

E-THEORY FOR L^p ALGEBRAS AND THE DUAL NOVIKOV CONJECTURE

By

Fan Fei Chong

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Approved:

Ph.D. Committee Members:

Professor Gennadi Kasparov (Chair)

Professor Guoliang Yu

Professor Ioana Suvaina

Professor Kalman Varga

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CHAPTER I

INTRODUCTION

Kasparov [Kas81] invented KK -theory, a bivariant K -theory for C^* -algebra which generalizes Hirzebruch-Atiyah's topological K -theory and its dual theory (K -homology), at the beginning of 1980s. Since its introduction, the theory has found deep applications in geometry and topology, including index theorems, homotopy invariance of higher signatures (e.g. Novikov conjecture), problem of positive scalar curvature and the Baum-Connes Conjecture[Bla98, Section 24].

As of today, there are various variants of the bivariant K -theory available, each invented with its application in mind and therefore comes with its limitation. In the study of Baum-Connes Conjecture (which is a priori a result concerning K -theory of reduced group C^* -algebras), it was discovered that it may be advantageous to cast the problem in a slightly more general setting, i.e. to consider not only unitary representations but also isometric representations in Banach spaces. This moves us from the realm of C^* -algebras to the more general realm of Banach algebras. Lafforgue [Laf02] introduced Banach KK -theory in his 2002 paper and got good results. Nevertheless, the generality of his theory comes at a price - it is lacking some essential features of bivariant K -theory, namely product and exact sequences.

In this paper, we develop a bivariant theory for Banach L^p -algebras. Our theory is closely related to Connes-Higson's E -theory (which was defined for C^* -algebras), but also contains construction specifically designed to overcome challenges in this more general setting. (C^* -algebra can be considered as algebra of operators on Hilbert space, hence is a L^2 -algebra). We showed that our theory has product, Bott periodicity, six-term and Mayer-Vietoris exact sequences. We also proved a result related to Novikov Conjecture with our theory. The approach to the Novikov conjecture that we use has its origin in two joint works of Kasparov and Yu [KY05; KY12]. In [KY12], the Strong Novikov conjecture was proved for groups acting on L^p -spaces. What we proved in this paper is for (Dual) L^p Novikov conjecture - a related but different conjecture. The relation will be explained in greater detail in the last section of this paper.

CHAPTER II

PRELIMINARIES

II.1 Banach space theory

The first concept we would like to recall is that of *Fredholm operator* and *Fredholm Index*. It is a great example of the interaction between operator theory and algebraic topology. We will follow the notation in [Pal65].

Definition II.1.1. If X and Y are Banach space, an element T of $L(X, Y)$ (Banach algebra of bounded linear operator from X to Y) is called a *Fredholm operator* from X to Y if

1. $\ker T = T^{-1}(0)$ is finite dimensional;
2. $\operatorname{coker} T = Y/T(X)$ is finite dimensional.

We denote the set of Fredholm operators from X to Y by $F(X, Y)$ and define the *index function*:

$$\operatorname{ind} : F(X, Y) \rightarrow \mathbb{Z}$$

by $\operatorname{ind}(T) = \dim \ker T - \dim \operatorname{coker} T$.

Theorem II.1.2 (Atkinson's Theorem). [Pal65, Thm 2] *Let X and Y be Banach spaces, $T \in L(X, Y)$ is a Fredholm operator if and only if T is invertible modulo compact operator.*

More precisely, if there exists $S, S' \in L(Y, X)$ such that $ST - I_X \in \mathbb{K}(X, X)$ and $TS' - I_Y \in \mathbb{K}(Y, Y)$ then $T \in F(X, Y)$. Conversely, if $T \in F(X, Y)$, there is in fact $S \in L(Y, X)$ such that $ST - I_X$ and $TS - I_Y$ have finite rank, and hence belong to $\mathbb{K}(X, X)$ and $\mathbb{K}(Y, Y)$ respectively. (compact operators are by definition norm limits of finite rank operators)

A corollary of the above characterization of Fredholm operator is the following:

Corollary II.1.3. *If $T \in F(X, Y)$ and $k \in \mathbb{K}(X, Y)$, then $T + k \in F(X, Y)$. If we have additionally that $S \in F(Y, Z)$, then $ST \in F(X, Z)$.*

What gives Fredholm operator an algebraic topology taste is the following result about the stability of Fredholm index:

Theorem II.1.4. [Pal65, Thm 4] *$\operatorname{ind}(T + k) = \operatorname{ind}(T)$ for any $T \in F(X, Y)$ and $k \in \mathbb{K}(X, Y)$. We have also that $\operatorname{ind} : F(X, Y) \rightarrow \mathbb{Z}$ is constant on a continuous path of in $F(X, Y)$ and hence is constant on each component of $F(X, Y)$.*

The following classical result from functional analysis will be useful for proving the boundedness of infinite sum of projections in the construction of quasicentral approximate unit for \mathbb{K} in the section of Bott Periodicity.

Theorem II.1.5 (Uniform Boundedness Principle). *Let X be a Banach space and Y be a normed vector space. Suppose that F is a collection of continuous linear operators from X to Y . If for all x in X one has $\sup_{T \in F} \|T(x)\|_Y < \infty$ then $\sup_{T \in F} \|T\|_{B(X,Y)} < \infty$.*

Our construction of algebra $\mathcal{A}(X)$ in Section IV.4 requires that the Banach space X possess an increasing sequence of finite dimensional subspaces with some good property. If X is a Hilbert space, we will just take the linear span of the first n elements of the orthonormal basis. In general Banach spaces, it is not so easy to determine such a sequence. Here we would like to introduce the notion of *Schauder Basis* for Banach space.

Definition II.1.6. [Sch27] Let V denote a Banach space over the field F . A *Schauder basis* is a sequence $\{e_n\}$ of elements of V such that for every element $v \in V$ there exists a unique sequence $\{\alpha_n\}$ of scalars in F so that

$$v = \sum_{n=0}^{\infty} \alpha_n e_n$$

where the convergence is understood with respect to the norm topology. A Schauder basis $\{e_n\}$ is *unconditional* if there exists a constant C such that

$$\left\| \sum_{k=0}^n \varepsilon_k \alpha_k e_k \right\|_V \leq C \left\| \sum_{k=0}^n \alpha_k b_k \right\|_V$$

for all integer n , all scalar coefficients $\{\alpha_k\}$ and all signs $\varepsilon_k = \pm 1$.

Not all Banach space has a Schauder basis, but most of the classical spaces do. The well-known example of $L^p(0, 1)$ for $p \in (1, \infty)$ has an unconditional (Schauder) basis [Sch] consisting of Haar functions defined below:

Definition II.1.7. Let $T = \{(n, j) : n \in \mathbb{N}_0, j = 0, 1, \dots, 2^n - 1\} \cup \{0\}$. We will define the Haar basis $(h_t)_{t \in T}$ and the normalized Haar basis $(h_t^{(p)})_{t \in T}$ in $L^p[0, 1]$ as follows:

1. $h_0 = h_0^{(p)} \equiv 1$ on $[0, 1]$
2. For $n \in \mathbb{N}_0$ and $j = 0, 1, 2, \dots, 2^n - 1$, $h_{(n,j)}$ is defined as subtracting the characteristic function of $[(j + \frac{1}{2})2^{-n}, (j + 1)2^{-n})$ from that of $[j2^{-n}, (j + \frac{1}{2})2^{-n})$
3. $h_{(n,j)}^{(p)} = 2^{n/p} h_{(n,j)}$

II.2 K-Theory

There are at least three different types of K-theory: topological, operator algebraic, algebraic. We will be most interested in the second type in this paper. It can be viewed as some sort of 'homology' theory on the category of (local) Banach algebra. K-theory for Operator Algebra has its root in the topological K-theory (a cohomological theory on the space) by Atiyah-Hirzebruch. Let us first recall the central definition of latter.

If X is a compact Hausdorff space, then $K^0(X)$ is the abelian group generated by the isomorphism classes of complex vector bundles over X , subject to the relations:

$$[E] + [F] = [E \oplus F]$$

for vector bundles E and F . A (finite-dimensional) complex vector bundles over X may be described as the range of a continuous projection-valued map $p : X \rightarrow M_n(\mathbb{C})$ for some n sufficiently large. ($p(x)^2 = p(x) \quad \forall x \in X$) By a standard classification theorem, in the limit as $n \rightarrow \infty$, there is a one-to-one correspondence between isomorphism classes of vector bundles over X and homotopy class of projection-valued functions. So, $K^0(X)$ may be reformulated as the abelian group generated by homotopy classes of maps from X to the space of projections in $M_n(\mathbb{C})$, as n runs over the natural numbers. Since a continuous map from X to the space of projections in $M_n(\mathbb{C})$ is equivalent to a projection in the Banach algebra $M_n(C(X))$ of $n \times n$ matrices over $C(X)$. It is then natural to introduce the following definition of K-theory for unital Banach algebra.

Definition II.2.1. [HR01, Def 4.1.1] Let A be a unital Banach algebra. Denote by $K_0(A)$ the abelian group with one generator, $[p]$, for each projection p in each matrix algebra $M_n(A)$, and the following relations:

- (a) if both p and q are projections in $M_n(A)$, for some n , and if p and q are joined by a continuous path of projections in $M_n(A)$, then $[p] = [q]$,
- (b) $[0] = 0$, for any size of (square) zero matrix, and
- (c) $[p] + [q] = [p \oplus q]$, for any sizes of projection matrices p and q

$K_0(A)$ has the following properties: [Bla98, Sec 5.2, 5.6]

1. **FUNCTORIALITY:** If $\phi : A \rightarrow B$ is a continuous homomorphism of Banach algebras, then ϕ induces a map $\phi_* : K_0(A) \rightarrow K_0(B)$ by applying ϕ coordinate-wise to elements in $M_\infty(A) = \varinjlim M_n(A)$
2. **HOMOTOPY INVARIANCE:** If $\phi, \psi : A \rightarrow B$ are homotopic, then $\phi(e) \sim_h \psi(e)$ for any idempotents in $M_\infty(A)$ and hence $\phi_* = \psi_*$.
3. **DIRECT SUMS:** If $A = A_1 \oplus A_2$, then $K_0(A) = K_0(A_1) \oplus K_0(A_2)$.
4. **INDUCTIVE LIMITS:** If $A = \varinjlim (A_i, \phi_{ij})$, then $K_0(A) \cong \varinjlim K_0(A_i)$

If A is non-unital, then we define $K_0(A)$ to be the kernel of π_* induced by the map $\pi : A^+ \rightarrow \mathbb{C}$ where A^+ is the unitization of A .

The functor K_0 is not exact, but is HALF-EXACT.

Theorem II.2.2. [Bla98, Thm 5.6.1] If J is a closed two-sided ideal in A , then the sequence $K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J)$ is exact in the middle, i.e. $\ker(\pi_*) = \text{im}(i_*)$.

This will later allow us to convert a short exact sequence of Banach algebras into a long exact sequence of K-theory groups.

To obtain a homology theory on category of Banach algebras, = [we would need the higher K-groups. Let the *suspension* of A , denoted ΣA , be $C_0((0,1),A)$ the algebra of continuous functions from $(0,1)$ to A which vanishes at the endpoints. We define $K_n(A) = K_0(\Sigma^n A)$. It can be shown that $K_1(A)$ can also be defined by the group of homotopy classes of invertible elements in matrix algebra over A^+ which $= 1_n \pmod{M_n(A)}$.

We can define a connecting map $\partial : K_1(A/J) \rightarrow K_0(A)$ which makes a long exact sequence

$$K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{\pi_*} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J)$$

∂ is called the *index map* because in the special case of $A = \mathcal{B}(H), J = \mathbb{K}(H)$, and $K_0(\mathbb{K})$ is identified with \mathbb{Z} in the standard way, the map ∂ is exactly the map which sends a unitary in the Calkin algebra to its Fredholm index. We can obtain connecting maps from $K_{n+1}(A/J)$ to $K_n(J)$ for each n by suspension, and an infinite long exact sequence:

$$\cdots \xrightarrow{\partial} K_n(J) \xrightarrow{i_*} K_n(A) \xrightarrow{\pi_*} K_n(A/J) \xrightarrow{\partial} K_{n-1}(J) \xrightarrow{i_*} \cdots \xrightarrow{\pi_*} K_0(A/J).$$

Last but not least, we have to mention the important result of Bott Periodicity, which roughly says that only K_0 and K_1 groups matter because the higher K-groups are all duplicates of them.

Theorem II.2.3 (Bott Periodicity). *[Bla98, Thm 9.2.1] $K_0(A)$ is naturally isomorphic to $K_1(SA)$ and hence to $K_2(A)$ via the Bott homomorphism:*

$$\beta_A : K_0(A) \rightarrow K_1(SA)$$

defined by $\beta_A([e] - [p_n]) = [f_e f_{p_n}^{-1}]$ where $f_e(z) = ze + (1 - e) \in C(S^1, GL_n(A^+))$ for any idempotent $e \in M_n(A^+)$.

CHAPTER III

E-THEORY FOR L^P -ALGEBRAS

III.1 Asymptotic Morphisms

Definition III.1.1. Let A and B be Banach algebras. A *pre-asymptotic morphism* from A to B is a family $\{\phi_t\}(t \in [1, \infty))$ of maps from A to B with the following properties:

- (i) $t \rightarrow \phi_t(a)$ is continuous and norm-bounded for every $a \in A$
- (ii) The set $\{\phi_t\}$ is asymptotically linear and multiplicative:

$$\lim_{t \rightarrow \infty} \|\phi_t(a + \lambda b) - (\phi_t(a) + \lambda \phi_t(b))\| = 0$$

$$\lim_{t \rightarrow \infty} \|\phi_t(ab) - (\phi_t(a)\phi_t(b))\| = 0$$

for all $a, b \in A, \lambda \in \mathbb{C}$.

From the definition above, we see that a *pre-asymptotic morphism* from A to B defines an algebra homomorphism from A to $B_\infty = C_b([1, \infty), B)/C_0([1, \infty), B)$. Unlike in the C^* -algebra case, we do not have the map A to B_∞ automatically continuous in the Banach algebra case. For technical reason regarding composition, we will consider only *pre-asymptotic morphisms* where the induced map from A to B_∞ is continuous (and is hence a Banach algebra homomorphism). We call this type of maps *asymptotic morphisms*.

Two asymptotic morphisms $\{\phi_t\}$ and $\{\psi_t\}$ are *equivalent* if for all $a \in A$

$$\lim_{t \rightarrow \infty} \|\phi_t(a) - \psi_t(a)\| = 0$$

A *homotopy* between asymptotic morphisms $\{\phi_t^{(0)}\}$ and $\{\phi_t^{(1)}\}$ from A to B is an asymptotic morphism $\{\phi_t\}$ from A to $C([0, 1], B)$ such that the evaluations of $\phi_t(a)$ at 0 and 1 are equal to $\phi_t^{(0)}(a)$ and $\phi_t^{(1)}(a)$ respectively for all a and t .

Denote the set of homotopy classes of asymptotic morphisms from A to B by $[[A, B]]$

Two asymptotic morphisms define the same Banach algebra homomorphism if and only if they are equivalent. Conversely, if $\phi : A \rightarrow B_\infty$ is a Banach algebra homomorphism, any set-theoretic cross section for ϕ is an asymptotic morphism from A to B , and any two such are equivalent.

By *Bartle-Graves Selection Theorem* [Micr, Prop 7.2], we get a continuous section¹ to the continuous, onto quotient map from $C_b([1, \infty), B)$ to B_∞ . Composing this with the Banach algebra homomorphism defined by any asymptotic morphism, we obtain a continuous map A to $C_b([1, \infty), B)$. This gives an asymptotic morphism between A and B . An asymptotic morphism coming from a continuous map from A to $C_b([1, \infty), B)$ is called *uniform* in the literature ([Bla98, p. 25.1.5c], [Con94, Appendix B, β]). The above

¹not necessarily linear, but it can be chosen to be homogeneous of degree 1, i.e. $f(\alpha x) = \alpha f(x)$ for all scalars α . This fact is important for computation in the Long Exact Sequence section.

argument shows that every asymptotic morphism is equivalent to a *uniform* asymptotic morphism. This is an important property that is needed in the definition of composition of asymptotic morphisms.

III.2 Tensor Products and Suspensions

We will be mostly interested in L^p -algebras in this work. A is an L^p -algebra means that A is closed subalgebra of $\mathcal{L}(L^p(Z, \mu))$, the Banach algebra of all bounded linear operators on $L^p(Z, \mu)$, Banach space of all p -integrable functions on Z .

Thanks to the following special case of a result of Beckner[Bec75], we may define the L^p -tensor product for two L^p -algebras. Let $S_i \in \mathcal{L}(L^p(Z_i, \mu_i))$. For $f_i \in L^p(Z_i, \mu_i)$, define $S_1 \otimes S_2(f_1 \otimes f_2) := S_1(f_1) \otimes S_2(f_2)$, we have:

Proposition III.2.1. $\|S_1 \otimes S_2 : L^p(Z_1, \mu_1) \otimes L^p(Z_2, \mu_2) \rightarrow L^p(Z_1, \mu_1) \otimes L^p(Z_2, \mu_2)\| = \|S_1\| \|S_2\|$

This in particular means that the algebraic tensor product of two L^p -algebras (A_i) may be identified as a subalgebra of $\mathcal{L}(L^p(Z_1 \times Z_2, \mu_1 \times \mu_2))$. Take completion with respect to this norm and we obtain the L^p -tensor products of L^p -algebras.

Unlike in the C^* -algebra case, where all cross norms on tensor products of two C^* -algebras with one of them nuclear coincides. This is in general not true for Banach algebras or L^p -algebras. We choose the L^p -tensor product to make our bounded estimate for "connecting asymptotic morphism" work. There is however a positive result that will be useful later:

Lemma III.2.2. $A \subset \mathcal{L}(L^p(Z))$ is a L^p -algebra for σ -finite Z , then $\Sigma \otimes A \subset \mathcal{L}(L^p((0, 1) \times Z))$ is isometrically isomorphic to $\Sigma A = C_0((0, 1), A)$.

Sketch of proof. We will use the elementary result that for Y, Z σ -finite measure space, we have $L^p(Y \times Z) \cong L^p(Y, L^p(Z))$. Noticing that the representation of $C_0(Y)$ and of $\mathcal{L}(L^p(Z))$ commute with each other, we may then apply Lemma III.6.3 and obtain the said result. \square

There is a tensor product functor on the homotopy category of asymptotic morphisms which associate to the asymptotic morphisms ϕ_i (from A_i to B_i , $i = 1, 2$) a tensor product asymptotic morphism from $A_1 \otimes A_2$ to $B_1 \otimes B_2$ where $\phi_1 \otimes \phi_2(a_1 \otimes a_2) = \phi_1(a_1) \otimes \phi_2(a_2)$. A reference of this may be found in [Bla98, p. 25.2].

In particular, if we denote by Σ the algebra $C_0(0, 1)$, then for any ϕ , an asymptotic morphism from A to B , there is a well-defined suspension $\Sigma\phi = id_\Sigma \otimes \phi$ from $\Sigma A = \Sigma \otimes A$ to $\Sigma B = \Sigma \otimes B$

III.3 Composition

We will define the composition of asymptotic morphisms. The key result here is essentially the same as in [Bla98, p. 25.3.1]. The analogous statement for Banach algebra is posted below for the convenience of the reader:

Theorem III.3.1. *a) A, B, C are separable Banach algebras, and $\{\phi_t\}, \{\psi_t\}$ are uniform asymptotic morphisms from A to B and B to C respectively, then for any increasing $[1, \infty)$ to $[1, \infty)$ function r growing*

sufficiently quickly the family $\{\psi_{r(t)} \circ \phi_t\}$ is an asymptotic morphism from A to C .

b) The resulting asymptotic morphism depends up to homotopy only on the homotopy class of $\{\phi_t\}$ and $\{\psi_t\}$, and thus defines a "composition" $[[A, B]] \times [[B, C]] \rightarrow [[A, C]]$.

c) Composition is associative, commutes with tensor products, and agrees with ordinary composition for homomorphisms.

The bulk of the argument in the proof for this result translates just fine to the Banach algebra case. Nevertheless, there are also two major differences.

The first one is that the automatic norm-decreasing property for *-homomorphisms between C^* -algebras got replaced by requiring the induced homomorphism between Banach algebras to be continuous. We will define some notations, state two elementary lemmas and give an example how the new requirement may be used in the proof of this result.

Notation 1. 1. Unless otherwise specified, A and B are always separable Banach algebras. Let B_∞ denote the quotient algebra $C_b([1, \infty), B)/C_0([1, \infty), B)$.

2. Let $f \in C_b([1, \infty), B)$. We will denote the norm of f in $C_b([1, \infty), B)$ and the norm (of its equivalence class) in B_∞ by $\|f\|_b$ and $\|f\|_\infty$ respectively.

3. Let $\{\phi_t\}$ be an asymptotic morphism from A to B . We will denote the induced map from A to $C_b([1, \infty), B)$ by $\tilde{\phi}$ ². The induced Banach algebra homomorphism from A to B_∞ will be denoted by ϕ .

With the notations defined, we will introduce two lemmas which are useful in understanding the proof for Theorem III.3.1.

Lemma III.3.2. *Let $f \in C_b([1, \infty), B)$, then $\|f\|_\infty = \limsup_{t \rightarrow \infty} \|f(t)\|$*

The proof amounts to an elementary argument examining the definition for norm of quotient space and limsup and realizing both as infimum. The second lemma deals with the property of uniform asymptotic morphisms. Notice that, norm of the induced Banach algebra homomorphism $\|\phi\|$ is involved in the third property below.

Lemma III.3.3. *Let $\{\phi_t\} : A \rightarrow B$ be a uniform asymptotic morphism, i.e. the induced map $\tilde{\phi} : A \rightarrow C_b([1, \infty), B)$ is continuous. Then for any K , compact subset of A , given any $\varepsilon > 0$, there exists $t_0 > 0$ such that*

$$1. \|\phi_t(x + \lambda y) - \phi_t(x) - \lambda \phi_t(y)\| < \varepsilon$$

$$2. \|\phi_t(xy) - \phi_t(x)\phi_t(y)\| < \varepsilon$$

$$3. \|\phi_t(x)\| < \|\phi\| \cdot \|x\| + \varepsilon$$

for all $x, y \in K$, $t > t_0$ and $|\lambda| \leq 1$

²Note that $\tilde{\phi}$ is not necessarily continuous. When it is, we call $\{\phi_t\}$ a *uniform* asymptotic morphism.

Proof for property 3 uses Lemma III.3.2, continuity of ϕ and $\tilde{\phi}$ and a standard argument involving taking finite subcover of an open cover of the compact set K . Proof for the other two properties are similar, just that the fact $\{\phi_t\}$ being an asymptotic morphism is used in place of Lemma III.3.2.

The full argument may be found in [Bla98, p. 25.3.1]. It starts with first choosing a dense σ -compact subalgebra of A (e.g. the polynomials in a countable generating set). Write $A_0 = \cup K_n$, where K_n is compact, $K_n + K_n \subset K_{n+1}$, $K_n K_n \subset K_{n+1}$, $\lambda K_n \subset K_{n+1}$ for $|\lambda| \leq n$. Inductively choose $t_n \geq t_{n-1}$, $t_n \geq n$, such that ϕ_t satisfies the conditions of the previous lemma with $\varepsilon = 1/n$ and $K = K_n$ for all $t \geq t_n$. Let $K'_n = \{\phi_t(a) : a \in K_n, t \leq t_{n+1}\}$. K'_n is a compact subset of B . Let $K''_1 = K'_1$, and inductively let $K''_{n+1} = K'_{n+1} \cup (K''_n + K''_n) \cup (K''_n K''_n) \cup (\lambda K''_n : |\lambda| \leq n)$. Choose r_n such that ψ_t satisfies the conditions of the previous lemma with $\varepsilon = 1/n$ and $K = K''_{n+2}$. Then let $r(t)$ be any increasing function with $r(t_n) \geq r_n$ for all n . We provide here a proof for the boundedness of the composition $\{\psi_{r(t)} \circ \phi_t(x)\}$ for a fixed $x \in A_0$, which was not needed in the C^* -algebra case.

First, the continuity of $\{\psi_{r(t)} \circ \phi_t(x)\}$ in t is easy to see. After we have continuity, because continuous image of compact set is compact (in particular bounded), we just need to show

Claim 1. $\limsup_t \|\psi_{r(t)} \circ \phi_t(x)\| < \infty$

Proof. Take $x \in A_0$, then $x \in K_n$ for some n . Given $\varepsilon > 0$, there exists $N > n$ such that $\frac{\|\psi\|+1}{N} < \varepsilon$. By definition of $\{\phi_t\}$, we have $\|\phi_t(x)\| \leq \|\phi\| \cdot \|x\| + 1/N$ for all $t > t_N$.

For any $i \in \mathbb{N}$, when $t \in (t_{N+i-1}, t_{N+i}]$, we have $\phi_t(x) \in K'_{N+i-1} \subset K'_{N+i+1}$. By the defining property of r , we have $r(t) \geq r(t_{N+i-1}) \geq r_{N+i-1}$, therefore

$$\begin{aligned} \|\psi_{r(t)} \circ \phi_t(x)\| &\leq \|\psi\| \cdot \|\phi_t(x)\| + 1/(N+i-1) \\ &\leq \|\psi\| \cdot \|\phi\| \cdot \|x\| + \frac{\|\psi\|+1}{N} \\ &\leq \|\psi\| \cdot \|\phi\| \cdot \|x\| + \varepsilon \end{aligned}$$

□

This proof actually shows more than what was claimed. By Lemma III.3.2, we have

$$\|(\psi \circ \phi)^3(x)\|_\infty \leq \|\psi\| \cdot \|\phi\| \cdot \|x\| \text{ for } x \in A_0$$

This shows that $\{\psi_{r(t)} \circ \phi_t\}$ defines a continuous algebra homomorphism from A_0 to C_∞ with norm less than or equal to $\|\psi\| \cdot \|\phi\|$, hence extends to a Banach algebra homomorphism from A to C_∞ . This is the crucial bit needed to extend the composition of two uniform asymptotic morphisms from the dense subalgebra A_0 to the whole algebra A .

³for lack of a better notation, we are using $\psi \circ \phi$ to denote the algebra homomorphism $A \rightarrow C_\infty$ induced by $\{\psi_{r(t)} \circ \phi_t\}$ the composition of the asymptotic morphisms

III.4 Additive Structure and Definition of $E(A, B)$

In this section, we will explore two possible notions of addition for asymptotic morphisms. We will arrive at a definition of $E(A, B)$ which gives an additive structure on asymptotic morphisms. From this point onward, \mathbb{K} will always mean $\mathbb{K}(L^p(Z, \mu))$ - the Banach algebra of compact operators on the L^p space of p -summable functions on the space Z with a σ -finite measure μ .

The first notion is for $[[A, B \otimes \mathbb{K}]]$. To define it, we will first fix an isomorphism $M_2 \otimes \mathbb{K} \cong \mathbb{K}$. For example, we can take the one induced from isomorphism $L^p[0, 1] \oplus L^p[0, 1] \cong L^p[0, 1/2] \oplus L^p[1/2, 1] \cong L^p[0, 1]$ where the first isomorphism is obtained by a "domain-squeezing" map.

Then given any two asymptotic morphisms $\phi, \psi : A \rightarrow B \otimes \mathbb{K}$, we may take the orthogonal sum

$$A \rightarrow (B \otimes \mathbb{K}) \oplus (B \otimes \mathbb{K}) \hookrightarrow M_2(B \otimes \mathbb{K}) \cong B \otimes \mathbb{K}$$

The resulting sum is well-defined up to homotopy and makes $[[A, B \otimes \mathbb{K}]]$ an abelian semi-group. It is in general not a group.

The second notion is for $[[A, \Sigma B]]$. Note that the interval $(0, 1)$ can be continuously deformed to any of its open sub-interval. Hence given any two asymptotic morphisms $\phi, \psi : A \rightarrow \Sigma B$, by modifying the S part of the map alone, we may obtain a homotopy to asymptotic morphisms ϕ', ψ' "supported" on $(0, 1/2)$ and $(1/2, 1)$ respectively. A sample homotopy on the S part is as follows:

$$f(x) \in S = C_0(0, 1); \text{ then for } s \in [0, 1], f_s(x) = \begin{cases} f(x \cdot \frac{1}{0.5(s+1)}) & : x \in (0, 0.5(s+1)) \\ 0 & : \text{otherwise} \end{cases}$$

This notion of "addition" is easily seen to be associative, with proof involving homotopies similar to that in the concatenation of loops for fundamental group. $[[A, \Sigma B]]$ under this "addition" is a group, with the inverse of an asymptotic morphism ϕ is $\phi \circ (\rho \otimes id)$, where $\rho : S \rightarrow S$ is given by $(\rho(f))(s) = f(1-s)$ [Bla98, Prop 25.4.3]. However, this "addition" is in general not commutative.

So far, we have explored two notions of "addition", each lacking some essential feature for addition. With some exercise in homotopy, we can see that the two notions agree on $[[A, \Sigma B \otimes \mathbb{K}]]$ [Bla98, Prop 25.4.3]. This gives an additive structure on $[[A, \Sigma B \otimes \mathbb{K}]]$ and turns it into an abelian group. Now we want to make sure the structure we define allows for composition of asymptotic morphisms, which leads to the following definition.

Definition III.4.1. If A and B be separable Banach algebras, then $E(A, B) := [[\Sigma A \otimes \mathbb{K}, \Sigma B \otimes \mathbb{K}]]$

With this, one may obtain a composition $E(A, B) \times E(B, C) \rightarrow E(A, C)$.

In some situations, it may be more convenient to define the group $E(A, B)$ in an "asymmetric" manner: $E(A, B) = [[\Sigma A, \Sigma B \otimes \mathbb{K}]]$. There exists a natural map from this definition to ours, given by tensoring on the identity map on \mathbb{K} then composing with the isomorphism between $\mathbb{K} \otimes \mathbb{K}$ and \mathbb{K} . This is a bijection since the map from \mathbb{K} to $\mathbb{K} \otimes \mathbb{K}$ given by $x \rightarrow x \otimes e_{11}$ is homotopic to an isomorphism. [Bla98, p. 25.4.1]

III.5 Existence of Quasiceutral Approximate Unit

In the C^* -algebra case, one of the most important examples of asymptotic morphisms is the "connecting morphism" associated to a short exact sequence of C^* -algebra. We would like to emulate the construction here. However, there is a major difficulty for Banach algebra: unlike ideals in separable C^* -algebra [Voi76], ideals in separable Banach algebra do not automatically have a quasiceutral approximate unit. In this section, we will show that with a certain assumption concerning our Banach algebras, we may produce a quasiceutral approximate unit.

Let us start with recalling the definition of *quasiceutral approximate unit*.

Definition III.5.1. Let J be a closed two-sided ideal in a Banach algebra B . Then a net of elements $(h_\lambda) \subset J$ of norm $\leq C$, indexed by a directed set $\lambda \in \Lambda$ is called a *quasiceutral bounded approximate unit* in J with respect to B if:

1. $\lim \|h_\lambda y - y\| = \lim \|y h_\lambda - y\| = 0$ for all $y \in J$
2. $\lim \|h_\lambda x - x h_\lambda\| = 0$ for all $x \in B$

We will make the following assumption concerning our Banach algebras.

Definition III.5.2. A Banach algebra J is called σ -*unital* if it possesses a countable bounded approximate unit contained in a closed subalgebra isomorphic to $C_0(Y)$ for some locally compact space Y . In the case with an action of a locally compact group G on J , the subalgebra $C_0(Y)$ must be G -invariant and the G -action on $C_0(Y)$ must be induced by the G -action on Y .

In the case of C^* -algebras, this coincides with the usual definition of σ -unitality (cf. [AK69]). The following proposition shows that this assumption is exactly what we need for the existence of quasiceutral approximate unit. We will describe the explicit construction of a quasiceutral approximate unit from a given approximate unit of a σ -unital ideal.

Proposition III.5.3. *Let J be a closed two-sided ideal in a Banach algebra B . Assume that J is σ -unital and the quotient B/J is separable. Then there is a quasiceutral bounded approximate unit in J with respect to B .*

If additionally, B is acted upon in a pointwise norm-continuous manner by a locally compact, σ -compact topological group G by automorphism, the approximate unit will be asymptotically G -invariant in the sense that for any K , compact subset of G , we have:

$$\lim_{n \rightarrow \infty} \sup_{g \in K} \|g u_n - u_n\| = 0$$

Proof. According to the previous definition, let $\{v_i\}$ be a bounded approximate unit for J contained in the subalgebra $C_0(Y) \subset J$. We will consider elements v_i as functions on Y . We may assume that all v_i are compactly supported on Y .

Under these assumptions, we may choose another approximate unit for J consisting of positive functions with compact support on Y and sup-norm 1. Indeed, for any $a \in J$, if $\|v_i a - a\| < \varepsilon$ and $\text{supp}(v_i) \subset K$ (a

compact set), then for any function $w \in C_0(Y)$ which is equal to 1 on K , one has: $\|wa - a\| \leq \|w(a - v_i a)\| + \|v_i a - a\| \leq (\|w\| + 1)\varepsilon$. The existence of w is guaranteed by the complete regularity of Y , which is a consequence of its being both Hausdorff and locally compact.

Next, we choose a compact subset $D \subset B$ such that the closure of its linear span modulo J is equal to B/J . This is possible because of our assumption that B/J is separable and that any countable dense subset may be re-normalized to become a compact generating (as linear space) set with limit point at 0.

Then we define inductively a sequence of integers $\{k_i\}$ and $\{u_m\}$ our quasiceutral approximate unit. Let $k_1 = 1$. Suppose k_1, \dots, k_m are already defined, choose k_{m+1} so that for any $d \in D$ and $i \leq m$, one has:

1. $(1 - v_{k_{m+1}}) \cdot v_{k_i} = 0$,
2. $\|(1 - v_{k_{m+1}}) \cdot d \cdot v_{k_i}\| = \|(1 - v_{k_{m+1}}) \cdot (dv_{k_i})\| \leq 2^{-(m+1)}$, and
3. $\|v_{k_i} \cdot d \cdot (1 - v_{k_{m+1}})\| = \|(v_{k_i} d) \cdot (1 - v_{k_{m+1}})\| \leq 2^{-(m+1)}$

The compactness of D and the fact that $\{v_i\}$ is an approximate unit for J are used to guarantee the existence of k_{m+1} which satisfies conditions 2 and 3. Define

$$u_m = \sum_{i=m+1}^{2m} \frac{v_{k_i}}{m}$$

It is clear that $\{u_m\}$ is again an approximate unit for J . We will prove that $\{u_m\}$ is quasiceutral with respect to B , by proving the following two claims.

Claim 2. Given any $d \in D$, we have $du_m - u_m d \rightarrow 0$ as $m \rightarrow \infty$.

Claim 3. Given any $a \in B$, we have $au_m - u_m a \rightarrow 0$ as $m \rightarrow \infty$.

Proof to Claim 3 is a straightforward exercise using the density of $\text{span}(D)/J$ in B/J once we have Claim 2, so we will focus on proving Claim 2 for the remainder of this proof.

We have

$$du_m - u_m d = \sum_{i=m+1}^{2m} \frac{dv_{k_i} - v_{k_i} d}{m} \quad (\text{III.1})$$

Setting $b_i = v_{k_i} - v_{k_{i-1}}$, $i > 1$, we can rewrite (III.1) by using the following relations:

1. $v_{k_i} = \sum_{j=1}^i b_j$
2. $1 = \sum_{j=1}^{\infty} b_j$ (in the sense of "strict topology", i.e. $y = y(\sum_{j=1}^{\infty} b_j) = (\sum_{j=1}^{\infty} b_j)y$ for all $y \in J$, which follows from definition of approximate unit)

And (III.1) becomes:

$$\frac{1}{m} \sum_{r=1}^{\infty} \sum_{i=m+1}^{2m} \sum_{j=1}^i (b_r db_j - b_j db_r)$$

After rearranging and collecting terms, we manage to eliminate one of the indices and obtain

$$\sum_{r=1}^{\infty} \sum_{j=1}^{m+1} \left(\frac{m}{m}\right) (b_r \cdot d \cdot b_j - b_j \cdot d \cdot b_r) + \sum_{r=1}^{\infty} \sum_{j=m+2}^{2m} \left(2 - \frac{j-1}{m}\right) (b_r \cdot d \cdot b_j - b_j \cdot d \cdot b_r)$$

Noticing that all terms with both r and j less than or equal to $m + 1$ exist in pairs with opposite coefficients, so they cancel with each other and we are left with:

$$\sum_{r=m+2}^{\infty} \sum_{j=1}^{m+1} \binom{m}{m} (b_r \cdot d \cdot b_j - b_j \cdot d \cdot b_r) + \sum_{r=1}^{\infty} \sum_{j=m+2}^{2m} \left(2 - \frac{j-1}{m}\right) (b_r \cdot d \cdot b_j - b_j \cdot d \cdot b_r) \quad (\text{III.2})$$

We then divide the terms in (III.2) into two groups according to their indices:

Group 1 $|r - j| \geq 2$

Group 2 $|r - j| = 1$

To take care of Group 1, we will need the following estimate derived from the assumptions used in defining k_i 's.

Claim 4. If $r - j \geq 2$, then $\|b_r \cdot d \cdot b_j\| \leq \frac{3}{2^{r-1}}$

Proof.

$$\begin{aligned} \|b_r \cdot d \cdot b_j\| &= \|((v_r - v_{r-1})d(v_j - v_{j-1}))\| \\ &\leq \|(v_r - 1)dv_j\| + \|(1 - v_{r-1})dv_j\| + \|(v_r - 1)dv_{j-1}\| + \|(1 - v_{r-1})dv_{j-1}\| \\ &\leq \frac{2}{2^r} + \frac{2}{2^{r-1}} = \frac{3}{2^{r-1}} \end{aligned} \quad \square$$

Using this estimate, we see that all the terms in Group 1 can be estimated to have magnitude

$$\begin{aligned} &< \sum_{r=m+2}^{\infty} (m+1) \cdot \frac{3}{2^{r-2}} + \sum_{r=m+4}^{\infty} 2 \cdot \frac{3}{2^{r-2}} \cdot (m-1) + \sum_{j=m+2}^{2m} 2 \cdot \frac{3}{2^{j-2}} \cdot (2m-2) \\ &< \frac{1}{2^m} (33m - 21) \end{aligned}$$

Now we will take care of Group 2. The sum is easily seen to be equal to

$$\frac{1}{m} \cdot \sum_{i=m+1}^{2m} (b_{i+1} \cdot d \cdot b_i - b_i \cdot d \cdot b_{i+1})$$

It is enough to show that the norm $\|\sum_{i=m+1}^{2m} (b_{i+1} \cdot d \cdot b_i - b_i \cdot d \cdot b_{i+1})\|$ is bounded when $m \rightarrow \infty$.

To evaluate the norm of $s = \sum_{i=m+1}^{2m} b_i \cdot d \cdot b_{i+1}$, we express it as the sum of four sums $\sum_{k=0}^3 \tilde{s}_k$, where $\tilde{s}_k = \sum_j b_{4j+k} \cdot d \cdot b_{4j+k+1}$, and the summation over j goes in the interval $m+1 \leq 4j+k \leq 2m$. Each of the \tilde{s}_k 's is then approximated by the corresponding product $s_k = (\sum_j b_{4j+k}) \cdot d \cdot (\sum_j b_{4j+k+1})$. The difference $\tilde{s}_k - s_k$ consists of terms of the form $b_{4j_1+k} \cdot d \cdot b_{4j_2+k+1}$ where $j_1 \neq j_2$, i.e. the indices for the two b 's are more than 2 apart, and hence the estimate in Claim 4 may apply. We see that $\tilde{s}_k - s_k \sim O(\frac{m^2}{2^m})$ which converges to 0 as $m \rightarrow \infty$. Each of the four products s_k is clearly bounded:

$$\|s_k\| \leq \left\| \sum_j b_{4j+k} \right\| \cdot \|d\| \cdot \left\| \sum_j b_{4j+k+1} \right\| \leq \|d\|$$

The last inequality follows from the fact that all v_i 's are assumed to have sup-norm 1 and the first assumption in our procedure of choosing $\{k_i\}$. The other sum $\sum_{i=m+1}^{2m} b_{i+1} \cdot d \cdot b_i$ is evaluated similarly.

In the case of G -action, we require an additional condition in choosing k_{m+1} :

$$4. \|(1 - v_{k_{m+1}}) \cdot (g v_{k_i})\| \leq 2^{-(m+1)} \text{ for all } g \in G_{m+1}$$

where $\{G_m\}$ is an ascending exhaustive sequence of compact subset of G . Because G acts on B by automorphism, $g \cdot 1 = 1$ for all elements in G . We may write $g u_n - u_n$ as $1 \cdot g u_n - u_n \cdot (g \cdot 1)$ and similar calculation as in the no G -action case (following III.1) would give our desired result. \square

The above proposition gives $\{u_m\}_{m \in \mathbb{N}}$, a countable bounded approximate unit of J . We may extend it to a continuous approximate unit $\{u_t\}$ by interpolating between the integers, namely $u_{sn+(1-s)(n+1)} := s u_n + (1-s) u_{n+1}, s \in [0, 1]$. This will be useful for our upcoming construction.

III.6 Asymptotic morphism associated to an extension

Having shown the existence of quascentral approximate unit for σ -unital Banach algebra, we are ready to introduce an important class of asymptotic morphisms - "connecting morphism" associated to an exact sequence of σ -unital Banach algebra. Unlike in the C^* -algebra case, ΣA (projective tensor product) is not necessarily isomorphic to ΣA (injective tensor product) for A Banach algebra. This creates certain trouble because the original approach of Connes-Higson employs the maximal tensor product definition, but we need ΣA to formulate a reasonable Bott periodicity and Mayer-Vietoris exact sequence. Additionally, algebra homomorphism between Banach algebras is not automatically bounded. Extra caution is needed to define "connecting morphism" associated to an extension of Banach algebras.

As noted in Section 2 of this chapter, ΣA is isomorphic to the L^p -tensor product $\Sigma \otimes A$. We managed to extend the construction to σ -unital extensions of L^p -algebras, i.e. Banach algebras which are isomorphic to some subalgebra of the algebra of bounded operators on some $L^p(Z, \mu)$.

Proposition III.6.1. [Bla98, c.f. Prop 25.5.1]

(a) Let $0 \rightarrow J \rightarrow A \xrightarrow{q} A/J \rightarrow 0$ be an exact sequence of separable L^p algebras. Suppose J is σ -unital (call such extension **σ -unital extension**) and hence has a continuous approximate unit $\{u_t\}$ which is quascentral for A (its existence is guaranteed by Proposition III.5.3). Then we have an asymptotic morphism from $\Sigma(A/J)$ to J .

(b) The class ε_q of this asymptotic morphism in $[[\Sigma(A/J), J]]$ is independent of the choice of σ and $\{u_t\}$.

As in the C^* -algebra case, a lemma dealing with property of $f(u_t)$ is essential to the proof. We provide a Banach algebra analogue and extend it slightly to better suit for results like "the asymptotic morphism associated to a split exact sequence is homotopic to the 0 asymptotic morphism". The proof is completely analogous.

Lemma III.6.2. [Bla98, Lemma 25.5.2] Let D be a Banach algebra with $\{u_\lambda\} \subset C_0(Y) \subset D$ a net of positive elements with norm ≤ 1 . Let $x \in D$ and $f \in C_0(0, 1], g \in C_0(0, 1)$.

1. if $\lim_{\lambda \rightarrow \infty} [u_\lambda, x] = 0$, then $\lim_{\lambda \rightarrow \infty} [f(u_\lambda), x] = 0$
2. if $\lim_{\lambda \rightarrow \infty} u_\lambda x = x$, then $\lim_{\lambda \rightarrow \infty} g(u_\lambda)x = 0$. It is not necessary that $\lim_{\lambda \rightarrow \infty} f(u_\lambda)x = 0$

We will break up the proof of the above Proposition into the following four steps:

1. an estimate for the norm of finite sums
2. define a norm-decreasing map from A to J_∞
3. show that the map is linear and multiplicative
4. $\Sigma A/J \cong \Sigma A/\Sigma J$

Step 1: An estimate for the norm of Finite Sums

Lemma III.6.3. [Kas75, Lemma 3.1] Let $A \subset \mathcal{L}(L^p(Z))$ be an L^p -algebra. If we have $\{\chi_i\}_{i=1}^n \subset C_0(Z)$, $\{x_i\}_{i=1}^n \subset A$ such that $\sum_{i=1}^n \chi_i \leq 1$ and $\chi_i \geq 0$ for all i , then $\|\sum \chi_i x_i\| \leq \max_i \|x_i\| + \sum_i \|\chi_i^{1/p}, x_i\|$

Proof. Let $M = \max_i \|x_i\|$. Using the fact that the dual space of $L^p(Z)$ is isomorphic to $L^q(Z)$ (where $1/p + 1/q = 1$), we have:

$$\begin{aligned}
\|\sum \chi_i x_i\| &= \sup_{\eta \in S(L^p), \xi \in S(L^q)} \left| \left(\sum_i \chi_i x_i \eta, \xi \right) \right| \\
&= \sup_{\eta \in S(L^p), \xi \in S(L^q)} \left| \left(\sum_i (\chi_i^{1/q} x_i \chi_i^{1/p} + \chi_i^{1/q} [\chi_i^{1/p}, x_i]) \eta, \xi \right) \right| \\
&\leq \sup_i \left| \sum_i (x_i \chi_i^{1/p} \eta, \chi_i^{1/q} \xi) \right| + \sum_i \|\chi_i^{1/p}, x_i\| && (0 \leq \chi_i^{1/p} \leq 1) \\
&\leq M \sup_i \left(\|\chi_i^{1/p} \eta\|_p \cdot \|\chi_i^{1/q} \xi\|_q \right) + \dots && \text{(Hölder's Inequality for } L^p \text{ and } L^q) \\
&\leq M \sup_i \left(\sum_i \|\chi_i^{1/p} \eta\|_p^p \right)^{1/p} \cdot \left(\sum_i \|\chi_i^{1/q} \xi\|_q^q \right)^{1/q} + \dots && \text{(Hölder's Inequality for } l^p \text{ and } l^q) \\
&= M \sup_i \left(\sum_i \int \chi_i |\eta|^p d\mu \right)^{1/p} \left(\sum_i \int \chi_i |\xi|^q d\mu \right)^{1/q} + \dots && \text{(by definition)} \\
&= M \sup_i \|\eta\|_p \cdot \|\xi\|_q + \dots = \max_i \|x_i\| + \sum_i \|\chi_i^{1/p}, x_i\|
\end{aligned}$$

□

Step 2: Define a Norm-Decreasing Map from ΣA to J_∞

For simplicity, we will fix a sequence of open coverings of $[0,1]$, $\{U_{k,i}\}$ (where the index k means k^{th} -level), such that:

1. k^{th} -covering consists of exactly 2^k elements,
2. $(k+1)^{\text{th}}$ -covering is a refinement of the k^{th} -covering, with every two of $(k+1)$ -elements coming from the same k -element, and with less than $2/3$ of the length of latter. $U_{k,l} = U_{k+1,2l-1} \cup U_{k+1,2l}$
3. in each level, 0 and 1 is covered only by $U_{k,1}$ and $U_{k,2^k}$, and no other elements of the covering

4. in each level, all the even-indexed elements are disjoint from each other; the same is true for all the odd-indexed ones

After such a sequence of coverings is chosen, we will also fix a sequence of partition of unity, $\{\phi_{k,i}\}$, subordinated to the coverings and a sequence of points $\{s_{k,i}\}$ such that $s_{k,i} \in U_{k,i}$ and $s_{k,1} = 0, s_{k,2^k} = 1$ for all k ⁴. Additionally, to simplify computation later, we require

$$\phi_{k,l} = \phi_{k+1,2l-1} + \phi_{k+1,2l}$$

Given any $f \in \Sigma A$, define $f_k(s) = \sum_{i=1}^{2^k} \phi_{k,i}(s) \cdot f(s_{k,i})$. Define

$$\varepsilon_{f,k} = \max_i \left\{ \sup_{x,y \in U_{k,i}} \|f(x) - f(y)\| \right\}$$

Because f is uniformly continuous and $\text{length}(U_{k,i}) \rightarrow 0$ as $k \rightarrow \infty$, we have $\varepsilon_{f,k} \rightarrow 0$ and hence $\|f_k - f\| \rightarrow 0$ as $k \rightarrow \infty$.

Again for simplicity, we will fix a sequence of $\{t_k\}$. We will prove a lemma later to show that the image of f in J_∞ is independent of the sequence of $\{t_k\}$ we choose. We define:

$$t_k = \inf \left\{ T : \forall t > T \max(\|[\phi_{k,i}^{1/p}(u_t), f(I)]\|, \|[\phi_{k+1,i}^{1/p}(u_t), f(I)]\|) < \frac{1}{2^{3(k+1)}}, \right. \\ \left. \max(\|[\phi_{k,i}(u_t), f(I)]\|, \|[\phi_{k+1,i}(u_t), f(I)]\|) < \frac{1}{2^{3(k+1)}} \quad \forall t > T \right\}$$

This definition is made possible by Lemma III.6.2 and a simple compactness argument on $[0, 1]$.

Next we will define the map $\Phi : \Sigma A \rightarrow J_\infty$. Define $\tilde{\Phi}_t(f) \in C_b([1, \infty), J)$ as follows:

$$t \in [1, t_1] \quad \sum_{i=1}^2 \phi_{1,i}(u_t) f(s_{1,i}) = 0 \text{ due to our choice of } s_{1,1} \text{ and } s_{1,2}$$

$$t \in [t_k, t_{k+1}] \quad \frac{t_{k+1} - t}{t_{k+1} - t_k} \sum_{i=1}^{2^k} \phi_{k,i}(u_t) f(s_{k,i}) + \frac{t - t_k}{t_{k+1} - t_k} \sum_{i=1}^{2^{k+1}} \phi_{k+1,i}(u_t) f(s_{k+1,i})$$

It is clear that we get a continuous path of elements in J with the above definition, the boundedness will be taken care of by the following computation. Denote the image in the quotient algebra, J_∞ by $\Phi_t(f)$, then

⁴this is to ensure that we won't have to worry about $f(u_t)$ for $f \notin C_0(0, 1)$ later

we have the following norm estimate:

$$\begin{aligned}
\|\Phi_t(f)\| &= \limsup_{t \rightarrow \infty} \|\tilde{\Phi}_t(f)\| && \text{(by Lemma III.3.2)} \\
&\leq \limsup_{t \rightarrow \infty, k \rightarrow \infty} \left\| \sum_{i=1}^{2^k} \phi_{k,i}(u_t) f(s_{k,i}) \right\| \\
&\leq \|f\| + \limsup \sum_{i=1}^{2^k} \|\phi_{k,i}^{1/p}(u_t), f(s_{k,i})\| && \text{(by Lemma III.6.3)} \\
&= \|f\|
\end{aligned}$$

This shows that Φ is norm-decreasing.

Step 3: Show that the map is Linear and Multiplicative

We will state and prove a lemma which shows that $\Phi(f)$ is independent of the choice of $\{t_k\}$. This will be helpful for simplifying proof of linearity and multiplicativity.

Lemma III.6.4. *If we replace $\{t_k\}$ by $\{\tilde{t}_k\}$ such that $\tilde{t}_k > \tilde{t}_{k-1}$ and $\tilde{t}_k > t_k$, then the corresponding paths in J are asymptotic equivalent. In other words, they yield the same element of J_∞ .*

Sketch of proof. Let $t \in (t_{k+m}, t_{k+m+1}) \cap (\tilde{t}_k, \tilde{t}_{k+1})$, then the norm of the difference between the two paths involve three terms with may be estimated similar to the following one.

$$\begin{aligned}
&\left\| \sum_i \phi_{k,i}(u_t) f(s_{k,i}) - \sum_j \phi_{k+m,j}(u_t) f(s_{k+m,j}) \right\| \\
&= \left\| \sum_j \phi_{k+m,j}(u_t) [f(s_{k, \lceil j/2^m \rceil}) - f(s_{k+m,j})] \right\| \\
&\leq \max_j \|f(s_{k, \lceil j/2^m \rceil}) - f(s_{k+m,j})\| + 2 \sum_j \|\phi_{k+m,j}^{1/p}(u_t), f(I)\| && \text{(Lemma III.6.3)} \\
&\leq \varepsilon_{f,k} + 2 \cdot 2^{k+m} \cdot \frac{1}{2^{3(k+m+1)}}
\end{aligned}$$

This goes to 0 as $k \rightarrow \infty$ ($t \rightarrow \infty$). □

Proposition III.6.5. *Φ is multiplicative.*

Sketch of proof. By the above lemma(III.6.4), up to replacing with another sequence $\{t'_k\}$ where $t'_k \geq t_k \forall k$, we may assume that t_k for the functions f, g and fg match up. This simplifies the situation and we need only verify the difference $\Phi(fg) - \Phi(f) \cdot \Phi(g)$ for at most on consecutive levels. For $t \in (t_k, t_{k+1})$, a sample

calculation is the following:

$$\begin{aligned}
& \left\| \sum_{i=1}^{2^k} \phi_{k,i}(u_t) [f(s_{k,i})g(s_{k,i})] - \sum_{i=1}^{2^k} \phi_{k,i}(u_t) f(s_{k,i}) \sum_{j=1}^{2^{k+1}} \phi_{k+1,j}(u_t) g(s_{k+1,j}) \right\| \\
&= \left\| \sum_{i,j} \phi_{k,i}(u_t) \phi_{k+1,j}(u_t) f(s_{k,i}) [g(s_{k,i}) - g(s_{k+1,j})] + \sum_{i,j} \phi_{k,i}(u_t) [\phi_{k+1,j}(u_t), f(s_{k,i})] g(s_{k+1,j}) \right\| \\
&\leq \max_{i,j: U_{k,i} \cap U_{k+1,j} \neq \emptyset} \|f(s_{k,i})[g(s_{k,i}) - g(s_{k+1,j})]\| + 2 \sum_{i,j} \|[\phi_{k,i}(u_t)^{1/p} \phi_{k+1,j}(u_t)^{1/p}, f(I)g(I)]\| \\
&+ (2^k \cdot 2^{k+1}) \cdot 1 \cdot \frac{1}{(2^{k+1})^3} \cdot \|g\|
\end{aligned}$$

It is easy to see that the first and the third term goes to 0 as $t \rightarrow \infty$. The middle term can be estimated using the following elementary commutator identity:

$$[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B$$

to be $\leq 2 \cdot 2^k \cdot 2^{k+1} \cdot 4 \max\{\|f\|, \|g\|\} \cdot \frac{1}{2^{3(k+1)}} \cdot 1 = \max\{\|f\|, \|g\|\} / 2^{k-1}$ which again goes to 0 as t (and hence k) approaches ∞ . \square

Step 4: $\Sigma(A/J) \cong \Sigma A / \Sigma J$

Lemma III.6.6. *Let J be a closed two-sided ideal in a Banach algebra A , then we have $\Sigma(A/J) \cong \Sigma A / \Sigma J$.*

Proof. Let $Q : A \rightarrow A/J$ be the natural quotient map. Consider the map $\Sigma Q : \Sigma A \rightarrow \Sigma(A/J)$ given by $f \mapsto Q \circ f$. It is easy to see that $\ker \Sigma Q = \Sigma J$. Left to show that ΣQ is a surjective map.

Identify $(0, 1)$ as a subspace of S^1 by the map $t \mapsto e^{2\pi i t}$. Given any $\bar{f} \in \Sigma(A/J)$, it can be viewed as an element of $C(S^1, A/J)$ whose value at 1 is 0. By the fact that "injective tensor product with $C(K)$ respects metric surjections" ([DF93, pp. 4.4, 4.5]), there exists $f' \in C(S^1, A)$ such that $Q \circ f' = \bar{f}$ and $f'(1) \in J$. Let $f = f' - f'(1)$, then we have $Q \circ f = \bar{f}$, $f(1) = 0$ and hence can be viewed as an element of ΣA . This shows that ΣQ is surjective and hence our result. \square

Lemma III.6.7. *The map Φ (which we defined in Step 2) is 0 on ΣJ .*

Given any element of ΣJ , this is a consequence of Lemma III.6.2, by possibly replacing $\{t_k\}$ with another increasing sequence of $\{t'_k\}$ (where $t'_k \geq t_k \forall k$) which would not impact the resulting element in J_∞ (Lemma III.6.4).

Combining the above two lemmas, we see that the map Φ factors through $\Sigma A / \Sigma J \cong \Sigma(A/J)$. This concludes the four steps and gives an asymptotic morphism from $\Sigma(A/J)$ to J_∞ .

III.7 Bott Periodicity and Exact Sequences

In this section, we will prove that our E-theory has six-term exact sequences. We will need three ingredients for its proof, with the first one being Bott Periodicity.

Definition III.7.1. Given $f \in L^p(\mathbb{R})$ ($1 < p < \infty$), define its Hilbert Transform to be the function

$$Hf(x) := \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|x-y| > \varepsilon} \frac{1}{x-y} f(y) dy$$

Stein shows that the Hilbert Transform is a bounded linear operator on $L^p(\mathbb{R})$ ([Ste71, II.2 Theorem 1]) for $1 < p < \infty$. Furthermore, $H(H(f)) = -f$ for f a sufficiently regular function ([Tit48, p.120]). Since the algebra of compactly supported smooth function is dense in $L^p(\mathbb{R})$, we have that $H^2 = -1$ on $L^p(\mathbb{R})$ by the boundedness of H .

We will show one additional property of H and use that to construct an asymptotic morphism to prove Bott Periodicity. Before that, let's recall an estimate regarding the norm of integral operators from L^p to L^q spaces (it is not necessary that $1/p + 1/q = 1$).

Lemma III.7.2. [Lev, Lemma 2.3] Let $p, q \in [1, \infty]$ (p^*, q^* are the conjugates of p and q respectively), let kernel k satisfies the bound

$$\|k\|_{L^{q^*}(d\nu; L^{p^*}(d\mu))} = \left(\left(\int |k(x, y)|^{p^*} d\mu(x) \right)^{q^*/p^*} d\nu(y) \right)^{1/q^*} < \infty$$

Then for every $u \in L^p(d\mu)$ and $v \in L^q(d\nu)$ we have

$$\int \int |k(x, y) u(x) \overline{v(y)}| d\mu(x) d\nu(y) \leq \|k\|_{L^{q^*}(L^{p^*})} \|u\|_{L^p} \|v\|_{L^q}.$$

Since the left-hand-side is bounded below by $|\langle Ku, v \rangle|$, this implies in particular that $\|K\|_{\mathcal{L}(L^p, L^{q^*})} \leq \|k\|_{L^{q^*}(L^{p^*})}$

Proposition III.7.3. Given any $f \in C_0(\mathbb{R})$, we have $[H, f] \in \mathbb{K}(L^p(\mathbb{R}))$ - the Banach algebra of compact operators on $L^p(\mathbb{R})$.

Proof. If $f \in C_c^\infty(\mathbb{R})$, then $[H, f]$ is an integral operator with kernel

$$h(x, y) f(y) - f(x) h(x, y) = \frac{f(y) - f(x)}{y - x}$$

which can be made into the smooth kernel $k(x, y)$ by removing the singularity on the diagonal:

$$k(x, y) = \begin{cases} \frac{f(y) - f(x)}{y - x} & x \neq y \\ f'(x) & x = y \end{cases}$$

If $k(x, y)$ is compactly supported, it is approximable by functions of the form $\sum_{i=1}^n f_i(x) g_i(y)$ where $f, g \in C_c(\mathbb{R})$ in 'Iterated Norm' (Lemma III.7.2). Integral operator with kernel of the above form is a finite-rank operator. This shows that integral operator with compactly supported smooth kernel is norm limit of finite-rank operator, hence a compact operator.

Let χ_N be a cut-off function supported on $[-N, N] \times [-N, N]$. Define $k_N = k \cdot \chi_N$. If $\|k\|_{L^p(L^p)} < \infty$, then $\lim_{N \rightarrow \infty} \|k - k_N\|_{L^p(L^p)} = 0$ and hence $K_N \rightarrow K$ in $\mathcal{L}(L^p, L^p)$. The last thing to show is that $\|k\|_{L^p(L^p)} < \infty$.

This can be done by using Fubini-Tonelli Theorem to estimate the integral on the 4 infinite bands of form similar to $[-R, R] \times [R, \infty)$.

We have that $[H, f]$ is compact for f any smooth compactly supported function on \mathbb{R} . And it is one easy step to generalize to $f \in C_0(\mathbb{R})$. \square

Next, we will define a Dirac asymptotic morphism following the procedure we used to define the asymptotic morphism associated to a σ -unital extension of L^p -algebras. Let

$$P = \frac{iH + 1}{2}$$

where i is the square root of -1 . We have $P^2 = P \in \mathcal{L}(L^p(\mathbb{R}))$. Let B be the closed linear subspace of $\mathcal{L}(L^p(\mathbb{R}))$ generated by $\mathbb{K}(L^p(\mathbb{R}))$ (will be denoted by \mathbb{K} from this point onwards) and the set $\{fP : f \in C_0(\mathbb{R}) \text{ or } f = 1^5\}$. It is straightforward to verify that B is an algebra by the commutation property of H above. We have also that $\mathbb{K} \trianglelefteq B$ and that B/\mathbb{K} is separable. The latter follows from the separability of $C_0(\mathbb{R})$, due to the fact that it is a subspace of the separable metric space $C(S^1)$. If \mathbb{K} is σ -unital, then by Proposition III.5.3, we can obtain an approximate unit of \mathbb{K} that is quasi-central with respect to B . In our case, we have a slightly weaker condition than σ -unital, but can nevertheless still obtain the quasi-central approximate unit.

Proposition III.7.4. *\mathbb{K} possesses a countable bounded approximate unit that is contained in a closed commutative subalgebra, and is quasentral with respect to B .*

Proof. To start with, we will define $P_{k,n}$ as projection onto the characteristic function $\chi_{k,n}$ of the segment $[k/2^n, (k+1)/2^n]$, for all integers k and all positive numbers n . Let

$$u_n = \sum_{-n \cdot 2^n}^{n \cdot 2^n - 1} P_{k,n}$$

which is the sum of all projections onto characteristic functions onto subintervals of length $1/2^n$ in $[-n, n]$.

Claim: $\{u_n\}$ forms an approximate unit of \mathbb{K} .

We will first define for any $f \in L^p(\mathbb{R})$, $g \in \mathcal{L}^q(\mathbb{R})$ (where $1/p + 1/q = 1$) the rank-one operator $\theta_{f,g} \in \mathbb{K}$:

$$\theta_{f,g}(\xi) = f\left(\int g\xi\right) \quad \forall \xi \in L^p(\mathbb{R})$$

Since any compact operators may be approximated in norm by finite rank operators, and the algebra of continuous function of compact support is dense in any $L^p(\mathbb{R})$ for $p \in (1, \infty)$, it suffices to prove that for f, g compactly supported continuous functions;

$$\|u_n \theta_{f,g} - \theta_{f,g}\| \rightarrow 0 \text{ and } \|\theta_{f,g} u_n - \theta_{f,g}\| \rightarrow 0$$

⁵we require our approximate unit to have good commutation with P so that a calculation involving operators F and G (which we are to define later) will work out in the proof of Bott Periodicity

The former is an easy consequence of the definition, so we will focus on proving the latter, which is equivalent to

$$\lim_{n \rightarrow \infty} \sup_{\|\xi\|_p=1} \|\theta_{f,g} u_n(\xi) - \theta_{f,g}(\xi)\| = 0$$

Pick n large enough such that $\text{supp } g \subset [-m, m]$ and $m < n$, we have the following estimate:

$$\begin{aligned} & \sup_{\|\xi\|_p=1} \langle g, u_n \xi - \xi \rangle \\ &= \sup_k \sum_k \langle g \chi_{k,n}, P_{k,n} \xi - \xi \chi_{k,n} \rangle \\ &= \sup_k \sum_k \langle g \chi_{k,n}, (2^n \int \xi \chi_{k,n}) \chi_{k,n} - \xi \chi_{k,n} \rangle \\ &= \sup_k \sum_k [(2^n \int g \chi_{k,n}) (\int \xi \chi_{k,n}) - (\int g \xi \chi_{k,n})] \\ &= \sup_k \sum_k [\int_{(k,n)} (\bar{g}_{k,n} - g) \xi] \quad (\bar{g}_{k,n} = \text{“average value” of } g \text{ over } [k/2^n, (k+1)/2^n]) \\ &\leq \sup \int_{[-m,m]} |(\bar{g}_{k,n} - g) \xi| \\ &\leq \sup \|\bar{g}_{k,n} - g\|_q \|\xi\|_p \quad (\text{H\"older's inequality}) \end{aligned}$$

Because $\|\bar{g}_{k,n} - g\| \rightarrow 0$ uniformly for all k as $n \rightarrow \infty$, we have obtained the said result.

Claim: $\{u_n\}$ is a bounded approximate unit, with norm uniformly bounded by 1 for $p \in (1, \infty)$

We notice that $\|\chi_{k,n}\|_p = \frac{1}{2^{n/p}}$ and $P_{k,n} \xi = \chi_{k,n} \cdot 2^n \langle \chi_{k,n}, \xi \rangle$. Now for $\xi \in L^p(\mathbb{R})$, we have the following estimate:

$$\begin{aligned} \|u_n \xi\|_p &= \left\{ \int \left| \sum_k \left(\int_{k,n} \xi \right) 2^n \chi_{k,n} \right|^p \right\}^{1/p} \\ &= 2^n \left\{ \sum_k \left(\int_{k,n} 1^p \right) \left| \int_{k,n} \xi \right|^p \right\}^{1/p} \\ &\leq 2^n \left\{ \sum_k \frac{1}{2^k} \left(\int_{k,n} |\xi \cdot 1|^p \right) \right\}^{1/p} \\ &\leq 2^{n(1-1/p)} \left\{ \sum_k \left(\int_{k,n} |\xi|^p \right) \cdot \left(\int_{k,n} 1^q \right)^{p/q} \right\}^{1/p} \quad (\text{H\"older's Inequality}) \\ &= \frac{2^{n(1-1/p)}}{2^{n/q}} \left(\int_{[-n,n]} |\xi|^p \right)^{1/p} \\ &\leq \|\xi\|_p \end{aligned}$$

Claim: $[u_n, u_{n+l}] = 0$ (they commute with each other)

We take arbitrary $\xi \in L^p(\mathbb{R})$. Since u_n 's are linear operators, if $u_n \cdot u_{n+l}$ and $u_{n+l} \cdot u_n$ agree on "segments"

of ξ they will agree when applied to ξ . Choosing the segment $[k/2^m, (k+1)/2^m]$, suppose we have

$$\chi_{k,m} = \sum_{j=1}^{2^l} \chi_{\bar{k}+j,m+l}$$

Computing $u_m u_{m+l}(\xi \chi_{k,m})$ and $u_{m+l} u_m(\xi \chi_{k,m})$, it is easy to see that they are both equal to $2^m \cdot \left(\int_{k/2^m}^{(k+1)/2^m} \xi \right) \chi_{k,m}$ or $2^m \cdot \left(\int_{k/2^m}^{(k+1)/2^m} \xi \right) \left(\sum_{j=1}^{2^l} \chi_{\bar{k}+j,m+l} \right)$.

Claim: The construction in the proof for Proposition III.5.3 works with our weaker condition on \mathbb{K} .

By the previous claim, we know that all the u_n 's commute with each other, and they generate a commutative subalgebra of \mathbb{K} . What we don't know is that this subalgebra is isomorphic to $C_0(Y)$ for some locally compact space Y . However, if we fix a number n and restrict ourselves to the first n elements of our approximate unit, taking the characteristic function for intervals of length $1/2^n$ in $[-n, n]$ as basis, we may obtain matrix representation of the first n elements. Recalling the fact that a set of commuting diagonalizable matrices is simultaneously diagonalizable - i.e. there exists a basis for the finite dimensional linear space of characteristic functions we mentioned above, in which our first n elements can all be represented as diagonal matrices. The set of diagonal matrices is isomorphic to the space of (continuous) function on finite number (equal to dimension of the matrix) of points, and hence we may repeat this procedure for each v_j to obtain a corresponding w .

Another delicate point happens near the end of the proof to Proposition III.5.3, when we estimate $\|s_k\|$ by sums like $\|\sum_j b_{4j+k}\|$. Because we do not assume that u_n are functions on the same locally compact space Y , we would have to further analyze the property of b_k . We see that b_k are projections onto different parts of Haar basis, an unconditional Schauder basis on $L^p(\mathbb{R})$. Because of the unconditional property, we have that given any combination of plus and minus signs, $\sum \pm b_n \xi$ converges for any ξ . This in particular means that $\sup_k \|\sum_{n=1}^k \pm b_n \xi\|$ is finite for all ξ . By Uniform Boundedness Principle, we have that $\sup_k \|\sum_{n=1}^k \pm b_n\|$ is finite. $\sum_j b_{4j+k}$ can be obtained by taking the average of two such sums of b_n 's, each uniformly bounded in norm, hence is itself bounded.

The rest of the proof follows without problem. □

The above proposition gives us a countable approximate unit of \mathbb{K} . WLOG, we may assume $\{u_n\}$ is quasi-central with respect to B and we will define a Dirac asymptotic morphism out of it.

Proposition III.7.5. *There is an asymptotic morphism $\eta : \Sigma C_0(\mathbb{R}) \rightsquigarrow \mathbb{K}$, which we call the Dirac asymptotic morphism.*

Sketch of proof. To define the asymptotic morphism $\Sigma^2 \rightarrow \mathbb{K}$, take any $f = f(x, y) \in \Sigma^2$, and let P be the projection of the Hilbert transform. We do not need to approximate f with $\sum_i \phi_i f_i(y)$. Just take the approximate unit v_i for $\mathbb{K}(L^p[0, 1])$, choose the sequence $\{k_j\}$ such that $\|(1 - v_{k_j})Pv_{k_j}\| < 2^{-j}$, and also $\|(1 - v_{k_j}(y))f(x, y)\| < 2^{-j}$ for any $x \in [0, 1]$. Then we put as usual $u_j = v_{k_j}, b_j = u_j - u_{j-1}$. Our asymptotic morphism is $t \mapsto f(u_t, y)P$, where $u_t = \sum_{m+1}^{2m} u_j/m$, and $m \rightarrow \infty$ with $t \rightarrow \infty$.

It is easy to calculate that $f(u_t, y)P = \sum_{m+1}^{2m} f(\lambda_j, y)b_jP$, where $\lambda_j = (2m - j + 1)/m$ for $m + 1 \leq j \leq 2m$. Since $b_j = u_j - u_{j-1}$, we get: $f(u_t, y)P = \sum_{m+1}^{2m} (f(\lambda_j, y) - f(\lambda_{j+1}, y))u_jP$. Note that the sum $\sum_{m+1}^{2m} (f(\lambda_j, y) - f(\lambda_{j+1}, y))$ is equal to 0. Therefore, in view of the condition $\|(1 - v_{k_j}(y))f(x, y)\| < 2^{-j}$, all sums $f(u_t, y)P = \sum_{m+1}^{2m} (f(\lambda_j, y) - f(\lambda_{j+1}, y))u_jP$ will be bounded in norm uniformly in m . \square

Theorem III.7.6 (Bott Periodicity; c.f. Blackadar 25.5.9). *For any separable L^p algebras B_1 and B_2 , there are canonical isomorphisms $E(B_1, B_2) \cong E(\Sigma^2 B_1, B_2) \cong E(B_1, \Sigma^2 B_2) \cong E(\Sigma B_1, \Sigma B_2)$, which are natural in B_1 and B_2 .*

Specifically, the asymptotic morphism η given in the previous proposition defines an element d of $E_0(C_0(\mathbb{R}^2), \mathbb{C})$. There exists an element b of $E_0(\mathbb{C}, C_0(\mathbb{R}^2))$ such that $b \circ d = [id_{C_0(\mathbb{R}^2)}]$ and $d \circ b = [id_{\mathbb{C}}]$. Then the isomorphism from $E(B_1, B_2)$ to $E(\Sigma^2 B_1, B_2)$ is given by composing on the right by $[d \otimes id_{\Sigma B_1}]$, and the isomorphism $E(B_1, B_2) \cong E(B_1, \Sigma^2 B_2)$ is given by composing on the left by $[b \otimes id_{B_2}]$.

Proof. We will show that there exists an element $b \in E(\mathbb{C}, C_0(\mathbb{R}^2))$ such that $\eta \circ b = 1_{\mathbb{C}} \in E(\mathbb{C}, \mathbb{C})$. Instead of working directly with $E(\mathbb{C}, C_0(\mathbb{R}^2)) = [[\Sigma, \Sigma C_0(\mathbb{R}^2) \otimes \mathbb{K}]]$ we define $b \in [[\mathbb{C}, C_0(\mathbb{R}^2) \otimes \mathbb{K}]]$ (we can get an element of $E(\mathbb{C}, C_0(\mathbb{R}^2))$ by tensoring with id_{Σ}). We then realize $\eta \circ b$ as a $(0, \infty)$ -path of projection in $\mathcal{B}(L^p(Z))/\mathbb{K}(L^p(Z))$, which gives an element in $K_0(\mathcal{B}(L^p(Z))/\mathbb{K}(L^p(Z)) \otimes C_0(0, \infty))$. Identifying $C_0(0, \infty)$ and Σ , we obtain by Bott Periodicity of K-theory an element in $K_1(\mathcal{B}(L^p(Z))/\mathbb{K}(L^p(Z)))$. Finally, applying the half-exactness of K-theory to the exact sequence:

$$0 \rightarrow \mathbb{K}(L^p(Z)) \rightarrow \mathcal{B}(L^p(Z)) \rightarrow \mathcal{B}(L^p(Z))/\mathbb{K}(L^p(Z)) \rightarrow 0$$

we obtain by the boundary map in the associated long exact sequence in K-theory an element in $K_0(\mathbb{K}(L^p(Z)))$. We will show that this is the element $1 \in K_0(\mathbb{K}) \cong \mathbb{Z}$. We would use Atiyah's Rotation Trick to obtain identity for the other composition.

After this, we note that tensoring by id_{Σ} gives an isomorphism $[[A \otimes \mathbb{K}, B \otimes \mathbb{K}]] \rightarrow [[\Sigma A \otimes \mathbb{K}, \Sigma B \otimes \mathbb{K}]]$. This can be proved by first showing that tensoring by id_{Σ^2} is an isomorphism for any L^p -algebras A, B through the following commutative diagram:

$$\begin{array}{ccc} \Sigma^2 A \otimes \mathbb{K} & \xrightarrow{\eta \otimes 1} & A \otimes \mathbb{K} \\ \downarrow 1 \otimes \alpha & & \downarrow 1 \otimes \alpha \\ \Sigma^2 B \otimes \mathbb{K} & \xrightarrow{\eta \otimes 1} & B \otimes \mathbb{K} \end{array}$$

This provides the justification for our initial choice to work with $[[\mathbb{C}, C_0(\mathbb{R}^2) \otimes \mathbb{K}]]$ instead of $E(\mathbb{C}, \mathbb{C})$ (dropped a Σ).

We will give more details to the first part of the proof. This is essentially the same as the proof to Lemma 7.5 in [HK01]. We adapt it into the context of E -theory.

Step 1: Define $b \in [[\mathbb{C}, \tilde{\Sigma}^2 \otimes M_2 \otimes \mathbb{K}]]$ ⁶

⁶For convenience of notation, we are working in unitized algebra $\tilde{\Sigma}^2 \cong C(S^2)$. This can be related back to the non-unitized case via the split exact sequence $0 \rightarrow \Sigma^2 \rightarrow C(S^2) \rightarrow \mathbb{C} \rightarrow 0$ and the split-exactness in both variables of E -theory.

b is given by the homomorphism which sends $1 \in \mathbb{C}$ to a projection valued function (in complex coordinate of the Riemann sphere) $z \mapsto \begin{pmatrix} \frac{1}{1+|z|^2} & \frac{z}{1+|z|^2} \\ \frac{\bar{z}}{1+|z|^2} & \frac{|z|^2}{1+|z|^2} \end{pmatrix}$.

Step 2: Compute $\eta \circ b$

Starting with $\eta : \Sigma C_0(\mathbb{R}) \rightsquigarrow \mathbb{K}$, we first define $\eta_t = t\eta_1$ for $t \leq 1$; we then extend η_t unittally to unitalized algebra $\tilde{\Sigma}^2$ ⁷; and then pass to matrices to obtain a unital map

$$\{\eta_t^+\} : M_2(\tilde{\Sigma}^2) \rightarrow M_2(\tilde{\mathbb{K}} \otimes C_0(0, \infty)).$$

To simplify notation, the following proof is given using the Connes-Higson definition of connecting morphism (instead of our stepwise approach introduced earlier using partition of unity), which defines the map on simple tensors and extend to the completion in maximal tensor norm. This is justifiable because both Σ and $C_0(\mathbb{R})$ are C^* -algebras and the maximal tensor norm and the injective tensor norm coincide for the pair.

Note that in the construction for η given in the previous proposition, we are identifying Σ^2 with $C_0((0, 1) \times \mathbb{R})$. Let $f = 1 \in C(S^2) = C_0((0, 1) \times \mathbb{R})$, we are to map f to the corresponding unit in $\tilde{\mathbb{K}}$, hence we would need to add $1 - P$ in our definition of η to obtain η^+ :

$$\tilde{\eta}_t^+(f \otimes g) = f(u_t)[gP + \lim_{y \rightarrow \infty} g(y)(1 - P)]$$

Now to compute the composition $\eta \circ b$, we would need to first identify Σ^2 with the C_0 - functions on the product $(0, 1) \times \mathbb{R}$. Writing the projection-valued function we have from the previous step in polar coordinate, we have

$$(r, \theta) \mapsto \begin{pmatrix} \frac{1}{1+r^2} & \frac{re^{i\theta}}{1+r^2} \\ \frac{re^{-i\theta}}{1+r^2} & \frac{r^2}{1+r^2} \end{pmatrix}$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi]$ Setting $x = \frac{1}{1+r^2}$, $y = \tan \frac{\theta - \pi}{2}$, we have the function on $(0, 1) \times \mathbb{R}$:

$$(x, y) \mapsto \begin{pmatrix} x & \sqrt{x-x^2} \exp(i(2 \tan^{-1} y + \pi)) \\ \sqrt{x-x^2} \exp(-i(2 \tan^{-1} y + \pi)) & 1-x \end{pmatrix}$$

Passing η^+ to matrices and applying to this matrix, we obtain an asymptotic morphism which maps $1 \in \mathbb{C}$ to the following path of elements in $M_2(\tilde{\mathbb{K}})$:

$$(x, y) \mapsto \begin{pmatrix} u_t & \sqrt{u_t - u_t^2}[g(y)P + (1 - P)] \\ \sqrt{u_t - u_t^2}[(1/g(y))P + (1 - P)] & 1 - u_t \end{pmatrix} \quad t \geq 0$$

where $g(y) = \exp(2 \tan^{-1} y + \pi) = \frac{y^2 - 1}{y^2 + 1} - \frac{2y}{y^2 + 1}i$. Here we extend the definition of u_t from $t \geq 1$ to $t \geq 0$ by setting $u_t = tu_1$ for $t \leq 1$. If we let $F = g(y)P + (1 - P)$ and $G = (1/g(y))P + (1 - P)$, it will be verified in Step 3 that it forms a Fredholm pair for all $p \in [2, \infty)$, which means that there exists $h, k \in \mathbb{K}(L^p(\mathbb{R}))$ such

⁷In reality, we are extending the definition to $C([0, 1] \times S^1)$

that:

$$\begin{cases} FG = 1 - h \\ GF = 1 - k \end{cases}$$

By a simple computation, we see that the difference between the square of the matrix and itself forms a norm continuous path (over t) in \mathbb{K} . Norm of the difference is determined by the magnitude of

$$[F, \sqrt{u_t - u_t^2}], [G, \sqrt{u_t - u_t^2}], [F, u_t], [G, u_t], (u_t - u_t^2)h, (u_t - u_t^2)k$$

Now, F, G are both contained in the algebra B (defined in the remark after Proposition III.7.3), with respect to which our approximate unit $\{u_t\}$ was chosen to be quasentral, and hence $\|[F, u_t]\|, \|[G, u_t]\|$ both approach 0 as $t \rightarrow \infty$. $\sqrt{x-x^2}, x-x^2$ are both $C_0(0, 1)$ functions, hence by Lemma III.6.2, we have all six terms approach 0 in norm as $t \rightarrow \infty$. Since $u_t = tu_1$ for $t < 1$, we have $u_t \rightarrow 0$ in norm as $t \rightarrow 0$ and hence all six terms also approach 0 in norm as $t \rightarrow 0$. This means that the difference is an element of $\mathbb{K}(L^p(\mathbb{R})) \otimes C_0(0, \infty)$, and by the definition of K-theory, we have an element of $K_0(\mathcal{B}(L^p(\mathbb{R}))/\mathbb{K}(L^p(\mathbb{R})) \otimes C_0(0, \infty))$.

If now we define

$$w_t = \begin{cases} t, & t \leq 1 \\ 1, & t \geq 1 \end{cases}$$

then the straight line homotopy from u_t to w_t gives a homotopy of K-theory element to the family

$$\left(\begin{array}{cc} w_t & \sqrt{w_t - w_t^2} [g(y)P + (1-P)] \\ \sqrt{w_t - w_t^2} [(1/g(y))P + (1-P)] & 1 - w_t \end{array} \right)_{t \geq 0}$$

This homotopy allows us to identify this with an element of $K_0(\mathbb{K})$ given by the index of $[g(y)P + (1-P)]$ through the boundary map in long exact sequence in K-theory. [Bla98, Def 8.3.1]

Step 3: Show that $d \circ b$ gives an index 1 operator in $M_2(\mathbb{K})$ We have Bott periodicity for E-theory of C^* -algebras, which are subalgebra for the algebra of bounded operator on some $L^2(Z)$. In this case, we know that $d \circ b$ will give an index 1 element. To use this result, we will first prove a proposition about the index of a continuous (over p) family of Fredholm operators on $L^p(Z)$. We will then show that $[g(y)P + (1-P)]$ gives a continuous family of Fredholm operator for $p \in [2, p_0]$ and hence has the same index as its $p = 2$ representative. This finishes the proof.

Proposition III.7.7. *Let F_p be a family of Fredholm operators in $L^p(Z)$ for $p \in [2, p_0]$, such that there exists another family $G_p \in \mathcal{B}(L^p(Z))$ such that*

$$\begin{cases} F_p G_p = 1 - h_p \\ G_p F_p = 1 - k_p \end{cases} \quad \text{where } h_p, k_p \in \mathbb{K}(L^p(Z))$$

Additionally, h_p, k_p are assumed to be continuous in the sense that given any $p_1 \in [2, p_0], \varepsilon > 0$, there exists a neighborhood of p_1 such that h_p, k_p are approximable to within ε (in L^p -norm) by finite rank operator involving at most first n basis element. Then $\text{index}(F_p) = \dim \ker F_p - \dim \ker G_p$ is constant over p .

Proof. Recall that Fredholm index is invariant under compact perturbation. In other words, if F is a Fredholm operator, T any compact operator, then $\text{index}(F + T) = \text{index}(F)$. We will replace F_p, G_p by a family of their compact perturbation F'_p, G'_p , such that $F'_p G'_p$ will be a finite codimension projection for all $p \in [2, p_0]$ and G'_p, F'_p will be a continuous family of projection and hence the rank should be constant over the whole path. This shows that index of F_p is constant.

Fix a family of basis $\{e_n\}$ across p , this is possible because separable non-atomic $L^p(Z)$ are all isometric to $L^p(0, 1)$ where Hahn basis forms a monotonic basis. Let q_n = projection on $\text{span}\{e_1, \dots, e_n\}$. By our continuity assumption, for each $p_1 \in [2, p_0]$, there exists n such that $\|(1 - q_n)h_p(1 - q_n)\| < \varepsilon$ for p in a neighborhood of p_1 . By the compactness of $[2, p_0]$, there exists n such that $\|(1 - q_n)h_p(1 - q_n)\| < 1/2$ for all $p \in [2, p_0]$. Let $\tilde{h}_p = (1 - q_n)h_p(1 - q_n)$. We have $q_n \tilde{h}_p = \tilde{h}_p q_n = 0$ and $(1 - q_n)\tilde{h}_p = \tilde{h}_p(1 - q_n) = \tilde{h}_p$.

What follows is similar to that on p.556 of [Kas81]. Let $l = (1 - q_n)(1 - \tilde{h}_p)^{-1/2}$ where $(1 - \tilde{h}_p)^{-1/2}$ is given by the binomial series with radius of convergent 1. Note that l commutes with $(1 - q_n)$ because \tilde{h}_p does. Finally, we may define $G'_p = G_p l (1 - q_n)$ and $F'_p = (1 - q_n) l F_p$. This is a compact perturbation of G_p and F_p because $(1 - q_n)l = (1 - q_n)(1 - \text{convergent series involving } h_p)$ and both q_n and h_p are compact operators.

We verify that:

$$\begin{aligned} F'_p G'_p &= (1 - q_n) l F_p G_p l (1 - q_n) \\ &= l (1 - q_n) (1 - h_p) (1 - q_n) l \\ &= l (1 - q_n - \tilde{h}_p) l \\ &= l (1 - \tilde{h}_p) l - l q_n l \\ &= 1 - q_n - 0 \end{aligned}$$

So $F'_p G'_p$ is a fixed projection across p .

We check also that $(G'_p F'_p)^2 = G'_p (F'_p G'_p) F'_p = G_p l (1 - q_n) (1 - q_n) (1 - q_n) l F_p = G'_p F'_p$. Let $\tilde{k}_p = 1 - G'_p F'_p$. \tilde{k}_p is a projection because $G'_p F'_p$ is. Additionally, F'_p and G'_p are compact perturbation of the Fredholm pair F_p, G_p means that $\tilde{k}_p \in \mathbb{K}(L^p(Z))$, so \tilde{k}_p is a continuous family of finite rank projection and thus has constant rank. This gives that $\dim \ker F'_p G'_p - \dim \ker G'_p F'_p = \text{rk } q_n - \text{rk } \tilde{k}_p$ is also constant over p . \square

Proposition III.7.8. $F_p = g(y)P + (1 - P)$ and $G_p = (1/g(y)P) + (1 - P)$ forms a family of Fredholm operators which satisfy the hypothesis of the previous proposition.

This can be easily verified by writing $g(y) = 1 + g_1(y)$, $1/g(y) = 1 + g_2(y)$, then $g_i(y) \in C_0(\mathbb{R})$ and $[g_i(y), P] \in \mathbb{K}(L^p(\mathbb{R}))$ (Lemma III.7.3).

Now that we have proved one direction of the composition, namely $d \circ b = [id_{\mathbb{C}}]$. The other composition would follow from Atiyah's Rotation Trick. [HKT98, pp. 2.16-2.18] The main idea is captured by the

following diagram (which can be checked to commute asymptotically):

$$\begin{array}{ccccc}
\Sigma^2 & \xrightarrow{1 \otimes \eta} & \mathbb{K} & \xrightarrow{b \otimes 1} & \Sigma^2 \otimes \mathbb{K} \\
\downarrow b \otimes 1 & & & & \downarrow flip \\
\Sigma^2 \Sigma^2 & \xrightarrow{flip} & \Sigma^2 \Sigma^2 & \xrightarrow{\eta \otimes 1} & \mathbb{K} \otimes \Sigma^2
\end{array}$$

The top row represents the composition $b \circ d$. Going down and across can be shown to induce isomorphism on K -theory by a computation which is detailed in [HKT98, Lem 2.18], of which our earlier result of $d \circ b = [id_{\mathbb{C}}]$ is a central ingredient. \square

Next we will introduce the second ingredient: half-exactness of E-theory. We will start with recalling the definition of mapping cone.

Definition III.7.9. Let $\phi : A \rightarrow B$ be a Banach algebra homomorphism. Let $CB = C_0([0, 1], B)$ ⁸, $\tilde{C}B = C_0((0, 1], B)$. Let $\pi_0 : CB \rightarrow B$ be evaluation at 0. Then the pullback of (A, CB) along (ϕ, π_0) is called the *mapping cone* of ϕ , denote C_ϕ . In other words, an element of C_ϕ is a pair $(a, (b(s))_{s \in [0, 1]})$ with $b(0) = \phi(a)$.

Here is our main theorem:

Theorem III.7.10 (c.f. Blackadar 25.5.10). *E is half-exact: for any σ -unital (defined at the beginning of this section) extension $0 \rightarrow J \xrightarrow{i} A \xrightarrow{q} B \rightarrow 0$ of separable L^p algebras, and every separable Banach algebra D , the sequences*

$$\begin{aligned}
E(D, J) &\xrightarrow{i_*} E(D, A) \xrightarrow{q_*} E(D, B), \\
E(B, D) &\xrightarrow{q^*} E(A, D) \xrightarrow{i^*} E(J, D)
\end{aligned}$$

are exact in the middle, where i_* [resp. i^*] is composition on the right [resp. on the left] with $[i] \in E(J, A)$, etc.

We need the following two lemmas for its proof. If $0 \rightarrow J \xrightarrow{i} A \xrightarrow{q} B \rightarrow 0$ is a σ -unital exact sequence as above, let $0 \rightarrow \Sigma J \rightarrow CA \xrightarrow{p} C_q \rightarrow 0$ be the associated mapping cone sequence. Let α be the projection map from C_q to A .

Lemma III.7.11 (c.f. Blackadar 25.5.11). $[\Sigma i \circ \varepsilon_p] = [\Sigma \alpha]$ in $[[\Sigma C_q, \Sigma A]]$.

In the above Lemma, recall that $\Sigma A = C_0((0, 1), A)$. If h_t is an asymptotic morphism between two Banach algebras A and B , then $(\Sigma h)_t$ maps $f \in \Sigma A$ to the function $s \mapsto h_t(f(s))$ for all $s \in (0, 1)$. It can be proved by some routine calculation that

- a) Σh gives an asymptotic morphism between ΣA and ΣB
- b) $\Sigma(h \circ i) = \Sigma h \circ \Sigma i$ for any asymptotic morphism (in particular, Banach algebra homomorphism) i
- c) if $[h] = [\tilde{h}] \in [[A, B]]$, then $[\Sigma h] = [\Sigma \tilde{h}] \in [[\Sigma A, \Sigma B]]$

⁸Some authors may prefer to have the function vanishes at 0 instead of 1, the choice here makes the proof of Lemma III.7.11 more convenient – not having to reverse the orientation of the function.

Proof to this lemma is almost the same as the one in Blackadar, except one minor change: our "connecting morphism", ε_p , is defined explicitly on the middle term (CA in our case) of the sequence. To compare the two maps, we will first need to take a lift from ΣC_q to ΣCA . On the lifted algebra, $\Sigma\alpha$ will be replaced by Σev_0 because given $x \in C_q$, take any lift $\tilde{x} \in CA$, $\tilde{x}(0)(= ev_0(\tilde{x}))$ will always be equal to $\alpha(x)$ by definition of mapping cone. The rest of the proof, which is essentially by a sequence of homotopy, goes as in the C^* -algebra case.

Lemma III.7.12 (c.f. Blackadar 25.5.12). *Let $0 \rightarrow J \xrightarrow{i} A \xrightarrow{q} B \rightarrow 0$ be a σ -unital extension of separable L^p algebras, and D a separable Banach algebra.*

1. *If h is an asymptotic morphism from D to A , and $[q \circ h] = [0]$ in $[[D, B]]$, then there is an asymptotic morphism k from ΣD to ΣJ such that $[\Sigma i \circ k] = [\Sigma h]$ in $[[\Sigma D, \Sigma A]]$.*
2. *If h is an asymptotic morphism from A to D , and $[h \circ i] = [0]$ in $[[J, D]]$, then there is an asymptotic morphism k from $\Sigma^2 B$ to $\Sigma^2 D$ such that $[k \circ \Sigma^2 q] = [\Sigma^2 h]$ in $[[\Sigma^2 A, \Sigma^2 D]]$.*

Sketch of proof. The proof for the first statement goes exactly as in the C^* -algebra case. One fact that is used in both statements is that: if $[\phi] = [0] \in [[A, B]]$, this means that ϕ is homotopic to the 0 asymptotic morphism, or in other words, there exist an asymptotic morphism Φ from A to CB such that $\Phi^{(0)} = \phi$ and $\Phi^{(1)} = 0$.

For the first statement, we obtain from the hypothesis, Φ , an asymptotic morphism in $[[D, CB]]$ such that $\Phi^{(0)} = q \circ h$ and $\Phi^{(1)} = 0$. Define $\Psi = h \oplus \Phi$ an asymptotic morphism in $[[D, C_q]]$, then $[\alpha \circ \Psi] = [h]$. Taking Σ on both sides and apply Lemma III.7.11, we have $[\Sigma i] \circ [\varepsilon_p \circ \Sigma \Psi] = [\Sigma h]$. In other words, we can take k any asymptotic morphism in the homotopy class of $\varepsilon_p \circ \Sigma \Psi$.

For the second statement, the proof in Blackadar works with a slight change in the definition of Ψ_t^r due to the fact that we are using L^p -tensor product (or equivalently the injective tensor product if Σ is involved) instead of projective tensor product. We will sketch the two parts of the proof here.

First, assuming $[h \circ \alpha] = [0]$ in $[[C_q, D]]$, we obtain $\Phi \in [[C_q, CD]]$ as indicated in the first paragraph above. By restricting Φ to ΣB , an ideal of C_q , we obtain an element of $[[\Sigma B, \Sigma D]]$ due to the fact that α is 0 on ΣB . We denote this restricted Φ by k . The claim is that $[k \circ \Sigma q] = [\Sigma h] \in [[\Sigma A, \Sigma D]]$. This can be done by showing that the two asymptotic morphisms are homotopic via $\Psi_t^{(r)} \in [[\Sigma A, C([0, 1], \Sigma D)]]$. Notice that the 0-end of Φ contains h , we will try to construct a map to bridge that with Σh . Define $\beta_r : [r, 1] \rightarrow [0, 1]$ be a scaling homeomorphism. Then we have for $f \in \Sigma A$:

$$[\Psi_t^{(r)}(f)](s) = \begin{cases} h_t(f(s)) & s \leq r \\ \Phi_t^{\beta_r(s)}(f(r) \oplus (\Sigma q)(f \circ \beta_r^{-1})) & s > r \end{cases}$$

Continuity in s at $s = r$ for fixed r and continuity in r as $r \rightarrow 1$ for fixed f is a routine computation.

Second, we would have to relate the assumption with the hypothesis given. We have that $[\Sigma h \circ \Sigma i] = [\Sigma(h \circ i)] = 0$, so $[\Sigma h \circ \Sigma \alpha] = [\Sigma h] \circ [\Sigma i \circ \varepsilon_p] = 0$ by Lemma III.7.11. By the first part of the argument, we get $k \in [[\Sigma^2 B, \Sigma^2 D]]$ with the desired property. \square

Proof of Theorem III.7.10 follows immediately from Lemma III.7.12 and the Bott Periodicity (Theorem III.7.6) result that $E(D, J) \cong E(\Sigma D, \Sigma J)$ and $E(B, D) \cong E(\Sigma^2 B, \Sigma^2 D)$.

The last ingredient to exact sequences for E is a general result about half-exact and homotopy invariant functor.

Theorem III.7.13. [Kas81, p. 564][Bla98, Lem 21.4.1-21.4.3] *Let F be a homotopy-invariant, half-exact covariant functor from \mathcal{S} , the category of separable Banach algebras, to \mathbf{Ab} , the category of abelian groups. Then F has long exact sequences: if $0 \rightarrow J \xrightarrow{j} A \xrightarrow{q} B \rightarrow 0$ is a short exact sequence in \mathcal{S} , then there is a natural long exact sequence*

$$\dots \xrightarrow{j_*} F(\Sigma A) \xrightarrow{q_*} F(\Sigma B) \xrightarrow{\partial} F(J) \xrightarrow{j_*} F(A) \xrightarrow{q_*} F(B).$$

Analogous results hold if F is a contravariant instead of covariant functor from \mathcal{S} to \mathbf{Ab} .

Sketch of proof. First step of the proof is showing that j_* gives an isomorphism between $F(J)$ and $F(A)$ when B is contractible. Assuming B is contractible, we obtain from half-exactness applied to the original exact sequence the surjectivity of j_* . To obtain injectivity, we introduce the algebra $Z = \{f : [0, 1] \rightarrow A \mid f(1) \in J\}$. It can be shown that k , the constant embedding of J into Z is a homotopic equivalence, and hence induces isomorphism between $F(J)$ and $F(Z)$. Now apply half-exactness to the following exact sequences: $0 \rightarrow CJ \rightarrow Z \xrightarrow{\pi} C_q \rightarrow 0$ and $0 \rightarrow \Sigma B \rightarrow C_q \xrightarrow{p} A \rightarrow 0$. We obtain that π_* and p_* are both injective. Since $j = p \circ \pi \circ k$, we have the map $j_* = p_* \circ \pi_* \circ k_*$ is also injective.

For the general case, apply the preliminary result to the exact sequence $0 \rightarrow J \xrightarrow{e} C_q \rightarrow CB \rightarrow 0$. This shows that the map $e(x) = (x, 0)$ induces an isomorphism between $F(J)$ and $F(C_q)$. Set $\partial = e_*^{-1} \circ i_*$ (i is the inclusion map of ΣB into C_q).

Because of Lemma III.6.6, we have that Σ takes short exact sequence to short exact sequence. It remains to check that the sequence is exact at $F(\Sigma B)$ and $F(J)$. The exactness at $F(J)$ comes from half-exactness applied to the exact sequence $0 \rightarrow \Sigma B \rightarrow C_q \rightarrow A \rightarrow 0$ and the isomorphism between $F(J)$ and $F(C_q)$ given by e_* . The exactness at $F(\Sigma B)$ follows from half-exactness applied to

$$0 \rightarrow C_0((-1, 0), A) \rightarrow T \rightarrow C_q \rightarrow 0$$

where T is the algebra defined by $T = \{(f, g) : g(0) = q(f(0))\} \subset C_0((-1, 0], A) \oplus C_0([0, 1), B)$. A useful fact for this last bit is that the canonical embedding $k : \Sigma B \rightarrow T$ induces an isomorphism because the quotient $T/k(\Sigma B)$ is isomorphic to CA by First Isomorphism Theorem.

The proof for contravariant functor is analogous. □

Combining all three ingredients together and we have the desired six-term exact sequences.

Corollary III.7.14 (c.f. Blackadar 25.5.13). *E -theory for L^p -algebras has six-term exact sequences in each variable for σ -unital⁹ extensions of separable L^p -algebras. Specifically, if $0 \rightarrow J \xrightarrow{j} A \xrightarrow{q} B \rightarrow 0$ is a short*

⁹While Theorem III.7.13 gives a long exact sequence for functor F for any short exact sequence of separable Banach algebras, E -theory for L^p -algebra is only half-exact for σ -unital extensions (Theorem III.7.10), hence we need the σ -unitality assumption here.

exact sequence of separable L^p -algebras where J is σ -unital, and D is any separable Banach algebras, then the following six-term sequences are exact:

$$\begin{array}{ccccc}
E(B, D) & \xrightarrow{q^*} & E(A, D) & \xrightarrow{i^*} & E(J, D) \\
\uparrow \varepsilon_q^* & & & & \downarrow \varepsilon_q^* \\
E(\Sigma J, D) & \xleftarrow{i^*} & E(\Sigma A, D) & \xleftarrow{q^*} & E(\Sigma B, D)
\end{array}$$

$$\begin{array}{ccccc}
E(D, J) & \xrightarrow{i_*} & E(D, A) & \xrightarrow{q_*} & E(D, B) \\
\uparrow \varepsilon_{q_*} & & & & \downarrow \varepsilon_{q_*} \\
E(D, \Sigma B) & \xleftarrow{q_*} & E(D, \Sigma A) & \xleftarrow{i_*} & E(D, \Sigma J)
\end{array}$$

III.8 Mayer-Vietoris Exact Sequence

Equipped with the results about exact sequences in the previous section, we are ready to derive the Mayer-Vietoris Exact Sequence for E-theory for Banach algebras:

Theorem III.8.1. [HRY93] *Let A and B be closed, two-sided ideals in a Banach algebra M , such that both $A \cap B$ and M are σ -unital Banach algebras. Assume that $A + B = M$, then there is a Mayer-Vietoris sequence in both variables of E -theory. I.e. let D be any separable Banach algebras, we have the following long-exact sequence:*

$$\begin{aligned}
\cdots \rightarrow E_n(D, A \cap B) &\xrightarrow{j_*} E_n(D, A) \oplus E_n(D, B) \xrightarrow{i_*} E_n(D, M) \xrightarrow{\partial} E_{n-1}(D, A \cap B) \rightarrow \cdots \\
\cdots \rightarrow E_n(M, D) &\xrightarrow{i^*} E_n(A, D) \oplus E_n(B, D) \xrightarrow{j^*} E_n(A \cap B, D) \xrightarrow{\partial} E_{n+1}(M, D) \rightarrow \cdots
\end{aligned}$$

Notation 2. 1. Let D be a Banach algebra, let $D[0, 1] := C([0, 1], D)$, $D(0, 1) := C_0([0, 1], D) = CD$ (Def III.7.9) and $D(0, 1] = \tilde{C}D$.

2. Let A, B , and M be as above, define $\mathcal{C} = \{f \in M[0, 1] : f(0) \in A, f(1) \in B\}$, $\mathcal{T} = \{f \in (M/(A \cap B))[0, 1] : f(0) \in A/(A \cap B), f(1) \in B/(A \cap B)\}$

Lemma III.8.2. \mathcal{T} is contractible.

Proof. Since $A + B = M$, by elementary algebra, we have $A/(A \cap B) \oplus B/(A \cap B) \cong M/(A \cap B)$. Thus we have $(M/(A \cap B))[0, 1] = (A/(A \cap B))[0, 1] \oplus (B/(A \cap B))[0, 1]$ by projection to the two components. This implies that any $f \in (M/(A \cap B))[0, 1]$ has a unique decomposition into $f_A + f_B$ where $f_A \in (A/(A \cap B))[0, 1]$, $f_B \in (B/(A \cap B))[0, 1]$.

Now since $(A/(A \cap B)) \cap (B/(A \cap B)) = 0$, for any $f \in \mathcal{T}$, we have $f_B(0) = 0$ and $f_A(1) = 0$. This shows that in fact $f_B \in \tilde{C}(B/(A \cap B))$, $f_A \in C(A/(A \cap B))$ and $\mathcal{T} = C(A/(A \cap B)) \oplus \tilde{C}(B/(A \cap B))$ – sum of two contractible Banach algebras, hence contractible. \square

Proof of Theorem. We will prove the covariant case. The contravariant case is analogous.

Consider the map $\mathcal{C} \rightarrow A \oplus B$ by evaluation at the endpoints 0 and 1. This is a surjective continuous homomorphism with kernel ΣM , which is a closed two-sided ideal in \mathcal{C} . ΣM is σ -unital because M is (by assumption). This gives the following σ -unital extension:

$$0 \rightarrow \Sigma M \xrightarrow{i} \mathcal{C} \xrightarrow{q} A \oplus B \rightarrow 0$$

By Theorem III.7.13, for separable Banach algebra D , we have the following long exact sequence

$$\cdots \rightarrow E_n(D, \mathcal{C}) \xrightarrow{q_*} E_n(D, A) \oplus E_n(D, B) \xrightarrow{\partial} E_{n-1}(D, \Sigma M) \xrightarrow{i_*} E_{n-1}(D, \mathcal{C}) \rightarrow \cdots$$

Next we will consider the natural quotient map from $M[0, 1]$ to $(M/(A \cap B))[0, 1]$. This is a surjection because of the fact that injective tensor product with $C(K)$ (in our case $K = [0, 1]$) preserves quotient operator¹⁰ [Rya02, Prop 3.5]. When we restrict this map to \mathcal{C} , it is easy to see that the image is all of \mathcal{T} and the kernel is $(A \cap B)[0, 1]$. This gives the following short exact sequence:

$$0 \rightarrow (A \cap B)[0, 1] \rightarrow \mathcal{C} \rightarrow \mathcal{T} \rightarrow 0$$

By the assumption that $A \cap B$ is σ -unital, we have that $(A \cap B)[0, 1]$ is also σ -unital. By Theorem III.7.13 and Lemma III.8.2, we have that $E_n(D, (A \cap B)[0, 1]) \cong E_n(D, \mathcal{C})$. Similarly, by considering the following σ -unital extension:

$$0 \rightarrow (A \cap B)(0, 1] \rightarrow (A \cap B)[0, 1] \rightarrow A \cap B \rightarrow 0$$

we have $E_n(D, (A \cap B)[0, 1]) \cong E_n(D, A \cap B)$. Applying these two results to the long exact sequence of E-theory above, and by Bott Periodicity (Theorem III.7.6), which gives $E_{n-1}(D, \Sigma M) \cong E_n(D, M)$, we obtain the following long exact sequence:

$$\cdots \rightarrow E_n(D, A \cap B) \xrightarrow{q_*} E_n(D, A) \oplus E_n(D, B) \xrightarrow{\partial} E_n(D, M) \xrightarrow{i_*} E_{n-1}(D, A \cap B) \rightarrow \cdots$$

□

¹⁰ $Q: Z \rightarrow Y$ is a *quotient operator* is equivalent to the fact that Y is isometrically isomorphic to the quotient space $Z/\ker Q$ [Rya02, p.18]

CHAPTER IV

APPLICATION: DUAL NOVIKOV CONJECTURE

IV.1 Equivariant E-Theory

In this section, we will discuss changes necessary to make the E-theory we defined earlier suitable for the study of Dual Novikov Conjecture (to be defined in the next section), where group action is involved. As mentioned briefly in the Introduction, this approach to the Novikov conjecture that we use here has its origin in the two joint work of Kasparov and Yu, [KY05; KY12]. In particular, algebras similar to $\mathcal{A}(X)$, which we are to define in Section IV.4, were also used in these references and works of others [HK01; HKT98].

In the following, G will be a fixed locally compact σ -compact topological group. A separable Banach algebra with a pointwise norm-continuous action of G by automorphisms will be called a G -algebra. Let A and B be G -algebras.

First, we will define the object of our study - *equivariant asymptotic morphisms*. They are asymptotic morphisms $\{\phi_t\}$ which are (pointwise) asymptotically equivariant, i.e. satisfying the additional condition:

$$\lim_{t \rightarrow \infty} \|\phi_t(g \cdot a) - g \cdot \phi_t(a)\| = 0, \quad g \in G, a \in A$$

Recall that in the non-equivariant case, we define composition of asymptotic morphisms on their uniformly continuous representatives (obtained from Bartle-Graves Selection Theorem). Uniformly continuous estimate are also important for defining composition of equivariant asymptotic morphisms.

Definition IV.1.1. [Tho, Def 1.1.9] Let A , B and C be G -algebras, and let $\phi = \{\phi_t\}_{t \in [1, \infty)} : A \rightarrow B$ and $\psi = \{\psi_t\}_{t \in [1, \infty)} : B \rightarrow C$ be uniformly continuous asymptotic equivariant homomorphisms. A *composition pair* for ψ and ϕ , is a dense subset $Y \subset A$ which is the union of a sequence of compact subsets containing 0 and a parametrization $r : [1, \infty) \rightarrow [1, \infty)$ such that

- i) $\lim_{t \rightarrow \infty} \sup_{s \geq r(t)} \|\psi_s \circ \phi_t(ab) - \psi_s \circ \phi_t(a)\psi_s \circ \phi_t(b)\| = 0$ for all $a, b \in Y$,
- ii) $\lim_{t \rightarrow \infty} \sup_{s \geq r(t)} \|\psi_s \circ \phi_t(a + \lambda b) - \psi_s \circ \phi_t(a) - \lambda \psi_s \circ \phi_t(b)\| = 0$ for all $a, b \in Y$ and all $\lambda \in \mathbb{C}$,
- iii) $\lim_{t \rightarrow \infty} \sup_{s \geq r(t)} \|\psi_s \circ \phi_t(a)\| \leq \|\psi\| \|\phi\| \|a\|$ for all $a \in Y$,
- iv) for every compact subset $K \subset G$, every pair $a, b \in Y$ and every $\varepsilon > 0$ there is a $t_0 \in [1, \infty)$ such that

$$\sup_{s \geq r(t)} \|g \cdot \psi_s \circ \phi_t(a) - \psi_s \circ \phi_t(h \cdot b)\| \leq \|g \cdot a - h \cdot b\| + \varepsilon$$

for all $g, h \in K$ when $t \geq t_0$

Note that if (Y, r) is a composition pair for ϕ and ψ , then so is (Y, r') for any parametrization r' such that $r' \geq r$.

Proposition IV.1.2. [Tho, Prop 1.1.10] Let A, B and C be G -algebras. Let $\phi : A \rightsquigarrow B$ and $\psi : B \rightsquigarrow C$ be uniformly continuous equivariant asymptotic morphisms, and let $Y \subset A$ be a dense subset which is the union of a countable family of compact sets containing 0 .

- a) There is then a parametrization $r : [1, \infty) \rightarrow [1, \infty)$ such that (Y, r) is a composition pair for ψ and ϕ .
- b) When $K_1 \subset K_2 \subset K_3 \subset \dots$ are compact subsets of X with $\cup_n K_n = Y$, we can arrange that the following holds: For all n and all $\varepsilon > 0$ there is a $t_0 \in [1, \infty)$ such that

$$\sup_{s \geq r(t)} \|\psi_s \circ \phi_t(a) - \psi_s \circ \phi_t(b)\| \leq \|a - b\| + \varepsilon$$

for all $a, b \in K_n$ and all $t \geq t_0$

Proof of the proposition goes exactly the same as for the corresponding result in the non-equivariant case (Proposition III.3.1), by recursively picking t_n to approximate to increasing precision on the sequence of exhausting compact subset K_n .

Next, we will discuss equivariant asymptotic morphisms associated to extension of G -algebras. An extension $0 \rightarrow J \xrightarrow{j} E \xrightarrow{p} A \rightarrow 0$ of Banach algebras, where A, E and J are G -algebras, is said to be an *extension of G -algebras* (or just a *G -extension*) when j and p are G -equivariant. We need first a result about the existence of quascentral approximate unit that is compatible with group action, which we call *asymptotically G -invariant approximate unit*. It is a quascentral approximate unit $\{u_t\}$ with the additional condition:

$$\limsup_{t \rightarrow \infty} \sup_{g \in K} \|g \cdot u_t - u_t\| = 0 \text{ for every compact subset } K \subset G$$

This is given in the second part of Proposition III.5.3 for J σ -unital, closed two-sided G -ideal in a G -algebra B such that the quotient B/J is separable.

We observe that our construction for "connecting morphism" (Proposition III.6.1) yields a equivariant asymptotic homomorphism for a σ -unital extension of G - L^p -algebras when $\{u_t\}$ is a continuous asymptotically G -invariant approximate unit.

Let A and B be separable G - L^p -algebras. We define $[[A, B]]_G$ to be the semi-group consisting of homotopy classes of G -equivariant asymptotic morphisms from A to B . We define

$$E_G(A, B) = [[\Sigma A \otimes \mathbb{K}(L^p(Z, L^p(G))), \Sigma B \otimes \mathbb{K}(L^p(Z, L^p(G)))]_G$$

IV.2 RE-theory

In this section, we will define *RE*-theory. The notation is inspired by the definition of $RKK^G(X; A, B)$ in [Kas88], which generalizes both $KK^G(A, B)$ and the representable K -functor group $RK^0(X)$ of a locally compact, σ -compact space X . According to the remark in the proof to Proposition 2.20 in [Kas88], any pair

$(\mathcal{E}, T) \in RKK^G(X; A, B)$ gives rise to a continuous field of Hilbert B -modules \mathcal{E}_x and operators T_x on \mathcal{E}_x over X .

Definition IV.2.1. We define a *continuous family of (equivariant) asymptotic morphism over Z* to be a family of asymptotic morphism over Z :

$$\phi_{t,z} : A \rightsquigarrow B$$

such that:

- a) For each a, t , the map $z \mapsto \phi_{t,z}(a)$ is norm continuous
- b) For each $a \in A$, $t \mapsto \phi_{t,z}(a)$ is uniformly continuous on compact subset of Z . This means that for K compact subset of Z , given $t_0, \varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in B(t_0, \delta)$:

$$\sup_{z \in K} \|\phi_{t,z}(a) - \phi_{t_0,z}(a)\| < \varepsilon$$

- c) For each $a_1, a_2 \in A$, $\lambda \in \mathbb{C}$, K compact subset of Z , we have:

$$\limsup_{t \rightarrow \infty} \sup_{z \in K} \|\phi_{t,z}(a_1 + \lambda a_2) - (\phi_{t,z}(a_1) + \lambda \phi_{t,z}(a_2))\| = 0$$

$$\limsup_{t \rightarrow \infty} \sup_{z \in K} \|\phi_{t,z}(a_1 a_2) - \phi_{t,z}(a_1) \phi_{t,z}(a_2)\| = 0$$

$$\limsup_{t \rightarrow \infty} \sup_{z \in K} \|\phi_{t,z}(g \cdot a_1) - g \cdot \phi_{t,z}(a_1)\| = 0 \quad (\text{additional condition for equivariant case})$$

$RE_G(Y; A, B)$ is defined as group of homotopy classes of continuous family of equivariant asymptotic morphism over Y from $\Sigma A \otimes \mathbb{K}(L^p(Z, L^p(G)))$ to $\Sigma B \otimes \mathbb{K}(L^p(Z, L^p(G)))$ while the non-equivariant version $RE(Y; A, B)$ is defined as the group of homotopy classes of continuous family of asymptotic morphism over Y from $\Sigma A \otimes \mathbb{K}(L^p(Z))$ to $\Sigma B \otimes \mathbb{K}(L^p(Z))$

Note that $RE(Y; A, B)$ is isomorphic to $E(A, C_0(Y, B))$ when Y is a compact space. This will be a useful fact in our proof of the dual Novikov conjecture. We note also that $RE(B\Gamma; \mathbb{C}, \mathbb{C})$ can be identified as $RK^0(B\Gamma)$, the representable K -theory group of the classifying space of group Γ . And more generally, $RE(Y; \mathbb{C}, \mathbb{C}) \cong RK^0(Y)$ for locally compact set Y .

IV.3 Clifford algebras of L^p space

Let X be $L^p(Z, \mu)$, the space of real-valued p -integrable function over a μ -measureable space Z . X is a Banach space over \mathbb{R} for $p \geq 1$. We know that X^* , its dual space, is isomorphic to the real Banach space $L^q(Z, \mu)$ where $1/p + 1/q = 1$. Denote by $\otimes_{alg}^n X$ the vector space tensor product $\overbrace{X \otimes \cdots \otimes X}^n$ for $n \geq 1$, and also put $\otimes^0 X = \mathbb{R}$.

The norm on $\otimes^0 X$ is the standard norm. For $n \geq 1$, we give $\otimes_{alg}^n X$ the L^p tensor norm. Given $L^p(Z_1, \mu)$ and $L^p(Z_2, \nu)$ two L^p Banach space, their L^p tensor product is defined as completion of the algebraic tensor

product in a tensor norm such that:

$$L^p(Z_1, \mu) \otimes L^p(Z_2, \nu) \cong L^p(Z_1 \times Z_2, \mu \times \nu)$$

Define $\otimes^n X$ to be the completion of $\otimes_{alg}^n X$ with respect to the above norm.

Let

$$T(X) = \left\{ \bigoplus_{n=0}^{\infty} a_n : a_n \in \otimes^n X, \sum_{n=0}^{\infty} \|a_n\|^p < +\infty \right\}.$$

Then $T(X)$ is a Banach space over \mathbb{R} with the norm $\| \bigoplus_{n=0}^{\infty} a_n \|^p = \sum_{n=0}^{\infty} \|a_n\|^p$.

If we have a bilinear pairing $q : X \times X \rightarrow \mathbb{R}$, we can define a Clifford algebra as follows. Let $I_q(X)$ be the closed two-sided ideal in $T(X)$ generated by all elements of the form: $x_1 \otimes x_2 + x_2 \otimes x_1 - q(x_1, x_2)$, $x_1, x_2 \in X$. The Clifford algebra $Cl(X, q)$ is defined as the complexified quotient Banach algebra: $Cl(X, q) = (T(X)/I_q(X)) \otimes \mathbb{C}$. Note that there is a bounded linear map $X \rightarrow T(X) \rightarrow Cl(X)$.

If we have two L^p spaces X_1 and X_2 equipped with bilinear pairings q_1 and q_2 respectively, we can define the Clifford algebra $Cl(X_1 \oplus X_2, q_1 \oplus q_2)$. There are uniquely defined natural homomorphisms $Cl(X_1, q_1) \rightarrow Cl(X_1 \oplus X_2, q_1 \oplus q_2)$ and $Cl(X_2, q_2) \rightarrow Cl(X_1 \oplus X_2, q_1 \oplus q_2)$.

Moreover, we can define a Banach algebra $Cl(X_1, q_1) \hat{\otimes} Cl(X_2, q_2)$ as the L^p tensor product of Banach spaces $Cl(X_1, q_1)$ and $Cl(X_2, q_2)$ with the product given by $(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{\deg a_2 \cdot \deg b_1} a_1 b_1 \otimes a_2 b_2$. There is a natural homomorphism $Cl(X_1, q_1) \hat{\otimes} Cl(X_2, q_2) \rightarrow Cl(X_1 \oplus X_2, q_1 \oplus q_2)$ which extends the two above homomorphisms of $Cl(X_1, q_1)$ and $Cl(X_2, q_2)$.

Let us now apply this construction to the Banach space $W = X \oplus X^*$. We define a pairing $q : W \times W \rightarrow \mathbb{R}$ by $q(w_1, w_2) = (x_1, y_2) + (x_2, y_1)$ for all $w_1 = x_1 \oplus y_1, w_2 = x_2 \oplus y_2 \in W = X \oplus X^*$. The Clifford algebra $Cl(W, q)$ will be denoted $Cl(W)$. In the case of a finite-dimensional space V , the algebra $Cl(V \oplus V^*)$ is isomorphic to $\mathcal{L}(\Lambda_{\mathbb{C}}^*(V))$, the full matrix algebra of the exterior algebra space $\Lambda_{\mathbb{C}}^*(V)$.

IV.4 Banach algebra of a l_p space.

In this section, we will define an algebra $A(X)$ associated to a Banach space X . We will first define $A(V)$ for finite-dimensional subspace V of X , and then define $A(X)$ using the direct limit construction as in the case of a Hilbert space X [HK01]. A necessary condition for his construction is that X has property π [Cas01], which means that there exists an increasing sequence of finite dimensional subspace $\{V_n\}$ of X , such that the corresponding projections $P_n : X \rightarrow V_n$ satisfies:

1. $\cup V_n$ is dense in X
2. $\sup_n \|P_n\| < \infty$

In the following discssion, we will assume that X is $l^p(\mathbb{N})$ (also known as l_p), the space of p -summable sequence of real numbers, where the norm for an element $v = (a_n)$ is given by

$$\|v\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}$$

Its dual space X^* is given by $l^q(\mathbb{N})$ where $1/p + 1/q = 1$. The pairing between X and X^* is given by:

$$((a_n), (b_n)) = \sum_{n=1}^{\infty} a_n b_n \text{ for } (a_n) \in X, (b_n) \in X^*$$

If we denote by e_n the sequence which is 1 at position n and 0 everywhere else. Then e_n is a unit vector in $l^p(\mathbb{N})$ for any $p \geq 1$ for each n . For clarity, we will use the notation e_n^* to distinguish a unit vector in X^* from its counterpart in X in this paper.

We will start our construction by defining a certain subalgebra \mathcal{S} of $C_0(\mathbb{R})$.

Denote by Cl_1 the complex Clifford algebra generated by the elements $\{1, \varepsilon\}$ with the relation: $\varepsilon^2 = 1$. The norm is: $\|a + b\varepsilon\| = |a| + |b|$. Our algebra \mathcal{S} will be a completion of an algebra of compactly supported continuous functions $f : \mathbb{R}_+ \rightarrow Cl_1$ in the norm given below. The functions of the algebra have the form: $f(s) = f_0(s^2) + \varepsilon s f_1(s^2)$, where both f_0 and f_1 are complex-valued continuous functions on \mathbb{R}_+ , vanishing at ∞ . We will consider $f_0(s^2)$ and $s f_1(s^2)$ as the even and the odd part of a continuous function on \mathbb{R} .

We define the norm on \mathcal{S} by $\|f\| = \sup_{s \in \mathbb{R}_+} (|f_0(s^2)| + |s f_1(s^2)|)$. The algebra \mathcal{S} has a natural grading induced by the grading of Cl_1 . If we consider the algebra \mathcal{S} as ungraded, it is isomorphic to $C_0(\mathbb{R})$ (the isomorphism is not isometric).

We define a sequence of finite-dimensional subspaces of X and X^* . Let $V_j = \text{span}\{e_1, e_2, \dots, e_j\} \subset X$ and $W_j = \text{span}\{e_1^*, e_2^*, \dots, e_j^*\} \subset X^*$. We also define a "star" map ($*$) between X and X^* which maps V_j to the corresponding W_j .

$$* : X \rightarrow X^*; \sum a_n e_n \mapsto \sum a_n |a_n|^{p-2} e_n^*$$

It is easy to verify that the image sequence belongs to l^q by using the relation $1/p + 1/q = 1$. We have also that $(v, v^*) = \|v\|_p^p$. To simplify notation below, we will define the notation $a_n^* = a_n |a_n|^{p-2}$. Define $\psi : X \times X \rightarrow Cl(X \oplus X^*)$ by:

$$((a_n), (b_n)) \mapsto ((a_n + b_n i)) \oplus \left(\left((a_n^* - b_n^* i) \frac{a_n a_n^* + b_n b_n^*}{(a_n a_n^* + b_n b_n^*) + i(b_n a_n^* - a_n b_n^*)} \right) \right)$$

Assuming norm of the complexified X (respectively X^*) is given simply by taking the sum of p^{th} (respectively q^{th}) power of the modulus of each complex number in the sequence. This is a *reasonable complexification* as defined in [MST99] and hence is equivalent to any other *reasonable complexification norms* on X and X^* (see [MST99, Prop 3]).

We will show that the image is contained in complexified $X \oplus X^*$. First note that, $(a_n), (b_n) \in l_p$ implies that $(|a_n|), (|b_n|) \in l_p$ because they have the same l_p norm. Because l_p is a linear space, this means that $(|a_n| + |b_n|) \in l_p$ and hence $(a_n + b_n i) \in l_p$ (complexified) because $|a_n + b_n i| \leq |a_n| + |b_n|$ by triangle inequality. On the other hand, noticing that $|(a_n a_n^* + b_n b_n^*) + i(b_n a_n^* - a_n b_n^*)| \geq |(a_n a_n^* + b_n b_n^*)|$ (modulus of any complex number is larger than absolute value of its real part), similar argument as above shows that the second component indeed belongs to complexified l_q .

We also compute $[\psi(v, w)]^2$, by definition of the Clifford algebra, we have that it is equal to:

$$\left(((a_n + b_n i), \left((a_n^* - b_n^* i) \frac{a_n a_n^* + b_n b_n^*}{(a_n a_n^* + b_n b_n^*) + i(b_n a_n^* - a_n b_n^*)} \right) \right)$$

which is equal to $\sum_n (a_n a_n^* + b_n b_n^*) = \sum_n (|a_n|^p + |b_n|^p)$. Notice that $[\psi(v, w)]^2 \rightarrow \infty$ as $v \rightarrow \infty$ or $w \rightarrow \infty$. This is important for guaranteeing that the Bott map we are to define below maps to elements which vanish at infinity of the input vector space.

For any V finite dimensional subspace of X , we define $A(V) = \mathcal{S} \hat{\otimes} C_0(V \times V, Cl(V \oplus V^*))$. Let $V_{j_1} \subset V_{j_2} \subset X$ and $W_{j_1} \subset W_{j_2} \subset X^*$, we will use the identifications $V_k^* \simeq W_k$. We have the map $Cl(V_{j_1} \oplus W_{j_1}) \rightarrow Cl(V_{j_2} \oplus W_{j_2})$. Vectors in $V_{j_2} \times V_{j_2}$ can be mapped to $V_{j_1} \times V_{j_1}$ via projection. This gives the map from $C_0(V_{j_1} \times V_{j_1}, Cl(V_{j_1} \oplus W_{j_1}))$ to the multiplier algebra of $C_0(V_{j_2} \times V_{j_2}, Cl(V_{j_2} \oplus W_{j_2}))$. We will illustrate this with a function $f \in C_0(V_{j_1} \times V_{j_1}, Cl(V_{j_1} \oplus W_{j_1}))$. We may define $\tilde{f} \in \mathcal{M}(C_0(V_{j_2} \times V_{j_2}, Cl(V_{j_2} \oplus W_{j_2})))$ by:

$$\tilde{f} : V_{j_2} \times V_{j_2} \xrightarrow{\text{projection}} V_{j_1} \times V_{j_1} \xrightarrow{f} Cl(V_{j_1} \oplus W_{j_1}) \rightarrow Cl(V_{j_2} \oplus W_{j_2})$$

Now let $Y_1 \subset V_{j_2}$ be the orthogonal complement of W_{j_1} in V_{j_2} and $Z_1 \subset W_{j_2}$ the orthogonal complement of V_{j_1} in W_{j_2} . Recall that (e_n) and (e_n^*) form a biorthogonal system, so there is a neat representation for both of the orthogonal complement as the span of some finitely many members of (e_n) or (e_n^*) . We have direct sum decompositions: $V_{j_2} = V_{j_1} \oplus Y_1, W_{j_2} = W_{j_1} \oplus Z_1$, which allow us to write $Cl(V_{j_2} \oplus W_{j_2})$ as $Cl(V_{j_1} \oplus W_{j_1}) \hat{\otimes} Cl(Y_1 \oplus Z_1)$.

Let \mathbb{R}_+ be the positive half-line $\{s \in \mathbb{R} | s \geq 0\}$. For any $f \in \mathcal{S}$, we define a multiplier of $A(V_{j_2})$ by the formula

$$(s, v, w) \mapsto f_0(s^2 + \psi(v, w)^2) + f_1(x^2 + \psi(v, w)^2)(s \mathcal{E} \hat{\otimes} 1 + 1 \hat{\otimes} \psi(v, w))$$

where $v \in Y_1, w \in Z_1$. This gives the Bott map $\mathcal{S} \rightarrow A(Y_1)$. In fact, the Bott map $\mathcal{S} \rightarrow A(V_n)$ is defined for any $V_n \subset X$ by the same formula.

Finally, the homomorphism $\rho_{j_1, j_2} : A(V_{j_1}) \rightarrow A(V_{j_2})$ is defined by the formula:

$$f \hat{\otimes} h \mapsto f_0(s^2 + \psi(v, w)^2) 1 \hat{\otimes} h \hat{\otimes} 1 + f_1(s^2 + \psi(v, w)^2)(s \mathcal{E} \hat{\otimes} h \hat{\otimes} 1 + 1 \hat{\otimes} h \hat{\otimes} \psi(v, w))$$

where $f \hat{\otimes} h \in \mathcal{S} \hat{\otimes} C_0(V_{j_1} \times V_{j_1}, Cl(V_{j_1} \oplus W_{j_1}))$.

The system of transition maps $\{\rho_{ij} : A(V_i) \rightarrow A(V_j) | i < j\}$ defined above is transitive (similar calculation as that in [HKT98, Prop 3.2], by comparing the two maps on e^{-x^2} and $x e^{-x^2}$, the two generators of \mathcal{S}), hence we have an inductive system $\{A(V_n), \rho_{ij}\}$.

We may then define $A(X)$ as the inductive limit $\varinjlim A(V_n)$. The K-theory of the finite dimensional algebra $A(V)$ is known to be isomorphic to that of \mathcal{S} via the Bott map defined above, so the same is true for $A(X)$.

As in [HK01, Prop 4.9], $A(X)$ contains a large commutative subalgebra $A_0(X)$ such that $A_0(X) \cdot A(X)$ is dense in $A(X)$. We will explain its construction and describe its maximal ideal space, which will be helpful for our argument later in the proof of Dual Novikov Conjecture. Let $C_X = \mathbb{R}_+ \times X \times X$.

Define $A_0(V_n)$ to be the center of $A(V_n)$. It is equal to $\mathcal{S}^{ev} \otimes C_0(V_n \times V_n)$, where \mathcal{S}^{ev} is the subalgebra of all even functions in \mathcal{S} . This algebra is isomorphic to the algebra of continuous functions, vanishing at infinity, on the locally compact space $[0, \infty) \times V_n \times V_n$. The maps of our inductive system, $\rho_{ij} : A(V_i) \rightarrow A(V_j)$, corresponding to the embedding $V_i \subset V_j$, carry these subalgebras into each other: $A_0(V_i) \rightarrow A_0(V_j)$. So we can form the direct limit $A_0(X)$. It has the property that $A_0(X) \cdot A(X)$ is dense in $A(X)$. The maximal ideal space of the algebras get mapped in the opposite direction and form an inverse system of topological space:

$$\mathbb{R}_+ \leftarrow \mathbb{R}_+ \times V_1 \times V_1 \leftarrow \mathbb{R}_+ \times V_2 \times V_2 \leftarrow \dots$$

Proposition IV.4.1. *The space $C_X = [0, \infty) \times X \times X$ endowed with the weakly topology in which all functions $\{s^2 + \psi(v - a, w - b)^2 | a \in V_n, b \in W_n \text{ for some } n \in \mathbb{N}\}$ are continuous, is a dense subspace of the maximal ideals space M_X of $A_0(X)$. Any subset of C_X which is bounded in the metric of C_X (i.e. any subset on which the function $x^2 + \psi(v, w)^2$ is bounded) has compact closure in M_X , and vice versa: the intersection of any compact subset of M_X with C_X is bounded in the metric of C_X .*

Proof. The first assertion follows from the fact that the functions $\{(s^2 + \psi((v, w) - (a, b))^2) | a \in V_n, b \in W_n\}$ separate points in C_X . The second assertion follows from the fact that the same set of functions are bounded exactly on bounded subsets of C_X , and neighborhoods of infinity of the maximal ideal space of $A_0(X)$ are complements of the compact sets of the type $\{(s^2 + \psi((v, w) - (a, b))^2) \leq \text{const}\}$ \square

IV.5 Dual Novikov Conjecture

In this section, we will apply the machinery developed in the previous section to prove a result related to the Novikov Conjecture. Let's begin with the statement of Novikov conjecture.

Let M^n be an oriented, closed, smooth manifold. Denote by $L_*(M^n) \in H^*(M^n; \mathbb{Q})$ its Pontrjagin-Hirzebruch characteristic class. Then the Hirzebruch signature theorem [Hir] states that:

$$\text{signature}(M) = \langle L_*(M^n), [M^n] \rangle$$

this shows that the value of $L_*(M^n)$ on the fundamental cycle $[M^n]$ of M^n is a homotopy invariant of M^n . Fixing a countable discrete group π and denote by $B\pi$ its classifying space. For any continuous map $f : M^n \rightarrow B\pi$ and any element $x \in H^*(B\pi; \mathbb{Q})$, S.P. Novikov defined the *higher signatures of M^n* :

$$\langle L_*(M^n) \cup f^*(x), [M^n] \rangle$$

and conjectured that [Nov70] for any π, f and x , they depend only on a homotopy type of M^n . When translated into the language of K -theory, this is a consequence of the Strong Novikov Conjecture [Kas75], which asserts that the following *assembly map* is injective:

$$\beta \otimes \mathbb{Q} : RK_*(B\pi) \otimes \mathbb{Q} \rightarrow K_*(C^*(\pi)) \otimes \mathbb{Q}$$

In the C^* -algebra case, there is a natural pairing between $K_*(C^*(\pi))$ and $R^*(\pi)$ - the ring of Fredholm

representations of the group π , which relates the assembly map above with the following natural map:

$$\alpha : R^*(\pi) \rightarrow RK^*(B\pi)$$

When $B\pi$ is compact, the surjectivity of α would imply the injectivity of $\beta \otimes \mathbb{Q}$ because they are dual. Hence the surjectivity of α is called the Dual Novikov Conjecture.

The example of application of the above asymptotic E-theory for Banach L^p -algebras presented here will be an l_p -version of the dual Novikov conjecture which naturally leads to the l_p Novikov conjecture. The Strong Novikov conjecture (but not an L^p Novikov conjecture) was proved for groups acting on L^p -spaces in [KY12].

Theorem IV.5.1. *Let X be an l_p Banach space. Assume that a discrete torsion-free group Γ acts on X isometrically and metrically properly. Then the map*

$$v : E_\Gamma^*(\mathbb{C}, \mathbb{C}) \rightarrow RE_\Gamma^*(E\Gamma; \mathbb{C}, \mathbb{C}) \simeq RK_\Gamma^*(E\Gamma) \simeq RK^*(B\Gamma),$$

where $E\Gamma$ means the universal free Γ -space, is surjective.

The RE group on the right is defined as the group of continuous families of E -theory elements parametrized by the space $E\Gamma$. The space $E\Gamma$ is realized as a locally compact space. The group Γ acts both on the parameter space and on the algebras involved.

The main diagram:

$$\begin{array}{ccc} E_\Gamma^*(\mathcal{A}(X), \mathbb{C}) & \longrightarrow & RE_\Gamma^*(E\Gamma; \mathcal{A}(X), \mathbb{C}) \simeq RE^*(B\Gamma; \mathcal{A}(X), \mathbb{C}) \\ \downarrow & & \downarrow \\ E_\Gamma^*(\mathcal{S}, \mathbb{C}) & \longrightarrow & RE_\Gamma^*(E\Gamma; \mathcal{S}, \mathbb{C}) \simeq RE^*(B\Gamma; \mathcal{S}, \mathbb{C}). \end{array}$$

In this diagram, the horizontal arrows map a single E -theory element into the family which is constant over $E\Gamma$. The vertical arrows are induced by the Bott maps.

We have to prove that the upper horizontal arrow and the right vertical arrow are isomorphisms. The surjectivity of the bottom horizontal arrow will follow.

Let's recall Milnor's \lim^1 exact sequence [Mil62].

Lemma IV.5.2. *Let \mathcal{W} be the category of all pairs (X, A) such that both X and A have homotopy type of a CW-complex; and all continuous maps between such pairs. Consider a telescope of CW-complexes:*

$$K_1 \subset K_2 \subset K_3 \subset \dots$$

with union K , each K_i subcomplex of K . Let H^* be an additive cohomology theory on \mathcal{W} , then we have the following exact sequence

$$0 \rightarrow \lim_{\leftarrow}^1 (H^{n-1}(K_i)) \rightarrow H^n(K) \rightarrow \lim_{\leftarrow} (H^n(K_i)) \rightarrow 0$$

Lemma IV.5.3. *The right vertical arrow is an isomorphism.*

Proof. First of all, the isomorphisms of the RE^* -groups on the right of the diagram follow from the fact that Γ acts on $E\Gamma$ freely and properly. In fact, any non-equivariant asymptotic morphism parametrized by $B\Gamma$ naturally lifts to an equivariant one. In the opposite direction, we use averaging over Γ of an asymptotic morphism parametrized by $E\Gamma$ after multiplying it by a cut-off function on $E\Gamma$.

The Bott map $\mathcal{S} \rightarrow \mathcal{A}(X)$ naturally induces the homomorphism

$$RE^*(B\Gamma; \mathcal{A}(X), \mathbb{C}) \rightarrow RE^*(B\Gamma; \mathcal{S}, \mathbb{C}).$$

To show that it is an isomorphism, we will use a realization of the space $B\Gamma$ as a telescope of its compact CW -subspaces. More specifically, we recall the *infinite join construction of Milnor*[Mil56], which states that $E\Gamma = \Gamma * \Gamma * \Gamma * \dots$ and $B\Gamma = E\Gamma/\Gamma$. Since Γ is discrete, we may assume that its element to be $\{x_n\}_{n=1}^\infty$. We may define K_n to be the space of orbits of the n -fold join of the set $\{x_1, \dots, x_n\}$:

$$(\{x_1, \dots, x_n\} * \dots * \{x_1, \dots, x_n\})\Gamma/\Gamma$$

We have that K_n is compact CW for each n and that $\lim K_n = B\Gamma$.

It is enough to prove that for any compact CW -space Z , the homomorphism $RE^*(Z; \mathcal{A}(X), \mathbb{C}) \rightarrow RE^*(Z; \mathcal{S}, \mathbb{C})$ is an isomorphism. This follows from exact sequences in the second argument, using induction on dimension of Z and the 5-lemma. The assertion for $B\Gamma$ will follow from Milnor's \lim^1 exact sequence.

We first note that for any compact space Z , we can rewrite our RE^* -groups as $E^*(\mathcal{A}(X), C(Z))$ and $E^*(\mathcal{S}, C(Z))$ respectively. Define $HE_A(Z) := E(A(X), C_0(Z))$ and $HE_S(Z) := E(\mathcal{S}, C_0(Z))$. $HE_A(Z)$ and $HE_S(Z)$ are both additive, homotopy invariant contravariant functors and have Mayer-Vietoris exact sequence. So the proof for Milnor's \lim^1 exact sequence applies and if we have a telescope of compact CW -complexes $\{K_n\}$ and isomorphism between the two functors on each K_n as follows:

$$\begin{array}{ccccccc} \dots & \longrightarrow & HE_A^*(K_n) & \longrightarrow & HE_A^*(K_{n-1}) & \dots & \longrightarrow & HE_A^*(K_1) \\ & & \downarrow \cong & & \downarrow \cong & & & \downarrow \cong \\ \dots & \longrightarrow & HE_S^*(K_n) & \longrightarrow & HE_S^*(K_{n-1}) & \dots & \longrightarrow & HE_S^*(K_1) \end{array}$$

we have the following natural maps between the corresponding Milnor's \lim^1 sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim_{\leftarrow}^1 (HE_A^{n-1}(K_i)) & \longrightarrow & HE_A^n(K) & \longrightarrow & \lim_{\leftarrow} (HE_A^n(K_i)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \lim_{\leftarrow}^1 (HE_S^{n-1}(K_i)) & \longrightarrow & HE_S^n(K) & \longrightarrow & \lim_{\leftarrow} (HE_S^n(K_i)) & \longrightarrow & 0 \end{array}$$

Because the left two and right two vertical arrows are isomorphisms, we have the isomorphism in the middle by the five lemma. \square

Lemma IV.5.4. *The upper horizontal arrow is an isomorphism.*

Before we go into the proof of the second lemma, let's recall some definitions about different notions of *properness*.

The affine isometric action of Γ on X can be written in the following form [HK01]:

$$g \cdot v = \pi(g)v + \kappa(g)$$

where π is a linear isometric representation of Γ on X and κ is a one-cocycle on Γ with values in X . Thus κ is a continuous function from Γ into X such that $\kappa(g_1g_2) = \pi(g_1)\kappa(g_2) + \kappa(g_1)$.

Definition IV.5.5. [HK01; Kas88; KY05] The affine isometric action of Γ on X is *metrically proper* if $\lim_{g \rightarrow \infty} \|\kappa(g)\| = \infty$.

Action of a group G on a locally compact Z is called *proper* if for any two compact subsets K and L in Z the set $\{g \in G | g(K) \cap L \neq \emptyset\}$ is compact in G . If G is discrete, then the said set would be finite.

A , a Banach algebra with continuous action of G by automorphism, is called a *proper G -algebra* if there is a locally compact, second-countable, proper G -space Z and an equivariant Banach algebra homomorphism from $C_0(Z)$ into the center of the multiplier algebra of A such that $C_0(Z) \cdot A$ is dense in A .

Because Γ acts metrically proper on X , it acts metrically proper on the space C_X as well. By Proposition IV.4.1, Γ acts properly on the M_X , the maximal ideal space of $\mathcal{A}_0(X)$. Additionally, $\mathcal{A}_0(X) \cdot \mathcal{A}(X)$ is dense in $\mathcal{A}(X)$, so $\mathcal{A}(X)$ is a proper Banach Γ -algebra.

Proof. Let Y be an open Γ -invariant subspace of the maximal ideal space of the algebra $\mathcal{A}_0(X)$. The intersection of Y with C_X is open in C_X . The subset Y defines a closed two-sided ideal in $\mathcal{A}(X)$, namely, $C_0(Y) \cdot \mathcal{A}(X)$. Call it $\mathcal{A}(X)_Y$.

Note that since the group Γ is torsion-free by our assumption and the action of Γ on X is metrically proper, any subset Y as above is a free and proper locally compact Γ -space. We can choose an exhaustive countable system $\{Y_i\}$ of such open subsets of the maximal ideal space of $\mathcal{A}_0(X)$ so that each $Y_{i+1} = Y_i \cup \tilde{U}_i$, where \tilde{U}_i is a product $U_i \times \Gamma$, U_i is open and the action of Γ on U_i is trivial.

Note that the algebra $\mathcal{A}(X)$ is the inductive limit of its subalgebras $\mathcal{A}((X)_{Y_i})$. Using the Mayer-Vietoris exact sequence and induction on i , we can show that the homomorphisms $E_\Gamma^*(\mathcal{A}(X)_{Y_i}, \mathbb{C}) \rightarrow RE_\Gamma^*(E\Gamma; \mathcal{A}(X)_{Y_i}, \mathbb{C})$ are all isomorphisms if the homomorphisms $E_\Gamma^*(\mathcal{A}(X)_{\tilde{U}_i}, \mathbb{C}) \rightarrow RE_\Gamma^*(E\Gamma; \mathcal{A}(X)_{\tilde{U}_i}, \mathbb{C})$ are isomorphisms. Here in order to obtain the Mayer-Vietoris exact sequence for the groups $RE_\Gamma^*(E\Gamma; \mathcal{A}(X)_{Y_i}, \mathbb{C})$, one can use the same trick as in the proof of the previous lemma: first get rid of the Γ -action: $RE_\Gamma^*(E\Gamma; \mathcal{A}(X)_{Y_i}, \mathbb{C}) \simeq RE^*(B\Gamma; \mathcal{A}(X)_{Y_i}, \mathbb{C})$, then reduce the assertion to the case of a compact CW space Z instead of $B\Gamma$, and then replace the group $RE^*(Z; \mathcal{A}(X)_{Y_i}, \mathbb{C})$ with $E^*(\mathcal{A}(X)_{Y_i}, C(Z))$.

Next, we show that $E_\Gamma^*(\mathcal{A}(X)_{\tilde{U}_i}, \mathbb{C}) \rightarrow RE_\Gamma^*(E\Gamma; \mathcal{A}(X)_{\tilde{U}_i}, \mathbb{C})$ are isomorphisms. In this case, there is an isomorphism $E_\Gamma^*(\mathcal{A}(X)_{\tilde{U}_i}, \mathbb{C}) \simeq E^*(\mathcal{A}(X)_{U_i}, \mathbb{C})$, and similarly for $RE_\Gamma^*(E\Gamma; \mathcal{A}(X)_{\tilde{U}_i}, \mathbb{C})$. Because $E\Gamma$ is contractible, the assertion is clear.

Finally, we use Milnor's \lim^1 sequence to show that $E_\Gamma^*(\mathcal{A}(X)_{Y_i}, \mathbb{C}) \simeq RE^*(B\Gamma; \mathcal{A}(X)_{Y_i}, \mathbb{C})$ implies $E_\Gamma^*(\mathcal{A}(X), \mathbb{C}) \simeq RE^*(B\Gamma; \mathcal{A}(X), \mathbb{C})$. \square

BIBLIOGRAPHY

- [AK69] J. F. Aarnes and R. V. Kadison. “Pure states and approximate identities”. In: *Proc. Amer. Math. Soc.* 21 (1969), pp. 749–752.
- [Bec75] W. Beckner. “Inequalities in Fourier Analysis”. In: *Annals of Math.* 102 (1975), pp. 159–182.
- [Bla98] B. Blackadar. *K-Theory for Operator Algebras*. Cambridge University Press, 1998.
- [Cas01] Peter G. Casazza. “Approximation Properties”. In: *Handbook of the Geometry of Banach Spaces, Volume 1* (2001), pp. 271–316.
- [Con94] A. Connes. *Noncommutative Geometry*. Acad. Press, 1994.
- [DF93] A. Defant and K. Floret. *Tensor Norms and Operator Ideals*. North-Holland, 1993.
- [Hir] F. Hirzebruch. “The signature theorem: reminiscences and recreation”. In: *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, *Ann. of Math. Studies* 70 (), pp. 3–31.
- [HK01] Nigel Higson and Gennadi Kasparov. “ E -theory and KK -theory for groups which act properly and isometrically on Hilbert space”. In: *Invent. Math.* 144 (2001), pp. 23–74.
- [HKT98] Nigel Higson, Gennadi Kasparov, and Jody Trout. “A Bott periodicity theorem for infinite-dimensional Euclidean space”. In: *Advances in Math.* 135 (1998), pp. 1–40.
- [HR01] Nigel Higson and John Roe. *Analytic K-homology*. Oxford University Press, 2001.
- [HRY93] Nigel Higson, John Roe, and Guoliang Yu. “A coarse Mayer-Vietoris sequence”. In: *Math. Proc. Camb. Phil. Soc* 114 (1993), pp. 85–97.
- [Kas75] G. Kasparov. “Topological invariants of elliptic operators. I. K -homology”. In: *Mathematics USSR - Izvestiya* 9 (1975), pp. 751–792.
- [Kas81] G. Kasparov. “The Operator K -functor and Extensions of C^* -algebras”. In: *Mathematics USSR - Izvestiya* 16.3 (1981), pp. 513–572.
- [Kas88] G. Kasparov. “Equivariant KK -theory and the Novikov conjecture”. In: *Invent. math.* 91 (1988), pp. 147–201.
- [KY05] G. Kasparov and G. Yu. “Groups with coarse positive duality and the Novikov conjecture (preprint)”. 2005.
- [KY12] G. Kasparov and G. Yu. “The Novikov conjecture and geometry of Banach space”. In: *Geometry and Topology* 16 (2012), pp. 1859–1880.
- [Laf02] V. Lafforgue. “ KK -théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes”. In: *Invent. math.* 149 (2002), pp. 1–95.
- [Lev] C. David Levermore. *Integral Operators*. URL: <http://www2.math.umd.edu/~lvrmr/2008-2009-S/Classes/AMSC674/IntOps.pdf> (visited on 04/27/2014).

- [Micr] E. Michael. “Continuous Selections. I”. In: *The Annals of Mathematics, 2nd Series* 63.2 (Mar. 1956), pp. 361–382.
- [Mil56] John Milnor. “Construction of universal bundles, I”. In: *The Annals of Mathematics, 2nd Series* 63.2 (1956), pp. 272–284.
- [Mil62] John Milnor. “On axiomatic homology theory”. In: *Pacific J. Math.* 12.1 (1962), pp. 337–341.
- [MST99] Gustavo A. Muoz, Yannis Sarantopoulos, and Andrew Tonge. “Complexifications of real Banach spaces, polynomials and multilinear maps”. In: *Studia Math.* 134.1 (1999), pp. 1–33.
- [Nov70] S. P. Novikov. “Algebraic construction and properties of Hermitian analogues of K-theory over rings with involution from the viewpoint of the Hamiltonian formalism. Applications to differential topology and the theory of characteristic classes”. In: *Izv. Akad. Nauk SSSR, Ser. Mat.* 34 (1970), pp. 253–288, 475–500.
- [Pal65] Richard S. Palais. “Fredholm Operators”. In: *Seminar on Atiyah Singer Index Theorem* (1965), pp. 119–124.
- [Rya02] Raymond A. Ryan. *Introduction to Tensor Products of Banach Spaces*. Springer, 2002.
- [Sch] Thomas Schlumprecht. *Appendix B: Some facts about $L^p[0, 1]$ and $L^p(\mathbb{R})$* . URL: http://www.math.tamu.edu/~schlump/lecture7_9and10_12c.pdf (visited on 04/27/2014).
- [Sch27] Juliusz Schauder. “Zur Theorie stetiger Abbildungen in Funktionalräumen”. In: *Mathematische Zeitschrift (in German)* 26 (1927), pp. 47–65.
- [Ste71] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1971.
- [Tho] Klaus Thomsen. “Asymptotic Equivariant E-theory I (preprint)”.
- [Tit48] E. Titchmarsh. *Introduction to the theory of Fourier integrals (2nd ed.)* Oxford University: Clarendon Press, 1948.
- [Voi76] Dan Voiculescu. “A non-commutative Weyl-von Neumann theorem.” In: *Rev. Roumaine Math. Pures Appl.* 21.1 (1976), pp. 97–113.