

ON GROUPS OF LARGE EXPONENTS  $N$  AND  $N$ -PERIODIC PRODUCTS

By

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Dissertation

Submitted to the Faculty of the  
Graduate School of Vanderbilt University  
in partial fulfillment of the requirements  
of the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August, 2005

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## ACKNOWLEDGMENTS

I am indebted to my doctoral advisor Alexander Olshanskiy for many years of guidance, advice and encouragement. It has been a privilege to have been introduced to Combinatorial Group Theory by him and I have benefitted immensely from his enthusiasm and mathematical insight.

I owe a great deal to my professors at Moscow State University, especially to Alfred Shmelkin, who had a great influence on my mathematical forming. I wish to express my gratitude to Mark Sapir for many interesting courses he taught and numerous fruitful discussions we had. It is a pleasure to thank Ralph McKenzie and Mike Mihalik who each introduced me to new ideas which have changed my perspective on mathematics. I would like to thank Sergei Ivanov, insightful discussions with whom were beneficial for various aspects of this work. Jeremy Spinrad kindly agreed to serve on my thesis examination committee. I am grateful to him for the time and effort he spent on this.

I am grateful to the Department of Mathematics of Vanderbilt University for providing an exciting and supportive environment during my graduate years. During my study here I enjoyed many courses given by the faculty and visiting professors. I would like to thank former and current graduate students and post-docs for their friendship and support.

Last but not least, I would like to thank my family for their support, love and encouragement.

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# CHAPTER I

## INTRODUCTION

Problems about existence of infinite finitely generated torsion groups posed by Burnside [8] gave rise to a large number of questions concerning periodic type relations in group theory.

In 1964 Golod [9] constructed examples of infinite finitely generated  $p$ -groups. Later series of other examples appeared in the papers of Aleshin [4], Grigorchuk [11], Sushchanskii [38] and of other authors. There is no bound on the orders of elements of mentioned above examples, but it was conjectured that, for sufficiently large  $n$ , there would exist infinite finitely generated groups of exponent  $n$ .

A breakthrough was made in 1968 by Novikov and Adian in the fundamental series of papers [25], where the existence of finitely-generated infinite groups satisfying the identity  $x^n = 1$  for sufficiently large  $n$  was proved, thus giving the negative solution to the restricted Burnside problem for odd exponents  $n \geq 4361$  (later Adian [1] lowered the estimate to odd  $n \geq 665$ ).

Using the geometric interpretation of deducing relations in groups, Olshanskiy [28], [29], [31] gave a new, considerably shorter solution of the restricted Burnside problem for odd  $n > 10^{10}$ . The case of even exponent  $n$  turned out to be more difficult to deal with. The principal difference can be illustrated comparing the following two results of Olshanskiy [27] and Held [12]. On one hand, for every odd  $n \gg 1$  there are infinite finitely generated groups of exponent  $n$  having only cyclic proper subgroups ([27]). On the other hand, any infinite 2-group contains infinite abelian subgroups ([12]).

The Burnside conjecture for even exponents  $n \gg 1$  was settled in 1994: using geometric techniques, Ivanov [14] constructed infinite finitely generated groups satisfying the law  $x^n = 1$  for either odd or divisible by  $2^9$  exponent  $n \geq 2^{48}$ , thus solving the restricted Burnside problem in negative for almost all exponents. Another solution of the Burnside problem in the case of even exponent was given by Lysenok [21] in 1996.

The geometric approach used in [17] and in mentioned above [28], [29], [31] proved to have far-reaching ramifications and was effective to answer many other questions in group theory.

The purpose of this thesis is further development of the geometric method of analyzing relations of periodic type and application of this method to obtain some new results in group theory.

The thesis consists of four chapters. In Chapter I we recall some basic definitions and known facts that we will need. In section 1.1 we introduce the concepts of graded presentation of a group, a diagram over presentation, and formulate lemmas of van Kampen and Schupp, which are the fundamental tools in the geometric method of studying relations in groups. In section 1.2 we consider periodic presentations, define condition  $A$  for maps and give definitions concerning construction of graded presentations of groups of odd exponent. The concepts discussed in Chapter 1 are described in details in [31], Chapters 4-8.

In Chapter II we present an embedding of the free infinitely generated Burnside group  $\mathbf{B}(\infty, n)$  of odd exponent  $n \gg 1$  into  $\mathbf{B}(2, n)$  so that the image of  $\mathbf{B}(\infty, n)$  satisfies the Congruence Extension Property in  $\mathbf{B}(2, n)$ . The construction repeats with some alterations the CEP-embedding of  $\mathbf{B}(\infty, n)$  into  $\mathbf{B}(m, n)$  ( $m \gg 1$ ) given by Olshanskiy and Sapir ([33], Section 4). It

was later shown by Ivanov [16] that  $\mathbf{B}(\infty, n)$  can be CEP-embedded into any non-cyclic subgroup of  $\mathbf{B}(2, n)$ . CEP-subgroups arise naturally in proofs of various embedding theorems ([13], [24], [32]). As a corollary to our result, any countable group satisfying the identity  $x^n = 1$  (for odd  $n \gg 1$ ) can be embedded into two-generated group satisfying the same identity. The latter is known as Obraztsov embedding theorem [26].

The last two chapters of the thesis are dedicated to constructions of graded presentations that involve aperiodic relations together with relations of type  $A^n = 1$  for large even  $n$ . The investigation is heavily dependent on the technique exposed in Ivanov's paper [14]. Construction of Ivanov is modified in Chapter III to present a large family of non-isomorphic 2-generated groups  $G_T$  of even exponent  $n \gg 1$  and to estimate the number of non-isomorphic 2-generated members of the Burnside variety  $\mathfrak{B}_n$ . Neumann [23] showed that the set of all pairwise non-isomorphic 2-generated groups is of cardinality continuum. It is known that so is the set of pairwise non-isomorphic 2-generated simple groups, and moreover, for any sufficiently large prime number  $p$  the set of pairwise non-isomorphic 2-generated simple groups satisfying the identity  $x^p = 1$  is of cardinality continuum (for a detailed discussion see [31] and [5]). In Chapter III we prove that for almost all values of  $n$  the same is true about the set of pairwise non-isomorphic 2-generated simple groups of exponent  $n$ .

The fourth chapter is dealing with graded presentations and diagrams over free products. The study of periodic products was initiated by Adian, who defined  $n$ -products of groups without involutions for odd exponents  $n \geq 665$  ([2], [3]), thus giving the examples of associative Mal'cev operations on the class of groups without involutions (see [22], [10], [31]). The problem about existence of associative Mal'cev operations (other than the free and the direct product)

on the class of all groups was solved by Olshanskiy [30]. The operations constructed by Olshanskiy involve relations of type  $A^n = 1$  for  $n \geq 10^{10}$ , and in the case when  $n$  is odd and the factors are without involutions they essentially coincide with the  $n$ -products of Adian (This was shown by Ivanov [15], who also found a non-inductive characterization of periodic products). It should be noted that in the case of even exponents the groups constructed in [30] are not torsion even if all the factors are. Here we construct periodic  $n$ -products of groups without involutions for even exponents  $n \gg 1$ , and give a criterion for simplicity of  $n$ -products. Remark that the  $n$ -products constructed in [2], [3] and in Chapter 4 coincide with the free product inside the Burnside variety of the corresponding exponent  $n$  in the case when the factors satisfy the identity  $x^n = 1$ .

### 1.1. Graded presentations and diagrams

Denote by  $F(\mathfrak{A})$  the free group with basis  $\mathfrak{A}$ . Its elements are words (more precisely, equivalence classes of words) over the group alphabet  $\mathfrak{A} \cup \mathfrak{A}^{-1}$ . Length (i.e. number of letters) of a word  $W$  is denoted by  $|W|$ . Let a group  $G$  be given by defining relations  $R = 1$ ,  $R \in \mathcal{R}$ , where  $\mathcal{R}$  is a set of defining words in the group alphabet (we assume here that none of the words  $R \in \mathcal{R}$  is equal to the identity element in the free group  $F(\mathfrak{A})$ ):

$$G = \langle \mathfrak{A} \mid R = 1; R \in \mathcal{R} \rangle. \quad (1)$$

We say that the group  $G$  is given by a presentation (1). A diagram  $\Delta$  over presentation (1) is a planar map (i.e. a finite connected planar 2-complex; its

$0-$ ,  $1-$  and  $2-$ cells are referred to as vertices, edges and cells respectively) equipped with a labelling function  $\phi$  which assignes an element from  $\mathfrak{A} \cup \mathfrak{A}^{-1}$  to each oriented edge such that edges with opposite orientations are labelled by mutual inverses ( $\phi(e^{-1}) = \phi(e)^{-1}$ ), and the boundary labels of each of its cells are cyclic conjugates of elements from  $\mathcal{R}$  or their inverses. For an oriented edge  $e$  we denote by  $e_-$  and  $e_+$  its initial and terminal vertices respectively. A path  $p$  in a diagram is a finite sequence of oriented edges  $e_1, \dots, e_k$  such that  $(e_{i+1})_- = (e_i)_+$  for every  $i = 1, \dots, k - 1$ . Label  $\phi(p)$  of a path  $p$  is defined to be the product of labels of edges  $e_1, \dots, e_k$ :  $\phi(p) = \phi(e_1) \dots \phi(e_k)$ . The length  $|p|$  is equal to the number of edges of  $p$ . A path of length 0 has no edges by definition. A path in a diagram is called geodesic if its length does not exceed length of any other path homotopic to it.

The importance of studying diagrams in combinatorial group theory is illustrated by the well-known van Kampen lemma (see [19], [20], [31]):

*Let  $W$  be a nontrivial group word over the alphabet  $\mathfrak{A}$ . Then  $W$  is equal to 1 in the group  $G$  given by presentation (1) if and only if there is a disk diagram over presentation (1) such that the label of its boundary is letter-for-letter equal to  $W$ .*

The following lemma due to Schupp gives geometric interpretation of the conjugacy relation ([34], [20], [31]):

*Two nontrivial words  $U$  and  $W$  are conjugated in the group  $G$  given by presentation (1) if and only if there is an annular diagram over presentation (1) labels of whose contours are letter-for-letter equal to  $U$  and  $W^{-1}$ .*

Let now the set of defining words  $\mathcal{R}$  be graded, that is

$$\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{S}_i$$

is a disjoint union of its subsets  $\mathcal{S}_i$ , which are referred to as the sets of defining words of rank  $i$ , such that no element from  $\mathcal{S}_i$  is equal to a cyclic conjugate or an inverse of a cyclic conjugate of an element of  $\mathcal{S}_j$  for  $j \neq i$ . In this case presentation (1) is called a graded presentation. Denote  $\mathcal{R}_i = \bigcup_{j=1}^i \mathcal{S}_j$  for  $i = 1, 2, \dots$ , and consider the sequence of groups  $G(0), G(1), G(2), \dots$ , where  $G(0) = F(\mathfrak{A})$  is the free group, and

$$G(i) = \langle \mathfrak{A} \mid R = 1, R \in \mathcal{R}_i \rangle \quad (2)$$

for  $i \geq 1$ . The group  $G$  is identified with the inductive limit  $G(\infty)$  of the sequence of groups  $\{G(i)\}$  with respect to the natural epimorphisms  $G(i) \rightarrow G(i+1)$ :

$$G = G(\infty) = \langle \mathfrak{A} \mid R = 1, R \in \bigcup_{i=1}^{\infty} \mathcal{R}_i \rangle$$

In what follows, the phrase "in rank  $i$ " means "in the group  $G(i)$ ". The symbol " $\equiv$ " stands for letter-for letter equality of two words. A diagram over presentation (2) is called a diagram of rank  $i$ . According to the grading of the set of the defining words, we assign rank to each cell in a diagram of rank  $i$ : a cell  $\Pi$  is said to be of rank  $j \leq i$  if starting from some vertex of the boundary  $\partial\Pi$  of the cell  $\Pi$  we read a word from  $\mathcal{S}_j$  going in some direction along  $\partial\Pi$ . By  $r(\Pi)$  we denote rank of a cell  $\Pi$ , by  $r(\Delta)$  - maximum of ranks of cells contained in a diagram  $\Delta$ . Certain cells and pairs of cells in a diagram will be termed reducible (these concepts will be defined later). By taking out

reducible subdiagrams and filling the obtained holes by cells of smaller ranks, in a finite number of steps we can obtain a reduced diagram (that is a diagram which does not contain reducible cells and pairs of cells) without changing the boundary labels of the original one (the process of reducing a diagram is explained below).

The following "graded" versions of van Kampen and Schupp lemmas will be used without special reference:

*Let  $W$  be a nontrivial group word over the alphabet  $\mathfrak{A}$ . Then  $W$  is equal to 1 in the group  $G(i)$  if and only if there is a disk reduced diagram  $\Delta$  of rank  $i$  such that the label of its boundary  $\phi(\partial\Delta)$  is letter-for-letter equal to  $W$ ;*

*Two nontrivial words  $U$  and  $W$  are conjugated in the group  $G(i)$  if and only if there is an annular reduced diagram of rank  $i$  labels of whose contours are letter-for-letter equal to  $U$  and  $W^{-1}$ .*

## 1.2. Periodic presentations and Condition A

Dealing with relations of type  $A^n = 1$  we consider graded periodic presentations. Reduced diagrams over such presentations possess certain properties useful for understanding the structure of the corresponding groups. Certain important characteristics of diagrams (such as lengths of the relators, lengths of paths, contiguity degrees, etc.) are estimated using some auxiliary parameters. The list of these parameters ordered by seniority is the following :

$$\alpha \succ \beta \succ \gamma \succ \delta \succ \varepsilon \succ \zeta \succ \eta \succ \iota \quad (3)$$

We also set

$$\bar{\alpha} = \frac{1}{2} + \alpha, \quad \bar{\beta} = 1 - \beta, \quad \bar{\gamma} = 1 - \gamma, \quad n = \iota^{-1},$$

where  $n$  is assumed to be an odd integer. Consistency of system of inequalities appearing in [31, Chapters 5 - 8] and Chapter 2 of the present thesis follows from the Lower Parameter Principle. According to LPP, every inequality can be made true by choosing a sufficiently small value of the lowest (according to (3)) parameter appearing in it. Thus, choosing the values of the parameters in the order given above, all inequalities appearing in the proof can be satisfied. Notice that all inequalities are valid for all sufficiently large values of parameter  $n$  which corresponds to  $\iota$  and is chosen after the values of all other parameters already fixed.

Construction of graded presentation of the free Burnside group  $\mathbf{B}(\mathfrak{A}, n)$  of large odd exponent  $n$  is inductive. Let  $G(0) = F(\mathfrak{A})$  be the free group over  $\mathfrak{A}$ . Suppose that the group  $G(i-1)$  is already defined and certain sets  $X_j$  of words (called periods of rank  $j$ ) are constructed for every  $j \leq i-1$ . A word  $A$  is called simple in rank  $i-1$  if it is not conjugate in rank  $i-1$  of a power of a period of rank  $j \leq i-1$  and not conjugate in rank  $i-1$  to a power of any word  $C$ , where  $|C| < |A|$ . In particular, simple in rank 0 words are just simple (cyclically reduced and not equal to a proper power in the free group) words. The set  $X_i$  of periods of rank  $i$  is a subset of the set of simple in rank  $i-1$  words of length  $i$ , maximal with respect to the property that if  $A, B \in X_i$ ,  $A \not\equiv B$ , then  $A$  is not conjugate in rank  $i-1$  to  $B^{\pm 1}$ . For every  $A \in X_i$  we introduce a relation  $A^n = 1$ . Thus, the set of defining words of rank  $i$  is given by  $\mathcal{S}_i = \{A^n | A \in X_i\}$ , and the inductive construction is complete.

Before discussing properties of diagrams over the constructed graded pre-

sentation we need to make a few remarks. First, it is convenient to allow the labelling function  $\phi$  to take values in the extended group alphabet  $\mathfrak{A} \cup \mathfrak{A}^{-1} \cup \{1\}$  (here we assume that  $1^{-1} = 1$ ). Reading labels of paths in a diagram we omit symbols 1. Edges labelled by 1 are termed 0-edges; all other edges are termed  $\mathfrak{A}$ -edges. In addition to cells responsible for the relations a diagram may contain cells whose boundaries are labelled by empty word or a word  $aa^{-1}$ ,  $a \in \mathfrak{A}$  (if symbols 1 are omitted). Such cells are said to be of rank 0 and called 0-cells. Over a diagram one can perform an operation of 0-refinement, that is insertion in a certain way of some 0-cells (this operation is described in details in [31, §11]). A pair of cells  $\Pi_1, \Pi_2$  of rank  $j > 0$  in a diagram  $\Delta$  is called a reducible pair (or  $j$ -pair) if in some 0-refinement  $\Delta'$  of  $\Delta$  there are vertices  $o_1 \in \partial\Pi'_1$ ,  $o_2 \in \partial\Pi'_2$  of the contours of the images  $\Pi'_1, \Pi'_2$  of  $\Pi_1, \Pi_2$  connected by a simple path  $t$ ,  $\phi(t) \stackrel{j-1}{=} 1$ , such that the labels of  $\Pi'_1, \Pi'_2$  are mutually inverse in the free group (if we read  $\phi(\partial\Pi'_1)$  starting from  $o_1$  clockwise and  $\phi(\partial\Pi'_2)$  - starting from  $o_2$  counterclockwise). Such  $j$ -pair can be cancelled ([31, §13]), i.e. substituted in  $\Delta$  by cells of ranks  $< j$  without changing the boundary labels of  $\Delta$ . A diagram of rank  $i$  without  $j$ -pairs for every  $j \leq i$  is called reduced.

The concept of an  $A$ -map is central in analizing properties of diagrams over the graded presentation of  $\mathbf{B}(\mathfrak{A}, n)$ , as illustrated in the following lemma [31, Lemma 19.4]:

**Lemma 19.4[31]** *Every reduced diagram over the graded presentation of  $\mathbf{B}(\mathfrak{A}, n)$  is an  $A$ -map.*

A map  $\Delta$  is called an  $A$ -map if the following conditions are satisfied:

- A1 The boundary path of every cell  $\Pi$  or rank  $j$  is cyclically reduced (i.e. does not contain subpaths of the form  $ee^{-1}$ ) and  $|\partial\Pi| \geq nj$ .

A2 Every subpath of length  $\leq \max(j, 2)$  of the boundary of an arbitrary cell of rank  $j$  in  $\Delta$  is geodesic in  $\Delta$ .

A3 If  $\pi, \Pi \in \Delta(2)$  and  $\Gamma$  is a contiguity submap of  $\pi$  to  $\Pi$  with  $(\pi, \Gamma, \Pi) \geq \varepsilon$ , then  $|\Gamma \wedge \Pi| < (1 + \gamma)k$  where  $k = r(\Pi)$ .

A (cyclic) section of a boundary of an  $A$ -map  $\Delta$  is called a smooth section of rank  $k > 0$  (we write  $r(q) = k$ ) if:

- 1) Every subpath of length  $\leq \max(k, 2)$  of  $q$  is geodesic in  $\Delta$ ;
- 2) For every contiguity submap  $\Gamma$  of a cell  $\pi$  to  $q$  satisfying  $(\pi, \Gamma, q) \geq \varepsilon$ , we have  $|\Gamma \wedge q| < (1 + \gamma)k$ .

Two  $\mathfrak{A}$ -edges  $e$  and  $f$  in a diagram  $\Delta$  are called adjacent if they coincide or there is a sequence of  $\mathfrak{A}$ -edges  $e = e_1, e_2, \dots, e_k = f$  such that every two consecutive edges of this sequence belong to the same 0-cell. Consider sections  $q_1$  and  $q_2$  of contours of  $\mathcal{R}$ -cells or of the contour of  $\Delta$ , and let  $e$  and  $f$  be adjacent edges that belong to  $q_1$  and  $q_2$  respectively. It follows that there is a disk subdiagram  $E$  in  $\Delta$  (or in some 0-refinement  $\Delta'$  of  $\Delta$ ) with contour  $pes^{-1}f$ , where  $|p| = |s| = 0$ , that is the paths  $p$  and  $s$  consist of 0-edges. The diagram  $E$  is called a 0-bond.

An important concept of  $k$ -contiguity subdiagram (submap)  $\Gamma$  of a cell  $\Pi_1$  to a cell  $\Pi_2$  is defined inductively. We start with a definition of 0-contiguity subdiagram. Let  $e_1, f_1$  and  $e_2, f_2$  be two pairs of adjacent edges where  $e_1, e_2$  belong to  $\partial\Pi_1$  and  $f_1, f_2$  belong to  $\partial\Pi_2$ . Denote the 0-bonds corresponding to the pairs  $e_1, f_1$  and  $e_2, f_2$  by  $E_1$  and  $E_2$  respectively. If  $E_1 = E_2$  (that is  $e_1 = e_2$  and  $f_1 = f_2$ ), then set  $\Gamma = E_1$ . In this case the edges  $e = e_1 = e_2$  and  $f = f_1 = f_2$  are called contiguity arcs of  $\Gamma$ . Let now  $E_1$  and  $E_2$  be different

0-bonds with contours  $z_1e_1w_1f_1^{-1}$  and  $z_2e_2w_2f_2^{-1}$  respectively. Suppose there are subpaths  $y_1$  and  $y_2$  of  $\partial\Pi_1$  and  $\partial\Pi_2$  respectively such that  $z_1y_1w_2y_2$  (or  $z_2y_1w_1y_2$ ) is a contour of a disk subdiagram  $\Gamma$  which does not contain cells  $\Pi_1$  and  $\Pi_2$ . Then the subdiagram  $\Gamma$  is called a 0-contiguity subdiagram (defined by the 0-bonds  $E_1$  and  $E_2$ ), the paths  $y_1$ ,  $y_2$  are called the contiguity arcs (we write  $y_1 = \Gamma \wedge \Pi_1$  and  $y_2 = \Gamma \wedge \Pi_2$ ), the paths  $z_1$  and  $w_2$  (or  $z_2$  and  $w_1$ ) are called the side arcs. The ratio  $|y_1|/|\partial\Pi_1|$  (or  $|y_2|/|\partial\Pi_2|$ ) is called the degree of contiguity of  $\Pi_1$  to  $\Pi_2$  (or  $\Pi_2$  to  $\Pi_1$ ) and is denoted by  $(\Pi_1, \Gamma, \Pi_2)$  (or  $(\Pi_2, \Gamma, \Pi_1)$ ). Notice that in the case when  $\Pi_1 = \Pi_2 = \Pi$ ,  $(\Pi, \Gamma, \Pi)$  is a pair of numbers. Two contiguity subdiagrams are called disjoint if they do not have common cells, the contiguity arcs do not have common points, and also the side arcs do not have common points.

Suppose that the concepts of  $j$ -bond and  $j$ -contiguity subdiagram are already defined for  $j = 0, 1, \dots, k-1$ . Consider two cells  $\Pi_1$  and  $\Pi_2$  (possibly  $\Pi_1 = \Pi_2$ ), a cell  $\pi$  of rank  $k$ , and assume that the following conditions hold:

- 1)  $r(\Pi_1) > k$ ,  $r(\Pi_2) > k$ ;
- 2) there exist disjoint subdiagrams  $\Gamma_1$  and  $\Gamma_2$  of  $j_1$ -contiguity of  $\pi$  to  $\Pi_1$  and of  $j_2$ -contiguity of  $\pi$  to  $\Pi_2$  respectively, with  $j_1 < k$ ,  $j_2 < k$ , such that  $\Pi_1$  is not contained in  $\Gamma_2$  and  $\Pi_2$  is not contained in  $\Gamma_1$ ;
- 3)  $(\pi, \Gamma_1, \Pi_1) \geq \varepsilon$ ,  $(\pi, \Gamma_2, \Pi_2) \geq \varepsilon$  (where, as in the case of 0-contiguity,  $(\pi, \Gamma, \Pi)$  stands for the contiguity degree of  $\pi$  to  $\Pi$ ).

Then the minimal subdiagram  $E$  containing  $\pi$ ,  $\Gamma_1$  and  $\Gamma_2$  is called a  $k$ -bond between  $\Pi_1$  and  $\Pi_2$  defined by the contiguity subdiagrams  $\Gamma_1$  and  $\Gamma_2$  with principal cell  $\pi$ . A  $k$ -contiguity subdiagram is defined using two bonds (a

$k$ -bond and a  $j$ -bond with  $j \leq k$ ) in the same way as a 0-contiguity subdiagram was defined using two 0-bonds. The concepts of contiguity arcs, side arcs and of contiguity degree are defined analogously. Contiguity subdiagrams of a cell to a section of the contour of the diagram are defined in a similar way.

We will refer to the following structure theorem of  $A$ -maps:

**Theorem 16.2[31]** *If  $\Delta$  is a disk or annular  $A$ -map of positive rank with its contour subdivided in at most 4 sections, then there exists a cell  $\Pi$ ,  $r(\Pi) > 0$ , and a contiguity submap to one of the sections  $q$  of the contour of  $\Delta$  such that  $r(\Delta) = 0$  and  $(\Pi, \Gamma, q) \geq \varepsilon$ .*

## CHAPTER II

### CEP SUBGROUPS OF FREE BURNSIDE GROUPS

We say that a subgroup  $H$  of a group  $G$  satisfies *congruence extension property* (CEP) in  $G$  if for any normal subgroup  $N$  in  $H$  there is a normal subgroup  $L$  in  $G$  such that  $L \cap H = N$ . In this case  $H$  is called a *CEP-subgroup* of  $G$ .

The main result of this chapter is the following theorem:

**Theorem 2.1.** *For sufficiently large odd exponent  $n$  there exists a CEP-subgroup isomorphic to a free Burnside group  $\mathbf{B}(\infty, n)$  with infinite number of generators in the free Burnside group  $\mathbf{B}(2, n)$  on two generators.*

As it was mentioned in [32] for a class of hyperbolic groups, this theorem has an immediate corollary that states "SQ-universality inside Burnside variety":

**Corollary.** *Any countable group satisfying the identity  $x^n = 1$  for sufficiently large odd  $n$  can be embedded into two-generated group satisfying the same identity.*

*Proof.* Any countable group, satisfying the identity  $x^n = 1$ , is isomorphic to some quotient of  $\mathbf{B}(\infty, n)$ . Since  $\mathbf{B}(2, n)$  has a CEP-subgroup isomorphic to  $\mathbf{B}(\infty, n)$ , any such quotient can be embedded into some quotient of  $\mathbf{B}(2, n)$ .

□

The result stated in this corollary was first obtained (using different arguments) by V.N.Obraztsov, [26].

## 2.1. Aperiodic semigroup homomorphisms

For integer  $k \geq 2$  a reduced word  $u$  is called *k-aperiodic* if it contains no non-empty subwords of type  $v^k$ . Let  $S, T$  be free semigroups,  $f : S \rightarrow T$  - a homomorphism. Mapping  $f$  is said to be *k-aperiodic* if  $f(u)$  is *k*-aperiodic whenever  $u$  is *k*-aperiodic.

Let  $S_k$  be a free semigroup on  $k$  generators ( $k$  - finite or infinite). A monomorphism  $h : S_\infty \rightarrow S_3$ , which is *k*-aperiodic for all  $k \geq 2$ , was constructed in [6] (there such homomorphisms were called *k*th power-free). We will need only the restriction of this monomorphism on  $S_m$  ( $m$  is finite, its meaning will become clear later) and further in this paper we will denote this restriction as  $h'$ . Denote  $\{a_1, \dots, a_m\}$  - free generators of the semigroup  $S_m$ . Next two properties mentioned in [6] for the original homomorphism  $h$  remain true for its restriction:

- (i)  $a_i = a_j$  whenever  $h'(a_i)$  is a subword of  $h'(a_j)$ ;
- (ii) If  $a, a_{j_1}, \dots, a_{j_k} \in \{a_1, \dots, a_m\}$  and  $h'(a_{j_1} \dots a_{j_k}) = X h'(a) Y$ , then  $a = a_{j_i}$ ,  $X = h'(a_{j_1} \dots a_{j_{i-1}})$  and  $Y = h'(a_{j_{i+1}} \dots a_{j_k})$ .

It will be useful to obtain an upper bound for the ratio  $r = \frac{\max |h'(a_i)|}{\min |h'(a_i)|}$ . In order to do it let's turn to the construction of  $h$  in [6]. The homomorphism  $h$  was created in [6] as a composition:  $h = h_2 \circ h_1$  where  $h_1 : S_\infty \rightarrow S_5$ ,  $h_2 : S_5 \rightarrow S_3$ . The images of the generators of  $S_5$  under  $h_2$  have almost equal lengths: the corresponding ratio is less than  $\frac{3}{2}$ . Homomorphism  $h_1$  is defined on generators of  $S_\infty$ :  $h_1(a_i) = W_i d W_i e$ , where  $\{W_i, i = 1, 2, \dots\}$  is a nonrepetitive list of 2-aperiodic words in alphabet  $\{a, b, c\}$ ,  $\{a, b, c, d, e\}$  is a free generating set of

$S_5$ . Since  $h'$  is a restriction of  $h$ , we have  $h' = h_2 \circ h'_1$ , where  $h'_1 : S_m \rightarrow S_5$  is a restriction of  $h_1$  to  $S_m$ . The number of 2-aperiodic words of length  $k$  in 3-letter alphabet is exponential on  $k$  (see, for example, [7]), so we can take  $m$  such words (one for each free generator of the semigroup  $S_m$ ) of the same length. As a result, images of generators of  $S_m$  under  $h'_1$  will have equal lengths in  $S_5$ . It follows that  $r = \frac{\max |h'(a_i)|}{\min |h'(a_i)|} < \frac{3}{2}$ .

**Lemma 2.1.** *Let  $w = w(ab^{10}a, a^3b^6a^3, a^5b^2a^5)$  be a reduced word in generators  $ab^{10}a, a^3b^6a^3, a^5b^2a^5$ ,  $w \neq 1$  in  $F_2$ . Suppose that  $w$  has a subword of type  $A^l$ , where  $A = A(a, b)$ ,  $l \geq 11$ . Then  $A$  is conjugated in the free group to some word  $\bar{A} = \bar{A}(ab^{10}a, a^3b^6a^3, a^5b^2a^5)$ .*

*Proof.* Case 1. Let  $|A| \geq 12$ . Denote  $u_1 = ab^{10}a, u_2 = a^3b^6a^3, u_3 = a^5b^2a^5$ . Let  $w = u_{i_1}^{\delta_1}u_{i_2}^{\delta_2}\dots u_{i_k}^{\delta_k} = v_{i_1}v_{i_2}\dots v_{i_k}$ , where  $\delta_j = \pm 1$ ,  $v_{i_j}$ 's are the remaining parts of the corresponding  $u_{i_j}^{\delta_j}$ 's after possible cancellations, and none of them can be empty if  $w = u_{i_1}^{\delta_1}u_{i_2}^{\delta_2}\dots u_{i_k}^{\delta_k}$  is a reduced form. It is easy to see from the construction of the words  $u_i, i = 1, 2, 3$ , that such  $v_{i_j}$ 's can be recognized canonically in any non-empty product of type  $\prod u_{i_j}^{\pm 1}$  (cancellations can not touch powers of  $b$ , and every maximal such power points on a unique  $u_i^{\pm 1}$  that could contain it).

Since  $|v_{i_j}| \leq 12 < |A|$  for all  $j \in \{1, \dots, k\}$ , one can find a cyclic shift  $A'$  of the word  $A$  which contains some  $v_{i_j}$  as a subword. Let  $W$  be the maximal product of the remaining parts of  $u_{i_j}^{\delta_j}$ 's inside  $A'$ . It follows that there is a unique (possibly empty) word  $v_{i_s}$  between every two consecutive occurrences of  $W$ , and hence  $A'' = v_{i_s}W$  - is a cyclic shift of  $A'$  (and of  $A$  as well). There is a (possibly empty) prefix  $X$  of  $u_{i_s}^{\delta_s}$  such that  $Xv_{i_s}$  is also prefix of  $u_{i_s}^{\delta_s}$  ( $Xv_{i_s}$  may coincide with  $u_{i_s}^{\delta_s}$ ). Then the word  $\bar{A} = Xv_{i_s}WX^{-1}$  (which is a conjugated to

$A$ ) can be written as  $\bar{A} = \bar{A}(ab^{10}a, a^3b^6a^3, a^5b^2a^5)$ .

Case 2. Let  $|A| < 12$ . Notice first that the case  $|A| = 1$  is impossible. Let now

$|A| \geq 2$ . The word  $w$  has a form  $a^{n_1}b^{m_1}a^{n_2}b^{m_2}\dots a^{n_k}b^{m_k}a^{n_{k+1}}$  when considered as a word in an alphabet  $\{a, b, a^{-1}, b^{-1}\}$ . Notice that all  $m_i$ 's and  $n_j$ 's for  $j = 2, \dots, k$  are even. It follows that  $|A|$  has to be even. Consider a cyclic shift  $A'$  of the word  $A$  which starts with  $b$  and ends with  $a$ . The power of a letter  $b$  in the beginning of  $A'$  points on the unique  $u_{i_0}^{\pm 1}$  since cancellations do not touch  $b$ 's.

It is easy to see that the case  $|A| = 2$  is impossible. Let us consider the case when  $|A| = 4$ . Then  $A'$  starts with  $b^{\pm 2}$ . Now one can check all nontrivial products  $(a^5b^2a^5)^{\pm 1}u_i^{\delta_i}$  and obtain that the word of length 4 to the right of  $A'$  can not be equal to  $A'$ . Thus,  $w$  can not contain  $A^{11}$  as a subword. The other cases ( $|A| = 6, 8$  or  $10$ ) can be checked analogously. Lemma is proved.  $\square$

Extend  $h'$  to  $\bar{h} : F_m \rightarrow F_3$  ( $F_k$  is a free group on  $k$  free generators). Let  $g : F_3 \rightarrow F_2$  be a homomorphism which maps generators  $\{x, y, z\}$  of  $F_3$  as follows:  $g(x) = ab^{10}a$ ,  $g(y) = a^3b^6a^3$ ,  $g(z) = a^5b^2a^5$ , where  $a, b$  are free generators of  $F_2$ . Notice that the restriction of  $g$  to the free semigroup  $S_3$  satisfies properties analogous to (i) and (ii). Denote  $f = g \circ \bar{h}$ . Let  $L = \max(|f(a_i)|)$ ,  $l = \min(|f(a_i)|)$ . We have  $\frac{L}{l} < \frac{3}{2}$ . The images under  $f$  of free generators  $\{a_1, \dots, a_m\}$  and their inverses we will call *blocks*.

## 2.2. Aperiodic words with small cancellation

Recall that a group word  $U$  over some alphabet is called positive (negative)

if it contains only positive (negative) powers of letters from. A word  $U$  is called  $A$ -periodic if it is a subword of some power  $A^k$ , where  $A$  is a reduced word,  $k > 0$ .

In [33], Lemma 4.2, it was shown that for given  $n$  and  $\varepsilon > 0$  there is a number  $m$  and a set of positive words  $S = \{A_1, A_2, \dots\}$  in an alphabet  $\{a_1, \dots, a_m\}$ , satisfying the following properties:

1. Every reduced product, whose factors belong to  $S$ , has no non-empty  $A$ -periodic subwords of length  $\geq (1 + \varepsilon)|A|$  unless the word  $A$  is freely conjugate to a product of some words of  $S$ ;
2. Suppose that  $A' \equiv UV'$  and  $A'' \equiv UV''$ , or  $A' \equiv V'U$  and  $A'' \equiv V''U$ , are distinct cyclic permutations of the words of  $S$  (that is,  $U$  is a common prefix or a common suffix). Then  $|U| < \frac{\varepsilon}{10} \min(|A'|, |A''|)$ . (this property is known as the small cancellation condition  $C'(\frac{\varepsilon}{10})$ ).
3.  $|A_i| \geq n, i = 1, 2, \dots$

Instead of condition 3 we will need the following condition

$$3'. |A_i| > L n \varepsilon^{-1}, i = 1, 2, \dots,$$

which can be satisfied by deleting some words from  $S$  and renumbering the remaining ones.

So we may assume that the words  $A_i$  constituting the set  $S$  satisfy conditions 1, 2, 3'. Denote  $B_i = f(A_i), i = 1, 2, \dots, T = \{B_i, i = 1, 2, \dots\}$ . It follows immediately that  $|B_i| > L n \varepsilon^{-1}$ .

The following two lemmas establish properties of the words  $B_i$ , similar to Properties 1 and 2 of the words  $A_i$ .

**Lemma 2.2.** *The symmetrized set obtained from the set  $T$  satisfies the small cancellation condition  $C'(\frac{\varepsilon}{5})$ .*

*Proof.* Proving by contradiction, suppose that  $B' \equiv XY'$  and  $B'' \equiv XY''$  are distinct cyclic permutations of some words of  $T$  having a common prefix  $X$  of length  $|X| > \frac{\varepsilon}{5} \min(|B'|, |B''|)$  (case with common suffix can be proved similarly). Since  $|X| > 24L$  there is a decomposition  $X \equiv X_1X_2X_3$ , where  $|X_1|, |X_3| < 12L$  ( $12L$  is maximum of lengths of  $f(a_i)$ ,  $i \in \{1, \dots, m\}$ ), such that cyclic permutations  $C' \equiv X_2X_3Y'X_1$  and  $C'' \equiv X_2X_3Y''X_1$  of the words  $B'$  and  $B''$  respectively are images under  $f$  of distinct cyclic permutations  $A'$  and  $A''$  of some words from  $S$ , and, moreover,  $X_2$  is image under  $f$  of a common prefix  $U$  of  $A'$  and  $A''$ .

In order to obtain contradiction we will need following inequalities:

$$\frac{\varepsilon}{5} - \frac{24}{ln\varepsilon^{-1}} > \frac{\varepsilon}{6}, \text{ (LPP, } n \gg \varepsilon^{-1}\text{); } |B_i| > l|A_i| > Lln\varepsilon^{-1} \text{ for all } i = 1, 2, \dots; \\ \frac{l}{L} > \frac{2}{3}.$$

It follows that

$$|U| \geq \frac{|X_2|}{L} > \frac{|X| - 24L}{L} > \frac{1}{L} \left( \frac{\varepsilon}{5} \min(|B'|, |B''|) - \frac{24}{ln^2} \min(|B'|, |B''|) \right) = \\ \frac{1}{L} \left( \frac{\varepsilon}{5} - \frac{24}{ln^2} \right) \min(|B'|, |B''|) > \frac{\varepsilon}{6L} \min(|B'|, |B''|) \geq \frac{\varepsilon}{6L} \min(l|A'|, l|A''|) = \\ \frac{\varepsilon l}{6L} \min(|A'|, |A''|) > \frac{\varepsilon}{10} \min(|A'|, |A''|)$$

Contradiction with Property 2 of words  $A_i$ . Lemma 2.2 is proved.  $\square$

**Lemma 2.3.** *Suppose that in some product of words  $B_i^{\pm 1}$  after all cancellations there is a  $B$ -periodic subword of length  $> (1 + \gamma)|B|$ . Then either  $B$  is freely conjugated to some product of words from  $T$  and their inverses, or*

$|B| < 2L\gamma^{-1}$ . In the later case the length of  $B$ -periodic subword is not greater than  $22L\gamma^{-1}$  (degree of  $B$  is less than or equal to 11.)

*Proof.* Assume that some product  $B_{i_1}^{\varepsilon_1} \dots B_{i_k}^{\varepsilon_k} = f(A_{i_1}^{\varepsilon_1} \dots A_{i_k}^{\varepsilon_k})$  has a  $B$ -periodic subword  $BB'$  of length greater than  $(1 + \gamma)|B|$ . Here  $B'$  is a prefix of  $B$ ,  $|B'| > \gamma|B|$ .

Suppose that  $B$  is not freely conjugated to a product of type  $\prod B_j^{\pm 1}$ .

**Claim.** *No cancellations could occur inside  $B'$ .*

*Proof of the claim.* Cancellations in products of words  $B_i$  and their inverses can occur only on junctions  $B_i^{\pm 1}B_j^{\mp 1}$ . Suppose that  $B'$  contains a subword  $U$  of some word  $B_i^{\pm 1}$  of length  $|U| > \frac{\varepsilon}{5}|B_i|$ . Consider a cyclic shift  $\bar{B}$  of  $B$  that starts (or ends) with  $U$ . By Lemma 2.2, the word  $U$  points on the unique occurrence  $B_{i_j}^{\varepsilon_j}$  from  $B_{i_1}^{\varepsilon_1} \dots B_{i_k}^{\varepsilon_k}$ . Thus, conjugating  $\bar{B}$ , if necessary, by the word that was cancelled on the junction of the occurrence of  $B_{i_j}^{\varepsilon_j}$  in the product  $B_{i_1}^{\varepsilon_1} \dots B_{i_k}^{\varepsilon_k}$ , we obtain a product of some  $B_i^{\pm 1}$  which is conjugated to  $B$ . In view of the assumption that  $B$  is not conjugated to any product of type  $\prod B_{i_j}^{\pm 1}$  it follows that there is at most one junction  $B_i^{\pm 1}B_j^{\mp 1}$  (and therefore at most one place where cancellations could occur) inside  $B'$ .

Assuming that there are cancellations inside  $B'$  we obtain a decomposition  $B' \equiv UV$ , where  $U$  is a negative(positive) word and  $V$  is positive (negative). Without loss of generality we may assume that  $U$  is negative and  $V$  is positive. Suppose that  $|V| \geq |U|$  and consider the leftmost occurrence of  $B'$  in  $BB'$  (remember that  $B'$  is a prefix of  $B$ ). It follows that either  $V$  is a subword of some  $B_{i_0}$  or  $V$  contains as a subword a remaining part of some  $B_{i_0}$  after cancellations. The later case in view of Lemma 2.2 and the above arguments implies that  $B$  is conjugate to some product  $\prod B_{i_j}^{\pm 1}$ . Suppose now that  $V$  is a

subword of  $B_{i_0}$ . Notice that the word that remained of  $B_{i_0}$  after cancellations cannot intersect the rightmost occurrence of  $B'$  in  $BB'$  ( $U$  is negative,  $B_{i_0}$  is positive). Then the following sequence of inequalities  $|V| \geq \frac{1}{2}|B'| > \frac{1}{2}\gamma|B| > \frac{1}{2}\gamma(1 - \frac{2\varepsilon}{5})|B_{i_0}| > \frac{\varepsilon}{5}|B_{i_0}|$  (LPP) and the arguments from the beginning of the proof of the claim imply that  $B$  is conjugated to some product  $\prod B_{i_j}^{\pm 1}$ . Contradiction. The case  $|U| \geq |V|$  can be considered analogously. The claim is proved.

Assume now that there is at least one block inside  $B'$  and denote  $W$  to be the maximal product of blocks inside  $B'$ ,  $B' = XWY$ . We have  $|X|, |Y| < \frac{L}{l}|W| < \frac{3}{2}|W|$ . Consider the word  $\bar{B}$  which is equal to a cyclic shift of the word  $B$  starting with  $W$ .

Without loss of generality we may assume that  $B'$  is positive. It means, in particular, that  $W$  is positive.

In view of the fact that  $W$  is a product of blocks, the properties (i), (ii) of  $h'$  and of the restriction of  $g$  to the free semigroup  $S_3$  imply that  $\bar{B}$  is a product of blocks. It means that we can find subwords  $\bar{A}$  and  $A_w$  in  $A_{i_1}^{\varepsilon_1} \dots A_{i_k}^{\varepsilon_k}$  which are preimages under  $f$  of the words  $\bar{B}$  and  $W$  respectively.

We obtained an  $\bar{A}$ -periodic subword  $\bar{A}A_w$  in the word  $A_{i_1}^{\varepsilon_1} \dots A_{i_k}^{\varepsilon_k}$ . Notice that  $\bar{A}$  is not a conjugate of a product of words from  $S$  and their inverses. Otherwise  $\bar{B}$  would be freely conjugated to a product of some  $B_i^{\pm 1}$  what, in turn, implies that  $B$  is freely conjugated to a product of type  $\prod B_j^{\pm 1}$ . Now it follows from the Property 1 of words  $A_i$  that  $|\bar{A}| > \varepsilon^{-1}|A_w|$ .

Let  $A_z$  be a suffix of  $\bar{A}$ :  $\bar{A} = A_wA_z$ . The previous inequality implies  $|A_z| > (\varepsilon^{-1}-1)|A_w|$ . Denote  $Z \equiv f(A_z)$ . It follows that  $|Z| > (\varepsilon^{-1}\frac{l}{L}-1)|W| > (\frac{2}{3}\varepsilon^{-1}-1)|W|$ . Let  $Z'$  be the word obtained from  $Z$  after cancellations. It was explained that no cancellations could occur on junctions  $WZ'$  and  $Z'W$ . In

view of Lemma 2.2 we conclude that  $|Z'| > (1 - \frac{4\varepsilon}{5})|Z|$ .

Hence

$$|Z'| > (1 - \frac{4\varepsilon}{5})|Z| > (1 - \frac{4\varepsilon}{5})(\frac{2}{3}\varepsilon^{-1} - 1)|W| > \frac{1}{2}\varepsilon^{-1}|W|.$$

Thus,

$$|\bar{B}| = |W| + |Z'| > \frac{1}{2}\varepsilon^{-1}|W|.$$

Then

$$\frac{1}{2}\varepsilon^{-1}|W| < |\bar{B}| = |B| < \gamma^{-1}|B'| = \gamma^{-1}(|X| + |W| + |Y|) < 4\gamma^{-1}|W|.$$

Finally,

$$\frac{1}{2}\varepsilon^{-1} < 4\gamma^{-1}.$$

We obtained a contradiction with LPP. Thus  $B'$  cannot contain a block as a subword and hence  $|B'| < 2L$ . Therefore  $|B| < 2L\gamma^{-1}$ . The fact that  $\bar{h}(A_i)$  are very long 2-aperiodic words ( $|\bar{h}(A_i)| > Ll n\varepsilon^{-1} \gg 2L\gamma^{-1}$ ) and Lemma 2.1 imply that maximal integer power of a word  $B$ ,  $|B| < 2L\gamma^{-1}$ , that can occur as a subword in a product of some product of  $B_i^{\pm 1}$ , is not greater than 10, and hence the length of  $B$ -periodic subword is less than  $22L\gamma^{-1}$ .  $\square$

### 2.3. Subgroups of free Burnside groups

The following analogue of Theorem 4.4 from [33] implies Theorem 2.1:

**Theorem 2.2.** *For sufficiently large odd exponent  $n$ , the subgroup  $H$  of the group  $\mathbf{B}(2, n)$ , generated by words  $\{B_i, i = 1, 2, \dots\}$ , satisfying conditions of*

*Lemmas 2.2, 2.3 and the condition on their lengths ( $> L n \varepsilon^{-1}$ ), is isomorphic to the group  $\mathbf{B}(\infty, n)$  and satisfies the congruence extension property in  $\mathbf{B}(2, n)$ .*

*Proof.* First we show that the words  $B_1, B_2, \dots$  freely generate a free Burnside subgroup of exponent  $n$  in the group  $\mathbf{B}(2, n)$ . Suppose that  $v$  is a reduced word in some variables  $x_1, \dots, x_s$ . The (cyclic) word  $v(B_1, \dots, B_s) = B_{i_1}^{\eta_1} \cdots B_{i_t}^{\eta_t}$  obtained by replacing  $B_i$  for  $x_i$  is called a (cyclic)  $H$ -word.

The proof proceeds along the lines of the proof of Lemma 4.5 from [33]. Consider the graded presentation of  $B(2, n)$ . Assume there is a non-trivial relation  $v(B_{i_1}, \dots, B_{i_k}) = 1$  that does not follow from the Burnside relations on  $\{B_1, B_2, \dots\}$ . Then there is a reduced diagram  $\Delta$  of some rank  $i > 0$  with boundary label  $\phi(\partial\Delta)$  being freely equal to a cyclically reduced  $H$ -word. Proving by contradiction, we may assume that the number of cells of  $\Delta$  is minimal among diagrams representing such relations.

By Theorem 16.2 [31], there is a cell  $\Pi$  of  $\Delta$  and a contiguity subdiagram  $\Gamma$  with  $(\Pi, \Gamma, \partial\Delta) \geq \varepsilon$ . According to the choice of the words  $B_1, B_2, \dots$ , Lemma 2.3 and the inequality  $n\varepsilon > 11$ , we see, that the period  $A$ ,  $n$ th power of which is written on  $\partial\Pi$ , is a cyclic shift of some  $H$ -word. By Lemma 2.3, the label of the contour of the diagram  $\Delta_1$  obtained from  $\Delta$  by cutting off the cell  $\Pi$  is freely equal to an  $H$ -word. By the minimality of  $\Delta$ , the relation  $v_1(B_{i_1}, \dots, B_{i_k}) = 1$  represented by  $\Delta_1$ , follows from the Burnside relations on  $\{B_1, B_2, \dots\}$ . Then the same is true about the relation  $v(B_{i_1}, \dots, B_{i_k}) = 1$ , contrary to the assumption.

Let us now fix an arbitrary normal subgroup  $N$  in  $H$ . We would like to define a graded presentation of the factor group of  $\mathbf{B}(2, n)$  by the normal closure of  $N$  in  $\mathbf{B}(2, n)$ .

We denote by  $T$  the set of all  $H$ -words (in the alphabet  $a, b$ ) representing elements of  $N$ , that are cyclically reduced as elements of the free subgroup  $H$  (of the absolutely free group) with generators  $B_1, B_2, \dots$ . In particular,  $T$  contains all powers of the form  $v(B_{i_1}, \dots, B_{i_k})^n$ . The following alteration of the definitions given in Section 1.2 (Section 18 of [31]) depends on  $N$ .

We define  $G(0) = F(a, b)$  to be the absolutely free group with the empty set  $\mathcal{R}_0$  of defining words. For every  $i \geq 0$  a non-trivial word  $A$  is called simple in rank  $i$  if it is conjugate in rank  $i$  neither to a power of a period  $B$  of rank  $k$ , where  $1 \leq k \leq i$ , nor to a power of a word  $C$ , where  $|C| < |A|$ , nor to an  $H$ -word. Every word of  $T$  is included in the system  $\mathcal{R}_1$  of relators of rank 1. We also include words  $a^n, b^n$  in the system  $\mathcal{R}_1$ . The words  $a$  and  $b$  of length 1 are, by definition, periods of rank 1, and  $G(1) = \langle a, b | \mathcal{R}_1 \rangle$ .

For ranks  $i \geq 2$ , the definitions of periods of rank  $i$  and the group  $G(i)$  given in Section 1.2 (Section 18 of [31]) remain valid. As usual, by  $G(\infty)$  denote the inductive limit of groups  $G(i)$ .

The outline of the proof that  $H$  satisfies CEP in  $\mathbf{B}(2, n)$  is similar to the corresponding part of the proof of Theorem 4.4 given in [33]. The principal idea of the latter argument is to show that reduced diagrams over the constructed graded presentation are  $A$ -maps. The only alterations are in the estimates and caused by the different aperiodic properties of words  $A_i$ 's and  $B_i$ 's (compare Lemma 4.2(1) [33] and Lemma 2.3). All the changes are discussed below.

In Lemma 4.12 [33], the inequality  $|q(0)| > |q|/2 > 2|A| > (1 + \varepsilon)|A|$  can be changed to the inequality  $|q(0)| > |q|/2 > 11|A|$ , which, using Lemma 2.3, leads to a contradiction as in [33].

In part of Lemma 4.15 [33], which deals with property A3, we have to consider just the case when  $\pi$  is a  $T$ -cell and  $\Pi$  is an  $R$ -cell. In this case

$r(\Gamma) = 0$ ,  $|q| > Lln\varepsilon^{-2} > 22L\gamma^{-1}$ . Lemma 2.3 implies now that  $|\Gamma \wedge \Pi| < (1 + \gamma)r(\Pi)$ .

In the end of the argument from part 2 of Lemma 4.16 [33] the estimate  $|q_2| < (1 + \gamma)(i+1)$  follows from Lemma 2.3 and the inequality  $(\pi, \Gamma, p) \geq \varepsilon$ .  $\square$

## CHAPTER III

### NON-ISOMORPHIC SIMPLE TORSION GROUPS

The goal of this chapter is to prove the following theorem:

**Theorem 3.1.** *For any sufficiently large exponent  $n$  the set of pairwise non-isomorphic 2-generated simple groups satisfying the identity  $x^n = 1$  is of cardinality continuum.*

In the proof of this theorem given in section 4 we assume that  $n \geq 2^{48}$  and  $n$  is either odd or divisible by  $2^9$ . The statement for any multiple of such  $n$  clearly follows.

#### 3.1. $\mathcal{T}$ -relators

This construction is based on the construction given by S.V. Ivanov in [14].

In the following table we list numerical values of auxiliary parameters that will be used in the estimates ([14], §2).

$$\alpha = 0.522, \quad \beta = 2^{-14}, \quad \gamma = 2^{-33}, \quad \delta = 1.005,$$

$$\varepsilon = 0.003, \quad \varepsilon_0 = 0.002, \quad \zeta = 0.01, \quad \eta = 0.923,$$

$$\theta = 0.99, \quad \mu = 1.3, \quad \xi = 2^{-26}, \quad \rho = 0.95,$$

$$\rho_0 = 0.89, \quad \chi = 0.999, \quad \omega = 2^{-22}, \quad n \geq 2^{48}.$$

For the fixed above values of  $n$  and  $\xi$  the following lemma follows from

Lemmas 2.2 and 2.3 of Chapter 2.

**Lemma 3.1.** *For given  $n$  and  $\xi$  there exists an infinite set of positive words  $\mathcal{T} = \{B_1, B_2, \dots\}$  in the alphabet  $\{a_1, a_2\}$ , satisfying the following properties:*

(a) *Suppose a cyclic shift of some word  $B_i^{\pm 1}$  contains a  $B$ -periodic subword*

*$U$  of length greater than  $(1 + \xi)|B|$ . Then  $|B| < \xi^{-3/2}$  and*

$$|U| < 11|B| < 11\xi^{-3/2}.$$

(b) *The symmetrized set obtained from the set  $\mathcal{T}$  satisfies the small cancellation condition  $C'(\frac{\xi}{10})$ .*

(c)  $|B_i| \geq n^2, i = 1, 2, \dots$

According to the choice of the auxiliary parameters ( $\beta n^2 > 11\xi^{-3/2}$ ,  $\beta > \xi$ ) the next lemma immediately follows from Lemma 3.1.

**Lemma 3.2.** *Let  $U$  be a  $B$ -periodic subword of a cyclic shift of some word  $B_i^{\pm 1}$  of length  $|U| > \beta|B_i|$ . Then  $|U| < (1 + \xi)|B|$  and  $|B_i| < (1 + \xi)\beta^{-1}|B|$ .*

Define  $G(0) = F(a_1, a_2)$  to be absolutely free group and set

$$G(1/2) = \langle a_1, a_2 \mid B = 1, B \in \mathcal{T} \rangle. \quad (4)$$

Arguing as for  $C'(1/8)$ -groups (see [20], Chapter V, Theorem 10.1), in view of Lemma 3.1 we obtain the following

**Lemma 3.3.** *The group  $G(1/2)$  is torsion free.*

### 3.2. Inductive construction of group $G_{\mathcal{T}}$

For every  $i$ ,  $i = 0, 1/2, 1, 2, \dots$ , we shall define the group  $G(i)$  of rank  $i$ .

The groups  $G(0)$  and  $G(1/2)$  are already defined. Following [14], the elements of  $F(a_1, a_2)$  and of its quotients are referred to as words (in the alphabet  $\{a_1^{\pm 1}, a_2^{\pm 1}\}$ ). Let ' $\prec$ ' be a total order on the set of words over  $\{a_1^{\pm 1}, a_2^{\pm 1}\}$ , such that  $|X| < |Y|$  implies  $|X| \prec |Y|$ , where  $|X|$  is the length of the word  $X$ .

Dealing with ranks, we agree that  $i - 1$  is equal to  $1/2$  (resp.  $0$ ) if  $i = 1$  (resp.  $(i = 1/2)$ ), and  $i + 1$  is equal to  $1$  (resp.  $1/2$ ) provided  $i = 1/2$  (resp.  $i = 0$ ).

Assuming that the group  $G(i - 1)$  for  $i \geq 1$  is defined, by the period  $A_i$  of rank  $i$  we mean the first (relative to the imposed order) of the words that have infinite order in  $G(i - 1)$ . Then the group  $G(i)$  ( $i \geq 1$ ) is defined by imposing the relation  $A_i^n = 1$  on  $G(i - 1)$ :

$$G(i) = \langle a_1, a_2 \mid \{B = 1, B \in \mathcal{T}\} \cup \{A_1^n = 1, \dots, A_i^n = 1\} \rangle . \quad (5)$$

A planar diagram over the presentation (4) (resp. (5)) is called a diagram of rank  $1/2$  (resp.  $i$ ). A cell  $\Pi$  of a diagram  $\Delta$  of rank  $i$  has rank  $1/2$  provided the label of its contour is a cyclic shift of  $B^{\pm 1}$  for some  $B \in \mathcal{T}$ . Following [31] and [33] any such cell is referred to as  $1/2$ -cell or a  $\mathcal{T}$ -cell while cells of rank  $j$ ,  $j \geq 1$ , are called  $\mathcal{R}$ -cells.

By a strict rank  $r(\Delta)$  of a diagram  $\Delta$  we mean maximum of ranks of the cells contained in it. By definition,  $r(\Delta) = 0$  if  $\Delta$  does not contain cells. The type  $\tau(\Delta)$  of a diagram  $\Delta$  of rank  $i$  is the sequence  $(n_i, \dots, n_{1/2})$ , where  $n_j$  is the number of cells of rank  $j$  in  $\Delta$ . The set of types is ordered in the usual

lexicographical way.

Diagrams will be usually considered with some fixed decomposition of its contours into products of their subpaths, which will be termed sections of the contour. Contours that are not decomposed into products of their subpaths will be regarded as oriented cyclic paths and called cyclic sections.

A section  $q$  of a boundary of  $\Delta$  is called  $\mathcal{T}$ -section if  $\phi(q)$  is freely equal to a subword of a cyclic shift of  $B^{\pm 1}$  for some  $B \in \mathcal{T}$ . In the case when the label  $\phi(q)$  is an  $A$ -periodic word the section  $q$  is called an  $A$ -periodic section.

A vertex  $o$  of the boundary  $\partial\Pi$  of a cell  $\Pi$  of rank  $j \geq 1$  is called a phase vertex if the label of the path  $\partial\Pi$  starting at  $o$  is equal to  $A_j^n$  or  $A_j^{-n}$ . Similarly we define phase vertices of  $A$ -periodic sections of a contour of a diagram.

Let each of the sections  $q_1$  and  $q_2$  be either a (cyclic) section of the contour of a diagram  $\Delta$  of rank  $i$  or the contour of a cell. Suppose that  $\phi(q_1)$  and  $\phi(q_2)$  are  $A_j^{\varepsilon_1}$ - and  $A_j^{\varepsilon_2}$ -periodic words respectively, where  $j \leq i$ ,  $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ . The sections  $q_1$  and  $q_2$  are called  $j$ -compatible in  $\Delta$  if one of the following two conditions holds:

- (A1) If  $\varepsilon_1\varepsilon_2 = -1$  (in particular  $q_1 \neq q_2$ ), then there are phase vertices  $o_1 \in q_1$ ,  $o_2 \in q_2$  and a simple path  $t = o_1 - o_2$  in  $\Delta$  such that  $|t| < \delta|A_j|$  and  $\phi(t) \stackrel{j-1}{=} T$ , where  $T \in \mathcal{F}(A_j)$  (the definition of the subgroup  $\mathcal{F}(A_j)$  is given below).
- (A2) If  $\varepsilon_1\varepsilon_2 = 1$ , then there are phase vertices  $o_1 \in q_1$ ,  $o_2 \in q_2$  and a simple path  $t = o_1 - o_2$  in  $\Delta$  such that  $|t| < \delta|A_j|$  and  $\phi(t) \stackrel{j-1}{=} T$ , where  $T$  is an  $\mathcal{F}(A_j)$ -involution (the definition of an  $\mathcal{F}(A_j)$ -involution is given below).

If for two sections  $q_1, q_2$  one of the properties (A1)-(A2) holds without the condition that  $|t| < \delta|A_j|$ , then  $q_1$  and  $q_2$  are called weakly  $j$ -compatible. If

sections  $\partial\Pi$  and  $q$  are (weakly)  $j$ -compatible, then we say that  $\Pi$  is (weakly)  $j$ -compatible with  $q$ , and, similarly, two cells  $\Pi_1$  and  $\Pi_2$  are called (weakly)  $j$ -compatible if their contours  $\partial\Pi_1$  and  $\partial\Pi_2$  are (weakly)  $j$ -compatible. A disk subdiagram  $\Gamma$  of  $\Delta$  that consists of a (weakly)  $j$ -compatible pair of cells of rank  $j \geq 1$  together with the corresponding path is called a (weak) reducible pair. A (weak) reducible pair of rank  $j$  can be cancelled (see [14]), i.e. substituted by a diagram consisting of cells of rank  $< j$ , without changing the boundary labels of the diagram. If the contour  $\partial\Pi$  of a cell  $\Pi$  of rank  $j \geq 1$  is  $j$ -compatible with  $\partial\Pi$ , then the cell  $\Pi$  is termed self-compatible. The annual diagram  $\Gamma$  of  $\Delta$  that consists of such a self-compatible cell  $\Pi$  of rank  $j$  and the path  $t$  (from the definition of compatibility) is called a reducible  $j$ -cell. A reducible  $j$ -cell can be also substituted by cells of smaller ranks without changing the labels of contours of  $\Delta$  (see [14]).

The definition of compatibility of two  $1/2$ -cells (or  $1/2$ -cell and a  $\mathcal{T}$ -section) is analogous to the definition of compatibility of two  $T$ -cells (resp.  $T$ -cell and an  $H$ -section) given in Section 4.2 of [33]. We only require that  $\phi(t) = 1$  in the free group and the label of the path  $q_1 t^{-1} p t q_2$  is freely equal to (a subword of) a cyclic shift of  $B^{\pm 1}$  for some  $B \in \mathcal{T}$ . Note that a pair of two compatible  $\mathcal{T}$ -cells (following [14] and [31], a pair of such cells is called a  $1/2$ -pair) can be substituted by a diagram without cells.

A disk diagram  $\Delta$  of rank  $i$  is called reduced if it does not contain reducible  $j$ -pairs for every  $j \leq i$ . A disk diagram  $\Delta$  of rank  $i$  is called strictly reduced if it contains no  $1/2$ -pairs and no weak reducible  $j$ -pairs as its subdiagrams for every  $1 \leq j \leq i$ . A non-disk diagram  $\Delta$  of rank  $i$  is termed reduced if  $\Delta$  does not contain reducible  $1/2$ -pairs, reducible  $j$ -pairs and reducible  $j$ -cells for every  $1 \leq j \leq i$ . A reduced non-disk diagram  $\Delta$  of rank  $i$  is called strictly reduced

provided it does not contain weak reducible  $j$ -pairs for every  $1 \leq j \leq i$ .

The concepts of  $k$ -contiguity subdiagram and  $k$ -bond are given inductively.

Notice that  $k$  takes on values  $0, 1/2, 1, 2, \dots$ .

Let  $\Delta$  be a diagram of rank  $i$  and each of the sections  $p$  and  $q$  be either a (cyclic) section of a contour of  $\Delta$  or a contour of some cell of  $\Delta$ . By  $H_p, H_q$  denote the holes (i.e. components of the complement of  $\Delta$  in  $\mathbf{R}^2$ ) or the cells of  $\Delta$  such that  $\partial H_p$  contains  $p$ ,  $\partial H_q$  contains  $q$ .

A 0-bond between  $p$  and  $q$  is a subdiagram  $E$  with contour  $\partial E = ee^{-1}$ , where  $e$  is an edge of the path  $p$  and  $e^{-1}$  belongs to  $q$ . A connecting line of the 0-bond  $E$  is a line joining some points  $o_1 \in H_p \setminus \partial H_p, o_2 \in H_q \setminus \partial H_q$  via the edge  $e$ . A 0-contiguity subdiagram  $\Gamma$  between  $p$  and  $q$  defined by a pair of 0-bonds  $E_1, E_2$  is a minimal disk subdiagram (if any) containing the bonds  $E_1, E_2$  such that  $\partial\Gamma = p_\Gamma q_\Gamma$  where  $p_\Gamma$  is a subpath of  $p$ ,  $q_\Gamma$  is a subpath of  $q$ . The paths  $p_\Gamma$  and  $q_\Gamma$  are denoted by  $p_\Gamma = \Gamma \wedge p, q_\Gamma = \Gamma \wedge q$  and called contiguity arcs of  $\Gamma$ . A connecting line of one of the bonds  $E_1, E_2$  is called a connecting line of  $\Gamma$ . Now suppose that  $j$ -bonds and  $j$ -contiguity subdiagrams are defined for all  $j \in \{0, 1/2, 1, 2, \dots\}, j < k$  and assume that the following conditions are satisfied:

- (B1)  $r(\Pi) = k$  and if  $p$  (respectively  $q$ ) is either  $\partial\Pi$ , where  $\Pi$  is a cell with  $r(\Pi) = l$ , or an  $A_l^{\pm 1}$ -periodic section of a contour of  $\Delta$  such that  $\Delta$  contains no cells of rank  $l$  which are  $l$ -compatible with  $p$  (or  $q$  respectively), then  $k < l$ .
- (B2) There exist subdiagrams  $\Gamma_1$  and  $\Gamma_2$  of  $j_1$ -contiguity of  $\partial\Pi$  to  $p$  and of  $j_2$ -contiguity of  $\partial\Pi$  to  $q$  respectively with  $\max(j_1, j_2) < k$ .
- (B3)  $\min(|\Gamma_1 \wedge \partial\Pi|, |\Gamma_2 \wedge \partial\Pi|) > \beta|\partial\Pi|$ .

- (B4) Subdiagrams  $\Gamma_1$  and  $\Gamma_2$  have no cells in common and the paths  $\Gamma_1 \wedge \partial\Pi$  and  $\Gamma_2 \wedge \partial\Pi$  have no edges in common.
- (B5) In the case  $p = q$ , the loop  $L$  obtained by joining within  $\Pi$  and  $H_p$  pairs of endpoints of connecting lines defined for  $\Gamma_1$  and  $\Gamma_2$  has the following property: if we replace the segment of  $L$  situated within  $H_p$  by any arc of  $\partial H_p$  with the same endpoints as those of the segment, then thus obtained loop  $L'$  is not contractible to a point within  $\Delta$ .

Then the minimal disk subdiagram  $E$  of  $\Delta$  containing  $\Pi$ ,  $\Gamma_1$ ,  $\Gamma_2$  is called a  $k$ -bond between  $p$  and  $q$  defined by  $\Gamma_1$ ,  $\Gamma_2$  and the principal cell  $\Pi$ . The contour of  $E$  is regarded with its standard decomposition  $\partial E = d_1 p_1 d_2 q_1$ , where the arcs  $p_1 = \Gamma_1 \wedge p$ ,  $q_1 = \Gamma_2 \wedge q$  are called contiguity arcs of  $E$ . To obtain the connecting line of  $E$  it suffices to join within  $\Pi$  the endpoints of the connecting lines of  $\Gamma_1$  and  $\Gamma_2$  that are contained inside  $\Pi$  and take a resulting line. Using two bonds between the same pair of sections (a  $k$ -bond  $E_1$  and a  $j$ -bond  $E_2$  ( $j \leq k$ )), we define  $k$ -contiguity subdiagram in the same way as 0-contiguity subdiagram was defined for two 0-bonds. A connecting line of a  $k$ -contiguity subdiagram is set to be one of the connecting lines of the corresponding bonds. The standard decomposition of the contour of a  $k$ -contiguity subdiagram  $\Gamma$  between sections  $p$  and  $q$  is of the form  $\partial\Gamma = d_1 p_\Gamma d_2 q_\Gamma$  where the paths  $p_\Gamma = \Gamma \wedge p$  and  $q_\Gamma = \Gamma \wedge q$  are called contiguity arcs. The ratio  $|\Gamma \wedge p|/|p|$  ( $|\Gamma \wedge q|/|q|$ ) is called the degree of contiguity of  $p$  to  $q$  (of  $q$  to  $p$  respectively) and is denoted by  $(p, \Gamma, q)$  (respectively  $(q, \Gamma, p)$ ).

A word  $A$  is called simple in rank  $i$  if  $A$  is cyclically reduced in rank  $i$  and  $A$  is not conjugate in rank  $i$  to a word of the form  $A_k^l F$ , where  $A_k$  is a period of some rank  $k \leq i$ ,  $l$  is an integer, and  $F$  is a word from  $\mathcal{F}(A_k)$  (the definition

of  $\mathcal{F}(A_k)$  is given below).

A (cyclic) section  $s$  of a contour of a diagram  $\Delta$  of rank  $i$  is called smooth if one of the following conditions (C1)-(C3) is satisfied:

- (C1)  $s$  is an  $A_j^{\pm 1}$ -periodic section and  $\Delta$  contains no cells of rank  $j$  that are  $j$ -compatible with  $s$ .
- (C2)  $s$  is an  $A$ -periodic section where  $A$  is a simple in rank  $i$  word.
- (C3)  $s$  is a  $\mathcal{T}$ -section and there are no  $\mathcal{T}$ -cells in  $\Delta$  which are compatible with  $s$ .

A section  $s$  satisfying condition (C3) is called smooth of rank  $1/2$ . Strictly (quasi)smooth sections of rank  $j$  are defined for  $j \geq 1$  only ([14]).

A reduced diagram  $\Delta$  of rank  $i$  is called tame provided it has the following properties:

- (D1) Let  $\Gamma$  be a contiguity subdiagram between sections  $p$  and  $q$ , where  $p = \partial\Pi_1$ ,  $q$  is either a smooth (cyclic) section of a contour of  $\Delta$  or  $q = \partial\Pi_2$  (perhaps  $\Pi_1 = \Pi_2$ ). Then  $r(\Gamma) < \min(r(p), r(q))$ .
- (D2) If  $\Pi$  is a cell in  $\Delta$  and  $e$ ,  $e^{-1} \in \partial\Pi$  for some edge  $e$ , then the subdiagram  $E$  given by  $\partial E = ee^{-1}$  is a bond between  $\partial\Pi$  and  $\partial\Pi$  in  $\Delta$ .

The subgroup  $\mathcal{F}(A_i)$  of the group  $G(i-1)$  is defined uniquely (see Lemma 18.5 [14]) as a finite group maximal with respect to the property that  $A_i$  normalizes  $\mathcal{F}(A_i)$ . A word  $J$  is called an  $\mathcal{F}(A_i)$ -involution associated with the period  $A_i$  if  $J$  normalizes the subgroup  $\mathcal{F}(A_i)$  of  $G(i-1)$ ,  $J^2 \in \mathcal{F}(A_i) \subseteq G(i-1)$ , and  $J^{-1}A_iJ \stackrel{i}{=} A_i^{-1}F$ , where  $F \in \mathcal{F}(A_i)$ .

The numeration of analogues of lemmas from [14] is organized as follows: we preserve the numbering from [14], attaching "[14]" at the end. The formulations and proofs of Lemmas 1.1[14] - 1.7[14] are the same as in [14]. Lemmas 3.4 - 3.8 together with Lemmas 3.1[14] - 20.3[14] are proved by simultaneous induction on the parameter  $i$ ,  $i = 0, 1/2, 1, 2, \dots$ . Fixing  $i$  and assuming that Lemmas 3.4 - 3.8, 3.1[14] - 20.3[14] hold in rank  $i - 1$ , we induct on the type  $\tau(\Delta)$  of a diagram  $\Delta$  of rank  $i$ .

In Lemmas 3.4 - 3.8 we establish some properties of bonds and contiguity subdiagrams in reduced diagrams of rank  $i$ .

**Lemma 3.4.** *Let  $\Gamma$  be a contiguity subdiagram of a cell  $\pi$  to a  $\mathcal{T}$ -cell  $\Pi$  in a reduced diagram  $\Delta$  of rank  $i$ . Then  $r(\Gamma) = 0$  and the contiguity degree of  $\pi$  to  $\Pi$  is less than  $\beta$ .*

*Proof.* We prove this lemma by contradiction. Assume that the triple  $(\pi, \Gamma, \Pi)$  is a counterexample with contiguity subdiagram  $\Gamma$  of minimal type. Let the standard contour of  $\Gamma$  be  $\partial\Gamma = d_1pd_2q$ , where  $p = \Gamma \wedge \Pi$ ,  $q = \Gamma \wedge \pi$ . The bonds defining  $\Gamma$  are 0-bonds (otherwise  $(\pi, \Gamma, \Pi)$  is not minimal). That means that  $|d_1| = |d_2| = 0$ . Assuming that  $\Gamma$  has cells, by Lemma 5.7[14] ( $\tau(\Gamma) < \tau(\Delta)$ ), there is a  $\theta$ -cell. However, by Lemma 3.5 applied to  $\Gamma$  ( $\tau(\Gamma) < \tau(\Delta)$ ), the degree of contiguity of any cell from  $\Gamma$  to  $p$  is less than  $\beta$  and the contiguity degree of any cell from  $\Gamma$  to  $q$  is less than  $\alpha$  by Lemmas 3.8 and 3.4[14] (again,  $\tau(\Gamma) < \tau(\Delta)$ ). The fact that  $\theta > \alpha + \beta$  implies that  $\Gamma$  does not have cells. Suppose that the contiguity degree of  $\pi$  to  $\Pi$  is greater than or equal to  $\beta$ . Lemma 3.1(b) and the fact that  $\Delta$  is reduced mean that  $\pi$  can not be a  $\mathcal{T}$ -cell. If  $\pi$  is an  $\mathcal{R}$ -cell, then we obtain a contradiction with Lemma 3.1(a) since  $\beta n^2 > 11$ .  $\square$

**Lemma 3.5.** *Let  $\Gamma$  be a contiguity subdiagram of a cell  $\pi$  to a  $\mathcal{T}$ -section  $q$  of the boundary of a reduced diagram  $\Delta$  of rank  $i$ . Assume there are no  $\mathcal{T}$ -cells compatible with  $q$  in  $\Delta$ . Then  $r(\Gamma) = 0$  and the contiguity degree of any cell from  $\Delta$  to  $q$  is less than  $\beta$  ( $q$  is a  $\beta$ -section of  $\partial\Delta$  in the terminology of [33]).*

*Proof.* The proof is similar to the proof of Lemma 3.4.  $\square$

**Lemma 3.6.** *Let  $\Gamma$  be a contiguity subdiagram of a  $\mathcal{T}$ -cell  $\pi$  to a geodesic section  $q$  in a reduced diagram  $\Delta$  of rank  $i$ . Then*

- (1)  $r(\Gamma) = 0$ ;
- (2) *The contiguity degree of  $\pi$  to  $q$  does not exceed  $1/2 < \alpha$ .*

*Proof.* **1.** Denote the standard contour of  $\Gamma$  by  $\partial\Gamma = d_1pd_2q_1$ , where  $p = \Gamma \wedge \pi$ ,  $q_1 = \Gamma \wedge q$ . By Lemma 3.4, the bonds defining  $\Gamma$  are 0-bonds, and therefore  $|d_1| = |d_2| = 0$ . Suppose  $r(\Gamma) \neq 0$ . Then Lemma 5.7[14] may be applied to  $\Gamma$  since  $\tau(\Gamma) < \tau(\Delta)$ , which guarantees that there is a cell  $\Pi \in \Gamma$ , such that the sum of the contiguity degrees of  $\Pi$  to  $p$  and to  $q_1$  is greater than  $\theta$ . Note that there are no  $\mathcal{T}$ -cells compatible with the section  $q_1$  in  $\Gamma$ , since otherwise  $\Delta$  would not be reduced. Thus, by Lemma 3.5, the degree of contiguity of  $\Pi$  to  $p$  is less than  $\beta$ . Lemmas 3.6 and 3.3[14] applied to  $\Gamma$  ( $\tau(\Gamma) < \tau(\Delta)$ ) mean that the degree of contiguity of  $\Pi$  to  $q$  is less than  $\alpha$ . Finally, the inequality  $\theta > \alpha + \beta$  provides a contradiction to the assumption that  $\Gamma$  has cells.

**2.** If the statement of part (2) was not true, by proven part (1) the section  $q$  would not be geodesic in  $\Delta$ .  $\square$

**Lemma 3.7.** *Let  $\Gamma$  be a contiguity subdiagram of a  $\mathcal{T}$ -cell  $\pi$  to an  $\mathcal{R}$ -cell  $\Pi$  in a reduced diagram  $\Delta$  of rank  $i$ . Then*

- (1)  $r(\Gamma) = 0$ ;
- (2) *The contiguity degree of  $\pi$  to  $\Pi$  does not exceed  $\alpha$ .*

*Proof.* **1.** Denote the standard contour of  $\Gamma$  by  $\partial\Gamma = d_1pd_2q$ , where  $p = \Gamma \wedge \Pi$ ,  $q = \Gamma \wedge \pi$ . By Lemma 3.4, the bonds defining  $\Gamma$  are 0-bonds. Therefore  $|d_1| = |d_2| = 0$ . Assume that  $\Gamma$  has cells. By Lemma 5.7[14] applied to  $\Gamma$  ( $\tau(\Gamma) < \tau(\Delta)$ ), there is a  $\theta$ -cell in  $\Gamma$ . There are no  $\mathcal{T}$ -cells compatible with the section  $q$  in  $\Gamma$  since  $\Delta$  is reduced. Therefore, by Lemma 3.5 applied to  $\Gamma$ , the degree of contiguity of any cell from  $\Gamma$  to  $q$  is less than  $\beta$ ; by Lemmas 3.8 and 3.4[14] applied to  $\Gamma$  ( $\tau(\Gamma) < \tau(\Delta)$ ), the degree of contiguity of any cell from  $\Gamma$  to  $p$  is less than  $\alpha$ . Since  $\theta > \alpha + \beta$ , the assumption that  $\Gamma$  has cells is wrong.

Part (1) is proved.

**2.** Assume that the degree of contiguity of a  $\mathcal{T}$ -cell  $\pi$  to an  $\mathcal{R}$ -cell  $\Pi$  of rank  $j \geq 1$  via  $\Gamma$  is greater than or equal to  $\alpha$ . By proven part (1) the subdiagram  $\Gamma$  does not have cells. This means that there is a common subpath  $u$  of  $\partial\pi$  and  $\partial\Pi$  of length at least  $\alpha|\partial\pi|$ . By Lemma 3.2 ( $\alpha > \beta$ ), we conclude that  $|u| < (1 + \xi)|A_j|$ . Denote  $\partial\pi = uv$ . If  $|u| \leq |A_j|$ , then  $A' \equiv \phi(u)A'' \stackrel{1/2}{=} \phi(v)^{-1}A''$  for some cyclic shift  $A'$  of  $A_j^{\pm 1}$ .

Consequently, the inequality  $|v| < \alpha^{-1}(1 - \alpha)|u| < |u|$  implies that the word  $A_j$  is conjugated in rank  $1/2 < j$  to a shorter word, contrary to the definition of  $A_j$ . Let now  $|A_j| < |u| < (1 + \xi)|A_j|$ . Denote  $u = u_1u_2$ , where  $|u_1| = |A_j|$ . Then for some cyclic shift  $A'$  of  $A_j$  one has

$$A' \equiv (\phi(u_1))^{\pm 1} \stackrel{1/2}{=} (\phi(u_2v))^{\mp 1}.$$

But

$$\begin{aligned} |(\phi(u_2v))| &= |v| + |u_2| < (1 - \alpha)\alpha^{-1}|u| + \xi|u_1| < \\ &< ((1 - \alpha)(1 + \xi)\alpha^{-1} + \xi)|u_1| < |u_1|, \end{aligned}$$

which means that  $A_j$  is conjugated in  $G(1/2)$  to a shorter word, contrary to the definition of  $A_j$ . Lemma 3.7 is proved.  $\square$

**Lemma 3.8.** *Let  $\Gamma$  be a contiguity subdiagram of a  $\mathcal{T}$ -cell  $\pi$  to a smooth section  $q$  in a reduced diagram  $\Delta$  of rank  $i$ . Then*

- (1)  $r(\Gamma) = 0$ ;
- (2) *The contiguity degree of  $\pi$  to  $q$  does not exceed  $\alpha$ .*

*Proof.* The case when  $q$  is a smooth section of rank  $1/2$  is taken care of in Lemma 3.5. If  $q$  is a smooth  $A_j$ -periodic section, where  $A_j$  is the period of rank  $j \leq i$ , than the proof is similar to the proof of Lemma 3.7. In the case when  $q$  is a smooth  $A$ -periodic section with a simple in rank  $i$  word  $A$  the proof also proceeds as in Lemma 3.7 except for instead of the definition of a period of rank  $i$  we now use the definition of a simple in rank  $i$  word.  $\square$

Below we discuss the changes needed be done in Lemmas 3.1 - 20.3 from [14] for the purpose of present paper.

In the formulation of Lemma 3.1[14] the section  $p_1$  ( $q_1$ ) is considered to be a smooth  $\mathcal{T}$ -section of  $\partial E$  provided  $p$  ( $q$ ) is either the contour of a cell  $\Pi_1$  ( $\Pi_2$ ) of rank  $j_1 = r(\Pi_1) = 1/2$  ( $j_2 = r(\Pi_2) = 1/2$ ), or a  $\mathcal{T}$ -section of a contour of the diagram  $\Delta$ . The conclusion of Lemma 3.1[14] does not change if none of  $p_1$ ,  $q_1$  is a  $\mathcal{T}$ -section. Otherwise  $\max(|d_1|, |d_2|) = 0$ .

Proving Lemma 3.1[14], we first note that the case when at least one of  $p_1$ ,  $q_1$  is a  $\mathcal{T}$ -section is taken care of in Lemmas 3.4 and 3.5: the bond  $E$  between  $p_1$  and  $q_1$  is a 0-bond, and therefore  $\max(|d_1|, |d_2|) = 0$ . Now assume that none of  $p_1$ ,  $q_1$  is a  $\mathcal{T}$ -section and let  $p_1$  ( $q_1$ ) be a smooth  $A$ - ( $B$ -) periodic section of  $\partial E$ . Consider the case when the principal cell  $\pi$  of the bond  $E$  is a  $\mathcal{T}$ -cell. By Lemmas 3.7, 3.8,  $r(\Gamma_1) = r(\Gamma_2) = 0$ , where  $\Gamma_1$  and  $\Gamma_2$  are the contiguity subdiagrams of  $\pi$  to  $p_1$  and  $q_1$  respectively, such that  $\pi$  together with  $\Gamma_1$  and  $\Gamma_2$  form the bond  $E$ . Therefore there are common subpaths of  $\partial\pi$  and each of the sections  $p_1$  and  $q_1$ , and, by the definition of a bond, the lengths of those subpaths are greater than  $\beta|\partial\pi|$ . Consequently, by Lemma 3.2,  $|\partial\pi| < (1 + \xi)\beta^{-1} \min(|A|, |B|)$ . Finally,

$$\max(|d_1|, |d_2|) < |\partial\pi| < 2\beta^{-1} \min(|A|, |B|) < \gamma \min(n|A|, n|B|).$$

In the formulation of Lemma 3.2[14] we require in addition that  $l = r(q) > 1/2$  (indeed, by Lemmas 3.4, 3.5, the degree of contiguity of any cell to a  $\mathcal{T}$ -cell or to a  $\mathcal{T}$ -section of the boundary is less than  $\beta$  in a reduced diagram of rank  $i$ ). In the conclusion of Lemma 3.2[14]  $|A_k|$  is substituted by  $n^{-1}|\partial\Pi|$  since we also have to consider the case  $r(\Pi) = 1/2$ . The proof of Lemma 3.2[14] does not change if  $r(\Pi) > 1/2$ . In the case  $r(\Pi) = 1/2$  Lemma 3.2[14] is straightforward from Lemmas 3.7, 3.8 and 3.2.

All definitions from Section 4 of [14] are preserved. The formulations and proofs of Lemmas 4.1[14] - 4.5[14] need not be changed. All estimates from the Lemmas 5.1[14] - 5.7[14] hold in view of Lemmas 3.4 - 3.8. In the proof of Lemma 5.1[14] notice that  $r(\Pi_2) \neq 1/2$  by Lemma 3.4. If  $r(\Pi_1) = 1/2$ , then  $|q_\Gamma| = |p^\Gamma|$  since  $r(\Gamma) = 0$  by Lemma 3.7, and, therefore, the inequality

(5.2)[14] is valid. In the proof of Lemma 5.4[14], if  $r(\Pi_k) = 1/2$ , where  $k = 1$  and/or 2, then  $r(\Gamma) = 0$  by Lemma 3.7. Hence  $|q_\Gamma| = |p^\Gamma|$  and the inequality (5.5)[14] follows. Proving Lemma 5.5[14], notice that case 1 is impossible by Lemma 3.4 if the considered ordinary cell  $\Pi$  is a  $\mathcal{T}$ -cell.

The formulation of Lemma 6.1[14] does not change. Proving this lemma in the case  $r(q) = 1/2$ , notice that a diagram  $\Delta'$  with contour  $p'q$  with  $p'$  being geodesic does not have cells (otherwise there must be a  $\theta$ -cell, but the degree of contiguity of any cell from  $\Delta'$  to  $q$  is less than  $\beta$  by Lemma 3.5 applied to  $\Delta'$ ; the degree of contiguity of any cell from  $\Delta'$  to  $p'$  is less than  $\alpha$  by Lemmas 3.6 and 3.3[14], but  $\theta > \alpha + \beta$ ). Then  $\rho|q| < |q| = |p'| \leq |p|$ .

Now consider the case when  $q$  is a smooth  $B$ -periodic section, where  $B \equiv A_l^{\pm 1}$  for some  $l \leq i$  or  $B$  is a simple word in rank  $i$ . The same proof as in [14] works if rank of the  $\theta$ -cell  $\Pi$  is greater than  $1/2$ . We need to consider the situation when the  $\theta$ -cell  $\Pi$  is of rank  $1/2$ . In the notation used in the proof of Lemma 6.1[14] we have  $|c_1| = |c_2| = |d_1| = |d_2| = 0$ ,  $v_1 = q_2^{-1}$ ,  $u_1 = p_2^{-1}$  by Lemmas 3.6 and 3.8 applied to  $\Delta$ . The inequality

$$\rho(|q_1| + |q_3|) < |p_1| + |p_3| + (1 - \theta)|\partial\Pi| \quad (6)$$

can be obtained in the same way as in [14].

By Lemma 3.8, the contiguity subdiagram  $\Gamma_q$  does not have cells. By Lemma 3.6 applied to the contiguity subdiagram  $\Gamma_p$  of  $\Pi$  to  $p$  and by the definition of a  $\theta$ -cell, the degree of contiguity of  $\Pi$  to  $q$  is greater than  $\theta - \alpha > \beta$ . Consequently, by Lemma 3.2 applied to  $\phi(q_2)$ , one has  $|q_2| < (1 + \xi)|B|$ .

Assume first that  $|q_2| \leq |B|$ . It follows that  $|q_2| \leq 1/2|\partial\Pi| \leq |u_2| + |v_2| + |p_2|$  since  $B$  is cyclically reduced in rank  $1/2$  and  $|p_2| > \theta|\partial\Pi| - |q_2| \geq (\theta - 1/2)|\partial\Pi|$ .

Therefore

$$\rho|q_2| < \rho|p_2| + \rho(1 - \theta)|\partial\Pi| < |p_2| + ((\rho - 1)(\theta - 1/2) + \rho(1 - \theta))|\partial\Pi| \quad (7)$$

Let now  $|q_2| > |B|$ . Denote  $q_2 = q'q''$ , where  $|q'| = |B|$ . Then  $|q''| < \xi|q'|$ . The fact that  $B$  is cyclically reduced implies that  $|q'| \leq 1/2|\partial\Pi| \leq |q''| + |u_2| + |v_2| + |p_2|$ . Therefore  $|q'| - |q''| < |u_2| + |v_2| + |p_2|$ . Consequently,

$$\frac{1 - \xi}{1 + \xi}|q_2| < (1 - \xi)|B| < |q'| - |q''| < |u_2| + |v_2| + |p_2| < |p_2| + (1 - \theta)|\partial\Pi|.$$

Using the fact that

$$|p_2| > \theta|\partial\Pi| - |q'| - |q''| > \left(\theta - \frac{1 + \xi}{2}\right)|\partial\Pi|,$$

we obtain

$$\begin{aligned} \rho|q_2| &< \frac{\rho(1 + \xi)}{1 - \xi}|p_2| + \frac{\rho(1 + \xi)}{1 - \xi}(1 - \theta)|\partial\Pi| < \\ &< |p_2| + \left(\left(\frac{\rho(1 + \xi)}{1 - \xi} - 1\right)\left(\theta - \frac{1 + \xi}{2}\right) + \frac{\rho(1 + \xi)}{1 - \xi}(1 - \theta)\right)|\partial\Pi|. \end{aligned} \quad (8)$$

Combining the inequalities (6) and (7) if  $|q_2| \leq |B|$ , and (6) and (8) if  $|q_2| > |B|$ , we complete the proof in the same way as in [14]. Lemma 6.1[14] is proved.

The formulations of Lemmas 6.2[14] - 6.5[14] do not change. Proving Lemma 6.3[14] in the case when  $\theta$ -cell  $\Pi$  is of rank 1/2, note that the bonds defining the contiguity subdiagrams  $\Gamma_p, \Gamma_q$  are 0-bonds by Lemma 3.4. Analogously, if  $r(\Pi) = 1/2$  in the proof of Lemma 6.5[14], then the bonds defining  $\Gamma_b, \Gamma_c$  are 0-bonds, while the contiguity subdiagrams  $\Gamma_p, \Gamma_q$  do not contain

cells by Lemmas 3.6 and 3.8. Thus, proofs of both of these lemmas in the case  $r(\Pi) = 1/2$  proceed in the same way as in [14]. We only need to replace  $\gamma$  by 0 in the estimates and refer to Lemmas 3.8 and 3.6 instead of Lemmas 3.4[14] and 3.3[14].

It is convenient to state the analogue of Lemma 6.2 from [14] as follows.

**Lemma 6.2[14].** *Let  $\Delta$  be a disk tame diagram of rank  $i$ . Then  $|\partial\Pi| \leq \rho^{-1}|\partial\Delta|$  for any cell  $\Pi \in \Delta$ . In particular, if  $|\partial\Delta| < \rho n|A_k|$  for some  $k \leq i$ , then  $r(\Delta) < k$ .*

All lemmas and estimates of Section 7 from [14] remain valid due to Lemmas 3.4 – 3.8. However, the presence of  $T$ -cells bring the need to consider several more diagrams of special sort in order to understand the bond structure of contiguity subdiagrams of rank  $i$ . We need to consider *degenerate special 8-gons of rank  $i$*  and *degenerate special 8'-gons of rank  $i$* . The definitions of these new types of diagrams follow.

A disk tame diagram  $\Delta$  of rank  $i$  is referred to as *degenerate special 8-gon of rank  $i$*  if it possesses the following properties:

- (DG1) The contour  $\partial\Delta$  of  $\Delta$  is considered to be decomposed into the product  $apbrcqds$ .
- (DG2) Each of the sections  $r$  and  $s$  is either smooth or geodesic.
- (DG3) The section  $p$  is smooth of some rank  $k$ ,  $1 \leq k \leq i$ ; the section  $q$  is smooth of rank  $1/2$ .
- (DG4)  $\max(|a|, |b|, |c|, |d|) < 2\beta^{-1}|A_k|$ .

(DG5) Let a vertex  $o_1$  be chosen on the section  $p$  and a vertex  $o_2$  be chosen on the section either (a)  $r$ , or (b)  $s$ . Denote  $p(\text{dec}, o_1) = p^1 p^2$  and (a)  $r(\text{dec}, o_2) = r^1 r^2$  or (b)  $s(\text{dec}, o_2) = s^1 s^2$ . Suppose that  $x = o_1 - o_2$  is a simple path joining the vertices  $o_1$  and  $o_2$  with  $|x| < 2\beta^{-1}|A_k|$ . Then the following is true:

- 1) the subdiagram  $\Delta_0$  with the contour (a)  $p^2 b r^1 x^{-1}$  or (b)  $p^1 x s^2 a$  contains no cells;
- 2) the inequality  $|x| \leq |b|$  implies  $x = b$  in case (a) or the inequality  $|x| \leq |a|$  implies  $x = a^{-1}$  in case (b).

(DG6) Let a vertex  $o_3$  be chosen on the section  $q$  and a vertex  $o_4$  be chosen on the section either (a)  $r$ , or (b)  $s$ . Denote  $q(\text{dec}, o_3) = q^1 q^2$  and (a)  $r(\text{dec}, o_4) = r^1 r^2$  or (b)  $s(\text{dec}, o_4) = s^1 s^2$ . Suppose that  $y = o_3 - o_4$  is a simple path joining the vertices  $o_3$  and  $o_4$  with  $|y| < 2\beta^{-1}|A_k|$ . Then the following is true:

- 1) the subdiagram  $\Delta_0$  with the contour (a)  $q^1 y r^2 c$  or (b)  $q^2 d s^1 y^{-1}$  contains no cells;
- 2) the inequality  $|y| \leq |c|$  implies  $y = c^{-1}$  in case (a) or the inequality  $|y| \leq |d|$  implies  $y = d$  in case (b).

(DG7) There are no bonds between sections  $r$  and  $s$  in  $\Delta$ .

A disk tame diagram  $\Delta$  of rank  $i$  is referred to as *degenerate special 8'-gon of rank i* if it possesses the following properties:

(DG1') The contour  $\partial\Delta$  of  $\Delta$  is considered to be decomposed into the product  $apbrcqds$ .

- (DG2') Each of the sections  $r$  and  $s$  is either smooth or geodesic.
- (DG3') The section  $p$  is smooth of some rank  $k$ ,  $1 \leq k \leq i$ ; the section  $q$  is smooth of rank  $1/2$ .
- (DG4')  $\max(|a|, |b|, |c|, |d|) < 2\beta^{-1}|A_k|$ .
- (DG5') There are no bonds between  $p$ ,  $r$  and between  $p$ ,  $s$  in  $\Delta$ .
- (DG6') There are no bonds between  $q$ ,  $r$  and between  $q$ ,  $s$  in  $\Delta$ .
- (DG7') There are no bonds between sections  $r$  and  $s$  in  $\Delta$ .

The study of degenerate special 8- and 8'-gons proceeds in the same way as the study of special 8- and 8'-gons in Sections 7, 8 in [14]. Lemma 7.1[14] remains valid if  $\Delta$  is a degenerate special 8-gon of rank  $i$ . Dealing with a degenerate special 8-gon of rank  $i$ , we consider the same cases as are considered in [14] for special 8-gon of rank  $i$ . Note that the section  $q$  is a  $\beta$ -section by Lemma 3.5. So the case (L4) is impossible. The analog of Lemma 7.2 from [14] for degenerate special 8-gon of rank  $i$  says that there are 16 cases to consider: (K1)&(L1), ..., (K4)&(L3), (K1)&(L5), ..., (K4)&(L5). Analog of Lemma 7.3 for degenerate special 8-gon of rank  $i$  says that the copy of  $\Delta$  with decomposition of its contour  $b^{-1}p^{-1}a^{-1}s^{-1}d^{-1}q^{-1}c^{-1}r^{-1}$  is also a degenerate special 8-gon of rank  $i$ . Thus, the cases (K1)&(L5), ..., (K4)&(L5) are symmetric to the cases (K1)&(L2), ..., (K4)&(L2). Therefore it suffices to consider 12 cases (K1)&(L1), ..., (K4)&(L3) only. Note that the same cases were considered in Lemmas 7.4 - 7.15 of [14] studying special 8-gons of rank  $i$ . Analogs of Lemmas 7.4 - 7.15 with some minor changes are valid for degenerate special 8-gon of rank  $i$ .

Thus, using arguments similar to those of Sections 7, 8 from [14] and Lemmas 3.4 - 3.8 one can prove the following new versions of Lemmas 8.7 and 8.8.

**Lemma 8.7[14].** *Let  $\Delta$  be a special 8-gon of rank  $i$ . Then*

$$|p| + |q| < 70\beta^{-1}M, \quad |r| + |s| < 102\beta^{-1}M, \quad |\partial\Delta| < 180\beta^{-1}M,$$

where  $M = \max(|A_k|, |A_l|)$ .

If  $\Delta$  is a degenerate special 8-gon of rank  $i$ , then the same estimates hold with  $M = |A_k|$ .

**Lemma 8.8[14].** *Let  $\Delta$  be a special 8'-gon of rank  $i$ . Then*

$$|apb| + |cqd| < 100\beta^{-1}M, \quad |r| + |s| < 130\beta^{-1}M, \quad |\partial\Delta| < 230\beta^{-1}M,$$

where  $M = \max(|A_k|, |A_l|)$ .

For a degenerate special 8'-gon  $\Delta$  of rank  $i$  the same estimates hold with  $M = |A_k|$ .

Here is the new version of Lemma 9.1.

**Lemma 9.1[14].** *Let  $\Delta$  be a tame diagram of rank  $i$  with the contour  $\partial\Delta = bpcq$ , where each of the sections  $p$  and  $q$  is either smooth or geodesic. Suppose also that  $\Delta$  itself is a contiguity diagram between  $p$  and  $q$ . Then there exists a system  $\mathcal{C}(\Delta)$  of bonds  $E_1, E_2, \dots, E_k$  between  $p$  and  $q$  in  $\Delta$  whose standard contours are  $\partial E_t = x_t p_t y_t q_t$ , where  $p_t = E_t \wedge p$ ,  $q_t = E_t \wedge q$ ,  $t = 1, 2, \dots, k$ , such that  $E_1, E_2, \dots, E_k$  pairwise have no cells in common and  $\mathcal{C}(\Delta)$  has the following properties:*

(a)  $x_1 = b$ ,  $y_k = c$  and  $p = p_1 r_1 \dots p_{k-1} r_{k-1} p_k$ ,  $q = q_k s_{k-1} q_{k-1} \dots s_1 q_1$  for some paths  $r_1, s_1, \dots, r_{k-1}, s_{k-1}$ .

(b) Denote the principal cell of the bond  $E_t$  by  $\pi_t$ ,  $t = 1, 2, \dots, k$ , (we set  $|\partial\pi_t| = 0$  if  $E_t$  is a 0-bond) and let the subdiagram  $\Delta_t$  be given by  $\partial\Delta_t = y_t^{-1} r_t x_{t+1}^{-1} s_t$ , where  $t = 1, 2, \dots, k-1$ . Then, in this notation, there are no bonds between the sections  $r_t$  and  $s_t$  in  $\Delta_t$  and the following inequalities hold for every  $t$ ,  $t = 1, 2, \dots, k-1$ :

$$|y_t| + |x_{t+1}| \leq 100\beta^{-1}n^{-1} \max(|\partial\pi_t|, |\partial\pi_{t+1}|),$$

$$|s_t| + |r_t| \leq 130\beta^{-1}n^{-1} \max(|\partial\pi_t|, |\partial\pi_{t+1}|),$$

$$|\partial\Delta_t| \leq 230\beta^{-1}n^{-1} \max(|\partial\pi_t|, |\partial\pi_{t+1}|),$$

provided both  $\pi_t, \pi_{t+1}$  are  $\mathcal{R}$ -cells, or

$$|y_t| + |x_{t+1}| \leq 100\beta^{-1}n^{-1}|\partial\pi|,$$

$$|s_t| + |r_t| \leq 130\beta^{-1}n^{-1}|\partial\pi|,$$

$$|\partial\Delta_t| \leq 230\beta^{-1}n^{-1}|\partial\pi|,$$

where  $\pi$  is the only  $\mathcal{R}$ -cell among  $\pi_t, \pi_{t+1}$ , or

$$|y_t| < \frac{\xi}{10} \min(|\partial\pi_t|, |\partial\pi_{t+1}|)$$

if there are no  $\mathcal{R}$ -cells among  $\pi_t, \pi_{t+1}$ .

In the latter case the subdiagram  $\Delta_t$  does not have cells,  $|s_t| = |r_t| = 0$

and  $y_t = x_{t+1}^{-1}$ .

(c) If  $r(\Delta) = j > 0$  and  $\Pi$  is a cell of rank  $j$  in  $\Delta$ , then  $\Pi$  is the principal cell  $\pi_t$  of  $E_t$  for some  $t$ ,  $t = 1, 2, \dots, k$ . In particular,

$$r(\Delta) = \max_{1 \leq t \leq k} (r(\pi_t)) .$$

*Proof.* The system of bonds  $\mathcal{C}(\Delta)$  is constructed in the same way as in [14].

Assume that for some  $t$ ,  $1 \leq t < k$  the bond  $E_{t+1}$  (or  $E_t$ ) consists of a  $\mathcal{T}$ -cell, and the bond  $E_t$  (resp.  $E_{t+1}$ ) is either a 0-bond or consists of a  $\mathcal{T}$ -cell. We will show that the subdiagram  $\Delta_t$  has no cells.

Assuming that  $\Delta_t$  contains at least one cell, by Lemma 5.7[14], there is a  $\theta$ -cell  $\Pi$  in  $\Delta_t$ . For the sum of the contiguity degrees of  $\Pi$  to the sections  $r_t, s_t$ , we have that  $(\Pi, \Gamma_r, r_t) + (\Pi, \Gamma_s, s_t) < \alpha + \beta$ , since each of the degrees does not exceed  $\alpha$  (following from the fact that  $p, q$  are smooth or geodesic) and if both the degrees of contiguity are at least  $\beta$  we would have a bond between  $r_t$  and  $s_t$ .

The sum of the degrees of contiguity of  $\Pi$  to the sections  $y_t$  and  $x_{t+1}$  does not exceed  $(\Pi, \Gamma_y, y_t) + (\Pi, \Gamma_x, x_{t+1}) < 2\beta$  since  $x_{t+1}$  is smooth of rank 1/2,  $y_t$  is smooth of rank 1/2 provided  $E_t$  consists of a  $\mathcal{T}$ -cell or  $|y_t| = 0$  provided  $E_t$  is a 0-bond (in the latter case the contiguity subdiagram  $\Gamma_y$  does not exist).

Thus

$$\theta < (\Pi, \Gamma_r, r_t) + (\Pi, \Gamma_s, s_t) + (\Pi, \Gamma_y, y_t) + (\Pi, \Gamma_x, x_{t+1}) < \alpha + 3\beta,$$

which is a contradiction. Thus,  $r(\Delta_t) = 0$ . The equalities  $|s_t| = |r_t| = 0$  and  $y_t = x_{t+1}^{-1}$  follow now from the fact that there are no bonds between sections

$s_t$  and  $r_t$ . In the case when both  $E_t$  and  $E_{t+1}$  are 1/2-bonds the inequality

$$|y_t| < \frac{\xi}{10} \min(|\partial\pi_t|, |\partial\pi_{t+1}|)$$

follows from Lemma 3.1(b) and the fact that  $\Delta$  is reduced.

In view of the above argument, if the subdiagram  $\Delta_t$  contains cells and if one of the bonds  $E_t, E_{t+1}$  consists of a  $\mathcal{T}$ -cell, then the other bond is a  $k$ -bond for some  $k \geq 1$ . Let  $E_t$  be a 1/2-bond and  $E_{t+1}$  be a  $k$ -bond,  $k \geq 1$  (the other case is similar). In the notations from [14],  $|b_t| = |c_t| = 0$  since  $r(\Gamma_p^t) = r(\Gamma_q^t) = 0$  by Lemmas 3.6, 3.8. The diagram  $\Delta_t$  with the decomposition of its contour

$$\partial\Delta_t = a_{t+1}^{-1}(v_1^{t+1})^{-1}d_{t+1}^{-1}s_tv_2^tr_t$$

has properties (DG1')-(DG7') of a degenerate special 8'-gon of rank  $i$  (if we set  $a = a_{t+1}^{-1}$ ,  $p = (v_1^{t+1})^{-1}$ ,  $b = d_{t+1}^{-1}$ ,  $r = s_t$ ,  $|c| = 0$ ,  $q = v_2^t$ ,  $|d| = 0$ ,  $s = r_t$ ). The remaining part of the proof proceeds along the lines of the proof of Lemma 9.1 given in [14] using the estimates from Lemmas 8.7[14] and 8.8[14].  $\square$

The formulation of Lemma 9.3[14] is preserved. Proving part (a) in the case when  $E_t$  is a 1/2-bond, the estimate  $|q_t| < (1 + \varepsilon)|B|$  is immediate from Lemmas 3.8 and 3.2 as  $\xi < \varepsilon$ . If in addition  $E_{t+1}$  is a 1/2-bond, then, by Lemmas 9.1[14](b) and 3.2,  $|r_t| = |s_t| = 0$  and

$$\begin{aligned} |\partial\Delta_t| &= |y_t| + |x_{t+1}| < \frac{\xi}{10} \min(|\partial\pi_t|, |\partial\pi_{t+1}|) < \\ &< \frac{\xi}{10}(1 + \xi)\beta^{-1}|B| < 0.001|B|. \end{aligned}$$

The remaining part of the proof of Lemma 9.3[14] goes in the same way as in [14].

The formulation of Lemma 9.4[14] is preserved. The proof proceeds as in [14] with some minor changes. The formulation of Lemma 9.5[14] is preserved. In the case when  $\Pi$  is an  $\mathcal{R}$ -cell there are no changes in the proof of 9.5[14](a). Assume that  $\Pi$  is a  $\mathcal{T}$ -cell.

First, let  $\Pi$  belong to a subdiagram  $\Delta_s$ ,  $m_1 \leq s \leq m_2 - 1$ , of  $\Delta(m_1, m_2)$ . By Lemmas 6.2[14], 9.1[14](b) and Lemma 9.5[14](a) for an  $\mathcal{R}$ -cell in  $\Delta(m_1, m_2)$ ,

$$|\partial\Pi| < 230\rho^{-1}\beta^{-1}n^{-1} \times 2.22|B| < 2.22|B| .$$

Let now  $\Pi$  be a principal cell of a bond  $E_t$ ,  $m_1 \leq t \leq m_2$ . The bond  $E_t$  coincides with the cell  $\Pi$ , and therefore  $\partial\Pi = x_tp_ty_tq_t$ . Applying Lemma 3.6 (if  $p$  is geodesic) or Lemma 3.8 (if  $p$  is smooth), one gets  $|p_t| < \alpha|\partial\Pi|$ . Therefore, in view of Lemma 9.3[14](a),

$$|\partial\Pi| < (1 - \alpha)^{-1}(|q_t| + |x_t| + |y_t|) < (1 - \alpha)^{-1}(1 + \varepsilon + 0.002)|B| < 2.22|B|,$$

as required.

Lemmas 10.1[14] - 10.9[14] are formulated and proved in the same way as the corresponding lemmas in [14].

The cells  $\Pi_1$ ,  $\Pi_2$  from the definitions of  $R$ - and  $S$ -diagrams are now required to be  $\mathcal{R}$ -cells. We define  $T$ -diagrams of rank  $i$  for  $i \geq 1$  only. The formulation of Lemma 12.1[14] is preserved. The proof is decomposed into cases in the same way as in [14]. Considering the case 1.1.1 in the proof of Lemma 12.1[14], the diagram  $\Delta_2$  is a degenerate special 8'-gon provided the bond  $E$  between  $r_1$

and  $r_2$  is a 1/2-bond. Therefore the required estimate can be obtained from Lemma 8.8[14].

The investigation of Cases 2.3 - 2.14 of the proof of Lemma 12.1[14] differs from the one in [14] if the  $\theta$ -cell  $\pi$  is a  $\mathcal{T}$ -cell.

**2.3.** By Lemma 3.4, the bonds defining  $\Gamma_a$  do not contain cells. Denote  $\partial\pi = uc_{12}vc_{22}$ , where  $c_1 = c_{11}c_{12}c_{13}$ ,  $c_2 = c_{21}c_{22}c_{23}$  and the 0-bonds  $E_1$  and  $E_2$  (between  $\pi$  and the sections  $c_1$  and  $c_2$  respectively), given by  $\partial E_1 = c_{12}c_{12}^{-1}$  and  $\partial E_2 = c_{22}c_{22}^{-1}$ , are chosen such that the sum  $|c_{13}| + |c_{21}|$  is minimal.

If there are no bonds between the sections  $v$  and  $q$ , then, by Lemma 6.5[14],  $|v| + |q| \leq \mu(|c_{13}| + |c_{21}|)$ . Therefore

$$|q| \leq \mu(|c_{13}| + |c_{21}|) < 4\mu|A_j| < 0.04n|A_j|.$$

Now assume that there is a bond between  $v$  and  $q$ . Consider the contiguity subdiagram  $\Gamma$  between sections  $v$  and  $q$ , maximal with respect to the sum  $|v_2| + |q_2|$ , where  $q = q_1q_2q_3$ ,  $v = v_1v_2v_3$ . By Lemma 3.8(1),  $r(\Gamma) = 0$  and therefore  $q_2 = v_2$ . By the choice of  $\Gamma$ , we can apply Lemma 6.5[14] to the diagrams  $\Delta_1$  and  $\Delta_2$  given by  $\partial\Delta_1 = c_{13}q_1v_1^{-1}$  and  $\partial\Delta_2 = c_{21}v_3^{-1}q_3$ . Hence  $|q_1| + |q_3| \leq \mu(|c_{13}| + |c_{21}|) < 4\mu|A_j|$ . In view of Lemma 3.1(a),  $|q_2| < 11|A_j|$ . Thus,

$$|q| = |q_1| + |q_2| + |q_3| < (11 + 4\mu)|A_j| < 0.04n|A_j| ,$$

as required.

Case 2.3 is complete.

**2.4.** By Lemmas 3.4 and 3.8(1), the bonds between  $\partial\pi$  and each of the sections  $r, q$  are 0-bonds, and the contiguity subdiagrams of  $\pi$  to the sections

$r$  and  $q$  are of rank 0. Denote  $\partial\pi = r_{12}uq_{21}v$ ,  $r = r_{11}r_{12}r_2$ ,  $q = q_1q_{21}q_{22}$ , where  $r_{12}$  (respectively  $q_{21}$ ) is the maximal common subpath of  $\partial\pi$  and  $r$  (respectively  $q$ ).

Assume that  $|q_{21}| > \beta|\partial\pi|$ . Then, by Lemma 3.2,  $|\partial\pi| < (1 + \xi)\beta^{-1}|A_j|$ . Thus,  $|u| < |\partial\pi| < (1 + \xi)\beta^{-1}|A_j| < 2\beta^{-1}|A_j|$ , giving a contradiction to the condition (T6).

Let now  $|q_{21}| \leq \beta|\partial\pi|$ . Then, by Lemma 3.8(2),

$$|v| > (\theta - (\alpha + 2\beta))|\partial\pi| > \frac{1}{3}|\partial\pi|.$$

Recall also that  $u < (1 - (\theta - \beta))|\partial\pi|$ . Consider the diagram  $\Delta_1$  given by  $\partial\Delta = c_2r_{11}v^{-1}q_{22}$ . Assuming that  $\Delta_1$  has cells, by Lemma 5.7[14], there is a  $\theta$ -cell  $\Pi$  in  $\Delta_1$ . The contiguity degree of  $\Pi$  to  $v$  is less than  $\beta$  by Lemma 3.4 applied to  $\Delta$ . One of the contiguity degrees of  $\Pi$  to  $r_{11}$  and  $q_{22}$  is less than  $\beta$  by Lemma 12.1.1[14]. In view of (T6), the arguments from the proof of Lemma 7.1[14](c) can be used to show that the sum of contiguity degrees of  $\Pi$  to  $r_{11}$  and  $c_2$ , as well as the sum of contiguity degrees of  $\Pi$  to  $q_{22}$  and  $c_2$ , is less than 0.8. Therefore the sum of the contiguity degrees of  $\Pi$  to all four sections of  $\partial\Delta_1$  is less than  $0.8 + 2\beta$ , contrary to the definition of a  $\theta$ -cell, as  $\theta > 0.8 + 2\beta$ . Therefore  $\Delta_1$  does not have cells, and  $|v| \leq |c_2|$  by the choice of  $r_{12}$  and  $q_{21}$ . It follows that  $|v| \leq 2\beta^{-1}|A_j|$ . Thus,

$$\frac{1}{3}|\partial\pi| < |v| \leq 2\beta^{-1}|A_j|.$$

Consequently,  $|\partial\pi| < 6\beta^{-1}|A_j|$  and

$$|u| < (1 - (\theta - \beta))|\partial\pi| < 6\beta^{-1}(1 - (\theta - \beta))|A_j| < 2\beta^{-1}|A_j| ,$$

contrary to (T6).

Case 2.4 is complete.

**2.5.** The bonds defining  $\Gamma_a$  are 0-bonds. Denote  $\partial\pi = c_{22}uc_{12}v$ , where  $c_{22}uc_{12} = \Gamma_a \wedge \partial\pi$ ,  $c_1 = c_{11}c_{12}c_{13}$ ,  $c_2 = c_{21}c_{22}c_{23}$ . By (T6) the paths  $c_{11}c_{12}$  and  $c_{22}c_{23}$  are geodesic in the disk subdiagram of  $\Delta$  consisting of  $\pi$  and  $\Gamma_a$ . Therefore, in view of Lemma 3.6(1), we may assume that  $c_{12}$  (resp.  $c_{22}$ ) is the maximal common subpath of  $\partial\pi$  and  $c_1$  (resp.  $c_2$ ). Consider the disk subdiagram  $\Delta_1$  of  $\Delta$  given by  $\partial\Delta_1 = c_{23}rc_{11}u$ .

First assume that there is a bond between  $u$  and  $r$  in  $\Delta_1$ . Let  $\Gamma$  be a contiguity subdiagram between  $u$  and  $r$  maximal with respect to the sum  $|u_2| + |r_2|$ , where  $u = u_1u_2u_3$ ,  $r = r_1r_2r_3$ . The diagram  $\Gamma$  does not have cells. It follows from Lemmas 3.5, 3.6, 3.3[14] and 5.7[14] provided  $\phi(r)$  is a reduced in rank  $i$  word, or from Lemmas 3.5, 3.8, 3.4[14] and 5.7[14] provided  $\phi(r)$  is  $A$ -periodic with a simple in rank  $i$  word  $A$ . Let  $\Delta_2$  be the diagram given by  $\partial\Delta_2 = r_3c_{11}u_3^{-1}$ . In view of the property (T6) of  $\Delta$  one can estimate the sum of contiguity degrees of a cell from  $\Delta_2$  to the sections  $r_3$  and  $c_{11}$ . Arguing as in the proof of Lemma 7.1[14](c), and using Lemma 5.7[14] and the fact that  $u_3$  is a  $\beta$ -section of  $\partial\Delta_2$ , one can show that  $\Delta_2$  does not have cells. Similarly, one gets that  $\Delta_3$  given by  $\partial\Delta_3 = u_1^{-1}c_{23}r_1$  does not have cells. Therefore  $\partial\pi = c_{22}c_{23}rc_{11}c_{12}v$ . Lemmas 3.6 and 3.8 imply that  $|r| < \alpha|\partial\pi|$ . Therefore

$|c_{22}| + |c_{23}| + |c_{11}| + |c_{12}| > (\theta - \beta - \alpha)|\partial\pi|$ . Thus,

$$|v| < (1 - (\theta - \beta))|\partial\pi| < (1 - (\theta - \beta))(\theta - \beta - \alpha)^{-1} \times 4\beta^{-1}|A_j| < 4\beta^{-1}|A_j|.$$

If there are no bonds between  $u$  and  $r$  in  $\Delta_1$ , then, by Lemma 6.5[14],

$$|u| < \mu(|c_{11}| + |c_{23}|) < 6\beta^{-1}|A_j| .$$

Consequently,

$$\begin{aligned} |v| &< (\theta - \beta)^{-1}(1 - (\theta - \beta))(|c_{22}| + |u| + |c_{12}|) < \\ &< (\theta - \beta)^{-1}(1 - (\theta - \beta)) \times 10\beta^{-1}|A_j| < 4\beta^{-1}|A_j|. \end{aligned}$$

Thus, regardless of whether there is a bond between  $u$  and  $r$  or there are no such bonds, the length of the path  $f = c_{21}vc_{11}$  in  $\Delta$  is estimated as follows:

$$|f| = |c_{21}| + |v| + |c_{11}| < 8\beta^{-1}|A_j|.$$

Repeating the arguments from [14] following the inequality (12.12), we obtain the estimate  $|q| < 0.02n|A_j|$ .

Case 2.5 is complete. The Case 2.6 is analogous to the Case 2.4, so we pass to the Case 2.7.

**2.7.** Consider the following subcases:

(2.7.1.) There is a bond between  $\pi, q$  and there is a bond between  $\pi, c_2$  in  $\Delta$ .

(2.7.2.) There are no bonds between  $\pi, q$  and there is a bond between  $\pi,$

$c_2$  in  $\Delta$ .

(2.7.3.) There is a bond between  $\pi$ ,  $q$  and there are no bonds between  $\pi$ ,  $c_2$  in  $\Delta$ .

(2.7.4.) There are no bonds between  $\pi$ ,  $q$  and between  $\pi$ ,  $c_2$  in  $\Delta$ .

**2.7.1.** The property (T6) of  $\Delta$  makes it possible to use arguments from the proof of Lemma 7.1[14](c) to show that  $\Gamma_a$  does not have cells in this case. Therefore  $|q| < 11|A_j|$  by Lemma 3.1(a).

**2.7.2.** Again referring to the proof of Lemma 7.1[14](c), we denote  $\partial\pi = uc_{12}vc_{22}r_1$ , where  $c_1 = c_{11}c_{12}c_{13}$ ,  $c_2 = c_{21}c_{22}$ ,  $r = r_1r_2$ . Applying Lemma 6.5[14] to the disk subdiagram  $\Delta_1$  given by  $\partial\Delta_1 = c_{13}qc_{21}v^{-1}$ , one gets that

$$|q| < \mu(|c_{13}| + |c_{21}|) < 6\beta^{-1}|A_j| .$$

**2.7.3.** Making use of the arguments from the proof of Lemma 7.1[14](c), we denote  $\partial\pi = uc_{12}q_1vr_2$ , where  $c_1 = c_{11}c_{12}$ ,  $q = q_1q_2$ ,  $r = r_1r_2r_3$  and there are no bonds between  $v$ ,  $q_2$ , between  $v$ ,  $r_1$  and between  $v$ ,  $c_2$ . By Lemma 3.1(a),  $|q_1| < 11|A_j|$ . By Lemma 12.1.1[14], there are no bonds between  $r_1$ ,  $q_2$ . Consider the disk subdiagram  $\Delta_1$  of  $\Delta$  given by  $\partial\Delta_1 = r_1v^{-1}q_2c_2$ . Assume that  $\Delta_1$  has cells. By Lemma 5.7[14], there is a  $\theta$ -cell  $\Pi$  in  $\Delta_1$ . The degree of contiguity of  $\Pi$  to one of the sections  $r_1$ ,  $q_2$  is less than  $\beta$  since there are no bonds between those sections. Let this section be  $r_1$  (the case when it is  $q_2$  is analogous). Arguing as in Lemma 7.1[14](c), we get that the sum of contiguity degrees of  $\Pi$  to the sections  $c_2$  and  $q_2$  is less than 0.8. The section  $v^{-1}$  is a  $\beta$ -section of  $\partial\Delta_1$ . So, the sum of contiguity degrees of  $\Pi$  to all four sections of  $\partial\Delta_1$  is less than  $0.8 + 2\beta$ . Thus, the inequality  $\theta > 0.8 + 2\beta$  means that  $\Delta_1$  does not have cells. It follows from the condition (T2) and the fact that there

are no bonds between  $r_1$ ,  $q_2$  and between  $v$ ,  $q_2$  that  $|q_2| < 2|A_j| + 2\beta^{-1}|A_j|$ .

Finally,  $|q| = |q_1| + |q_2| < (13 + 2\beta^{-1})|A_j|$ .

**2.7.4.** Denote  $\partial\pi = uc_{12}vr_2$ , where  $r = r_1r_2r_3$ ,  $c_1 = c_{11}c_{12}c_{13}$  and there are no bonds between  $v$ ,  $r_1$  and between  $v$ ,  $c_{13}$ . The disk subdiagram  $\Delta_1$  of  $\Delta$  given by  $\partial\Delta_1 = c_{13}qc_2r_1v^{-1}$  has properties (DG1')-(DG7') of a degenerate special 8'-gon of rank  $i$  (if we set  $a = c_{13}$ ,  $p = q$ ,  $b = c_2$ ,  $r = r_1$ ,  $|c| = 0$ ,  $q = v^{-1}$ ,  $|d| = |s| = 0$ ). Referring to Lemma 8.8[14] we get that  $|q| < 100\beta^{-1}|A_j|$ .

Case 2.7. is complete.

The remaining Cases 2.8 - 2.14 of the proof of Lemma 12.1[14] with the  $\theta$ -cell  $\pi$  of rank  $1/2$  can be considered in a similar manner.

In the definition of a  $U$ -diagram  $\Delta$  we require in addition  $r(\Delta) = j \geq 1$ . In the formulation of Lemma 13.1[14] we add an assumption  $j = r(\Delta) > 1/2$ . Similarly, in Lemmas 13.2[14], 13.4[14] we require that  $j = r(\Delta(1)) > 1/2$  and in Lemma 13.3[14] -  $j = r(\Delta) > 1/2$ .

The formulation and the proof of Lemma 14.1[14] do not change. Proving Lemma 14.2[14] we need to consider the case  $r(\Pi) = 1/2$ . If  $p_\Gamma$  is a subpath of  $q$ , then the needed inequalities are immediate from Lemma 3.2. Since the bonds defining  $\Gamma$  are 0-bonds, it is possible to obtain the same inequalities (14.1)-(14.3) modulo substitution 0 for  $\gamma$ . Arguing in the same way as in [14], instead of (14.4) we get  $|u| > \beta|\partial\Pi|$ .

If there are no bonds between sections  $y$  and  $u$  in  $\Delta_5$ , then, by Lemma 6.5[14],

$$|y| + |u| \leq \mu(|d_1| + |d_3|) < \mu(1 + \delta)|A_j|,$$

as  $|d_1| = 0$ ,  $|d_3| < (1 + \delta)|A_j|$ . Consequently,

$$|\partial\Pi| < \beta^{-1}|u| < \beta^{-1}\mu(1 + \delta)|A_j| < 4\beta^{-1}|A_j|$$

and

$$|p_\Gamma| = |y| + |q_{12}| + |t| < (1 + \mu)(1 + \delta)|A_j| < 5|A_j|.$$

If there are bonds between  $y$  and  $u$  in  $\Delta_5$ , then all these bonds are 0-bonds. Since a diagram with contour  $y'u'$  with  $y'$  being smooth and  $u'$  being smooth of rank  $1/2$  does not have cells, there are decompositions  $y = y_1y_2$ ,  $u = u_2u_1$ , where  $y_1 = u_1^{-1}$  is the longest common subpath of  $y$  and  $u^{-1}$ . By Lemma 6.5[14],  $|y_2| + |u_2| \leq \mu|d_3| < \mu(1 + \delta)|A_j|$ . The fact that  $y_1 = u_1^{-1}$  is a common subpath of a  $\mathcal{T}$ -section and an  $A_j$ -periodic section means that either  $|y_1| = |u_1| < (1 + \xi)|A_j|$ , or  $(1 + \xi)|A_j| < |y_1| = |u_1| < 11|A_j|$  and  $|A_j| < \xi^{-3/2}$ .

The latter case is impossible since otherwise

$$|\partial\Pi| < \beta^{-1}|u| = \beta^{-1}(|u_1| + |u_2|) < \beta^{-1}(\mu(1 + \delta) + 11)\xi^{-3/2} < n^2,$$

contrary to Lemma 3.1(3). Therefore

$$|y|, |u| < (1 + \xi + \mu(1 + \delta))|A_j|$$

and the inequalities sought follow as before.

The formulations and proofs of Lemmas 14.3[14] - 14.6[14] do not change. Proving Lemma 14.7[14], notice that the inequality (14.19) holds also in the case when  $\Pi$  is a  $1/2$ -cell. Indeed, in this case when  $\Gamma \wedge p_1$  is a subpath of  $q_1$ ,  $r(\Gamma) = 0$  by Lemma 3.8, and, in view of the inequality  $|p^\Gamma| = |q_\Gamma| > \beta|\partial\Pi|$ ,

Lemma 3.2 implies the estimate  $|p^\Gamma| < (1 + \xi)|A_{j_1}|$ . If  $\Gamma \wedge p_1$  is a subpath of  $t_1$ , then using Lemmas 14.1[14] and 3.6 we again obtain that  $r(\Gamma) = 0$ . Therefore  $|p_\Gamma| \leq |t_1| < \delta|A_{j_1}|$ . It remains to note that  $1 + \xi < \delta$  and the rest of the proof proceeds as in [14]. The formulations and proofs of Lemmas 14.8[14] and 14.9[14] do not change.

Proving Lemma 14.10[14] we need to consider the case when  $\Pi_1$  is a cell of rank  $1/2$ . In the notations of [14], if  $p^\Gamma$  is a subpath of either  $q_1$  or  $t_1$ , then  $r(\Gamma) = 0$  by Lemmas 3.8, 14.1[14] and 3.6. Therefore, by Lemma 3.1(a), provided  $p^\Gamma$  is a subpath of  $q_1$ , or, by Lemma 18.5[14](c), provided  $p^\Gamma$  is a subpath of  $t_1$ ,  $|p^\Gamma| < 11|A_{j_1}|$ . The obtained estimate implies the inequality (14.25) from [14].

Now suppose that  $p^\Gamma$  is a subpath of  $p_1$  but not a subpath of either  $q_1$  or  $t_1$ . By Lemma 3.4, the bonds defining  $\Gamma$  are 0-bonds. By Lemma 6.1[14] applied to  $\Gamma$ ,  $\rho|q_\Gamma| \leq |p^\Gamma|$ . As in the proof of Lemma 14.2[14] one can obtain a reduced diagram  $\Delta_1$  with  $\partial\Delta_1 = q_\Gamma y z$ , where  $y$  is a strictly smooth  $A_{j_1}$ -periodic section,  $|y| > |p^\Gamma| - (1 + \delta)|A_{j_1}|$ , and  $|z| < (1 + \delta)|A_{j_1}|$ . Arguing as in the proof of Lemma 14.2[14],  $|y| + |q_\Gamma| \leq \mu|z| < \mu(1 + \delta)|A_{j_1}|$  by Lemma 6.5[14] provided there are no bonds between sections  $y$  and  $q_\Gamma$  in  $\Delta_1$ . It follows that  $|p^\Gamma| < |y| + (1 + \delta)|A_{j_1}| < (\mu + 1)(1 + \delta)|A_{j_1}|$ . If there are bonds between  $y$  and  $q_\Gamma$  in  $\Delta_1$ , then, as in the proof of Lemma 14.2[14], we obtain the estimate  $|y| < (11 + \mu(1 + \delta))|A_{j_1}|$ . Therefore  $|p^\Gamma| < (11 + (\mu + 1)(1 + \delta))|A_{j_1}|$ . In both cases  $|p^\Gamma| < \beta^{-1}|A_{j_1}|$ , which forces the inequality (14.25).

If now  $p^\Gamma$  is a subpath of a contour of a cell  $\Pi_2$  in  $\Delta$ , then  $r(\Gamma) = 0$  by Lemma 3.4. Therefore  $|p^\Gamma| = |q_\Gamma|$  and the inequality (14.28) can be obtained as in [14].

There are no changes in the formulation and in the proof of Lemma 14.11[14].

Note that in the proof the weight of a cell  $\pi$  of rank  $1/2$  is given by the formula  $\nu(\pi) = |\partial\pi|^{2/3}$ . Lemmas 14.12[14] - 16.6[14] are formulated and proved without changes.

In the formulations of Lemmas 17.1[14], 17.2[14] we add the condition  $r(\Delta) > 1/2$ . The conclusion of Lemma 17.2[14] reads:

*Then  $\Delta_0$  contains either a  $16\beta^{-1}n^{-1}$ -contractile cell  $\Pi$  of rank  $r(\Pi) > 1/2$  or a cell  $\pi$  of rank  $j \leq i$ ,  $j > 1/2$  such that if  $o \in \partial\pi$  is a phase vertex and  $t = o - o$  is a simple path in  $\Delta_0$  homotopic as a cyclic path in  $\Delta_0$  to a contour of  $\Delta_0$ , then  $\phi(t) \stackrel{i}{=} T$ , where  $T$  is an  $\mathcal{F}(A_j)$ -involution.*

Proof of Lemma 17.2[14] proceeds in the same way as in [14]. Remark only that in the sequence of diagrams  $\Delta_0 = \Delta_0^{(1)}, \Delta_0^{(2)}, \dots$  the diagram  $\Delta_0^{(k)}$  (for every  $k > 1$ ) is obtained from  $\Delta_0^{(k-1)}$  by removal of a reducible  $j$ -pair (for some  $j$ ,  $1 \leq j \leq i$ ) and all reducible  $1/2$ -pairs.

The formulation of Lemma 17.3[14] and the scheme of proof of this lemma in the case  $r(\Delta) > 1/2$  coincide with those in [14]. The details of the proof that require additional consideration caused by presence of  $1/2$ -cells will be considered below. The equality  $r(\Delta) = 1/2$  means that  $\Delta$  consists of  $1/2$ -cells only. Therefore  $\Delta$  can be considered as a diagram over presentation considered in Section 4.2 [33] (see also Chapter 2) with additional assumption that the subgroup  $N$  of  $H$  is the whole  $H$ . It means that Lemma 4.12 from [33] (up to change of notations) may be applied to  $\Delta$  (indeed, the estimates on the lengths of the sections of  $\partial\Delta$  used in the formulation of Lemma 4.12 in [33] follow from the analogous estimates in the formulation of Lemma 17.3[14] and the fact that  $\Delta$  itself is a contiguity subdiagram between the sections  $p$  and  $q$ ; simplicity of the word  $A$  mentioned in the formulation of Lemma 17.3[14] implies that in context of [33] the word  $A$  is simple in some positive rank and

can be considered as a period of some rank; in the arguments from [33] (and [31]) that use the fact that the exponent is odd we refer to Lemma 3.3 instead). Thus, the formulation of Lemma 17.3[14] admits the following addition:

*If  $r(\Delta) = 1/2$ , then there are two phase vertices  $o_p \in p$ ,  $o_q \in q$  that can be joined in  $\Delta$  by a path whose label is  $T \stackrel{1/2}{=} 1$ , and the subgroup*

$$\langle A^{-l}TA^l \mid l = 0, 1, 2, 3 \rangle$$

of  $G(i)$  is trivial.

Proving Lemma 17.3[14] in the case when  $r(\Delta) = j > 1/2$ , as in [14], by Lemmas 9.4[14] and 9.5[14], we find a rigid subdiagram  $\Delta(m_1, m_2)$  in  $\Delta$ . Note that there might be  $\mathcal{T}$ -cells in  $\Delta(m_1, m_2)$  with boundaries longer than  $n|A_j|$ , however, by Lemma 9.5[14](a),  $|\partial\Pi| < 2.22|A|$  for any cell  $\Pi \in \Delta(m_1, m_2)$ . So, instead of the inequality (17.30) we use two inequalities:

$$|\partial\Pi| \leq n|A_j| < 2.22|A|, \quad \text{if } r(\Pi) > 1/2,$$

$$|\partial\Pi| < 2.22|A|, \quad \text{if } r(\Pi) = 1/2.$$

Proving lemma 17.3.1[14], notice that the cell  $\Pi$  is an  $\mathcal{R}$ -cell by Lemma 17.2[14]. In the case 4 considered in the proof of Lemma 17.3.1[14] the subdiagram  $\Gamma_q^t$  contains  $\Pi$  and therefore  $j_q^t = r(\Gamma_q^t) > 1/2$ . The inequality (17.39) is true for the length of contours of all subdiagrams  $\Delta_s(\Gamma_q^t)$ , and those of the subdiagrams  $E_s(\Gamma_q^t)$  that have an  $\mathcal{R}$ -cell as its principal cell.

In the formulation of Lemma 17.3.2[14] we remark that  $r(\pi_{t'}) = j > 1/2$ , and similarly,  $r(\pi_{m_3}) = j > 1/2$  in the definition of the diagram  $\Delta(m_3, m_4)$  and in the formulation of Lemma 17.3.3[14]. The proofs of these lemmas are

preserved.

The formulation of Lemma 17.3.4[14] does not change. Proving part (a) of this lemma, we need to consider the situation when the initial vertex  $(p_{m_3})_-$  of the path  $p_{m_3}$  belongs to a subpath  $\tilde{p}_t = \tilde{E}_t \wedge \tilde{p}(m_3, m_4)$  for some  $t$ ,  $m_3 \leq t \leq m_4$ , with the bond  $\tilde{E}_t$  consisting of a 1/2-cell  $\tilde{\pi}_t$ . As in [14], denote  $\tilde{p}_{t2}$  to be the subpath of  $\tilde{p}_t$  that connects vertex  $(p_{m_3})_-$  to the terminal vertex  $(\tilde{p})_+$  of the path  $\tilde{p}_t$ .

Assume that  $|\tilde{p}_{t2}| < \max(\beta|\partial\tilde{\pi}_t|, \beta n|A_j|)$ . Then, using Lemma 9.1[14](b) and the inequalities (17.30), the length of the path  $f_1 = \tilde{p}_{t2}\tilde{y}_t$  is estimated as follows

$$|f_1| < \beta \max(|\partial\tilde{\pi}_t|, n|A_j|) + \max\left(\frac{\xi}{10}|\partial\tilde{\pi}_t|, 100\beta^{-1}|A_j|\right) < 2.5\beta|A| .$$

Now assume that  $|\tilde{p}_{t2}| > \max(\beta|\partial\tilde{\pi}_t|, \beta n|A_j|)$  and, as in [14], consider two cases:

1.  $(\tilde{p})_+ \in p_{m_3}$ .
2.  $(\tilde{p})_+ \notin p_{m_3}$ .

**1.** Consider the diagram  $\Gamma_1$  given by

$$\partial\Gamma_1 = a_{m_3}\tilde{p}_{t2}h_2v.$$

Here  $|a_{m_3}| < 2\beta^{-1}|A_j|$ ,  $|h_2| < \beta^{-1}|A_j|$  by Lemmas 3.1[14] and 9.3[14](b), and  $v$  is an arc of  $\partial\pi_{m_3}$ . By Lemma 6.1[14] applied to  $\Gamma_1$

$$|v| \geq \rho|\tilde{p}_{t2}| - |a_{m_3}| - |h_2| > 0.94\beta n|A_j| .$$

In view of  $|\tilde{p}_{t2}| > \beta |\partial\tilde{\pi}_t|$  and Lemma 3.1(b) we can consider  $\tilde{p}_{t2}$  as a  $\mathcal{T}$ -section of  $\partial\Gamma_1$  and remove (if necessary) 1/2-cells in  $\Gamma_1$  compatible with  $\tilde{p}_{t2}$ . Thus we may assume that  $\tilde{p}_{t2}$  is a smooth section of rank 1/2 of  $\partial\Gamma_1$ .

Let  $\Gamma_2$  be a contiguity subdiagram between  $v$  and  $\tilde{p}_{t2}$  in  $\Gamma_1$  maximal with respect to the sum  $|v^2| + |\tilde{p}_{t2}^2|$ , where  $v^2 = \Gamma_2 \wedge v$ ,  $\tilde{p}_{t2}^2 = \Gamma_2 \wedge \tilde{p}_{t2}$ ,  $v = v^1 v^2 v^3$ ,  $\tilde{p}_{t2} = \tilde{p}_{t2}^1 \tilde{p}_{t2}^2 \tilde{p}_{t2}^3$ . By Lemma 3.5 (applied to the diagram consisting of  $\Gamma_1$  and  $\pi_{m_3}$ )  $r(\Gamma_2) = 0$ . It follows now from Lemma 6.5[14] and the choice of  $\Gamma_2$  that

$$|v^1| + |v^3| \leq \mu(|a_{m_3}| + |h_2|) < 5\beta^{-1}|A_j|.$$

Then the length of  $A_j^{\pm 1}$ -periodic subword  $\phi(\tilde{p}_{t2}^2)$  of  $\phi(\tilde{p}_{t2})$  can be estimated as follows:

$$|\phi(\tilde{p}_{t2}^2)| = |\phi(v^2)| > (0.94\beta n - 5\beta^{-1})|A_j| > 0.9\beta n|A_j|.$$

But this is impossible by Lemma 3.1(a).

**2.** Consider the subdiagram  $\Gamma$  of  $\Delta(l)$  consisting of  $\pi_{m_3}$ ,  $\tilde{\pi}_t$  and  $\Gamma_p^{m_3}$ . Assume that  $\Gamma$  is reduced. Then, following from rigidity of  $\Delta(m_3, m_4)$  and Lemma 3.4[14], the contiguity degree of  $\pi_{m_3}$  to  $\tilde{\pi}_t$  is greater than  $\chi - \alpha > \beta$ . But this contradicts Lemma 3.4. If  $\Gamma$  is not reduced, then the cell  $\tilde{\pi}_t$  with some cell from  $\Gamma_p^{m_3}$  form a 1/2-pair. Removing this 1/2-pair and taking out the cell  $\pi_{m_3}$  from  $\Gamma$ , we obtain a reduced diagram  $\Gamma_1$  with  $\partial\Gamma_1 = va_{m_3}p'b_{m_3}$ . Removing (if necessary) 1/2-cells in  $\Gamma_1$  compatible with the section  $p'$ , we may assume that  $p'$  is a smooth section of rank 1/2 of  $\partial\Gamma_1$ . The estimate  $|v| > 0.94\beta n|A_j|$  follows now from rigidity of  $\Delta(m_3, m_4)$  and Lemma 3.4[14]. Therefore one

can obtain a contradiction to Lemma 3.1(a) in the same way as it was done in case 1.

To prove part (b) of Lemma 17.3.4[14] one has to consider the case when  $(\tilde{y}_{m_4})_- \in p_t = E_t \wedge p(m_3, m_4)$  with the bond  $E_t$  consisting of a  $\mathcal{T}$ -cell  $\Pi$ . By Lemma 9.3[14](a),  $|p_t| < (1 + \xi)|A|$ . The remaining part of the proof of (b) as well as the proof of (c) proceed as in [14]. Lemma 17.3.4[14] is proved.

The formulation of Lemma 17.3.5[14] remains the same. The estimate in part (a) is valid if the bond  $E_{k^0+1}^0$  consists of a 1/2-cell:

Lemma 6.5[14] applied to  $\Delta_{k^0+1}^0$  and the inequalities (17.30), (17.63) imply that

$$|r_{k^0+1}^0| + |s_{k^0+1}^0| \leq \mu(|y_{k^0+1}^0| + |c^0(l)|) < \mu(2.22|A| + 0.663|A|) < 4|A|.$$

Considering case 1 of the proof of Lemma 17.3.5.1[14] with the additional condition that the bond  $E_1^0$  consists of a 1/2-cell, notice that the subdiagram  $\Delta_0^0$  is a degenerate special 8-gon of rank  $i$  (if we set  $a = d, p = v, b = e, r = r_0^0, q = (x_1^0)^{-1}, s = s_0^0$ ). Now, by Lemma 8.7[14],  $|\partial\Delta_0^0| < 180\beta^{-1}|A_j|$ , and therefore  $r(\Delta_0^0) < j$  by Lemma 6.2[14].

Dealing with case 2 of the proof of Lemma 17.3.5.1[14] in the situation when  $r(E_{k^0+1}^0) = 1/2$ , one needs to substitute the summand  $(1 + 2\gamma)n|A_j|$  by  $2.22|A|$  in the estimate of the length of  $\partial\Delta_{k^0+1}^0$ . The inequality  $|\partial\Delta_{k^0+1}^0| < 7|A|$  is still valid and the proof of Lemma 17.3.5.1[14], as well as of Lemma 17.3.5[14], can be completed in the same way as in [14].

The formulations and proofs of Lemmas 17.3.6[14] - 17.3.8[14] do not need any changes. In the formulation of Lemma 17.4[14] we require in addition that the rank  $m$  of the section  $\phi(q)$  is greater than  $1/2$ . If the diagram  $\Delta$  from

the condition of Lemma 17.4[14] satisfies  $r(\Delta) = 1/2$ , then the subdiagram  $\Delta(m_1, m_2)$  can be obtained from  $\Delta = \Delta(1, k)$  by removing those of the bonds  $E_1, E_k$  defining  $\Delta$ , that are not 0-bonds. The inequalities (17.92)-(17.94) follow from Lemmas 9.3[14](a) and 9.5[14].

Proving the existence of a short path connecting the vertex  $o_p^1 \in p_t$  with some vertex  $o_q^1 \in q(m_2, m_1)$  in the case when  $E_t = \pi_t$  is a cell of rank  $1/2$ , notice that  $|p_t| < \alpha|\partial\pi_t| < 2.22\alpha|A_m|$  by Lemma 3.8 and the inequality (17.94). Using Lemma 9.3[14](a) and the inequality (17.93), one can obtain a path connecting  $o_p^1$  with some vertex  $o_q^1 \in q(m_2, m_1)$  of length at most

$$\left(\frac{2.22\alpha}{2} + 0.003\right)|A_m| < 0.6|A_m|,$$

as required. The rest of the proof proceeds as in [14].

The formulation and the proof of Lemma 18.1[14] need not be changed.

The formulation of Lemma 18.2[14] is preserved. Instead of the word  $B$  considered in [14] we take the word  $\bar{B}$  obtained from  $B$  substituting every occurrence of the letter  $a_1$  by  $a_1^{-1}$ . It follows from the way  $B$  is constructed that the word  $\bar{B}$  is cyclically reduced,  $|B| = |\bar{B}|$ , and  $\bar{B}$  does not contain subwords of the form  $D^3$ ,  $|D| > 0$ . Moreover, no cyclic shift of  $\bar{B}^{\pm 1}$  contains a positive (negative) subword of length greater than 4. Therefore the maximal length of a common subword of a  $\bar{B}^{\pm 1}$ -periodic word and an element from  $\mathcal{T}$  is less than  $4 < \beta n^2$ . The remaining part of the proof proceeds as in [14].

In the formulation of Lemma 18.3[14] we need to consider only diagrams  $\Delta_k$  of strict rank greater than  $1/2$ . Moreover, we consider only those of diagrams  $\Delta_k$ , for which the diagrams  $\Gamma_k^l$  and  $\Gamma_k^l(m_1^k, m_2^k)$  constructed in the beginning of the proof of Lemma 18.3[14] and in Lemma 18.3.1[14] are of strict rank

greater than  $1/2$ . Indeed, by Lemma 17.3[14], applied to  $\Gamma_k^l(m_1^k, m_2^k)$  in the case  $r(\Gamma_k^l(m_1^k, m_2^k)) = 1/2$  and by construction of  $\Gamma_k^l(m_1^k, m_2^k)$  and  $\Delta_k^l$ , there are phase vertices  $\bar{o}_1^k \in p_k$  and  $\bar{o}_2^k \in q_k$  and a path  $r^k = \bar{o}_1^k - \bar{o}_2^k$  in  $\Delta_k$  such that  $\phi(r^k) \stackrel{1/2}{=} 1$  and the subgroup

$$\langle A_{i+1}^l \phi(r^k) A_{i+1}^{-l} \mid l = 0, 1, 2, 3 \rangle$$

of  $G(i)$  is trivial.

In the estimate for  $l$  in the formulation of Lemma 18.3.1[14] we substitute  $i$  by  $i+1$ . The length of the chain (18.13) is now bounded from above by  $i+1$ . The remarks about ranks of diagrams  $\Gamma_k^l(m_1^k, m_2^k)$  mean that  $j^k = r(\pi_{m_0^k}) > 1/2$  for the cell  $\pi_{m_0^k} \in \Delta_k(m_1^k, m_2^k)$  from the condition of Lemma 18.3.2[14]. The proof of Lemma 18.3.1[14] proceeds as in [14] with obvious changes caused by new estimate for  $l$ . The proof of Lemma 18.3.2[14] is preserved.

In the condition of Lemma 18.3.3[14]  $j^{k'} = r(\pi_{m_0^{k'}}) > 1/2$  by construction of  $\Delta_{k'}(m_1^{k'}, m_2^{k'})$ ; the proof remains unchanged. Notice that  $r(\Delta_k(g)) > 1/2$  provided claim (2) of Lemma 18.3.3[14] holds.

The formulations and proofs of Lemmas 18.3.4[14] - 18.3.6[14] and the argument completing the proof of Lemma 18.3[14] are preserved.

The formulations of Lemmas 18.4[14], 18.4.1[14] and 18.4.1.1[14] are preserved. The diagrams constructed in the proof of part (a) of Lemma 18.4[14] and in the beginning of the proof of part (b) are of strict rank greater than  $1/2$  since otherwise, by Lemma 17.3[14], the word  $A_{i+1}$  would be of finite order in  $G(i)$  provided  $S_k$  is nontrivial in  $G(i)$ .

In the condition of Lemma 18.4.3[14] we also allow the section  $q$  to be smooth  $\mathcal{T}$ -section of length  $|q| > \beta n^2$ . The proofs of Lemmas 18.4.2[14] and

18.4.3[14] do not change. In the case of a 1/2-cell  $\Pi$  considered in the proof of Lemma 18.4.2[14] every letter that occurs in  $\phi(\partial\Pi)$  occurs also in  $\phi(u_t)$ , since  $|u_t| > \beta|\partial\Pi| > \beta n^2$ , the alphabet consists of two letters and the word  $\phi(\partial\Pi)$  does not contain long periodic subwords.

The formulation of Lemma 18.5[14] does not change. In the proof of part (b) in the case when  $r(\Delta) = 1/2$  Lemma 17.3[14] implies that some phase vertices  $o_p \in p$  and  $o_q \in q$  can be connected by a path of zero length.

The formulation of Lemma 19.1[14] is preserved. Notice that the possibility of the diagram  $\Delta$  from the condition of Lemma 19.1[14] to be of strict rank 1/2 can be eliminated using similar arguments as in [31] (Lemma 18.9) as the group  $G(1/2)$  is torsion free. The same argument can be also applied to the rigid subdiagram  $\Delta(m_1, m_2)$  of  $\Delta$ . Thus we may assume that  $r(\Delta(m_1, m_2)) > 1/2$ . The formulation and proof of Lemma 19.1.1[14] need not be changed in view of the above remarks. If now the contiguity subdiagram  $\Delta_0$  happened to be of strict rank 1/2, then Lemma 17.3[14] imply the equality (19.16) with  $F_0 \stackrel{i}{=} 1$ . The proof of Lemma 19.1[14] can be completed as in [14]. Lemmas 19.2[14] - 19.6[14] are formulated and proved without any changes. The formulations of Lemmas 20.1[14] - 20.3[14] are preserved. Proving Lemma 20.1[14], the diagram  $\Delta(m_1, m_2)$  is constructed in the same way as in the proof of Lemma 17.4[14]. The remaining part of the proof of Lemma 20.1[14], as well as proofs of Lemmas 20.2[14] and 20.3[14], proceed as in [14].

The induction is now complete. By  $G_{\mathcal{T}}$  we denote the inductive limit of groups  $G(i)$ ,  $i = 0, 1/2, 1, 2, \dots$ :

$$G_{\mathcal{T}} = \langle a_1, a_2 \mid \{B = 1, B \in \mathcal{T}\} \cup \{A_i^n = 1\}_{i=1}^{\infty} \rangle .$$

### 3.3. Properties of groups $G_{\mathcal{T}}$

In Theorem 3.2 below we collect some properties of groups  $G_{\mathcal{T}}$ . Note, in particular, that finite subgroups of  $G_{\mathcal{T}}$  behave in the same way as those of a free Burnside group of the corresponding exponent (see [14], Theorem A(c)).

**Theorem 3.2.** *For every set  $\mathcal{T}$  (finite or infinite) satisfying conditions (a), (b) and (c) of Lemma 3.1 the group  $G_{\mathcal{T}}$  has the following properties:*

- (1)  *$G_{\mathcal{T}}$  is a 2-generated infinite group belonging to the Burnside variety of exponent  $n$ .*
- (2) *Let  $n = n_1 n_2$ , where  $n_1$  is the maximal odd divisor of  $n$ . Then every finite subgroup of  $G_{\mathcal{T}}$  is isomorphic to a subgroup of  $\mathbb{D}(2n_1) \times \mathbb{D}(2n_2)^k$ , for some  $k$ , where  $\mathbb{D}(2m)$  is a dihedral group of order  $2m$ .*
- (3) *The center of  $G_{\mathcal{T}}$  is trivial.*

Furthermore, the set of pairwise non-isomorphic groups among  $\{G_{\mathcal{T}'}\}$ ,  $\mathcal{T}' \subseteq \mathcal{T}$ , is of cardinality continuum provided  $\mathcal{T}$  is infinite. If  $\mathcal{T}'$  is a recursive subset of  $\mathcal{T}$ , then the word and conjugacy problems are solvable in  $G_{\mathcal{T}'}$ .

*Proof.* The claims that  $G_{\mathcal{T}}$  is infinite and satisfies the identity  $x^n = 1$  are deduced from Lemmas 10.4[14](a) and 18.2[14] in the same way as in [14]. The claim about the structure of finite subgroups of  $G_{\mathcal{T}}$  follows, as in [14], from Lemma 15.9[14].

Assuming that the center of  $G_{\mathcal{T}}$  is nontrivial, instead of the equality (21.6) we consider the equality

$$Z \bar{D} Z^{-1} \bar{D}^{-1} \stackrel{i}{=} 1 ,$$

where  $\bar{D}$  is obtained from the word  $D$  used in [14] by substituting every occurrence of  $a_1$  by  $a_1^{-1}$ . Using the remarks made in the proof of Lemma 18.2[14] one can prove Lemma 21.2[14] and complete the argument as in [14].

Passing to a subset  $\mathcal{T}'$  of  $\mathcal{T}$  we may assume that presentation of the group  $G_{\mathcal{T}'}$  is constructed. Let us show that the kernels of presentations of groups  $G_{\mathcal{T}_1}$  and  $G_{\mathcal{T}_2}$  are different for any two different subsets  $\mathcal{T}_1$  and  $\mathcal{T}_2$  of  $\mathcal{T}$ . Without loss of generality, there is a word  $w \in \mathcal{T}_1 \setminus \mathcal{T}_2$ . Thus,  $w = 1$  in  $G_{\mathcal{T}_1}$ . Assuming that  $w = 1$  in  $G_{\mathcal{T}_2}$  we obtain a disk reduced diagram  $\Delta$  of some rank  $i$  over the presentation of  $G_{\mathcal{T}_2}$  with the boundary label  $w$ . The diagram  $\Delta$  has cells since  $w$  is not equal to the identity in a free group. The word  $w$  is cyclically reduced. By Lemma 18.1[14], there is a cell  $\pi$  in  $\Delta$  such that the length of a common subpath of  $\partial\pi$  and  $\partial\Delta$  is greater than  $\beta|\partial\pi|$ . But this is impossible by Lemma 3.1 and the choice of  $w$ . Therefore  $w \neq 1$  in  $G_{\mathcal{T}_2}$ , and the kernels of presentations of  $G_{\mathcal{T}_1}$  and  $G_{\mathcal{T}_2}$  are different.

Thus, the groups  $G_{\mathcal{T}_1}$  and  $G_{\mathcal{T}_2}$  are quotients of a free group  $F(a_1, a_2)$  over different normal subgroups provided  $\mathcal{T}_1 \neq \mathcal{T}_2$ . Note that there is only countably many different homomorphisms of a finitely generated group  $F(a_1, a_2)$  onto a fixed countable group. So we conclude that the set of pairwise non-isomorphic groups among  $\{G_{\mathcal{T}'}\}$  is of cardinality continuum, since so is the set of different subsets of  $\mathcal{T}' \subseteq \mathcal{T}$  provided  $\mathcal{T}$  is infinite.

To show the solvability of the word and conjugacy problems in  $G_{\mathcal{T}'}$  in the case when  $\mathcal{T}'$  is a recursive subset of  $\mathcal{T}$ , we repeat the arguments from [14]. In the proof of Lemma 21.1[14] the  $\theta$ -cell  $\Pi$  may happen to be a 1/2-cell. In this situation Lemma 3.1[14] implies  $|d_1| = |d_2| = 0$ . Case 1 of the proof of Lemma 21.1[14] can be considered in the same way as in [14]. In Case 2 the subdiagram  $\Gamma$  given by  $\partial\Gamma = w_1 u_1^{-1}$  contains cells since  $r(\Gamma) = i + 1$ . By

Lemma 5.7[14], there is a  $\theta$ -cell  $\Pi'$  in  $\Gamma$ . It follows from the choice of  $\Gamma$  that the contiguity degree of  $\Pi'$  to  $w_1$  is less than  $\alpha$ . The contiguity degree of  $\Pi'$  to the section  $u_1^{-1}$  is less than  $\beta$  by Lemma 3.4 applied to  $\Delta$ . The inequality  $\theta > \alpha + \beta$  means that Case 2 is impossible.  $\square$

### 3.4. Proof of Theorem 3.1

Pick a subset  $\mathcal{S}$  of  $\mathcal{T}$  and obtain a new set of words  $\mathcal{T}'$  in the following way. In every word from  $\mathcal{S}$  we delete an arbitrary occurrence of a letter ( $a_1$  or  $a_2$ ). Denote the set thus obtained by  $\bar{\mathcal{S}}$  and set  $\mathcal{T}' = (\mathcal{T} \setminus \mathcal{S}) \cup \bar{\mathcal{S}}$ . The set  $\mathcal{T}'$  thus obtained has properties similar to the properties of the set  $\mathcal{T}$  listed in Lemma 3.1.

More precisely, no cyclic shift of (an inverse of) an element of  $\mathcal{T}'$  contains a  $B$ -periodic subword  $U$  of length greater than  $(1 + 3\xi)|B|$  unless  $|B| < \xi^{-3/2}$  and  $|U| < 23|B| < 23\xi^{-3/2}$ ; the symmetrized set obtained from the set  $\mathcal{T}'$  satisfies the small cancellation condition  $C'(\frac{3\xi}{10})$ ; and any word from  $\mathcal{T}'$  is a positive word of length at least  $n^2 - 1$ . It follows that for any set  $\mathcal{T}'$  obtained from the set  $\mathcal{T}$  in the way described above one can use the scheme of Sections 3.1, 3.2 to construct the group  $G_{\mathcal{T}'}$ .

Now we are ready to make some alterations to the set  $\mathcal{T}$ . Consider the set  $\mathcal{T}$  decomposed into pairs of words:  $\{u_i, v_i\}$ ,  $i = 1, 2, \dots$ . For  $i = 1, 2, \dots$  denote by  $u'_i$  (resp.  $v'_i$ ) the word obtained by deleting an arbitrary occurrence of  $a_1$  (resp.  $a_2$ ) from  $u_i$  (resp.  $v_i$ ). For any sequence  $\alpha = (\alpha_i)_{i=1}^\infty$  of 0's and 1's the set  $\mathcal{T}_\alpha$  is constructed as follows: for every  $i$ ,  $i = 1, 2, \dots$ , the words  $u_i, v_i$  are included in  $\mathcal{T}_\alpha$  if  $\alpha_i = 0$ , and the words  $u'_i, v'_i$  are included in  $\mathcal{T}_\alpha$  otherwise.

So, every set  $\mathcal{T}_\alpha$  contains as a subset exactly one of the pairs  $\{u_i, v_i\}$  or  $\{u'_i, v'_i\}$  for every  $i$ ,  $i = 1, 2, \dots$ . The above remarks allow us to assume that the groups  $G_{\mathcal{T}_\alpha}$  are constructed, and for every sequence  $\alpha$  the group  $G_{\mathcal{T}_\alpha} = F/N_\alpha$  ( $F = F(a_1, a_2)$  is a free group) satisfies the identity  $x^n = 1$ . Note that  $N_\alpha N_\beta = F$  for any two different sequences  $\alpha$  and  $\beta$ . Indeed, for some index  $i$  the subgroup  $N_\alpha N_\beta$  contains the words  $u_i, v_i, u'_i$  and  $v'_i$ . It follows that  $a_1 = 1$  and  $a_2 = 1$  in the quotient  $F/N_\alpha N_\beta$ , and therefore  $N_\alpha N_\beta = F$ .

Consider a quotient  $F/M_\alpha$  of the group  $G_{\mathcal{T}_\alpha}$  over its maximal proper normal subgroup. The group  $F/M_\alpha$  is simple, and  $M_\alpha \neq M_\beta$  for any two different sequences  $\alpha$  and  $\beta$ . Indeed,  $N_\alpha \subseteq M_\alpha$  for every sequence  $\alpha$ . Therefore  $M_\alpha M_\beta = F$  and consequently  $M_\alpha \neq M_\beta$  since both are proper subgroups of  $F$ .

Thus, the set of different kernels  $M_\alpha$  of homomorphisms of a free group  $F$  of rank 2 onto simple groups of exponent  $n$  is of cardinality continuum, and therefore so is the set of pairwise non-isomorphic groups in the collection  $\{F/M_\alpha\}$ .

## CHAPTER IV

### N-PERIODIC PRODUCTS

In this chapter we construct periodic products of groups without involutions for a given large even exponent. Namely, given a family of groups without involutions  $G_\iota$ ,  $\iota \in I$ , where  $I$  is arbitrary set of indices, and given an even number  $n \geq 2^{48}$  divisible by  $2^9$ , we construct a group

$$G = \prod_{\iota \in I} G_\iota,$$

such that every factor  $G_\iota$  is embedded into  $G$ ,  $G$  is generated by its subgroups  $G_\iota$ ,  $\iota \in I$  and any element  $x \in G$  either is conjugate to an element of some group  $G_\iota$  or satisfies  $x^n = 1$ .

#### 4.1. Presentations of periodic products

The present construction of periodic products is based on the machinery developed by S.V. Ivanov in [14]. In what follows we discuss a way to extend technique from [14] to presentations over free products of groups without involutions.

Let  $F = *_\iota G_\iota$  be a free product of groups without involutions, where  $I$  is some set of indices,  $\text{Card}(I) \geq 2$ . Elements of  $F$  are considered as words over an alphabet  $\mathfrak{A}$ , which is a union of all nontrivial elements of all the factors  $G_\iota$ . Any word  $W$  in the alphabet  $\mathfrak{A}$  can be written uniquely (up to relations in the groups  $G_\iota$ ) in the form  $W_1 \dots W_m$ , where  $W_k$  is a nontrivial element of some

$G_{\iota_k}$  and  $\iota_k \neq \iota_{k+1}$  for  $k = 1, \dots, m - 1$ . The form  $W_1 \dots W_m$  is cyclically reduced if additionally  $\iota_1 \neq \iota_m$ . By  $|W|$  we denote the length of the word  $W$  which is set to be equal to the number  $m$  of the syllables  $W_k$ .

We well order the elements of  $\mathfrak{A}$  and introduce the lexicographic order " $\prec$ " on the set of all words over  $\mathfrak{A}$  of length at least 2.

Set

$$G(0) = F = \langle \mathfrak{A} \mid Z = 1; Z \in \mathcal{Z} \rangle,$$

where  $\mathcal{Z}$  is the list of lefthandsides of all relations between the elements from  $\mathfrak{A}$  in the groups  $G_\iota$ ,  $\iota \in I$  (relations of type  $Z = 1$ ,  $Z \in \mathcal{Z}$  are referred to as 0-relations), and define the group  $G(i)$  inductively as follows. Set  $\mathcal{R}_0 = \emptyset$ . For a nonlimit ordinal  $i$  we set the period  $A_i$  of rank  $i$  to be the first (relative to the order " $\prec$ ") of the words that have infinite order in the group  $G(i-1)$ . Then  $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{A_i^n\}$  and the group  $G(i)$  is defined by imposing the relation  $A_i^n = 1$  on  $G(i-1)$ . For a limit ordinal  $i$  we do not introduce any periods and set  $\mathcal{R}_i = \bigcup_{j < i} \mathcal{R}_j$ . Now define the group  $G(i)$  as a limit of groups  $G(j)$  for  $j < i$ . In both cases:

$$G(i) = \langle \mathfrak{A} \mid R = 1, Z = 1; R \in \mathcal{R}_i, Z \in \mathcal{Z} \rangle.$$

A cell  $\Pi$  of a diagram  $\Delta$  of rank  $i$  has rank  $j$ ,  $1 \leq j \leq i$ , if the label  $\phi(\partial\Pi)$  of its contour is a cyclic shift of  $A_j^n$  or  $A_j^{-n}$ . Remark that  $j$  is a nonlimit ordinal here. To indicate that  $\Pi$  is a cell of rank  $j$  we write  $r(\Pi) = j$ . Also we would encounter cells responsible for 0-relations. Following the notations of [31] any such cell is referred to as a cell of rank 0 or just a 0-cell while the cells of nonzero rank are referred to as  $\mathcal{R}$ -cells. As in [14], the maximum

$\max\{r(\Pi) \mid \Pi \in \Delta\}$  is called the strict rank of  $\Delta$  and denoted by  $r(\Delta)$ . If  $\Delta$  does not contain any  $\mathcal{R}$ -cells (i. e.  $\Delta$  is a diagram over the free product  $*_{\iota \in I} G_\iota$ ) we set  $r(\Delta) = 0$ . Note that  $|\partial\Pi| = n|A_j|$  if  $j = r(\Pi)$  is nonzero and  $|\partial\Pi| = 1$  if  $r(\Pi) = 0$ .

By a type of a diagram  $\Delta$  of strict rank  $r(\Delta) = i$  we mean the sequence  $((i, n_i), (i_1, n_{i_1}), \dots, (i_k, n_{i_k}))$  where  $\{i, i_1, \dots, i_k\}$  is the list of ranks of  $\mathcal{R}$ -cells in  $\Delta$ ,  $i > i_1 > \dots > i_k > 0$ , and  $n_j$  is the number of cells of rank  $j$  in  $\Delta$ . The set of types is viewed as ordered in the lexicographical way using the lexicographical ordering of the set of pairs  $(j, n_j)$ .

For a nonlimit ordinal  $j > 0$  the concepts of (weak)  $j$ -compatibility, (weak) reducible pair of rank  $j$  and reducible  $j$ -cell are the same as those given in Chapter 3. Remark only that instead of letter-for-letter equality of words over  $\mathfrak{A}$  we use syllable-for-syllable equality, that is equality modulo the relations of the free product.

A concept of 0-bond needs the following modification. A 0-bond between sections  $p$  and  $q$  is a subdiagram  $E$  with  $\partial E = e_1e_2$ , where  $e_1 \in p$  is an edge and the edge  $e_2 \in q$  and either  $e_1 = e_2^{-1}$  or  $E$  is a 0-cell. Remark, following from the definition of the set  $\mathcal{Z}$ , that a subdiagram consisting of two 0-cells  $\Pi_1$  and  $\Pi_2$  contours of which have an edge in common can be substituted by a diagram consisting of just one 0-cell provided  $\Pi_1$  and  $\Pi_2$  are not mirror images of each other and by a diagram without cells otherwise. This modification implies obvious changes in the definition of contiguity subdiagrams and definition of tame diagram. Speaking about contiguity subdiagram of a cell to a section of contour (or of a cell to another cell) by "cell" we mean a cell of positive rank. It also applies to formulations of analogs of lemmas from [14] that state the existence of cells with certain properties (Lemma 5.7[14], Lemma 18.1[14]). In

the definition of  $W$ -diagram (see [14], §14) it is convenient to allow the lengths of contours  $p_2$  and  $p_3$  to be equal to 1 (i.e. labels of  $p_2$  and  $p_3$  may represent some elements of subgroups  $G_\iota$ ).

A word of length  $\geq 2$   $A$  is called simple in rank  $i$  provided  $A$  is cyclically reduced in rank  $i$  and  $A$  is neither conjugate in rank  $i$  to a word of the form  $A_k^l F$ , where  $A_k$  is the period of some rank  $k \leq i$ ,  $l$  is an integer and  $F$  is an element of  $\mathcal{F}(A_k)$  (the definition of  $\mathcal{F}(A_k)$  is given below), nor conjugate in rank  $i$  to an element of a group  $G_\iota$  for any  $\iota \in I$ .

For a nonlimit ordinal  $i$  the subgroup  $\mathcal{F}(A_i)$  of the group  $G(i-1)$  is defined uniquely as a finite group maximal with respect to the property that  $A_i$  normalizes  $\mathcal{F}(A_i)$ . A word  $J$  is called an  $\mathcal{F}(A_i)$ -involution associated with the period  $A_i$  if  $J$  normalizes the subgroup  $\mathcal{F}(A_i)$  of  $G(i-1)$ ,  $J^2 \in \mathcal{F}(A_i) \subseteq G(i-1)$ , and  $J^{-1}A_iJ \stackrel{i}{=} A_i^{-1}F$ , where  $F \in \mathcal{F}(A_i)$ .

For  $j \leq i$  the full finite subgroup  $\mathcal{K}(A_j) \subseteq G(i)$ , associated with the period  $A_j$  is defined to be the factor group  $\langle J, A_j, \mathcal{F}(A_j) \rangle / \langle A_j^n \rangle$  if there are  $\mathcal{F}(A_j)$ -involutions, and  $\langle A_j, \mathcal{F}(A_j) \rangle / \langle A_j^n \rangle$  if there are no  $\mathcal{F}(A_j)$ -involutions.

Analogously to Lemma 10.2[14], any element  $W$  of finite order in  $G(i)$ , which is not conjugate in  $G(i)$  to an element from any  $G_\iota$ ,  $\iota \in I$ , is conjugate in  $G(i)$  to a word of the form  $A_j^k F$  for some  $j \leq i$ ,  $0 < k < n$  and  $F \in \mathcal{F}(A_j)$ . The rank  $j$  here is defined uniquely (Lemma 10.2[14]) and is referred to as the height  $h(W)$ . If  $W$  is conjugate in  $G(i)$  to an element of  $G_\iota$  for some  $\iota \in I$  and has finite order in the group  $G(i)$ , then the height  $h(W)$  is set to be equal to  $1/2$ . In view of this addition we need the following lemma:

**Lemma 4.1.** *Height is well defined.*

*Proof.* We have to check only that a word of height  $1/2$  can not be conjugated

to a word of type  $A_j^l F$ , where  $F \in \mathcal{F}(A_j)$ . Proving on the contrary, note first that any element of finite order  $a \in G_\iota$  is of odd order, while, by Lemma 10.4[14], a word  $A_j^l F$  (with  $F \in \mathcal{F}(A_j)$ ) is of odd order only in the case when  $F = 1$ . Conjugacy of elements  $a$  and  $A_j^l$  means that there is an annular reduced diagram  $\Delta$  with contours  $p$  and  $q$ , where  $\phi(p) = a$ ,  $\phi(q) = A_j^l$ . This diagram is tame by Lemma 9.2[14],  $q$  is a smooth contour. Therefore Lemma 6.1[14] implies to  $\Delta$  and we obtain that  $\rho|q| \leq |p|$  which is a contradiction since  $|A_j| \geq 2$ .  $\square$

By the height  $h(\mathcal{H})$  of a finite subgroup  $\mathcal{H}$  of  $G(i)$  we mean the maximum height  $h(W)$  for all  $W \in \mathcal{H}$ .

Lemma 15.2[14] must be reformulated as follows.

**Lemma 4.2.** *Suppose  $\mathcal{G}$  is a finite subgroup of  $G(i)$ . Then either  $\mathcal{G}$  is conjugate to a finite subgroup of  $G_\iota$  for some  $\iota \in I$ , or for some  $j \leq i$   $\mathcal{G}$  is conjugate to a subgroup  $\mathcal{H}$  of the group  $\mathcal{K}(A_j) \subseteq G(i)$  such that  $j = h(\mathcal{H})$ .*

*Proof.* The case when  $\mathcal{H}$  does not contain elements of height  $1/2$  can be done in the similar way as in [14]. The remaining cases are:

1.  $\mathcal{G}$  contains only elements of height  $1/2$ .
2.  $\mathcal{G}$  contains both elements of height  $1/2$  and elements of larger height.

1. Let  $U$  and  $V$  be nontrivial elements of  $\mathcal{G}$ . There exist some words  $X_1$ ,  $X_2$  such that  $U \stackrel{i}{=} X_1 a X_1^{-1}$ ,  $V \stackrel{i}{=} X_2 b X_2^{-1}$  where  $a \in G_{\iota_1}$ ,  $b \in G_{\iota_2}$ ,  $\iota_1, \iota_2 \in I$  and  $\iota_1 \neq \iota_2$ . The word  $V^{-1}U^{-1}$  being an element of  $\mathcal{G}$  has height  $1/2$  also. Conjugating  $\mathcal{G}$  if necessary, we can assume that  $V^{-1}U^{-1} \stackrel{i}{=} c$ , where  $c \in G_{\iota_3}$  for some  $\iota_3 \in I$ . Consider the equation

$$c X_1 a X_1^{-1} X_2 b X_2^{-1} \stackrel{i}{=} 1$$

Let  $\Delta$  be a disk diagram of rank  $i$  for this equation,

$$\partial\Delta = q_1x_1q_2\bar{x}_1x_2q_3\bar{x}_2 ,$$

where  $\phi(q_1) = c$ ,  $\phi(x_1) = \phi(\bar{x}_1)^{-1} = X_1$ ,  $\phi(q_2) = a$ ,  $\phi(x_2) = \phi(\bar{x}_2)^{-1} = X_2$ ,  $\phi(q_3) = b$ . By gluing the paths  $x_1$  and  $\bar{x}_1^{-1}$ ,  $x_2$  and  $\bar{x}_2^{-1}$ , we obtain a diagram  $\Delta'$  with two holes whose contours are  $q_1$ ,  $q_2$  and  $q_3$ .

After removing reducible pairs and reducible cells we will obtain a reduced diagram  $\Delta_0$ , which is tame by Lemma 9.2[14]. If  $\Delta_0$  had  $\mathcal{R}$ -cells, Lemma 5.7[14] would assure the existence of a  $\theta$ -cell  $\Pi$  in  $\Delta_0$ . By Lemmas 3.1[14], 6.1[14] applied to the contiguity subdiagrams between  $\Pi$  and contours of  $\Delta_0$  we would obtain that  $\rho\theta|\partial\Pi| \leq 4 + 8\gamma|\partial\Pi|$ , which is a contradiction. It means that  $\Delta_0$  does not contain  $\mathcal{R}$ -cells, and therefore it is a diagram over  $G(0)$ . It follows now from properties of free product that  $\iota_1 = \iota_2 = \iota_3$  and the group  $\mathcal{G}$  is conjugate to a subgroup of  $G_{\iota_1}$ .

Case 1 is complete.

**2.** Let  $j$  be the maximal height of an element of  $\mathcal{G}$ , and let  $U$  be an element of  $\mathcal{G}$  of height  $j$ . Conjugating  $\mathcal{G}$  (and raising  $V$  to an appropriate power), if necessary, we can assume that  $U \stackrel{i}{=} A_j^l F$ , where  $\frac{n}{3} \leq |l| \leq \frac{2n}{3}$ ,  $F \in \mathcal{F}(A_j)$ . Let  $V$  be an element of  $\mathcal{G}$  of height  $1/2$ : for some  $X_1$ ,  $V \stackrel{i}{=} X_1 a X_1^{-1}$ , where  $a \in G_\iota$  for some  $\iota \in I$ . Assume first that the element  $V^{-1}U^{-1} \in \mathcal{G}$  is of height  $j_1 > 1/2$ . By Lemma 10.2(a)[14], we have with some  $X_2$  that  $V^{-1}U^{-1} \stackrel{i}{=} X_2 A_{j_1}^{l_1} F_1 X_2^{-1}$ , where  $0 < l_1 < n$ ,  $F_1 \in \mathcal{F}(A_{j_1})$  and  $j_1 \leq j$  by the choice of  $j$ .

Consider a disk diagram  $\Delta$  of rank  $i$  for the equation

$$A_j^l F X_1 a X_1^{-1} X_2 A_{j_1}^{l_1} F_1 X_2^{-1} \stackrel{i}{=} 1 ,$$

whose contour is

$$\partial\Delta = q'_1 t_1 x_1 q_2 \bar{x}_1 x_2 q'_3 t_3 \bar{x}_2 ,$$

where  $\phi(q'_1) = A_j^l$ ,  $\phi(t_1) = F$ ,  $\phi(x_1) = \phi(\bar{x}_1)^{-1} = X_1$ ,  $\phi(q_2) = a$ ,  $\phi(x_2) = \phi(\bar{x}_2)^{-1} = X_2$ ,  $\phi(q'_3) = A_{j_1}^{l_1}$ ,  $\phi(t_3) = F_1$ .

As in the case 1, we glue the paths  $x_1$  and  $\bar{x}_1^{-1}$ ,  $x_2$  and  $\bar{x}_2^{-1}$ , thus obtaining a diagram  $\Delta'$  with contours  $p'_1 = q'_1 t_1$ ,  $p_2 = q_2$ ,  $p'_3 = q'_3 t_3$ . After removing from  $\Delta'$  weak reducible pairs, reducible cells and cells of ranks  $j$ ,  $j_1$  which are weakly  $j$ -,  $j_1$ -compatible with sections  $q'_1$ ,  $q'_3$  respectively, we obtain a strictly reduced diagram  $\Delta_0$  with two holes whose contours are  $p_1 = q_1 t_1$ ,  $p_2 = q_2$ ,  $p_3 = q_3 t_3$ . Using the fact that either  $|q_1| = |q'_1|$  or  $|q_1| = n|A_j| - |q'_1|$ , we have that

$$\frac{n}{3}|A_j| \leq |q_1| \leq \frac{2n}{3}|A_j| .$$

In the same way,  $0 < |q_3| < n|A_{j_1}|$ . Therefore  $\Delta_0$  is a  $W$ -diagram of rank  $i$ , and Lemma 14.13[14] is applicable. Each alternative from Lemma 14.13[14] means that the element  $a$  is conjugate in rank  $i$  to an element of  $\mathcal{K}(A_j)$  which can not be the case as it was shown in Lemma 4.1. Similar arguments lead to a contradiction if the height of  $V^{-1}U^{-1}$  is equal to  $1/2$ . It follows that  $\mathcal{G}$  does not contain any elements of height  $1/2$ . Case 2 is complete.  $\square$

The following lemma is the analogue of Lemma 18.2[14]:

**Lemma 4.3.** *The period  $A_{i+1}$  of rank  $i+1$  exists and is a simple in rank  $i$ . If  $i$  is a nonlimit ordinal then  $|A_{i+1}| \leq |A_i| + 2$ . If  $i$  is a limit ordinal then  $|A_{i+1}| \leq \max_j(|A_j|) + 2$  where maximum is taken over all nonlimit ordinals  $j < i$ .*

Denote the limit of sequence of ranks as  $\lambda$ . By the definition of the periods  $A_i$ , Lemma 4.1 and Lemma 10.4(a)[14] we obtain the following

**Theorem 4.1.** *Any element  $x$  of the group  $G(\lambda)$  either satisfies  $x^n = 1$  or is conjugate to an element of group  $G_\iota$  for some  $\iota \in I$ .*

For any relation in  $G(\lambda)$  between elements of a group  $G_\iota$  one can construct a reduced disk diagram  $\Delta$  of some strict rank  $i$ . Since the contour  $\partial\Delta$  is labelled by an element of  $G_\iota$  it follows that  $|\partial\Delta| = 1$  which implies (in view of Lemma 6.2[14]) that  $i = r(\Delta) = 0$ . It means that every group  $G_\iota$ ,  $\iota \in I$ , embeds into  $G(\lambda)$ . Clearly  $G(\lambda)$  is generated by its subgroups  $G_\iota$ .

Thus we set  $G = G(\lambda) = \prod_{\iota \in I} {}^n G_\iota$ .

In the case when all factors  $G_\iota$ ,  $\iota \in I$ , satisfy the identity  $x^n = 1$ , the constructed periodic product coincides with the free product inside the Burnside variety  $\mathfrak{B}_n$  of exponent  $n$  in view of the following universal property. For any group  $H \in \mathfrak{B}_n$  and any system of homomorphisms  $\phi_\iota : G_\iota \rightarrow H$ ,  $\iota \in I$ , there exists a homomorphism  $\psi : G \rightarrow H$ , such that  $\psi|_{G_\iota} = \phi_\iota$ . Indeed, by construction of  $G$ , all relations between elements from different factors  $G_\iota$  in  $G$  follow from the identity  $x^n = 1$ . Existence of the homomorphism  $\psi$  now follows from the fact that  $H$  satisfies the same identity.

#### 4.2. A simplicity criterion for periodic products

**Theorem 4.2.** *Let  $n \geq 2^{48}$  be an integer divisible by  $2^9$ . For any family of nontrivial groups without involutions  $\{G_\iota\}_{\iota \in I}$ ,  $\text{Card}(I) \geq 2$ , in order for the periodic product*

$$G = \prod_{\iota \in I} {}^n G_\iota,$$

of exponent  $n$  to be a simple group, it is necessary and sufficient that every group  $G_\iota$  is generated by the  $n$ th powers of its elements.

*Proof.* First we establish the sufficiency. Assume that a normal subgroup  $N$  of  $G$  contains a nontrivial element  $a$  from  $G_{\iota_0}$  for some  $\iota_0 \in I$ . For any nontrivial  $b \in G_\iota$ ,  $\iota \neq \iota_0$  the element  $ab$  can not be conjugated in  $G$  to an element  $c$  of some group  $G_{\iota_1}$ ,  $\iota_1 \in I$  (otherwise Lemmas 9.2[14] and 6.4[14] applied to a reduced diagram of this conjugacy would mean that that  $ab$  and  $c$  are conjugated in the free product  $*_{\iota \in I} G_\iota$  which is not the case). By Theorem 4.1, it follows that  $(ab)^n = 1$  in  $G$ . Therefore  $b^n \in N$ . Since every factor  $G_\iota$  is generated by the  $n$ th powers of its elements we see that  $N$  contains all nontrivial elements from the groups  $G_\iota$ ,  $\iota \neq \iota_0$ . Using now any of these elements we show in the same way that  $G_{\iota_0} \subseteq N$ . Thus,  $N = G$ .

It remains to show that any normal subgroup  $N$  of  $G$  contains a nontrivial element from some factor  $G_\iota$ . By Lemma 10.2(a)[14], we may assume that  $N$  contains some  $A_j^l F$ , where  $A_j$  is the period of rank  $j$ ,  $F \in \mathcal{F}(A_j)$  and (raising to an appropriate power, if necessary, we obtain that)  $\frac{n}{3} \leq |l| \leq \frac{2n}{3}$ . By the hypothesis of the theorem there is an element  $a \in G_{\iota_0}$  such that  $a^n \neq 1$  in  $G_{\iota_0}$ . Assuming that the word  $A_j^l Fa$  is conjugate to an element of some factor  $G_\iota$ , one can construct a diagram with two holes and obtain a contradiction in the same way as it was done proving Lemma 4.2. It follows that  $(A_j^l Fa)^n = 1$  in  $G$  which implies that  $a^n \in N$ . The sufficiency part is now complete.

To prove the necessity, fix  $\iota_0 \in I$  and denote the group obtained by imposing on  $G$  all the relations  $a = 1$  for all  $a \in G_\iota$ ,  $\iota \neq \iota_0$ , by  $G'$ . It was shown above

that  $(ab)^n = 1$  in  $G$  for any such  $a$  and for any  $b \in G_{\iota_0}$ . It means that  $b^n = 1$  in  $G'$  and  $G'$  can be given by relative presentation  $\langle G_{\iota_0} \mid b^n = 1; b \in G_{\iota_0} \rangle$ . But the assumption that  $G$  is simple implies that  $G'$  is the identity group. The necessity part is proved.  $\square$

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