Recall that, by the definition of a piecewise analytic curve, each arc of L with z_k as one of its endpoints is part of a longer analytic simple arc containing z_k as an interior point.

Let $w_k := \phi(z_k)$, $1 \leq k \leq s$, so that $|w_1| = |w_2| = \cdots = |w_s| = 1$. The mapping ψ has an asymptotic expansion about w_k involving terms of the form

$$c_{i,j,m}^{(k)} (w - w_k)^{i + j\lambda_k} (\log(w - w_k))^m, \quad c_{i,j,m}^{(k)} \in \mathbb{C}, \quad i, j, m \in \mathbb{Z}_+. \tag{17}$$

The exact meaning of this expansion is explained in Section II.4.1 below (see also [13]), but for the purpose of stating our main results, a weaker version is sufficient.

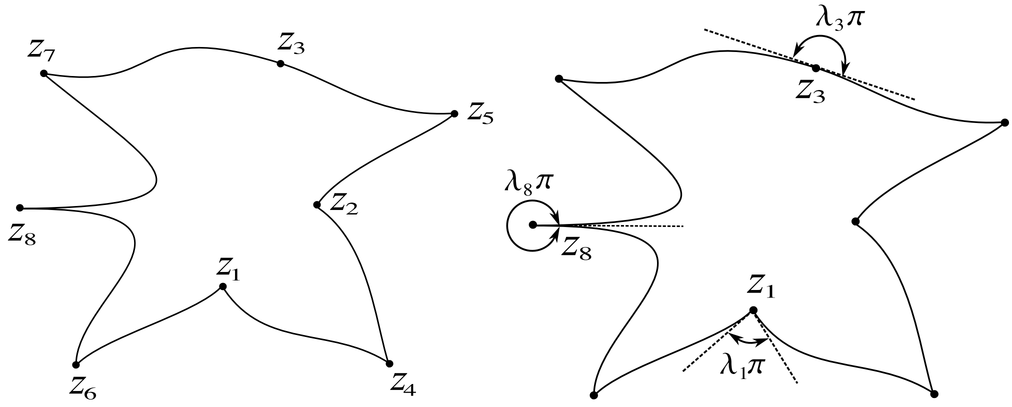


Figure 3: A piecewise analytic Jordan curve L with eight corners $z_1, \cdots, z_8$.

If $0 < \lambda_k < 2$, $\lambda_k \neq 1$, then as $w \to w_k$ from the exterior of the unit circle,

$$\psi(w) = z_k + A_k(w - w_k)^{\lambda_k} (1 + o(1)), \quad A_k \neq 0. \tag{18}$$

If $\lambda_k \in \{1, 2\}$, then terms of the form (17) with $m \geq 1$ may occur in the expansion of ψ about w_k, and **we assume L is such that this will always be the case if $\lambda_k \in \{1, 2\}$**[1].

Then, if $\lambda_k \in \{1, 2\}$, there are positive integers r_k, m_k, with $r_k \geq \lambda_k$, $1 \leq m_k \leq \lfloor r_k/\lambda_k \rfloor$, such that as $w \to w_k$ within the exterior of the unit circle,

$$\begin{aligned}\psi(w) = z_k + \sum_{i=0}^{r_k-1} c_{i,1,0}^{(k)} (w - w_k)^{i+\lambda_k} \\ + A_k (w - w_k)^{r_k+\lambda_k} \left(\log(w - w_k)^{m_k}\right) (1 + o(1)), \quad A_k \neq 0\,.\end{aligned} \tag{19}$$

Thus, A_k is the coefficient of the first term in the expansion (18)-(19) having a singularity at w_k. It is known (see [22], p. 59) that if $\lambda_k = 1$, then $r_k = 1$ if, and only if, the curvatures of the arcs that meet at z_k coincide. In this case, $m_k = 1$.

The branches of the functions of the form $(w - w_k)^\beta$, $(\log(w - w_k))^m$ that appear

[1]Suppose $\lambda_k \in \{1, 2\}$. If a term of the form (17) with $m \geq 1$ occurs in the expansion of ψ about w_k, then necessarily w_k is a singularity of ψ. We do not know whether the converse of this statement is also true.

in these expansions can be taken arbitrarily. However, for the sake of definiteness, we will specify them as follows. Let

$$\theta_k := \arg(w_k), \quad 0 \leq \theta_k < 2\pi, \quad 1 \leq k \leq s\,.$$

Then, the values of said functions along the half line $\{w = re^{i\theta_k} : 1 < r < \infty\}$ are defined to be

$$(w - w_k)^\beta = (r-1)^\beta e^{i\beta\theta_k}, \quad (\log(w - w_k))^m = (\log(r-1) + i\theta_k)^m\,. \tag{20}$$

From the expansion (18)-(19) of ψ about w_k, we associate to each z_k the number A_k, the numbers r_k and m_k whenever $\lambda_k \in \{1, 2\}$, and the following pair:

$$(\Lambda_k, M_k) := \begin{cases} (\lambda_k, 0), & \text{if } \lambda_k \notin \{1,2\}, \\ (r_k + \lambda_k, m_k - 1), & \text{if } \lambda_k \in \{1,2\}. \end{cases} \tag{21}$$

Observe that $\Lambda_k \geq 2$ if $\lambda_k \in \{1, 2\}$.

If (a_1, b_1), (a_2, b_2) are two pairs of positive real numbers, we will say that $(a_1, b_1) < (a_2, b_2)$ if either $a_1 < a_2$, or $a_1 = a_2$ and $b_1 > b_2$. Therefore, $(a_1, b_1) < (a_2, b_2)$ if, and only if, $n^{-a_1}(\log n)^{b_1}$ decreases slower than $n^{-a_2}(\log n)^{b_2}$ as $n \to \infty$.

Hereafter, we suppose that the z_k's have been numbered in such a way that

$$(\Lambda_1, M_1) = \cdots = (\Lambda_u, M_u) < (\Lambda_{u+1}, M_{u+1}) \leq \cdots \leq (\Lambda_s, M_s)\,,$$

with $u \in \{1, 2, \ldots, s\}$.

Recall that Ω denotes the exterior of the curve L. We will denote its interior domain by G (instead of by G_1, as we had done in the previous section).

Then, with this notation we have:

Theorem II.2.1. *(a) Uniformly on any closed set $E \subset \overline{\Omega} \setminus \{z_k : \lambda_k \neq 1\}$:*

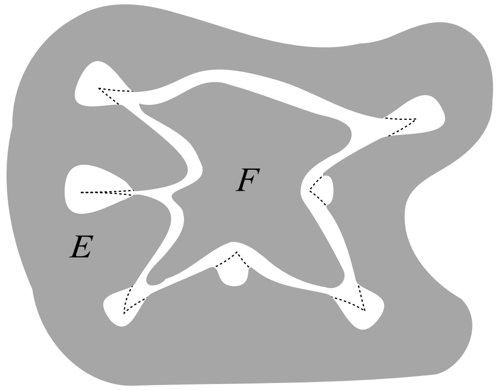


Figure 4: Closed sets E and F as in Theorems II.2.1 and II.2.4.

$$F_n(z) = [\phi(z)]^n + \mathcal{O}\left(\frac{(\log n)^M}{n^\Lambda}\right) \quad (n \to \infty), \tag{22}$$

where (Λ, M) *is the smallest element of the set*

$$\{(\Lambda_1, M_1)\} \cup \{(r_k, M_k) : z_k \in E,\ \lambda_k = 1\}.$$

(b) For every non-smooth corner z_j $(\lambda_j \neq 1)$

$$F_n(z_j) = \lambda_j[\phi(z_j)]^n + \begin{cases} \mathcal{O}\left(n^{-\lambda_1}\right), & \text{if } \lambda_j \neq 2, \\ \mathcal{O}\left((\log n)^{M_j^*}/n^{r_j^*}\right), & \text{if } \lambda_j = 2, \end{cases} \tag{23}$$

where $(\Lambda_j^*, M_j^*) = \min\{(\Lambda_1, M_1), (r_j, M_j)\}$.

Remark II.2.2. (a): If the closed set E in Theorem II.2.1(a) contains no corners of L, then $(\Lambda, M) = (\Lambda_1, M_1)$ and the convergence order given in (22) is sharp: if E has more than $s-1$ points, then there is a constant $\alpha(E) > 0$ such that

$$\max_{z \in E} |F_n(z) - [\phi(z)]^n| \geq \alpha(E) \left(\frac{(\log n)^{M_1}}{n^{\Lambda_1}}\right) \qquad \forall\ n \geq 0. \tag{24}$$

In fact, there are closed sets E for which "max" can be replaced by "min" in (24).

(b): The estimates (22) (even if E contains smooth corners) and (23) are sharp in the sense that there are curves L and closed sets E for which the order given is exact. This can be seen from the finer estimates (79) and (83) obtained in Section II.4.3 (see Example II.2.3 below).

Example II.2.3. In this example we show that there are curves for which the estimate in (22) is exact. Let $z_1 = -i$, $z_2 = 2/\sqrt{3} + i$, $z_3 = i$, and let L be the Jordan curve consisting of the segment connecting z_1 with z_2, plus the one connecting z_2 with z_3, plus the half of the unit circle lying in the left half plane (see Figure 5 below).

For this curve we have $\lambda_1 = \Lambda_1 = 4/3$, $\lambda_2 = \Lambda_2 = 5/3$, $\lambda_3 = 1$, $\Lambda_3 \geq 2$. Hence, $(\Lambda_1, M_1) = (4/3, 0)$ and (22) tell us that for a closed set E containing no corners of L,

$$\max_{z\in E} |F_n(z) - [\phi(z)]^n| = \mathcal{O}\left(n^{-4/3}\right). \quad (n \to \infty)$$

Now, the expansion of ψ at z_3 has the form (19), and since the arcs meeting at z_3 have different curvatures, it follows that $r_3 = m_3 = 1$, and therefore $(r_3, M_3) = (1, 0) < (\Lambda_1, M_1)$. Thus, if the closed set E contains the smooth corner z_3, then this time (22) gives us the worse estimate

$$\max_{z\in E} |F_n(z) - [\phi(z)]^n| = \mathcal{O}\left(n^{-1}\right). \quad (n \to \infty)$$

However, this is exact, since by the equality (79) proven in Section II.4.3 below, we have

$$\begin{aligned} F_n(z_3) = [\phi(z_3)]^n &+ \frac{1}{n^{4/3}}\left(\frac{\mathcal{C}_1 A_1 w_1^{n+4/3}}{z_3 - z_1} + o(1)\right) + \frac{1}{n^{5/3}}\left(\frac{\mathcal{C}_2 A_2 w_2^{n+5/3}}{z_3 - z_2} + o(1)\right) \\ &- \frac{1}{n}\left(\frac{A_3 w_3^{n+1}}{c_{0,1,0}^{(3)}} + o(1)\right). \end{aligned}$$

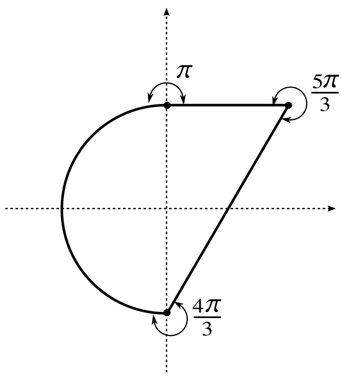


Figure 5: A curve providing sharpness for estimate (22).

Theorem II.2.4. *For $z \in G$,*

$$\frac{\Gamma(n+\Lambda_1+1)F_n(z)}{n!\,(\log n)^{M_1}} = \mathcal{C}_1 \sum_{k=1}^{u} \frac{A_k e^{i(n+\Lambda_1)\theta_k}}{z - z_k} + R_n(z), \tag{25}$$

where $R_n(z)$ converges uniformly to zero on any compact set $F \subset G$, and

$$\mathcal{C}_1 := \begin{cases} 1/\Gamma(-\Lambda_1), & \text{if } \Lambda_1 < 2, \\ (-1)^{\Lambda_1+M_1+1}(M_1+1)\Lambda_1!\,, & \text{if } \Lambda_1 \geq 2. \end{cases} \tag{26}$$

Remark II.2.5. (a): The rate of decay of the functions $R_n(z)$ in Theorem II.2.4 is at least that of the dominant terms in the equality

$$R_n(z) = \sum_{k=1}^{u} r_{k,n}(z) + \sum_{k=u+1}^{s} \mathcal{O}\left(\frac{(\log n)^{M_k-M_1}}{n^{\Lambda_k-\Lambda_1}}\right), \tag{27}$$

where

$$r_{k,n}(z) = \begin{cases} \mathcal{O}\left(n^{-\lambda_k}\right), & \text{if } 0 < \lambda_k < 1, \\ \mathcal{O}\left(n^{-1}\right) & \text{if } 1 < \lambda_k < 2, \\ \mathcal{O}\left(\log^{-1} n\right) & \text{if } \lambda_k \in \{1,2\},\ M_k \geq 1, \\ \mathcal{O}\left(n^{-1}(\log n)^{\lfloor (r_k+1)/\lambda_k \rfloor - 1}\right) & \text{if } \lambda_k \in \{1,2\},\ M_k = 0, \end{cases}$$

uniformly on compact subsets of G as $n \to \infty$.

(b): In particular, if no $\lambda_k \in \{1,2\}$, then

$$R_n(z) = \begin{cases} \mathcal{O}\left(n^{-\min\{\lambda_1, \lambda_{u+1}-\lambda_1\}}\right) & if\ 0 < \lambda_1 < 1, \\ \mathcal{O}\left(n^{-\min\{1, \lambda_{u+1}-\lambda_1\}}\right) & if\ 1 < \lambda_1 < 2. \end{cases} \tag{28}$$

These estimates are best possible in the following sense: if $0 < \lambda_1 < 1$, $\lambda_1 \neq 1/2$, then for every compact set $F \subset G$ containing more than $u + s - 1$ points, there is a constant $\beta(F) > 0$ such that

$$\max_{z \in F} |R_n(z)| \geq \beta(F)\, n^{-\min\{\lambda_1, \lambda_{u+1}-\lambda_1\}} \qquad \forall\, n \geq 0, \tag{29}$$

and, in fact, there are compact sets $F \subset G$ for which "max" can be replaced by "min" in (29).

The corresponding statement for the case $1 < \lambda_1 < 2$ (substitute "$u + s - 1$ points" by "$s - 1$ points") also holds true provided that $c_{1,1,0}^{(k)} \neq 0$ for at least one k ($1 \leq k \leq u$), where $c_{1,1,0}^{(k)}$ is the coefficient of the $(w - w_k)^{1+\lambda_k}$-term appearing in the expansion of ψ about w_k:

$$\psi(w) = z_k + A_k(w - w_k)^{\lambda_k} + c_{1,1,0}^{(k)}(w - w_k)^{1+\lambda_k} + o((w - w_k)^{1+\lambda_k}).$$

II.3 The zeros of F_n

In this section we discuss some of the implications that Theorems II.2.1 and II.2.4 have concerning the zeros of F_n.

We say that $t \in \overline{\mathbb{C}}$ is an accumulation point of the zeros of the Faber polynomials F_n, if for every open neighborhood U of t, there are infinitely many polynomials F_n having a zero in U. Denoting by $\mathcal{Z}$ the set of all such accumulation points t, and by

$\mathcal{Z}_n$ the set of zeros of F_n, we obviously have

$$\mathcal{Z} = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n \geq k} \mathcal{Z}_n} \,.$$

Recall that u is defined to be that positive integer for which

$$(\Lambda_1, M_1) = \cdots = (\Lambda_u, M_u) < (\Lambda_{u+1}, M_{u+1}) \leq \cdots \leq (\Lambda_s, M_s) \,.$$

Corollary II.3.1. *For any closed set $E \subset \overline{\Omega} \setminus \{z_k : \lambda_k \neq 1\}$ there is a positive integer N_E such that if $n > N_E$, then $F_n(z) \neq 0$ for all $z \in E$.*

For any compact set $F \subset G$, there is a positive integer N_F such that if $n > N_F$, then $F_n(z)$ has at most $u - 1$ zeros in F (counting multiplicities).

Proof. The first assertion of Corollary II.3.1 is a straightforward consequence of (22) (notice that, by the definition of F_n, the function $F_n(z) - [\phi(z)]^n$ is analytic at ∞).

Consider now a compact set $F \subset G$, and suppose there is a subsequence $n_1 < n_2 < \cdots < n_\ell < \cdots$ such that, for all $i \geq 1$, $F_{n_\ell}(z)$ has more than $u - 1$ zeros on an open set U such that $F \subset U \subset \overline{U} \subset G$. Without loss of generality, we can assume that there exist $\widehat{\theta}_1, \ldots, \widehat{\theta}_u$ such that

$$\lim_{\ell \to \infty} e^{i n_\ell \theta_k} = e^{i \widehat{\theta}_k} \quad \forall\, 1 \leq k \leq u \,.$$

Then, by Theorem II.2.4,

$$\lim_{\ell \to \infty} \frac{\Gamma(n_\ell + \Lambda_1 + 1) F_{n_\ell}(z)}{n_\ell! \, (\log n_\ell)^{M_1}} = \mathcal{C}_1 \sum_{k=1}^{u} \frac{A_k e^{i\left(\widehat{\theta}_k + \Lambda_1 \theta_k\right)}}{z - z_k}, \tag{30}$$

uniformly in $z \in U$. The rational function in the right-hand side of (30) is not identically zero and has at most $u - 1$ zeros in U. By Hurwitz's Theorem, if ℓ is sufficiently large, F_{n_ℓ} has at most $u-1$ zeros in U too, contradicting our assumption. $\square$

We already knew from the discussion held in the introduction to this chapter that $\mathcal{Z} \cap \Omega = \emptyset$, and it is a consequence of Ullman's results in [32] that $L \subset \mathcal{Z}$.

Let ν_{F_n} be the normalized counting measure of the zeros of F_n; that is, if $z_{n1}, \dots,$ z_{nn} are the zeros of F_n (counting multiplicities) and $\delta_{z_{ni}}$ is the unit mass Dirac measure at z_{ni}, then

$$\nu_{F_n} := \frac{1}{n} \sum_{i=1}^{n} \delta_{z_{ni}} \,.$$

It is said that the sequence $\{\nu_{F_n}\}_{n=1}^{\infty}$ converges in the weak*-topology to the finite measure μ if for every continuous function f defined on $\overline{\mathbb{C}}$,

$$\lim_{n\to\infty} \int_{\overline{\mathbb{C}}} f \nu_{F_n} = \int_{\overline{\mathbb{C}}} f d\mu \,.$$

The equilibrium measure μ_L of the curve L is the measure defined on any Borel set $\mathfrak{B} \subset L$ by

$$\mu_L(\mathfrak{B}) = \frac{1}{2\pi} \int_{\phi(\mathfrak{B})} |dt| \,.$$

In [12] (see Theorem 1.3), Kuijlaars and Saff proved that there exists a subsequence of $\{\nu_{F_n}\}_{n=1}^{\infty}$ that converges in the weak*-topology to the measure μ_L. Moreover, from the proof of this result, it follows that if μ is any other measure supported on L that is a weak*-limit point of the sequence $\{\nu_{F_n}\}_{n=1}^{\infty}$, then necessarily $\mu = \mu_L$. We use this fact and Corollary II.3.1 to deduce the following

Corollary II.3.2. *The sequence $\{\nu_{F_n}\}_{n=1}^{\infty}$ converges in the weak*-topology to the equilibrium measure of the curve L.*

Proof. By Corollary II.3.1, for any open disk $D \subset \mathbb{C} \setminus L$, $\nu_{F_n}(D) \leq (u-1)/n$ if n is sufficiently large. Hence, if μ is a weak*-limit point of the sequence $\{\nu_{F_n}\}_{n=1}^{\infty}$, then

$\mu(D) = 0$. Since this implies that μ is supported on L, we conclude that μ must be μ_L. $\square$

We finish this section with the following characterization of the set $\mathcal{Z} \cap G$.

Corollary II.3.3. *The point $t \in G$ belongs to $\mathcal{Z}$ if, and only if, there exist a subsequence $n_1 < n_2 < \cdots < n_\ell < \cdots$ and real numbers $\widehat{\theta}_1, \ldots, \widehat{\theta}_u$ such that*

$$\lim_{\ell \to \infty} e^{in_\ell \theta_k} = e^{i\widehat{\theta}_k}, \quad 1 \le k \le u\,, \tag{31}$$

and

$$\sum_{k=1}^{u} \frac{A_k e^{i\left(\widehat{\theta}_k + \Lambda_1 \theta_k\right)}}{t - z_k} = 0\,.$$

In particular, if θ_k/π is rational for all $1 \le k \le u$, then $\mathcal{Z} \cap G$ is a finite set.

Proof. The "if" part follows by using the argument employed in the proof of Corollary II.3.1. For the "only if" part, suppose $t \in \mathcal{Z} \cap G$. Then, there is a subsequence $n_1 < n_2 < \cdots < n_\ell < \cdots$ and a corresponding sequence of points $\{t_{n_1}, t_{n_2}, \ldots\} \subset G$ such that

$$\lim_{\ell \to \infty} t_{n_\ell} = t, \quad \text{and} \quad F_{n_\ell}(t_{n_\ell}) = 0 \quad \forall\, \ell \ge 1\,. \tag{32}$$

Without loss of generality, we can assume that (31) holds, so that by Theorem II.2.4,

$$\lim_{\ell \to \infty} \frac{\Gamma(n_\ell + \Lambda_1 + 1) F_{n_\ell}(z)}{n_\ell!\,(\log n_\ell)^{M_1}} = \mathcal{C}_1 \sum_{k=1}^{u} \frac{A_k e^{i\left(\widehat{\theta}_k + \Lambda_1 \theta_k\right)}}{z - z_k} \not\equiv 0\,, \tag{33}$$

uniformly for values of z lying in any open disk centered at t and contained in G. This and (32) force t to be a zero of the rational function in (33).

If θ_k/π is rational for all $1 \le k \le u$, say

$$\theta_k = \frac{2\pi p_k}{q_k}, \quad 0 \le p_k < q_k, \quad p_k,\, q_k \in \mathbb{Z},$$

then for every $n \geq 1$, $e^{in\theta_k} = e^{i2\pi s_{nk}/q_k}$, where $0 \leq s_{nk} < q_k$, $s_{nk} \in \mathbb{Z}$. Hence, there are finitely many $e^{i\widehat{\theta}_1}, \ldots, e^{i\widehat{\theta}_u}$ for which a subsequence $n_1 < n_2 < \cdots < n_\ell < \cdots$ can be found so that (31) holds. This completes the proof of Corollary II.3.3. $\square$

II.4 Proofs of the results in Section II.2

II.4.1 Development of $\psi(w)$ near w_k

Because $z_k = \psi(w_k)$ is a regular point of the two analytic arcs of L meeting at it, the function ψ can be analytically continued by the reflection principle onto the entire logarithmic Riemann surface with branch point at w_k. The function ψ is analytic for w sufficiently close to w_k, say $0 < |w - w_k| < \epsilon$, on any finite sector $\theta_1 \leq \arg(w - w_k) \leq \theta_2$ of this Riemann surface, where ϵ depends, in general, on the sector.

In what follows, we abbreviate by putting $y = w - w_k$. R. S. Lehman [13, Thm. 1] proved that ψ has the following asymptotic expansion: if λ_k is *irrational*, then as $w \to w_k$ with $\theta_1 \leq \arg(w - w_k) \leq \theta_2$,

$$\psi(w) = \psi(w_k) + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} c_{i,j,0}^{(k)} y^{i+j\lambda_k}, \quad c_{0,1,0}^{(k)} \neq 0; \tag{34}$$

if $\lambda_k = p/q$ is a *fraction* reduced to lowest terms, then

$$\psi(w) = \psi(w_k) + \sum_{i=0}^{\infty} \sum_{j=1}^{q} \sum_{m=0}^{\lfloor i/p \rfloor} c_{i,j,m}^{(k)} y^{i+j\lambda_k} (\log y)^m, \quad c_{0,1,0}^{(k)} \neq 0. \tag{35}$$

The terms in the above series are assumed to be arranged in an order such that a term of the form $y^{i+j\lambda_k}(\log y)^m$ precedes one of the form $y^{i'+j'\lambda_k}(\log y)^{m'}$ if either $i + j\lambda_k < i' + j'\lambda_k$ or $i + j\lambda_k = i' + j'\lambda_k$ and $m > m'$.

The precise meaning of these expansions is the following: if ordered as explained above, either (34) or (35) can be written as

$$\psi(w) = \psi(w_k) + \sum_{n=1}^{\infty} \chi_n(y),$$

where for all $N \geq 1$,

$$\psi(w) - \psi(w_k) - \sum_{n=1}^{N} \chi_n(y) = o\left(\chi_N(y)\right)$$

as $w \to w_k$ within any finite sector $\theta_1 \leq \arg(w - w_k) \leq \theta_2$.

As in Section II.2, we fix the branches of the functions of the form $(w - w_k)^\beta$, $(\log(w - w_k))^m$ that appear in these expansions by defining them along the half line $\{w = re^{i\theta_k} : 1 < r < \infty\}$ via (20).

As pointed out by Lehman, the expansions of the derivatives of ψ are obtained from (34) and (35) by rearranging the terms obtained after termwise differentiation.

For a more detailed description of these expansions we separate in two cases:

Case $\boldsymbol{0 < \lambda_k < 2,\ \lambda_k \notin \{1, 2\}}$: as in Section II.2, we put $A_k := c^{(k)}_{0,1,0} \neq 0$, and it follows from (34) and (35) that if $\upsilon > 0$ is sufficiently small, say

$$0 < \upsilon < \begin{cases} \min\{\lambda_k, 1 - \lambda_k\}, & if\ 0 < \lambda_k < 1, \\ 2 - \lambda_k & if\ 1 < \lambda_k < 2, \end{cases}$$

then,

$$\psi(w) = z_k + A_k y^{\lambda_k} + c^{(k)}_{0,2,0} y^{2\lambda_k} + \mathcal{O}\left(y^{2\lambda_k + \upsilon}\right), \quad \text{if } 0 < \lambda_k < 1, \tag{36}$$

$$\psi(w) = z_k + A_k y^{\lambda_k} + c^{(k)}_{1,1,0} y^{1+\lambda_k} + c^{(k)}_{0,2,0} y^{2\lambda_k} + \mathcal{O}\left(y^{2\lambda_k + \upsilon}\right), \quad \text{if } 1 < \lambda_k < 2\,. \tag{37}$$

(If $1 < \lambda_k = p/q < 2$, then $p \geq 3$, $q \geq 2$, and no log-terms correspond to $i = 0, 1, 2$.)

Case $\boldsymbol{\lambda_k = p,\ p = 1, 2}$: here $q = 1$ and there is a smallest integer r_k, $p = \lambda_k \leq r_k < \infty$ for which a log-term of the form $y^{r_k + \lambda_k} (\log y)^{m_k}$, $1 \leq m_k \leq \lfloor r_k/\lambda_k \rfloor$, appears in the expansion of ψ, so that in this case

$$\psi(w) = z_k + \sum_{i=0}^{r_k-1} c_{i,1,0}^{(k)} y^{i+\lambda_k} + A_k y^{\Lambda_k} (\log y)^{m_k} + B_k y^{\Lambda_k} (\log y)^{m_k - 1} \tag{38}$$
$$+ \begin{cases} \mathcal{O}\left(y^{\Lambda_k} (\log y)^{m_k-2}\right), & \text{if } m_k \geq 2, \\ C_k y^{\Lambda_k+1} (\log y)^{\lfloor (r_k+1)/\lambda_k \rfloor} + \mathcal{O}\left(y^{\Lambda_k+1} (\log y)^{\lfloor (r_k+1)/\lambda_k \rfloor - 1}\right), & \text{if } m_k = 1, \end{cases}$$

where $c_{0,1,0}^{(k)} \neq 0$ and

$$\Lambda_k := r_k + \lambda_k, \quad A_k := c_{r_k,1,m_k}^{(k)} \neq 0, \quad B_k := c_{r_k,1,(m_k-1)}^{(k)}, \quad C_k := c_{(r_k+1),1,\lfloor (r_k+1)/\lambda_k \rfloor}^{(k)} .$$

Thus, setting

$$Q_k(w) := z_k + \sum_{i=0}^{r_k-1} c_{i,1,0}^{(k)} y^{i+\lambda_k} = z_k + \mathcal{O}\left(y^{\lambda_k}\right), \tag{39}$$

we have

$$\psi'(w) = Q_k'(w) + A_k \Lambda_k y^{\Lambda_k - 1} (\log y)^{m_k} + D_k y^{\Lambda_k - 1} (\log y)^{m_k - 1} \tag{40}$$

$$+ \begin{cases} \mathcal{O}\left(y^{\Lambda_k-1} (\log y)^{m_k-2}\right), & \text{if } m_k \geq 2, \\ C_k (1+\Lambda_k) y^{\Lambda_k} (\log y)^{\lfloor (r_k+1)/\lambda_k \rfloor} + \mathcal{O}\left(y^{\Lambda_k} (\log y)^{\lfloor (r_k+1)/\lambda_k \rfloor - 1}\right), & \text{if } m_k = 1, \end{cases}$$

where $D_k = A_k m_k + B_k \Lambda_k$.

II.4.2 Auxiliary lemmas

Lemma II.4.1. *For every two integers $m \geq 1$, $\ell \geq 0$,*

$$\int_0^1 x^n (1-x)^\ell (\log(1-x))^m \, dx = \frac{\ell! \, n! (-\log n)^m \left[1 + \mathcal{O}\left(\log^{-1} n\right)\right]}{(n+\ell+1)!} \tag{41}$$

as $n \to \infty$.

Proof of Lemma II.4.1. The proof is by induction. We shall prove the equivalent statement

$$\int_0^1 x^n(1-x)^\ell \left(\log(1-x)\right)^m dx = \frac{\ell!\, n!(-\log(n+\ell+1))^m \left[1+\mathcal{O}\left(\log^{-1}(n+\ell+1)\right)\right]}{(n+\ell+1)!}. \tag{42}$$

By using the same computations that follow below, one can see that

$$\int_0^1 x^n \log(1-x)dx = -\frac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{n+1}, \quad n \geq 0,$$

so that (42) holds for $m=1$, $\ell=0$. Now assume that (42) holds for some $m \geq 1$ and $\ell = 0$. Then

$$\begin{aligned}
&\int_0^1 x^n \left(\log(1-x)\right)^{m+1} dx \\
&= \left.\frac{(x^{n+1}-1)\left(\log(1-x)\right)^{m+1}}{n+1}\right|_0^1 - \frac{m+1}{n+1}\int_0^1 \frac{x^{n+1}-1}{x-1}\left(\log(1-x)\right)^m dx \\
&= -\frac{m+1}{n+1}\sum_{j=0}^n \int_0^1 x^j \left(\log(1-x)\right)^m dx \\
&= \frac{(-1)^{m+1}(m+1)}{n+1}\left(\sum_{j=0}^n \frac{(\log(j+1))^m}{j+1} + \mathcal{O}\left(\sum_{j=0}^n \frac{(\log(j+1))^{m-1}}{j+1}\right)\right).
\end{aligned} \tag{43}$$

Now, for $m \geq 0$, the function $(\log x)^m/x$ decreases in (e^m, ∞), so that if $n_0 > e^m$ is a fixed integer, then for all $n \geq n_0$

$$\int_{n_0}^{n+1} \frac{(\log(x+1))^m\, dx}{x+1} < \sum_{j=n_0}^n \frac{(\log(j+1))^m}{j+1} < \int_{n_0}^{n+1} \frac{(\log x)^m dx}{x}.$$

Hence,

$$\frac{(\log(n+2))^{m+1}}{m+1}+\mathcal{O}(1)<\sum_{j=0}^{n}\frac{(\log(j+1))^m}{j+1}<\frac{(\log(n+1))^{m+1}}{m+1}+\mathcal{O}(1),$$

which implies that

$$\sum_{j=0}^{n}\frac{(\log(j+1))^m}{j+1}=\frac{(\log(n+1))^{m+1}}{m+1}+\mathcal{O}(1)\,.$$

Plugging this above in (43), we obtain that (42) is also valid for $m+1$, $\ell=0$.

Finally, assume (42) holds for some $\ell\geq 0$ and all $m\geq 1$. Integration by parts yields

$$\begin{aligned}
&\int_0^1 x^n(1-x)^{\ell+1}(\log(1-x))^m dx\\
&=\frac{\ell+1}{n+1}\int_0^1 x^{n+1}(1-x)^{\ell}(\log(1-x))^m dx\\
&\quad+\frac{m}{n+1}\int_0^1 x^{n+1}(1-x)^{\ell}(\log(1-x))^{m-1} dx\\
&=\frac{(\ell+1)!n!(-\log(n+\ell+2))^m}{(n+\ell+2)!}+\mathcal{O}\left(\frac{n!\log^{m-1}(n+\ell+2)}{(n+\ell+2)!}\right).
\end{aligned}$$

$\square$

Some standard notation that we will use from this point on is the following. For any $0<r<\infty$,

$$\mathbb{T}_r:=\{w:|w|=r\},\quad \mathbb{D}_r:=\{w:|w|<r\},\quad \Delta_r:=\{w:|w|>r\}.$$

Throughout the remaining of this chapter, D_{w_k} will denote an element of some collection $\{D_{w_k}\}_{k=1}^{s}$ of pairwise disjoint small open disks such that D_{w_k} is centered at w_k and its boundary intersects $\mathbb{T}_1$ at the points $w_k^+=e^{i(\theta_k+\epsilon_k)}$, $w_k^-=e^{i(\theta_k-\epsilon_k)}$ for some $\epsilon_k>0$ (see Fig. 6 below).

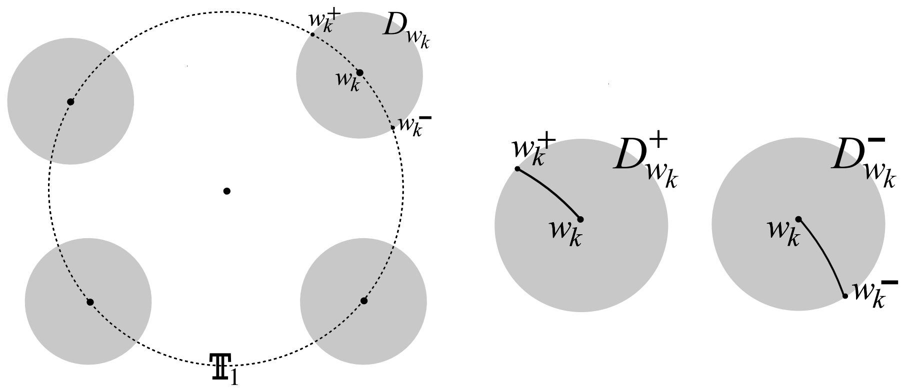


Figure 6: A collection of disks $\{D_{w_k}\}_{k=1}^s$.

Let us define the half-open circular arcs

$$[w_k, w_k^+)^\frown := \{e^{i\theta} : \theta_k \le \theta < \theta_k + \epsilon_k\},$$

$$(w_k^-, w_k]^\frown := \{e^{i\theta} : \theta_k - \epsilon_k < \theta \le \theta_k\},$$

together with their corresponding open sets (see Figure 6 above)

$$D_{w_k}^+ := D_{w_k} \setminus [w_k, w_k^+)^\frown, \quad D_{w_k}^- := D_{w_k} \setminus (w_k^-, w_k]^\frown.$$

If D_{w_k} is sufficiently small, then ψ has analytic continuations ψ_+, ψ_- from Δ_1 onto $D_{w_k}^+$, $D_{w_k}^-$, respectively. Hereon, we assume that every element of a given collection $\{D_{w_k}\}_{k=1}^s$ has this property.

For every $0 < \sigma < 1$, we define $\sigma_k := \sigma e^{i\theta_k}$, $1 \le k \le s$, and the contour (see Figure 7 below)

$$\Gamma_\sigma := \mathbb{T}_\sigma \cup (\cup_{k=1}^s (\sigma_k, w_k]).$$

(we use $(\sigma_k, w_k]$ to denote the half-open segment joining σ_k with w_k. A similar meaning is attached to $[\sigma_k, w_k]$ and $[\sigma_k, w_k)$.)

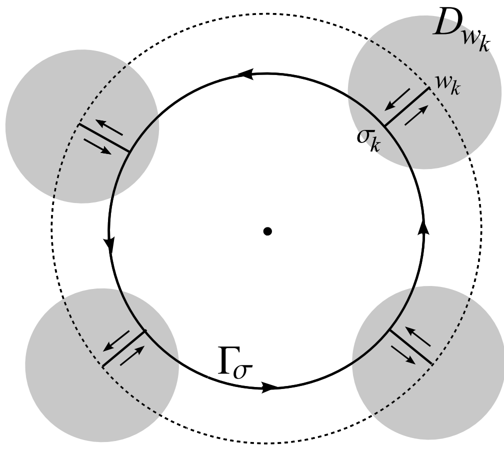


Figure 7: A positively oriented contour Γ_σ, $0 < \sigma < 1$.

The exterior of the contour Γ_σ is understood to be

$$\mathrm{ext}(\Gamma_\sigma) := \Delta_\sigma \setminus \left(\cup_{k=1}^{s}[\sigma_k, w_k]\right).$$

Recall that for every k such that $\lambda_k \in \{1,2\}$, Q_k is defined by (39).

Lemma II.4.2. *Let $\epsilon > 0$ be given. Suppose D_{w_k} and σ_k are such that*

$$\overline{\psi_\pm(D^{\pm}_{w_k})} \subset \{z : |z - z_k| < \epsilon\}, \quad \sigma_k \in D_{w_k},$$

and, that in case $\lambda_k \in \{1,2\}$, we also have $Q_k\left(D_{w_k}\right) \subset \{z : |z - z_k| < \epsilon\}$. Then,

$$\frac{1}{2\pi i}\int_{\sigma_k}^{w_k}\left(\frac{t^n\psi_+'(t)}{\psi_+(t)-\xi} - \frac{t^n\psi_-'(t)}{\psi_-(t)-\xi}\right)dt = \frac{n!(\log n)^{M_k}}{\Gamma(n+\Lambda_k+1)}\left(\frac{\mathcal{C}_k A_k w_k^{n+\Lambda_k}}{\xi - z_k} + r_{k,n}(\xi)\right)$$

with

$$\mathcal{C}_k := \begin{cases} 1/\Gamma(-\Lambda_k), & \textit{if } \Lambda_k < 2 \ (\textit{i.e., } \lambda_k \notin \{1,2\}), \\ (-1)^{\Lambda_k+M_k+1}(M_k+1)\Lambda_k!\,, & \textit{if } \Lambda_k \geq 2 \ (\textit{i.e., } \lambda_k \in \{1,2\}), \end{cases} \tag{44}$$

and $r_{k,n}(\xi)$ *converging uniformly to zero on* $\{\xi : |\xi - z_k| \geq \epsilon\}$ *as* $n \to \infty$ *with the following rate:*

$$r_{k,n}(\xi) = \begin{cases} \mathcal{O}\left(n^{-\lambda_k}\right), & \textit{if } 0 < \lambda_k < 1, \\ \mathcal{O}\left(n^{-1}\right), & \textit{if } 1 < \lambda_k < 2, \\ \mathcal{O}\left(\log^{-1} n\right), & \textit{if } \lambda_k \in \{1,2\},\ M_k \geq 1, \\ \mathcal{O}\left(n^{-1}(\log n)^{\lfloor (r_k+1)/\lambda_k \rfloor - 1}\right), & \textit{if } \lambda_k \in \{1,2\},\ M_k = 0. \end{cases}$$

The constants involved in the $\mathcal{O}$ *terms are independent of* ξ*, and the rate of convergence is exact when* $0 < \lambda_k < 1$*,* $\lambda_k \neq 1/2$*, provided that* $\xi \neq z_k - A_k^2/2c^{(k)}_{0,2,0}$*, and when* $1 < \lambda_k < 2$ *provided that* $c^{(k)}_{1,1,0} \neq 0$*.*

Proof of Lemma II.4.2. Suppose first that z_k is a corner of the curve L with $\lambda_k \notin \{1,2\}$. Then, it follows from (36) and (37) that uniformly in $\{\xi : |\xi - z_k| \geq \epsilon\}$,

$$\frac{1}{\psi(w) - \xi} = \frac{1}{z_k - \xi} - \frac{A_k y^{\lambda_k}}{(z_k - \xi)^2} + \begin{cases} \mathcal{O}\left(y^{2\lambda_k}\right), & 0 < \lambda_k < 1, \\ \mathcal{O}\left(y^{\lambda_k+1}\right), & 1 < \lambda_k < 2, \end{cases}$$

as $w \to w_k$.

Hence, if $0 < \lambda_k < 1$, then

$$\frac{\psi'(w)}{\psi(w) - \xi} = \frac{A_k \lambda_k y^{\lambda_k - 1}}{z_k - \xi} + \frac{\lambda_k \left[2c^{(k)}_{0,2,0}(z_k - \xi) - A_k^2\right] y^{2\lambda_k - 1}}{(z_k - \xi)^2} + \mathcal{O}\left(y^{2\lambda_k + \upsilon - 1}\right), \tag{45}$$

while if $1 < \lambda_k < 2$, then

$$\frac{\psi'(w)}{\psi(w) - \xi} = \frac{A_k \lambda_k y^{\lambda_k - 1}}{z_k - \xi} + \frac{c^{(k)}_{1,1,0}(1 + \lambda_k) y^{\lambda_k}}{z_k - \xi} + \mathcal{O}\left(y^{2\lambda_k - 1}\right). \tag{46}$$

Let us denote by $(w-w_k)^{\beta}_{\pm}$, $\log_{\pm}(w-w_k)$, the analytic continuation of $(w-w_k)^{\beta}$ and $\log(w-w_k)$ onto the open set $D^{\pm}_{w_k}$. For any real $\beta > -1$,

$$\int_{\sigma_k}^{w_k} (t-w_k)^{\beta}_{\pm} t^n dt = e^{\mp i\beta\pi} \int_{\sigma_k}^{w_k} (w_k-t)^{\beta}_{\pm} t^n dt = e^{\mp i\beta\pi} w_k^{n+1+\beta} \int_{\sigma}^{1} (1-x)^{\beta} x^n dx,$$

and

$$\begin{aligned}\int_{\sigma}^{1} (1-x)^{\beta}\, x^n dx &= \int_{0}^{1} (1-x)^{\beta}\, x^n dx - \int_{0}^{\sigma} (1-x)^{\beta}\, x^n dx \\ &= \frac{\Gamma(\beta+1)\Gamma(n+1)}{\Gamma(n+\beta+2)} + \mathcal{O}\left(\sigma^n\right) \sim \frac{\Gamma(\beta+1)}{n^{\beta+1}}.\end{aligned}$$

Hence,

$$\int_{\sigma_k}^{w_k} (t-w_k)^{\beta}_{\pm} t^n dt = \frac{e^{\mp i\beta\pi} w_k^{n+1+\beta}\Gamma(\beta+1)\Gamma(n+1)}{\Gamma(n+\beta+2)} + \mathcal{O}\left(\sigma^n\right), \tag{47}$$

and

$$\int_{\sigma_k}^{w_k} \mathcal{O}\left((t-w_k)^{\beta}_{\pm} t^n\right) dt = \mathcal{O}\left(\int_{\sigma}^{1} (1-t)^{\beta} t^n dt\right) = \mathcal{O}\left(\frac{1}{n^{\beta+1}}\right). \tag{48}$$

Therefore, we get from (45), (47) and (48) that if $0 < \lambda_k < 1$, then

$$\begin{aligned}&\int_{\sigma_k}^{w_k} \left(\frac{t^n \psi_+'(t)}{\psi_+(t)-\xi} - \frac{t^n \psi_-'(t)}{\psi_-(t)-\xi}\right) dt \\ &= \frac{A_k\lambda_k}{z_k-\xi} \cdot \frac{2i\sin(\pi\lambda_k) w_k^{n+\lambda_k}\Gamma(\lambda_k) n!}{\Gamma(n+\lambda_k+1)} \\ &\quad + \frac{\lambda_k\left[2c_{0,2,0}^{(k)}(z_k-\xi) - A_k^2\right]}{(z_k-\xi)^2} \cdot \frac{2i\sin(2\pi\lambda_k) w_k^{n+2\lambda_k}\Gamma(2\lambda_k) n!}{\Gamma(n+2\lambda_k+1)} + \mathcal{O}\left(\frac{1}{n^{2\lambda_k+\upsilon}}\right),\end{aligned}$$

so that, using the identity $-\lambda_k\Gamma(-\lambda_k)\Gamma(\lambda_k) = \pi/\sin(\lambda_k\pi)$, we finally get for $0 < \lambda_k < 1$ (with the understanding $\Gamma(-2\lambda_k) = \infty$ if $\lambda_k = 1/2$):

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\sigma_k}^{w_k} \left(\frac{t^n \psi_+'(t)}{\psi_+(t)-\xi} - \frac{t^n \psi_-'(t)}{\psi_-(t)-\xi}\right) dt \\ &= \frac{n!}{\Gamma(n+\lambda_k+1)}\left(\frac{A_k w_k^{n+\lambda_k}}{\Gamma(-\lambda_k)(\xi-z_k)} + \frac{\left[2c_{0,2,0}^{(k)}(\xi-z_k) + A_k^2\right] w_k^{n+2\lambda_k}}{2\Gamma(-2\lambda_k)(\xi-z_k)^2}\frac{(1+o(1))}{n^{\lambda_k}}\right).\end{aligned} \tag{49}$$

Similarly, if $1 < \lambda_k < 2$, then we get from (46), (47) and (48) that

$$\frac{1}{2\pi i}\int_{\sigma_k}^{w_k}\left(\frac{t^n\psi_+'(t)}{\psi_+(t)-\xi}-\frac{t^n\psi_-'(t)}{\psi_-(t)-\xi}\right)dt$$
$$=\frac{n!}{\Gamma(n+\lambda_k+1)}\left(\frac{A_k w_k^{n+\lambda_k}}{\Gamma(-\lambda_k)(\xi-z_k)}+\frac{c_{1,1,0}^{(k)}w_k^{n+\lambda_k+1}}{\Gamma(-\lambda_k-1)(\xi-z_k)}\frac{(1+o(1))}{n}\right). \tag{50}$$

Thus, Lemma II.4.2 for a non-smooth corner follows from (49) and (50).

Next, let us consider the case $\lambda_k \in \{1,2\}$. From (38) and (40) we see that, uniformly in $\{\xi : |\xi - z_k| \geq \epsilon\}$ as $w \to w_k$,

$$\begin{aligned}
&\frac{\psi'(w)}{\psi(w)-\xi}\\
&=\frac{Q_k'(w)}{Q_k(w)-\xi}+\frac{A_k\Lambda_k y^{\Lambda_k-1}(\log y)^{m_k}}{z_k-\xi}+\frac{D_k y^{\Lambda_k-1}(\log y)^{m_k-1}}{z_k-\xi} \qquad (51)\\
&+\begin{cases}\mathcal{O}\left(y^{\Lambda_k-1}(\log y)^{m_k-2}\right), & \text{if } m_k\geq 2,\\ \dfrac{C_k(1+\Lambda_k)y^{\Lambda_k}(\log y)^{\lfloor (r_k+1)/\lambda_k\rfloor}}{z_k-\xi}+\mathcal{O}\left(y^{\Lambda_k}(\log y)^{\lfloor (r_k+1)/\lambda_k\rfloor-1}\right), & \text{if } m_k=1.\end{cases}
\end{aligned}$$

Now, if $\ell \geq 0$, $m \geq 1$ are integers, then we get from Lemma II.4.1 that

$$\begin{aligned}
&\int_{\sigma_k}^{w_k} t^n(t-w_k)_\pm^\ell(\log_\pm(t-w_k))^m dt\\
&=(-1)^\ell w_k^{n+1+\ell}\int_\sigma^1 x^n(1-x)^\ell\left(\log(1-x)+i(\theta_k\mp\pi)\right)^m dx\\
&=(-1)^\ell w_k^{n+1+\ell}\int_0^1 x^n(1-x)^\ell\left(\log(1-x)\right)^m dx+\mathcal{O}(\sigma^n)\\
&\quad+\frac{i\,m(\theta_k\mp\pi)(-1)^{\ell+m-1}w_k^{n+1+\ell}\ell!n!(\log n)^{m-1}}{(n+\ell+1)!}\\
&\quad+(1-\delta_{m,1})\mathcal{O}\left(\frac{n!(\log n)^{m-2}}{(n+\ell+1)!}\right),
\end{aligned} \tag{52}$$

and

$$\int_{\sigma_k}^{w_k}\mathcal{O}\left(t^n(t-w_k)_\pm^\ell(\log_\pm(t-w_k))^m\right)dt=\mathcal{O}\left(\frac{n!(\log n)^m}{(n+\ell+1)!}\right). \tag{53}$$

Thus, we get from (51), (52) and (53) that if $\lambda_k \in \{1, 2\}$, then

$$\frac{1}{2\pi i}\int_{\sigma_k}^{w_k}\left(\frac{t^n\psi_+'(t)}{\psi_+(t)-\xi}-\frac{t^n\psi_-'(t)}{\psi_-(t)-\xi}\right)dt$$
$$=\frac{n!\,(\log n)^{m_k-1}}{(n+\Lambda_k)!}\left(\frac{A_k\Lambda_k!m_kw_k^{n+\Lambda_k}}{(-1)^{\Lambda_k+m_k}(\xi-z_k)}+\begin{cases}\mathcal{O}\left(\dfrac{1}{\log n}\right), & \text{if } m_k\geq 2,\\ \mathcal{O}\left(\dfrac{(\log n)^{\lfloor (r_k+1)/\lambda_k\rfloor-1}}{n}\right), & \text{if } m_k=1.\end{cases}\right)$$

This completes the proof of Lemma II.4.2. □

Lemma II.4.3. *Let $E\subset\overline{\Omega}\setminus\{z_k:\lambda_k\neq 1\}$ be a closed set and let $F\subset G$ be a compact set (see Fig. 3 above). There exist $0<\epsilon<\operatorname{dist}(E\cup F,\{z_k\notin E\})$ and a collection $\{D_{w_k}\}_{k=1}^s$ such that for every contour Γ_σ with $\sigma_k\in D_{w_k}$, $1\leq k\leq s$, the following statements hold simultaneously:*

(a) for every non-smooth corner z_k ($\lambda_k\neq 1$),

$$\overline{\psi_\pm\left(D_{w_k}^\pm\right)}\subset\{z:|z-z_k|<\epsilon\},\quad z_k\notin\psi_\pm\left(D_{w_k}^\pm\right), \tag{54}$$

while for every smooth corner z_k, the three arcs

$$\psi_+([\sigma_k,w_k)),\quad \psi_-([\sigma_k,w_k)),\quad Q_k([\sigma_k,w_k))$$

lie entirely on G and

$$\left|\frac{\psi_\pm(t)-z_k}{\psi_\pm(t)-\xi}\right|<2,\quad \left|\frac{Q_k(t)-z_k}{Q_k(t)-\xi}\right|<2, \tag{55}$$

for all $t\in[\sigma_k,w_k)$, $\xi\in\overline{\Omega}\cap\{\xi:|\xi-z_k|<\epsilon\}$;

(b) ψ has an analytic continuation from Δ_1 onto $\operatorname{ext}(\Gamma_\sigma)$ with continuous boundary values on Γ_σ when viewing each $[\sigma_k,w_k]$ as having two sides; ψ' has continuous, integrable boundary values on $\Gamma_\sigma\setminus\{w_1,\ldots,w_s\}$. Moreover, ψ is one-to-one on

$\overline{\Delta}_1 \cup U$ *for any open, connected component* U *of*

$$\mathbb{D} \cap \left(\mathrm{ext}(\Gamma_\sigma) \setminus \left(\cup_{\lambda_k \neq 1} \overline{D}_{w_k}\right)\right)$$

and $\psi\left(\overline{\mathrm{ext}(\Gamma_\sigma)}\right) \cap F = \emptyset$.

Proof of Lemma II.4.3. (a): By the continuity of $\psi_\pm$, and the fact that

$$\frac{\psi(w) - z_k}{(w - w_k)^{\lambda_k}} \underset{w \to w_k}{\longrightarrow} c_{0,1,0}^{(k)} \neq 0, \quad \forall\, 1 \leq k \leq s\,,$$

the conditions (54) are trivially satisfied by taking the disks D_{w_k} sufficiently small.

Suppose z_k is a smooth corner ($\lambda_k = 1$) which is the common endpoint of the two analytic arcs $\gamma_k^+ = \psi([w_k, w_k^+)^\frown)$, $\gamma_k^- = \psi((w_k^-, w_k]^\frown)$. By assumption, γ_k^+ is part of a larger analytic simple arc containing z_k as an interior point, which we denote by γ_k^+ as well. Notice that γ_k^+ and L share the same tangent line at z_k. Since, as $w \to w_k$,

$$\psi(w) = z_k + c_{0,1,0}^{(k)}(w - w_k) + o\left(w - w_k\right), \quad c_{0,1,0}^{(k)} \neq 0\,,$$

the arc $\psi((w_k, 1/\overline{\sigma}_k])$ lies entirely in Ω and is perpendicular to L. If σ_k is sufficiently close to w_k, then by the Schwarz reflection principle for analytic arcs (see [3]), $\psi_-([\sigma_k, w_k))$ is the reflection of $\psi((w_k, 1/\overline{\sigma}_k])$ across γ_k^+, which is perpendicular to L and lies therefore entirely on G. In fact, Inequality (55) for ψ_- is also a consequence of the perpendicularity between $\psi_-([\sigma_k, w_k))$ and L. Clearly, similar considerations apply to the arc γ_k^-. Now, because

$$Q_k(w) = z_k + c_{0,1,0}^{(k)}(w - w_k) + o\left(w - w_k\right),$$

Q_k maps a small circular arc of $\mathbb{T}_1$ centered at z_k onto an analytic arc tangent to L at z_k, and therefore $Q_k([\sigma_k, w_k))$ also lies entirely on G and is perpendicular to L, provided that σ_k is sufficiently close to w_k. Hence, (55) for Q_k follows.

Finally, from the Schwarz reflection principle and the fact that

$$\psi'(w) = \mathcal{O}\left((w - w_k)^{\lambda_k - 1}\right)$$

as $w \to w_j$, it is easy to see that by making σ sufficiently close to 1 we can, in addition, satisfy the conditions stated in Lemma II.4.3(b). $\square$

II.4.3 Proof of Theorems II.2.1 and II.2.4

We shall prove Theorems II.2.1, II.2.4, and the statements in Remarks II.2.2, II.2.5 simultaneously.

Proof of Theorem II.2.4 and Remark II.2.5(a): Let $E \subset \overline{\Omega} \setminus \{z_k : \lambda_k \neq 1\}$ be a closed set and let $F \subset G$ be a compact set. Lemma II.4.3 allows us to choose $0 < \epsilon < \operatorname{dist}(E \cup F, \{z_k : z_k \notin E\})$, a collection $\{D_{w_k}\}_{k=1}^{s}$ and a contour Γ_σ with $\sigma_k \in D_{w_k}$ for all $1 \leq k \leq s$, for which all the statements listed in Lemma II.4.3 hold true.

First, suppose $\xi \in E$ is not a corner of L. The function $\psi'(w)/[\psi(w) - \xi]$ in the variable w is analytic on $\operatorname{ext}(\Gamma_\sigma) \setminus \{\phi(\xi)\}$, with a simple pole at $\phi(\xi)$ and integrable boundary values on Γ_σ. We choose a small circle C_ξ centered at $\phi(\xi)$ that lies on a neighborhood of $\phi(\xi)$ on which ψ is univalent. Then, in the integral of (12) with $R > |\phi(\xi)|$, the circle $\mathbb{T}_R$ can be deformed without altering the value of the integral, to the positively oriented contour $\Gamma_\sigma \cup C_\xi$ to obtain

$$\begin{aligned} F_n(\xi) &= \frac{1}{2\pi i} \oint_{\mathbb{T}_R} \frac{t^n \psi'(t) dt}{\psi(t) - \xi} = \frac{1}{2\pi i} \oint_{C_\xi} \frac{t^n \psi'(t) dt}{\psi(t) - \xi} + \frac{1}{2\pi i} \oint_{\Gamma_\sigma} \frac{t^n \psi'(t) dt}{\psi(t) - \xi} \\ &= [\phi(\xi)]^n + \frac{1}{2\pi i} \oint_{\Gamma_\sigma} \frac{t^n \psi'(t) dt}{\psi(t) - \xi} \quad \forall \xi \in E \setminus \{z_k : \lambda_k = 1\}. \end{aligned} \tag{56}$$

Similarly, since $\psi'(w)/[\psi(w)-\xi]$ is analytic on $\mathrm{ext}(\Gamma_\sigma)$ for all $\xi \in F$, we have

$$F_n(\xi) = \frac{1}{2\pi i}\oint_{\Gamma_\sigma} \frac{t^n \psi'(t)dt}{\psi(t)-\xi} \quad \forall \, \xi \in F. \tag{57}$$

Now, for all $\xi \in F \cup E \setminus \{z_k : \lambda_k = 1\}$,

$$\oint_{\Gamma_\sigma} \frac{t^n \psi'(t)dt}{\psi(t)-\xi} = \oint_{\mathbb{T}_\sigma} \frac{t^n \psi'(t)dt}{\psi(t)-\xi} + \sum_{k=1}^{s} \int_{\sigma_k}^{w_k} \left(\frac{t^n \psi'_+(t)}{\psi_+(t)-\xi} - \frac{t^n \psi'_-(t)}{\psi_-(t)-\xi} \right) dt, \tag{58}$$

so that by Lemma II.4.2,

$$\oint_{\Gamma_\sigma} \frac{t^n \psi'(t)dt}{\psi(t)-\xi} = \mathcal{O}(\sigma^n) + 2\pi i \sum_{k=1}^{s} \frac{n!(\log n)^{M_k}}{\Gamma(n+\Lambda_k+1)} \left(\frac{\mathcal{C}_k A_k w_k^{n+\Lambda_k}}{\xi - z_k} + r_{k,n}(\xi) \right) , \tag{59}$$

uniformly in $\xi \in F \cup (E \setminus \cup_{z_k \in E}\{z : |z-z_k| < \epsilon\})$ as $n \to \infty$. Thus, Theorem II.2.4 and Remark II.2.5(a) follows from (57), (59) and the definition of $\mathcal{C}_k$ in Lemma II.4.2.

Proof of Remark II.2.5(b): Indeed, from the more precise expressions provided by (49) and (50) for $r_{k,n}(\xi)$, we obtain in the case in which no $\lambda_k \in \{1,2\}$ that if $0 < \lambda_1 < 1$ (recall the convention $\Gamma(-1) = \infty$), then

$$\begin{aligned} R_n(\xi) = & \frac{1}{2\Gamma(-2\lambda_1)n^{\lambda_1}} \left(\sum_{k=1}^{u} \frac{\left[2c_{0,2,0}^{(k)}(\xi - z_k) + A_k^2\right] w_k^{n+2\lambda_1}}{(\xi - z_k)^2} + o(1) \right) \\ & + \frac{\mathcal{C}_{u+1}}{n^{\lambda_{u+1}-\lambda_1}} \left(\sum_{\lambda_k = \lambda_{u+1}} \frac{A_k w_k^{n+\lambda_{u+1}}}{\xi - z_k} + o(1) \right) , \end{aligned} \tag{60}$$

while if $1 < \lambda_1 < 2$, then

$$\begin{aligned} R_n(\xi) = & \frac{1}{\Gamma(-\lambda_1 - 1)\, n} \left(\sum_{k=1}^{u} \frac{c_{1,1,0}^{(k)} w_k^{n+\lambda_1+1}}{\xi - z_k} + o(1) \right) \\ & + \frac{\mathcal{C}_{u+1}}{n^{\lambda_{u+1}-\lambda_1}} \left(\sum_{\lambda_k = \lambda_{u+1}} \frac{A_k w_k^{n+\lambda_{u+1}}}{\xi - z_k} + o(1) \right) . \end{aligned} \tag{61}$$

Hence, if $0 < \lambda_1 < 1$, $\lambda_1 \neq 1/2$, there is $\mathcal{I} \subset \{1, 2, \ldots, s\}$ such that

$$n^{\min\{\lambda_1, \lambda_{u+1}-\lambda_1\}} R_n(\xi) = \sum_{k\in\mathcal{I}} w_k^n \mathcal{H}_k(\xi) + o(1) \tag{62}$$

where the $\mathcal{H}_k$'s are different elements of the system of linearly independent functions

$$\left\{ \frac{2c_{0,2,0}^{(k)}(\xi - z_k) + A_k^2}{2w_k^{-2\lambda_1}\Gamma(-2\lambda_1)(\xi - z_k)^2} \right\}_{k=1}^{u} \cup \left\{ \frac{\mathcal{C}_{u+1} A_k w_k^{\lambda_{u+1}}}{\xi - z_k} \right\}_{\lambda_k = \lambda_{u+1}} .$$

Clearly, from every subsequence of $\{\sum_{k\in\mathcal{I}} w_k^n \mathcal{H}_k(\xi)\}_{n=0}^{\infty}$ one can extract a subsequence that converges uniformly on compact subsets of G to a not identically zero rational function with numerator of degree at most $u + s - 1$. Hence, if the compact set $F \subset G$ is composed of more than $u + s - 1$ points, there is a constant $\alpha(F) > 0$ such that

$$\max_F |R_n(\xi)| \geq \alpha(F) n^{-\min\{\lambda_1, \lambda_{u+1}-\lambda_1\}} \quad \forall\, n \geq 0\,. \tag{63}$$

To see that there are compact sets F for which "max" can be replaced by "min" in (63), simply notice that, clearly, there is $\delta > 0$ such that for every $\xi \in G$ that is sufficiently close to any of the corners z_k, $k \in \mathcal{I}$,

$$\left| \sum_{k\in\mathcal{I}} w_k^n \mathcal{H}_k(\xi) \right| \geq \delta > 0.$$

Similarly, from (61) we obtain in case $1 < \lambda_1 < 2$ and $c_{1,1,0}^{(k)} \neq 0$ for at least one k ($1 \leq k \leq u$), that if the compact set F contains more than $s - 1$ points, then there is a constant $\beta(F)$ for which

$$\max_F |R_n(\xi)| \geq \beta(F) n^{-\min\{1, \lambda_{u+1}-\lambda_1\}} \quad \forall\, n \geq 0\,.$$

Proof of Theorem II.2.1(a) and Remark II.2.2(a): We also obtain from (56) and (59) that

$$F_n(\xi) = [\phi(\xi)]^n + \frac{(\log n)^{M_1}}{n^{\Lambda_1}} \left(\sum_{(\Lambda_k, M_k)=(\Lambda_1, M_1)} \frac{\mathcal{C}_1 A_k w_k^{n+\Lambda_k}}{\xi - z_k} + o(1) \right) \tag{64}$$

uniformly in $\xi \in E \setminus \cup_{z_k \in E} \{z : |z - z_k| < \epsilon\}$; that is, Theorem II.2.1(a) holds for E containing no smooth corners.

The proof of Remark II.2.2(a) follows by proceeding as in the proof of Remark II.2.5(b), because of the form of the terms appearing under the Σ sign in (64).

Let us complete now the proof of Theorem II.2.1(a) by assuming that E contains some smooth corners. We still get from (56) and (59) that if z_j is a *smooth* corner of E, then

$$\begin{aligned} F_n(\xi) = [\phi(\xi)]^n + \mathcal{O}(\sigma^n) + \sum_{k \neq j} \frac{(\log n)^{M_k}}{n^{\Lambda_k}} \left(\frac{\mathcal{C}_k A_k w_k^{n+\Lambda_k}}{z_j - z_k} + o(1) \right) \\ + \frac{1}{2\pi i} \int_{\sigma_j}^{w_j} \left(\frac{t^n \psi_+'(t)}{\psi_+(t) - \xi} - \frac{t^n \psi_-'(t)}{\psi_-(t) - \xi} \right) dt \end{aligned} \tag{65}$$

uniformly in $\xi \in E \cap \{z : 0 < |z - z_j| < \epsilon\}$ as $n \to \infty$.

To get a similar estimate for $\xi = z_j$, choose a small closed simple path ℓ_j encircling the segment $(\sigma_j, w_j]$, whose only common point with $\mathbb{D}_\sigma \cup \Gamma_\sigma$ is σ_j. Let $R > 1$ be such that ℓ_j lies interior to $\mathbb{T}_R$. Then, since the function $\psi'(w)/[\psi(w) - z_j]$ is analytic on $\mathrm{ext}(\Gamma_\sigma)$ with continuous boundary values on $\Gamma_\sigma \setminus \{w_j\}$, we obtain from (12) and Lemma II.4.2 that

$$\begin{aligned} 2\pi i\, F_n(z_j) &= \oint_{\mathbb{T}_R} \frac{t^n \psi'(t) dt}{\psi(t) - z_j} \\ &= \oint_{\mathbb{T}_\sigma} \frac{t^n \psi'(t) dt}{\psi(t) - z_j} + \sum_{k \neq j} \int_{\sigma_k}^{w_k} \left(\frac{t^n \psi_+'(t)}{\psi_+(t) - z_j} - \frac{t^n \psi_-'(t)}{\psi_-(t) - z_j} \right) dt + \oint_{\ell_j} \frac{t^n \psi'(t) dt}{\psi(t) - z_j} \\ &= \mathcal{O}(\sigma^n) + \sum_{k \neq j} \frac{(\log n)^{M_k}}{n^{\Lambda_k}} \left(\frac{\mathcal{C}_k A_k w_k^{n+\Lambda_k}}{z_j - z_k} + o(1) \right) + \oint_{\ell_j} \frac{t^n \psi'(t) dt}{\psi(t) - z_j}\,. \end{aligned} \tag{66}$$

To be able to estimate simultaneously the last integral in (66) and the integral in (65), we use the following identity:

$$\frac{\psi'(w)}{\psi(w)-\xi} = \frac{Q_j'(w)}{Q_j(w)-\xi} + \frac{\psi'(w)-Q_j'(w)}{Q(w)-\xi} - \frac{[\psi(w)-Q_j(w)]\psi'(w)}{(Q_j(w)-\xi)^2} + \frac{[\psi(w)-Q_j(w)]^2\psi'(w)}{(Q_j(w)-\xi)^2(\psi(w)-\xi)}. \tag{67}$$

Since $Q_j([\sigma_j,w_j))\subset G$, we can assume that ℓ_j was chosen so close to $[\sigma_j,w_j]$ that $Q_j(w)-z_j\neq 0$ for all $w\neq w_j$ in $\ell_j\cup\operatorname{int}(\ell_j)$. Then, integrating over ℓ_j equality (67) for $\xi=z_j$ after it has been multiplied by w^n, and expressing the last three integrals obtained in this way over the two-sided segment $[\sigma_j,w_j]$, we obtain

$$\oint_{\ell_j}\frac{t^n\psi'(t)dt}{\psi(t)-z_j} = 2\pi i(w_k)^n + \oint_{[\sigma_j,w_j]}\frac{t^n[\psi'(t)-Q_j'(t)]dt}{Q_j(t)-z_j} - \oint_{[\sigma_j,w_j]}\frac{t^n[\psi(t)-Q_j(t)]\psi'(t)dt}{(Q_j(t)-z_j)^2} + \oint_{[\sigma_j,w_j]}\frac{t^n[\psi(t)-Q_j(t)]^2\psi'(t)dt}{(Q_j(t)-z_j)^2(\psi(t)-z_j)}. \tag{68}$$

Thus, from (65), (67), (66) and (68) we see that

$$\begin{aligned}2\pi iF_n(\xi) = {} & 2\pi i[\phi(\xi)]^n + \mathcal{O}(\sigma^n) + \sum_{k\neq j}\mathcal{O}\left(\frac{(\log n)^{M_k}}{n^{\Lambda_k}}\right) \\ & + \oint_{[\sigma_j,w_j]}\frac{t^n[\psi'(t)-Q_j'(t)]dt}{Q_j(t)-\xi} - \oint_{[\sigma_j,w_j]}\frac{t^n[\psi(t)-Q_j(t)]\psi'(t)dt}{(Q_j(t)-\xi)^2} \\ & + \oint_{[\sigma_j,w_j]}\frac{t^n[\psi(t)-Q_j(t)]^2\psi'(t)dt}{(Q_j(t)-\xi)^2(\psi(t)-\xi)},\end{aligned} \tag{69}$$

uniformly in $\xi\in E\cap\{z:|z-z_j|<\epsilon\}$ as $n\to\infty$.

We proceed to estimate the integrals appearing in (69). For this, we first observe that if $\ell\geq 0$, $m\geq 1$ are integers, and $\{\mathcal{F}(\cdot,\xi):\xi\in E\cap\{z:|z-z_j|<\epsilon\}\}$ is a uniformly bounded family of measurable functions on $[\sigma_j,w_j]$, then Lemma II.4.1

yields

$$
\begin{aligned}
&\int_{\sigma_j}^{w_j} \mathcal{F}(t,\xi) t^n (t-w_j)_\pm^\ell (\log_\pm(t-w_j))^m dt \\
&= (-1)^\ell w_j^{n+1+\ell} \int_\sigma^1 \mathcal{F}(w_j x,\xi) x^n (1-x)^\ell \left(\log(1-x) + i(\theta_j \mp \pi)\right)^m dx \\
&= \mathcal{O}(\sigma^n) + (-1)^\ell w_j^{n+1+\ell} \int_0^1 \mathcal{F}(w_j x,\xi) x^n (1-x)^\ell \left(\log(1-x)\right)^m dx \\
&\quad + \frac{i\, m(\theta_j \mp \pi)(-1)^{\ell+m-1} w_j^{n+1+\ell} \ell!\, n! (\log n)^{m-1}\, \mathcal{O}(1)}{(n+\ell+1)!} \\
&\quad + (1-\delta_{m,1}) \mathcal{O}\left(\frac{n!(\log n)^{m-2}}{(n+\ell+1)!}\right), \qquad (70)
\end{aligned}
$$

uniformly in $\xi \in E \cap \{z : |z-z_j| < \epsilon\}$, where the $\mathcal{O}(1)$ factor appearing in the first fraction of (70) is independent of the sign $\pm$ and can be replaced by 1 for a ξ such that $\mathcal{F}(\cdot,\xi) \equiv 1$. Also,

$$
\int_{\sigma_j}^{w_j} \mathcal{O}\left(t^n (t-w_j)_\pm^\ell (\log_\pm(t-w_j))^m\right) dt = \mathcal{O}\left(\frac{n!(\log n)^m}{(n+\ell+1)!}\right). \qquad (71)
$$

Now, recall that with $y = w - w_j$ (see (38)-(39)-(40)),

$$
Q_j(w) := z_j + c_{0,1,0}^{(j)} y^{\lambda_j} + \mathcal{O}(y^{\lambda_j+1}), \quad Q_j'(w) := c_{0,1,0}^{(j)} \lambda_j y^{\lambda_j-1} + \mathcal{O}(y^{\lambda_j}), \qquad (72)
$$

$$
\begin{aligned}
\psi(w) - Q_j(w) &= A_j y^{\Lambda_j} (\log y)^{m_j} + B_j y^{\Lambda_j} (\log y)^{m_j-1} \\
&\quad + \begin{cases} \mathcal{O}(y^{\Lambda_j} (\log y)^{m_j-2}) & \text{if } m_j \geq 2, \\ \mathcal{O}(y^{\Lambda_j+1} (\log y)^{\lfloor (r_j+1)/\lambda_j \rfloor}) & \text{if } m_j = 1, \end{cases} \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
\psi'(w) - Q_j'(w) &= A_j \Lambda_j y^{\Lambda_j-1} (\log y)^{m_j} + D_j y^{\Lambda_j-1} (\log y)^{m_j-1} \\
&\quad + \begin{cases} \mathcal{O}(y^{\Lambda_j-1} (\log y)^{m_j-2}) & \text{if } m_j \geq 2, \\ \mathcal{O}(y^{\Lambda_j} (\log y)^{\lfloor (r_j+1)/\lambda_j \rfloor}) & \text{if } m_j = 1, \end{cases} \qquad (74)
\end{aligned}
$$

where $c_{0,1,0}^{(j)} \neq 0$, $A_j \neq 0$, B_j and D_j are certain constants.

If we set $\mathcal{F}(t,\xi) := (Q_j(t)-z_j)/(Q_j(t)-\xi)$, then by (55) in Lemma II.4.3, $\{\mathcal{F}(t,\xi) : \xi \in E \cap \{\xi : |\xi - z_j| < \epsilon\}\}$ is uniformly bounded for $t \in [\sigma_j, w_j)$, and we get from (72), (74) and the equality

$$(Q_j(t)-z_j)^{-1} = {c_{0,1,0}^{(j)}}^{-1}(t-w_j)^{-\lambda_j} + \mathcal{O}((t-w_j)^{1-\lambda_j})$$

that

$$\begin{aligned}
&\int_{\sigma_j}^{w_j} \frac{t^n[\psi'_\pm(t) - Q'_j(t)]dt}{Q_j(t)-\xi} \\
&= \int_{\sigma_j}^{w_j} \frac{\mathcal{F}(t,\xi)t^n[\psi'_\pm(t) - Q'_j(t)]dt}{Q_j(t)-z_j} \\
&= \frac{A_j\Lambda_j}{c_{0,1,0}^{(j)}} \int_{\sigma_j}^{w_j} \mathcal{F}(t,\xi)t^n(t-w_j)_\pm^{r_j-1}(\log_\pm(t-w_j))^{m_j}dt \\
&\quad + \frac{D_j}{c_{0,1,0}^{(j)}} \int_{\sigma_j}^{w_j} \mathcal{F}(t,\xi)t^n(t-w_j)_\pm^{r_j-1}(\log_\pm(t-w_j))^{m_j-1}dt \\
&\quad + \begin{cases} \int_{\sigma_j}^{w_j} \mathcal{O}\left(t^n(t-w_j)_\pm^{r_j-1}(\log_\pm(t-w_j))^{m_j-2}\right), & m_j \geq 2, \\ \int_{\sigma_j}^{w_j} \mathcal{O}\left(t^n(t-w_j)_\pm^{r_j}(\log_\pm(t-w_j))^{\lfloor (r_j+1)/\lambda_j \rfloor}\right), & m_j = 1. \end{cases}
\end{aligned}$$

Combining this with (70), (71), we see that, uniformly in $\xi \in E \cap \{\xi : |\xi - z_j| < \epsilon\}$ as $n \to \infty$,

$$\oint_{[\sigma_j,w_j]} \frac{t^n[\psi'(t)-Q'_j(t)]dt}{Q_j(t)-\xi} = \frac{(\log n)^{m_j-1}}{n^{r_j}} \left(\frac{2\pi i\, A_j\Lambda_j r_j! m_j w_j^{n+r_j}\mathcal{O}(1)}{(-1)^{r_j+m_j-1}r_j\, c_{0,1,0}^{(j)}} + o(1) \right), \quad (75)$$

where the $\mathcal{O}(1)$ factor in (75) can be replaced by 1 if $\xi = z_j$.

Similarly, we get from (73), (74) and the equality

$$(Q_j(t)-z_j)^{-2} = {c_{0,1,0}^{(j)}}^{-2}(t-w_j)^{-2\lambda_j} + \mathcal{O}(t-w_j)\,,$$

that

$$\begin{aligned}
&\int_{\sigma_j}^{w_j} \frac{t^n[\psi(t)_\pm - Q_j(t)]\psi'_\pm(t)dt}{(Q_j(t)-\xi)^2} \\
&= \int_{\sigma_j}^{w_j} \frac{\mathcal{F}^2(t,\xi)t^n[\psi(t)_\pm - Q_j(t)]\psi'_\pm(t)dt}{(Q_j(t)-z_j)^2} \\
&= \frac{A_j\lambda_j}{c_{0,1,0}^{(j)}} \int_{\sigma_j}^{w_j} \mathcal{F}^2(t,\xi)t^n(t-w_j)_\pm^{r_j-1}(\log_\pm(t-w_j))^{m_j}dt \\
&\quad + \frac{\lambda_j B_j}{c_{0,1,0}^{(j)}} \int_{\sigma_j}^{w_j} \mathcal{F}^2(t,\xi)t^n(t-w_j)_\pm^{r_j-1}(\log_\pm(t-w_j))^{m_j-1}dt \\
&\quad + \begin{cases} \int_{\sigma_j}^{w_j} \mathcal{O}\left(t^n(t-w_j)_\pm^{r_j-1}(\log_\pm(t-w_j))^{m_j-2}\right), & m_j \geq 2, \\ \int_{\sigma_j}^{w_j} \mathcal{O}\left(t^n(t-w_j)_\pm^{r_j}(\log_\pm(t-w_j))^{2m_j}\right), & m_j = 1. \end{cases}
\end{aligned}$$

Hence, uniformly in $\xi \in E \cap \{\xi : |\xi - z_j| < \epsilon\}$ as $n \to \infty$,

$$\oint_{[\sigma_j,w_j]} \frac{t^n[\psi(t)_\pm - Q_j(t)]\psi'_\pm(t)dt}{(Q_j(t)-\xi)^2} = \frac{(\log n)^{m_j-1}}{n^{r_j}} \left(\frac{2\pi i\, A_j\lambda_j r_j! m_j w_j^{n+r_j}\mathcal{O}(1)}{(-1)^{r_j+m_j-1} r_j\, c_{0,1,0}^{(j)}} + o(1) \right), \tag{76}$$

where the $\mathcal{O}(1)$ factor in (76) can be replaced by 1 if $\xi = z_j$.

As for the last integral in (69), it follows directly from (73), (74), (55) and (70) that

$$\begin{aligned}
&\int_{\sigma_j}^{w_j} \frac{t^n[\psi_\pm(t) - Q_j(t)]^2\psi'_\pm(t)dt}{(Q_j(t)-\xi)^2(\psi(t)_\pm - \xi)} \\
&= \int_{\sigma_j}^{w_j} \left(\frac{\psi_\pm(t)-z_j}{\psi_\pm(t)-\xi}\right) \frac{\mathcal{F}^2(t,\xi)[\psi_\pm(t) - Q_j(t)]^2\psi'_\pm(t)dt}{(Q_j(t)-z_j)^2(\psi_\pm(t)-z_j)} \\
&= \int_{\sigma_j}^{w_j} \mathcal{O}\left(t^n(t-w_j)_\pm^{2r_j-1}(\log_\pm(t-w_j))^{2m_j}\right)dt \\
&= \mathcal{O}\left(\frac{(\log n)^{2m_j-1}}{n^{2r_j}}\right).
\end{aligned} \tag{77}$$

Finally, combining (69) with (75), (76) and (77), and since $(r_j, M_j) < (\Lambda_j, M_j)$, we obtain that uniformly in $\xi \in E \cap \{\xi : |\xi - z_j| < \epsilon\}$ as $n \to \infty$,

$$\begin{aligned} F_n(\xi) &= [\phi(\xi)]^n + \mathcal{O}(\sigma^n) + \sum_{k \neq j} \mathcal{O}\left(\frac{(\log n)^{M_k}}{n^{\Lambda_k}}\right) + \mathcal{O}\left(\frac{(\log n)^{M_j}}{n^{r_j}}\right) \\ &= [\phi(\xi)]^n + \mathcal{O}\left(\frac{(\log n)^{M_j^*}}{n^{\Lambda_j^*}}\right), \end{aligned} \tag{78}$$

with $(\Lambda_j^*, M_j^*) = \min\{(\Lambda_1, M_1), (r_j, M_j)\}$. Moreover,

$$\begin{aligned} F_n(z_j) = [\phi(z_j)]^n &+ \sum_{k \neq j} \frac{(\log n)^{M_k}}{n^{\Lambda_k}} \left(\frac{\mathcal{C}_k A_k w_k^{n+\Lambda_k}}{z_j - z_k} + o(1)\right) \\ &+ \frac{(\log n)^{M_j}}{n^{r_j}} \left(\frac{A_j r_j! m_j (-1)^{r_j + m_j - 1} w_j^{n+r_j}}{c_{0,1,0}^{(j)}} + o(1)\right) + \mathcal{O}(\sigma^n)\,. \end{aligned} \tag{79}$$

It is now clear that Theorem II.2.1(a) is a consequence of (64) and (78).

Proof of Theorem II.2.1(b): Suppose z_j is such that $\lambda_j \neq 1$. Choose a contour Γ_σ as the one provided by Lemma II.4.1. Let $\ell_j \subset D_{w_j}$ be a small closed simple path encircling the segment $(\sigma_j, w_j]$, whose only common point with $\mathbb{D}_\sigma \cup \Gamma_\sigma$ is σ_j. As before, we obtain from (12) and Lemma II.4.2 that

$$F_n(z_j) = \mathcal{O}(\sigma^n) + \sum_{k \neq j} \frac{(\log n)^{M_k}}{n^{\Lambda_k}} \left(\frac{\mathcal{C}_k A_k w_k^{n+\Lambda_k}}{z_j - z_k} + o(1)\right) + \frac{1}{2\pi i} \oint_{\ell_j} \frac{t^n \psi'(t) dt}{\psi(t) - z_j}\,. \tag{80}$$

If $\lambda_j = 2$, we can assume D_{w_j} is so small that $Q_j(w) - z_j \neq 0$ for all $D_{w_j} \setminus \{w_j\}$. Then, in exactly the same way we deduced (79) (for a smooth corner) from (66) and (67), we find that (79) is also valid for every corner z_j for which $\lambda_j = 2$, whence Theorem II.2.1(b) for the case $\lambda_j = 2$ follows.

If $\lambda_j \neq 2$, then we have in virtue of (36) and (36) that for all $\upsilon > 0$ small enough,

$$\psi(w) = z_j + A_j y^{\lambda_j} + c_{0,2,0}^{(j)} y^{2\lambda_j} + \mathcal{O}\left(y^{2\lambda_j + \upsilon}\right), \quad \text{if } 0 < \lambda_j < 1,$$

$$\psi(w) = z_j + A_j y^{\lambda_j} + c_{1,1,0}^{(j)} y^{1+\lambda_j} + c_{0,2,0}^{(j)} y^{2\lambda_j} + \mathcal{O}\left(y^{2\lambda_j+\upsilon}\right), \quad \text{if } 1 < \lambda_j < 2.$$

Hence, if $0 < \upsilon < \min\{\lambda_j, 2-\lambda_j\} < 1$, then

$$\frac{\psi'(w)}{\psi(w)-z_j} = \frac{\lambda_j}{y} + \frac{c_{0,2,0}^{(j)}\lambda_j y^{\lambda_j-1}}{A_j} + \mathcal{O}\left(y^{\lambda_j+\upsilon-1}\right), \quad \text{if } 0 < \lambda_j < 1, \tag{81}$$

$$\frac{\psi'(w)}{\psi(w)-z_j} = \frac{\lambda_j}{y} + \frac{c_{1,1,0}^{(j)}}{A_j} + \frac{c_{0,2,0}^{(j)}\lambda_j y^{\lambda_j-1}}{A_j} + \mathcal{O}\left(y^{\lambda_j+\upsilon-1}\right), \quad \text{if } 1 < \lambda_j < 2, \tag{82}$$

so that

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\ell_j}\frac{t^n\psi'(t)dt}{\psi(t)-z_j} &= \lambda_j[\phi(z_j)]^n + \frac{c_{0,2,0}^{(j)}\lambda_j}{2\pi i A_j}\oint_{[\sigma_j,w_j]} t^n(t-w_k)^{\lambda_j-1}dt \\ &\quad + \oint_{[\sigma_j,w_j]}\mathcal{O}\left(t^n(t-w_k)^{\lambda_j+\upsilon-1}\right)dt \\ &= \lambda_j[\phi(z_j)]^n - \frac{n!}{\Gamma(n+\lambda_j+1)}\left(\frac{c_{0,2,0}^{(j)}w_j^{n+\lambda_j}}{\Gamma(-\lambda_j)A_j} + o(1)\right),\end{aligned}$$

which combined with (77) yields

$$\begin{aligned}F_n(z_j) = \lambda_j[\phi(z_j)]^n &+ \sum_{k\neq j}\frac{(\log n)^{M_k}}{n^{\Lambda_k}}\left(\frac{\mathcal{C}_k A_k w_k^{n+\Lambda_k}}{z_j-z_k} + o(1)\right) \\ &- \frac{1}{n^{\lambda_j}}\left(\frac{c_{0,2,0}^{(j)}w_j^{n+\lambda_j}}{\Gamma(-\lambda_j)A_j} + o(1)\right) + \mathcal{O}(\sigma^n).\end{aligned} \tag{83}$$

Since $\Lambda_1 = \lambda_1 \leq \lambda_j < 2 \leq \Lambda_k$ if $\lambda_k \in \{1,2\}$, Theorem II.2.1(b) for the case $\lambda_j \neq 2$ follows immediately from (83).

CHAPTER III

POLYNOMIALS ORTHOGONAL OVER REGIONS BOUNDED BY ANALYTIC JORDAN CURVES

III.1 Introduction

Let L be an analytic Jordan curve in the complex plane $\mathbb{C}$ and let G be its interior domain. Applying the standard Gram-Schmidt orthogonalization procedure to the system of linearly independent functions $\{1, z, z^2, \ldots, z^n, \ldots\}$, one can construct a unique sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ with positive leading coefficients that are orthonormal with respect to area measure over G, i.e., they satisfy

$$P_n(z) = \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n \geq 0,$$

$$\int_G P_n(z)\overline{P_m(z)}\, dxdy = \delta_{n,m}, \quad n, m \geq 0,$$

where $dxdy$ is the two-dimensional Lebesgue measure.

These polynomials are sometimes called *Carleman* or *Bergman* polynomials for G. The problem we investigate in this chapter is that of describing the asymptotic behavior of the polynomials P_n and their zeros.

The aim of this introduction is to present those known results that are relevant to our investigation. We start by introducing some notation that will be used throughout the chapter.

For any $0 \leq r < \infty$, we define

$$\mathbb{T}_r := \{w : |w| = r\}, \quad \Delta_r := \{w : |w| > r\}, \quad \mathbb{D}_r := \{w : |w| < r\}.$$

Let $\Omega = \text{ext}(L)$ be the exterior domain of L. By the Riemann mapping theorem, there is a unique conformal map $\psi(w)$ of the exterior of the unit circle Δ_1 onto Ω such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. Since the curve L is analytic, there is a smallest number $0 \leq \rho < 1$ such that ψ has an analytic and *univalent* continuation (also denoted by ψ) to Δ_ρ. For every $\rho \leq r < \infty$, put

$$\Omega_r := \{z = \psi(w) : |w| > r\}, \quad L_r := \partial\Omega_r, \quad G_r := \mathbb{C} \setminus \overline{\Omega}_r \,.$$

Thus, if $r > \rho$, then L_r is an analytic Jordan curve and Ω_r, G_r are, respectively, the exterior and interior domains of L_r.

There are essentially two results that reflect the interrelation between the polynomials P_n and the canonical conformal maps of the domains Ω and G. The first states that if $\rho < r < 1$, then for all $z \in \Omega_\rho$,

$$P_n(z) = \sqrt{\frac{n+1}{\pi}}\, \phi'(z)[\phi(z)]^n \left(1 + h_n(z)\right), \tag{84}$$

where

$$h_n(z) = \begin{cases} \mathcal{O}\left(n^{1/2}\rho^n\right), & z \in \overline{\Omega}, \\ \mathcal{O}\left(n^{-1/2}\right)(\rho/r)^n, & z \in \overline{\Omega}_r, \end{cases} \tag{85}$$

and the constants involved in the $\mathcal{O}$-terms are independent of z.

Formula (84) was first established by T. Carleman in [1], but only for values of $z \in \overline{\Omega}$ and with the estimate $h_n(z) = \mathcal{O}(n^{1/2}q^n)$ for some $\rho < q < 1$. Its validity for points $z \in G \cap \Omega_\rho$ was independently established by E. R. Johnston [10] and Korovkin [11][1], but with estimates for $h_n(z)$ worse than (85). The statement presented here has been taken from the book of Gaier [5], which seems to provide the best estimate obtained so far.

[1]Indeed, Korovkin obtained a stronger result. Using Carleman's method he proved a generalization of formula (84) for polynomials orthogonal over G with respect to weights of a specific type.

Thus, the behavior of P_n as $n \to \infty$ on Ω_ρ is rather simple:

$$\lim_{n\to\infty} \frac{P_n(z)}{\sqrt{(n+1)/\pi}\,[\phi(z)]^n} = \phi'(z)\,, \quad z \in \Omega_\rho.$$

Moreover, since this limit holds uniformly on compact subsets of Ω_ρ and ϕ' never vanishes in Ω_ρ, it follows that for all $\rho < r < \infty$, there exists a natural number N_r such that if $n > N_r$, then $P_n(z) \neq 0$ for all $z \in \overline{\Omega}_r$. Hence, asymptotically, the zeros of P_n accumulate on $\mathbb{C} \setminus \Omega_\rho$.

The second result connects the polynomials P_n with an interior conformal map of G onto the unit disk $\mathbb{D}_1$ and it is based on the completeness of the system $\{P_n\}_{n=0}^{\infty}$.

Consider the vector space

$$\mathcal{B}^2(G) := \left\{ f \text{ analytic on } G\ :\ \int_G |f(z)|^2 dxdy < \infty \right\},$$

endowed with the inner product and corresponding norm

$$\langle f|g\rangle := \int_G f(z)\,\overline{g(z)}\,dxdy\,, \quad \|f\| := \sqrt{\langle f|f\rangle}\,.$$

The system $\{P_n(z)\}_{n=0}^{\infty}$ is complete in $\mathcal{B}^2(G)$ (cf. [5, §3, Thm. 1]), meaning that if $f \in \mathcal{B}^2(G)$ and

$$\alpha_{n,f} := \langle f|P_n\rangle, \quad n \geq 0$$

are the Fourier coefficients of f, then

$$f(z) = \sum_{n=0}^{\infty} \alpha_{n,f} P_n(z), \tag{86}$$

where the convergence of the series is understood in the $\mathcal{B}^2(G)$-norm sense.

Moreover, setting

$$\tau(f) := \sup\{\tau : f(z) \text{ has an analytic continuation to } G_\tau\}, \quad f \in \mathcal{B}^2(G), \tag{87}$$

we have ([34, pp. 130-131])

$$\limsup_{n\to\infty} |\alpha_{n,f}|^{1/n} = \frac{1}{\tau(f)}, \tag{88}$$

and the series on the right-hand side of (86) also converges to f (or more precisely, to its analytic continuation) locally uniformly on $G_{\tau(f)}$.

For $\xi \in G$ fixed, let φ_ξ be that conformal map of G onto the unit disk $\mathbb{D}_1$ such that $\varphi_\xi(\xi) = 0$, $\varphi_\xi'(\xi) > 0$. It is well-known that

$$K_\xi(z) = \frac{\varphi_\xi'(\xi)\varphi_\xi'(z)}{\pi}$$

is the reproducing kernel (or Bergman kernel) of the space $\mathcal{B}^2(G)$; in other words, for $\xi \in G$ fixed, $K_\xi(z)$ is the unique function of $\mathcal{B}^2(G)$ satisfying

$$f(\xi) = \langle f | K_\xi(\cdot)\rangle \quad \forall\, f \in \mathcal{B}^2(G)\,. \tag{89}$$

Thus, according to (86) and (89),

$$K_\xi(z) = \frac{\varphi_\xi'(\xi)\varphi_\xi'(z)}{\pi} = \sum_{n=0}^{\infty} \overline{P_n(\xi)}P_n(z), \tag{90}$$

uniformly (in the variable z for fixed ξ) on compact subsets of $G_{\tau(\varphi_\xi)}$.

Equality (90) suggests that there might be a closer relationship between the interior conformal maps of G and the polynomials P_n, although some doubts might appear if one is aware that said equality is not a unique feature of the P_n's, since for any other complete orthonormal system $\{\chi_n(z)\}_{n=0}^{n}$ in $\mathcal{B}^2(G)$, the equality

$$\frac{\varphi_\xi'(\xi)\varphi_\xi'(z)}{\pi} = \sum_{n=0}^{\infty} \overline{\chi_n(\xi)}\chi_n(z) \tag{91}$$

also holds both in the metric of $\mathcal{B}^2(G)$ and uniformly on compact subsets of G.

However, what is certainly more specific to the system $\{P_n\}_{n=0}^{\infty}$ is the Cauchy-Hadamard type of formula (88), which combined with (90) gives

$$\limsup_{n\to\infty} |P_n(\xi)|^{1/n} = \frac{1}{\tau(\varphi_\xi)} \quad \forall\, \xi \in G\,. \tag{92}$$

Summarizing, the asymptotic behavior of the polynomials P_n in the region of the complex plane Ω_ρ is completely understood and given by (84). Trying to understand what could happen on its complement $\mathbb{C} \setminus \Omega_\rho$ is the core objective of this chapter. Formula (92) is rather weak, but provides some information on the behavior of $P_n(\xi)$ for points $\xi \in G$.

In Section III.2 we present our main results. Combining (84) with (90) we derive an integral representation (Theorem III.2.1) for the polynomials $P_n(z)$ valid for all $z \in G$, and apply it to the concrete situation when $L_\rho = \partial\Omega_\rho$ is a piecewise analytic Jordan curve without smooth corners or cusps. In this case, we are able to describe the behavior of $P_n(z)$ at every point z of the complex plane (Theorems III.2.2 and III.2.4). In particular, the domain of validity of Carleman's formula (84) is extended to include the analytic portions of L_ρ. Immediate corollaries of these theorems about the zeros of P_n are presented in Section III.3. The remaining sections are devoted to proving the results. It is worth noticing that the theorems and corollaries that follow are of a similar flavor to those previously obtained for Faber polynomials.

III.2 Asymptotic behavior of P_n

We have seen in Chapter II how much could be derived from the integral representation (12) for the Faber polynomials, which is nothing but the Cauchy integral formula for the coefficients of their generating function. It would therefore be very convenient to have a generating function for the P_n's that leads to an integral representation for them. Indeed, we have such a generating function, namely, the kernel function

$$\frac{\varphi'_\xi(\xi)\varphi'_\xi(z)}{\pi} = \sum_{k=0}^{\infty} \overline{P_k(\xi)} P_k(z). \tag{93}$$

Of course, the inconvenience of this generating function is that $\overline{P_k(\xi)}$ is the coefficient of the same polynomial $P_k(z)$, whose behavior is precisely what we want to find out. However, we already know how P_n behaves in some part of the complex plane, because Carleman's formula tells us that for all $z \in \Omega_\rho$

$$P_k(z) \sim \sqrt{(k+1)/\pi}\, \phi'(z)[\phi(z)]^k. \tag{94}$$

Let us make an informal substitution to get from (93) and (94) that for fixed $\xi \in G$ and all $z \in L \subset \Omega_\rho$,

$$\frac{\varphi'_\xi(\xi)\varphi'_\xi(z)}{\pi} \sim \sum_{k=0}^{\infty} \overline{P_k(\xi)} \sqrt{(k+1)/\pi}\, \phi'(z)[\phi(z)]^k. \tag{95}$$

In order to extricate the coefficient $\overline{P_n(\xi)}$ from this last series, we can multiply (95) by $[\phi(z)]^{-(n+1)}/2\pi i$ and integrate it over L with respect to dz. This yields

$$\overline{P_n(\xi)} \sim \frac{\varphi'_\xi(\xi)}{\sqrt{\pi(n+1)}} \cdot \frac{1}{2\pi i} \oint_L \varphi'_\xi(z)[\phi(z)]^{-(n+1)} dz\,, \quad \xi \in G.$$

We have just obtained (albeit very informally) an integral representation for the polynomials P_n. Its correct statement is the content of the following theorem.

Theorem III.2.1. *Let φ be a conformal map of G onto $\mathbb{D}_1$, and let $\rho < r < 1$. Then, for any fixed δ such that $r\rho < \delta < 1$,*

$$\sqrt{\pi(n+1)}\, P_n(\xi) = \frac{\varphi'(\xi)}{2\pi i} \oint_L \frac{\varphi'(z)[\phi(z)]^{n+1} dz}{[\varphi(z) - \varphi(\xi)]^2} + \mathcal{O}(\delta^n), \tag{96}$$

uniformly in $\xi \in \overline{G}_r$ as $n \to \infty$.

As it is to expect from Theorem III.2.1, the expression

$$\frac{\varphi'(\xi)\varphi'(z)}{[\varphi(z) - \varphi(\xi)]^2}$$

does not depend on the choice of the conformal map φ. This can be easily verified by using the fact that any other conformal map φ_1 of G onto $\mathbb{D}_1$ is of the form

$$\varphi_1(z) = e^{i\vartheta} \frac{\varphi(z) - \varphi(\xi)}{1 - \overline{\varphi(\xi)}\varphi(z)}, \tag{97}$$

where ξ is the point of G mapped to 0 by φ_1.

The integral representation (96) has several features that should be highlighted. First, it holds uniformly in $\xi \in \mathbb{C}\setminus\Omega_\rho$, and so there is hope it could lead to new results on the behavior of P_n. Second, it combines in its integrand the exterior mapping ϕ and the interior mapping φ, suggesting that the behavior of these polynomials could actually depend on both mappings. Notice also that the $\mathcal{O}$ term involved decays faster than ρ^n as $n \to \infty$, and therefore, it is reasonable to expect it will be negligible.

To illustrate how Theorem III.2.1 quickly leads to new results, let us consider a concrete situation. The reader should compare the results and methods of proof that follow with those of Chapter II for Faber polynomials.

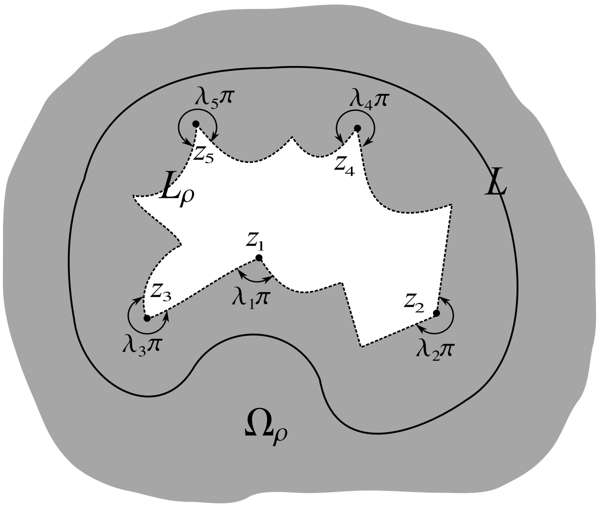


Figure 8: The case of a piecewise analytic L_ρ.

Suppose $L_\rho = \partial\Omega_\rho$ is a piecewise analytic Jordan curve with corners at the points $z_1, z_2, \ldots, z_s$, $s \geq 1$, such that for all $1 \leq k \leq s$, the two analytic arcs that meet at z_k form an *exterior* angle $\lambda_k\pi$, $0 < \lambda_k < 2$, $\lambda_k \neq 1$ (see Figure 8 above). Recall that, by the definition of a piecewise analytic curve, each arc of L_ρ with z_k as one of its endpoints is part of a longer analytic simple arc containing z_k in its interior. A curve admitting smooth corners ($\lambda_k = 1$) and outer cusps ($\lambda_k = 2$) can be handled as we did for the Faber polynomials in Chapter II, but we have decided to exclude this case for the sake of simplicity.

We assume that the corners z_k's have been numbered in such a way that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_u < \lambda_{u+1} \leq \cdots \leq \lambda_s$$

with $u \in \{1, 2, \ldots, s\}$.

Since L_ρ is a Jordan curve, ϕ has a one-to-one continuous extension to $\overline{\Omega}_\rho$. Let $w_k := \phi(z_k)$, $1 \leq k \leq s$, so that $|w_1| = |w_2| = \cdots = |w_s| = \rho$. Let φ be an arbitrary conformal map of G onto $\mathbb{D}_1$. Since φ is conformal, φ maps the curve L_ρ onto a piecewise analytic curve lying inside the unit disk, with corners at $\varphi(z_1), \ldots, \varphi(z_s)$, and exterior angle $\lambda_k\pi$ at $\varphi(z_k)$. By the quoted result of Lehman ([13, Thm. 1]),

$\varphi(\psi(w))$ has an asymptotic expansion about w_k (see Section III.4.2), which cut at its second term gives the following: if $0 < \lambda_k < 2$, $\lambda_k \neq 1$, then as $w \to w_k$ from the exterior of the unit circle,

$$\varphi(\psi(w)) = \varphi(z_k) + A_k(w - w_k)^{\lambda_k} \left(1 + o(1)\right), \quad A_k \neq 0. \tag{98}$$

Clearly,

$$\frac{\varphi'(z_k)}{A_k} = \lim_{\substack{z \to z_k \\ z \in \Omega_\rho}} \frac{[\phi(z) - \phi(z_k)]^{\lambda_k}}{z - z_k}. \tag{99}$$

Although the branch of the function $(w - w_k)^{\lambda_k}$ may be taken arbitrarily (with the expense of changing the coefficient A_k), we will fix it as follows. Let

$$\theta_k := \arg(w_k), \quad 0 \leq \theta_k < 2\pi, \quad 1 \leq k \leq s\,.$$

Then, the value of said function at $e^{i\theta_k}$ is defined to be

$$\left(e^{i\theta_k} - w_k\right)^{\lambda_k} := (1 - \rho)^{\lambda_k} e^{i\lambda_k\theta_k}\,. \tag{100}$$

Because ϕ maps each analytic arc of L_ρ onto a circular arc, by the reflection principle, ϕ can be analytically continued from Ω_ρ onto a connected open set $U \supset \overline{\Omega}_\rho \setminus \{z_1, \ldots, z_s\}$. We denote such a continuation by ϕ as well.

Theorem III.2.2. *(a) Uniformly on any compact set $E \subset G \cap \overline{\Omega}_\rho \setminus \{z_1, \ldots, z_s\}$:*

$$P_n(z) = \sqrt{(n+1)/\pi}\, \phi'(z)[\phi(z)]^n + \mathcal{O}\left(\frac{\rho^n}{n^{\lambda_1 + 1/2}}\right) \quad (n \to \infty). \tag{101}$$

(b) For every corner z_j,

$$\lim_{n\to\infty} \sqrt{\frac{\pi}{n+1}} \cdot \frac{\Gamma(\lambda_j)\Gamma(n+2-\lambda_j)P_n(z_j)}{n!\,[\phi(z_j)]^{n+1-\lambda_j}} = \lim_{\substack{z \to z_j \\ z \in \Omega_\rho}} \frac{[\phi(z) - \phi(z_j)]^{\lambda_j}}{z - z_j}. \tag{102}$$

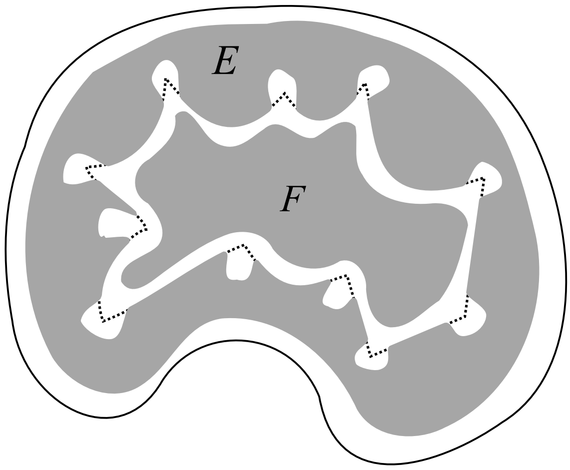


Figure 9: Compact sets E and F as in Theorems III.2.2 and III.2.4.

Theorem III.2.2(a) tells us that when L_ρ is a piecewise analytic Jordan curve as described above, the domain of validity of Carleman's formula (84) can be extended to include the analytic portions of L_ρ. Moreover, taking $E = L_r$ with $\rho < r < 1$ and applying the maximum modulus principle for analytic functions, we see that

$$P_n(z) = \sqrt{(n+1)/\pi}\,\phi'(z)[\phi(z)]^n \left[1 + \mathcal{O}\left(n^{-\lambda_1-1}\right)\left(\frac{\rho}{r}\right)^n\right],$$

uniformly in $z \in \overline{\Omega}_r$ as $n \to \infty$. That is, the rate of decay of the functions $h_n(z)$ in (85) is improved as well.

Remark III.2.3. The estimate (101) is sharp: if E has more than $2(s-1)$ points, then there is a constant $\alpha(E) > 0$ such that

$$\max_{z\in E}\left|P_n(z) - \sqrt{\tfrac{n+1}{\pi}}\,\phi'(z)[\phi(z)]^n\right| \geq \alpha(E)\left(\frac{\rho^n}{n^{\lambda_1+1/2}}\right) \qquad \forall\, n \geq 0. \tag{103}$$

In fact, there are compact sets E for which "max" can be replaced by "min" in (103).

The behavior of P_n inside G_ρ is given next.

Theorem III.2.4. *Let φ be a conformal map of G onto $\mathbb{D}_1$. Then, with the notations above,*

$$\sqrt{\frac{\pi}{n+1}}\cdot\frac{\Gamma(n+\lambda_1+2)P_n(z)}{n!\rho^{n+1+\lambda_1}}=-\frac{\varphi'(z)}{\Gamma(-\lambda_1)}\sum_{k=1}^{u}\frac{A_k e^{i(n+1+\lambda_1)\theta_k}}{[\varphi(z)-\varphi(z_k)]^2}+R_n(z),$$

where

$$R_n(z)=\begin{cases}\mathcal{O}\left(n^{-\min\{\lambda_1,\lambda_{u+1}-\lambda_1\}}\right), & if\ 0<\lambda_1<1,\\ \mathcal{O}\left(n^{-\min\{1,\lambda_{u+1}-\lambda_1\}}\right), & if\ 1<\lambda_1<2,\end{cases}\tag{104}$$

uniformly on any compact set $F\subset G_\rho$.

Theorem III.2.4 is very appealing because it shows that once we are in the interior domain of L_ρ, the interior mapping φ starts playing a predominant role.

Remark III.2.5. The estimate (104) is best possible whenever $\lambda_1\neq 1/2$: if $0<\lambda_1<1$, $\lambda_1\neq 1/2$, then for every compact set $F\subset G_\rho$ containing more than $u+2(s-1)$ points, there is a constant $\beta(F)>0$ such that

$$\max_{z\in F}|R_n(z)|\geq\beta(F)\,n^{-\min\{\lambda_1,\lambda_{u+1}-\lambda_1\}}\qquad\forall\ n\geq 0,\tag{105}$$

and, in fact, there are compact sets $F\subset G_\rho$ for which "max" can be replaced by "min" in (105).

The corresponding statement for the case $1<\lambda_1<2$ (substitute "$u+2(s-1)$ points" by "$2(s-1)$ points") also holds true provided that $b_{1,1,0}^{(k)}\neq 0$ for at least one k ($1\leq k\leq u$), where $b_{1,1,0}^{(k)}$ is the coefficient of the $(w-w_k)^{1+\lambda_k}$-term appearing in the expansion of $\varphi(\psi(w))$ about w_k:

$$\varphi(\psi(w))=\varphi(z_k)+A_k(w-w_k)^{\lambda_k}+b_{1,1,0}^{(k)}(w-w_k)^{1+\lambda_k}+o\left((w-w_k)^{1+\lambda_k}\right).$$

III.3 The zeros of P_n

We shall now examine the location and distribution of the zeros of P_n under the same assumption of piecewise analyticity of L_ρ . The corollaries that follow are direct consequences of Theorems III.2.2 and III.2.4. The proofs of Corollaries III.3.1 and III.3.2 below proceed along the same lines as the corresponding Corollaries II.3.1 and II.3.2 for Faber polynomials; therefore we omit it.

We shall denote by $\mathcal{Z}$ the set of all accumulation points of the zeros of the polynomials P_n; that is, the set of those points $t \in \overline{\mathbb{C}}$ such that for every open neighborhood U of t, it is possible to find infinitely many polynomials P_n having a zero in U.

Recall that u is defined to be that positive integer for which

$$\lambda_1 = \cdots = \lambda_u < \lambda_{u+1} \leq \cdots \leq \lambda_s.$$

Corollary III.3.1. *For any closed set $E \subset \overline{\Omega}_\rho \backslash \{z_1, \ldots, z_s\}$, there is a positive integer N_E such that if $n > N_E$, then $P_n(z) \neq 0$ for all $z \in E$.*

For any compact set $F \subset G_\rho$, there is a positive integer N_F such that if $n > N_F$, then $P_n(z)$ has at most $2(u-1)$ zeros in F (counting multiplicities).

Corollary III.3.2. *The point $t \in G_\rho$ belongs to $\mathcal{Z}$ if, and only if, there exist a subsequence $n_1 < n_2 < \cdots < n_\ell < \cdots$ and real numbers $\widehat{\theta}_1, \ldots, \widehat{\theta}_u$ such that*

$$\lim_{\ell \to \infty} e^{in_\ell \theta_k} = e^{i\widehat{\theta}_k}, \quad 1 \leq k \leq u\,, \tag{106}$$

and

$$\sum_{k=1}^{u} \frac{A_k e^{i\left(\widehat{\theta}_k + (\lambda_1+1)\theta_k\right)}}{[\varphi(t) - \varphi(z_k)]^2} = 0\,.$$

In particular, if θ_k/π is rational for all $1 \leq k \leq u$, then $\mathcal{Z} \cap G_\rho$ is a finite set.

Let ν_{P_n} be the normalized counting measure of the zeros of P_n; that is, if $z_{n1}, \ldots,$ z_{nn} are the zeros of P_n (counting multiplicities) and $\delta_{z_{ni}}$ is the unit mass Dirac measure at z_{ni}, then

$$\nu_{P_n} := \frac{1}{n}\sum_{i=1}^{n} \delta_{z_{ni}} \,.$$

The sequence $\{\nu_{P_n}\}_{n=1}^{\infty}$ is said to converge in the weak*-topology to the finite measure μ, if for every continuous function f defined on $\overline{\mathbb{C}}$,

$$\lim_{n\to\infty} \int_{\overline{\mathbb{C}}} f\nu_{P_n} = \int_{\overline{\mathbb{C}}} f d\mu \,.$$

The equilibrium measure μ_{L_ρ} of the curve L_ρ is the measure defined on any Borel set $\mathfrak{B} \subset L_\rho$ by

$$\mu_{L_\rho}(\mathfrak{B}) = \frac{1}{2\pi\rho}\int_{\phi(\mathfrak{B})} |dt| \,.$$

Corollary III.3.3. *The sequence $\{\nu_{P_n}\}_{n=1}^{\infty}$ converges in the weak*-topology to the equilibrium measure of the curve L_ρ. Therefore, $L_\rho \subset \mathcal{Z}$.*

Proof. The proof makes use of Lemma V.4.3 of Chapter V. The (normalized) conformal map of Ω_ρ onto Δ_1 is $\phi(z)/\rho$, and the logarithmic capacity of L_ρ is $\rho/\phi'(\infty)$. Then, by a well-known result relating the logarithmic potential of the equilibrium measure of a Jordan curve with the exterior conformal map, the logarithmic potential $U^{\mu_{L_\rho}}(z)$ of μ_{L_ρ} is given by

$$U^{\mu_{L_\rho}}(z) = \begin{cases} \log|\phi'(\infty)/\phi(z)|, & \text{if } z \in \overline{\Omega}_\rho, \\ \log\left(\phi'(\infty)/\rho\right), & \text{if } z \in G_\rho. \end{cases} \tag{107}$$

Let κ_n be the leading coefficient of P_n an'd let $q_n(z) := P_n(z)/\kappa_n$. By the regularity of the area measure of G (see Section V.4), we have

$$\lim_{n\to\infty} \kappa_n^{1/n} = \phi'(\infty), \tag{108}$$

which combined with Theorem III.2.2 yields

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = |\phi(z)/\phi'(\infty)| \qquad \forall\, z \in L_\rho. \tag{109}$$

Applying Lemma V.4.3 to the sequence $\{q_n\}_{n=0}^{\infty}$, the set $E = \mathbb{C} \setminus \Omega_\rho$ and the function $g(z) = \phi(z)/\phi'(\infty)$, we find that, with the notation of said lemma, $\mu_g = \mu_{L_\rho}$ and

$$\limsup_{n\to\infty} |q_n(z)|^{1/n} \le \frac{\phi'(\infty)}{\rho} = e^{-U^{\mu_g}(z)} \quad \forall\, z \in G_\rho\,.$$

Moreover, according to Lemma V.4.3(b), if for some $z_0 \in G_\rho$,

$$\lim_{n\to\infty} |q_n(z_0)|^{1/n} = \frac{\phi'(\infty)}{\rho}, \tag{110}$$

then $\{\nu_{P_n}\}_{n=1}^{\infty}$ converges in the weak*-topology to $\mu_g = \mu_{L_\rho}$.

But, obviously, there exist constants $0 < m < M < \infty$ such that for any $z_0 \in G_\rho$ that is sufficiently closed to z_1,

$$m \le \frac{|A_1|}{|\varphi(z_0) - \varphi(z_1)|^2} - \sum_{k=2}^{u} \frac{|A_k|}{|\varphi(z_0) - \varphi(z_k)|^2} \le \left| \sum_{k=1}^{u} \frac{A_k e^{i(n+1+\lambda_1)\theta_k}}{[\varphi(z_0) - \varphi(z_k)]^2} \right| \le M,$$

and so by Theorem III.2.4 and (108), we conclude that for any of these z_0, (110) holds. The proof is complete. □

Observe that every piecewise analytic Jordan curve γ is the set L_ρ corresponding to some analytic curve L. To see this, consider the conformal map $g(z)$ of the exterior of the curve γ onto Δ_1 satisfying that $g(\infty) = \infty$, $g'(\infty) > 0$. Fix $1 < \eta < \infty$ and define $L := \{z : |g(z)| = \eta\}$. Then, the conformal map ϕ of the exterior of L onto Δ_1 is $g(z)/\eta$, and therefore, $\rho = 1/\eta$ and Ω_ρ is the exterior of γ.

Let us finish this section with an example.

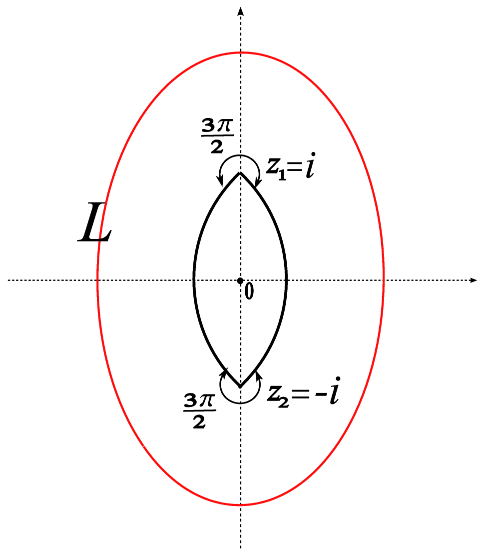


Figure 10: Curves γ and L of Example III.3.4.

Example III.3.4. Choose γ to be the curve formed by the union of two circular arcs of equal radius that meet at i and $-i$ forming an exterior angle of value $3\pi/2$. Fix $1 < \eta < \infty$ and define $L := \{z : |g(z)| = \eta\}$. Then, $\rho = 1/\eta$ and $L_\rho = \gamma$. Putting $z_1 := i$, $z_2 := -i$, we have $\lambda_1 = \lambda_2 = 3/2$ (see Figure 10 above). We then consider the sequence $\{P_n\}_{n=0}^{\infty}$ of orthonormal polynomials with respect to area measure over the interior of L.

Let φ be a conformal map of the interior of L onto Δ_1 satisfying $\varphi(0) = 0$. Then, from the symmetry of the curve γ, it follows that $\varphi(z) = -\varphi(-z)$, $\phi(z) = -\phi(-z)$, $\phi(i) = \eta^{-1}e^{i\pi/2}$, $\phi(-i) = \eta^{-1}e^{i3\pi/2}$, so that $\theta_1 = \pi/2$, $\theta_2 = 3\pi/2$. Since

$$A_k = \varphi'(z_k) \lim_{z \to z_k} \frac{z - z_k}{[\phi(z) - \phi(z_k)]^{\lambda_k}}, \quad k = 1, 2,$$

it is easily seen that

$$A_1 e^{i(\lambda_1+1)\theta_1} = A_2 e^{i(\lambda_2+1)\theta_2},$$

and so, by Theorem III.2.4,

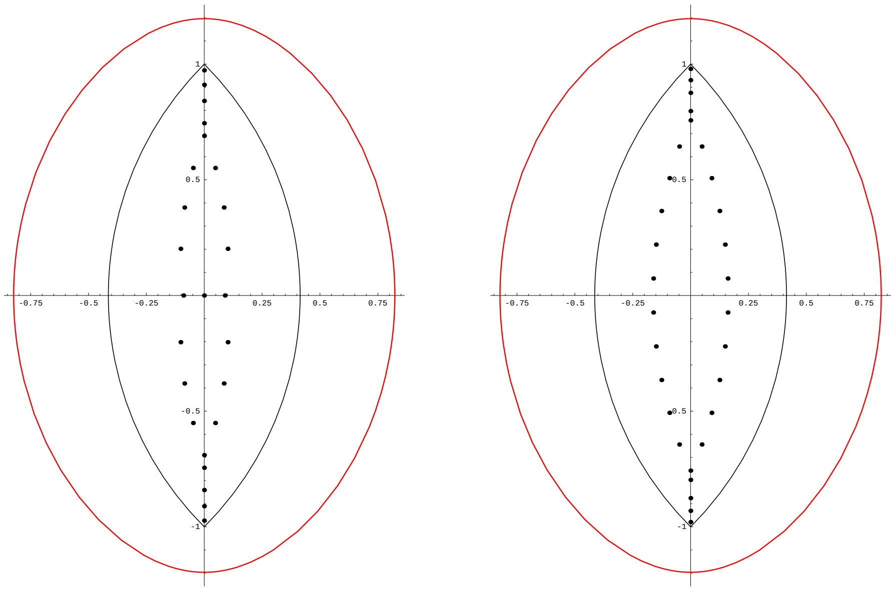


Figure 11: Zeros of the orthonormal polynomials $P_{25}(z)$ (left) and $P_{30}(z)$ (right) for the curve L corresponding to $\eta = 1.5$ in Example III.3.4.

$$\frac{n^2\,P_n(z)}{[\phi(i)]^{n+5/2}} = -\frac{A_1\varphi'(z)}{\sqrt{\pi}\,\Gamma(-3/2)}\left(\frac{1}{[\varphi(z)-\varphi(i)]^2}+\frac{e^{in\pi}}{[\varphi(z)+\varphi(i)]^2}\right)+o(1),$$

uniformly on compact subsets of $G_\rho = \mathrm{int}(\gamma)$ as $n\to\infty$. Hence, for $n = 2m+1$ odd,

$$\frac{n^2\,P_n(z)}{[\phi(i)]^{n+5/2}} = -\frac{4A_1}{\sqrt{\pi}\,\Gamma(-3/2)}\cdot\frac{\varphi(i)\varphi'(z)\varphi(z)}{[\varphi^2(z)-\varphi^2(i)]^2}+o(1), \quad \text{as } m\to\infty,$$

while for $n = 2m$ even,

$$\frac{n^2\,P_n(z)}{[\phi(i)]^{n+5/2}} = -\frac{2A_1}{\sqrt{\pi}\,\Gamma(-3/2)}\cdot\frac{\varphi'(z)\,[\varphi^2(z)+\varphi^2(i)]}{[\varphi^2(z)-\varphi^2(i)]^2}+o(1), \quad \text{as } m\to\infty.$$

Thus, once the sequence $\{P_n\}$ has been properly normalized, the subsequence corresponding to even degrees converges to a zero free function, while the one corresponding to odd degrees converges to a (different) function with 0 as its only zero in G_ρ ($\varphi(0) = 0$). Hence, 0 is the only point of G_ρ that is an accumulation point of the zeros of the P_n's. Indeed, it is easy to see that $P_n(z)$ has a simple zero at 0 for all n odd. Figure 11 above shows the plots of the zeros of P_n, $n = 25$, $n = 30$, corresponding to $\eta = 1.5$.

III.4 Proofs of the results in Section III.2

III.4.1 Proof of Theorem III.2.1

Recall that $\varphi_\xi(z)$ is the conformal map of G onto the unit disk $\mathbb{D}_1$ such that $\varphi_\xi(\xi) = 0$, $\varphi_\xi'(\xi) > 0$. Fix $\rho < r < 1$ and suppose $\xi \in \overline{G}_r$. Because $L = \partial G$ is an analytic Jordan curve, φ_ξ has an analytic and univalent continuation from G onto $G_{1/r}$, which we denote by φ_ξ as well. Indeed, by the Schwarz reflection principle [3], φ_ξ satisfies the equality

$$\varphi_\xi(z) = \frac{1}{\overline{\varphi_\xi\left(\psi\left(1\big/\overline{\phi(z)}\right)\right)}}, \qquad z \in \Omega_r \cap G_{1/r}\,. \tag{111}$$

Then, we obtain from (90) in the introduction to this chapter (since $1/r \le \tau(\varphi_\xi)$) that

$$\frac{\varphi_\xi'(\xi)\varphi_\xi'(z)}{\pi} = \sum_{k=0}^{\infty} \overline{P_k(\xi)} P_k(z), \tag{112}$$

with the series converging uniformly on compact subsets of $G_{1/r}$.

Combining equations (84) and (112) we see that for every $\xi \in \overline{G}_r$, $z \in L$,

$$\begin{aligned}\frac{\varphi_\xi'(\xi)\varphi_\xi'(z)}{\pi} &= \sum_{k=0}^{\infty} \overline{P_k(\xi)} \sqrt{\frac{k+1}{\pi}}\phi'(z)[\phi(z)]^k \\ &\quad + \sum_{k=0}^{\infty} \overline{P_k(\xi)} \sqrt{\frac{k+1}{\pi}}\phi'(z)[\phi(z)]^k h_k(z).\end{aligned} \tag{113}$$

To see that the two series in the right-hand side of (113) converge uniformly in $z \in L$, observe that in view of (84) and the maximum modulus principle for analytic functions, there are certain constants C_r, C'_r (that only depend on r) for which

$$\left|\overline{P_k(\xi)}\sqrt{\frac{k+1}{\pi}}\phi'(z)[\phi(z)]^k\right| \le C_r k\, r^k\,, \quad \forall\, \xi \in \overline{G}_r, \quad \forall\, z \in L, \tag{114}$$

$$\left|\overline{P_k(\xi)}\sqrt{\frac{k+1}{\pi}}\phi'(z)[\phi(z)]^k h_k(z)\right| \le C'_r k^{3/2}\,(r|\phi(z)|\rho)^k, \quad \forall\, \xi \in \overline{G}_r,\ \forall\, z \in \overline{\Omega}. \tag{115}$$

Multiplying (113) by $[\phi(z)]^{-(n+1)}$ and integrating over L yields

$$\begin{aligned}\frac{\varphi'_\xi(\xi)}{\pi}\cdot\frac{1}{2\pi i}\oint_L \frac{\varphi'_\xi(z)\,dz}{[\phi(z)]^{n+1}} = \sum_{k=0}^{\infty}\overline{P_k(\xi)}\sqrt{\frac{k+1}{\pi}}\cdot\frac{1}{2\pi i}\oint_L \frac{[\phi(z)]^k\phi'(z)\,dz}{[\phi(z)]^{n+1}}\\ +\frac{1}{2\pi i}\oint_L \frac{H(z,\xi)\,dz}{[\phi(z)]^{n+1}}\end{aligned} \tag{116}$$

where

$$H(z,\xi) := \sum_{k=0}^{\infty}\overline{P_k(\xi)}\sqrt{\frac{k+1}{\pi}}\phi'(z)[\phi(z)]^k h_k(z).$$

Now, observe first that

$$\frac{1}{2\pi i}\oint_L \frac{[\phi(z)]^k\phi'(z)\,dz}{[\phi(z)]^{n+1}} = \delta_{k,n}, \quad k,n \ge 0,$$

and second, that in view of (115), $\{H(\cdot,\xi)\}_{\xi\in\overline{G}_r}$ is a family of uniformly bounded analytic functions on compact subsets of $G_{1/r\rho}\cap\overline{\Omega}$. Therefore, we get from (116) by deforming the path of integration that for any $r\rho < \delta < 1$,

$$\begin{aligned}\frac{\varphi'_\xi(\xi)}{\pi}\cdot\frac{1}{2\pi i}\oint_L \frac{\varphi'_\xi(z)\,dz}{[\phi(z)]^{n+1}} &= \sqrt{\frac{n+1}{\pi}}\,\overline{P_n(\xi)} + \frac{1}{2\pi i}\oint_{L_{1/\delta}} \frac{H(z,\xi)\,dz}{[\phi(z)]^{n+1}}\\ &= \sqrt{\frac{n+1}{\pi}}\,\overline{P_n(\xi)} + \mathcal{O}(\delta^n)\,.\end{aligned} \tag{117}$$

To finish the proof, observe that with the notation $\overline{f}(z) := \overline{f(\overline{z})}$, (111) gives

$$\varphi_\xi(\psi(t)) = \frac{1}{\overline{\varphi_\xi\left(\psi\left(1/\overline{t}\right)\right)}} = \frac{1}{\overline{\varphi_\xi}\left(\overline{\psi}\left(1/t\right)\right)}, \quad r < |t| < 1/r,$$

so that with $z = \psi(t)$, $t \in \mathbb{T}_1$, we have

$$\varphi'_\xi(z)dz = \frac{\overline{\varphi'_\xi}\left(\overline{\psi}\,(1/t)\right)\overline{\psi'}\,(1/t)}{\left[\overline{\varphi_\xi\left(\psi\left(1/\overline{t}\right)\right)}\right]^2} \cdot \frac{dt}{t^2} = \frac{\overline{\varphi'_\xi(z)\psi'(t)}}{\left[\overline{\varphi_\xi(z)}\right]^2} \cdot \frac{i|dt|}{t} .$$

Hence,

$$\overline{\oint_L \frac{\varphi'_\xi(z)\,dz}{[\phi(z)]^{n+1}}} = -\oint_{\mathbb{T}_1} \frac{\varphi'_\xi(z)[\phi(z)]^{n+1}\psi'(t)dt}{\varphi^2_\xi(z)} = -\oint_L \frac{\varphi'_\xi(z)[\phi(z)]^{n+1}dz}{\varphi^2_\xi(z)} ,$$

which together with (117) yields (96), because by (97), for any conformal map of G onto $\mathbb{D}_1$, we have

$$\frac{\varphi'_\xi(\xi)\varphi'_\xi(z)}{\varphi^2_\xi(z)} = \frac{\varphi'(\xi)\varphi'(z)}{[\varphi(z) - \varphi(\xi)]^2} .$$

The proof of Theorem III.2.1 is complete.

III.4.2 Development of $\varphi(\psi(w))$ near w_k

This subsection is essentially a repetition of Subsection II.4.1, so we just state the facts the we will use later.

Let φ be a conformal map of G onto $\mathbb{D}_1$. Since φ is conformal, φ maps the curve L_ρ onto a piecewise analytic curve lying inside the unit disk, with corners $\varphi(z_1), \dots, \varphi(z_s)$, and exterior angle $\lambda_k\pi$ at $\varphi(z_k)$. The function $\varphi(\psi(w))$ maps the annulus $\{\rho < |w| < 1\}$ onto those points of the unit disk that lie exterior to $\varphi(L_\rho)$. The function $\varphi(\psi(w))$ can be analytically continued by the reflection principle onto the entire logarithmic Riemann surface with branch point at w_k, being analytic for w sufficiently close to w_k, say $0 < |w - w_k| < \epsilon$, on any finite sector $\theta_1 \le \arg(w - w_k) \le \theta_2$ of this Riemann surface, where ϵ depends, in general, on the sector. By the quoted result of Lehman ([13, Thm. 1]), $\varphi(\psi(w))$ has an asymptotic expansion about w_k of the following form.

Set $y = w - w_k$. If λ_k is *irrational*, then as $w \to w_k$ from any finite sector $\theta_1 \leq \arg(w - w_k) \leq \theta_2$,

$$\varphi(\psi(w)) = \varphi(\psi(w_k)) + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} b_{i,j,0}^{(k)} y^{i+j\lambda_k}, \quad b_{0,1,0}^{(k)} \neq 0;$$

if $\lambda_k = p/q$ is a *fraction* reduced to lowest terms, then

$$\varphi(\psi(w)) = \varphi(\psi(w_k)) + \sum_{i=0}^{\infty}\sum_{j=1}^{q}\sum_{m=0}^{\lfloor i/p \rfloor} b_{i,j,m}^{(k)} y^{i+j\lambda_k} (\log y)^m, \quad b_{0,1,0}^{(k)} \neq 0.$$

Hence, if $\upsilon > 0$ is sufficiently small, say

$$0 < \upsilon < \begin{cases} \min\{\lambda_k, 1 - \lambda_k\}\,, & if\ 0 < \lambda_k < 1, \\ 2 - \lambda_k\,, & if\ 1 < \lambda_k < 2, \end{cases}$$

then, as $w \to w_k$ from any finite sector $\theta_1 \leq \arg(w - w_k) \leq \theta_2$,

$$\varphi(\psi(w)) = \varphi(z_k) + A_k y^{\lambda_k} + b_{0,2,0}^{(k)} y^{2\lambda_k} + \mathcal{O}\left(y^{2\lambda_k+\upsilon}\right), \quad \text{if } 0 < \lambda_k < 1, \tag{118}$$

$$\varphi(\psi(w)) = \varphi(z_k) + A_k y^{\lambda_k} + b_{1,1,0}^{(k)} y^{1+\lambda_k} + b_{0,2,0}^{(k)} y^{2\lambda_k} + \mathcal{O}\left(y^{2\lambda_k+\upsilon}\right), \quad \text{if } 1 < \lambda_k < 2\,, \tag{119}$$

where $A_k := b_{0,1,0}^{(k)} \neq 0$.

If $0 < \lambda_k < 1$, then uniformly for ξ on compact subsets of $G \setminus \{z_k\}$ as $w \to w_k$,

$$\begin{aligned} \frac{1}{\varphi(\psi(w)) - \varphi(\xi)} = {} & \frac{1}{\varphi(z_k) - \varphi(\xi)} - \frac{A_k y^{\lambda_k}}{[\varphi(z_k) - \varphi(\xi)]^2} \\ & - \frac{\left[b_{0,2,0}^{(k)}(\varphi(z_k) - \varphi(\xi)) - A_k^2\right] y^{2\lambda_k}}{[\varphi(z_k) - \varphi(\xi)]^3} + \mathcal{O}\left(y^{2\lambda_k+\upsilon}\right). \end{aligned} \tag{120}$$

If $1 < \lambda_k < 2$, then

$$\frac{1}{\varphi(\psi(w)) - \varphi(\xi)} = \frac{1}{\varphi(z_k) - \varphi(\xi)} - \frac{A_k y^{\lambda_k}}{[\varphi(z_k) - \varphi(\xi)]^2} - \frac{b^{(k)}_{1,1,0} y^{1+\lambda_k}}{[\varphi(z_k) - \varphi(\xi)]^2} + \mathcal{O}\left(y^{2\lambda_k}\right). \tag{121}$$

Finally, for $\xi = z_k$,

$$\frac{1}{\varphi(\psi(w)) - \varphi(z_k)} = \begin{cases} \dfrac{1}{A_k y^{\lambda_k}} - \dfrac{b^{(k)}_{0,2,0}}{A_k^2} + \mathcal{O}\left(y^{\upsilon}\right), \quad 0 < \lambda_k < 1, \\ \dfrac{1}{A_k y^{\lambda_k}} - \dfrac{b^{(k)}_{1,1,0}}{A_k^2 y^{\lambda_k - 1}} + \mathcal{O}\left(1\right), \;\; 1 < \lambda_k < 2. \end{cases} \tag{122}$$

III.4.3 Auxiliary lemmas

Throughout the remaining of this chapter, D_{w_k} will denote an element of some collection $\{D_{w_k}\}_{k=1}^{s}$ of pairwise disjoint small open disks such that D_{w_k} is centered at w_k and its boundary intersects $\mathbb{T}_\rho$ at the points $w_k^+ = \rho e^{i(\theta_k + \epsilon_k)}$, $w_k^- = \rho e^{i(\theta_k - \epsilon_k)}$ for some $\epsilon_k > 0$.

Let us define the half-open circular arcs

$$[w_k, w_k^+)^\frown := \left\{\rho e^{i\theta} : \theta_k \le \theta < \theta_k + \epsilon_k\right\},$$

$$(w_k^-, w_k]^\frown := \{\rho e^{i\theta} : \theta_k - \epsilon_k < \theta \le \theta_k\},$$

together with their corresponding open sets

$$D^+_{w_k} := D_{w_k} \setminus [w_k, w_k^+)^\frown, \quad D^-_{w_k} := D_{w_k} \setminus (w_k^-, w_k]^\frown.$$

If D_{w_k} is sufficiently small, then ψ has analytic continuations ψ_+, ψ_- from Δ_ρ onto $D^+_{w_k}$, $D^-_{w_k}$, respectively, and $\psi_\pm(D^\pm_{w_k}) \subset G$. Hereafter, we assume that every element of a given collection $\{D_{w_k}\}_{k=1}^{s}$ has this property. Notice that $\varphi(\psi_\pm(w))$ is the analytic

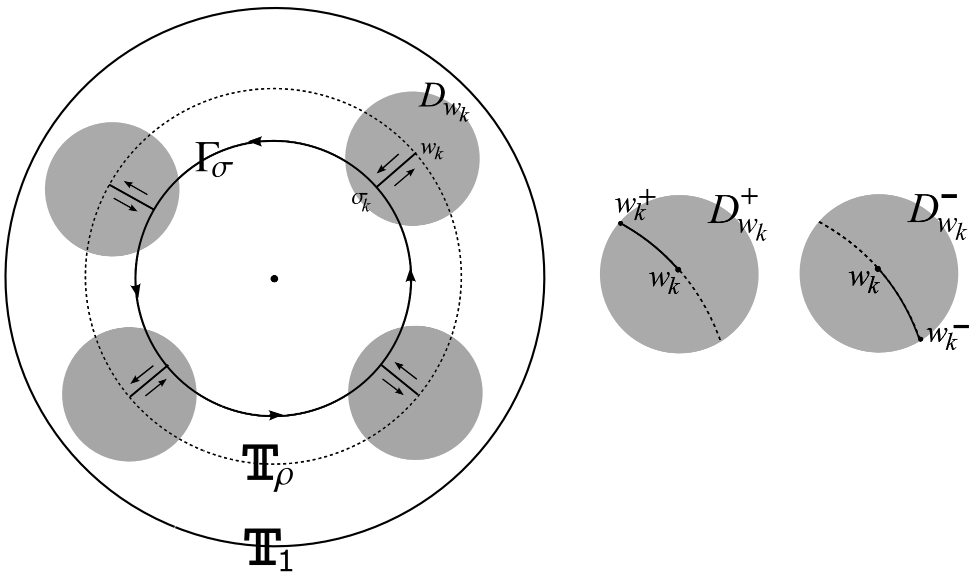


Figure 12: A positively oriented contour Γ_σ, $0 < \sigma < \rho$.

continuation of $\varphi(\psi(w))$ from Δ_ρ onto $D^{\pm}_{w_k}$.

For every $0 < \sigma < \rho$, we define $\sigma_k := \sigma e^{i\theta_k}$, $1 \le k \le s$, and the contour

$$\Gamma_\sigma := \mathbb{T}_\sigma \cup \left(\cup_{k=1}^{s}(\sigma_k, w_k]\right).$$

The exterior of the contour Γ_σ is understood to be

$$\mathrm{ext}(\Gamma_\sigma) := \Delta_\sigma \setminus \left(\cup_{k=1}^{s}[\sigma_k, w_k]\right).$$

Lemma III.4.1. *Suppose $\epsilon > 0$, D_{w_k} and σ_k are such that*

$$\overline{\psi_\pm(D^{\pm}_{w_k})} \subset \{z : |z - z_k| < \epsilon\} \subset G, \quad \sigma_k \in D_{w_k}.$$

Then

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\sigma_k}^{w_k}\left(\frac{t^n}{\varphi(\psi_-(t)) - \varphi(\xi)} - \frac{t^n}{\varphi(\psi_+(t)) - \varphi(\xi)}\right)dt \\
&= \frac{n!\rho^{n+\lambda_k+1}}{\Gamma(n+\lambda_k+2)}\left(\frac{A_k e^{i(n+\lambda_k+1)\theta_k}}{\Gamma(-\lambda_k)[\varphi(\xi) - \varphi(z_k)]^2} + r_{k,n}(\xi)\right),
\end{aligned}$$

with $r_{k,n}(\xi)$ *converging uniformly to zero on* $G \cap \{\xi : |\xi - z_k| \geq \epsilon\}$ *as* $n \to \infty$ *with the following rate:*

$$r_{k,n}(\xi) = \begin{cases} \mathcal{O}\left(n^{-\lambda_k}\right), & if\ 0 < \lambda_k < 1, \\ \mathcal{O}\left(n^{-1}\right), & if\ 1 < \lambda_k < 2. \end{cases}$$

The constants involved in the $\mathcal{O}$ *terms are independent of* ξ*, and the rate of convergence is exact when* $0 < \lambda_k < 1$*,* $\lambda_k \neq 1/2$*, provided that* $\varphi(\xi) \neq \varphi(z_k) - A_k^2/b_{0,2,0}^{(k)}$*, and when* $1 < \lambda_k < 2$ *provided that* $b_{1,1,0}^{(k)} \neq 0$*.*

Proof. Let us denote by $(w - w_k)_{\pm}^{\beta}$ the analytic continuation of $(w - w_k)^{\beta}$ onto the open set $D_{w_k}^{\pm}$. For any $\beta > 0$,

$$\int_{\sigma_k}^{w_k} (t - w_k)_{\pm}^{\beta} t^n dt = e^{\mp i\beta\pi} \int_{\sigma_k}^{w_k} (w_k - t)_{\pm}^{\beta} t^n dt = e^{\mp i\beta\pi} w_k^{n+1+\beta} \int_{\sigma/\rho}^{1} (1 - x)^{\beta} x^n dx,$$

and

$$\begin{aligned} \int_{\sigma/\rho}^{1} (1 - x)^{\beta} x^n dx &= \int_0^1 (1 - x)^{\beta} x^n dx - \int_0^{\sigma/\rho} (1 - x)^{\beta} x^n dx \\ &= \frac{\Gamma(\beta + 1)\Gamma(n + 1)}{\Gamma(n + \beta + 2)} + \mathcal{O}\left(\frac{\sigma^n}{\rho^n}\right) \sim \frac{\Gamma(\beta + 1)}{n^{\beta+1}}. \end{aligned}$$

Hence,

$$\int_{\sigma_k}^{w_k} (t - w_k)_{\pm}^{\beta} t^n dt = \frac{e^{\mp i\beta\pi} w_k^{n+1+\beta} \Gamma(\beta + 1)\Gamma(n + 1)}{\Gamma(n + \beta + 2)} + \mathcal{O}\left(\sigma^n\right), \tag{123}$$

and

$$\int_{\sigma_k}^{w_k} \mathcal{O}\left((t - w_k)_{\pm}^{\beta} t^n\right) dt = \mathcal{O}\left(\int_{\sigma}^{\rho} (\rho - t)^{\beta} t^n dt\right) = \mathcal{O}\left(\frac{\rho^n}{n^{1+\beta}}\right). \tag{124}$$

Therefore, we get from (120), (123) and (124) that

$$\int_{\sigma_k}^{w_k}\left(\frac{t^n}{\varphi(\psi_-(t))-\varphi(\xi)}-\frac{t^n}{\varphi(\psi_+(t))-\varphi(\xi)}\right)dt$$
$$=-\frac{A_k}{[\varphi(\xi)-\varphi(z_k)]^2}\cdot\frac{2i\sin(\lambda_k\pi)w_k^{n+\lambda_k+1}\Gamma(\lambda_k+1)n!}{\Gamma(n+\lambda_k+2)}$$
$$-\frac{\left[b_{0,2,0}^{(k)}(\varphi(z_k)-\varphi(\xi))-A_k^2\right]2i\sin(2\lambda_k\pi)w_k^{n+2\lambda_k+1}\Gamma(2\lambda_k+1)n!}{\left[\varphi(z_k)-\varphi(\xi)\right]^3\Gamma(n+2\lambda_k+2)}+\mathcal{O}\left(\frac{\rho^n}{n^{2\lambda_k+\upsilon+1}}\right),$$

so that, using the identity $\Gamma(-\lambda_k)\Gamma(\lambda_k+1)=-\pi/\sin(\lambda_k\pi)$, we finally get for $0<\lambda_k<1$ (with the understanding $\Gamma(-2\lambda_k)=\infty$ if $\lambda_k=1/2$):

$$\frac{1}{2\pi i}\int_{\sigma_k}^{w_k}\left(\frac{t^n}{\varphi(\psi_-(t))-\varphi(\xi)}-\frac{t^n}{\varphi(\psi_+(t))-\varphi(\xi)}\right)dt \tag{125}$$
$$=\frac{n!}{\Gamma(n+\lambda_k+2)}\left(\frac{A_kw_k^{n+\lambda_k+1}}{\Gamma(-\lambda_k)[\varphi(\xi)-\varphi(z_k)]^2}+\frac{\left[b_{0,2,0}^{(k)}(\varphi(\xi)-\varphi(z_k))+A_k^2\right]w_k^{n+2\lambda_k+1}}{\Gamma(-2\lambda_k)[\varphi(\xi)-\varphi(z_k)]^3n^{\lambda_k}(1+o(1))}\right).$$

Similarly, if $1<\lambda_k<2$, then we obtain from (121), (123) and (124) that

$$\frac{1}{2\pi i}\int_{\sigma_k}^{w_k}\left(\frac{t^n}{\varphi(\psi_-(t))-\varphi(\xi)}-\frac{t^n}{\varphi(\psi_+(t))-\varphi(\xi)}\right)dt \tag{126}$$
$$=\frac{n!}{\Gamma(n+\lambda_k+2)}\left(\frac{A_kw_k^{n+\lambda_k+1}}{\Gamma(-\lambda_k)[\varphi(z_k)-\varphi(\xi)]^2}+\frac{b_{1,1,0}^{(k)}w_k^{n+\lambda_k+2}(1+o(1))}{\Gamma(-\lambda_k-1)[\varphi(\xi)-\varphi(z_k)]^2n}\right).$$

□

The following lemma is just a reformulation of Lemma II.4.3 of Subsection II.4.2 (even simpler, since this time no $\lambda_k\in\{1,2\}$). Therefore, we omit its proof.

Lemma III.4.2. *Let $E\subset\left(G\cap\overline{\Omega}_\rho\right)\setminus\{z_1,\dots,z_s\}$, $F\subset G_\rho$ be two fixed compact sets. There exist $0<\epsilon<\operatorname{dist}(E\cup F,\{z_1,\dots,z_s\})$ and a collection $\{D_{w_k}\}_{k=1}^s$ such that for every contour Γ_σ with $\sigma_k\in D_{w_k}$, $1\le k\le s$, the following statements hold simultaneously:*

(a) for every corner z_k,

$$\overline{\psi_\pm\left(D^\pm_{w_k}\right)} \subset \{z : |z - z_k| < \epsilon\} \subset G\,, \quad z_k \notin \psi_\pm\left(D^\pm_{w_k}\right); \tag{127}$$

(b) ψ, ψ' have analytic continuations from Δ_ρ onto $\mathrm{ext}(\Gamma_\sigma)$ with continuous boundary values on $\Gamma_\sigma \setminus \{z_1, \ldots, z_s\}$ when viewing each $[\sigma_k, w_k]$ as having two sides. Moreover, ψ is one-to-one on $\overline{\Delta}_\rho \cup U$ for any open, connected component U of

$$\mathbb{D} \cap \left(\mathrm{ext}(\Gamma_\sigma) \setminus \left(\cup_{k=1}^s \overline{D}_{w_k}\right)\right)$$

and $\psi\left(\overline{\mathrm{ext}(\Gamma_\sigma)}\right) \cap F = \emptyset$.

III.4.4 Proof of Theorems III.2.2 and III.2.4

We shall prove Theorems III.2.2, III.2.2, and the statements in Remarks III.2.3, III.2.5 simultaneously.

Let $E \subset \left(G \cap \overline{\Omega}_\rho\right) \setminus \{z_1, \ldots, z_s\}$, $F \subset G_\rho$ be two fixed compact sets. Let $\rho < r < 1$ be such that $E \subset \overline{G}_r$. Then, according to Theorem III.2.1, for $0 < \eta < 1 - r$,

$$\sqrt{\pi(n+1)}\, P_n(\xi) = \frac{\varphi'(\xi)}{2\pi i} \oint_{\mathbb{T}_1} \frac{[\varphi(\psi(t))]' t^{n+1} dt}{[\varphi(\psi(t)) - \varphi(\xi)]^2} + \mathcal{O}(\rho^n (r+\eta)^n), \tag{128}$$

uniformly in $\xi \in \overline{G}_r \supset E \cup F$ as $n \to \infty$.

Lemma III.4.2 allows us to choose $0 < \epsilon < \mathrm{dist}(E \cup F, \{z_1, \ldots, z_s\})$, a collection $\{D_{w_k}\}_{k=1}^s$ and a contour Γ_σ with $\sigma_k \in D_{w_k}$ for all $1 \le k \le s$, for which all the statements listed in Lemma III.4.2 hold true.

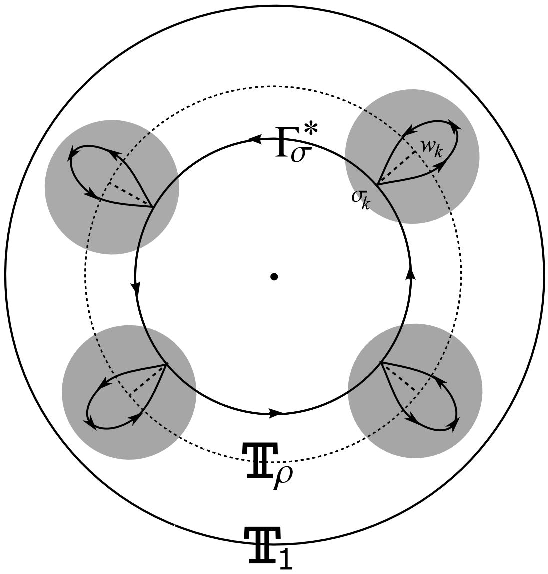


Figure 13: A positively oriented contour Γ_σ^*, $0 < \sigma < \rho$.

For each $1 \leq k \leq s$, let $\ell_k \subset D_{w_k}$ be a positively oriented closed simple path encircling the segment $(\sigma_k, w_k]$, whose only common point with $\mathbb{D}_\sigma \cup \Gamma_\sigma$ is σ_k. Define

$$\Gamma_\sigma^* := \mathbb{T}_\sigma \cup \left(\cup_{k=1}^s \ell_k\right), \quad \operatorname{ext}(\Gamma_\sigma^*) := \Delta_\sigma \setminus \cup_{k=1}^s \left(\ell_k \cup \operatorname{int}(\ell_k)\right) .$$

Let $\xi \in E$. The function $\varphi(\psi(w))' / \left[\varphi(\psi(w)) - \varphi(\xi)\right]^2$ in the variable w is analytic on $\mathbb{D}_1 \cap \operatorname{ext}(\Gamma_\sigma^*) \setminus \{\phi(\xi)\}$, with a double pole at $\phi(\xi)$ and continuous boundary values on $\mathbb{T}_1 \cup \Gamma_\sigma^*$. We choose a small circle $C_\xi \subset \mathbb{D}_1 \cap \operatorname{ext}(\Gamma_\sigma^*)$ centered at $\phi(\xi)$ that lies on a neighborhood of $\phi(\xi)$ on which ψ is univalent. Then, in the integral of (128), the circle $\mathbb{T}_1$ can be deformed without altering the value of the integral, to the positively oriented contour $\Gamma_\sigma^* \cup C_\xi$ to obtain

$$\begin{aligned}\oint_{\mathbb{T}_1} \frac{[\varphi(\psi(t))]' t^{n+1} dt}{\left[\varphi(\psi(t)) - \varphi(\xi)\right]^2} &= \oint_{C_\xi} \frac{[\varphi(\psi(t))]' t^{n+1} dt}{\left[\varphi(\psi(t)) - \varphi(\xi)\right]^2} + \oint_{\mathbb{T}_\sigma} \frac{[\varphi(\psi(t))]' t^{n+1} dt}{\left[\varphi(\psi(t)) - \varphi(\xi)\right]^2} \\ &\quad - \sum_{k=1}^{s} \oint_{\ell_k} \left(\frac{1}{\varphi(\psi(t)) - \varphi(\xi)}\right)' t^{n+1} dt. \qquad (129)\end{aligned}$$

Making the change of variables $t = \phi\left(\varphi^{-1}(v)\right)$ and applying the Cauchy integral formula we obtain

$$\oint_{C_\xi} \frac{[\varphi(\psi(t))]' t^{n+1} dt}{[\varphi(\psi(t)) - \varphi(\xi)]^2} = \oint_{\varphi(\psi(C_\xi))} \frac{[\phi(\varphi^{-1}(v))]^{n+1} dv}{[v - \varphi(\xi)]^2} = \frac{2\pi i (n+1)[\phi(\xi)]^n \phi'(\xi)}{\varphi'(\xi)},$$

which combined with (129) yields

$$\begin{aligned} \oint_{\mathbb{T}_1} \frac{[\varphi(\psi(t))]' t^{n+1} dt}{[\varphi(\psi(t)) - \varphi(\xi)]^2} = {} & \frac{2\pi i (n+1)[\phi(\xi)]^n \phi'(\xi)}{\varphi'(\xi)} \\ & - \sum_{k=1}^{s} \oint_{\ell_k} \left(\frac{1}{\varphi(\psi(t)) - \varphi(\xi)} \right)' t^{n+1} dt + \mathcal{O}(\sigma^n)\,, \end{aligned} \tag{130}$$

uniformly in $\xi \in E$ as $n \to \infty$.

Similarly, if $\xi \in F$, the function $\varphi(\psi(w))' / \left[\varphi(\psi(w)) - \varphi(\xi)\right]^2$ in the variable w is analytic on $\mathbb{D}_1 \cap \operatorname{ext}(\Gamma_\sigma^*)$ with continuous boundary values on $\mathbb{T}_1 \cup \Gamma_\sigma^*$, and we get from (128) by deforming the path of integration from $\mathbb{T}_1$ to Γ_σ^* that

$$\oint_{\mathbb{T}_1} \frac{[\varphi(\psi(t))]' t^{n+1} dt}{[\varphi(\psi(t)) - \varphi(\xi)]^2} = - \sum_{k=1}^{s} \oint_{\ell_k} \left(\frac{1}{\varphi(\psi(t)) - \varphi(\xi)} \right)' t^{n+1} dt + \mathcal{O}(\sigma^n)\,, \tag{131}$$

uniformly in $\xi \in F$ as $n \to \infty$.

We proceed to estimate the integrals under the Σ sign in (130) and (131). Integrating by parts over ℓ_k we get

$$\begin{aligned} & \oint_{\ell_k} \left(\frac{1}{\varphi(\psi(t)) - \varphi(\xi)} \right)' t^{n+1} dt \\ & = \frac{\sigma_k^{n+1}}{\varphi(\psi_-(\sigma_k)) - \varphi(\xi)} - \frac{\sigma_k^{n+1}}{\varphi(\psi_+(\sigma_k)) - \varphi(\xi)} - (n+1) \oint_{\ell_k} \frac{t^n dt}{\varphi(\psi(t)) - \varphi(\xi)} \\ & = (n+1) \int_{\sigma_k}^{w_k} \left(\frac{t^n}{\varphi(\psi_-(t)) - \varphi(\xi)} - \frac{t^n}{\varphi(\psi_+(t)) - \varphi(\xi)} \right) dt + \mathcal{O}(\sigma^n)\,, \end{aligned} \tag{132}$$

uniformly in $\xi \in E \cup F$ as $n \to \infty$.

Thus, we get from (128), (130), (131), (132) and Lemma III.4.1 that, uniformly in $\xi \in F$ as $n \to \infty$,

$$\sqrt{\pi(n+1)}\, P_n(\xi) = \frac{(n+1)!\rho^{n+\lambda_1+1}}{\Gamma(n+\lambda_1+2)} \left(-\sum_{k=1}^{u} \frac{\varphi'(\xi) A_k e^{i(n+\lambda_1+1)\theta_k}}{\Gamma(-\lambda_1)[\varphi(\xi)-\varphi(z_k)]^2} + o(1) \right), \quad (133)$$

while, uniformly in $\xi \in E$ as $n \to \infty$,

$$P_n(\xi) = \sqrt{\tfrac{n+1}{\pi}}\, \phi'(\xi)[\phi(\xi)]^n + \frac{\rho^{n+\lambda_1+1}}{n^{\lambda_1+1/2}} \left(-\frac{\varphi'(\xi)}{\sqrt{\pi}} \sum_{k=1}^{u} \frac{A_k e^{i(n+\lambda_1+1)\theta_k}}{\Gamma(-\lambda_1)[\varphi(\xi)-\varphi(z_k)]^2} + o(1) \right). \quad (134)$$

So, Theorem III.2.2(a) follows from (134), while Theorem III.2.4 follows from (133). Indeed, from the more precise expressions provided by (125) and (126) for $r_{k,n}(\xi)$, we see that if $0 < \lambda_1 < 1$, then the $o(1)$ term in (133) (which is $R_n(\cdot)$ in Theorem III.2.4) has the form (recall the convention $\Gamma(-1) = \infty$)

$$\begin{aligned} R_n(\xi) = {} & \frac{1}{n^{\lambda_1}} \left(-\sum_{k=1}^{u} \frac{\varphi'(\xi) \left[b^{(k)}_{0,2,0} \left(\varphi(\xi)-\varphi(z_k)\right) + A_k^2 \right] e^{i(n+2\lambda_1+1)\theta_k}}{\Gamma(-2\lambda_1)[\varphi(\xi)-\varphi(z_k)]^3} + o(1) \right) \\ & + \frac{1}{n^{\lambda_{u+1}-\lambda_1}} \left(-\sum_{\lambda_k=\lambda_{u+1}} \frac{\varphi'(\xi) A_k e^{i(n+\lambda_{u+1}+1)\theta_k}}{\Gamma(-\lambda_{u+1})[\varphi(\xi)-\varphi(z_k)]^2} + o(1) \right), \end{aligned} \quad (135)$$

while if $1 < \lambda_1 < 2$, then

$$\begin{aligned} R_n(\xi) = {} & \frac{1}{n} \left(-\sum_{k=1}^{u} \frac{\varphi'(\xi) b^{(k)}_{1,1,0} e^{i(n+\lambda_1+2)\theta_k}}{\Gamma(-\lambda_1-1)[\varphi(\xi)-\varphi(z_k)]^2} + o(1) \right) \\ & + \frac{1}{n^{\lambda_{u+1}-\lambda_1}} \left(-\sum_{\lambda_k=\lambda_{u+1}} \frac{\varphi'(\xi) A_k e^{i(n+\lambda_{u+1}+1)\theta_k}}{\Gamma(-\lambda_{u+1})[\varphi(\xi)-\varphi(z_k)]^2} + o(1) \right). \end{aligned} \quad (136)$$

Thus, the estimate for $R_n(\cdot)$ in equality (104) of Theorem III.2.4 follows directly from (135) and (136). Moreover, the statement of Remark III.2.5 is derived by an argument analogous to the one employed in Subsection II.4.3 to deduce Remark II.2.5(b) from (60) and (61).

It only remains to prove Theorem III.2.1(b). Let z_j be a corner of L_ρ. Since $z_j \in \overline{G}_r$ for all $\rho < r < 1$, we still get from (128) that as $n \to \infty$,

$$\sqrt{\pi(n+1)}\,P_n(z_j) = -\frac{\varphi'(z_j)}{2\pi i}\sum_{k=1}^{s}\oint_{\ell_k}\left(\frac{1}{\varphi(\psi(t))-\varphi(z_j)}\right)' t^{n+1}dt$$
$$+\mathcal{O}(\sigma^n)+\mathcal{O}(\rho^n(r+\eta)^n)\,.$$

Integrating by parts over ℓ_k (just as we did to get (132)) yields

$$\oint_{\ell_k}\left(\frac{1}{\varphi(\psi(t))-\varphi(z_j)}\right)' t^{n+1}dt = \mathcal{O}(\sigma^n)-(n+1)\oint_{\ell_k}\frac{t^n dt}{\varphi(\psi(t))-\varphi(z_j)}. \tag{137}$$

If $k \neq j$, this last integral can be taken over the two-sided segment $[\sigma_k, w_k]$ to get from Lemma III.4.1 that for all $1 \le k \le s$, $k \neq j$,

$$\begin{aligned}
&\oint_{\ell_k}\left(\frac{1}{\varphi(\psi(t))-\varphi(z_j)}\right)' t^{n+1}dt \\
&= (n+1)\int_{\sigma_k}^{w_k}\left(\frac{t^n}{\varphi(\psi_-(t))-\varphi(z_j)}-\frac{t^n}{\varphi(\psi_+(t))-\varphi(z_j)}\right)dt+\mathcal{O}(\sigma^n) \\
&= \frac{2\pi i(n+1)!\rho^{n+\lambda_k+1}}{\Gamma(n+\lambda_k+2)}\left(\frac{A_k e^{i(n+\lambda_k+1)\theta_k}}{\Gamma(-\lambda_k)[\varphi(z_j)-\varphi(z_k)]^2}+o(1)\right).
\end{aligned} \tag{138}$$

If $0 < \lambda_j < 1$, we can deform directly ℓ_j to the two-sided segment $[\sigma_k, w_k]$ and get

$$\begin{aligned}
\oint_{\ell_j}\frac{t^n dt}{\varphi(\psi(t))-\varphi(z_j)} &= \int_{\sigma_k}^{w_k}\left(\frac{t^n}{\varphi(\psi_+(t))-\varphi(z_j)}-\frac{t^n}{\varphi(\psi_-(t))-\varphi(z_j)}\right)dt \\
&= \frac{1}{A_j}\left(\int_{\sigma_k}^{w_k}(t-w_j)_+^{-\lambda_j}t^n dt-\int_{\sigma_k}^{w_k}(t-w_j)_-^{-\lambda_j}t^n dt\right) \\
&\quad+\int_{\sigma_k}^{w_k}\mathcal{O}\left((t-w_j)_+^{\upsilon}t^n\right)dt-\int_{\sigma_k}^{w_k}\mathcal{O}\left((t-w_j)_-^{\upsilon}t^n\right)dt \\
&= -\frac{2i\sin(-\lambda_j\pi)\Gamma(1-\lambda_j)n!w_j^{n+1-\lambda_j}}{A_j\Gamma(n+2-\lambda_j)}+\mathcal{O}\left(\frac{\rho^n}{n^{1+\upsilon}}\right) \\
&= \frac{2\pi i n! w_j^{n+1-\lambda_j}}{A_j\Gamma(\lambda_j)\Gamma(n+2-\lambda_j)}+\mathcal{O}\left(\frac{\rho^n}{n^{1+\upsilon}}\right),
\end{aligned} \tag{139}$$

while if $1 < \lambda_j < 2$, we need to integrate by parts one more time:

$$
\begin{aligned}
&\oint_{\ell_j} \frac{t^n dt}{\varphi(\psi(t)) - \varphi(z_j)} \\
&= \frac{1}{A_j} \oint_{\ell_j} (t - w_j)^{-\lambda_j} t^n + \int_{\sigma_k}^{w_k} \mathcal{O}\left((t - w_j)_+^{1-\lambda_j} t^n\right) dt - \int_{\sigma_k}^{w_k} \mathcal{O}\left((t - w_j)_-^{1-\lambda_j} t^n\right) dt \\
&= \frac{\sigma_j^n}{A_j(1-\lambda_j)} \left((\sigma_j - w_j)_-^{1-\lambda_j} - (\sigma_j - w_j)_+^{1-\lambda_j}\right) \\
&\quad - \frac{n}{A_j(1-\lambda_j)} \left(\int_{\sigma_k}^{w_k} (t - w_j)_+^{1-\lambda_j} t^{n-1} dt - \int_{\sigma_k}^{w_k} (t - w_j)_-^{1-\lambda_j} t^{n-1} dt\right) + \mathcal{O}\left(\frac{\rho^n}{n^{2-\lambda_j}}\right) \\
&= \frac{2\pi i n! w_j^{n+1-\lambda_j}}{A_j \Gamma(\lambda_j) \Gamma(n+2-\lambda_j)} + \mathcal{O}\left(\frac{\rho^n}{n^{2-\lambda_j}}\right).
\end{aligned} \tag{140}
$$

Then, combining (137), (138), (139) and (140) we conclude that

$$
\begin{aligned}
\sqrt{\pi(n+1)}\, P_n(z_j) = &- \sum_{k \neq j}^{s} \frac{\rho^n}{n^{\lambda_k}} \left(\frac{A_k \varphi'(z_j) e^{i(n+\lambda_k+1)\theta_k}}{\Gamma(-\lambda_k)[\varphi(z_j) - \varphi(z_k)]^2} + o(1)\right) \\
&\frac{\varphi'(z_j)(n+1)! w_j^{n+1-\lambda_j}}{A_j \Gamma(\lambda_j) \Gamma(n+2-\lambda_j)} + \mathcal{O}\left(\frac{\rho^n}{n^{\upsilon}}\right) + \mathcal{O}(\sigma^n) + \mathcal{O}(\rho^n (r+\eta)^n),
\end{aligned}
$$

which together with (99) finishes the proof of Theorem III.2.2(b).

CHAPTER IV

POLYNOMIALS ORTHOGONAL OVER THE INTERIOR OF SOME SPECIAL LEMNISCATES

IV.1 Introduction and main result

We first recall part of the notation introduced in Section III.1.

Let L be an analytic Jordan curve in $\mathbb{C}$. The interior and exterior domains of L are denoted by G and Ω, respectively. For any $0 \leq r < \infty$,

$$\mathbb{T}_r := \{w : |w| = r\}, \quad \Delta_r := \{w : |w| > r\}, \quad \mathbb{D}_r := \{w : |w| < r\} .$$

Let ψ be that conformal map of Δ_1 onto Ω such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. Let $0 \leq \rho < 1$ be the smallest number such that ψ has an analytic and univalent continuation to Δ_ρ. Define $\Omega_\rho := \psi(\Delta_\rho)$, $L_\rho := \partial\Omega_\rho$ and $G_\rho := \mathbb{C} \setminus \overline{\Omega}_\rho$.

In Section III.3 we obtained some results on the location and distribution of the zeros of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ that are orthonormal with respect to area measure over G for the case when L_ρ is a piecewise analytic Jordan curve without smooth corners or cusps. In particular, we saw that if F is a compact set lying in the interior of L_ρ, then (Corollary III.3.1) $P_n(z)$ has at most a finite (independent of n) number of zeros in F (counting multiplicities). Moreover (Corollary III.3.3), the sequence $\{\nu_{P_n}\}_{n=1}^{\infty}$ of normalized counting measures of the zeros of the P_n's converges in the weak*-topology to the equilibrium measure of L_ρ.

The purpose of this chapter is to show that these statements do not necessarily hold if L_ρ is a piecewise analytic curve such that G_ρ is not connected (and therefore, L_ρ is not a Jordan curve). We provide an example of a curve L for which the zeros of $P_n(z)$ remain fixed for all n varying through a specific subsequence (say $n \in \mathcal{N} \subset \mathbb{N}$), and so, the corresponding sequence of normalized zero counting measures $\{\nu_{P_n}\}_{n\in\mathcal{N}}$ converges in the weak*-topology to a discrete measure whose support is contained in G_ρ. However, the complementary subsequence $\{\nu_{P_n}\}_{n\notin\mathcal{N}}$ will converge to the equilibrium measure of L_ρ.

Let $K \in \mathbb{N} \setminus \{1\}$ and $r > 1$ be two given numbers. Consider the lemniscate

$$L := \left\{z : |z^K - 1| = r^K\right\}.$$

It is easy to see that L is an analytic Jordan curve for which $\rho = 1/r$ and

$$L_\rho = \left\{z : |z^K - 1| = 1\right\}.$$

Thus, L_ρ is a piecewise analytic curve consisting of "K congruent petals" (see Figure 14 below). Any two consecutive arcs of L_ρ meet at 0 forming an exterior angle of value π/K. In particular, when $K = 2$, L_ρ is the Bernoulli lemniscate.

Notice that $G_\rho = \left\{z : |z^K - 1| < 1\right\}$ is not connected; it has K connected components.

Theorem IV.1.1. *Let $\{P_n\}_{n=0}^\infty$ be the sequence of polynomials orthonormal with respect to area measure over the interior $G = \left\{z : |z^K - 1| < r^K\right\}$ of the lemniscate L. Then,*

(a)

$$P_{Km+K-1}(z) = \sqrt{\frac{K(m+1)}{\pi}}\, r^{-K(m+1)} z^{K-1}(z^K - 1)^m, \quad m \geq 0; \tag{141}$$

(b) for all $0 \leq j < K-1$,

$$\lim_{m\to\infty} \frac{(-1)^m r^{Km+j+1} P_{Km+j}(z)}{m^{(1+j-2K)/K}\sqrt{(Km+j+1)/\pi}} = \frac{z^{j-K}}{\Gamma\big((1+j-K)/K\big)} \left[\frac{r^K - r^{-K}}{r^K - r^{-K}(1-z^K)}\right]^{(j+1)/K}$$

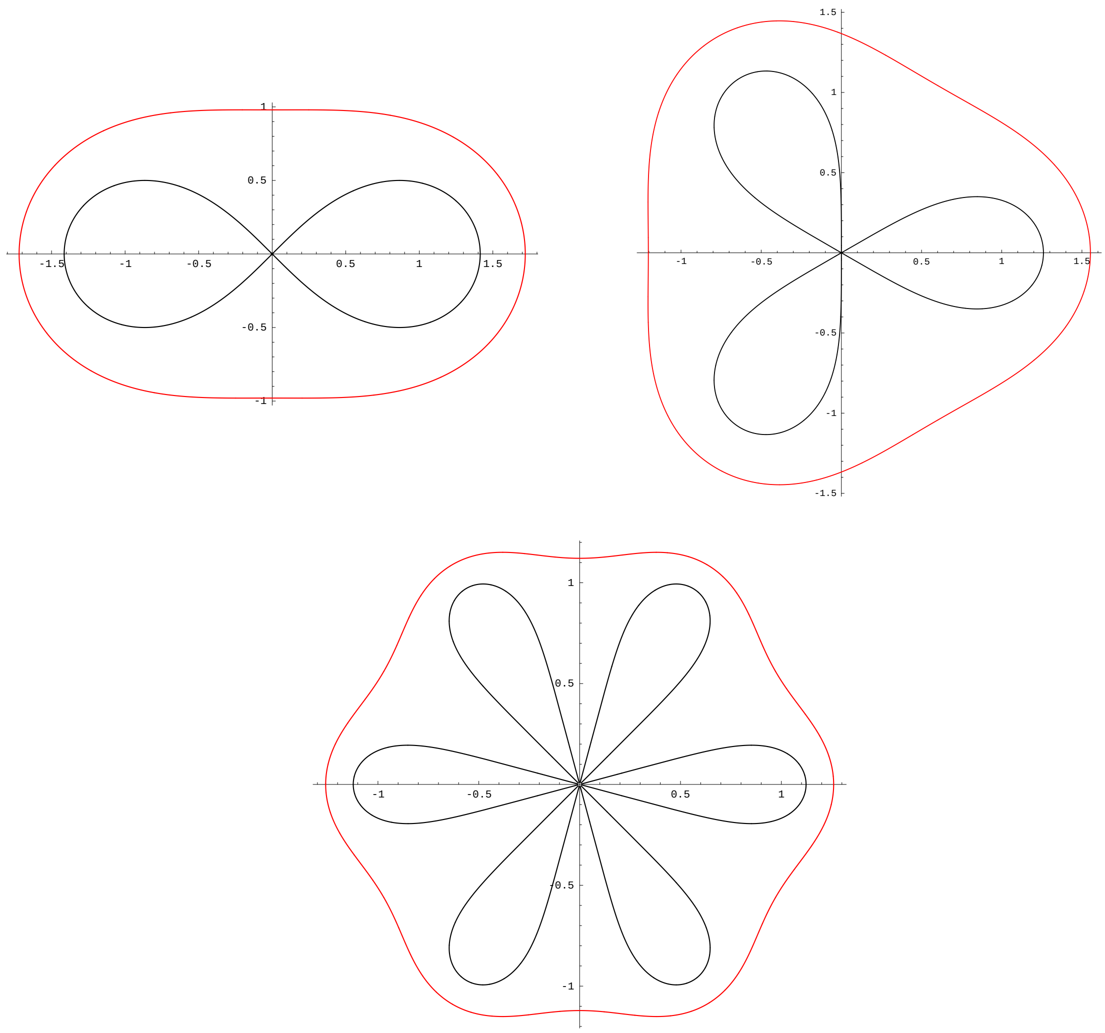


Figure 14: Curves L and L_ρ for $K = 2$, $K = 3$ and $K = 6$.

uniformly on compact subsets of $G_\rho = \left\{z : |z^K - 1| < 1\right\}$. *In particular, for any compact set* $F \subset G_\rho$*, there is a number* $N_F \in \mathbb{N}$ *such that if* $n > N_F$ *and* $n \neq K - 1 \bmod(K)$*, then* P_n *has no zeros on* F*.*

Recall that the behavior of the polynomials $P_n(z)$ in $\Omega_\rho = \left\{z : |z^K - 1| > 1\right\}$ is already given by Carleman's formula (84).

Let $\omega_{K,1}, \omega_{K,2}, \ldots, \omega_{K,K}$ be the K roots of the unit. We see from Theorem IV.1.1(a) that if $n = Km + K - 1$, then $P_n(z)$ has a zero at 0 of multiplicity $K - 1$ and a zero

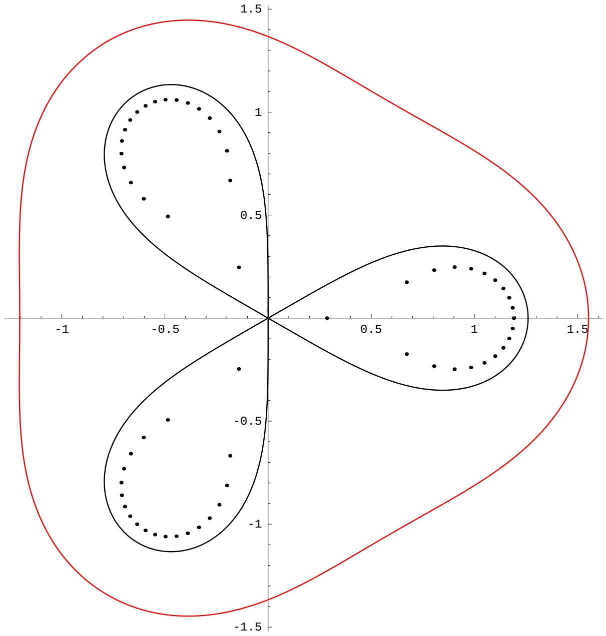


Figure 15: Zeros of $P_{60}(z)$ for the lemniscate L corresponding to $K = 3$ and $r = 1.4$.

at each $\omega_{K,i}$ of multiplicity m, so that

$$\nu_{P_n} = \frac{(K-1)}{n}\delta_0 + \frac{m}{n}\sum_{i=1}^{K}\delta_{\omega_{K,i}} \qquad n = Km + K - 1,\ m \geq 0.$$

It follows that the subsequence $\left\{\nu_{P_{Km+K-1}}\right\}_{m=0}^{\infty}$ converges in the weak*-topology to the measure $\frac{1}{K}\sum_{i=1}^{K}\delta_{\omega_{K,i}}$.

On the other hand, Theorem IV.1.1(b) and Carleman's formula imply that every weak*-limit point of the subsequence $\{\nu_{P_n}\}_{n\not\equiv K-1\,\mathrm{mod}(K)}$ must be supported on L_ρ (see Figures 15 and 16), which forces said subsequence to converge in the weak*-topology to the equilibrium measure of L_ρ (this assertion can be proven by using standard arguments of potential theory similar to those in the proof of Corollary III.3.3).

We prove Theorem IV.1.1 by expressing the polynomials P_n in terms of orthogonal polynomials on the unit circle with respect to some particular weights that we specify

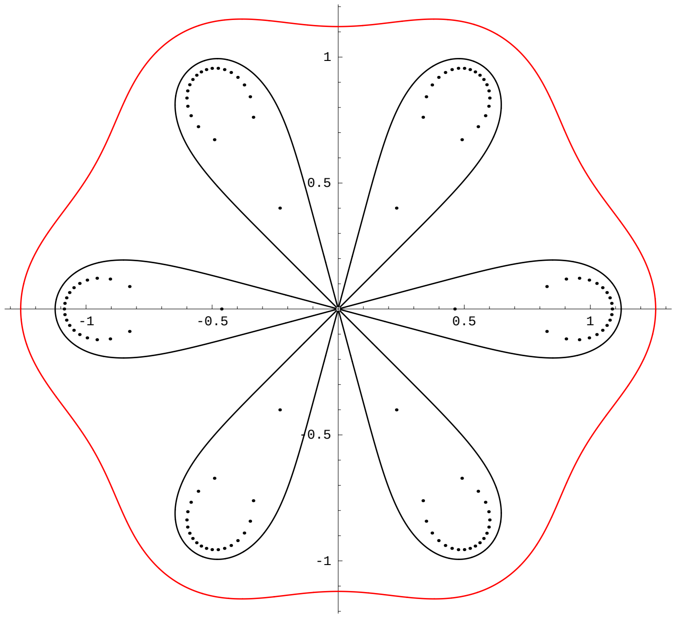


Figure 16: Zeros of $P_{120}(z)$ for the lemniscate L corresponding to $K = 6$ and $r = 1.2$.

next. For fixed $0 \leq j \leq K-1$, consider the measure

$$|r^K w + 1|^{-2(K-1-j)/K} |dw|, \quad |w| = 1\,,$$

on the unit circle $\mathbb{T}_1$, and let $S_{j,n}(w)$ be the nth monic orthogonal polynomial with respect to this measure; that is, they satisfy the defining properties

$$S_{j,n}(w) = w^n + \text{lower degree terms}, \quad n \geq 0,$$

$$\int_{\mathbb{T}_1} S_{j,n}(w)\overline{S_{j,m}(w)} |r^K w + 1|^{-2(K-1-j)/K} |dw| = \delta_{n,m}, \quad n, m \geq 0. \tag{142}$$

Let $\gamma_{j,n} > 0$ be such that $\gamma_{j,n} S_{j,n}$ is orthonormal. Then, we have

Proposition IV.1.2. *For all* $0 \leq j \leq K-1$, $m \geq 0$,

$$P_{Km+j}(z) = z^j Q_{j,m}(w)\,, \quad w = \frac{z^K - 1}{r^K}\,, \tag{143}$$

where

$$Q_{j,m}(w) := \kappa_{j,m} r^{Km} \cdot \frac{S_{j,m+1}(w) - \dfrac{S_{j,m+1}(-r^K)}{S_{j,m}(-r^K)} S_{j,m}(w)}{w + r^K} \tag{144}$$

and

$$\kappa_{j,m} := \frac{\gamma_{j,m}}{r^{Km}} \sqrt{-\frac{2(Km+j+1)S_{j,m}(-r^K)}{r^K S_{j,m+1}(-r^K)}} \sim \frac{1}{r^{Km+j+1}} \sqrt{\frac{Km+j+1}{\pi}}. \tag{145}$$

Hence, $\kappa_{j,m}$ *is the leading coefficient of* P_{Km+j}.

It is a well-known fact that the zeros of polynomials that are orthogonal on the unit circle have all their zeros lying in the open unit disk (see [31, Thm. 11.4.1]). Therefore, $S_{j,m}(-r^K) \neq 0$ and $Q_{j,m}(w)$ is well-defined by (144).

Now suppose that the proposition above is true. It is clear from (142) that

$$S_{K-1,m}(z) = z^m, \qquad \gamma_{K-1,m} = 1/\sqrt{2\pi}\,, \tag{146}$$

and so Theorem IV.1.1(a) follows immediately from (143), (144) and (145).

Theorem IV.1.1(b) is a statement that only concerns points $z \in \left\{z : |z^K - 1| < 1\right\}$, and in (143) we have $w = (z^K - 1)/r^K$. So that in order to prove part (b) of Theorem IV.1.1 from Proposition IV.1.2, it suffices to know the behavior as $m \to \infty$ of $\gamma_{j,m}$, $S_{j,m}(-r^K)$, and $S_{j,m}(w)$ in $|w| < r^{-K}$.

Now, by a well-known theorem of Szegő (see (153), (157) and the explanations in Section IV.2 below),

$$\lim_{m\to\infty} \frac{\gamma_{j,m} S_{j,m}(w)}{w^m} = (2\pi)^{-1/2} \left(r^K + w^{-1}\right)^{(K-j-1)/K}, \quad |w| > 1, \tag{147}$$

where the function in the right-hand side of (147) is positive at $w = \infty$.

In particular,

$$\lim_{m\to\infty}\frac{S_{j,m+1}(-r^K)}{S_{j,m}(-r^K)}=-r^K,\qquad \lim_{m\to\infty}\gamma_{j,m}=\frac{r^{K-j-1}}{\sqrt{2\pi}}\,. \tag{148}$$

The behavior of $S_{j,m}(w)$ in $|w|<r^{-K}$ when $0\le j<K-1$ is given by Theorem IV.2.1 of Section IV.2. Applying said theorem with $s=1$, $a_1=-1/r^K$, $\lambda_1=-(K-1-j)/K$ and $h\equiv 1$ we get

$$\lim_{m\to\infty}\left(-r^K\right)^{m+1}m^{1-\lambda_1}S_{j,m}(w)=-\frac{\left(1-r^{-2K}\right)^{\lambda_1}}{\Gamma(\lambda_1)\left(1+r^{-K}w\right)^{\lambda_1}\left(w+r^{-K}\right)}\,, \tag{149}$$

uniformly on compact subsets of $\left\{w:|w|<r^{-K}\right\}$.

Having (148) and (149) at hand, it is then very easy to deduce Theorem IV.1.1(b) from Proposition IV.1.2. We then pass to the proof of Proposition IV.1.2.

IV.1.1 Proof of Proposition IV.1.2

First, notice that because L is invariant under a rotation of angle $2\pi/K$ about the origin, if we set $G_1:=\{z\in G:-\pi/K\le\arg(z)\le\pi/K\}$, then for any two nonnegative integers ℓ and s

$$\begin{aligned}\int_G z^\ell\overline{z}^s dxdy &= \left(\sum_{j=0}^{K-1}e^{2\pi j(\ell-s)i/K}\right)\int_{G_1}z^\ell\overline{z}^s dxdy\\ &=\begin{cases}K\int_{G_1}z^\ell\overline{z}^s dxdy, & \text{if } \ell-s=0\,\mathrm{mod}(K)\,,\\ 0, & \text{otherwise}\,.\end{cases}\end{aligned} \tag{150}$$

Now, with $w=(z^K-1)/r^K$, $Q_{j,m}(w)$ is a polynomial in z of exact degree Km containing solely powers of z^K. By (150), $z^jQ_{j,m}$ is then orthogonal with respect to area measure on G to all powers of the form $z^{Km'+j'}$, $0\le m'\le m$, $0\le j'<j$.

If $L_1 := L \cap \partial G_1$, then

$$L_1 = \left\{ z = \left(r^K w + 1\right)^{1/K},\ w = e^{i\theta} : -\pi \le \theta < \pi \right\} \quad \text{with} \quad -\pi \le \arg(r^K e^{i\theta}+1) < \pi\,.$$

Applying Green's formula (see e.g., [20, Formula (106), p. 241]) we get for powers of the form $z^{Km'+j}$, $0 \le m' \le m$,

$$\int_G z^j Q_{j,m}\left(\frac{z^K-1}{r^K}\right)\overline{\kappa_{j,m} z^{Km'+j}}\,dxdy$$

$$= \frac{\kappa_{j,m} r^K}{2(Km'+j+1)} \oint_L z^j Q_{j,m}\left(\frac{z^K-1}{r^K}\right)\overline{z^{Km'+j+1}}\,dz$$

$$= \frac{K\kappa_{j,m} r^K}{2(Km'+j+1)} \oint_{L_1} z^j Q_{j,m}\left(\frac{z^K-1}{r^K}\right)\overline{z^{Km'+j+1}}\,dz$$

$$= \frac{\kappa_{j,m} r^K}{2(Km'+j+1)i} \oint_{\mathbb{T}_1} \left(r^K w+1\right)^{j/K} Q_{j,m}\left(w\right) \frac{\overline{(r^K w+1)^{(Km'+j+1)/K}}\,dw}{(r^K w+1)^{(K-1)/K}}$$

$$= \frac{\kappa_{j,m} r^K}{2(Km'+j+1)} \oint_{\mathbb{T}_1} Q_{j,m}\left(w\right)\overline{(r^K w+1)^{m'}}\, \frac{\left|r^K w+1\right|^{2j/K}\overline{(r^K w+1)^{1/K}}\,w|dw|}{(r^K w+1)^{(K-1)/K}}$$

$$= \frac{\kappa_{j,m} r^K}{2(Km'+j+1)} \oint_{\mathbb{T}_1} Q_{j,m}\left(w\right)\overline{(r^K w+1)^{m'}}\, \frac{\overline{(r^K w+1)}w|dw|}{|r^K w+1|^{2(K-j-1)/K}}$$

$$= \frac{\kappa_{j,m}^2 r^{K(m+1)}}{2(Km'+j+1)} \oint_{\mathbb{T}_1} \left(S_{j,m+1}(w) - \frac{S_{j,m+1}(-r^K)}{S_{j,m}(-r^K)} S_{j,m}(w)\right) \frac{\overline{(r^K w+1)^{m'}}|dw|}{|r^K w+1|^{2(K-j-1)/K}}$$

$$= -\frac{\kappa_{j,m}^2 r^{K(m+1)} S_{j,m+1}(-r^K)}{2(Km'+j+1) S_{j,m}(-r^K)} \oint_{\mathbb{T}_1} S_{j,m}(w)\overline{(r^K w+1)^{m'}} \frac{|dw|}{|r^K w+1|^{2(K-j-1)/K}}$$

$$= \begin{cases} 0, & \text{if } 0 \le m' < m, \\ -\dfrac{\kappa_{j,m}^2 r^{K(2m+1)} S_{j,m+1}(-r^K)}{2(Km+j+1)\gamma_{j,m}^2 S_{j,m}(-r^K)} = 1, & \text{if } m' = m\,. \end{cases}$$

The proof is complete.

IV.2 Orthogonal polynomials on the unit circle

The theory of orthogonal polynomials on the unit circle is rather well-developed. An extensive account of the existing results can be found in the classical books [31], [4], [7], and more recently [27]. In this chapter we will restrict our attention to a special class of absolutely continuous orthogonality measures, and we will be concerned with the asymptotic behavior of the corresponding orthogonal polynomials.

Let $W(z) \geq 0$ be a measurable function (also called a "weight") defined on the unit circle $\mathbb{T}_1$, and such that

$$0 < \int_{\mathbb{T}_1} W(z)|dz| < \infty\,.$$

Under these conditions, there is a unique sequence of polynomials $\{s_n(z)\}_{n=0}^{\infty}$ with positive leading coefficients that are orthonormal with respect to the weight W:

$$s_n(z) = \gamma_n z^n + \cdots\,, \quad \gamma_n > 0, \quad n \geq 0,$$

$$\int_{\mathbb{T}_1} s_n(z)\overline{s_m(z)}W(z)|dz| = \delta_{n,m}\,, \quad n,m \geq 0.$$

One of the earliest results in the theory is a theorem proven by Szegő that gives the asymptotic behavior of the polynomials $s_n(z)$ in the exterior of the unit circle for a weight W satisfying the so-called Szegő condition:

$$\int_{\mathbb{T}_1} \log W(z)|dz| > -\infty\,. \tag{151}$$

To state that theorem we need to introduce first the Szegő functions.

If W satisfies (151), then the function

$$\exp\left(\frac{1}{4\pi}\int_{\mathbb{T}_1}\log W(t)\frac{t+z}{t-z}|dt|\right) \tag{152}$$

is well-defined for all $z \in \overline{\mathbb{C}} \setminus \mathbb{T}_1$. Its restriction to the unit disk $\mathbb{D}_1$ is the *interior* Szegő function for W and we denote it by $D_i(z;W)$ (see [31, Chap X, §10.2]). The restriction of the same function (152) to the exterior of the unit circle Δ_1 is called the *exterior* Szegő function for W, and we denote it by $D_e(z;W)$. These two functions are analytic in their respective domains and they are related by the equality

$$D_e(z;W) = \frac{1}{\overline{D_i(1/\overline{z}\,;W)}}\,.$$

It is worth having in mind the multiplicative property of the Szegő functions: if W, V are weights satisfying the Szegő condition, then

$$D_e(z;WV) = D_e(z;W)D_e(z;V), \quad D_i(z;WV) = D_i(z;W)D_i(z;V).$$

Szegő proved that if W satisfies (151), then

$$s_n(z) = (2\pi)^{-1/2} z^n D_e(z;W)\,(1+o(1)) \tag{153}$$

uniformly on closed subsets of Δ_1 as $n \to \infty$. Moreover, if $0 \le \varrho \le 1$ is the smallest number such that $D_e(z;W)$ admits an analytic continuation to Δ_ϱ, then (153) also holds uniformly on closed subsets of Δ_ϱ.

If $S_n(z) := s_n(z)/\gamma_n$ is the nth *monic* orthogonal polynomial with respect to W, then it follows directly from (153) that

$$\lim_{n\to\infty} \sqrt{2\pi}\,\gamma_n = D_e(\infty;W) = D_i(0;W)^{-1}, \tag{154}$$

and

$$S_n(z) = z^n \frac{D_e(z;W)}{D_e(\infty;W)} (1 + o(1)), \tag{155}$$

uniformly on closed subsets of Δ_ϱ as $n \to \infty$.

We now restrict our attention to a special class of weights. Let $0 < \varrho < 1$ and let $a_1, a_2, \ldots, a_s$ be s distinct complex numbers all lying on the circle $\mathbb{T}_\varrho$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$ be s given numbers such that $\lambda_k \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ for all $1 \leq k \leq s$. Consider a weight of the form

$$W(z) := \left(\prod_{k=1}^{s} |z - a_k|^{2\lambda_k} \right) h(z), \quad z \in \mathbb{T}_1, \tag{156}$$

where $h(z)$ is an arbitrary *strictly positive* weight defined on $\mathbb{T}_1$ that coincides with an analytic function on some annular open neighborhood of $\mathbb{T}_1$. Further, we assume that the exterior Szegő function $D_e(z;h)$ for h is analytic on $|z| \geq \varrho$ and satisfies

$$D_e(a_k\,; h) \neq 0, \quad 1 \leq k \leq s\,.$$

The exterior Szegő function $D_e(z) = D_e(z;W)$ for W has an analytic continuation to Δ_ϱ with a singularity at each a_k, $1 \leq k \leq s$. Likewise, the interior Szegő function $D_i(z) = D_i(z;W)$ is analytic on $\mathbb{D}_{1/\varrho}$. Indeed, these functions are given by

$$D_e(z) = D_e(z;h) \prod_{k=1}^{s} \left(\frac{z}{z - a_k} \right)^{\lambda_k}, \quad -\pi < \arg\left(\frac{z}{z - a_k} \right) < \pi, \quad z \in \Delta_1\,, \tag{157}$$

$$D_i(z) = D_i(z;h) \prod_{k=1}^{s} (1 - \overline{a}_k z)^{\lambda_k}, \quad -\pi < \arg(1 - \overline{a}_k z) < \pi, \quad z \in \mathbb{D}_1\,. \tag{158}$$

Values of $\lambda_k \in \{0, -1, -2, \ldots\}$ are purposely excluded because their corresponding factors $(z - a_k)^{\lambda_k}$ do not create a singularity (but a zero) for $D_e(z)$ at a_k, and therefore, these factors may be simply regarded as being part of the function $h(z)$.

Let $\{S_n\}_{n=0}^{\infty}$ be the sequence of monic orthogonal polynomials on $\mathbb{T}_1$ with respect to W. Then the behavior of S_n on Δ_ϱ is given by (155). An asymptotic representation for $S_n(z)$ holding for values of $z \in \mathbb{D}_\varrho$ has been recently given in [15, Thm. 3], but for the case when $\lambda_k \in \mathbb{N}$, $1 \le k \le s$ (that is, when each a_k is a polar singularity of $D_e(z)$).

The proof is based on the following integral representation given in the same paper [15, Formula (38), p. 12][1]: for any $\varrho < r < 1$ fixed,

$$2\pi i D_i(0)^{-1} D_i(z) S_n(z) = \oint_{\mathbb{T}_1} \frac{\mathcal{F}(t) t^n dt}{t - z} + \mathcal{O}(r^{3n}) \tag{159}$$

uniformly on compact subsets of $\mathbb{D}_r$ as $n \to \infty$, where

$$\mathcal{F}(z) := D_e(z) D_i(z)\,, \quad \varrho < |z| < 1/\varrho.$$

We shall equally use (159) to find the behavior of $S_n(z)$ in $\mathbb{D}_\varrho$ for a weight W as general as in (156). Our main motivation has been to be able to manage the polynomials orthogonal over the lemniscates considered in Section IV.1, where the problem of finding their asymptotic behavior was reduced to a similar one for orthogonal polynomials on the unit circle with respect to weights of the form $W(z) = |r^K z + 1|^{-2(K-1-j)/K}$, with $r > 1$, $K \in \mathbb{N} \setminus \{1\}$, $0 \le j \le K$.

Define

$$\vartheta_k := \arg(a_k), \quad 0 \le \vartheta_k < 2\pi,$$

$$\widehat{D_e}(a_k) = \lim_{z \to a_k} D_e(z) \Big/ \left(\frac{z}{z - a_k}\right)^{\lambda_k}, \quad 1 \le k \le s\,.$$

With the assumptions and notation above, we have

[1] The results in [15] go far beyond (159): a full asymptotic expansion valid in all $\mathbb{C}$ is given for polynomials orthogonal with respect to an analytic and positive weight on $\mathbb{T}_1$.

Theorem IV.2.1. *Let* $\{S_n\}_{n=0}^{\infty}$ *be the sequence of monic orthogonal polynomials on* $\mathbb{T}_1$ *with respect to the weight in (156), and let* $u \in \{1, 2, \ldots, s\}$ *be such that*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_u > \lambda_{u+1} \geq \lambda_{u+2} \geq \cdots \geq \lambda_s.$$

Then,

$$\frac{\Gamma(\lambda_1)\Gamma(n+2-\lambda_1)S_n(z)}{\varrho^{n+1}\Gamma(n+1)} = \frac{D_i(0)}{D_i(z)} \sum_{k=1}^{u} \frac{D_i(a_k)\widehat{D}_e(a_k)e^{i(n+1)\vartheta_k}}{a_k - z} + R_n(z), \tag{160}$$

with

$$R_n(z) = \begin{cases} \mathcal{O}(\delta^n) & \text{if } \lambda_1 = 1 \text{ and } u = s\,, \\ \mathcal{O}\left(n^{-(\lambda_1 - \lambda_{u+1})}\right) & \text{if } \lambda_1 = 1 \text{ but } u < s\,, \\ \mathcal{O}\left(n^{-\min\{1,\, \lambda_1 - \lambda_{u+1}\}}\right) & \text{if } \lambda_1 \neq 1\,, \end{cases}$$

uniformly as $n \to \infty$ *on any compact set* $E \subset \mathbb{D}_\varrho$*, where* $0 < \delta < 1$ *is a constant depending on* E*.*

Proof of Theorem IV.2.1. Fix a compact set $E \subset \mathbb{D}_\varrho$ and take $r = \varrho^{1/2}$ in (159) to get that

$$2\pi i D_i(0)^{-1} D_i(z) S_n(z) = \oint_{\mathbb{T}_1} \frac{\mathcal{F}(t)t^n dt}{t - z} + \mathcal{O}(\varrho^{3n/2}) \tag{161}$$

uniformly in $z \in E$ as $n \to \infty$.

Fix a number $0 < \sigma < \varrho$ such that $E \subset \mathbb{D}_\sigma$ and $D_e(z; h)$ is analytic on $\{z : |z| \geq \sigma\}$. For each $1 \leq k \leq s$, put $\sigma_k := \sigma e^{i\vartheta_k}$ and let ℓ_k be a simple, positively oriented contour contained in the annulus $\{z : \sigma \leq |z| < 1\}$, containing the segment $(\sigma_k, a_k]$ in its interior domain, and such that $\ell_k \cap \mathbb{T}_\sigma = \{\sigma_k\}$. We assume that any two ℓ_k's are disjoint. Define the positively oriented contour

$$\Gamma_\sigma := \mathbb{T}_\sigma \cup \left(\cup_{\lambda_k > 0} \ell_k\right) \cup \left(\cup_{\lambda_k < 0} [\sigma_k, a_k]\right)\,.$$

The exterior $\mathrm{ext}(\Gamma_\sigma)$ of the contour Γ_σ is understood to be

$$\mathrm{ext}(\Gamma_\sigma) := \Delta_\sigma \setminus \left(\left(\cup_{\lambda_k>0}\overline{\mathrm{int}(\ell_k)}\right) \cup \left(\cup_{\lambda_k<0}\left(\sigma_k, a_k\right]\right)\right).$$

Recall that $\mathcal{F}(z) = D_e(z)D_i(z)$ where $D_e(z)$ and $D_i(z)$ are given by (157) and (158). Define

$$\mathcal{F}_k(z) := \mathcal{F}(z)\Big/\left(\frac{z}{z-a_k}\right)^{\lambda_k}, \quad 1 \le k \le s.$$

The function $\mathcal{F}(z)$ is analytic on $\mathbb{D}_{1/\varrho} \cap \mathrm{ext}(\Gamma_\sigma)$. Moreover, $\mathcal{F}(z)$ is continuous up to Γ_σ when viewing the segment $[\sigma_k, a_k]$ as having two sides.

Applying the Cauchy theorem, we can deform $\mathbb{T}_1$ into Γ_σ so that the integral in (161) is expressed as an integral over the contour Γ_σ:

$$\begin{aligned}\oint_{\mathbb{T}_1}\frac{\mathcal{F}(t)t^n}{t-z}dt = \oint_{\mathbb{T}_\sigma}\frac{\mathcal{F}(t)t^n}{t-z}\,dt + \sum_{\lambda_k>0}\oint_{\ell_k}\frac{\mathcal{F}(t)t^n}{t-z}\,dt \\ + \sum_{\lambda_k<0}\left(e^{i2\pi\lambda_k}-1\right)\int_{\sigma_k}^{a_k}\frac{\mathcal{F}^+(t)t^n}{t-z}\,dt + \mathcal{O}\left(\varrho^{3n/2}\right),\end{aligned} \tag{162}$$

where $\mathcal{F}^+$ denotes the continuous extension of $\mathcal{F}$ to the left side of the segment $[\sigma_k, a_k]$:

$$\mathcal{F}^+(t) := \lim_{\substack{z\to t\\ \mathfrak{Im}\left(ze^{-i\vartheta_k}\right)>0}} \mathcal{F}(z), \quad t \in [\sigma_k, a_k]\,.$$

We need to estimate the integrals in (162). Since $|\mathcal{F}(t)|$ is bounded on $\mathbb{T}_\sigma$, we have

$$\int_{\mathbb{T}_\sigma}\frac{\mathcal{F}(t)t^n}{t-z}\,dt = \mathcal{O}(\sigma^n), \tag{163}$$

uniformly in $z \in E$ as $n \to \infty$.

To estimate the other integrals, let us abbreviate by putting

$$G_k(t,z) := \frac{\mathcal{F}_k(t)t^{\lambda_k}}{t-z}\,,$$

so that

$$\oint_{\ell_k} \frac{\mathcal{F}(t)t^n}{t-z}\,dt = \oint_{\ell_k} \left(\frac{t}{t-a_k}\right)^{\lambda_k} \frac{\mathcal{F}_k(t)t^n}{t-z}\,dt = \oint_{\ell_k} (t-a_k)^{-\lambda_k} G_k(t,z)t^n dt, \quad (164)$$

with $-\pi+\vartheta_k < \arg(t) < \vartheta_k+\pi$, $-\pi+\vartheta_k < \arg(t-a_k) < \pi+\vartheta_k$ for $t\in\ell_k$.

For any $p\in\mathbb{N}\cup\{0\}$, we have for $z\in E$ fixed (differentiating with respect to t),

$$\begin{aligned}
[G_k(t,z)t^n]^{(p)} &= \sum_{i=0}^{p} \binom{p}{i} G_k(t,z)^{(p-i)} t^{n-i} \prod_{v=1}^{i}(n+1-v) \qquad (165)\\
&= t^{n-p}\prod_{v=1}^{p}(n+1-v)\left(G_k(t,z)+\begin{cases}0, & \text{if } p=0\\ \mathcal{O}(1/n), & \text{if } p>0\end{cases}\right) = \mathcal{O}(n^p t^n)\,,
\end{aligned}$$

uniformly in $z\in E$ as $n\to\infty$.

Thus, if $\lambda_k>0$ is an integer, then by the Cauchy integral formula,

$$\begin{aligned}
\oint_{\ell_k}(t-a_k)^{-\lambda_k}G_k(t,z)t^n dt &= \frac{2\pi i}{(\lambda_k-1)!}\,[G_k(t,z)t^n]^{(\lambda_k-1)}\Big|_{t=a_k} \qquad (166)\\
&= \frac{2\pi i G_k(a_k,z)a_k^{n+1-\lambda_k}n!}{(\lambda_k-1)!(n+1-\lambda_k)!} + \begin{cases}0, & \text{if } \lambda_k=1,\\ \mathcal{O}\left(\dfrac{\varrho^n}{n^{2-\lambda_k}}\right), & \text{if } \lambda_k>1.\end{cases}
\end{aligned}$$

uniformly in $z\in E$ as $n\to\infty$.

Now, consider a $\lambda_k > 0$ that is not an integer. Then, integration by parts yields

$$\oint_{\ell_k} (t-a_k)^{-\lambda_k} G_k(t,z) t^n \, dt$$

$$= \sum_{j=1}^{\lfloor \lambda_k \rfloor} \frac{(-1)^{j-1} \left[G_k(t,z)t^n\right]^{(j-1)}}{(t-a_k)^{\lambda_k - j} \prod_{i=1}^{j}(-\lambda_k + i)} \Bigg|_{\substack{t\to\sigma_k \\ \arg(t-a_k)\to -\pi+\vartheta_k}}^{\substack{t\to\sigma_k \\ \arg(t-a_k)\to \pi+\vartheta_k}}$$

$$+ \frac{(-1)^{\lfloor \lambda_k \rfloor}}{\prod_{i=1}^{\lfloor \lambda_k \rfloor}(-\lambda_k + i)} \oint_{\ell_k} \frac{\left[G_k(t,z)t^n\right]^{(\lfloor \lambda_k \rfloor)} dt}{(t-a_k)^{\{\lambda_k\}}}$$

$$= \sum_{j=1}^{\lfloor \lambda_k \rfloor} \mathcal{O}\left(n^{j-1}\sigma^n\right)$$

$$+ \frac{2ie^{i\pi\{\lambda_k\}} \sin(\pi\{\lambda_k\}) \prod_{i=1}^{\lfloor \lambda_k \rfloor}(n+1-i)}{\prod_{i=0}^{\lfloor \lambda_k \rfloor - 1}(\{\lambda_k\}+i)} \int_{\sigma_k}^{a_k} \frac{t^{n-\lfloor \lambda_k \rfloor}[G_k(t,z) + \mathcal{O}(1/n)]dt}{(t-a_k)^{\{\lambda_k\}}}$$

$$= [G_k(a_k,z) + \mathcal{O}(1/n)] \cdot \frac{2i \sin(\pi\{\lambda_k\}) \prod_{i=0}^{\lfloor \lambda_k \rfloor - 1}(n+1-\lfloor \lambda_k \rfloor + i)}{\prod_{i=0}^{\lfloor \lambda_k \rfloor - 1}(\{\lambda_k\}+i)} \int_{\sigma_k}^{a_k} \frac{t^{n-\lfloor \lambda_k \rfloor}dt}{(a_k-t)^{\{\lambda_k\}}}$$

$$- \frac{2i \sin(\pi\{\lambda_k\}) \prod_{i=0}^{\lfloor \lambda_k \rfloor - 1}(n+1-\lfloor \lambda_k \rfloor + i)}{\prod_{i=0}^{\lfloor \lambda_k \rfloor - 1}(\{\lambda_k\}+i)} \int_{\sigma_k}^{a_k} [G_k(a_k,z) - G_k(t,z)] \frac{t^{n-\lfloor \lambda_k \rfloor}dt}{(a_k-t)^{\{\lambda_k\}}}$$

$$+ \mathcal{O}\left(n^{\lfloor \lambda_k \rfloor - 1}\sigma^n\right), \tag{167}$$

with $\arg(a_k - t) = \vartheta_k$, $t \in [\sigma_k, a_k]$.

Putting $G_k(t,z) = G_{k,1}(t,z) + iG_{k,2}(t,z)$, we see that for every $t \in [\sigma_k, a_k]$, there are $t_1, t_2 \in [t, a_k]$ such that (again, derivatives taken with respect to t)

$$G_k(a_k,z) - G_k(t,z) = [G'_{k,1}(t_1,z) + iG'_{k,2}(t_2,z)](a_k - t). \tag{168}$$

Now

$$\int_{\sigma_k}^{a_k} (a_k-t)^{-\{\lambda_k\}} t^{n-\lfloor \lambda_k \rfloor} dt = a_k^{n+1-\lambda_k} \int_{\sigma/\varrho}^{1} (1-x)^{-\{\lambda_k\}} x^{n-\lfloor \lambda_k \rfloor} dx, \tag{169}$$

and

$$\begin{aligned}\int_{\sigma/\varrho}^{1}(1-x)^{-\{\lambda_k\}}x^{n-\lfloor\lambda_k\rfloor}dx &= \int_0^1(1-x)^{-\{\lambda_k\}}x^{n-\lfloor\lambda_k\rfloor}dx - \int_0^{\sigma/\varrho}(1-x)^{-\{\lambda_k\}}x^{n-\lfloor\lambda_k\rfloor}dx \\ &= \frac{\Gamma(1-\{\lambda_k\})\Gamma(n+1-\lfloor\lambda_k\rfloor)}{\Gamma(n+2-\lambda_k)} + \mathcal{O}\left(\frac{\sigma^n}{\varrho^n}\right) \\ &\sim \frac{\Gamma(1-\{\lambda_k\})}{n^{1-\{\lambda_k\}}}. \end{aligned} \tag{170}$$

Thus, combining (167), (168), (169), (170) and taking into account at the same time the properties of the Gamma function

$$\Gamma(n+1-\lfloor\lambda_k\rfloor)\prod_{i=0}^{\lfloor\lambda_k\rfloor-1}(n+1-\lfloor\lambda_k\rfloor+i) = \Gamma(n+1),$$

$$\Gamma(\{\lambda_k\})\Gamma(1-\{\lambda_k\}) = \frac{\pi}{\sin(\pi\{\lambda_k\})}, \quad \Gamma(\{\lambda_k\})\prod_{i=0}^{\lfloor\lambda_k\rfloor-1}(\{\lambda_k\}+i) = \Gamma(\lambda_k),$$

we obtain that if $\lambda_k > 0$ is not an integer, then

$$\oint_{\ell_k}(t-a_k)^{-\lambda_k}G_k(t,z)t^n dt = \frac{2\pi i G_k(a_k,z)a_k^{n+1-\lambda_k}\Gamma(n+1)}{\Gamma(n+2-\lambda_k)\Gamma(\lambda_k)} + \mathcal{O}\left(\frac{\varrho^{n+1}}{n^{2-\lambda_k}}\right), \tag{171}$$

uniformly in $z \in E$ as $n \to \infty$.

Similarly, we treat the case $\lambda_k < 0$ not an integer.

$$\begin{aligned}\left(e^{i2\pi\lambda_k}-1\right)\int_{\sigma_k}^{a_k}\frac{\mathcal{F}^+(t)t^n}{t-z}dt &= 2i\sin(\pi\lambda_k)e^{i\pi\lambda_k}\int_{\sigma_k}^{a_k}\frac{\mathcal{F}_k(t)}{t-z}\left(\frac{t}{t-a_k}\right)^{\lambda_k}t^n\,dt \\ &= 2i\sin(\pi\lambda_k)G_k(a_k,z)\int_{\sigma_k}^{a_k}(a_k-t)^{-\lambda_k}t^n\,dt \\ &\quad -2i\sin(\pi\lambda_k)\int_{\sigma_k}^{a_k}\left[G_k(a_k,z)-G_k(t,z)\right](a_k-t)^{-\lambda_k}t^n\,dt \\ &= \frac{2iG_k(a_k,z)a_k^{n+1-\lambda_k}\sin(\pi\lambda_k)\Gamma(1-\lambda_k)\Gamma(n+1)}{\Gamma(n+2-\lambda_k)} \\ &\quad +\mathcal{O}\left(\frac{\varrho^n}{n^{2-\lambda_k}}\right) \\ &= \frac{2\pi i G_k(a_k,z)a_k^{n+1-\lambda_k}\Gamma(n+1)}{\Gamma(n+2-\lambda_k)\Gamma(\lambda_k)} + \mathcal{O}\left(\frac{\varrho^{n+1}}{n^{2-\lambda_k}}\right). \end{aligned} \tag{172}$$

Therefore, we get from (162), (163), (166), (171) and (172) that

$$\frac{1}{2\pi i}\int_{\mathbb{T}_1}\frac{\mathcal{F}(t)t^n}{t-z}dt=\sum_{k=1}^{s}\frac{\varrho^{n+1}\Gamma(n+1)e^{i(n+1)\vartheta_k}\mathcal{F}_k(a_k)}{\Gamma(n+2-\lambda_k)\Gamma(\lambda_k)(a_k-z)} + \sum_{\lambda_k\neq 1}\mathcal{O}\left(\frac{\varrho^{n+1}}{n^{2-\lambda_k}}\right)+\mathcal{O}(\sigma^n)+\mathcal{O}(\varrho^{3n/2}), \tag{173}$$

uniformly in $z \in E$ as $n \to \infty$.

Recall that $1 \le u \le s$ is an integer such that

$$\lambda_1=\lambda_2=\cdots=\lambda_u>\lambda_{u+1}\ge\lambda_{u+2}\ge\cdots\ge\lambda_s.$$

From (173) we obtain the following relations: if $\lambda_1 = 1$ and $u = s$, then

$$\frac{\Gamma(n+2-\lambda_1)\Gamma(\lambda_1)}{2\pi i\varrho^{n+1}\Gamma(n+1)}\int_{\mathbb{T}_1}\frac{\mathcal{F}(t)t^n}{t-z}dt=\sum_{k=1}^{u}\frac{e^{i(n+1)\vartheta_k}\mathcal{F}_k(a_k)}{a_k-z}+\mathcal{O}\left(\frac{\sigma^n}{\varrho^n}\right)+\mathcal{O}\left(\varrho^{n/2}\right). \tag{174}$$

If $\lambda_1 = 1$ but $u < s$, then

$$\frac{\Gamma(n+2-\lambda_1)\Gamma(\lambda_1)}{2\pi i\varrho^{n+1}\Gamma(n+1)}\int_{\mathbb{T}_1}\frac{\mathcal{F}(t)t^n}{t-z}dt=\sum_{k=1}^{u}\frac{e^{i(n+1)\vartheta_k}\mathcal{F}_k(a_k)}{a_k-z}+\sum_{\lambda_k\neq 1}\mathcal{O}\left(\frac{1}{n^{\lambda_1-\lambda_k}}\right). \tag{175}$$

If $\lambda_1 \neq 1$, then

$$\frac{\Gamma(n+2-\lambda_1)\Gamma(\lambda_1)}{2\pi i\varrho^{n+1}\Gamma(n+1)}\int_{\mathbb{T}_1}\frac{\mathcal{F}(t)t^n}{t-z}dt=\sum_{k=1}^{u}\frac{e^{i(n+1)\vartheta_k}\mathcal{F}_k(a_k)}{a_k-z}+\mathcal{O}\left(\frac{1}{n}\right)+\sum_{k=u+1}^{s}\mathcal{O}\left(\frac{1}{n^{\lambda_1-\lambda_k}}\right). \tag{176}$$

The theorem follows from (161) and relations (174), (175) and (176). $\square$

CHAPTER V

ZEROS OF POLYNOMIALS ORTHOGONAL OVER JORDAN REGIONS WITH WEIGHTS

V.1 Introduction

Let G be the interior of a closed Jordan curve $L = \partial G$ in the complex plane, and let m denote the two-dimensional Lebesgue measure. For a function $w : \mathbb{C} \to \mathbb{C}$, analytic and not identically zero on G that satisfies the integrability condition

$$\int_G |w(z)|^2 \, dm(z) < \infty \, , \tag{177}$$

we consider the space

$$\mathcal{B}^2_w(G) := \left\{ f \text{ analytic on } G \, : \, \int_G |f(z)|^2 |w(z)|^2 \, dm(z) < \infty \right\}, \tag{178}$$

endowed with the inner product and corresponding norm

$$\langle f|g \rangle_w := \int_G f(z) \, \overline{g(z)} \, |w(z)|^2 \, dm(z) \, , \qquad \|f\|_{\mathcal{B}^2_w(G)} := \sqrt{\langle f|f \rangle_w} \, . \tag{179}$$

Let $\{P_n(z; w)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the measure $|w|^2 dm|_G$. This is the sequence of polynomials,

$$P_n(z; w) = \kappa_n^w z^n + \cdots \, , \qquad \kappa_n^w > 0, \qquad n = 0, \, 1, \, 2, \ldots ,$$

that are orthonormal with respect to the inner product $\langle \cdot | \cdot \rangle_w$.

The aim of this chapter is to investigate the zero distribution of the sequence of polynomials $\{P_n(z;w)\}_{n=0}^{\infty}$. Namely, we address the following question: *given a domain G and a function w as described above, where do the zeros of the P_n's accumulate as $n \to \infty$?*

This question has been previously studied to some extent for the case when $w \equiv 1$ (see [17], [14]). In [14], the authors found that the zero distribution of the polynomials $P_n(z;1)$ is related to the analytic continuation properties of a conformal mapping φ of G onto the unit disk $\mathbb{D}$. For example, a simplified version of their main result (Thm. 2.1) is the following: *if the mapping φ has a singularity on the boundary L of G, then every point of L attracts zeros of the $P_n(z;1)$'s.*

If the map φ can be analytically continued across the Jordan curve L, then either L is analytic, or L is a finite union of analytic arcs joining at corners having interior angles of the form π/N, $N \geq 2$ an integer.

If L is analytic, then Carleman's formula (84) implies that the zeros of the $P_n(z;1)$'s must accumulate on the compact set $\mathbb{C} \setminus \Omega_\rho \subset G$ (see Section III.1 for definitions). In particular, if the interior G_ρ of $\mathbb{C} \setminus \Omega_\rho$ is empty, then it is easy to see (e.g., applying Lemma V.4.3 of Section V.4 below) that the sequence of normalized counting measures of the zeros of these polynomials converges in the weak*-topology to the equilibrium measure of the compact set $\partial\Omega_\rho$. Thus, the most interesting case is when $G_\rho \neq \emptyset$, which we have analyzed in Subsection III.3 under the additional assumption that $\partial\Omega_\rho$ is a piecewise analytic curve without smooth corners or cusps. We have also seen with the example provided by the lemniscates in Chapter IV that different results should be expected according to whether or not G_ρ is connected.

The existing results for the situation when L is not analytic are limited to the analysis of some particular cases. It seems that in this case only the corners of L attract zeros of the polynomials $P_n(z;1)$. For instance, in [17] the authors took G to be either the interior of an equilateral triangle or the interior of a square. In both cases they showed that the zeros lie on (and are dense in) the segments joining the center of G with the vertices of the triangle/square. A similar result that we will discuss later was obtained in [14] for a lens-shaped domain bounded by two circular arcs.

The purpose of this chapter if to investigate what could happen if one introduces a weight w like the one described at the beginning of this introduction. A key role in our investigation is played by the reproducing kernel of the space $\mathcal{B}^2_w(G)$, which is the unique function

$$K_w(z,\zeta) : G \times G \to \mathbb{C} \tag{180}$$

such that

$$K_w(\cdot,\zeta) \in \mathcal{B}^2_w(G) \quad \forall\, \zeta \in G, \quad \text{and} \quad f(\zeta) = \langle f | K_w(\cdot,\zeta)\rangle_w \quad \forall\, f \in \mathcal{B}^2_w(G). \tag{181}$$

When w is a function as described above, we find that the zero distribution of the $P_n(\cdot\,; w)$'s depends on the analytic continuation properties of the family of functions $\{K_w(\cdot,\zeta) : \zeta \in G\}$. For example, Theorem V.2.1 of Section V.2 below, which extends Thm. 2.1 of [14], can be roughly stated as follows:

If w is such that the polynomials are dense in $\mathcal{B}^2_w(G)$, and if for some $\zeta \in G$, $K_w(\cdot,\zeta)$ has a singularity on the boundary ∂G of G, then every point of ∂G attracts zeros of the $P_n(z; w)$'s (a converse of this statement is valid in some sense as well).

The relevance of this result is strengthened by the fact that we have formulas that express $K_w(z,\zeta)$ in terms of the weight w and a conformal mapping φ of G onto the unit disk $\mathbb{D}$, which help us to determine the singularities of $K_w(\cdot,\zeta)$, and in particular, whether or not this kernel has a singularity on ∂G. For instance, it is well-known that if $w(z) \neq 0$ for all $z \in G$, then (see [30], p. 37)

$$K_w(z,\zeta) = \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{\pi\, w(z)\,\overline{w(\zeta)}\left[1 - \varphi(z)\overline{\varphi(\zeta)}\right]^2}\,.$$

It is clear from this formula that certain properties of φ and w will guarantee that $K_w(\cdot,\zeta)$ has a singularity on ∂G. Possibly the simplest is that w has a zero at a point $z_0 \in \partial G$ in a neighborhood of which $|\varphi'|$ is bounded below.

Much more interesting is the situation when w has zeros in G. In this paper we derive formulas for $K_w(z,\zeta)$ when the number of these zeros is finite. We use a well-known iterative procedure that, given a zero $a \in G$ of w, allows one to construct $K_w(z,\zeta)$ from the kernel corresponding to the weight $w(z)/(z-a)$ (see Proposition V.3.4). Applying this procedure we derive Lemma V.3.6 of Section V.3, which gives a representation of the kernel in terms of w and φ. If the zeros of w inside G are simple, then a simple determinant representation for $K_w(z,\zeta)$ is given in Proposition V.3.5.

To gain insight into what can happen in the less transparent situation where $K_w(\cdot,\zeta)$ can be analytically continued across ∂G for every $\zeta \in G$, we analyze in detail two specific cases. First, we let G be the unit disk, and take w to be meromorphic with no poles in $\overline{G}$. We prove that the zeros of the $P_n(z\,;w)$'s accumulate on a disk of radius $r \leq 1$, and each point of the boundary of this disk attracts zeros of the polynomials. The radius r is determined by the zeros and the poles of w.

In the second case, G is a domain bounded by two circular arcs that meet at $-i$ and i with opening angle π/N, $N \in \mathbb{N}$, $N \geq 2$. The weight w is taken to be an entire function. This case was studied in [14] for $N = 2$, $w \equiv 1$, and it was shown that the zeros of Bergman polynomials for these lens-shaped domains accumulate on an arc Γ that connects the vertices $-i$, i (see Figure 17(a)). The same result is true for $N > 2$.

For a general entire function w, we find that the zeros of the $P_n(\cdot\,;w)$'s accumulate on a compact set consisting of two subarcs of the same curve Γ and a "bubble" connecting these two subarcs (see Figure 17(b)). This bubble is determined by the zeros of w, and each boundary point of it, as well as each point of the two subarcs, attracts zeros of the polynomials.

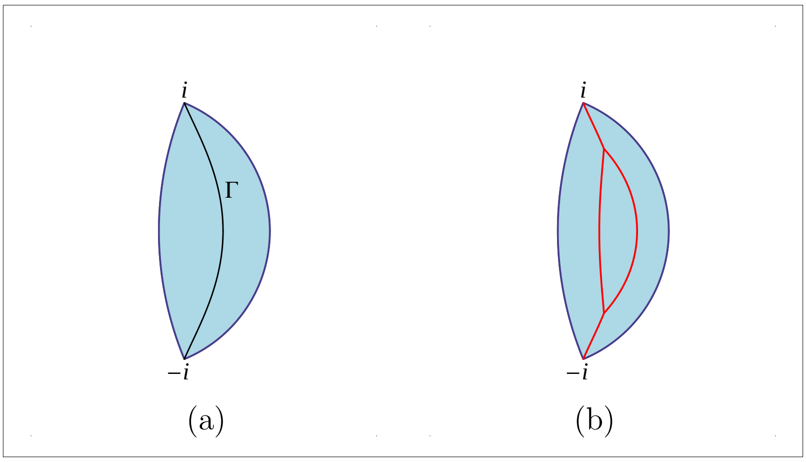


Figure 17: (a) Zeros accumulate on Γ in unweighted case and (b) on bubble with subarcs of Γ in weighted case for w entire.

We remark that in both of the above cases one can consider more general functions w. As long as we are able to determine the singularity of $K_w(\cdot, \zeta)$ that is closest (in some sense) to G, our method of proof will yield similar results. For example, the same phenomenon is observed in the case of a lens if one considers meromorphic weights. However, we restrict ourselves to the case of w entire for the sake of simplicity.

The rest of the chapter is organized as follows. In Section V.2 we introduce some notation and present the main results. In Section V.3 we establish the existence of the kernel function as well as some of its properties and formulas. In Section V.4 we derive a basic relation between the orthogonal polynomials and the kernel function (Corollary V.4.2), and give (in Lemma V.4.3) the general argument that is employed in Section V.5 to prove the zero distribution results.

The results of the present chapter have been published in [19] and were obtained in collaboration with Professors Edward B. Saff and Nikos S. Stylianopoulos.

V.2 Main results

Throughout this chapter, (G, w) will denote a pair formed by a bounded Jordan domain G and a function $w : \mathbb{C} \to \mathbb{C}$ that is analytic and not identically zero on G, and that satisfies (177). In each theorem, it will be clearly stated whether any other property of G or w is assumed, and $P_n(z) := P_n(z; w)$ will denote the n-orthonormal polynomial with respect to the measure $|w|^2 dm|_G$ corresponding to the domain G and weight w so specified. The letter $\mathbb{D}$ will stand for the open unit disk and $\mathbb{D}_r$ for the open disk $\{z : |z| < r\}$.

For any G under consideration,

$$\Phi : \overline{\mathbb{C}} \setminus \overline{G} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \tag{182}$$

will denote the exterior conformal map from $\overline{\mathbb{C}} \setminus \overline{G}$ onto $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, normalized so that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. This map Φ can be naturally extended to a homeomorphism (also denoted by Φ) between $L := \partial G$ and the unit circle $\mathbb{T} := \partial\mathbb{D}$. Then, the *equilibrium measure* μ_L of the compact set L can be defined as the preimage by Φ of the normalized arclength measure $|dz|/2\pi$ on $\mathbb{T}$, that is,

$$\mu_L(A) := \frac{1}{2\pi} \int_{\Phi(A)} |dz|$$

for any Borel set $A \subset L$. We refer the reader to [24] or [26] for the definition of the equilibrium measure of more general compact sets and also for the related notion of *logarithmic capacity* of a set E, which we denote by $\mathrm{cap}(E)$.

If Q is a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$ (listed according to multiplicity), *the normalized counting measure of the zeros of* Q is denoted by ν_Q and defined by

$$\nu_Q := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k} \,,$$

where δ_z denotes the unit mass at the point z.

We say that the sequence of Borel measures $\{\sigma_n\}$ *converges in the weak*-sense* to a measure σ, symbolically $\sigma_n \xrightarrow{*} \sigma$, if

$$\lim_{n\to\infty} \int_{\overline{\mathbb{C}}} f d\sigma_n = \int_{\overline{\mathbb{C}}} f d\sigma \,,$$

for every function f continuous on the extended complex plane $\overline{\mathbb{C}}$.

Recall that $K_w(z,\zeta)$, defined by (180) and (181), is the reproducing kernel of the space $\mathcal{B}^2_w(G)$ introduced in (178). The existence of this kernel, as well as some of its properties, will be established in Section V.3. With this notation, we have the following basic theorem:

Theorem V.2.1. *For any (G,w) as above, if*

(a) *there exists a subsequence $\mathcal{N} \subset \mathbb{N}$ such that*

$$\nu_{P_n} \xrightarrow{*} \mu_L \quad as \quad n \to \infty \,,\; n \in \mathcal{N} \,,$$

then

(b) *there exists a point $\zeta \in G$ for which $K_w(\cdot,\zeta)$ has a singularity on the boundary L of G.*

Moreover, if w is such that the polynomials are dense in $\mathcal{B}^2_w(G)$, then **(b)** $\Rightarrow$ **(a)***; that is,* **(a)** *and* **(b)** *are equivalent.*

Remark V.2.2. There are several results giving conditions that ensure the completeness of the system of polynomials in Banach spaces of analytic functions on a domain G whose norm is given by an integral over G with respect to a weight function. For example, see the survey [18] and the papers [8], [9], as well as the references therein. Here, we just mention that when w is analytic in $\overline{G}$ (which is the case in Theorems V.2.3 and V.2.7 below), the polynomials are dense in $\mathcal{B}_w^2(G)$. This assertion is easy to verify with the help of Thm. 2 of [8].

We drop the subscript w and write $K(z,\zeta)$ for the kernel corresponding to $w \equiv 1$, which is the so-called *Bergman kernel function* of G. For the practical determination of the singularities of $K_w(\cdot,\zeta)$, one can use formula (196) of Section V.3 for a weight $w \neq 0$. When w has finitely many zeros on G, the iterative procedure given in Proposition V.3.4 of Section V.3 can be used to find $K_w(z,\zeta)$ in terms of the weight w and a conformal map φ of G onto $\mathbb{D}$. Notice that in the Bergman case $w \equiv 1$, the possibility of continuing $K(\cdot,\zeta)$ analytically across L is independent of ζ since, as easily follows from (196), $K(\cdot,\zeta)$ has a singularity on L if and only if an interior conformal map φ has a singularity on L.

We now consider the particular case in which the boundary of G is as nice as possible, namely, the unit circle, and the weight w is a meromorphic function.

Theorem V.2.3. *Let $w \not\equiv 0$ be a meromorphic function in $\mathbb{C}$ that is analytic in $\overline{\mathbb{D}}$. Let*

$$\begin{aligned} \{a_1,\ldots,a_\ell\} &= set\ of\ zeros\ of\ w\ in\ \mathbb{D}\,, \\ \{b_1,b_2,\ldots\} &= set\ of\ zeros\ of\ w\ in\ \mathbb{C}\setminus\mathbb{D}\,, \\ \{c_1,c_2,\ldots\} &= set\ of\ poles\ of\ w\,; \end{aligned}$$

and let

$$\mathcal{A} := \{|a_i| : 1/\overline{a}_i = c_j\ for\ some\ j\ and\ \mathrm{mult}(c_j) \geq \mathrm{mult}(a_i)+1\}\,,$$

where $\mathrm{mult}(c_j)$ *and* $\mathrm{mult}(a_i)$ *denote the respective orders of the pole* c_j *and the zero* a_i*. Set*

$$r := \max\left(\left\{0, |a_1|, \ldots, |a_\ell|, |b_1|^{-1}, |b_2|^{-1}, \ldots\right\} \setminus \mathcal{A}\right) .$$

Then, for all but countably many $z \in \mathbb{D}$*,*

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = \begin{cases} |z| \text{ if } |z| > r \\ r \text{ if } |z| \le r \end{cases} , \tag{183}$$

which implies that:

(a) *if* $r = 0$*, then*

$$\nu_{P_n} \xrightarrow{*} \delta_0 \quad as \quad n \to \infty ,$$

where δ_0 *denotes the unit point mass at* 0*;*

(b) *if* $r > 0$*, then any measure that is a weak*-limit point of the sequence* $\{\nu_{P_n}\}$ *is supported in* $\overline{\mathbb{D}}_r := \{z : |z| \le r\}$*. Let* $\mathcal{N} \subset \mathbb{N}$ *be a subsequence (which indeed exists) such that the* $\limsup$ *in* (183) *is realized for some* $z \in \mathbb{D}_r$*. Then*

$$\nu_{P_n} \xrightarrow{*} \mu_r \quad as \quad n \to \infty , \quad n \in \mathcal{N} ,$$

where $\mu_r := |dz|/2\pi r$ *is the normalized arclength measure on the circle* $\mathbb{T}_r := \{z : |z| = r\}$*.*

Example V.2.4. Let $w(z) := (z-a)^v/(1-z\overline{a})^\lambda$, where $0 < |a| < 1$ and $v \ge 1$, $\lambda \ge 0$ are integers. Then, according to Theorem V.2.3, when $\lambda < v+1$, $\{\nu_{P_n}\}$ has at least a subsequence converging weakly* to $\mu_{|a|}$. However, if $\lambda \ge v+1$, the entire sequence $\{\nu_{P_n}\}$ converges weakly* to δ_0. Figure 18 illustrates the case $\lambda = 0$, $v = 1$. Figure 19 illustrates the case $\lambda = 2$, $v = 1$. Another example is discussed after the proof of Theorem V.2.3 in Section V.5.

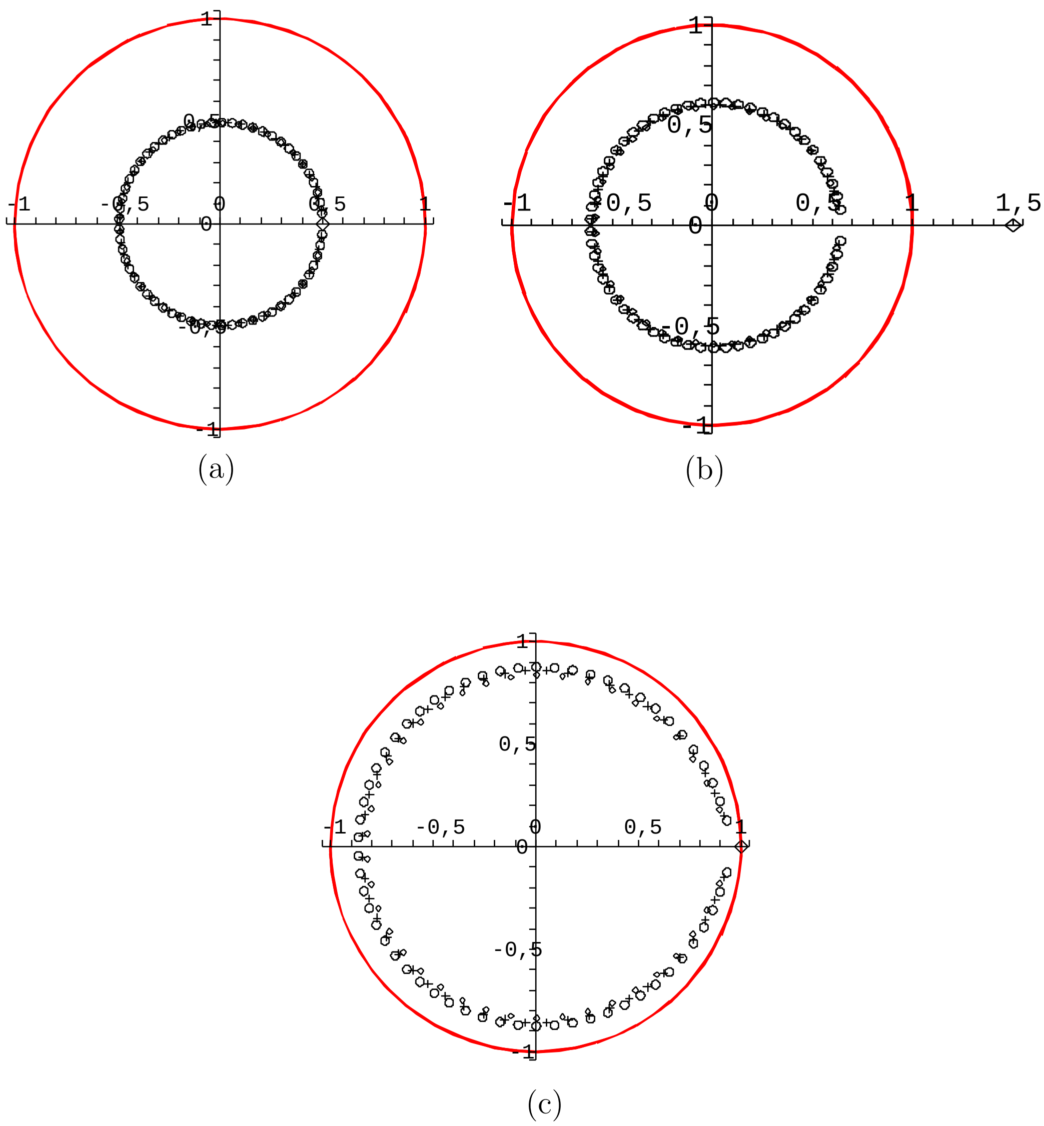


Figure 18: Zeros of P_n, $n = 40(\diamond)$, $50(+)$, $60(\circ)$, for $G = \mathbb{D}$ and (a) $w(z) = z - 1/2$, (b) $w(z) = z - 3/2$, (c) $w(z) = z - 1$.

Remark V.2.5. The ideas involved in the proof of Theorem V.2.3 can be applied to other functions w not necessarily meromorphic. For example, the function

$$w(z) = \prod_{i=1}^{\ell}(z - a_i) \prod_{j=1}^{m} e^{1/(z-d_j)}\,, \quad a_i \in \mathbb{D}\,, \quad d_j \in \mathbb{C} \setminus \mathbb{D}\,,$$

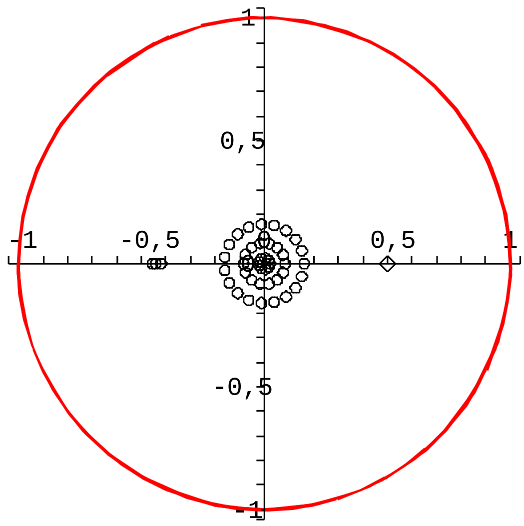


Figure 19: Zeros of P_n, $n = 10, 15, 20$, for $w(z) = (z - \frac{1}{2})/(2 - z)^2$ and $G = \mathbb{D}$.

has essential singularities at each d_j, and for this function the conclusions of Theorem V.2.3 hold with

$$r := \max \left(\left\{ 0, |a_1|, \ldots, |a_\ell|, |d_1|^{-1}, \ldots, |d_m|^{-1} \right\} \right).$$

Remark V.2.6. We note that a result similar to Theorem V.2.3 is known for orthogonal polynomials on the unit circle $\mathbb{T}$. Let ψ_n be the n-th orthonormal polynomial with respect to a measure σ in the Szegő class of $\mathbb{T}$, and let $0 \leq \rho \leq 1$ be the smallest number such that the reciprocal of the interior Szegő function for σ'

$$D(\sigma', z)^{-1} := \exp \left\{ -\frac{1}{4\pi} \int_0^{2\pi} \log \sigma'(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta \right\}$$

is analytic in $\mathbb{D}_{1/\rho} := \{z : |z| < 1/\rho\}$. In [16] it was shown that for some subsequence $\mathcal{N} \subset \mathbb{N}$,

$$\nu_{\psi_n} \xrightarrow{*} \mu_\rho\,, \quad \text{as} \quad n \to \infty\,, \quad n \in \mathcal{N}\,,$$

where μ_ρ is the arc-measure $|dz|/2\pi\rho$ on $\mathbb{T}_\rho$ if $\rho > 0$, or $\mu_\rho = \delta_0$ if $\rho = 0$. Hence, if $w(z)$ is as in Theorem V.2.3 then

$$D(|w|^2, z)^{-1} = \frac{h(0)}{|h(0)|h(z)(1 - z\overline{a}_1)^{v_1} \cdots (1 - z\overline{a}_\ell)^{v_\ell}},$$

where

$$v_i = \mathrm{mult}(a_i) \text{ and } h(z) = w(z)/[(z - a_1)^{v_1} \cdots (z - a_\ell)^{v_\ell}].$$

Thus

$$\rho = \max\left(\left\{0, |a_1|, \ldots, |a_\ell|, |b_1|^{-1}, |b_2|^{-1}, \ldots\right\} \setminus \mathcal{A}^*\right),$$

where

$$\mathcal{A}^* := \{|a_i| : 1/\overline{a}_i = c_j \text{ for some } j \text{ and } \mathrm{mult}(c_j) \geq \mathrm{mult}(a_i)\}.$$

So, if w is as in Theorem V.2.3, the zeros of the ψ_n's and the zeros of the P_n's accumulate on the same circle, except possibly when $\mathcal{A} \neq \mathcal{A}^*$. Indeed, for $w(z) = (z - 1/2)/(2 - z)$ the Szegő polynomials have all zeros at the origin (since $|w(z)| \equiv 2$ for $|z| = 1$), while the weighted Bergman polynomials have zeros accumulating on $|z| = 1/2$.

We now consider a special class of domains bounded by a piecewise analytic Jordan curve. Let $N \geq 2$ be a natural number and let G be a lens-shaped domain whose boundary L consists of two circular arcs L_α and L_β (L_α being to the left of L_β) meeting at i and $-i$ with opening angle π/N. Let α and β be the angles formed by L_α and L_β with the segment $[-i, i]$, respectively. Notice that L_α and L_β are arcs of circles centered, respectively, at $a := \cot\alpha$, $b := -\cot\beta$, with corresponding radii $\rho_\alpha := 1/\sin\alpha$, $\rho_\beta := 1/\sin\beta$. In the limit case when either α or $\beta = 0$, one of these circles becomes the imaginary axis.

For any point $z \in \overline{G}$, let

$$z_\alpha = \frac{a\overline{z}+1}{\overline{z}-a}\,, \quad z_\beta = \frac{b\overline{z}+1}{\overline{z}-b} \tag{184}$$

be the reflections of z with respect to L_α and L_β, respectively. The following facts are stated without proof, since they can be obtained by using the method employed for $N = 2$ in Section 4 of [14].

The set

$$\Gamma := \left\{z \in \overline{G} : |\Phi(z_\alpha)| = |\Phi(z_\beta)|\right\}. \tag{185}$$

is an analytic Jordan arc that lies on G, except for its two endpoints i, $-i$. Define

$$G_\alpha := \operatorname{int}(L_\alpha \cup \Gamma)\,, \quad G_\beta := \operatorname{int}(L_\beta \cup \Gamma)\,.$$

Then, by the reflection principle, the function

$$\widehat{\Phi}(z) := \begin{cases} \Phi(z) & \text{if } z \in \overline{\mathbb{C}} \setminus G \\ 1\Big/\overline{\Phi(z_\alpha)} & \text{if } z \in G_\alpha \cup \Gamma \\ 1\Big/\overline{\Phi(z_\beta)} & \text{if } z \in G_\beta \end{cases} \tag{186}$$

is analytic in $\mathbb{C}\setminus\Gamma$, and $|\widehat{\Phi}|$ is continuous in $\overline{\mathbb{C}}$. If $p_\Gamma := \Gamma \cap \{\mathfrak{Im}\, z = 0\}$ is the midpoint of Γ, then

$$0 < R_\Gamma := |\widehat{\Phi}(p_\Gamma)| < |\widehat{\Phi}(z)| \qquad \forall\, z \neq p_\Gamma\,. \tag{187}$$

For any $R_\Gamma \leq r < \infty$, consider the level set

$$\gamma_r := \left\{z : |\widehat{\Phi}(z)| = r\right\}. \tag{188}$$

When $R_\Gamma < r < 1$, γ_r is a Jordan curve that intersects Γ at two conjugate points, and it is such that $\gamma_r \setminus \Gamma$ consists of two analytic simple arcs, one contained in G_α,

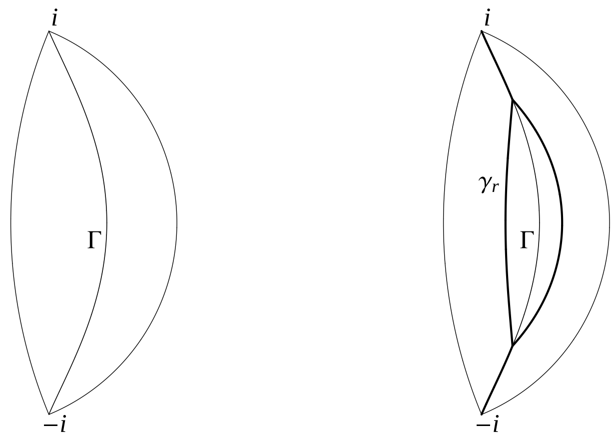


Figure 20: Curves Γ and γ_r for $N = 2$, $\alpha = \pi/8$, $r = 5/7$. Here, $R_\Gamma \approx 0.58731$.

the other in G_β. Notice that $\gamma_{R_\Gamma} = \{p_\Gamma\}$, $\gamma_1 = L$, and that for $r > 1$, γ_r is a standard level curve of the exterior mapping Φ (see Figure 20 above).

Theorem V.2.7. *Let G be a lens-shaped domain with opening angle π/N, and let $w \not\equiv 0$ be an entire function. Let $\{a_1, \ldots, a_\ell\}$ and $\{b_1, b_2, \ldots\}$ be the sets of zeros of w in G and $\mathbb{C} \setminus G$, respectively, and define r as the largest number of the set*

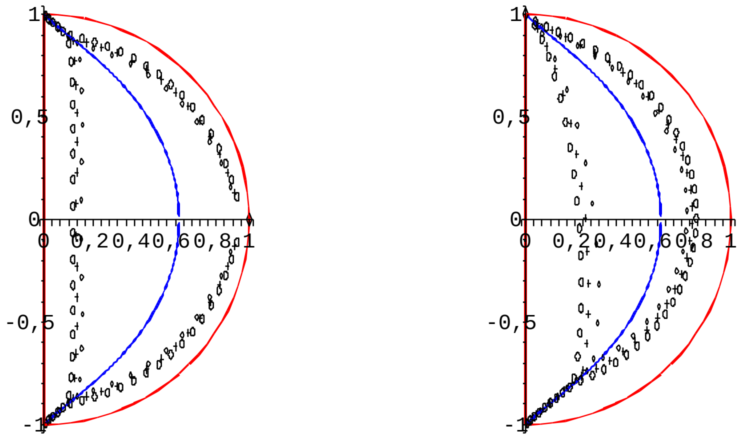
$$\left\{R_\Gamma, |\widehat{\Phi}(a_1)|, \ldots, |\widehat{\Phi}(a_\ell)|\right\} \cup \left\{|\widehat{\Phi}(b_k)|^{-1} : b_k \notin \{-i, i\} \text{ or } \operatorname{mult}(b_k) > N-1\right\}.$$

Then, for all but countably many $z \in G$,

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = \begin{cases} |\widehat{\Phi}(z)| & if\ \ z \in \operatorname{ext}(\gamma_r) \\ r & if\ \ z \in \gamma_r \cup \operatorname{int}(\gamma_r) \end{cases}, \tag{189}$$

which implies that any weak-limit point σ of the measures ν_{P_n} is supported in $\Gamma \cup \gamma_r \cup \operatorname{int}(\gamma_r)$, and every point of $\Gamma \setminus \operatorname{int}(\gamma_r)$ belongs to $\operatorname{supp}(\sigma)$. Moreover, there is a measure μ_r whose support coincides with $(\Gamma \cup \gamma_r) \setminus \operatorname{int}(\gamma_r)$ such that*

(a) *if $r = R_\Gamma$ (i.e. if $\gamma_r = \{p_\Gamma\}$), then $\nu_{P_n} \xrightarrow{\ *\ } \mu_r$ as $n \to \infty$;*



(a) (b)

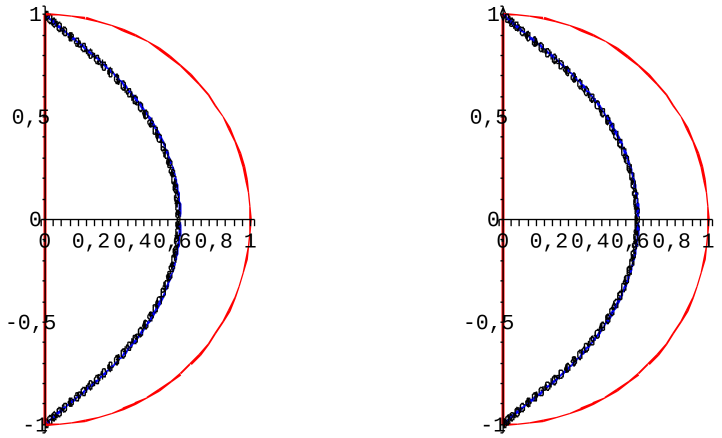
Figure 21: Zeros of P_n, $n = 40, 50, 60$, for lens parameters $N = 2$, $\alpha = 0$, and (a) $w(z) = z - 1$, (b) $w(z) = (z - i)^2$.

(b) *if* $r > R_\Gamma$ *and for some* $z \in \mathrm{int}(\gamma_r)$ *the* $\limsup$ *in* (189) *is realized through a subsequence* $\mathcal{N} \subset \mathbb{N}$, *then*

$$\nu_{P_n} \xrightarrow{\ *\ } \mu_r \quad as \quad n \to \infty\,, \quad n \in \mathcal{N}\,.$$

It is of help to discuss Theorem V.2.7 for the simplest case when $w(z) = (z - a)^v$, $v \in \mathbb{N}$, has a zero in a single point. If $a \in L \setminus \{-i, i\}$, or $a \in \{-i, i\}$ and $v > N - 1$, we see that γ_r coincides with the boundary L of G, and every point of L attracts zeros of the P_n's (see Figure 21 above). If $a \in \{-i, i\}$ but $v \leq N - 1$, or $a \in \mathbb{C} \setminus \overline{G}$ is sufficiently far from the lens (in the sense $|\Phi(a)| \geq 1/R_\Gamma$), or a coincides with the midpoint p_Γ of Γ, then γ_r shrinks to the point p_Γ and the zeros of the P_n's accumulate on the whole of Γ (see Figure 22 below). If none of these things happens, then a “proper” bubble bounded by γ_r and joining two subarcs of Γ is formed, and every point of γ_r, as well as of the subarcs, attracts zeros of the polynomials (see Figure 23 below).

From the proof of Theorem V.2.7 one can see that the measure μ_r in that theorem can be characterized in different ways. For example, if μ_{R_Γ} is the limiting measure



(a) (b)

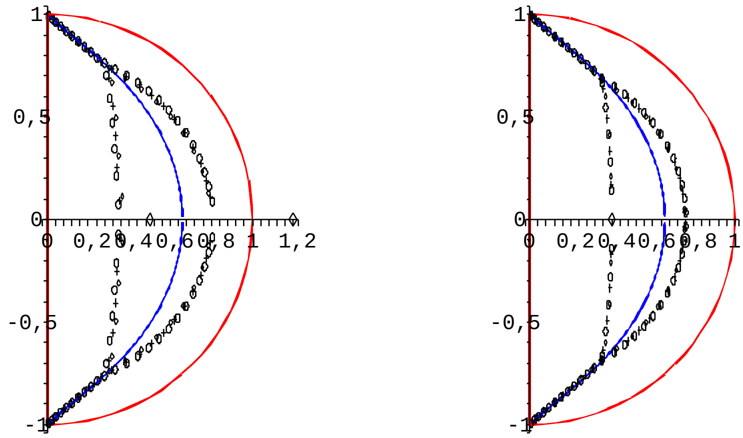
Figure 22: Zeros of P_n, $n = 40$, 50, 60, for lens parameters $N = 2$, $\alpha = 0$, and (a) $w(z) = z - 4$, (b) $w(z) = z - i$.

corresponding to the value $r = R_\Gamma$, which is supported on Γ, then for any other $R_\Gamma < r \leq 1$, μ_r is the measure supported on $(\Gamma \cup \gamma_r) \setminus \operatorname{int}(\gamma_r)$ that coincides with μ_{R_Γ} on the two subarcs $\Gamma \setminus \operatorname{int}(\gamma_r)$, and that equals the balayage of the restriction of μ_{R_Γ} to $\Gamma \cap \operatorname{int}(\gamma_r)$ onto γ_r. Alternatively, μ_r can also be characterized as the unique measure whose logarithmic potential U^{μ_r} is

$$U^{\mu_r}(z) = \begin{cases} -\log|\operatorname{cap}(L)\widehat{\Phi}(z)|, & \text{if } z \in \operatorname{ext}(\gamma_r) \\ -\log[\operatorname{cap}(L)r], & \text{if } z \in \gamma_r \cup \operatorname{int}(\gamma_r). \end{cases} \tag{190}$$

V.3 The reproducing kernel $K_w(z, \zeta)$

For any (G, w), we have introduced in (178) and (179) the space $\mathcal{B}^2_w(G)$ together with its inner product $\langle \cdot | \cdot \rangle_w$ and norm $\| \cdot \|_{\mathcal{L}^2_w(G)}$. When $w \equiv 1$, we simply write $\mathcal{L}^2(G)$. Although the notation (G, w) assumes that $L = \partial G$ is a Jordan curve, all the results stated in this section are also valid for any bounded simply-connected domain G.



(a) (b)

Figure 23: Zeros of P_n, $n = 40, 50, 60$, for lens parameters $N = 2$, $\alpha = 0$, and (a) $w(z) = (z - 1.2)^2$, (b) $w(z) = z - 0.4$.

Here, we establish the existence of the kernel function $K_w(z, \zeta)$, state some of its basic properties, and give some formulas for it.

Lemma V.3.1. *Let $z \in G$ be such that $w(z) \neq 0$. Then, for every $f \in \mathcal{B}_w^2(G)$ we have*

$$|f(z)| \leq \frac{\|f\|_{\mathcal{B}_w^2(G)}}{\sqrt{\pi}\,|w(z)|\,d_z}\,, \tag{191}$$

where

$$d_z := \operatorname{dist}(z, L) = \inf_{\zeta \in L} |\zeta - z|\,.$$

Consequently, for any compact set $K \subset G$, we can find a constant C_K such that

$$|f(z)| \leq C_K \|f\|_{\mathcal{B}_w^2(G)}, \quad \forall\ f \in \mathcal{B}_w^2(G)\,, \quad z \in K\,. \tag{192}$$

Proof. Inequality (191) follows at once by applying Lemma 1 on p. 4 of [6] to fw. Now, given any compact set $K \subset G$, one can find a Jordan curve $\Gamma_K \subset G$ surrounding K on which w has no zeros. Then from (191) we get that (192) holds for all $f \in \mathcal{B}_w^2(G)$ and $z \in \Gamma_K$, where $C_K^{-1} = \sqrt{\pi} \times \min\{|w| \text{ on } \Gamma_K\} \times \min\{d_z : z \in \Gamma_K\} > 0$. Then, by the maximum modulus principle for analytic functions, the same estimate holds for all $z \in K$. $\square$

With the help of (192) one can easily extend some results that are already known to be valid for the Bergman case $w \equiv 1$. For example, paraphrasing the proof of Thm. 1 on p. 5 of [6], we get

Lemma V.3.2. *The space $\mathcal{B}^2_w(G)$ is a Hilbert space with respect to the inner product $\langle \cdot | \cdot \rangle_w$. Moreover, if $\{f_n\}_{n=0}^{\infty} \subset \mathcal{L}^2_w(G)$ and $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{L}^2_w(G)} = 0$ for some $f \in \mathcal{L}^2_w(G)$, then $f_n(z) \to f(z)$ uniformly on compact subsets of G.*

Inequality (192) shows that for every $\zeta \in G$, the linear functional that assigns to each $f \in \mathcal{B}^2_w(G)$ the value $f(\zeta)$ is bounded. Therefore, by the Riesz representation theorem, there is a unique function $K_w(\cdot, \zeta) \in \mathcal{B}^2_w(G)$ having the reproducing property

$$f(\zeta) = \int_G \overline{K_w(z,\zeta)}\, f(z)\, |w(z)|^2\, dm(z) = \langle f | K_w(\cdot,\zeta) \rangle_w\,, \quad \forall\, f \in \quad \forall\, f \in \mathcal{B}^2_w(G)\,. \tag{193}$$

That is, $K_w(z, \zeta)$ is the *kernel function* for the space the space $\mathcal{B}^2_w(G)$. For $w \equiv 1$, we write which is the so-called *Bergman kernel function* for G. The following basic properties of $K_w(z, \zeta)$, which we state without proof, are consequences of its reproducing property (193).

Lemma V.3.3. (i) *For all z, ζ, $a \in G$,*

$$K_w(z,\zeta) = \overline{K_w(\zeta, z)} \quad \textit{and} \quad K_w(a,a) = \|K_w(\cdot, a)\|^2_{\mathcal{B}^2_w(G)} > 0\,;$$

(ii) *If $\{S_n\}_{n=1}^{\infty}$ is an orthonormal system of functions in the space $\mathcal{B}^2_w(G)$, then $\{S_n\}_{n=1}^{\infty}$ is complete if and only if for every $\zeta \in G$,*

$$K_w(\cdot, \zeta) = \sum_{n=1}^{\infty} \overline{S_n(\zeta)} S_n(\cdot)$$

in the $\mathcal{B}^2_w(G)$*-norm.*

Let $\varphi(z)$ be any conformal mapping of G onto the unit disk $\mathbb{D}$. Then it is well-known (see [6], p. 33) that the Bergman kernel function for the space $\mathcal{B}^2(G)$ is the space $\mathcal{B}^2(G)$ is given by

$$K(z,\zeta) = \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{\pi[1-\varphi(z)\overline{\varphi(\zeta)}]^2}\,. \tag{194}$$

It is straightforward to check that if $h(z)$ is analytic and never zero in G, and such that $h(z)w(z)|_G \in \mathcal{B}^2(G)$, then

$$K_{wh}(z,\zeta) = \frac{K_w(z,\zeta)}{h(z)\,\overline{h(\zeta)}}\,. \tag{195}$$

In particular, if $w(z) \neq 0\ \forall\, z \in G$, then

$$K_w(z,\zeta) = \frac{K(z,\zeta)}{w(z)\,\overline{w(\zeta)}} = \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{\pi\, w(z)\,\overline{w(\zeta)}\left[1-\varphi(z)\overline{\varphi(\zeta)}\right]^2}\,. \tag{196}$$

We call the reader's attention to the following simple fact: suppose (G,w) is such that w vanishes at each element of the set $\{a_1,a_2,\ldots,a_n\} \subset G$ (repetition allowed). Put $h(z) := w(z)/\prod_{i=1}^n (z-a_i)$. Then the reproducing kernel of the Hilbert space

$$\left\{g \in \mathcal{B}^2_h(G) : g(a_i) = 0,\ \ 1 \le i \le n\right\} = \left\{f(z)\prod_{i=1}^{n}(z-a_i) : f \in \mathcal{B}^2_w(G)\right\}$$

is $K_w(z,\zeta)/\prod_{i=1}^n (z-a_i)(\overline{\zeta}-\overline{a}_i)$. Thus, it is essentially known and easy to verify (see [20], Ex. 11, p. 262) that if $w(z) = (z-a)h(z)$ then

$$K_w(z,\zeta) := \frac{K_h(z,\zeta) - \dfrac{K_h(a,\zeta)K_h(z,a)}{K_h(a,a)}}{(z-a)\overline{(\zeta-a)}}\,. \tag{197}$$

By reiterating this formula one arrives to the following proposition. As usual, any empty product of the form $\prod_{i=1}^{0} \cdots$ is understood to equal 1.

Proposition V.3.4. *Let (G, w) be such that w has exactly $n \geq 0$ zeros in G, counting multiplicity. Write w as $w(z) = h(z) \prod_{i=1}^{n}(z - a_i)$, with $\{a_1, a_2, \ldots, a_n\} \subset G$ and $h(z) \neq 0$ for $z \in G$ (the a_i's not necessarily distinct). Then*

$$K_w(z, \zeta) = \frac{H_n(z, \zeta)}{h(z)\,\overline{h(\zeta)}} \,, \tag{198}$$

where $H_n(z, \zeta)$ is constructed from the sequence $\{a_1, a_2, \ldots, a_n\}$ by using the following iterative procedure:

$$H_0(z, \zeta) := K(z, \zeta)\,;$$

if $H_i(z, \zeta)$ is already defined for all $z, \zeta \in G$, put

$$H_{i+1}(z, \zeta) := \frac{H_i(z, \zeta) - \dfrac{H_i(a_{i+1}, \zeta) H_i(z, a_{i+1})}{H_i(a_{i+1}, a_{i+1})}}{(z - a_{i+1})\overline{(\zeta - a_{i+1})}} \,, \qquad \forall\, z,\, \zeta \in G \setminus \{a_{i+1}\} \,,$$

and

$$H_{i+1}(a_{i+1}, \zeta) := \lim_{z \to a_{i+1}} H_{i+1}(z, \zeta)\,, \qquad \forall\, \zeta \in G \setminus \{a_{i+1}\} \,,$$

$$H_{i+1}(z, a_{i+1}) := \lim_{\zeta \to a_{i+1}} H_{i+1}(z, \zeta)\,, \qquad \forall\, z \in G\,.$$

When the zeros a_i's of w are simple we have the following determinant representation:

Proposition V.3.5. *If in Proposition V.3.4, the a_i's are all distinct, then the kernel function $K_w(z,\zeta)$ for the space $\mathcal{B}^2_w(G)$ is given by*

$$K_w(z,\zeta) = \frac{\begin{vmatrix} K(a_1,a_1) & K(a_2,a_1) & \cdots & K(a_n,a_1) & K(z,a_1) \\ K(a_1,a_2) & K(a_2,a_2) & \cdots & K(a_n,a_2) & K(z,a_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K(a_1,a_n) & K(a_2,a_n) & \cdots & K(a_n,a_n) & K(z,a_n) \\ K(a_1,\zeta) & K(a_2,\zeta) & \cdots & K(a_n,\zeta) & K(z,\zeta) \end{vmatrix}}{h(z)\overline{h(\zeta)}Q_n(z)\overline{Q_n(\zeta)}\,A_n} \tag{199}$$

where $Q_n(z) = \prod_{i=1}^n (z-a_i)$ and $A_n > 0$ is the $n\times n$ principal minor of the determinant above.

Proof. The proposition can be derived by using Proposition V.3.4 and Silvester's determinant identity. However, here we give a more straightforward proof. Again by (195), it suffices to prove the proposition for the case $h(z)\equiv 1$, that is, when $w(z) = Q_n(z) = \prod_{i=1}^n (z-a_i)$. It is easy to see that the system of functions $\{K(z,a_1),\ldots,K(z,a_n)\}$ is linearly independent. The Grammian of this system is precisely A_n. Hence $A_n > 0$.

Let us denote by $D_{Q_n}(z,\zeta)$ the right-hand side of (199) and let $\zeta \notin \{a_i\}_{i=1}^n$ be fixed. Then $D_{Q_n}(z,\zeta)$ is well defined for all $z\in G$. Moreover, if we develop the determinant in (199) by its last column, we see that $D_{Q_n}(\cdot,\zeta)\in \mathcal{B}^2_{Q_n}(G)$ and, for certain constants C_i,

$$\begin{aligned}
&\int_G \overline{D_{Q_n}(z,\zeta)}f(z)|Q_n(z)|^2dm(z)\\
&= \int_G \left(\frac{\overline{K(z,\zeta)} + \sum_{i=1}^n \overline{C}_i\,\overline{K(z,a_i)}}{\overline{Q_n(z)}Q_n(\zeta)}\right) f(z)|Q_n(z)|^2dm(z)\\
&= \frac{f(\zeta)Q_n(\zeta)}{Q_n(\zeta)} + \sum_{i=1}^n \overline{C}_i \frac{f(a_i)Q_n(a_i)}{Q_n(\zeta)} = f(\zeta)\,.
\end{aligned}$$

Therefore, for $\zeta \notin \{a_i\}_{i=1}^{n}$, $D_{Q_n}(z,\zeta) = K_{Q_n}(z,\zeta)$. But then for $1 \le i \le n$,

$$D_{Q_n}(z, a_i) := \lim_{\zeta \to a_i} K_{Q_n}(z,\zeta) = K_{Q_n}(z, a_i)\,.$$

□

In order to find the singularities of $K_w(\cdot,\zeta)$ we need a description of these formulas that reflects the dependence of the kernel on the conformal mapping φ and the weight w. For this purpose we provide a useful lemma. Suppose that

$$K(z,\zeta) = \frac{f(z)\overline{g(\zeta)}}{\pi\left[1 - t(z)\overline{s(\zeta)}\right]^2}\,, \quad z,\zeta \in G\,, \tag{200}$$

where f, t, g, s are analytic functions in G, and moreover, that t and s are one-to-one in G and

$$1 - t(z)\overline{s(\zeta)} \neq 0\,, \quad \forall\, z,\, \zeta \in G\,. \tag{201}$$

In view of (194), a representation like (200) is always possible. Notice that, from Lemma V.3.3(i), $f(z)\overline{g(\zeta)} \neq 0$ for all $z, \zeta \in G$.

Lemma V.3.6. *With the above notation we have*

(a) *for* $w(z) = \omega_a^v(z) := (z-a)^v$, $v \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} K_w(z,\zeta) = K(z,\zeta) \times \frac{[t(z)-t(a)]^v\left[\overline{s(\zeta)} - \overline{s(a)}\right]^v}{(z-a)^v(\overline{\zeta} - \overline{a})^v} \\ \times \frac{\left[1 - t(a)\overline{s(\zeta)}\right]\left[1 - t(z)\overline{s(a)}\right] + v\left[1 - t(a)\overline{s(a)}\right]\left[1 - t(z)\overline{s(\zeta)}\right]}{\left[1 - t(a)\overline{s(\zeta)}\right]^{v+1}\left[1 - t(z)\overline{s(a)}\right]^{v+1}}\,; \end{aligned} \tag{202}$$

(b) *for* $w(z) = (z-a_1)^{v_1}(z-a_2)^{v_2}\cdots(z-a_n)^{v_n}$*,* $v_i \in \mathbb{N}$*,* $1 \le i \le n$*,* a_i*'s distinct,*

$$K_w(z,\zeta) = K(z,\zeta) \times \frac{\prod_{i=1}^n [t(z)-t(a_i)]^{v_i}\left[\overline{s(\zeta)}-\overline{s(a_i)}\right]^{v_i}}{\prod_{i=1}^n (z-a_i)^{v_i}(\overline{\zeta}-\overline{a}_i)^{v_i}} \times \frac{Q_w\left(t(z),\overline{s(\zeta)}\right)}{\prod_{i=1}^n \left[1-t(a_i)\overline{s(\zeta)}\right]^{v_i+1}\left[1-t(z)\overline{s(a_i)}\right]^{v_i+1}}, \tag{203}$$

where $Q_w(\tau,\xi)$ *is a polynomial in the two variables* τ *and* ξ *(of degree* $\le n$ *in each independent variable) satisfying:*

(i) $Q_w\left(t(a),\overline{s(a)}\right) \neq 0 \quad \forall\, a \in G$*;*

(ii) *if* $\xi \neq 0$*, then*

$$Q_w(1/\xi,\xi) \neq 0 \Leftrightarrow \xi \notin \left\{1/t(a_1),\ldots,1/t(a_n),\overline{s(a_1)},\ldots,\overline{s(a_n)}\right\};$$

(iii) *for every* $1 \le i \le n$*,*

$$Q_w\left(1/\overline{s(a_i)}\,,\,\cdot\right) \not\equiv 0 \quad \text{if } \overline{s(a_i)} \neq 0,$$

$$Q_w(\cdot\,,\,1/t(a_i)) \not\equiv 0 \quad \text{if } t(a_i) \neq 0;$$

(iv) *for every* $1 \le i \le n$*,*

$$Q_w\left(\tau,\overline{s(a_i)}\right) = \left[1-\tau\overline{s(a_i)}\right] S_i^w(\tau)$$

and

$$Q_w(t(a_i),\xi) = [1-t(a_i)\xi]\; T_i^w(\xi),$$

with

$$S_i^w\left(1/\overline{s(a_i)}\right) \neq 0 \quad \text{if } \overline{s(a_i)} \neq 0$$

and

$$T_i^w(1/t(a_i)) \neq 0 \quad \text{if } t(a_i) \neq 0.$$

Consequently, from (i), $S_i^w(t(a_i)) = T_i^w\left(\overline{s(a_i)}\right) \neq 0$ *for all* $1 \leq i \leq n$.

Proof. Given a point a, let us define the iteration I_a by

$$I_a\left(H(z,\zeta)\right) := H(z,\zeta) - \frac{H(a,\zeta)H(z,a)}{H(a,a)}, \tag{204}$$

which applies to any function $H(z,\zeta)$ for which (204) makes sense. Then, **(a)** follows without major complications by induction on the number v, since by Proposition V.3.4,

$$K_{\omega_a^{v+1}}(z,\zeta) = \frac{I_a\left(K_{\omega_a^v}(z,\zeta)\right)}{(z-a)(\overline{\zeta}-\overline{a})}.$$

The computations involved can be simplified by observing that if

$$H(z,\zeta) = r(z)l(\zeta)H_1(z,\zeta)$$

with $r(a)l(a) \neq 0$, then

$$I_a(H(z,\zeta)) := r(z)l(\zeta)I_a(H_1(z,\zeta)). \tag{205}$$

We now prove **(b)**. If $w(z) = \omega_{a_1}^{v_1}(z) = (z-a_1)^{v_1}$, $v_1 \geq 1$, then $K_w(z,\zeta)$ is given by formula (202), so that in this case

$$Q_w(\tau,\xi) = Q_{\omega_{a_1}^{v_1}}(\tau,\xi) = [1 - t(a_1)\xi]\left[1 - \tau\overline{s(a_1)}\right] + v_1\left[1 - t(a_1)\overline{s(a_1)}\right][1-\tau\xi]$$

and properties (i)-(iv) are trivially satisfied (property (i) is a consequence of Lemma V.3.3(i)). Thus, we only need to prove **(b)** for a w that has zeros in $n \geq 2$ points. We proceed by induction. Let $w(z) = (z-a_1)^{v_1}(z-a_2)^{v_2}\cdots(z-a_n)^{v_n}$ be such that

$n \geq 2$, and let $m := v_1 + \cdots + v_n$. Assume that **(b)** holds for any other w such that the sum of the multiplicities of its zeros is $\leq m-1$. For all $1 \leq i \leq n$, define

$$w_i(z) := (z-a_i)^{v_i-1} \prod_{\substack{1\leq j\leq n \\ j\neq i}} (z-a_j)^{v_j},$$

so that by the induction hypothesis $K_{w_i}(z,\zeta)$ has the following form:

$$\frac{f(z)\overline{g(\zeta)}\,[t(z)-t(a_i)]^{v_i-1}\left[\overline{s(\zeta)}-\overline{s(a_i)}\right]^{v_i-1}\prod_{j\neq i}^{n}[t(z)-t(a_j)]^{v_j}\left[\overline{s(\zeta)}-\overline{s(a_j)}\right]^{v_j}}{\pi\left[1-t(z)\overline{s(\zeta)}\right]^2(z-a_i)^{v_i-1}(\overline{\zeta}-\overline{a}_i)^{v_i-1}\prod_{j\neq i}^{n}(z-a_j)^{v_j}(\overline{\zeta}-\overline{a}_j)^{v_j}}$$

$$\times\frac{\widehat{Q}_{w_i}\left(t(z),\overline{s(\zeta)}\right)}{\left[1-t(a_i)\overline{s(\zeta)}\right]^{v_i}\left[1-t(z)\overline{s(a_i)}\right]^{v_i}\prod_{j\neq i}^{n}\left[1-t(a_j)\overline{s(\zeta)}\right]^{v_j+1}\left[1-t(z)\overline{s(a_j)}\right]^{v_j+1}},$$

where

$$\widehat{Q}_{w_i}(\tau,\xi) = \begin{cases} Q_{w_i}(\tau,\xi) & \text{if } v_i \geq 2 \\ \\ Q_{w_i}(\tau,\xi)\,[1-t(a_i)\xi]\left[1-\tau\overline{s(a_i)}\right] & \text{if } v_i = 1 \end{cases}$$

is a polynomial in the two variables τ and ξ (of degree $\leq n$ in each independent variable) that satisfies

(i′) $\widehat{Q}_{w_i}\left(t(a),\overline{s(a)}\right) \neq 0 \quad \forall\, a \in G$;

(ii′) if $\xi \neq 0$, then

$$\widehat{Q}_{w_i}(1/\xi,\xi) \neq 0 \Leftrightarrow \xi \notin \left\{1/t(a_1),\ldots,1/t(a_n),\overline{s(a_1)},\ldots,\overline{s(a_n)}\right\};$$

(iv′) for every $1 \leq j \leq n$,

$$\widehat{Q}_{w_i}\left(\tau,\overline{s(a_j)}\right) = \left[1-\tau\overline{s(a_j)}\right]\widehat{S}_j^{w_i}(\tau)$$

and

$$\widehat{Q}_{w_i}\left(t(a_j),\xi\right) = [1 - t(a_j)\xi]\, \widehat{T}_j^{w_i}(\xi),$$

with $\widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0$ if $\overline{s(a_j)} \neq 0$, and $\widehat{T}_j^{w_i}\left(1/t(a_i)\right) \neq 0$ if $t(a_j) \neq 0$. (It then follows from (i′) that $\widehat{S}_j^{w_i}(t(a_j)) = \widehat{T}_j^{w_i}\left(\overline{s(a_j)}\right) \neq 0$ for all $1 \leq i \leq n$.)

Properties (i′) and (ii′) are obvious. As for (iv′), notice that if $v_i = 1$ then

$$\widehat{S}_j^{w_i}(\tau) = \begin{cases} \left[1 - t(a_i)\overline{s(a_j)}\right]\left[1 - \tau\overline{s(a_i)}\right] S_j^{w_i}(\tau) \text{ if } j \neq i \\ \\ \left[1 - t(a_i)\overline{s(a_i)}\right] Q_{w_i}\left(\tau, \overline{s(a_i)}\right) \qquad \text{if } j = i \end{cases},$$

and

$$\widehat{T}_j^{w_i}(\xi) = \begin{cases} \left[1 - t(a_j)\overline{s(a_i)}\right][1 - t(a_i)\xi]\, T_j^{w_i}(\xi) \text{ if } j \neq i \\ \\ \left[1 - t(a_i)\overline{s(a_i)}\right] Q_{w_i}(t(a_i), \xi) \qquad \text{if } j = i \end{cases},$$

so that $\widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0$ and $\widehat{T}_j^{w_i}\left(1/t(a_j)\right) \neq 0$ for all $\overline{s(a_j)}$, $t(a_j) \neq 0$, $1 \leq j \leq n$, since $t(z)$ and $s(\zeta)$ are one-to-one and $Q_{w_i}(\tau, \xi)$ satisfies (ii) and (iv). Observe that the degrees of $\widehat{S}_j^{w_i}(\cdot)$ and $\widehat{T}_j^{w_i}(\cdot)$ are $\leq n-1$.

Thus, according to Proposition V.3.4 and taking (205) into account, we have that, for all $1 \leq i \leq n$,

$$\begin{aligned} K_w(z,\zeta) &= \frac{I_{a_i}\left(K_{w_i}(z,\zeta)\right)}{(z - a_i)\left(\overline{\zeta} - \overline{a}_i\right)} \\ &= \frac{f(z)\overline{g(\zeta)}\,\prod_{j=1}^n [t(z) - t(a_j)]^{v_j}\left[\overline{s(\zeta)} - \overline{s(a_j)}\right]^{v_j}}{\pi\left[1 - t(z)\overline{s(\zeta)}\right]^2 \prod_{j=1}^n (z - a_j)^{v_j}(\overline{\zeta} - \overline{a}_j)^{v_j}} \\ &\quad \times \frac{Q_w\left(t(z), \overline{s(\zeta)}\right)}{\prod_{j=1}^n \left[1 - t(a_j)\overline{s(\zeta)}\right]^{v_j+1}\left[1 - t(z)\overline{s(a_j)}\right]^{v_j+1}}, \end{aligned} \tag{206}$$

where

$$Q_w\left(t(z),\overline{s(\zeta)}\right):=\frac{\left[1-t(z)\overline{s(\zeta)}\right]^2\left[1-t(a_i)\overline{s(\zeta)}\right]\left[1-t(z)\overline{s(a_i)}\right]}{[t(z)-t(a_i)]\left[\overline{s(\zeta)}-\overline{s(a_i)}\right]}$$
$$\times I_{a_i}\left(\widehat{Q}_{w_i}\left(t(z),\overline{s(\zeta)}\right)\left[1-t(z)\overline{s(\zeta)}\right]^{-2}\right).$$

On expanding the last term and replacing $t(z)$ by τ and $s(\zeta)$ by ξ, we get

$$Q_w(\tau,\xi)\,\widehat{Q}_{w_i}\left(t(a_i),\overline{s(a_i)}\right)[\tau-t(a_i)]\left[\xi-\overline{s(a_i)}\right]$$

$$\begin{aligned}&=[1-t(a_i)\xi]\left[1-\tau\overline{s(a_i)}\right]\widehat{Q}_{w_i}\left(t(a_i),\overline{s(a_i)}\right)\widehat{Q}_{w_i}(\tau,\xi)\\&\quad-[1-\tau\xi]^2\left[1-t(a_i)\overline{s(a_i)}\right]^2\widehat{S}_i^{w_i}(\tau)\widehat{T}_i^{w_i}(\xi).\end{aligned}\tag{207}$$

This shows that $Q_w(\tau,\xi)$ is a polynomial with the degree in each independent variable no greater than n, and so we see from (206) that $K_w(z,\zeta)$ has the form (203). Notice that the representation for $Q_w(\tau,\xi)$ given by (207) is valid for every $1\le i\le n$. Also, since $K_w(a,a)>0$ for all $a\in G$ (see Lemma V.3.3(i)), we must have $Q_w\left(t(a),\overline{s(a)}\right)\neq 0$ for all $a\in G$. Hence, property (i) holds.

Further, it follows from (207) that for every $\xi\neq 0$

$$Q_w(1/\xi,\xi)=\frac{\widehat{Q}_{w_i}(1/\xi,\xi)\,[1-t(a_i)\xi]\left[1-\overline{s(a_i)}/\xi\right]}{[1/\xi-t(a_i)]\left[\xi-\overline{s(a_i)}\right]}=\widehat{Q}_{w_i}(1/\xi,\xi)\,.$$

Thus, in view of (ii′), property (ii) also holds.

To prove (iii), suppose that $1 \le i \le n$ is such that $\overline{s(a_i)} \neq 0$. Then by (207) and (iv′),

$$\begin{aligned}
&\lim_{\xi \to \overline{s(a_i)}} \frac{Q_w\left(1/\overline{s(a_i)}, \xi\right)}{\xi - \overline{s(a_i)}} \\
&= \lim_{\xi \to \overline{s(a_i)}} \frac{-\left[1 - \xi/\overline{s(a_i)}\right]^2 \left[1 - t(a_i)\overline{s(a_i)}\right]^2 \widehat{S}_i^{w_i}\left(1/\overline{s(a_i)}\right) \widehat{T}_i^{w_i}(\xi)}{\widehat{Q}_{w_i}\left(t(a_i), \overline{s(a_i)}\right)\left[1/\overline{s(a_i)} - t(a_i)\right]\left[\xi - \overline{s(a_i)}\right]^2} \\
&= \frac{-\widehat{S}_i^{w_i}\left(1/\overline{s(a_i)}\right)}{\overline{s(a_i)}} \neq 0\,.
\end{aligned}$$

Similarly, we find for $t(a_i) \neq 0$,

$$\lim_{\tau \to t(a_i)} \frac{Q_w\left(\tau, 1/t(a_i)\right)}{\tau - t(a_i)} = \frac{-\widehat{T}_i^{w_i}(1/t(a_i))}{t(a_i)} \neq 0\,,$$

from which (iii) follows.

Finally, we prove (iv). For any $1 \le j \le n$, choose $a_i \neq a_j$ (this is possible because $n \ge 2$). Then, with the notation of (iv′), we have

$$\widehat{Q}_{w_i}\left(\tau, \overline{s(a_j)}\right) = \left[1 - \tau\overline{s(a_j)}\right] \widehat{S}_j^{w_i}(\tau),$$

and therefore we get from (207)

$$Q_w\left(\tau, \overline{s(a_j)}\right) = \left[1 - \tau\overline{s(a_j)}\right] S_j^{w}(\tau),$$

where

$$\begin{aligned}
&S_j^w(\tau)\, \widehat{Q}_{w_i}\left(t(a_i), \overline{s(a_i)}\right)\left[\tau - t(a_i)\right]\left[\overline{s(a_j)} - \overline{s(a_i)}\right] \\
&\quad = \left[1 - t(a_i)\overline{s(a_j)}\right]\left[1 - \tau\overline{s(a_i)}\right] \widehat{Q}_{w_i}\left(t(a_i), \overline{s(a_i)}\right) \widehat{S}_j^{w_i}(\tau) \\
&\qquad - \left[1 - \tau\overline{s(a_j)}\right]\left[1 - t(a_i)\overline{s(a_i)}\right]^2 \widehat{S}_i^{w_i}(\tau)\widehat{T}_i^{w_i}\left(\overline{s(a_j)}\right).
\end{aligned}$$

Hence, if $\overline{s(a_j)} \neq 0$,

$$S_j^w\left(1/\overline{s(a_j)}\right) = \frac{\left[1 - t(a_i)\overline{s(a_j)}\right]\left[1 - \overline{s(a_i)}/\overline{s(a_j)}\right]\widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right)}{\left[1/\overline{s(a_j)} - t(a_i)\right]\left[\overline{s(a_j)} - \overline{s(a_i)}\right]}$$
$$= \widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0,$$

by (iv$'$). Similarly, we find that

$$T_j^w(1/t(a_j)) = \widehat{T}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0.$$

□

V.4 Orthogonal polynomials and the kernel function

Recall that for any (G, w), $P_n(z) := P_n(z; w) = \kappa_n^w z^n + \cdots$ denotes the polynomial of degree n and positive leading coefficient κ_n^w that is orthonormal with respect to the measure $|w|^2 dm|_G$. It is well-known that the logarithmic capacity cap(L) of $L = \partial G$ is given by

$$\mathrm{cap}(L) = 1/\Phi'(\infty)\,, \tag{208}$$

where, as before,

$$\Phi : \overline{\mathbb{C}} \setminus \overline{G} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \tag{209}$$

is the exterior conformal map associated with G, normalized so that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$.

In the sense of Definition 3.1.2 of [28], the measure $|w|^2 dm|_G$ belongs to the **Reg** class, that is,

$$\lim_{n\to\infty} \left(\kappa_n^w\right)^{1/n} = [\mathrm{cap}(L)]^{-1}\,. \tag{210}$$

To see that this is true, first notice that since L is a regular set with respect to the Dirichlet problem in $\mathbb{C} \setminus \overline{G}$, (210) is equivalent to (see Thm. 3.2.3 of [28])

$$\lim_{n\to\infty} \|P_n\|_{L_\infty(\overline{G})}^{1/n} = 1\,. \tag{211}$$

To show that (211) holds, one can proceed as in the proof of the corresponding result (Lemma 4.3 of [21]) for the case $w \equiv 1$, using (192) instead of inequality (4.4) of [21].

We say that a property $\mathcal{P}$ holds for *quasi-every* $z \in \Omega$, or that $\mathcal{P}$ holds *quasi-everywhere* on Ω (briefly, $\mathcal{P}$ *q.e.* $z \in \Omega$), if

$$\mathrm{cap}(\{z \in \Omega : \mathcal{P} \text{ does not hold for } z\}) = 0.$$

Another relation that is equivalent to (210) and that will be used in this paper is the following (see Thm. 3.1.1 of [28]):

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = 1 \quad q.e.\ \ z \in L. \tag{212}$$

For each $r > 1$, set

$$l_r := \{z \,:\, |\Phi(z)| = r\} \tag{213}$$

and $l_1 := L = \partial G$. If g is an analytic function on G, define

$$\rho(g) := \sup\{r : g \text{ is analytic on } \mathrm{int}(l_r)\}\,. \tag{214}$$

Then $1 \le \rho(g) \le \infty$, and if $\mathcal{P}_w^2(G)$ denotes the closure of the set of polynomials in $\mathcal{B}_w^2(G)$, we polynomials in $\mathcal{B}_w^2(G)$, we have

Lemma V.4.1. *Let $g \in \mathcal{B}^2_w(G)$ and let $a_n := \langle g | P_n(\cdot\,; w)\rangle_w$, $n = 0, 1, \ldots$. Then*

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{\rho(g)}. \tag{215}$$

Moreover, if $g \in \mathcal{P}^2_w(G)$, then equality holds in (215) *and*

$$g(z) = \sum_{n=0}^{\infty} a_n P_n(z)$$

locally uniformly on $\operatorname{int}\left(l_{\rho(g)}\right)$.

With Lemma V.3.2 and (211) at hand, the proof of Lemma V.4.1 is essentially the same as that given by J.L. Walsh in [34], pp. 130–131 (see also [21], p. 336).

We can apply the above lemma to estimate $|P_n(\zeta)|$ for $\zeta \in G$. Indeed, since by (193), $\overline{P_n(\zeta)} = \langle K_w(\cdot, \zeta) | P_n\rangle_w$, it follows that for each $\zeta \in G$ fixed,

$$\sum_{n=1}^{\infty} \overline{P_n(\zeta)} P_n(\cdot) =: L_w(\cdot, \zeta)$$

represents a function of the space $\mathcal{B}^2_w(G)$. By Lemma V.3.3(ii), $\mathcal{P}^2_w(G) = \mathcal{L}^2_w(G)$ if and only if

$$L_w(\cdot, \zeta) = K_w(\cdot, \zeta)\,, \qquad \forall\, \zeta \in G\,.$$

Of course, we also have $\overline{P_n(\zeta)} = \langle L_w(\cdot, \zeta) | P_n\rangle_w$, so that by applying Lemma V.4.1 to $g = L_w(\cdot, \zeta)$ and $g = K_w(\cdot, \zeta)$ we get

Corollary V.4.2. *For every $\zeta \in G$,*

$$\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} = \frac{1}{\rho\left(L_w(\cdot, \zeta)\right)} \le \frac{1}{\rho\left(K_w(\cdot, \zeta)\right)}\,. \tag{216}$$

Furthermore, if $\mathcal{P}^2_w(G) = \mathcal{B}^2_w(G)$, then equality holds in (216) *and, therefore,*

$$\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} = 1$$

if and only if $K_w(\cdot,\zeta)$ has a singularity on $L = \partial G$.

Corollary V.4.2 describes a basic relationship between the orthogonal polynomials and the kernel function which will play an essential role in deriving our zero distribution results. We shall also apply the next lemma which involves the logarithmic potential of a measure, as well as the notion of harmonic majorant. While somewhat more general, it is similar to results of Walsh (see Remark V.4.5 below).

For any finite, positive Borel measure σ with compact support $\operatorname{supp}(\sigma) \subset \mathbb{C}$, we denote by U^σ its *logarithmic potential* defined by

$$U^\sigma(z) := \int_{\mathbb{C}} \log \frac{1}{|z-t|}\, d\sigma(t)\,, \qquad z \in \mathbb{C}\,.$$

Notice that if q_n is a monic polynomial of degree n, then the logarithmic potential of the counting measure ν_{q_n} is

$$U^{\nu_{q_n}}(z) = n^{-1} \log |q_n(z)|^{-1}.$$

Lemma V.4.3. *Let $E \neq \emptyset$ be a compact subset of $\mathbb{C}$ such that both $\overline{\mathbb{C}} \setminus E$ and $\mathring{E} := \operatorname{int}(E)$ are connected (see Figure 24). Let $g : \overline{\mathbb{C}} \setminus \mathring{E} \to \overline{\mathbb{C}}$ be such that g is analytic in $\mathbb{C} \setminus E$, $|g|$ is continuous and never zero in $\overline{\mathbb{C}} \setminus \mathring{E}$, $g(\infty) = \infty$ and $g'(\infty) = 1$. Let $\{q_n\}_{n=1}^\infty$ be a sequence of monic polynomials of respective degrees $n = 1, 2, \ldots$, such that ∞ is not an accumulation point of the set of zeros of the q_n's. Further, assume that*

$$\limsup_{n\to\infty} |q_n(z)|^{1/n} \le |g(z)| \qquad q.e. \quad z \in \partial E\,. \tag{217}$$

Then, any measure σ that is a weak-limit point of the sequence $\{\nu_{q_n}\}_{n=1}^\infty$ is supported on E and*

$$U^\sigma(z) = \log |g(z)|^{-1} \qquad \forall\, z \in \mathbb{C} \setminus \mathring{E}\,. \tag{218}$$

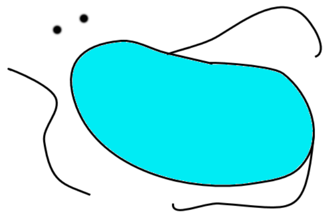


Figure 24: A set E satisfying the hypotheses of Lemma V.4.3.

Moreover, there is a unique measure μ_g *supported on* ∂E *such that* (218) *holds with* $\sigma = \mu_g$. *For such a measure, we have*

$$\limsup_{n\to\infty} |q_n(z)|^{1/n} \le e^{-U^{\mu_g}(z)} \quad \forall\, z \in \mathbb{C}\,, \tag{219}$$

and

(a) *if* $\overset{\circ}{E} = \emptyset$, *then* $\nu_{q_n} \xrightarrow{*} \mu_g \quad as \quad n \to \infty$;

(b) *if* $\overset{\circ}{E} \neq \emptyset$ *and for some* $z_0 \in \overset{\circ}{E}$ *and a subsequence* $\mathcal{N} \subset \mathbb{N}$

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} |q_n(z_0)|^{1/n} = e^{-U^{\mu_g}(z_0)}, \tag{220}$$

then

$$\nu_{q_n} \xrightarrow{*} \mu_g \quad as \quad n \to \infty, \quad n \in \mathcal{N}\,. \tag{221}$$

Conversely, if μ_g *is a weak*-star limit point of the sequence* $\{\nu_{q_n}\}$, *then equality holds in* (219) *for quasi-every* $z \in \mathbb{C}$.

Proof. Observe that (217) is equivalent to

$$\liminf_{n\to\infty} U^{\nu_{q_n}}(z) \ge \log|g(z)|^{-1} \qquad q.e. \quad z \in \partial E\,. \tag{222}$$

Let σ be a weak*-limit point of the sequence $\{\nu_{q_n}\}_{n=1}^{\infty}$, so that for some subsequence $\mathcal{N} \subset \mathbb{N}$

$$\nu_{q_n} \xrightarrow{*} \sigma \quad \text{as} \quad n \to \infty\,, \quad n \in \mathcal{N}\,.$$

Then σ is a probability measure and, by (222) and the Lower Envelope Theorem ([26], Thm. I.6.9), we have

$$U^{\sigma}(z) = \liminf_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z) \geq \liminf_{n\to\infty} U^{\nu_{q_n}}(z) \geq \log|g(z)|^{-1} \quad q.e. \quad z \in \partial E\,. \tag{223}$$

By the assumptions on g, the function

$$F^{\sigma}(z) := U^{\sigma}(z) - \log|g(z)|^{-1}\,, \quad z \in \mathbb{C}\setminus E\,,$$

is superharmonic and lower bounded in $\mathbb{C}\setminus E$, harmonic and equal to zero at ∞, and in view of (223) and the lower semicontinuity of U^{σ}, it also satisfies for *quasi-every* $z' \in \partial E$

$$\liminf_{\substack{z\to z' \\ z\in\mathbb{C}\setminus E}} F^{\sigma}(z) \geq \liminf_{z\to z'} U^{\sigma}(z) - \lim_{\substack{z\to z' \\ z\in\mathbb{C}\setminus E}} \log|g(z)|^{-1} \geq U^{\sigma}(z') - \log|g(z')|^{-1} \geq 0\,.$$

Then, by the generalized minimum principle for superharmonic functions ([26], Thm. I.2.4) we conclude that $F^{\sigma} \equiv 0$, which implies that (218) holds in $\mathbb{C}\setminus E$. It also implies that U^{σ} is harmonic in $\mathbb{C}\setminus E$ and therefore, in view of the unicity theorem (see e.g. [26], Thm. II.2.1), supp(σ) must be contained in E. Since the boundary of the domain $\mathbb{C}\setminus E$ in the fine topology (i.e. the coarsest topology that makes every logarithmic potential continuous) coincides with its boundary in the Euclidean topology (see [26], Cor. I.5.6), we see that (218) is also valid in $\mathbb{C}\setminus \mathring{E}$.

It is a direct consequence of Carleson's Unicity Theorem (see [26], Thm. II.4.13) that there can be at most one measure μ_g supported on ∂E that satisfies (218) with $\sigma = \mu_g$. To see that such a μ_g actually exists, choose any measure σ that is a weak*-star limit point of the sequence $\{\nu_{q_n}\}_{n=1}^{\infty}$. (This is possible in view of Helly's Theorem ([26], Thm. 0.1.3) because, by assumption, all the zeros of the q_n's lie in a fixed compact subset of $\mathbb{C}$.) Let σ_1 be the restriction of σ to $\mathring{E}$, and let $\widehat{\sigma}_1$ be the balayage of σ_1 onto $\partial \overset{\circ}{E}$. Then, $\mu_g := \sigma - \sigma_1 + \widehat{\sigma}_1$ is the measure we are looking for, since it easily follows from the properties of balayage measures (see [26], Thm. II.4.1) that this μ_g satisfies

$$U^{\mu_g}(z) = U^{\sigma}(z) \quad \forall\, z \in \mathbb{C} \setminus E, \quad U^{\sigma}(z) \geq U^{\mu_g}(z) \quad \forall\, z \in \mathbb{C}\,. \tag{224}$$

Accordingly, when $\overset{\circ}{E} = \emptyset$, the measure μ_g is the unique weak*-limit point of $\{\nu_{q_n}\}$, so that **(a)** takes place.

Now, for any $z \in \mathbb{C}$ fixed, choose a subsequence $\mathcal{N} \subset \mathbb{N}$ through which the lim sup in (219) is realized. We can assume that also $\nu_{q_n} \xrightarrow{*} \sigma$ as $n \to \infty$, $n \in \mathcal{N}$. Then, by the principle of descent ([26], Thm. I.6.8) and (224),

$$\liminf_{n\to\infty} U^{\nu_{q_n}}(z) = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z) \geq U^{\sigma}(z) \geq U^{\mu_g}(z)\,,$$

which proves (219).

Let us now prove (**b**). Suppose (220) holds, and let σ_0 be an arbitrary weak*-limit point of $\{\nu_{q_n}\}_{n\in\mathcal{N}}$. Because U^{μ_g} is harmonic in $\overset{\circ}{E}$, we get from (224) and the minimum principle for superharmonic functions that $U^{\sigma_0}(z) > U^{\mu_g}(z)\; \forall\, z \in \overset{\circ}{E}$, unless $U^{\sigma_0} \equiv U^{\mu_g}$ on $\overset{\circ}{E}$. But from (220) and the principle of descent, we have that

$$U^{\mu_g}(z_0) = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z_0) = \liminf_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z_0) \geq U^{\sigma_0}(z_0)\,.$$

Therefore, $U^{\sigma_0} \equiv U^{\mu_g}$ is harmonic in $\overset{\circ}{E}$, and consequently $\mathrm{supp}(\sigma_0) \subset \partial E$. By the uniqueness of μ_g, $\sigma_0 = \mu_g$, and since σ_0 is arbitrary, (221) must hold.

Finally, suppose that conversely, (221) takes place for some subsequence $\mathcal{N} \subset \mathbb{N}$. Then, by the Lower Envelope Theorem, we have for *quasi-every* $z \in \mathbb{C}$

$$U^{\mu_g}(z) = \liminf_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z) \geq \liminf_{n\to\infty} U^{\nu_{q_n}}(z) \geq U^{\mu_g}(z);$$

that is, we have equality in (219) *quasi-everywhere* on $\mathbb{C}$. $\square$

Remark V.4.4. (i) By arguing as in the proof of Lemma V.4.3, one readily sees that if the inequality in (217) is satisfied *quasi-everywhere* on $\mathbb{C} \setminus E$, then the conclusions of that lemma remain true, even if g has zeros on ∂E. One can also verify that if $z_0 \in \mathbb{C} \setminus E$ has a neighborhood on which q_n has no zeros for n large enough, then

$$\lim_{n\to\infty} |q_n(z_0)|^{1/n} = |g(z_0)| \,. \tag{225}$$

Hence, equality holds in (217) *quasi-everywhere* on $\mathbb{C} \setminus E$.

(ii) A well-known result by Fejér asserts that the zeros of orthogonal polynomials with respect to a compactly supported measure σ are contained in the closed convex hull of $\mathrm{supp}(\sigma)$ (see e.g. [25]). Thus, if the q_n's in Lemma V.4.3 are orthogonal, it is already guaranteed that all their zeros are uniformly bounded in $\mathbb{C}$. We will be using this fact in all the applications of Lemma V.4.3.

Remark V.4.5. The fact that a condition like (217) has consequences on the zero distribution of the sequence $\{q_n\}_{n=1}^{\infty}$ is well-known. For example, from (225) (see [36], Thm. 1) it follows that for every continuum $Q \subset \mathbb{C} \setminus E$ containing more that one point

$$\limsup_{n\to\infty} \|q_n\|_{L_\infty(Q)}^{1/n} = \|g\|_{L_\infty(Q)}. \tag{226}$$

In the terminology of [36] (see also p. 635 of [35]), this is expressed by saying that $\log|g(z)|$ is an *exact harmonic majorant* of the sequence $\left\{q_n^{1/n}\right\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus E$. We refer the reader to [36] for earlier results on the behavior of zeros of functions having an exact harmonic majorant. More recent results of a similar nature to that of Lemma V.4.3 can be found in Section III.4 of [26].

V.5 Proofs of the zero distribution results

Proof of Theorem V.2.1. Define $E := \overline{G}$, $q_n(z) := P_n(z)/\kappa_n^w$ and

$$g(z) := \mathrm{cap}(L)\Phi(z) \qquad z \in \mathbb{C} \setminus G\,.$$

With the help of (208), (210) and (212), it is easily seen that E, g and $\{q_n\}_{n=1}^{\infty}$ so defined satisfy the hypotheses of Lemma V.4.3. Then, with the notations of that lemma, we have $\mu_g = \mu_L$ (the equilibrium measure of L), since it is well-known that μ_L is supported on L and satisfies (218). Hence, Theorem V.2.1 is a direct consequence of (219), Lemma V.4.3(b), and Corollary V.4.2. □

Proof of Theorem V.2.3. Suppose for the moment that (183) holds. To prove (b), define $E := \overline{\mathbb{D}}_r$, $q_n := P_n/\kappa_n^w$ and $g(z) := z$ for all $|z| \geq r$. Since the capacity of the unit circle is 1, it follows from (210) and (183) that E, q_n and g so chosen satisfy the hypotheses of Lemma V.4.3. It is well-known that $\mu_r := |dz|/2\pi r$ is the equilibrium measure of the circle $\mathbb{T}_r$, and that its potential is given by

$$U^{\mu_r}(z) = \begin{cases} \log\left(1/|z|\right) \text{ if } |z| > r \\ \log\left(1/r\right) \quad \text{if } |z| \leq r \end{cases} .$$

This implies, with the notations of Lemma V.4.3, that $\mu_g = \mu_r$, and hence Theorem V.2.3(b) is just a consequence of statement (b) of that lemma (cf. also the paragraph preceding (218)).

Similarly, we prove (a). Define $E := \overline{\mathbb{D}}_\rho$, $(1 > \rho > 0)$, with q_n and g as above. Then (183) implies that (217) holds on $\mathbb{T}_\rho$, and so by Lemma V.4.3 any weak*-limit of $\nu_{q_n} = \nu_{P_n}$ is supported on $\overline{\mathbb{D}}_\rho$. Letting ρ go to zero we deduce Theorem V.2.3(a).

Thus, it remains to establish (183). Let us write the function w as

$$w(z) = h(z)\prod_{i=1}^{\ell}(z-a_i)^{v_i},$$

where $v_i = \mathrm{mult}(a_i)$. The exterior conformal mapping for $\mathbb{D}$ is simply $\Phi(z) = z$, so that in view of (194), Lemma V.3.6(b) with $f \equiv g \equiv 1$, $t(z) := z$, $s(\zeta) := \zeta$, and (195), the kernel function $K_w(z,\zeta)$ for the space $\mathcal{B}^2_w(\mathbb{D})$ space $\mathcal{B}^2_w(\mathbb{D})$ has the form

$$K_w(z,\zeta) = \frac{Q_w\left(z,\overline{\zeta}\right)}{\pi\left(1-z\overline{\zeta}\right)^2\left[\prod_{i=1}^{\ell}\left(1-a_i\overline{\zeta}\right)^{v_i+1}(1-z\overline{a}_i)^{v_i+1}\right]h(z)\overline{h(\zeta)}}, \tag{227}$$

where $Q_w(\cdot,\cdot)$ is a polynomial in two variables satisfying

(ii) if $\xi \neq 0$, then $Q_w(1/\xi,\xi) \neq 0 \Leftrightarrow \xi \notin \{1/a_1,\ldots,1/a_\ell,\overline{a}_1,\ldots,\overline{a}_\ell\}$;

(iii) for every $1 \leq i \leq \ell$, $Q_w(1/\overline{a}_i,\cdot) \not\equiv 0 \quad \text{if } \overline{a}_i \neq 0$.

It follows from (227) that for all $\zeta \in \mathbb{D}$, $K_w(\cdot,\zeta)$ is a meromorphic function in $\mathbb{C}$ whose possible poles are the elements of the set

$$\left\{1/\overline{\zeta}, 1/\overline{a}_1,\ldots,1/\overline{a}_\ell, b_1, b_2,\ldots\right\} \setminus \mathcal{A}^{-1}, \tag{228}$$

where (notice that each c_j is now a zero of $K_w(\cdot,\zeta)$)

$$\mathcal{A}^{-1} := \{1/\overline{a}_i : 1/\overline{a}_i = c_j \text{ for some } j \text{ and } \mathrm{mult}(c_j) \geq \mathrm{mult}(a_i)+1\}.$$

We shall show that for every $\zeta \in \mathbb{D}$ (except possibly countably many), the finite elements of the set (228) are, in fact, poles of $K_w(\cdot,\zeta)$.

First, we see from (ii) that if $\zeta \in \mathbb{D}$ and

$$\zeta \notin \{a_1,\ldots,a_\ell,0\} \cup \{1/\overline{c}_j \,:\, \mathrm{mult}(c_j) \geq 2,\ j \geq 1\},$$

then $K_w(\cdot,\zeta)$ has a pole at $z = 1\big/\overline{\zeta}$. Second, it is a consequence of (iii) that for all but finitely many $\zeta \in \mathbb{D}$, $K_w(\cdot,\zeta)$ has a pole at $1/\overline{a}_i$ if $a_i \neq 0$ and $1/\overline{a}_i \notin \mathcal{A}^{-1}$. And finally, if $b_k \notin \{1/\overline{a}_i : a_i \neq 0,\ 1 \leq i \leq \ell\}$, then again by (ii), $Q_w\,(b_k, 1/b_k) \neq 0$, so that $Q_w\,(b_k,\xi)$ is a polynomial in ξ not identically zero, and consequently, for all but finitely many $\zeta \in \mathbb{D}$, $K_w(\cdot,\zeta)$ has a pole at b_k.

Thus, according to (213) and (214), for all but countably many $\zeta \in \mathbb{D}$,

$$\rho(K_w(\cdot,\zeta)) = \min\left(|z| : z \in \left\{1/\overline{\zeta}, 1/\overline{a}_1, \ldots, 1/\overline{a}_\ell, b_1, b_2, \ldots\right\} \setminus \mathcal{A}^{-1}\right);$$

whence, by Corollary V.4.2 (recall Remark V.2.2), for all but countably many $\zeta \in \mathbb{D}$,

$$\begin{aligned}\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} &= \max\left(\left\{0, |\zeta|, |a_1|, \ldots, |a_\ell|, |b_1|^{-1}, |b_2|^{-1}, \ldots\right\} \setminus \mathcal{A}\right)\\ &= \max\{|\zeta|, r\} = \begin{cases} |\zeta| & \text{if } r < |\zeta| < 1 \\ r & \text{if } |\zeta| \leq r \end{cases},\end{aligned}$$

where

$$\mathcal{A} = \left\{|z|^{-1} : z \in \mathcal{A}^{-1}\right\} \quad \text{and} \quad r = \max\left(\left\{0, |a_1|, \ldots, |a_\ell|, |b_1|^{-1}, |b_2|^{-1}, \ldots\right\} \setminus \mathcal{A}\right).$$

$\square$

Example V.5.1. Let w be a meromorphic function on $\mathbb{C}$, that does not vanish, whose poles $c_1, c_2, \ldots$ all lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and each of them has multiplicity no less than 2. Since in this case the kernel function has the form

$$K_w(z,\zeta) = \frac{1}{\pi\left(1 - z\overline{\zeta}\right)^2 w(z)\overline{w(\zeta)}},$$

we see that $K_w(\cdot, 1/\overline{c}_j)$ is an entire function for all $1 \leq j < \infty$, and if $\zeta \notin \{1/\overline{c}_1, 1/\overline{c}_2,$ $\ldots\ \}$, then $K_w(\cdot,\zeta)$ is a meromorphic function with a double pole at $1/\overline{\zeta}$. Consequently, we have for all $\zeta \in \mathbb{D}$

$$\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} = \begin{cases} 0 & \text{if } \zeta \in \{1/\bar{c}_1, 1/\bar{c}_2, \ldots\} \\ |\zeta| & \text{otherwise} \end{cases}, \tag{229}$$

and according to Theorem V.2.3(a), this implies that $\nu_{P_n} \xrightarrow{*} \delta_0$ as $n \to \infty$. However, each point $1/\bar{c}_j$ is a limit point of the zeros of the P_n's, because if, to the contrary, there is a neighborhood V of $1/\bar{c}_j$ and a subsequence $\mathcal{N} \subset \mathbb{N}$ such that P_n has no zeros on V for $n \in \mathcal{N}$, then by the continuity of $\log |t - 1/\bar{c}_j|^{-1}$ in $\mathbb{C} \setminus V$, we would have

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{P_n}}(1/\bar{c}_j) = U^{\delta_0}(1/\bar{c}_j) = \log |c_j| \,,$$

contradicting (229).

Proof of Theorem V.2.7. Recall that the lens-shaped domain G, as well as its associated curves Γ, γ_r, and function $\widehat{\Phi}$ have been introduced in the paragraph preceding the statement of Theorem V.2.7. Assume that (189) is true for some r with $R_\Gamma \le r \le 1$. Set $E := \Gamma \cup \gamma_r \cup \operatorname{int}(\gamma_r)$, $q_n = P_n/\kappa_n^w$, and $g(z) := \operatorname{cap}(L)\widehat{\Phi}(z)$ for all $z \in \overline{\mathbb{C}} \setminus \operatorname{int}(\gamma_r)$. We see from (210) and (189) that E, q_n and g so defined satisfy the assumptions of Lemma V.4.3, and hence, any weak*-limit point σ of $\{\nu_{P_n}\} = \{\nu_{q_n}\}$ is supported in $\Gamma \cup \gamma_r \cup \operatorname{int}(\gamma_r)$. Let $\mu_r := \mu_g$ be the unique measure supported on $\partial E = (\Gamma \cup \gamma_r) \setminus \operatorname{int}(\gamma_r)$ that satisfies

$$U^{\mu_r}(z) = \log |g(z)|^{-1} = \log |\operatorname{cap}(L)\widehat{\Phi}(z)|^{-1} \quad \forall\, z \in \mathbb{C} \setminus \operatorname{int}(\gamma_r)\,.$$

Now, from the definition of γ_r in (188), and the lower semicontinuity of U^{μ_r}, we have that if $\operatorname{int}(\gamma_r) \ne \emptyset$, then

$$\liminf_{\substack{z\to z' \\ z\in\operatorname{int}(\gamma_r)}} U^{\mu_r}(z) \ge U^{\mu_r}(z') = \log[\operatorname{cap}(L) r]^{-1} \quad \forall\, z' \in \gamma_r\,, \tag{230}$$

and in view of (189) and (219), we have for some $z_0 \in \mathrm{int}(\gamma_r)$

$$\mathrm{cap}(L)\, r = \limsup_{n\to\infty} |q_n(z_0)|^{1/n} \le e^{-U^{\mu_r}(z_0)}\,. \tag{231}$$

Since (230), (231) and the minimum principle for superharmonic functions imply that

$$U^{\mu_r}(z) = \log\left[\mathrm{cap}(L)\, r\right]^{-1} \quad \forall\, z \in \mathrm{int}(\gamma_r),$$

the statements (a) and (b) of Theorem V.2.7 follow directly from their corresponding ones in Lemma V.4.3.

Let us now show that if σ is a weak*-limit point of the measures ν_{P_n}, then necessarily every point of $\Gamma \setminus \mathrm{int}(\gamma_r)$ belongs to $\mathrm{supp}(\sigma)$. Suppose that $z_0 \in \Gamma \setminus \overline{\mathrm{int}(\gamma_r)}$ is not in $\mathrm{supp}(\sigma)$ and let us derive a contradiction. Let $D_{z_0} \subset G \setminus \overline{\mathrm{int}(\gamma_r)}$ be a disk centered at z_0 and of radius so small that $\mathrm{supp}(\sigma) \cap D_{z_0} = \emptyset$. Then U^σ is harmonic in D_{z_0} and we have from (218) and (186)

$$U^\sigma(z) = \begin{cases} \log\left|\overline{\Phi(z_\alpha)}/\mathrm{cap}(L)\right| & z \in G_\alpha \cap D_{z_0} \\ \log\left|\overline{\Phi(z_\beta)}/\mathrm{cap}(L)\right| & z \in G_\beta \cap D_{z_0} \end{cases}. \tag{232}$$

But since the harmonic extension is unique, it follows that the first row of the right-hand side of (232) also represents U^σ in $G_\beta \cap D_{z_0}$, contradicting the obvious fact that

$$G_\beta = \{z \in G : |\Phi(z_\alpha)| > |\Phi(z_\beta)|\}\,.$$

Analogously, one can show that $\mathrm{supp}(\mu_r) = (\Gamma \cup \gamma_r) \setminus \mathrm{int}(\gamma_r)$.

We now turn to the proof of (189). Similar to the case of the unit disk, the argument is based on Corollary V.4.2. Therefore, our next task is to find the singularities of the kernel function $K_w(\cdot, \zeta)$ for the lens-shaped domain G and an entire weight function w. It is not difficult to see that for every $\zeta \in G$, the function

$$\varphi_\zeta(z) := \frac{\left(\dfrac{z-i}{z+i}\right)^N - \left(\dfrac{\zeta-i}{\zeta+i}\right)^N}{\left(\dfrac{z-i}{z+i}\right)^N - \left(\dfrac{\overline{\zeta}+i}{\overline{\zeta}-i}\right)^N \cdot e^{-2N\alpha i}} \tag{233}$$

maps G conformaly onto $\mathbb{D}$ in such a way that $\varphi_\zeta(\zeta) = 0$. Then, choosing $\varphi = \varphi_\zeta$ in formula (194) for each particular $\zeta \in G$, we obtain after some computations that

$$K(z,\zeta) = -\frac{4N^2}{\pi} \cdot \frac{\left[\left(\overline{\zeta}-i\right)\left(\overline{\zeta}+i\right)(z-i)(z+i)\right]^{N-1}}{\left[e^{N\alpha i}\left(\overline{\zeta}-i\right)^N (z-i)^N - e^{-N\alpha i}\left(\overline{\zeta}+i\right)^N (z+i)^N\right]^2}. \tag{234}$$

Let $\{a_1, \ldots, a_\ell\}$ be the set of zeros of w lying on G, and let $\{b_1, b_2, \ldots\}$ be the set of zeros of w lying on $\mathbb{C} \setminus G$. Write w as

$$w(z) := h(z) \prod_{j=1}^{\ell} (z-a_i)^{v_i},$$

where $v_i = \mathrm{mult}(a_i)$, $1 \le i \le \ell$, is the multiplicity of the zero a_i. Then, by (195) and Lemma V.3.6(b), we have the following representation for $K_w(z,\zeta)$ in terms of any f, g, t, and s satisfying (200):

$$\begin{aligned} K_w(z,\zeta) = {} & \frac{f(z)\overline{g(\zeta)}}{\pi\left[1 - t(z)\overline{s(\zeta)}\right]^2} \times \frac{\prod_{i=1}^{\ell}\left[t(z)-t(a_i)\right]^{v_i}\left[\overline{s(\zeta)} - \overline{s(a_i)}\right]^{v_i}}{h(z)\overline{h(\zeta)}\prod_{i=1}^{\ell}(z-a_i)^{v_i}(\overline{\zeta}-\overline{a}_i)^{v_i}} \\ & \times \frac{Q_w\left(t(z), \overline{s(\zeta)}\right)}{\prod_{i=1}^{\ell}\left[1 - t(a_i)\overline{s(\zeta)}\right]^{v_i+1}\left[1 - t(z)\overline{s(a_i)}\right]^{v_i+1}}, \end{aligned} \tag{235}$$

where $Q_w(\tau, \xi)$ is a polynomial in two variables (that depends on the choice of t and s) with the properties stated in Lemma V.3.6(b)(i)-(iv).

We first prove that

(I) *$K_w(\cdot,\zeta)$ is a meromorphic function in $\mathbb{C}$ such that $h(\cdot)K_w(\cdot,\zeta)$ is analytic in $\overline{G}$ and, for all but finitely many $\zeta \in G$, i and $-i$ are zeros of $h(\cdot)K_w(\cdot,\zeta)$ of multiplicity $N-1$.*

Let φ be a conformal map of G onto $\mathbb{D}$. By (194), we can set $f(z) = \varphi'(z)$, $g(\zeta) = \varphi'(\zeta)$, $t(z) = \varphi(z)$, and $s(\zeta) = \varphi(\zeta)$. Then, since φ is a rational function that has an analytic continuation (also denoted by φ) across ∂G, we see from (235) with the above choice of f, g, t, and s, that for all $\zeta \in G$, $K_w(\cdot,\zeta)$ is a meromorphic function in $\mathbb{C}$. On the other hand, since $|\varphi(\pm i)| = 1$, we have

$$\varphi(\pm i)^{-1} \notin \left\{1/\varphi(a_1), \ldots, 1/\varphi(a_\ell), \overline{\varphi(a_1)}, \ldots, \overline{\varphi(a_\ell)}\right\},$$

so that by Lemma V.3.6(b)(ii), $Q_w(\varphi(\pm i),\cdot) \not\equiv 0$. Thus, it follows from (235) that for all but finitely many $\zeta \in G$, $\pm i$ is a zero of $K_w(\cdot,\zeta)$ if and only if $\pm i$ is a zero of $K(\cdot,\zeta)$, so that the rest of (I) follows from (234).

Now, we see from (234) that for all $z \neq i$, $K(z,\zeta)$ can be expressed in the form of (200), this time with the choice of functions

$$f(z) = \frac{(z+i)^{N-1}}{(z-i)^{N+1}}, \quad \overline{g(\zeta)} = -\frac{4N^2 e^{-2N\alpha i}(\overline{\zeta}+i)^{N-1}}{(\overline{\zeta}-i)^{N+1}},$$

$$t(z) = \left(\frac{z+i}{z-i}\right)^N, \quad \overline{s(\zeta)} = \left(\frac{\overline{\zeta}+i}{\overline{\zeta}-i}\cdot e^{-2\alpha i}\right)^N. \tag{236}$$

Then, looking at the denominator of (235), we see that the possible poles of $K_w(\cdot,\zeta)$ are contained in the set

$$\mathcal{S} := \{b_1, b_2, \ldots\} \cup \mathcal{S}(\zeta) \cup \mathcal{S}(a_1) \cup \cdots \cup \mathcal{S}(a_\ell),$$

where $\mathcal{S}(\zeta)$, $\mathcal{S}(a_i)$ denote, respectively, the solution sets of the equations in the variable z

$$1 - t(z)\overline{s(\zeta)} = 0, \quad 1 - t(z)\overline{s(a_i)} = 0, \quad i = 1, \ldots, \ell; \tag{237}$$

Next, we show that

(II) *for all but countably many $\zeta \in G$, $z \in \mathcal{S}$ is* **not** *a pole of $K_w(\cdot, \zeta)$ if and only if there exists $1 \leq k < \infty$ such that $z = b_k$ and either one of the following statements holds:*

(II′) $b_k \in \{-i, i\}$ *and* $\mathrm{mult}(b_k) \leq N - 1$*;*

(II″) $b_k = c$ *for some c that is a zero of the rational function*

$$\prod_{i=1}^{\ell} [t(z) - t(a_i)]^{v_i}$$

with multiplicity $\geq \mathrm{mult}(b_k)$.

Suppose that z is a solution of any of the equations (237) that is not a pole of $K_w(\cdot, \zeta)$. Since obviously $z \neq \pm i$, we have that at least one of the equations

$$t(z) - t(a_i) = 0, \quad 1 \leq i \leq \ell\,, \tag{238}$$

$$Q_w\left(1/\overline{s(\zeta)}, \overline{s(\zeta)}\right) = 0, \quad Q_w\left(1/\overline{s(a_i)}, \overline{s(\zeta)}\right) = 0, \quad 1 \leq i \leq \ell\,, \tag{239}$$

must be satisfied. But for all $\eta, \lambda \in G$, $1 - t(\eta)\overline{s(\lambda)} \neq 0$, so that z cannot satisfy any of the equations (238). Also, since Q_w is a polynomial and $s(\zeta)$ is given by (236), Lemma V.3.6(b)(ii)(iii) implies that only a finite number of $\zeta \in G$ can be a solution to one of the equations (239). Hence, for all but finitely many $\zeta \in G$, every element of $\mathcal{S}(\zeta) \cup \mathcal{S}(a_1) \cup \cdots \cup \mathcal{S}(a_\ell)$ is a pole of $K_w(\cdot, \zeta)$.

Now, it follows from (I) that for all but finitely many $\zeta \in G$, $b_k \in \{-i, i\}$ is a pole of $K_w(\cdot, \zeta)$ if and only if $\mathrm{mult}(b_k) > N - 1$. On the other hand, for any $b \in$

$\mathbb{C} \setminus (G \cup \{i, -i\})$, the polynomial $Q_w(t(b), \cdot)$ is not identically zero (this is guaranteed by Lemma V.3.6(b)(i) if $t(b) = t(a_i)$ for some $1 \leq i \leq \ell$, by (iii) if $t(b) = 1/\overline{s(a_i)}$ for some $1 \leq i \leq \ell$, and by (ii) if $t(b)$ is otherwise). Thus, for all but finitely many $\zeta \in G$, the zero $b_k \in \mathbb{C} \setminus (G \cup \{i, -i\})$ of $h(z)$ is a pole of $K_w(\cdot, \zeta)$ unless $b_k = c$ for some c that is a solution to

$$\prod_{i=1}^{\ell} [t(z) - t(a_i)]^{v_i} = 0\,,$$

of multiplicity $\geq \mathrm{mult}(b_k)$. This completes the proof of (II).

Notice that according to the definition in (213) and (214),

$$1/\rho(K_w(\cdot, \zeta)) = \max\{|1/\Phi(z)| : z \text{ is a pole of } K_w(\cdot, \zeta)\}\ . \tag{240}$$

Now, it easily follows from (186) and (185) that for any $\zeta \in G$,

$$\max\{|1/\Phi(\zeta_\alpha)|, |1/\Phi(\zeta_\beta)|\} = \begin{cases} |1/\Phi(\zeta_\alpha)| \text{ if } \zeta \in G_\alpha \cup \Gamma \\ |1/\Phi(\zeta_\beta)| \text{ if } \zeta \in G_\beta \end{cases} = |\widehat{\Phi}(\zeta)| \tag{241}$$

and it is not difficult to verify by using the explicit expressions of t and s in (236) that

$$\mathcal{S}(\zeta) = \left\{ \frac{\overline{\zeta} \cot(\alpha - k\pi/N) + 1}{\overline{\zeta} - \cot(\alpha - k\pi/N)}\,,\ 1 \leq k \leq N \right\}\ .$$

In particular, $\zeta_\alpha, \zeta_\beta \in \mathcal{S}(\zeta)$ (cases $k = N$ and $k = 1$, respectively).

Suppose we have proven that

$$\max\{|1/\Phi(\eta)| : \eta \in \mathcal{S}(\zeta)\} = \max\{|1/\Phi(\zeta_\alpha)|, |1/\Phi(\zeta_\beta)|\}\ . \tag{242}$$

Then by (240), (II), (242) and (241), we get that for all but finitely many $\zeta \in G$,

$$1/\rho(K_w(\cdot, \zeta)) = \max\left\{|\widehat{\Phi}(\zeta)|, |\widehat{\Phi}(a_1)|, \ldots, |\widehat{\Phi}(a_\ell)|, |\Phi(b_1)|^{-1}, |\Phi(b_2)|^{-1}, \cdots\right\} \setminus \mathcal{B},$$

where $\mathcal{B} := \left\{|\Phi(b_k)|^{-1} : b_k \text{ satisfies either (II}'\text{) or (II}''\text{)}\right\}$. We will show, however, that

$$t(c) - t(a_i) = 0 \Rightarrow |\Phi(c)|^{-1} \leq |\Phi(a_i)|^{-1}, \tag{243}$$

and therefore (recall (187)), $1/\rho(K_w(\cdot,\zeta)) = \max\left\{|\widehat{\Phi}(\zeta)|, r\right\}$, where r is the largest number of the set

$$\left\{R_\Gamma, |\widehat{\Phi}(a_1)|, \ldots, |\widehat{\Phi}(a_\ell)|, |\Phi(b_1)|^{-1}, |\Phi(b_2)|^{-1}, \cdots\right\}$$
$$\setminus \left\{|\Phi(b_k)|^{-1} : b_k \in \{-i, i\} \text{ and } \operatorname{mult}(b_k) \leq N-1\right\}.$$

Then, the validity of relation (189) follows as a consequence of Corollary V.4.2 and the definition of γ_r in (188).

The above argument assumes that (242) and (243) were true. Let us verify that this is the case.

With G the lens-shaped domain described in the paragraph preceding Theorem V.2.7, the normalized exterior mapping $w = \Phi(z)$ is given by the composition of the following three transformations:

$$\xi(z) := e^{(\pi-\beta)i}\left(\frac{z-i}{z+i}\right), \tag{244}$$

$$t(\xi) = \xi^{N/(2N-1)}, \quad \arg\xi \in \left(-\tfrac{\pi}{N}, \tfrac{(2N-1)\pi}{N}\right), \tag{245}$$

$$w(t) := \frac{1-\lambda_\beta t}{t-\lambda_\beta}, \quad \lambda_\beta := e^{\frac{N(\pi-\beta)i}{2N-1}}. \tag{246}$$

Let us prove (242). If $\eta \in \mathcal{S}(\zeta)$, then by definition, $1 - t(\eta)\overline{s(\zeta)} = 0$ where t and s are given by (236). Hence, for some $1 \leq k \leq N$,

$$\frac{\eta - i}{\eta + i} = \frac{\overline{\zeta} + i}{\overline{\zeta} - i} \cdot e^{(2\pi k/N - 2\alpha)i},$$

so that $(\eta - i)/(\eta + i)$ lies on the circular arc

$$\mathcal{C}_\zeta := \left\{ \left| \frac{\overline{\zeta} + i}{\overline{\zeta} - i} \right| e^{i\theta} \, : \, \arg\left(\frac{\overline{\zeta} + i}{\overline{\zeta} - i} \right) + 2\beta \leq \theta \leq \arg\left(\frac{\overline{\zeta} + i}{\overline{\zeta} - i} \right) + 2\pi - 2\alpha \right\} .$$

Notice that the endpoints of $\mathcal{C}_\zeta$ correspond to the values $\eta = \zeta_\beta$, $\eta = \zeta_\alpha$. By (244), (245) and (246), $\{\Phi(\eta) : \eta \in \mathcal{S}(\zeta)\}$ is contained in the set

$$\mathcal{C}^*_\zeta := \left\{ (w \circ t)(\xi) : |\xi| = |(\overline{\zeta} + i)/(\overline{\zeta} - i)| \right\} ,$$

which is obviously a circle intersecting the unit circle at two points. Indeed,

$$\{\Phi(\eta) : \eta \in \mathcal{S}(\zeta)\}$$

is contained in the subarc $(w \circ t)\left(e^{(\pi-\beta)i}\mathcal{C}_\zeta\right)$ of $\mathcal{C}^*_\zeta$, which lies on $\{|w| > 1\}$ and connects the points $\Phi(\zeta_\alpha)$, $\Phi(\zeta_\beta)$. Consequently, $\Phi(\zeta_\alpha)$ and $\Phi(\zeta_\beta)$ are the nearest points of $(w \circ t)\left(e^{(\pi-\beta)i}\mathcal{C}_\zeta\right)$ to the origin, whence (242) follows.

Now, to prove (243), assume that $t(c) - t(a_j) = 0$. Then for some $1 \leq k \leq N - 1$,

$$\frac{c - i}{c + i} = \overline{\left(\frac{\overline{a}_j + i}{\overline{a}_j - i} \right)} \cdot e^{2k\pi i/N} ,$$

so that $(c - i)/(c + i)$ lies on the circular arc

$$\left\{ \left| \frac{\overline{a}_j + i}{\overline{a}_j - i} \right| e^{i\theta} \, : \, 2\pi - \arg\left(\frac{\overline{a}_j + i}{\overline{a}_j - i} \right) + 2\pi/N \leq \theta \leq 4\pi - \arg\left(\frac{\overline{a}_j + i}{\overline{a}_j - i} \right) - 2\pi/N \right\} .$$

By the argument given above to prove (242), it suffices to show that this arc is a subset of $\mathcal{C}_{a_j}$. But this is a trivial fact since $\alpha + \beta = \pi/N$ and

$$\pi - \beta < \arg\left(\frac{\overline{a}_j + i}{\overline{a}_j - i} \right) < \pi + \alpha .$$

The proof is complete. $\square$

BIBLIOGRAPHY

[1] T. Carleman: *Über die approximation analytischer funktionen durch lineare aggregate von vorgegebenen potenzen.* Archiv. för Math. Atron. och Fysik 17 (1922), No. 9, pp. 1-30.

[2] J. H. Curtiss: *Faber polynomials and the Faber series.* Amer. Math. Monthly 78 (1971), pp. 577-596.

[3] P. J. Davis: The Schwarz Function and Its Applications. The Mathematical Association of America, Buffalo, N. Y. 1974. The Carus Mathematical Monographs, No. 17.

[4] G. Freud: Orthogonal Polynomials. Akadémiai Kiadó, Budapest, 1971.

[5] D. Gaier: Lectures on Complex Approximation. Boston: Birkhäuser, 1987. Translated from German by Renate McLaughlin.

[6] D. Gaier: *On the decrease of Faber polynomials in domains with piecewise analytic boundary.* Analysis 21 (2001), 219-229.

[7] Ya. L. Geronimus: Orthogonal Polynomials: Estimates, Asymptotic Formulas, and Series of Polynomials Orthogonal on the Unit Circle and on an Interval. Consultants Bureau, New York, 1961.

[8] L. I. Hedberg: *Weighted mean approximation in Carathéodory regions.* Math. Scand. 23 (1968), 113-122.

[9] L. I. Hedberg: *Weighted mean square approximation in plane regions and generators of an algebra of analytic functions.* Ark. Mat. 5 (1965), 541-552.

[10] E. R. Johnston: A Study in Polynomial Approximation in the Complex Domain. Doctoral Dissertation, University of Minnesota, 1954.

[11] P. P. Korovkin: *The asymptotic representation of polynomials orthogonal over a region.* Dokl. Akad. Nauk. SSSR 58 (1947), 1883-1885 (in Russian).

[12] A. B. J. Kuijlaars and E. B. Saff: *Asymptotic distribution of the zeros of Faber polynomials.* Math. Proc. Camb. Phil. Soc. 118 (1995), 437-447.

[13] R. S. Lehman: *Development of the mapping function at an analytic corner.* Pacific J. Math. 7 (1957), 1437-1449.

[14] A. L. Levin, E. B. Saff and N. S. Stylianopoulos: *Zero distribution of Bergman orthogonal polynomials for certain planar domains.* Const. Approx. 19 (2003), 411-435.

[15] A. Martínez-Finkelshtein, K. T.-R. McLaughlin and E. B. Saff: *Szegő orthogo-*

nal polynomials with respect to an analytic weight: canonical representation and strong asymptotics. Const. Approx., 2006, in press.

[16] H. N. Mhaskar and E. B. Saff: *On the distribution of zeros of polynomials orthogonal on the unit circle.* J. Approx. Theory 63 (1990), No. 1, 30-38.

[17] V. Maymeskul and E. B. Saff: *Zeros of polynomials orthogonal over regular N-gons.* J. Approx. Theory 122 (2003), No. 1, 129-140.

[18] S. N. Mergelyan: *On the completeness of systems of analytic functions.* Amer. Math. Soc. Transl. 19 (1962), No. 2, 109-166.

[19] E. Miña-Díaz, E. B. Saff and N. S. Stylianopoulos: *Zero distributions for polynomials orthogonal with weights over certain planar regions.* CMFT 5 (2005), No. 1, 185-221.

[20] Z. Nehari: Conformal Mapping. McGraw-Hill Book Company, Inc., 1952.

[21] N. Papamichael, E. B. Saff and J. Gong: *Asymptotic behaviour of zeros of Bieberbach polynomials.* J. Comput. Appl. Math. 34 (1991), 325-342.

[22] Ch. Pommerenke: Boundary Behavior of Conformal Maps. Springer-Verlag Berlin Heidelberg New York, 1992.

[23] I. E. Pritsker: *On the local asymptotics of Faber polynomials.* Proc. Amer. Math. Soc. 127 (1999), 2953-2960.

[24] T. Ransford: Potential Theory in the Complex Plane. London Math. Soc. Student Texts 28, Cambridge University Press, 1995.

[25] E. B. Saff: *Orthogonal polynomials from a complex perspective.* In: P. Nevai. Ed., Orthogonal Polynomials: Theory and Practice (Kluwer, Dordrecht, 1990), 363-393.

[26] E. B. Saff, V. Totik: Logarithmic Potentials with External Fields. Berlin: Springer-Verlag, 1997.

[27] B. Simon: Orthogonal Polynomials on the Unit Circle I and II. AMS Colloq. Series. Amer. Math. Soc., Providence, RI, 2005.

[28] H. Stahl, V. Totik: General Orthogonal Polynomials. Cambridge: Cambridge University Press, 1992.

[29] P. K. Suetin: Polynomials Orthogonal over a Region and Bieberbach Polynomials. Proceedings of the Steklov Institute of Mathematics, Amer. Math. Soc., Providence, Rhode Island, 1975.

[30] P. K. Suetin: Series of Faber polynomials. Gordon and Breach Science Publications, Amsterdam, 1998.

[31] G. Szegő: Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ. 23. Amer. Math. Soc., Providence, RI, fourth edition, 1975.

[32] J. L. Ullman: *Studies in Faber polynomials I.* Trans. Amer. Math. Soc. 73 (1960), 515-528.

[33] J. L. Ullman: Studies on Faber polynomials. Doctoral Dissertation, Stanford University, 1960.

[34] J. L. Walsh: Interpolation and Approximation by Rational Functions in the Complex Domain. Amer. Math. Soc. Colloq. Publ. 20 (Amer. Math. Soc., Providence, RI, 5th ed., 1969.)

[35] J. L. Walsh: Selected Papers. T. J. Rivlin and E. B. Saff Editors. Springer-Verlag, 1991.

[36] J. L. Walsh: *Overconvergence, degree of convergence, and zeros of sequences of analytic functions.* Duke Math. J. 13 (1946), 195-234. Also in [35], pp. 545-584.

[37] H. Widom: *Extremal polynomials associated with a system of curves in the complex plane.* Adv. Math. 3 (1969), 127-232.