STABILITY OF COMPRESSED SENSING FOR DICTIONARIES AND ALMOST SURE CONVERGENCE RATE FOR THE KACZMARZ ALGORITHM

By

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To my dear parents,

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CHAPTER I

INTRODUCTION

This dissertation consists of two topics: compressed sensing and the Kaczmarz algorithm.

Compressed sensing addresses the problem of recovering an unknown signal $z_0 \in \mathbb{R}^d$ from a small number of linear measurements based on an underlying structure of sparsity or compressibility. There are generally two approaches for solving this problem. This dissertation will focus on the ℓ^q minimization approach. The classical result is that ℓ^q minimization can stably recover an almost sparse signal from its noisy measurements when the measurement matrix satisfies a so called restricted isometry property. Other conditions on measurement matrices are explored for stable recovery. We show that the null space property is a necessary and sufficient condition on the measurement matrix for stable recovery.

When the signal is sparse in an overcomplete dictionary, we have the compressed sensing problem in a dictionary. Some basic conditions are given for this problem to be meaningful. It is known that under an appropriate restricted isometry property for a dictionary, reconstruction methods based on ℓ^q minimization can provide an effective signal recovery tool even when the dictionary is coherent. We propose that a modified null space property for the dictionary is also sufficient to stably recover the signal. Perturbations on the measurement matrices and the dictionary are also considered.

The second part of this dissertation is concerned with the almost sure convergence rate of the Kaczmarz algorithm. The Kaczmarz algorithm is an iterative method for reconstructing a signal $x \in \mathbb{R}^d$ from an overcomplete collection of linear measurements $y_n = \langle x, \varphi_n \rangle$, $n \geq 1$. This algorithm is widely used in image processing and computer tomography. We prove quantitative bounds on the rate of almost sure exponential convergence in the Kaczmarz algorithm for suit-

able classes of random measurement vectors $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$. Refined convergence results are given for the special case when each φ_n has i.i.d. Gaussian entries and, more generally, when each $\varphi_n/\|\varphi_n\|$ is uniformly distributed on $\mathbb{S}^{d-1}$.

CHAPTER II

PRELIMINARIES AND NOTATIONS

II.1 Frames and dictionaries

A sequence $\{e_i, i \in I\}$ in a (separable) Hilbert space $\mathscr{H}$ is called a *frame* (for example, see [1]) if there exist positive constants $\alpha, \beta > 0$ such that for all $f \in \mathscr{H}$

$$\alpha \|f\| \leq \sqrt{\sum_{i \in I} |\langle f, e_i \rangle|^2} \leq \beta \|f\|. \tag{II.1}$$

If $\alpha = \beta$, then $\{e_i, i \in I\}$ is called a *tight frame*; If $\alpha = \beta = 1$, then $\{e_i, i \in I\}$ is called a *Parseval frame*.

This dissertation will focus on frames in finite dimensions, more specifically, when $\mathscr{H} = \mathbb{R}^d$. In this case, the frame will consist of $n (n \geq d)$ vectors spanning $\mathbb{R}^d$. We will use a matrix $D = [e_1, e_2, \ldots, e_n]$ to indicate this frame. We also call D a *dictionary* of size n.

If D is a Parseval frame, then $DD^* = I$, then identity matrix.

II.2 Sequences and sum of independent random variables

Definition II.2.1 (almost sure convergence)**.** The sequence $\xi_1, \xi_2, \ldots$ of random variables converges with probability one (almost surely) to the random variable ξ if

$$\mathbb{P}\{\omega : \xi_n \to \xi\} = 1$$

A necessary and sufficient condition that $\xi_n \to \xi$ almost surely is that

$$\lim_{n\to\infty} \mathbb{P}\left\{\bigcup_{k\geq n}\{|\xi_k-\xi|\geq\varepsilon\}\right\} \left(=\lim_{n\to\infty} \mathbb{P}\left\{\sup_{k\geq n}|\xi_k-\xi|\geq\varepsilon\right\}\right)=0.$$

This condition will be used in the proof of Theorem V.6.2.

The following three theorems are classical theorems about the sum of independent random variables, which will be used in the analysis of the convergence rate of the Kaczmarz algorithm. Different versions of these theorems are available, and the ones we collect here can be found in [2].

Let $\xi_1, \xi_2, \ldots$ be independent random variables, and $S_n = \xi_1 + \cdots + \xi_n$.

Theorem II.2.2 (Strong law of large numbers)**.** *Let $\xi_1, \xi_2, \ldots$ be independent random variables with finite fourth moments and let*

$$\mathbb{E}|\xi_n - \mathbb{E}\xi_n|^4 \leq C, n \geq 1,$$

for some constant C. Then as $n \to \infty$,

$$\frac{S_n - \mathbb{E}S_n}{n} \to 0 \textit{ almost surely.}$$

Theorem II.2.3 (Central Limit Theorem)**.** *Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables with finite second moments, then as $n \to \infty$,*

$$\mathbb{P}\left\{\frac{S_n - \mathbb{E}S_n}{\sqrt{\mathbb{V}S_n}} \leq x\right\} \to \Phi(x), \quad x \in \mathbb{R},$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du. \tag{II.2}$$

Theorem II.2.4 (Law of the Iterated Logarithm)**.** *Let $\xi_1, \xi_2, \dots$ be a sequence of i.i.d. random variables with $\mathbb{E}\xi_i = 0$ and $\mathbb{E}\xi_i^2 = \sigma^2 > 0$. Then*

$$\mathbb{P}\left\{\overline{\lim}\frac{S_n}{\psi(n)} = 1\right\} = 1,$$

where

$$\psi(n) = \sqrt{2\sigma^2 n \log\log n}.$$

II.3 Notations

Throughout this dissertation, $\|\cdot\|$ indicates a generic norm.

For $u \in \mathbb{R}^d$, $\|u\|_q = (|u_1|^q + |u_2|^q + \cdots + |u_d|^q)^{1/q}$. When $q \geq 1$, this is the ℓ^q norm. When $0 < q < 1$, this is a quasinorm. The inequality

$$\|u\|_p \leq \|u\|_q \leq N^{1/q-1/p}\|u\|_p, \quad 0 < q \leq p \leq \infty. \tag{II.3}$$

is useful in the proof in Section IV.4.3.

Let $T \subset \{1, 2, \dots, d\}$ be an index set, then T^c is the complement of T, and $|T|$ is the cardinality of T. For a vector $u \in \mathbb{R}^d$, denote by u_T the vector which is equal to u on T and vanishes on T^c; For a matrix M of d columns, denote by M_T the matrix whose columns are equal to those of M on T and vanish on T^c.

Given a matrix M, M^* is the transpose of M, and $\|M\|_{op}$ is the spectral norm of M.

The notation "log" indicates the logarithmic function with base e.

CHAPTER III

COMPRESSED SENSING IN A BASIS

Compressed sensing originated recently from questioning the current methodology in signal compression by Donoho [3]. The conventional scheme is to acquire the entire signal and then compress it. For instance, the way that the transform coder JPEG2000 works is that it acquires the full signal, computes the complete set of transform coefficients, encodes the largest coefficients and discards all the others. This process of massive data acquisition followed by compression is extremely wasteful [4]. The question is can we combine acquisition and compression. Recent work in Compressed Sensing has answered this question affirmatively and there is still a growing interest in this rapidly developing field.

Not only will it be economical to acquire and compress the signal simultaneously, but sometimes we simply have no other choices. For instance, in Magnetic Resonance Imaging, it is not conceivable to collect the same number of measurements as the unknown pixels. Also, in wideband radio frequency analysis, limitations in Analog-to-Digital converter technology prevents the acquisition of a full signal based on the Nyquist-Shannon paradigm [5].

Compressed Sensing offers a way to acquire just about what is needed, by sampling and compressing simultaneously and by providing efficient reconstruction algorithms. It has numerous applications including error correction, imaging, radar, and secure communication. Compressed sensing can be applied to build a single-pixel digital camera, because it combines sampling and compression into a single non-adaptive linear measurement process [6].

The problem is formulated in Section III.1. In Section III.2, we will motivate the problem more and introduce two major algorithmic approaches in the current literature, along with the main stability theorems. Three different conditions on measurement matrices for stable recovery are also being introduced and analyzed further in Section III.3. Our results are in Section III.4, and depend upon further investigation of the null space property. Indeed, we establish that the null space property is is a necessary and sufficient condition for stable recovery of signals via ℓ^q minimization, see Theorem III.4.1. We generalize ℓ^q minimization to F-minimization in Section III.4.2. More stability results are presented in Section III.4.3 when the measurement matrix is perturbed, providing a generalization of Section III.4.1.

III.1 Problem formulation and notations

Compressed sensing addresses the problem of recovering an unknown signal $z_0 \in \mathbb{R}^d$ from a small number of linear measurements based on an underlying structure of sparsity or compressibility. In this chapter, we will assume signals are sparse in an orthonormal basis. Furthermore, without loss of generality, we assume signals are sparse in the canonical basis of $\mathbb{R}^d$.

Definition III.1.1 (Sparsity in a basis)**.** We say a vector z in $\mathbb{R}^d$ is s-sparse if the number of its nonzero coordinates $\|z\|_0 \le s$. We will also call s the sparsity level of z. The symbol $\mathbf{\Sigma}_s$ will denote all the s-sparse vectors in $\mathbb{R}^d$.

The sparsity level s is taken to be far smaller than the dimension d for it to be meaningful, that is, z being sparse means most of its coordinates are zero.

Sparsity is a very reasonable assumption. In fact, JPEG2000 already relies on the fact that images have an "almost sparse" representation in a fixed basis.

Let z_0 be an s-sparse or almost sparse signal in $\mathbb{R}^d$, which we want to recover from a small number of linear measurements $y = Az_0 \in \mathbb{R}^m$. Therefore, A is a matrix of size $m \times d$, where m is much smaller than d. The compressed sensing problem can be stated as **Knowing A a priori, how can we find a reconstruction map Δ from $\mathbb{R}^m$ to $\mathbb{R}^d$, such that $\Delta(Az_0) = z_0$, where z_0 is s-sparse.**

III.1.1 Stability of a reconstruction map

In practice, signals are often not entirely sparse, rather, are "almost sparse" or compressible. Basically, this class of signals is very close to sparse signals and can be treated like sparse signals. The following notion of $\sigma_s(z)$ indicates how sparse a signal is.

Definition III.1.2. The error between a signal z and its best s term approximation associated with a norm $\|\cdot\|$ on $\mathbb{R}^d$ is defined as

$$\sigma_s(z) := \min\{\|z-w\|,\ w \in \Sigma_s\}.$$

We will use $\sigma_s(x)_2$ to denote the error under ℓ^2-norm and $\sigma_s(x)_q$ to denote the error under ℓ^q quasinorm.

Moreover, the measurement vector y is often perturbed due to measurement error as $y = Az_0 + e$ with noise level $\|e\|_2 \leq \varepsilon$.

We would like Δ to perform in a way such that the reconstruction error $\|\Delta(Az_0) - z_0\|_2$ is controlled by $\sigma_s(z)$ and the measurement noise level ε.

Definition III.1.3 (Stability with respect to measurement noise)**.** Given $z_0 \in \Sigma_s$, for a measurement vector y such that $\|y - Az_0\|_2 \leq \varepsilon$, the reconstruction map Δ is stable with respect to the measurement error if

$$\|\Delta(y) - z_0\| \leq C_1\varepsilon,$$

where C_1 is a constant.

Definition III.1.4 (Stability with respect to compressible signals)**.** Given $z_0 \in \mathbb{R}^d$, the reconstruction map Δ is stable with respect to compressible signals if

$$\|\Delta(Az_0) - z_0\| \leq C_2\sigma_s(z_0),$$

where C_2 is a constant.

We are seeking reconstruction maps Δ, as well as suitable measurement matrices A, such that Δ is stable with respect to the measurement noise and compressible signals, i.e.

$$\|\Delta(y) - z_0\| \leq C_1\varepsilon + C_2\sigma_s(z_0). \tag{III.1}$$

Notice when there is no noise and the signal z_0 is exactly s-sparse, we get exact reconstruction.

We will also consider stability with respect to measurement matrices in Section III.4.3.

III.1.2 Notations

Throughout this chapter, $z_0 \in \mathbb{R}^d$ will be the signal that we are trying to recover from the linear measurements $y \in \mathbb{R}^m$. We also call y the measurement vector. The signal z_0 is sparse or almost sparse, and s will be the sparsity level. With $A \in \mathscr{M}(m,d)$ being the measurement matrix (also called the sensing matrix), the measurement vector y can be expressed as $y = Ax + e$, where $\|e\|_2 \leq \varepsilon$. When $\varepsilon = 0$, there are no perturbations of the measurements. We also assume that q is a number such that $0 < q \leq 1$.

III.2 Major Algorithmic approaches

III.2.1 A naive approach

Let us first consider when z_0 is exactly sparse and there is no perturbation on the measurement vector y. It is obvious that we have to choose the measurement matrix A judiciously so it will extract enough information about z_0. One may already notice that, at the least, A has to be injective on Σ_s, since the recovery process only sees the measurement y. It is easy to prove that the injectivity of A on Σ_s is equivalent to

$$\Sigma_{2s} \cap \ker A = \{0\} \tag{NSP$_0$}$$

We call this property NSP$_0$ for a reason. See the comment after Theorem III.2.7. If A is indeed injective on Σ_s, then the only s-sparse vector that satisfies the equation $Az = y$ is the original signal z_0. So we can consider to solve the following ℓ^0 minimization problem:

$$\min \|z\|_0 \quad \text{s.t. } Az = y. \tag{P$_0$}$$

This ℓ^0 minimization has a unique solution x_0 provided A is injective on Σ_s. But this is not the end of the story because this minimization problem is considered to be NP hard in general and not numerically feasible (see section 9.2.2 in [7]). Indeed, to solve this problem we need to fix a support T, and then see if there is a solution for $Az = y$ where z is supported on T. Since there are $\binom{d}{s}$ many choices for T, this is computationally expensive.

III.2.2 ℓ^q minimization, $0 < q \leq 1$

Main ideas of Basis Pursuit

Suppose z_0 is s-sparse. S ince the problem (P_0) is not numerically feasible, one may wish to find a different approach. At first glance, one may consider solving:

$$\min \|z\|_2 \quad \text{s.t. } Az = y. \tag{III.2}$$

This is very efficient because we only need to solve a least squares problem, but it generally doesn't give us the sparse signal, as illustrated in Figure III.1(a). The solution of problem (III.2) is the first point at which the l_2 ball meets the hyperplane $Ax = y$ during its expansion. Almost surely, this point is not going to lie in any low-dimensional coordinate subspace. As this suggests, we need a "ball" that is more "pointy" towards the axes. Thus the l_1 ball is considered for this purpose. As we can see in Figure III.1(b), the geometry of the l_1 ball lends itself to detecting the sparsity.

Therefore, Candes, Tao, and Romberg proposed a Basis Pursuit (BP) method which relaxes ℓ^0 minimization to ℓ^1 minimization [8, 9]:

$$\min \|z\|_1 \quad \text{s.t. } Az = y. \tag{P_1}$$

Candes and Tao published a series of papers on ℓ^1 minimization [8–10] and proved that random matrices can recover sparse signals with very high probability. This ℓ^1 minimization is a convex problem, so we are able to solve it using linear programming with reasonable efficiency [11]. Moreover, it has nice stability features which we will mention in the next section. Overall, the ℓ^1 minimization works well because of its good balance between stability and efficiency.

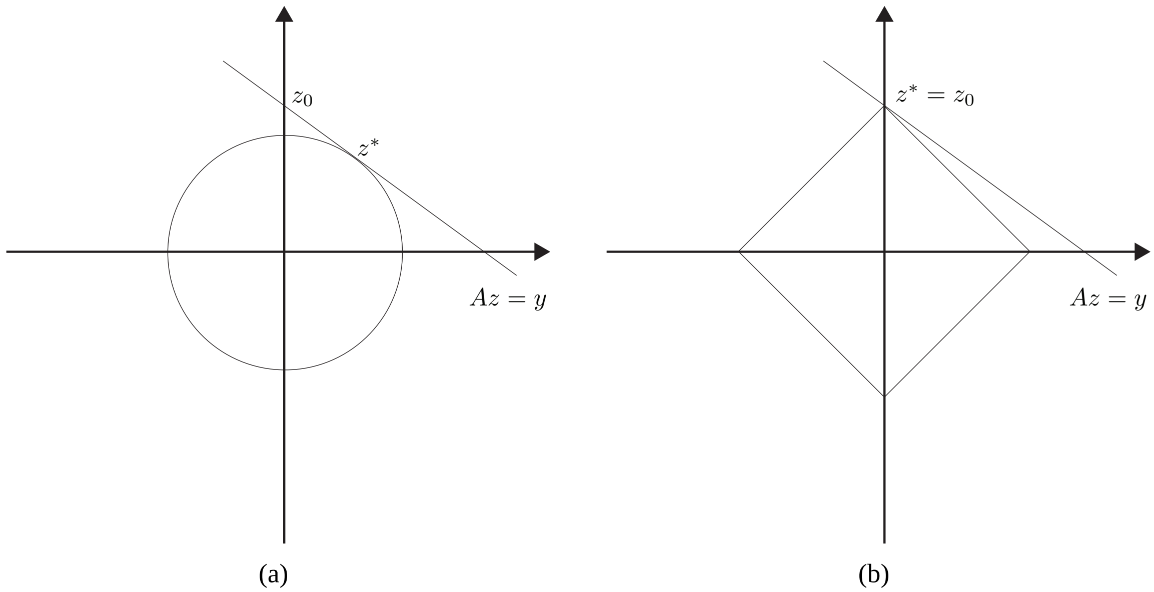

Figure III.1: The geometry of ℓ^2 and ℓ^1 minimizations.

Main theorems of Basis Pursuit

Candes and Tao proved that if A satisfies a certain quantitative property, then solving (P_1) is equivalent to solving (P_0) [10]. They showed that under a condition stronger than NSP_0, the minimizer of problem (P_1) is the original sparse signal and that, the recovery is stable.

Definition III.2.1 (Restricted Isometry Property (RIP) [9])**.** A matrix A has RIP if there exists $0 < \delta < 1$ such that

$$(1-\delta)\|z\|_2^2 \leq \|Az\|_2^2 \leq (1+\delta)\|z\|_2^2 \tag{III.3}$$

holds for any $z \in \Sigma_k$. The smallest δ that satisfies (III.3) is called the *Restricted Isometry Constant* (RIC) of A, denoted as $\delta_k(A)$ or sometimes δ_k when it is not ambiguous.

RIP essentially requires that every set of columns with cardinality less than or equal to k approximately behaves like an orthonormal system. The smaller δ_k is, the better A behaves. For example, $\delta_1 = 0$ if and only if every column of A has unit norm. The ideal situation is when $\delta_k = 0$, which is almost impossible for big k since A has far fewer rows than columns. Discussions about RIP related to random matrices will be further made in Theorem III.2.5.

Remark III.2.2. Note that RIP with $\delta_{2s} < 1$ implies NSP_0. On the other hand, if NSP_0 holds, then by compactness of $\Sigma_{2s} \cap B$ with B being the unit ball, we have

$$\alpha_{2s}\|z\|_2^2 \leq \|Az\|_2^2 \leq \beta_{2s}\|z\|_2^2, \text{ for all } z \in \Sigma_{2s} \tag{III.4}$$

holds for $z \in \Sigma_{2s}$, where $\alpha_{2s}, \beta_{2s} > 0$. This essentially means cA has RIP with δ_{2s} for some scale c. Indeed, choose c such that $c^2 = \dfrac{2}{\alpha_{2s} + \beta_{2s}}$ and then cA will have RIP with $\delta_{2s}(cA) = \dfrac{\beta_{2s} - \alpha_{2s}}{\beta_{2s} + \alpha_{2s}}$. We can rescale our measurement matrix because $cAz_0 = cy$ is equivalent to $Az_0 = y$.

As discussed earlier, we wish the reconstruction map, in this case, the ℓ^1 minimization, to be stable. Candes, Romberg and Tao showed in [9] that a version of Basis Pursuit indeed approximately recovers signals contaminated with noise. Obviously (P_1) no longer recovers the signal if there is noise, so it is modified slightly to allow for small perturbations. We consider this new minimization problem:

$$\min \|z\|_1 \quad \text{ s.t. } \|Az - y\|_2 \leq \varepsilon. \tag{$\mathrm{P}_{1,\varepsilon}$}$$

Theorem III.2.3. [Stability of BP [9]] *Suppose that x_0 is an arbitrary vector in $\mathbb{R}^d$, and A satisfies RIP with*

$$\delta_{3s}(A) + 3\delta_{4s}(A) < 2.$$

Then for any perturbation $e = Az_0 - y$ with $\|e\|_2 \leq \varepsilon$, any solution $\widetilde{z}$ to ($\mathrm{P}_{1,\varepsilon}$) *satisfies*

$$\|\widetilde{z} - z_0\|_2 \leq C_1 \cdot \varepsilon + C_2 \frac{\sigma_s(z_0)_1}{\sqrt{s}} \tag{III.5}$$

For reasonable values of δ_{4s}, the constants in (III.5) are well behaved; e.g. $C_1 \approx 12.04$ and

$C_2 \approx 8.77$ *for* $\delta_{4s} = 1/5$.

Candes sharpened Theorem III.2.3 in a later paper [12]. The new result only requires restrictions on δ_{2s} instead of δ_{4s}.

Theorem III.2.4. *[12] Suppose that* z_0 *is an arbitrary vector in* $\mathbb{R}^d$ *and the noise* $e = Ax_0 - y$ *satisfies* $\|e\|_2 \leq \varepsilon$. *If A satisfies RIP with*

$$\delta_{2s} < \sqrt{2} - 1 \approx 0.4142, \tag{III.6}$$

then any solution $\widetilde{z}$ *to* ($P_{1,\varepsilon}$) *obeys*

$$\|\widetilde{z} - z_0\|_2 \leq C_1 \cdot \varepsilon + C_2 \frac{\sigma_s(z_0)_1}{\sqrt{s}} \tag{III.7}$$

It is later shown that we can further loosen the inequality (III.6) and get better results. We only need $\delta_{2s} < 2/(3+\sqrt{2}) \approx 0.4531$ by Foucart and Lai (see Theorem III.2.9), $\delta_{2s} < 3/(4+\sqrt{6} \approx 0.4652$ by Foucart [13], and $\delta_{2s} < 2/(2+\sqrt{5}) \approx 0.4731$ by Cai, Wang and Xu [14]. Very recently, the bound has been even improved [15]. It is natural to conjecture that we only need $\delta_{2s} < 1/2$.

Pros and cons of Basis Pursuit

The beauty of Basis Pursuit is that ℓ^1 norm is a convex function, so we can use linear programming (LP) to solve (P_1). The problem (P_1) can be recast as the following linear program:

$$\min_{t_i, x_i} \sum_{i=1}^{d} t_i$$

with constraints

$$-t_i \leq z_i \leq t_i, \quad Az = y$$

The set of linear constraints forms a convex polyhedron. If an optima exists, it will be attained at a vertex of the polyhedron.

There are many ways to solve this LP. The simplex method and the interior point method are the two major ones [16]. The best bound currently attained on the runtime of an interior point method is $O(m^2 d^{1.5})$.

So far ℓ^1 minimization appears to be a very good approach, yet we cannot claim that the compressed sensing problem has been solved. There are two major drawbacks. One is its lack of efficiency. The other lies in the difficulty of deterministic construction of measurement matrices that satisfy RIP with small RIC when the dimension of the signal is too large.

Therefore mathematicians have been exploring probabilistic solutions. Using tools from random matrix theory, matrices constructed in certain random way can have RIP with high probability.

Theorem III.2.5 (BP using random matrices [5])**.** *Let* $0 < \delta < 1$ *and* A *be an* $m \times d$ *matrix that is drawn according to a probability distribution satisfying the concentration inequality*

$$\mathbb{P}(|\|Az\|_2^2 - \|z\|_2^2| \geq \delta \|z\|_2^2) \leq \exp(-c(\delta)m), \tag{III.8}$$

where $c(\delta)$ *is a constant depending only on* δ*. Then there exist constants* $c_0(\delta), c_1(\delta) > 0$ *depending on* δ *and the probability distribution such that* A *has RIP with* δ_k *with probability higher than* $1 - 2\exp(-c_0(\delta)m)$*, provided that*

$$m \geq c_1(\delta) \cdot k \cdot \log(\mathrm{ed/k}). \tag{III.9}$$

For example, a Gaussian random matrix satisfies the concentration inequality (III.8).

Theorem III.2.5 directly implies that a randomly distributed matrix allows the ℓ^1 minimization to stably recover sparse signals with high probability.

Main ideas of ℓ^q minimization

The following minimization problem, ℓ^q minimization, where $0 < q \le 1$, is a generalization of Basis Pursuit,

$$\min \|z\|_q^q \quad \text{s.t. } Az = y. \tag{P_q}$$

It comes very naturally because the shape of the ℓ^q ball is even more "pointy", looking very much like a star, which is in favor of finding sparse solutions. Another intuition for ℓ^q minimization is its attempt to approximate the sparsity $\|z\|_0$. Observe that the sparsity $\|z\|_0$ of a given vector can be approximated by the q-th power of its ℓ^q quasinorm :

$$\|z\|_q^q = \sum_{i=1}^{d} |z_i|^q \xrightarrow{q \to 0} \sum_{i=1}^{d} 1_{\{z_j \neq 0\}} = \|z\|_0.$$

It is worth pointing out that $\|\cdot\|_q^q$ induces a metric on $\mathbb{R}^d$ even if it is not a norm itself.

The ℓ^q minimization can be more powerful in the sense that it allows more matrices to be used for reconstructing sparse signals because of its geometry, but so far there is not a good algorithm to solve ℓ^q minimization. However, studying this topic from a theoretical perspective still offers benefit.

Similar to Basis Pursuit, when there is noise or the signal is almost sparse, we use the following minimization:

$$\min \|z\|_q^q \quad \text{s.t. } \|Az - y\| \le \varepsilon \tag{$\mathrm{P}_{q,\varepsilon}$}$$

Main theorems of ℓ^q minimization

In section III.2.2 we introduced RIP and stated that ℓ^1 minimization gives us stability if A satisfies RIP with small RIC. Now we are going to introduce another property of A that allows exact reconstruction via ℓ^q minimization. Since ℓ^q minimization is a generalization, all the theorems about ℓ^q minimization here apply to Basis Pursuit as well.

Definition III.2.6 (Null Space Property for ℓ^q minimization [5])**.** A matrix A has Null Space Property relative to ℓ^q with order s if

$$\|v_T\|_q^q < \|v_{T^c}\|_q^q, \text{ for } \forall\ v \in \ker A \backslash \{0\}, \forall \text{ index set } T \text{ such that } |T| \leq s \tag{NSP$_q$}$$

In fact, a simple argument using the compactness of the intersection of $\ker A$ and the unit ball shows that NSP$_q$ of order s is equivalent to the following [5]:

There exists c satisfying $0 < c < 1$ such that

$$\|v_T\|_q^q \leq c\|v_{T^c}\|_q^q, \text{ for } \forall\ v \in \ker A, \forall \text{ index set } T \text{ such that } |T| \leq s \tag{NSP$_q'$}$$

The smallest c that satisfies the last inequality is called the *Null Space Constant (NSC)*.

The second version of the Null Space Property is what we will use later.

Theorem III.2.7. [Exact recovery via ℓ^q minimization [5]] *NSP$_q$ is a necessary and sufficient condition for exact reconstruction of all s-sparse vectors via (P$_q$).*

This theorem characterizes the exact recovery of sparse signals from noiseless measurements via ℓ^q minimization. Recalling that $\Sigma_{2s} \cap \ker A = \{0\}$ is a necessary and sufficient condition for unique recovery via (P$_0$), so it is very natural for us to call it NSP$_0$ in order to be consistent with Theorem III.2.7.

In fact, NSP$_q$ implies something stronger than exact reconstruction of sparse signals via (P$_q$). It implies that ℓ^q minimization is stable with respect to compressible signals.

Theorem III.2.8 (Stability of ℓ^q minimization with respect to compressible signals [5])**.** *If A satisfies NSP$_q$, then for any vector $z_0 \in \mathbb{R}^d$,*

$$\|\widetilde{z} - z_0\|_q \leq C\sigma_s(x_0)_q,$$

where $\widetilde{z}$ is any minimizer from (P_q). The constant C depends on s, q and $\ker A$*.*

In section III.2.2, we learned that RIP gives stability of Basis Pursuit with respect to both measurement noise and compressible signals. It turns out this can be generalized to ℓ^q minimization. Consider the version of RIP in (III.4), and let $\gamma_k = \beta_k^2/\alpha_k^2$, then we have:

Theorem III.2.9 (Stability of ℓ^q minimization [17] via RIP)**.** *If A satisfies (III.4) such that*

$$\gamma_{2t} - 1 < 4(\sqrt{2}-1)(\frac{t}{s})^{1/q-1/2}, \quad \textit{for some integer } t \geq s,$$

then any solution $\widetilde{z}$ of $(\mathrm{P}_{q,\varepsilon})$ *satisfies*

$$\|\widetilde{z} - z_0\|_2 \leq C\frac{\sigma_s(x)_q}{t^{1/q-1/2}} + D\varepsilon.$$

In the case when $q = 1$, if we convert the RIP in (III.4) to the standard RIP, then Theorem III.2.9 implies that $\delta_{2s} < 0.4531$ is sufficient to give BP stability, hence this is an improvement compared to Theorem III.2.4.

Recently, Sun [18] has made a connection between NSP_q and NSP_0. It says that NSP_0 can imply NSP_q for some $0 < q < 1$.

Theorem III.2.10. *[18] If A is an $m \times d$ matrix with $m \leq d$ and $2s \leq m$, then A satisfies NSP_0 is equivalent to A has NSP_q with order s for some* $0 < q < 1$.

We discussed that NSP_0 is a necessary condition for recovery in section III.2.1. Theorem III.2.10 combined with Theorem III.2.8 indicates this necessary condition is in fact sufficient to stably reconstruct a sparse signal via ℓ^q minimization for some q with respect to compressible signals. This is quite interesting.

Moreover, the following theorem says something even stronger.

Theorem III.2.11. *[18] Let m,d and s be integers with $2s \le m \le d$, A be an $m \times d$ matrix satisfying RIP with $\delta_{2s}(A) \in (0,1)$. Let z_0 be any vector in $\mathbb{R}^d$ and $e = Az_0 - y$ be the error in measurement such that $\|e\|_2 \le \varepsilon$. Then if $q \in (0,1)$ satisfies [1] $a(q,\delta^*) < \delta^*$, we get any solution $\widetilde{z}$ of $(\mathrm{P}_{q,\varepsilon})$ obeys*

$$\|\widetilde{z} - z_0\|_2 \le C_0 s^{1/2-1/q} \sigma_s(z_0)_q + C_1 \varepsilon. \tag{III.10}$$

C_0, C_1 are positive constants independent on ε, z_0 and s.

Notice that the assumption in Theorem III.2.11 is equivalent to NSP_0 due to Remark III.2.2 if we allow to rescale the matrix A. This means that injectivity of A on Σ_s, which is a necessary condition to recover all s-sparse vectors, is in fact sufficient for stable recovery of ℓ^q minimization for some q. This is quite surprising.

These two theorems above help us understand the relationship between NSP_0 and NSP_q, and hence ℓ^0 and ℓ^q minimization as well. The stability result here is especially interesting. The fact that the weakest condition can provide stability of ℓ^q minimization implies that somehow ℓ^q minimization is self-stable. This motivates further work on the stability of ℓ^q minimization, which leads to the majority of this dissertation. Before we continue with the stability results, we need to introduce more algorithms and another property of measurement matrices.

III.2.3 Orthogonal Matching Pursuit and other greedy approaches

Main idea and description

Another approach to Compressed Sensing is Orthogonal Matching Pursuit (OMP), which is a very different approach from BP. OMP calculates the support of z_0 first. Once the support is fixed, it then computes the coordinates on the support by computing the pseudo-inverse of the measurement matrix restricted on the support.

[1] For the definitions of δ^* and $a(q,\delta)$, refer to Theorem III.3.3.

The first two approaches state that solving some specific problems (P_1, P_q) can recover a sparse signal, but it is not straightforward to see how one can actually implement these approaches. For (P_1), we convert it to a Linear Programming problem, and for (P_q), we still do not have a good algorithm. Unlike these two methods, OMP gives a direct and detailed procedure to reconstruct the signal.

Algorithm of OMP [19]:

Input:

- An $m \times d$ measurement matrix A;
- An m dimensional measurement vector y (coming from $y = Az_0$);
- The sparsity level s of the original signal.

Output:

- An estimate $\widetilde{z} \in \mathbb{R}^d$ for the original signal;
- A set T_s containing s elements from $\{1, 2, \cdots, d\}$ (support);
- An m dimensional residual r_s.

Procedure:

1) Initialize the residual $r_0 = y$, the index set $T_0 = \emptyset$, and the iteration counter $i = 1$.

2) Find the index of the coordinate which has the largest absolute value of the vector $A^* r_{i-1}$. Denote it as t_i. If the maximum occurs for multiple indices, break the tie deterministically.

3) Augment the index set $T_i = T_{i-1} \cup \{t_i\}$ and update the residual:

$$z_i = \text{argmin}_w \|y - A_{T_i} w\|_2; \quad r_i = y - A_{T_i} z_i.$$

The algorithm has only s iterations and $\widetilde{z}$ is defined as z_s.

Main theorems and implementation

The advantage of OMP is its low computational cost. OMP is especially efficient when the signal is highly sparse. But when the signal is not very sparse, OMP may be a poor choice because the cost of orthogonalization increases quadratically with the number of iterations.

Now we want to ask the two major questions for every algorithm in compressed sensing. What kind of matrices allows OMP to recover any s-sparse signals? Is OMP stable?

To answer the first question, we need to introduce a new property for matrices.

Definition III.2.12 (Coherence)**.** Suppose the columns of A are $\{a_1, \cdots, a_d\}$ and normalized, then the coherence of A, denoted by $\mu(A)$, is:

$$\mu(A) = \max_{i \neq j} |\langle a_i, a_j \rangle|$$

Theorem III.2.13. [Exact recovery via OMP [5]] *Let $z_0 \in \Sigma_s$ and A satisfy*

$$\mu(A) < \frac{1}{2s-1}. \tag{III.11}$$

Then OMP exactly reconstructs z_0 from the noiseless measurement $y = Az_0$.

In addition to this deterministic result, Tropp and Gilbert claim OMP can recover sparse signals with high probability if A is an admissible measurement matrix. For example, independently selecting each entry of A from the Gaussian distribution with mean 0 and variance $1/m$ makes an admissible measurement matrix. For the precise definition of admissible measurement matrices, see [19].

Theorem III.2.14. [OMP with admissible measurements [19]] *Fix $\delta \in (0, 0.36)$, and choose $m \geq Ks\log(d/\delta)$ where K is an absolute constant. Suppose that z_0 is an arbitrary s-sparse* *signal in* $\mathbb{R}^d$*, and draw a random* $m \times d$ *admissible measurement matrix* A *independently from the signal. Then OMP can reconstruct the signal with probability exceeding* $1-\delta$.

It is quite interesting to compare this result to Theorem III.2.5. Although both are using random matrices, they are fundamentally different. Theorem III.2.5 demonstrates that the measurement matrix can have RIP for small δ with high probability. Once the random measurement matrix does satisfy RIP, it will recover *ALL* signals stably. Theorem III.2.14 shows that OMP works with high probability for each fixed signal, i.e, for the same matrix, most of the sparse signals can be recovered, some not. The latter is not uniform with respect to signals.

The stability of OMP, unfortunately, is not quite yet established, because the strategies used to prove Theorem III.2.13 and III.2.14 depend heavily on the fact that the input signals are exactly sparse.

In light of the fact that BP and OMP present disjoint advantages and challenges, Needell and Vershynin developed new greedy algorithms which combine BP and OMP [20, 21]. These two methods are called Regularized Orthogonal Matching Pursuit and Compressive Sampling Matching Pursuit. They bridge the gap between BP and OMP, and provide good speed, stability and uniform guarantees. But they ask for slightly stronger requirements.

III.3 RIP, NSP and Coherence

We introduced RIP, NSP, and coherence respectively through the introduction of the algorithmic approaches for compressed sensing in the last section. This section will summarize and further discuss the connections between them.

III.3.1 RIP with $\delta_{2s} < 1$ implies NSP$_1$

This statement seems obvious, because it is a result of Theorem III.2.4 and Theorem III.2.7. RIP with $\delta_{2s} < \sqrt{2} - 1$ implies the exact recovery of sparse signals via ℓ^1 minimization by Theorem III.2.4 (let $\varepsilon = 0$), and this exact recovery is equivalent to NSP$_1$ by Theorem III.2.7.

However, we would like to provide a more direct and quantitative proof, which is inspired by [12]. It turns out that the null space constant of a matrix is controlled by its restricted isometry constant. This gives us one way to find matrices that have small NSC, eg. random matrices. Moreover, the proposition below only needs $\delta_{2s} < \frac{2}{3+\sqrt{2}}$, which is the same constant given by [17].

Proposition III.3.1. *Suppose A satisfies RIP with $\delta_{2s} < \dfrac{2}{3+\sqrt{2}} \approx 0.453$, then A has NSP$_q$ of order s with its null space constant $c \leq \dfrac{(1+\sqrt{2})\delta_{2s}}{2(1-\delta_{2s})}$ for any $q \in (0,1]$. For instance, if $\delta_{2s} < \dfrac{1}{2+\sqrt{2}} \approx 0.3$, then $c < 1/2$.*

Proof: Suppose A has RIP with δ_{2s}, we only need to show that A has NSP$_1$. Because it has been shown in [22, 23] that NSP$_1$ implies NSP$_q$ for $0 < q < 1$ with the same NSC.

Let $v \in \ker A \backslash \{0\}$ and S be the index set of its largest components in absolute value. We begin by dividing S^c into subsets of size s. S_1 is the first s largest components in S^c, S_2 is the next s, and so on so forth.

$$\begin{aligned}\|Av_{S+S_1}\|^2 &= \langle Av_{S+S_1}, Av_{S+S_1}\rangle = -\langle Av_{S+S_1}, \sum_{j\geq 2} Av_{S_j}\rangle = |\langle Av_{S+S_1}, \sum_{j\geq 2} Av_{S_j}\rangle| \\ &\leq |\langle Av_S, \sum_{j\geq 2} Av_{S_j}\rangle| + |\langle Av_{S_1}, \sum_{j\geq 2} Av_{S_j}\rangle| \\ &\leq \sum_{j\geq 2} \delta_{2s}\|v_S\|_2\|v_{S_j}\|_2 + \sum_{j\geq 2} \delta_{2s}\|v_{S_1}\|_2\|v_{S_j}\|_2 \\ &= \delta_{2s}(\|v_S\|_2 + \|v_{S_1}\|_2)\sum_{j\geq 2}\|v_{S_j}\|_2\end{aligned}$$

The second inequality is due to Lemma 2.1 in [12].

It then follows that

$$\|v_S\|_2^2 + \|v_{S_1}\|_2^2 = \|v_{S+S_1}\|_2^2 \leq \frac{1}{1-\delta_{2s}}\|Av_{S+S_1}\|_2^2 \leq \frac{\delta_{2s}}{1-\delta_{2s}}(\|v_S\|_2 + \|v_{S_1}\|_2)\sum_{j\geq 2}\|v_{S_j}\|_2 \quad \text{(III.12)}$$

Setting $\frac{\delta_{2s}}{1-\delta_{2s}}\sum_{j\geq 2}\|v_{S_j}\|_2 = \xi$, then (III.12) becomes

$$(\|v_S\|_2 - \xi/2)^2 + (\|v_{S_1}\|_2 - \xi/2)^2 \leq \xi^2/2,$$

which implies

$$\|v_S\|_2 - \xi/2 \leq \xi/\sqrt{2} \Rightarrow \|v_S\|_2 \leq \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\frac{\delta_{2s}}{1-\delta_{2s}}\sum_{j\geq 2}\|v_{S_j}\|_2. \quad \text{(III.13)}$$

Note

$$\|v_{S_{j+1}}\|_2 \leq s^{1/2}\|v_{S_{j+1}}\|_\infty \leq s^{-1/2}\|v_{S_j}\|_1,$$

and thus

$$\sum_{j\geq 2}\|v_{S_j}\|_2 \leq s^{-1/2}\sum_{j\geq 1}\|v_{S_j}\|_1 \leq s^{-1/2}\|v_{S^c}\|_1. \quad \text{(III.14)}$$

It follows from (III.13) and (III.14) that

$$\|v_S\|_1 \leq \sqrt{s}\|v_S\|_2 \leq \frac{(1+\sqrt{2})\delta_{2s}}{2(1-\delta_{2s})}\|v_{S^c}\|_1.$$

$\frac{(1+\sqrt{2})\delta_{2s}}{2(1-\delta_{2s})}$ needs to be less than 1 in order to achieve NSP_1, which requires $\delta_{2s} < \frac{2}{3+\sqrt{2}}$. $\square$

III.3.2 NSP_0 is equivalent to RIP with $\delta_{2s} < 1$

The equivalence between NSP_0 and RIP with $\delta_{2s}(A) < 1$ is proved in Remark III.2.2 if we allow rescaling of the matrix.

III.3.3 Incoherence implies NSP_1

To compute the RIC of a matrix is not an easy task. One way to compute $\delta_k(A)$ is to study the extremal eigenvalues of $A_T^* A_T$ for every index set T whose cardinality is less than k, which is not computationally feasible. It is just as hard to test if a matrix has NSP, though there is some numerical test of NSP available [24]. Coherence, on the other hand, is very easy to compute. A desirable feasure for a measurement matrix to have is small coherence (which we call, ***incoherence***), as suggested by Theorem III.2.13. Therefore, one idea is to see if incoherence is a sufficient condition for RIP or NSP.

Intuitively, the smaller the coherence is, the more spread out the columns of A are, as vectors in $\mathbb{R}^d$. This lets the measurement vector y extract as much information as possible from sampling.

Theorem III.3.2. *[5] Suppose that the $m \times d$ matrix A has a coherence satisfying*

$$\mu(A) < \frac{1}{2s-1}, \tag{III.15}$$

then the matrix A satisfies NSP$_1$ *of order s.*

Combining with Theorem III.2.13, we conclude that (III.15) is a sufficient condition for exact recovery of sparse signals from noiseless measurements via both Basis Pursuit and OMP.

It is natural to ask whether small coherence implies RIP with δ_{2s} being small enough. If the answer is yes, we would be able to determine if a measurement matrix A can provide stable recovery via ℓ^1 minimization by computing the coherence of A. However, no satisfactory answer has yet been given.

III.3.4 NSP$_0$ is equivalent to NSP$_q$ for some $0 < q < 1$

Let's go back to Theorem III.2.10 and give a complete version of it.

For $0 < q \le 1$ and $\delta \in (0,1)$, define

$$a(q,\delta) = \inf_{0<r_0<1} \max\left\{ \frac{1+r_0\delta}{(1+r_0^q\delta^q)^{1/q}}, \sup_{\sqrt{2}(1-r_0)\delta/2 \le y \le 1} \frac{2y}{(1+2^{-q/2}y^{2+q})^{1/q}}, \right. \tag{III.16}$$

$$\left. \sup_{\sqrt{2}(1-r_0)\delta/2 \le y \le 1} \frac{3y}{(1+y)^{1/q}}, \sup_{1 \le y} \frac{2y}{(1+y)^{1/q}} \right\}$$

Theorem III.3.3. *Let q be a positive number in* $(0,1]$*,* m,d *and s be integers with* $2s \le m \le d$*, A be an* $m \times d$ *matrix with* $\delta_{2s}(A) \in (0,1)$*, and set*

$$\delta^* := \left(\frac{1-\delta_{2s}(A)}{1+\delta_{2s}(A)}\right)^{1/2}$$

Then A has null space property of order s with respect to ℓ^q*, with a constant* $a(q,\delta^*)/\delta^*$*, i.e.,*

$$\|v_T\|_q \le a(q,\delta^*)/\delta^* \|v_{T^c}\|_q, \ \textit{for all } x \in \ker A \tag{III.17}$$

If $a(q,\delta^*)/\delta^* < 1$, then (III.17) implies the regular NSP$_q$ of order s. This inequality admits a certain range of q.

III.3.5 NSP_1 implies NSP_q for any $0 < q < 1$

Since the geometry of the ℓ^q ball makes it easier to detect sparse signals than the ℓ^1 ball, it is reasonable for us to believe that NSP_q is a weaker condition than NSP_1. This turns out to be true.

Proposition III.3.4. *[5] Define the set of reconstruction exponents*

$$\mathscr{Q}_s(A) := \{q \in (0,1] : x_0 \textit{ is the unique minimization of } (P_q) \textit{ for every } x_0 \in \Sigma_s\},$$

or by Theorem III.2.7, $\mathscr{Q}_s(A) = \{q \in (0,1] : A \textit{ has } NSP_q\}$*, then the set* $\mathscr{Q}_s(A)$ *is a - possibly empty - open interval* $(0, q_s^*(A))$ *if* $q_s^*(A) < 1$*; or* $(0,1]$ *if* $q_s^*(A) = 1$.

From the proof of this proposition, which we omit, we will see NSP_q implies NSP_p if $q \geq p \geq 1$, hence NSP_1 implies NSP_q for any $0 < q < 1$. Moreover, the set of reconstruction exponents is an open set in (0,1]. So if A satisfies NSP_q, then A actually satisfies NSP_p for p slightly bigger than q, which is interesting.

III.3.6 Summary

Finally, we summarize the relationships among the conditions we have discussed with a flow chart in Figure III.2.

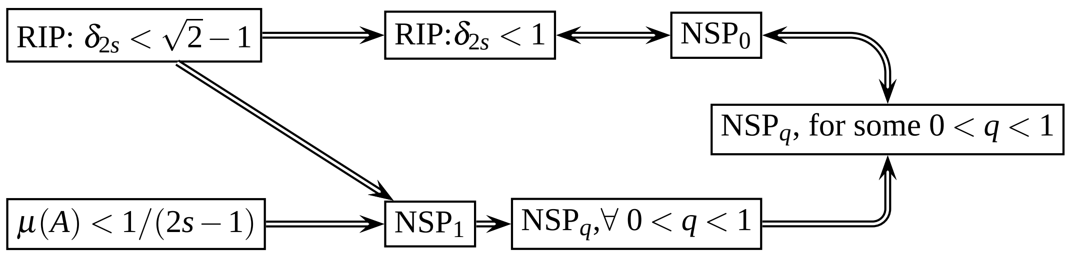

Figure III.2: The relationships among RIP, NSP and coherence.

III.4 Contributions

III.4.1 A necessary and sufficient condition for stability of ℓ^q minimization

We will state another stability result in this section. Unlike other stability results in the previous sections, this does not require the RIP condition. Moreover, we propose a necessary and sufficient condition for stability of ℓ^q minimization.

Recall the second version of the null space property: There exists $0 < c < 1$ such that

$$\|v_T\|_q^q \leq c\|v_{T^c}\|_q^q, \text{ for } \forall\ v \in \ker A, \forall \text{ index set } T \text{ such that } |T| \leq s \qquad (\mathrm{NSP}_q')$$

Theorem III.4.1 (Stability of ℓ^q minimization via NSP)**.** *A has NSP_q of order s is a necessary and sufficient condition for stability of ℓ^q minimization using A as the measurement matrix, i.e. given any vector z_0 in $\mathbb{R}^d$ and the measurement vector y such that $\|Az_0 - y\|_2 \leq \varepsilon$, we have*

$$\|\widetilde{z} - z_0\|_q \leq C_1\sigma_s(z_0)_q + C_2 d^{1/q-1/2}\varepsilon$$

where $\widetilde{z}$ is any minimizer of ($P_{q,\varepsilon}$). For the value of constants, please see (III.24).

This theorem also appeared as a corollary in [25]. The significance of Theorem III.4.1 is that, using NSP_q as a bridge, we get

Theorem III.4.2. *If ℓ^q minimization can exactly reconstruct any s-sparse signal from its noiseless measurements via (P_q), then it can stably recover any compressible signal from its perturbed measurement via ($P_{q,\varepsilon}$).*

Obviously, exact recovery of sparse vectors from its noiseless measurement is necessary to achieve stability. The above theorem is saying that it is in fact sufficient, which is quite surprising. So the ℓ^q minimization method gets stablility for free, that is, if ℓ^q minimization can recover sparse signals from its noiseless measurements, then it will be stable. Thus, fixing the measurement matrix, either ℓ^q minimization fails for some noiseless sparse signals, or it performs well when measurements are corrupted and signals are not exactly sparse. Therefore we don't need to worry about the ℓ^q minimization method in terms of stability, since if it can recover sparse signals, then it can also approximately recover compressible signals with perturbed observations.

Before proving Theorem III.4.1, we need to prove a lemma, which is essential in the proof.

Lemma III.4.3. *Suppose A is an $m \times d$ matrix where $m \leq d$, then any vector $h \in \mathbb{R}^d$ can be decomposed as $h = a + \eta$ with $a \in \ker A$, $\eta \perp \ker A$, and $\|\eta\|_2 \leq \frac{1}{s_A}\|Ah\|_2$, where s_A is the smallest positive singular value of A.*

Proof. Using the singular value decomposition, $A = U\Sigma V^*$, where U is an $m \times m$ unitary matrix whose columns are $u_1, \cdots, u_m$, Σ is an $m \times d$ diagonal matrix whose nonzero diagonal entries are $s_1, \cdots, s_r$ in descending order, and V is a $d \times d$ unitary matrix whose columns are $v_1, \cdots, v_d$. Then for any vector $x \in \mathbb{R}^d$.

$$Ax = \sum_{i=1}^{r} s_i \langle v_i, x\rangle u_i. \tag{III.18}$$

Since $\{v_i\}_{i=1}^d$ is an orthogonal basis for $\mathbb{R}^d$, $h = \sum_{j=1}^n \langle v_j, h\rangle v_j$.

Also, one can prove that

$$a := \sum_{j=r+1}^{d} \langle v_j, h\rangle v_j \in \ker A \text{ and } \eta := \sum_{j=1}^{r} \langle v_j, h\rangle v_j \perp \ker A$$

Indeed, by (III.18), $A(\sum_{j=r+1}^{d} \langle v_j, h\rangle v_j) = \sum_{i=1}^{r} s_i \langle v_i, \sum_{j=r+1}^{d} \langle v_j, h\rangle v_j\rangle u_i = 0$. Also, $\{v_i\}_{i=r+1}^{d}$ forms a basis for $\ker A$, so $\langle \sum_{j=1}^{r} \langle v_j, h\rangle v_j, v_i\rangle = 0$ for every $i = r+1, \cdots, d \Rightarrow \sum_{j=1}^{r} \langle v_j, h\rangle v_j \perp \ker A$.

So

$$\|Ah\|_2^2 = \sum_{i=1}^{r} s_i^2 \langle v_i, h\rangle^2 \geq s_r^2 \sum_{i=1}^{r} \langle v_i, h\rangle^2 = s_r^2 \|\eta\|_2^2 := s_A^2 \|\eta\|_2^2$$

$\square$

Proof of Theorem III.4.1: Let $h = \widetilde{z} - z_0$, so

$$\|Ah\|_2 = \|A\widetilde{z} - Az_0\|_2 \leq \|A\widetilde{z} - y\|_2 + \|y - Az_0\|_2 \leq 2\varepsilon. \tag{III.19}$$

Since $\widetilde{z}$ is a minimizer,

$$\|z_{0,T}\|_q^q + \|z_{0,T^c}\|_q^q = \|z_0\|_q^q \geq \|\widetilde{z}\|_q^q = \|h + z_0\|_q^q = \|h_T + z_{0,T}\|_q^q + \|h_{T^c} + z_{0,T^c}\|_q^q$$

$$\geq \|z_{0,T}\|_q^q - \|h_T\|_q^q + \|h_{T^c}\|_q^q - \|z_{0,T^c}\|_q^q.$$

This is true for any support T. If we choose T such that $\sigma_s(z_0)_q = \|z_{0,T^c}\|_q$, then we have

$$\|h_{T^c}\|_q^q \leq \|h_T\|_q^q + 2\sigma_s(z_0)_q^q. \tag{III.20}$$

Using Lemma III.4.3, we decompose h as $h = a + \eta$ where $a \in \ker A$ and

$$\|\eta\|_2 \leq \frac{1}{\nu_A} \|Ah\|_2. \tag{III.21}$$

It follows that

$$
\begin{aligned}
\|h_T\|_q^q = \|a_T + \eta_T\|_q^q \le \|a_T\|_q^q + \|\eta_T\|_q^q \le c\|a_{T^c}\|_q^q + \|\eta_T\|_q^q \\
\le c\|h_{T^c}\|_q^q + c\|\eta_{T^c}\|_q^q + \|\eta_T\|_q^q \le c\|h_{T^c}\|_q^q + \|\eta\|_q^q,
\end{aligned}
$$

which gives

$$
\|h_T\|_q^q \le \frac{2c}{1-c}\sigma_s(z_0)_q^q + \frac{1}{1-c}\|\eta\|_q^q. \tag{III.22}
$$

Eventually, from (III.20) and (III.22), we get the estimate

$$
\begin{aligned}
\|h\|_q^q &= \|h_T\|_q^q + \|h_{T^c}\|_q^q \le 2\|h_T\|_q^q + 2\sigma_s(z_0)_q^q \\
&\le \frac{4c}{1-c}\sigma_s(z_0)_q^q + \frac{2c}{1-c}\|w\|_q^q + \frac{2}{1-c}\|\eta\|_q^q + 2\sigma_s(z_0)_q^q \\
&\le \frac{2(1+c)}{1-c}\sigma_s(z_0)_q^q + \frac{2}{1-c}\|\eta\|_q^q.
\end{aligned}
$$

It follows that

$$
\|h\|_q \le 2^{1/q-1/2}(\frac{2(1+c)}{1-c})^{1/q}\sigma_s(z_0)_q + 2^{1/q-1/2}(\frac{2}{1-c})^{1/q}\|\eta\|_q. \tag{III.23}
$$

Using inequalities (III.19), (III.21), and $\|\eta\|_q \le d^{1/q-1/2}\|\eta\|_2$, we get

$$
\begin{aligned}
\|h\|_q &\le 2^{1/q-1/2}(\frac{2(1+c)}{1-c})^{1/q}\sigma_s(z_0)_q + 2^{1/q-1/2}(\frac{2}{1-c})^{1/q}\frac{d^{1/q-1/2}}{\nu_A}2\varepsilon \\
&= \frac{1}{\sqrt{2}}(\frac{4(1+c)}{1-c})^{1/q}\sigma_s(z_0)_q + (\frac{4}{1-c})^{1/q}\frac{2}{\sqrt{2\nu_A^2}}d^{1/q-1/2}\varepsilon
\end{aligned} \tag{III.24}
$$

□

In view of Theorem III.3.2, we get the following corollary.

Corollary III.4.4 (Stability of ℓ^q minimization via incoherence)**.** *If A has small coherence such* *that*

$$\mu(A) < \frac{1}{2s-1},$$

then given any vector $z_0 \in \mathbb{R}^d$ and the measurement vector y such that $\|Az_0 - y\|_2 \leq \varepsilon$, we have

$$\|\widetilde{z} - z_0\|_q \leq C_1 \sigma_s(z_0)_q + C_2 d^{1/q-1/2} \varepsilon,$$

where $\widetilde{z}$ is any minimizer of ($P_{q,\varepsilon}$).

As mentioned earlier, it is very easy to compute the coherence of a matrix. Therefore, this corollary is an easy way to test if a deterministic measurement matrix A can provide stability.

At the same time, Sun [26] proposed another condition, the *sparse approximation property*, that can stably recovery compressible signals from its noisy measurements via ℓ^q minimization. This property turns out to be basically equivalent to the null space property.

III.4.2 Recovery using F-minimization and NSP$_F$

Main idea

Inspired by ℓ^q minimization, we want to use a general function F to serve as a "norm", and recover sparse signals by minimizing this F function. One motivation of this is to find a function whose minimization problem can be solved by a feasible algorithm.

Definition III.4.5. Let $F(x) : \mathbb{R}^d \to \mathbb{R}^+ \cup \{0\}$ satisfy the following properties:

(i) Subadditivity: $F(x+y) \leq F(x) + F(y)$;

(ii) If $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$, then $F(x+y) = F(x) + F(y)$;

($\mathrm{supp}(x)$ is the index set on which x does not vanish.)

(iii) F(-x)=F(x).

then F is called a recovery cost function.

We wish to recover the original sparse signal by solving the following problem:

$$\min F(z) \quad \text{s.t. } Az = y \tag{P$_F$}$$

Similar to the ℓ^q minimization, we will give the null space property for F-minimization, and likewise, this NSP is equivalent to exact reconstruction of sparse signals from noiseless measurements via F-minimization.

Definition III.4.6. *[Null space property for F-minimization]* A matrix A has the null space property relative to F with order s if

$$F(v_T) < F(v_{T^c}), \text{ for } \forall\, v \in \ker A \backslash \{0\}, \forall\, |T| \leq s \tag{NSP$_F$}$$

Theorem III.4.7. *NSP_F is a necessary and sufficient condition for exact reconstruction of any s-sparse vector x via P_F.*

Proof. The proof is similar to the ℓ^1 version. Suppose A satisfies NSP$_F$, let $z \neq z_0$ and $Az = Az_0$,

$$\begin{aligned} F(z_0) &= F(z_0 - z_T + z_T) \leq F(z_0 - z_T) + F(z_T) = F((z_0 - z)_T) + F(z_T) \\ &< F((z_0 - z)_{T^c}) + F(z_T) = F(-z_{T^c}) + F(z_T) = F(z_{T^c}) + F(z_T) = F(z) \end{aligned}$$

Hence the solution of (P_F) is unique and it is z_0.

For the other direction, pick any $v \in \ker(A)/\{0\}, |T| \leq s$. By assumption, the solution of (P$_F$) with $y = Av_T$ is unique. Notice $A(-v_{T^c}) = Av_T$ and $-v_{T^c} \neq v_T$, so $F(v_T) < F(-v_{T^c}) = F(v_{T^c})$ since only v_T minimize $F(z)$. ☐

If $F(x) = \|x\|_q^q$, this is exactly ℓ^q minimization.

It is not easy to establish stability for F-minimization just using NSP$_F$, because in order to achieve robustness, it is essential to get $F(v_T) \leq cF(v_{T^c})$ for some $0 < c < 1$ from NSP$_F$.

However, this can not be achieved unless F has the property such that $F(av) = CF(v)$ where C is a constant depending on a.

Properties of a recovery cost function

Gribonval has some discussions in [22] about the properties of recovery cost functions. In this section, we will see a recovery cost function is actually a sum of subadditive functions defined on $\mathbb{R}$.

Let $\{e_k\}_{k=1}^{d}$ be the canonical basis of $\mathbb{R}^d$, and we can write $x = \sum_{k=1}^{d} x_k e_k$. Then

$$F(x) = F(\sum_{k=1}^{d} x_k e_k) = \sum_{k=1}^{d} F(x_k e_k)$$

since all of the $x_k e_k$ have disjoint support.

Define $f_k(t) = F(te_k), t \in \mathbb{R}, k = 1, \ldots, n$. Then $F(x) = \sum_{k=1}^{n} f_k(x_k)$.

Furthermore, from the properties of F, we can get

$$f_k(t) \geq 0,\ f_k(-t) = f_k(t),\ f_k(0) = 0 \text{ and } f_k(t+s) \leq f_k(t) + f_s(t). \tag{III.25}$$

On the other hand, if $F(x) = \sum_{k=1}^{d} f_k(x_k)$ with $f_k(t)$ satisfying (III.25), then F is a recovery cost function. So we get $F(x)$ is a recovery cost function if and only if $F(x) = \sum_{k=1}^{n} f_k(x_k)$ with $f_k(t)$ satisfying (III.25).

If we put one more condition on F, that is, $F(x) = F(y)$ if the coordinates of y are just a permutation of the coordinates of x, then it is easy to get $f_i(t) = f_j(t) := f(t), 1 \leq i, j \leq n$, so that

$$F(x) = \sum_{k=1}^{n} f(x_k). \text{ for some } f \text{ satisfying (III.25)}$$

This condition is reasonable if we wish not to have a bias on certain coordinates.

In summary, we have the following theorem:

Theorem III.4.8. *[27] (i)$F(x)$ is a recovery cost function if and only if $F(x) = \sum_{k=1}^{n} f_k(x_k)$ with $f_k(t)$ satisfying (III.25).*

(ii) If $F(x) = F(y)$ when y is a permutation of coordinates of x, then $F(x)$ is a recovery cost function if and only if $F(x) = \sum_{k=1}^{n} f(x_k)$ with $f(t)$ satisfying (III.25).

III.4.3 Stability with respect to perturbed measurement matrix

We now consider a model that involves perturbation of both measurement matrix A and measurement vector y. This has already been considered by works such as [28, 29]. The work [28] considers the stability of Basis Pursuit when dealing with a perturbed sensing matrix of the form $A = B + E$. It was shown that if A satisfies a certain restricted isometry property, then letting ε be a combined error to account for both measurement noise e and matrix perturbation E allows $(\mathrm{P}_{1,\varepsilon})$ to stably recover approximately sparse signals:

Our model is slightly different. During the measurement process, A is slightly perturbed so matrix B is actually used instead of A. Thus the measurement should be $u = Bz_0$. But in practice, the true measurement vector y that we get also differs from u because of an additive noise e (i.e., $y = Bz_0 + e$). Assuming we know that $\|y - u\|_2 \leq \varepsilon$, we will solve the minimization problem $(\mathrm{P}_{q,\varepsilon})$ with matrix A because A is the matrix that we "think" generates y, and we have knowledge of.

The perturbation $B - A$ of the measurement matrix can not be combined with the measurement vector error ε since it is correlated with the signal of interest as we will have $\|(B-A)z_0\|_2$, of which we have no control.

We will show that ℓ^q minimization is stable with respect to the perturbation on the measurement matrix, but again, using the NSP_q approach rather than RIP. Indeed, NSP_q is a weaker condition on A than RIP, so our result is stronger.

First we show that the set of all matrices that satisfy NSP_q of order s is open in the operator norm topology (this result first appeared in [27]). This means that if A satisfies NSP$_q$, then a little perturbation on A is allowed because $A+E$ will still have NSP$_q$ as long as $\|E\|_{op}$ is small enough.

Theorem III.4.9 (Openness of NSP$_q$)**.** *Let A be an $m\times d$ matrix that has NSP$_q$ of order s, then there exists $\rho>0$ such that for every matrix B that has the same size as A and $\|A-B\|_{op}<\rho$, B also has NSP$_q$ of order s. In fact, $\rho<\left(\frac{1-c}{2}\right)^{1/q}d^{1/2-1/q}\nu_A$, where c is the NSC and ν_A is the smallest nonzero singular value of A.*

This result is interesting by itself. For example, suppose that we know that A is a measurement matrix that allows us to recover compressible signals stably, but because of the physical constraints, we can only approximate A by a real physical measurement device B (e.g., the entries of A are quantized), then we have confidence in B for the reconstruction as long as it is sufficiently close to A.

Proof of Theorem III.4.9: Assume $\|A-B\|_{op}<\rho$, ρ to be determined later.

Let $b\in\ker B$ and T be any index set such that $|T|\le s$. By Lemma III.4.3, $b=a+\eta$ where $a\in\ker A$ and $\|Ab\|^2\ge\nu_A^2\|\eta\|^2$. Thus

$$\|\eta\|_2\le\frac{1}{\nu_A}\|Ab\|_2=\frac{1}{\nu_A}\|(A-B)b\|_2\le\frac{1}{\nu_A}\rho\|b\|_2,$$

which implies

$$\|\eta\|_q\le d^{1/q-1/2}\frac{1}{\nu_A}\rho\|b\|_2\le d^{1/q-1/2}\frac{1}{\nu_A}\rho\|b\|_q. \tag{III.26}$$

Since A has NSP$_q$, it follows that

$$\|a_T\|_q^q\le c\|a_{T^c}\|_q^q \text{ where } c<1. \tag{III.27}$$

Using the triangle inequality, (III.27) and (III.26), we get that

$$\begin{aligned}\|b_T\|_q^q &\le \|a_T\|_q^q + \|\eta_T\|_q^q \le c\|a_{T^c}\|_q^q + \|\eta_T\|_q^q \le c\|b_{T^c}\|_q^q + c\|\eta_{T^c}\|_q^q + \|\eta_T\|_q^q \\ &\le c\|b_{T^c}\|_q^q + \|\eta\|_q^q \le c\|b_{T^c}\|_q^q + \left(d^{1/q-1/2}\frac{1}{\nu_A}\rho\|b\|_q\right)^q \\ &= c\|b_{T^c}\|_q^q + \left(d^{1/q-1/2}\frac{1}{\nu_A}\rho\right)^q (\|b_T\|_q^q + \|b_{T^c}\|_q^q),\end{aligned}$$

from which we get

$$\|b_T\|_q^q \le \frac{c+\xi}{1-\xi}\|b_{T^c}\|_q^q, \text{ where } \xi = \left(d^{1/q-1/2}\frac{1}{\nu_A}\rho\right)^q.$$

In order for B to have NSP$_q$, we need $\frac{c+\xi}{1-\xi} < 1$, which leads to

$$\rho < \left(\frac{1-c}{2}\right)^{1/q} d^{1/2-1/q}\nu_A.$$

□

The following theorem shows that ℓ^q minimization is stable with respect to perturbations on measurement matrices if A satisfies NSP$_q$. Moreover, it also includes the error generated by measurement vectors or compressible signals, so it is a generalization of Theorem III.4.1.

Theorem III.4.10 (Stability of ℓ^q minimization with respect to perturbation on measurement matrices)**.** *Let z_0 be any vector in $\mathbb{R}^d$ and A, B be two $m \times d$ matrices with A satisfying NSP$_q$ of order s and A being full rank. Let the measurement vector y obey $\|y - Bz_0\|_2 \le \varepsilon$. If $\widetilde{z}$ is any minimizer of the minimization problem:*

$$\min \|z\|_q^q \quad s.t.\ \|Az - y\|_2 \le \varepsilon, \tag{$P_{q,\varepsilon}$}$$

then

$$\|\widetilde{z}-z_0\|_q \le C_1\sigma_s(z_0)_q + C_2 d^{1/q-1/2}\varepsilon + C_3 d^{1/q-1/2}\|(A-B)\|_{op}\|z_0\|_2.$$

See (III.34, when $A=B$) for the evaluation of the constants.

This shows that NSP_q, as a necessary condition, is sufficient for stability of ℓ^q minimization with respect to perturbations on measurement matrices. Again, using NSP_q as a bridge, we can see that ℓ^q minimization also gets this kind of stability for free, that is, if ℓ^q minimization can recover any s-sparse signal from its noiseless measurements, then it can recover any signal stably with respect to the measurement error, compressible signals, and perturbations on the measurement matrices.

Proof of Theorem III.4.10: The proof is very similar to the proof of Theorem III.4.1. The key is to find a vector that's feasible in $(\mathrm{P}_{q,\varepsilon})$ and close to z_0 since z_0 is no longer feasible, and Lemma III.4.3 will be used again to generate the term $\|(A-B)z_0\|_2$. Let $h=\widetilde{z}-z_0$, so

$$\|Ah\|_2 = \|A\widetilde{z}-Az_0\|_2 \le \|A\widetilde{z}-y\|_2 + \|y-Bz_0\|_2 + \|Bz_0-Az_0\|_2 \le 2\varepsilon + \|(A-B)z_0\|_2. \quad \text{(III.28)}$$

Since A has full rank, there exists w such that $Aw=(B-A)z_0$. Then $\|A(z_0+w)-y\|_2 = \|Bz_0-y\|_2 \le \varepsilon$, which means z_0+w is feasible in problem $(\mathrm{P}_{q,\varepsilon})$. So

$$\|z_{0,T}+w_T\|_q^q + \|z_{0,T^c}+w_{T^c}\|_q^q = \|z_0+w\|_q^q \ge \|\widetilde{z}\|_q^q = \|h+z_0\|_q^q = \|h_T+z_{0,T}\|_q^q + \|h_{T^c}+z_{0,T^c}\|_q^q$$

$$\ge \|z_{0,T}+w_T\|_q^q - \|h_T-w_T\|_q^q + \|h_{T^c}\|_q^q - \|z_{0,T^c}\|_q^q,$$

which gives

$$\|h_{T^c}\|_q^q \le \|h_T-w_T\|_q^q + \|z_{0,T^c}\|_q^q + \|z_{0,T^c}+w_{T^c}\|_q^q \le \|h_T\|_q^q + 2\|z_{0,T^c}\|_q^q + \|w\|_q^q.$$

This is true for any support T. If we choose T such that $\sigma_s(z_0)_q = \|z_{0,T^c}\|_q$, then we have

$$\|h_{T^c}\|_q^q \le \|h_T\|_q^q + 2\sigma_s(z_0)_q^q + \|w\|_q^q. \tag{III.29}$$

Let $h = a + \eta$ be the decomposition for h as in proof of Theorem III.4.1. It follows from the proof of Theorem III.4.1 and (III.29) that

$$\|h_T\|_q^q \le c\|h_{T^c}\|_q^q + \|\eta\|_q^q \le c\|h_T\|_q^q + 2c\sigma_s(z_0)_q^q + c\|w\|_q^q + \|\eta\|_q^q,$$

which gives

$$\|h_T\|_q^q \le \frac{2c}{1-c}\sigma_s(z_0)_q^q + \frac{c}{1-c}\|w\|_q^q + \frac{1}{1-c}\|\eta\|_q^q. \tag{III.30}$$

Eventually, from (III.29) and (III.30), we get the estimate

$$\begin{aligned}
\|h\|_q^q = \|h_T\|_q^q + \|h_{T^c}\|_q^q &\le 2\|h_T\|_q^q + 2\sigma_s(z_0)_q^q + \|w\|_q^q \\
&\le \frac{4c}{1-c}\sigma_s(z_0)_q^q + \frac{2c}{1-c}\|w\|_q^q + \frac{2}{1-c}\|\eta\|_q^q + 2\sigma_s(z_0)_q^q + \|w\|_q^q \\
&\le \frac{2(1+c)}{1-c}\sigma_s(z_0)_q^q + \frac{2}{1-c}\|\eta\|_q^q + \frac{1+c}{1-c}\|w\|_q^q.
\end{aligned}$$

It follows that

$$\|h\|_q \le 3^{1/q-1/2}\left(\frac{2(1+c)}{1-c}\right)^{1/q}\sigma_s(z_0)_q + 3^{1/q-1/2}\left(\frac{2}{1-c}\right)^{1/q}\|\eta\|_q + 3^{1/q-1/2}\left(\frac{1+c}{1-c}\right)^{1/q}\|w\|_q. \tag{III.31}$$

$\|\eta\|_q$ is estimated using (III.28) and

$$\|\eta\|_2 \le \frac{1}{\nu_A}\|Ah\|_2. \tag{III.32}$$

Hence, it only remains to estimate $\|w\|_q$. Notice there are many choices for w, and by Lemma III.4.3 again, we can choose w such that

$$\|w\|_2 \leq \frac{1}{\nu_A}\|Aw\|_2 = \frac{1}{\nu_A}\|(B-A)z_0\|_2. \tag{III.33}$$

Substituting the inequalities (III.28), (III.32) and (III.33) into (III.31) and using $\|\eta\|_q \leq d^{1/q-1/2}\|\eta\|_2$, we get

$$\begin{aligned}
\|h\|_q \leq\ & 3^{1/q-1/2}(\frac{2(1+c)}{1-c})^{1/q}\sigma_s(z_0)_q + 3^{1/q-1/2}(\frac{2}{1-c})^{1/q}\frac{d^{1/q-1/2}}{\nu_A}(2\varepsilon + \|(B-A)z_0\|_2) \\
& + 3^{1/q-1/2}(\frac{1+c}{1-c})^{1/q}\frac{d^{1/q-1/2}}{\nu_A}\|(B-A)z_0\|_2 \\
& = \frac{1}{\sqrt{3}}(\frac{6(1+c)}{1-c})^{1/q}\sigma_s(z_0)_q + (\frac{6d}{1-c})^{1/q}\frac{2}{\sqrt{3d\nu_A^2}}\varepsilon \\
& + 3^{1/q-1/2}\left[(\frac{2}{1-c})^{1/q} + (\frac{1+c}{1-c})^{1/q}\right]\frac{d^{1/q-1/2}}{\nu_A}\|(A-B)z_0\|_2 \\
& \leq \frac{1}{\sqrt{3}}(\frac{6(1+c)}{1-c})^{1/q}\sigma_s(z_0)_q + (\frac{6d}{1-c})^{1/q}\frac{2}{\sqrt{3d\nu_A^2}}\varepsilon + (\frac{6d}{1-c})^{1/q}\frac{2}{\sqrt{3d\nu_A^2}}\|(A-B)z_0\|_2 \\
& \leq \frac{1}{\sqrt{3}}(\frac{6(1+c)}{1-c})^{1/q}\sigma_s(z_0)_q + (\frac{6}{1-c})^{1/q}\frac{2}{\sqrt{3\nu_A^2}}d^{1/q-1/2}\varepsilon + (\frac{6}{1-c})^{1/q}\frac{2}{\sqrt{3\nu_A^2}}d^{1/q-1/2}\|(A-B)\|_{op}\|z_0\|_2
\end{aligned} \tag{III.34}$$

□

CHAPTER IV

MORE CONTRIBUTIONS: COMPRESSED SENSING IN A DICTIONARY

A recent direction of interest in compressed sensing concerns problems where signals are sparse in an overcomplete dictionary D instead of a basis, see [22, 25, 30–32]. This is motivated by the widespread use of overcomplete dictionaries in signal processing and data analysis.

Allowing the signal to be sparse with respect to a redundant dictionary adds a lot of flexibility and extends the range of signals significantly. Moreover, there may not be any sparsifying basis, for example, many EEG signals are represented as time-frequency atoms in a redundant dictionary of Gabor wavelets.

We will formulate the problem in Section IV.1. In Section IV.2, we would like to get some basic ideas of this problem for any reconstruction map, and list a few basic conditions for the measurement matrix A and the dictionary D. Subsequent sections focus on when the reconstruction map is ℓ^q minimization. Section IV.3 introduces some results of the behavior of the restricted isometry property or the coherence of AD under the condition that D is incoherent. Section IV.4 discusses the situation when we allow dictionaries to be coherent, which is a more desirable feature. Some discussions about the null space property associated with dictionaries are given in Section IV.4.1. We present a stability result when A satisfies a modified null space property, see Theorem IV.4.8. This is motivated by the work in [30]. Like in the basis case, Section IV.4.3 concerns stability with respect to perturbations on the measurement matrices, as well as dictionaries. We provide an extension of this to the case of sparsity with respect to a redundant dictionary, see Theorem IV.4.10 when A satisfies D-RIP, and Theorem IV.4.9 when A satisfies a null space property only.

IV.1 Problem formulation

Definition IV.1.1. We say a vector z is s-sparse in a dictionary D if

$$z \in D\Sigma_s := \{w \in \mathbb{R}^d : w = Dx \text{ for some } x \in \Sigma_s\},$$

x is called a representation of z in D if $z = Dx$.

Suppose we have $z_0 = Dx_0$, an s-sparse signal in D, and we want to recover this signal from a small set of linear measurements $y = Az_0 \in \mathbb{R}^m$. Therefore A is a matrix of size $m \times d$, where m is much smaller than d. The compressed sensing problem in dictionaries can be stated as **Knowing A and D a priori, how can we find a reconstruction map Δ from $\mathbb{R}^m$ to $\mathbb{R}^d$, such that $\Delta(ADx_0) = z_0$, where x_0 is s-sparse.**

In this problem, we are not only dealing with the measurement matrix A, we are also exploring how the dictionary D affects the reconstruction of a signal that can be sparsely represented. One way is to just let AD be the new measurement matrix and apply all the compressed sensing results for the basis case to AD, however, one loses the information of the interplay between A and D by doing that. So the idea is to **explore what conditions we should put on D in order to find a good measurement matrix A.** Here by "good", we want A to extract enough information from z_0 so that it can be recovered, and we want the number of measurements to be as small as possible.

IV.1.1 Stability

As in the basis case, we need to consider measurement noise or signals which are not exactly sparse. We would like the reconstruction map to perform stably in the sense of (III.1). More specifically, we will focus on ℓ^q minimization. The goal of this chapter is to generalize the stability results under the null space property in Chapter III to compressed sensing in dictionaries since a basis is a particular kind of dictionary. As we progress in this chapter, we will also consider perturbations on the measurement matrices, even dictionaries.

IV.1.2 Notations

Throughout this chapter, $z_0 \in \mathbb{R}^d$ will be the signal that we are trying to recover from the linear measurements $y \in \mathbb{R}^m$. We also call y the measurement vector. The signal z_0 is sparse or almost sparse in the dictionary D, and $D \in \mathscr{M}(d,n)$ meaning D is a dictionary for $\mathbb{R}^d$ of size n. Let s will be the sparsity level. In this chapter, when we say a signal is sparse, it is always sparse in a dictionary, unless otherwise specified.

With $A \in \mathscr{M}(m,n)$ being the measurement matrix, the measurement vector y can be expressed as $y = Az_0 + e$, where $\|e\|_2 \leq \varepsilon$. When $\varepsilon = 0$, there is no perturbation on the measurements. The model $y = Az_0 + e$ will be slightly modified in Section IV.4.3.

We also assume q is a number such that $0 < q \leq 1$.

IV.2 Basic conditions on the measurement matrix A and the dictionary D for any reconstruction map

IV.2.1 A necessary condition

For the case of a dictionary, A needs to be injective on $D\Sigma_s$ for unambiguous recovery since our signals are in $D\Sigma_s$ and the reconstruction map only sees the measurement y.

Theorem IV.2.1. *The following statements are equivalent:*

(i) A is injective on $D\Sigma_s$.

(ii) A satisfies

$$D\Sigma_{2s} \cap \ker A = \{0\}. \tag{NSP$_{D,0}$}$$

(iii) $D(\Sigma_{2s} \cap \ker(AD)) = \{0\}$.

(iv) $\operatorname{rank} D_T = \operatorname{rank} AD_T$*, for any index set* $|T| = 2s$.

Each of the four conditions above is necessary for any reconstruction map to successfully reconstruct sparse signals.

Following the notation for the basis case, we say A has $\mathrm{NSP}_{D,0}$ if $D\Sigma_{2s} \cap \ker A = \{0\}$, because it is equivalent to the injectivity of A on $D\Sigma_{2s}$.

Proof. (i)$\Rightarrow$(ii) Let $z \in D\Sigma_{2s} \cap \ker A$, so $Az = 0$ and $z = Dx$. Let $x = x_1 + x_2$ be the sum of two s-sparse vectors. So $ADx_1 = AD(-x_2)$. By injectivity of A on $D\Sigma_s$, we get $Dx_1 = D(-x_2) \iff z = 0$.

(ii)$\Rightarrow$(iii) Let $z \in D(\Sigma_{2s} \cap \ker(AD))$, so $z = Dx$ and $ADx = 0$, $x \in \Sigma_{2s}$. That exactly means $z \in D\Sigma_{2s} \cap \ker A$, hence $z = 0$ by assumption.

(iii)$\Rightarrow$(iv) We only need to show $\ker AD_T \subset \ker D_T$. Suppose $AD_T x' = 0$, if we let $x \in \mathbb{R}^n$ be the $2s$-sparse vector who equals to x' on T and vanishes on T^c, then $ADx = 0 \Rightarrow Dx \in$

$D(\Sigma_{2s} \cap \ker(AD)) = \{0\}$. So we get $Dx = 0$ by assumption. That is, $D_T x' = 0$.

(iv)$\Rightarrow$(i) Assume $ADx_1 = ADx_2$ where x_1, x_2 are both s-sparse. Let T be the support of $x_1 - x_2$ and assume $|T| = 2s$ (if $|T| < 2s$, then just choose some index set who contains T and has cardinality $2s$). $AD(x_1 - x_2) = 0$. Now if we let x' be a $2s$ length vector who is just the truncate of $x_1 - x_2$ on T, then we have $AD_T x' = 0$. The assumption of (iv) tells us $\ker AD_T \subset \ker D_T$, which means $D_T x' = 0$. This is equivalent to $D(x_1 - x_2) = 0$, so we get $Dx_1 = Dx_2$, which proves the injectivity. $\square$

We know injectivity of A on $D\Sigma_{2s}$ is necessary. Once we have this property, we can find the signal z_0 by an exhaustive search, or equivalently, solving the following minimization problem:

$$\min \|x\|_0 \quad \text{s.t. } ADx = y \tag{P$_{D,0}$}$$

If the signal has multiple sparse representations in D, we are no longer guaranteed to have a unique solution of this problem. We may find several minimizers $x_1, \cdots, x_k$ who all have the same sparsity level r, where $r \leq s$. NSP$_{D,0}$ will guarantee all these minimizers are representations of z_0, i.e. $Dx_i = z_0, i = 1, \cdots, k$. Since the signal itself, not the representation, is what we care about, the above minimization problem actually uniquely determines the signal. Summarizing the above we get the following theorem:

Theorem IV.2.2. *A has NSP$_{D,0}$ if and only if for every $z_0 \in D\Sigma_s$, the minimizers of (P$_{D,0}$) are all representations of z_0 in D, i.e.*

$$\textit{if } x^* \textit{ is a minimizer, then } Dx^* = z_0 \tag{IV.1}$$

For convenience, throughout this chapter, we will say a minimization problem has a *unique* solution if (IV.1) is satisfied. Even if it actually has multiple solutions, it is unique in the sense that all solutions give the same signal under D.

IV.2.2 Conditions for the dictionary

If D is the identity matrix, this coincides with the canonical basis case. We know we can always find A to satisfy NSP_0 in this case. Now that there is a dictionary involved, the first question to ask is with a redundant dictionary, can we still find A such that A satisfies the necessary condition $\mathrm{NSP}_{D,0}$.

We would first like to explore what kind of dictionaries will admit measurements matrices that satisfy $\mathrm{NSP}_{D,0}$. Furthermore, how to find these admissible measurement matrices, and what kind of conditions should we impose on them?

A signal always has more than one representations in a redundant dictionary, but a signal that has sparse representations in D may only have exactly one sparse representation. A simple argument will give us an equivalent condition to unique sparse representations:

Proposition IV.2.3. *Every signal that has an s-sparse representation in D has only one sparse representation if and only if D satisfies* NSP_0.

The question is "Do we want to require unique sparse representation on D?"

Let us start with a simple example:

Example IV.2.4. Let $D = \{e_1, e_2, e_3, e_4, e_1\}$, where $\{e_i\}_{i=1}^4$ is the canonical basis in $\mathbb{R}^4$. We want to recover all the 1-sparse signals in this dictionary. D doesn't satisfy NSP_0 for $s = 1$ because the first and the fifth columns are linearly dependent. Choose the measurement matrix $A = [e_1 + e_4, e_2 + e_4, e_3]^*$. For any $z_0 = Dx_0$ where $\|x_0\|_0 = 1$, solve the problem $\{w : \|w\|_0 = 1 \text{ and } ADw = ADx_0\}$. There are two cases, if $z_0 = te_i, i = 2, 3, 4$, then we get w is the vector supported on the ith coordinate with value t, multiplying w by D we get exactly z_0; if $z_0 = te_1$, then there are two minimizers: $w = \{t, 0, 0, 0, 0\}^T$ or $\{0, 0, 0, 0, t\}^T$, multiplied by D, both solutions give z_0.

This simple example shows that it is not important if we don't get a unique sparse representation in D, as long as the algorithm gives us one of the representations, we still get our original signal. This is the major difference between the dictionary case and basis case.

On the other hand, requiring unique sparse representation on D can make our problem simpler. One can imagine, if D has property NSP_0, which makes D injective on all the s-sparse signals, then this is more or less reducing this new problem to the compressed sensing problem in a basis with AD being the new measurement matrix, as the format of ($\mathrm{P}_{D,0}$) suggests.

We will show that if D satisfies NSP_0, there exists a matrix A such that $\Sigma_{2s} \cap \ker(AD) = \{0\}$, hence A satisfies $\mathrm{NSP}_{D,0}$ by using the equivalent condition (iii) in Theorem IV.2.1.

Theorem IV.2.5. *Given a dictionary D, whose size is $d \times n$, the following statements are equivalent:*

(i) $\Sigma_{2s} \cap \ker D = \{0\}$, and there exists a linear mapping $A : \mathbb{R}^d \to \mathbb{R}^{2s}$ and a reconstruction map $\Delta : \mathbb{R}^{2s} \to \mathbb{R}^n$ such that $\Delta(Az) = z$, for each z that has an s-sparse representation in D.

(ii) There exists a matrix A with size $2s \times d$, such that $\Sigma_{2s} \cap \ker(AD) = \{0\}$. (In fact, the proof shows that the probability of selecting such a matrix A is 1.)

(iii) $\Sigma_{2s} \cap \ker D = \{0\}$.

(iv) For any vector z that has an s-sparse representation in D, this sparse representation is unique.

Indeed, the work in [33] suggests that requiring D to have NSP_0 is a very reasonable assumption because such kind of frames "fills up" the space of all frames.

Before proving Theorem IV.2.5, we need a few lemmas first.

Lemma IV.2.6. *If a matrix M has size $d \times 2s$ where $d \geq 2s$, then M is not full column rank $\iff$ for any matrix A whose size is $2s \times d$, AM is not full rank.*

Proof. ($\Rightarrow$) Say the columns of M are $\alpha_1, \alpha_2, \ldots, \alpha_{2s}$, M is not full rank implies there exists

$c = \{c_1, c_2, \ldots c_{2s}\} \neq 0$ such that $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_{2s}\alpha_{2s} = 0$. Applying A to this equation we get $c_1 A\alpha_1 + c_2 A\alpha_2 + \cdots + c_{2s} A\alpha_{2s} = 0$, which means the columns of AM are linearly dependent, hence AM is not full rank.

($\Leftarrow$) By way of contradiction, suppose M is full rank, and without loss of generality, we can assume the first $2s$ rows of M are linearly independent. Choose $A = [I_{2s}|\ 0]$, where I_{2s} is the identity matrix with size $2s$. Then AM equals the matrix obtained by extracting the first $2s$ rows of M, which has full rank, but this contradicts to our original assumption. $\square$

Lemma IV.2.7. *A matrix M has NSP_0 if and only if every $2s$ columns of M are linearly independent.*

This lemma is easy to prove and we are going to use it frequently.

Lemma IV.2.8. *If $\Sigma_{2s} \cap \ker D = \{0\}$, then for every $d \times 2s$ submatrix of D, say D_{2s}, $\det(AD_{2s})$ as a polynomial of the entires of A (with size $2s \times d$), is a nonzero polynomial.*

Proof. By Lemma IV.2.7, $\Sigma_{2s} \cap \ker D = \{0\}$ implies D_{2s} has full rank, then by Lemma IV.2.6, there exists A_0 whose size is $2s \times d$, such that $A_0 D_{2s}$ is full rank. This means $\det(A_0 D_{2s}) \neq 0$, so $\det(AD_{2s})$ can not be constantly zero. $\square$

Proof of Theorem IV.2.5 (iii) $\Longleftrightarrow$ (iv) is obvious, so we just need to show the equivalent of (i), (ii), and (iii).

- (i)$\Rightarrow$(iii) is obvious.

- (iii)$\Rightarrow$(ii): define the set $\mathscr{S} = \{A \in M(2s, d) : \Sigma_{2s} \cap \ker(AD) = \{0\}\}$. We need to show

$\mathscr{S}$ is not empty. Let $\mathscr{S}^C$ be the complement of $\mathscr{S}$.

$$\mathscr{S}^C = \{A : \text{There exist } 2s \text{ columns of } AD \text{ that are linearly dependent}\} \tag{IV.2}$$

$$= \bigcup_{T=\{i_1,i_2,\ldots,i_{2s}\}} \{A : \det(AD_T) = 0\} \tag{IV.3}$$

$$= \{A : \prod_{T=\{i_1,i_2,\ldots,i_{2s}\}} \det(AD_T) = 0\} \tag{IV.4}$$

The first equation holds because of Lemma IV.2.7. The second equation uses the fact that $(AD)_T = A(D_T)$.

Suppose by contradiction that $\mathscr{S}^C$ is the whole space, then for any A, $\prod_{T=\{i_1,i_2,\ldots,i_{2s}\}} \det(AD_T)$ is a polynomial that's constantly zero (the variables of this polynomial are the entries of A). Since this is a product of polynomials, one of them must be a zero polynomial. (This is the property of integral domain. $\mathbb{R}$ is an integral domain makes the ring of polynomials on $\mathbb{R}$ also an integral domain. [34]) This contradicts to Lemma IV.2.8.

- (ii)⇒(i): Let $x \in \Sigma_{2s} \cap \ker D$, so $Dx = 0$, hence $ADx = 0$ as well, therefore $x \in \Sigma_{2s} \cap \ker AD = \{0\}$.

  For the second part, let the reconstruction map be ($\mathrm{P}_{D,0}$), (ii) implies $D(\Sigma_{2s} \cap \ker AD) = \{0\}$. By Theorem IV.2.1 and Theorem IV.2.2, this map uniquely recovers the sparse signals.

□

IV.3 Incoherent dictionaries

Consider the following minimization problem when $0 < q \leq 1$:

$$\min \|x\|_q^q \quad \text{s.t. } ADx = y. \tag{$P_{D,q}$}$$

Similar to the basis case, we want to explore under what condition solving ($P_{D,0}$) is equivalent to solving ($P_{D,q}$), and we are seeking the uniqueness of this problem. Once again, by uniqueness of ($P_{D,q}$), we mean the minimizers of it are all representations of z_0 in D.

Moreover, when the measurement vector y is perturbed, we consider a slightly different problem as usual:

$$\min \|x\|_q^q \quad \text{s.t. } \|ADx - y\|_2 \leq \varepsilon. \tag{$P_{D,q,\varepsilon}$}$$

In order to solve this compressed sensing problem in dictionaries, one major way is to apply all the reconstruction methods described in Chapter III to the matrix AD, as we can see from ($P_{D,q,\varepsilon}$). The work in [31] took this approach. They showed that if the dictionary D has RIP, then AD also has RIP if we choose A to be a random matrix.

Theorem IV.3.1. *[31] Let D be a dictionary of size n in $\mathbb{R}^d$ with RIC $\delta_s(D)$. Let A be a random matrix that satisfies the concentration inequality*

$$\mathbb{P}(|\|Av\|_2^2 - \|v\|_2^2| \geq \varepsilon \|v\|_2^2) \leq 2\exp(-cn\varepsilon^2/2), \tag{IV.5}$$

and assume $n \geq C\delta^{-2}(s\log(n/s) + \log(2e(1+12/\delta)) + t)$ for some $\delta \in (0,1)$ and $t > 0$. Then with probability at least $1 - \exp(-t)$, the matrix AD has RIC

$$\delta_s(AD) \leq \delta_s(D) + \delta(1 + \delta_s(D)). \tag{IV.6}$$

For the dictionary D to satisfy RIP, essentially it cannot be too redundant, or cannot be too coherent. In this section, we will explore how the coherence of D controls the coherence of AD.

Theorem IV.3.2. *Suppose that the i_0th and j_0th columns of AD have the largest correlation, i.e. $\mu(AD)$ equals the absolute value of inner product of its normalized i_0th and j_0th columns, then*

$$\mu(AD) \leq \frac{\mu(D) + \delta_2 IJ}{\sqrt{1-\delta_2 I^2}\sqrt{1-\delta_2 J^2}},$$

where $\delta_2 = \delta_2(A)$ is the RIC for A of order 2, and I, J are the ℓ^1 norm of the i_0th and j_0th columns of D respectively.

Proof. We assume the columns of D are normalized. $\mu(AD) = \frac{1}{\|ADe_{i_0}\|_2 \cdot \|ADe_{j_0}\|_2} |\langle ADe_{i_0}, ADe_{j_0}\rangle|$.

First we estimate the inner product of ADe_{i_0} and ADe_{j_0}:

$$\begin{aligned}
|\langle ADe_{i_0}, ADe_{j_0}\rangle| &= |\langle A(\sum_k d_{ki_0}\eta_k), A(\sum_{k'} d_{k'j_0}\eta_{k'})\rangle| \qquad \text{(Let } De_i = \textstyle\sum_k d_{ki}\eta_k, i = i_0, j_0\text{)} \\
&= |\sum_k \sum_{k'} d_{ki_0} d_{k'j_0} \langle A\eta_k, A\eta_{k'}\rangle| \\
&\leq |\sum_{k=k'}| + |\sum_{k\neq k'}| \\
&\leq |\sum_{k=k'} (d_{ki_0}d_{k'j_0} + \delta_1 |d_{ki_0}d_{k'j_0}|)| + \sum_{k\neq k'} |d_{ki_0}d_{k'j_0}|\delta_2 \\
&\leq \mu(D) + \delta_1 \sum_{k=k'} |d_{ki_0}d_{k'j_0}| + \delta_2 \sum_{k\neq k'} |d_{ki_0}d_{k'j_0}| \\
&\leq \mu(D) + \delta_2 \sum_k \sum_{k'} |d_{ki_0}d_{k'j_0}| \\
&= \mu(D) + \delta_2 IJ,
\end{aligned}$$

where $\{\eta_k\}_{k=1}^d$ is the canonical basis of $\mathbb{R}^d$.

Now we estimate the norm of ADe_{i_0}, in a very similar fashion:

$$\begin{aligned}
\langle ADe_{i_0}, ADe_{i_0}\rangle &= \langle A(\sum_k d_{ki_0}\eta_k), A(\sum_{k'} d_{k'i_0}\eta_{k'})\rangle \\
&= \sum_k \sum_{k'} d_{ki_0} d_{k'i_0} \langle A\eta_k, A\eta_{k'}\rangle \\
&= \sum_{k=k'} d_{ki_o} d_{ki_0} \|A\eta_k\|^2 + \sum_{k\neq k'} d_{ki_0} d_{k'i_0} \langle A\eta_k, A\eta_{k'}\rangle \\
&\geq \sum_{k=k'} d_{ki_0}^2 (1-\delta_1) - \sum_{k\neq k'} |d_{ki_0} d_{k'i_0}|\delta_2 \\
&= 1 - \delta_1 \sum_{k=k'} d_{ki_0}^2 - \delta_2 \sum_{k\neq k'} |d_{ki_0} d_{k'i_0}| && (\textstyle\sum_k d_{ki_0}^2 = 1) \\
&\geq 1 - \delta_2 \sum_k \sum_{k'} |d_{ki_0} d_{k'i_0}| && (\delta_1 \leq \delta_2) \\
&= 1 - \delta_2 I^2
\end{aligned}$$

Recall that $\sum_k d_{ki_0}^2 = 1$ because each column in the dictionary is normalized.

Combining these two inequalities, we have the desired result. □

Theorem IV.3.3. *With the same notation in Theorem IV.3.2, if* $\mu(D) \leq \dfrac{1}{2s-1}$, *then*

$$\delta_2(A) \leq \frac{-b + \sqrt{b^2 + 4(4s^2-2s)I^2J^2(1-(2s-1)^2\mu^2)}}{2(4s^2-4s)I^2J^2} \Rightarrow \mu(AD) \leq \frac{1}{2s-1}$$

where $b = 2\mu IJ(2s-1)^2 + I^2 + J^2, \mu = \mu(D)$.

Proof. From Theorem IV.3.2, if we can make

$$(2s-1)(\mu(D) + \delta_2 IJ) \leq \sqrt{1-\delta_2 I^2}\sqrt{1-\delta_2 J^2} \tag{IV.7}$$

then we have $\mu(D) \leq \dfrac{1}{2s-1}$. Square both sides of (IV.7) and move things around, we will get a quadratic inequality with respect to δ_2:

$$(4s^2-4s)I^2J^2\delta_2^2+(2\mu IJ(2s-1)^2+I^2+J^2)\delta_2+(2s-1)^2\mu^2-1\leq 0. \tag{IV.8}$$

The solution of (IV.8) is $\gamma_1 \leq \delta_2 \leq \gamma_2$ where γ_1, γ_2 are the roots of the quadratic equation. The only way to make γ_2 positive is to let $(2s-1)^2\mu^2-1<0$ which is exactly $\mu < \frac{1}{2s-1}$. Under this condition, we get

$$\gamma_2 = \frac{-b+\sqrt{b^2+4(4s^2-2s)I^2J^2(1-(2s-1)^2\mu^2)}}{2(4s^2-4s)I^2J^2}$$

where $b = 2\mu IJ(2s-1)^2+I^2+J^2$. $\square$

Remark IV.3.4. Theorem IV.3.3 tells us that if $\mu(D) < \frac{1}{2s-1}$, then we can find A such that $\mu(AD) < \frac{1}{2s-1}$ as well. This is quite interesting considering in Theorem IV.2.5, D having NSP_0 makes it possible to find A such that AD has NSP_0. It seems like all these nice properties that we are requiring on A in the basis case are required by the dictionary matrix now.

Corollary IV.3.5. *Suppose the columns of D have unit ℓ^2 norm (therefore $I,J \leq \sqrt{d}$), then together with Theorem IV.3.2, we have another estimate of $\mu(AD)$ which doesn't involve I,J:*

$$\mu(AD) \leq \frac{\mu(D)+\delta_2 d}{\sqrt{1-\delta_2 d}\sqrt{1-\delta_2 d}}.$$

Furthermore, we can get a result similar to Theorem IV.3.3 that doesn't involve I,J:

If $\mu(D) \leq \frac{1}{2s-1}$, then

$$\delta_2(A) \leq \frac{-\mu(2s-1)^2-1+\sqrt{(\mu(2s-1)^2+1)^2+(4s^2-4s)(1-(2s-1)^2\mu^2)}}{d(4s^2-4s)},$$

implies $\mu(AD) \leq \frac{1}{2s-1}$, *where* $\mu = \mu(D)$.

The purpose to make AD incoherent is obvious. If we regard AD as a measurement matrix in the basis case, then $\mu(AD) < \frac{1}{2s-1}$ allow both OMP and BP to exactly reconstruct the sparse representation x_0 (hence z_0 too) from noiseless measurements. Moreover, by Corollary III.4.4, the signal can be also stably recovered via BP when measurement noise is present.

IV.4 Coherent dictionaries

One loses the information of the interplay between A and D if one just consider AD as a measurement matrix, therefore making the compressed sensing problem in dictionaries less meaningful. Moreover, as shown in the last section, AD being a nice measurement matrix requires D to be incoherent and not too redundant. We wish to be able to use very coherent or redundant dictionaries.

IV.4.1 Null space property with dictionaries

Our goal is to develop a null space property with dictionaries in analogy to the null space property for a basis, such that $\text{NSP}_{D,0}$ is equivalent to the uniqueness of $(\text{P}_{D,0})$, as already shown in Theorem IV.2.2, $\text{NSP}_{D,q}$ is equivalent to the stable recovery of signals from noisy measurements via $(\text{P}_{D,q,\varepsilon})$. However, things get more complicated in the dictionary case. So far we cannot quite achieve the above goal.

The first issue is how to even define the null space property for the dictionary case. We will introduce three versions of this property throughout this section.

Definition IV.4.1 (A stronger NSP for dictionaries)**.** We say A has a *stronger* $\mathit{NSP}_{D,q}$ if

$$\|v_T\|_q^q < \|v_{T^c}\|_q^q, \text{ for } \forall\, v \in D^{-1}(\ker A \backslash \{0\}), |T| \leq s, \qquad (\text{NSP}_{D,q,S})$$

where D^{-1} means the pre-image of a set under the map D.

Theorem IV.4.2. *A has $\mathit{NSP}_{D,q,S}$ implies that for every $z_0 \in D\Sigma_s$, the minimizers of $(P_{D,q})$ are all representations of z_0 in D, i.e.*

$$\textit{if } x^* \in \textit{argmin}\{\|x\|_q^q : ADx = y\}, \textit{ then } Dx^* = z_0$$

Proof. Assume $z_0 = Dx_0$ where x_0 is s-sparse. Let x^* be a minimizer, suppose to the contrary that $Dx \neq Dx_0$. Let $v = x_0 - x^* \in D^{-1}(\ker A \backslash \{0\})$, then

$$\|x_0\|_q^q \leq \|x_{0,T} - x_T^*\|_q^q + \|x_T^*\|_q^q = \|v_T\|_q^q + \|x_T^*\|_q^q < \|v_{T^c}\|_q^q + \|x_T^*\|_q^q = \|x_{T^c}^*\|_q^q + \|x_T^*\|_q^q = \|x^*\|_q^q.$$

$\square$

Recall for the basis case, NSP$_q$ is an equivalent condition for exact recovery of sparse signals from noiseless measurements via (P$_q$) (Theorem III.2.7). However, we have only one direction here, which motivates us to find a weaker version of this null space property.

Definition IV.4.3 (A weaker NSP for dictionary)**.** We say A has a *weaker NSP$_{D,q}$* if

$$\forall |T| \leq s, \forall\, v \in D^{-1}(\ker A \backslash \{0\}), \exists\, u \text{ s.t. } Du = Dv \text{ and } \|u_T\|_q^q < \|u_{T^c}\|_q^q \qquad (\mathrm{NSP}_{D,q,W})$$

This version is obviously weaker because instead of requiring every vector in $D^{-1}(\ker A \backslash \{0\})$ has certain tail behavior, we only require one representation to have that behavior. This is treating vectors in $\mathbb{R}^n$ as in the quotient space $\mathbb{R}^d / \ker D$, which is in favor of D being coherent.

Theorem IV.4.4. *A has NSP$_{D,q,W}$ is a necessary condition for exact recovery of signals sparse in D from its noiseless measurements via* (P$_{D,q}$).

Proof. Take any support T such that $|T| \leq s$ and any $v \in D^{-1}(\ker A \backslash \{0\})$, let us try to solve (P$_{D,q}$) with the original signal being $z_0 = Dv_T$. Say w is a minimizer, then by assumption $Dw = Dv_T$ and

$$\|w\|_q^q < \|-v_{T^c}\|_q^q = \|v_{T^c}\|_q^q. \qquad \text{(IV.9)}$$

This is true because $-v_{T^c}$ is feasible but can not be a minimizer since $D(-v_{T^c}) \neq z_0$.

Notice $w+v_{T^c} \sim v$. Define $u = w+v_{T^c}$, we will prove $\|u_T\|_q^q < \|u_{T^c}\|_q^q$.

$$\begin{aligned}
\|u_{T^c}\|_q^q = \|w_{T^c}+v_{T^c}\|_q^q \geq \|v_{T^c}\|_q^q - \|w_{T^c}\|_q^q &= \|v_{T^c}\|_q^q - (\|w\|_q^q - \|w_T\|_q^q) \\
&= \|w_T\|_q^q + \|v_{T^c}\|_q^q - \|w\|_q^q \\
&> \|w_T\|_q^q && \text{(by (IV.9))} \\
&= \|u_T\|_q^q
\end{aligned}$$

□

However, we can't prove that this weaker NSP condition is sufficient for exact recovery, let alone for stable recovery.

IV.4.2 Stability of ℓ^q minimization by null space property

The work by Candes et al. [30] guarantees stable recovery of signals that are compressible in highly overcomplete and coherent dictionaries. It is believed to be the first to have this kind of result. They use a slightly different version of ℓ^q minimization:

$$\min \|D^*z\|_q^q \quad \text{subject to} \quad \|Az-y\|_2 \leq \varepsilon. \qquad (\mathrm{P2}_{D,q,\varepsilon})$$

D^*z_0 is the frame coefficients of z_0 with respect to the canonical dual frame. In particular, if D is a Parseval frame, i.e. $DD^* = I$, then $z = DD^*z$, which means D^*z is a representation for z.

Comparing ($\mathrm{P2}_{D,q,\varepsilon}$) with ($\mathrm{P}_{D,q,\varepsilon}$), ($\mathrm{P}_{D,q,\varepsilon}$) is minimizing the ℓ^q quasinorm of all representations of z_0, whereas ($\mathrm{P2}_{D,q,\varepsilon}$) is only minimizing the ℓ^q quasinorm of one particular kind of frame coefficients. Moreover, ($\mathrm{P}_{D,q,\varepsilon}$) is minimizing over the representations of the signal and ($\mathrm{P2}_{D,q,\varepsilon}$) is minimizing over the signals, so the first gives the representation of the signal but the second gives the signal directly.

The classical restricted isometry property was also modified for the setting of sparsity in a dictionary in [30].

Definition IV.4.5 (Restricted Isometry Property for dictionaries)**.** Let D be a given $d \times n$ matrix. The $m \times d$ matrix A satisfies the *restricted isometry property with respect to D* (DRIP) of order k if there exists a constant $\delta > 0$ such that

$$\forall\, x \in \Sigma_k, \quad (1-\delta)\|Dx\|_2^2 \leq \|ADx\|_2^2 \leq (1+\delta)\|Dx\|_2^2. \tag{IV.10}$$

The smallest value of $\delta > 0$ for which (IV.10) holds is denoted by δ_k.

Similar to the standard restricted isometry property, random matrices provide examples that satisfy D-RIP, see [30].

Theorem IV.4.6. *[30] Let D be an arbitrary Parseval frame and let A be a measurement matrix satisfying D-RIP with $\delta_{2s} < 0.08$. Then the solution $\widetilde{z}$ to* $(\mathrm{P2}_{D,q,\varepsilon})$ *satisfies*

$$\|\widetilde{z} - z_0\|_2 \leq C_0\varepsilon + C_1 s^{-1/2}\sigma_s(D^*z_0),$$

where the constants C_0 and C_1 may only depend on δ_{2s}.

This is saying that the reconstructed signal $\widetilde{z}$ is not far away from the original signal if D^*z_0 is almost sparse and ε is small. The setting here is stronger because we want the dual frame coefficients D^*z_0 to be compressible, instead of the existence of one particular almost sparse representations. The same is true for our main theorems, Theorem IV.4.8, Theorem IV.4.9, and Theorem IV.4.10, where the term $\sigma_s(D^*z_0)_q$ is involved.

The assumption that D^*z_0 is approximately sparse is justified in applications since practical signal classes often have sparse frame coefficients, for example, with respect to wavelets, curvelets, edgelets, shearlets, [35–37].

Inspired by [30], we would also like to examine the stability of $(\mathrm{P2}_{D,q,\varepsilon})$, but again under the null space property. We now introduce the third version of the null space property with dictionaries, which we show guarantees stable recovery.

Definition IV.4.7 (The null space property related to D^*)**.** Let D be a given $d \times n$ dictionary matrix. The matrix A satisfies the ℓ^q *null space property of order s relative to D* ($\mathrm{NSP}_{D,q}$) if

$$\forall\, z \in \ker A \backslash \{0\},\ \forall\ |T| \leq s, \qquad \|D_T^* z\|_q^q < \|D_{T^c}^* z\|_q^q. \tag{$\mathrm{NSP}_{D,q}$}$$

Here $D_T^* = (D^*)_T$. A simple compactness argument, e.g., see [5], shows that $\mathrm{NSP}_{D,q}$ is equivalent to the existence of a constant c, $0 < c < 1$, such that

$$\forall\, z \in \ker A,\ \forall\ |T| \leq s, \qquad \|D_T^* z\|_q^q \leq c\|D_{T^c}^* z\|_q^q. \tag{$\mathrm{NSP}'_{D,q}$}$$

The smallest value of the constant c in ($\mathrm{NSP}'_{D,q}$) is referred to as the *null space constant* (NSC).

Theorem IV.4.8 ($\mathrm{NSP}_{D,q}$ provides stability and robustness)**.** *Suppose that D is a $d \times n$ dictionary with frame constants $\beta \geq \alpha > 0$ and A is an $m \times d$ matrix satisfying $\mathrm{NSP}_{D,q}$. Let z_0 be a vector in $\mathbb{R}^d$ and the measurement vector y be such that $\|y - Az_0\| \leq \varepsilon$, then any solution $\widetilde{z}$ to ($P2_{D,q,\varepsilon}$) obeys*

$$\|\widetilde{z} - z_0\|_2 \leq C'\sigma_s(D^* z_0)_q + C'' n^{1/q-1/2}\varepsilon$$

for some constants C' and C'' (see (IV.12) *when $A = B$ and $D = \widetilde{D}$).*

This is a generalization of Theorem III.4.1. We omit the proof, because it is a special case of Theorem IV.4.9. Some discussion about the constants here can be found after Theorem IV.4.10.

IV.4.3 Stability with respect to perturbed measurement matrix and dictionaries

The effect of a perturbed measurement matrix A satisfying the restricted RIP has previously been considered in the classical case of sparsity with respect to a basis, see [28], as well as in

Theorem III.4.1. We also investigate a second, not previously considered, type of stability to address imprecision in the dictionary D. Our performance analysis for the ℓ^q recovery method $(\mathrm{P2}_{D,q,\varepsilon})$ will typically require that D is chosen to satisfy a design condition such as $\mathrm{NSP}_{D,q}$. However, in practice it may only be possible to use a perturbed version of D for which there are no a priori guarantees that the desired design condition holds. For example, D may be viewed as a real reconstruction device which in practice will differ from its exact specifications. We prove that ℓ^q minimization is stable with respect to imprecisions in the dictionary D.

In this section we describe our main stability theorems for ℓ^q recovery of signals that are sparse in a dictionary. We initially assume the following set-up:

- D is a $d \times n$ dictionary matrix for $\mathbb{R}^d$ (thus $n \geq d$),
- B is an $m \times d$ measurement matrix for $\mathbb{R}^d$,
- $D^* z_0$ is approximately s-sparse.

The assumption that $D^* z_0$ is approximately sparse is justified in the last section. At this point, one is given the noisy measurements $y = Bz_0 + e \in \mathbb{R}^m$ with noise level $\|e\|_2 \leq \varepsilon$, and one wishes to recover z_0 from y. We assume that one only has approximate knowledge of B, for example, due to a nonideal measurement device or because of computational limitations. We also assume perturbations of the dictionary D. For example, the intended D in $(\mathrm{P2}_{D,q,\varepsilon})$ might have been carefully designed to satisfy a hypothesis such as $\mathrm{NSP}_{D,q}$, but computational necessities, or quantization errors, could result in the use of a perturbed $\widetilde{D}$ as in the ℓ^q minimization in (IV.11) below. So, we further assume that:

- $\widetilde{D}$ is a $d \times n$ dictionary (perturbation of the intended dictionary D),
- A is an $m \times d$ full rank measurement matrix (our knowledge of the true matrix B).

The full rank condition is justified when redundant measurements are excluded. For fixed $0 < q \leq 1$, the following ℓ^q minimization problem reconstructs the approximation $\widetilde{z}$ to z_0 based on the noisy measurements y and the perturbations $\widetilde{D}$ and A of D and B, respectively

$$\min \|\widetilde{D}^* z\|_q^q \quad \text{subject to} \quad \|Az - y\|_2 \leq \varepsilon. \tag{IV.11}$$

The matrix A will satisfy hypotheses such as $\mathrm{NSP}_{D,q}$ or DRIP, but the perturbed matrix $\widetilde{D}$ used in (IV.11) introduces uncertainty and distortion into these hypotheses.

For Theorem IV.4.9 (therefore Theorem IV.4.8 as well), we assume that the dictionary D satisfies the frame inequality

$$\forall\, z \in \mathbb{R}^d, \quad \alpha\|z\|_2 \leq \|D^* z\|_2 \leq \beta\|z\|_2,$$

with frame constants $\beta \geq \alpha > 0$.

For direct comparision with [30], Theorem IV.4.10 assumes that the dictionary D satisfies the *Parseval frame* condition $DD^* = I$, but as noted in [30] there are extensions to general frames.

The following two theorems and their proofs first appeared in [25].

Theorem IV.4.9. *[25] Suppose that D is a $d \times n$ dictionary with frame constants $\beta \geq \alpha > 0$ and suppose that the $m \times d$ matrix A satisfies $NSP'_{D,q}$ with null space constant c. Moreover, suppose that the $d \times n$ matrix $\widetilde{D}$ satisfies $\|D^* - \widetilde{D}^*\|_{op} \leq \dfrac{5\alpha}{2^{1/q} n^{1/q-1/2}} \left(\dfrac{1-c}{10}\right)^{1/q}$ and that B is an $m \times d$ measurement matrix.*

If $z_0 \in \mathbb{R}^d$ and $y \in \mathbb{R}^m$ satisfy $\|y - Bz_0\|_2 \leq \varepsilon$ then any solution $\widetilde{z}$ to (IV.11) satisfies

$$\|\widetilde{z} - z_0\|_2 \leq \frac{2\beta}{5\nu_A} C_1 n^{1/q-1/2}\varepsilon + 2^{1/q} C_1 \sigma_s(D^* z_0)_q + 2^{1/q} C_1 n^{1/q-1/2} \|D^* - \widetilde{D}^*\|_{op} \|z_0\|_2$$
$$+ \frac{C_1}{\nu_A} n^{1/q-1/2} \left(\beta(1 + 2^{1/q}) + 2^{1/q} \|D^* - \widetilde{D}^*\|_{op} \right) \|A - B\|_{op} \|z_0\|_2. \qquad \text{(IV.12)}$$

Here ν_A is the smallest positive singular value of A. The constant C_1 is quantified in (IV.20) *and* (IV.25).

Theorem IV.4.10. *[25] Suppose that D is a $d \times n$ Parseval frame matrix and that the $d \times n$ matrix $\widetilde{D}$ satisfies $\|D^* - \widetilde{D}^*\|_{op} \leq \frac{1}{\sqrt{2}K_2} \left(\frac{n}{s}\right)^{1/2-1/q}$ for some constant K_2. Suppose that A and B are $m \times d$ matrices and that A satisfies D-RIP with $\delta_{7s} < \frac{6 - 3(2/3)^{2/q-2}}{6 - (2/3)^{2/q-2}}$.*

If $z_0 \in \mathbb{R}^d$ and $y \in \mathbb{R}^m$ satisfy $\|y - Bz_0\|_2 \leq \varepsilon$ then any solution $\widetilde{z}$ to (IV.11) *satisfies:*

$$\|\widetilde{z} - z_0\|_2 \leq C_5 \varepsilon + C_6 s^{1/2-1/q} \sigma_s(D^* z_0)_q + C_7 \left(\frac{n}{s}\right)^{1/q-1/2} \|D^* - \widetilde{D}^*\|_{op} \|z_0\|_2$$
$$+ \left(\frac{n}{s}\right)^{1/q-1/2} \frac{1}{\nu_A} \left(C_8 + C_9 \|D^* - \widetilde{D}^*\|_{op}\right) \|A - B\|_{op} \|z_0\|_2. \qquad \text{(IV.13)}$$

Here ν_A is the smallest positive singular value of A. Quantitative bounds on the constants C_5, C_6, C_7, C_8, C_9 and K_2 are contained in the proof, see (IV.41), (IV.42), (IV.43).

It is possible to formulate Theorems IV.4.9 and IV.4.10 using different choices of norms. Except for the term $\sigma_s(D^* z_0)_q$, the bounds in (IV.12) and (IV.13) are stated using the ℓ^2 norm and the associated operator norm and hence incur the discouraging constants $n^{1/q-1/2}$. Note that if we use $\sigma_s(D^* z_0)_2$ instead of the standard $\sigma_s(D^* z_0)_q$, we would also incur the constant $n^{1/q-1/2}$ in front of this term as well. Furthermore, $n^{1/q-1/2}$ is multiplied by the factor $1/\nu_A$ in the 4th term on the right hand side of (IV.12) and (IV.13) which is essentially $(\frac{m}{d})^{1/2}$. Indeed in the case where A is an $m \times d$ Gaussian random matrix with i.i.d. $\mathcal{N}(0, 1/m)$ entries, it is known that this choice of A satisfies D-RIP with high probability, see [30], when $m \gtrsim s \log(d/s)$. Moreover, the smallest singular value ν_A satisfies $\nu_A \gtrsim \left(\frac{d}{m}\right)^{1/2}$ with high probability greater than $1-2e^{-d/8}$, e.g., see Corollary 35 in [38].

Remark IV.4.11. We conclude this section with the following remarks:

(i) In the noise free case $\varepsilon = 0$, if A and D are exactly known (unperturbed), and D^*z_0 is exactly s-sparse, then $\widetilde{z}$ exactly reconstructs z_0, i.e., $\widetilde{z} = z_0$.

(ii) With no perturbations on the sensing matrix or the dictionary, and $q = 1$, we recover Theorem IV.4.6 and gain the same result. Furthermore, if D is the canonical basis, we obtain the now classical result in Theorem III.2.4.

(iii) When $D = I$ is the canonical basis and there are no perturbations of $D = I$, we obtain a result related to the one in [28].

(iv) If $D = I$, our proofs can be used to show Theorem III.4.9. However, if A satisfies NSP$_{D,q}$, we do not know yet whether A satisfies $\widetilde{D}$-NSP$_q$ even if $\|\widetilde{D} - D\|_{op}$ is small.

(v) We have shown that the third null space property NSP$_{D,q}$ is a sufficient condition for stability of a modified ℓ^q minimization when there is perturbation on the measurements, the measurement matrix, and the dictionary. It is natural to ask whether this condition is also necessary, like in the basis case. Unfortunately, the guess is that it is not necessary but we have not able to construct a counter example yet. Moreover, we are not even able to show whether this NSP$_{D,q}$ is a stronger or weaker condition than D-RIP. I believe more work needs to be done in the direction of the weaker NSP for dictionary.

Proof of Theorem IV.4.9:

Set $h = \widetilde{z} - z_0$. There are two main inqualities. One obtained from the null space property. The other from the ℓ^q minimization which is essentially the reverse of the null space property.

Combining these two, we obtain an upper bound on $\|D^*h\|_2$ in terms of the perturbations, and thus an upper bound for $\|h\|_2$ since D is a frame.

Step 1: Approximate NSP$_{D,q}$ for h. Note that h is expected to be almost in the null space of A. Thus we will decompose h as $h = a + \eta$ where $a \in \ker A$ and η small since, by Lemma III.4.3, $\|\eta\|_2 \leq \frac{1}{\nu_A}\|Ah\|_2$.

Since $a \in \ker A$ and A has NSP$_{D,q}$, let T be any index set such that $|T| \leq s$,

$$\|D_T^*h\|_q^q \leq \|D_T^*a\|_q^q + \|D_T^*\eta\|_q^q \leq c\|D_{T^c}^*a\|_q^q + \|D_T^*\eta\|_q^q \leq c\|D_{T^c}^*h\|_q^q + \|D^*\eta\|_q^q.$$

Thus, we get the approximate NSP$_{D,q}$ for h

$$\|D_T^*h\|_q^q \leq c\|D_{T^c}^*h\|_q^q + \|D^*\eta\|_q^q. \tag{IV.14}$$

Step 2: An approximate reversed inequality for h from ℓ^q minimization. Since A is a perturbation of B, $\|y - Az_0\|_2$ is not necessarily less than ε, i.e., z_0 is not necessarily feasible for program (IV.11). However, we can find a vector $z_0 + w$ that is very close to z_0 and is feasible. Specifically, since A is full rank by assumption, there exists w such that $Aw = (B - A)z_0$. Thus $\|A(z_0 + w) - y\|_2 = \|Bz_0 - y\|_2 \leq \varepsilon$, and $z_0 + w$ is feasible in Program (IV.11). Moreover, w is small since, by Lemma III.4.3, we can pick w such that

$$\|w\|_2 \leq \frac{1}{\nu_A}\|Aw\|_2 = \frac{1}{\nu_A}\|(B - A)z_0\|_2. \tag{IV.15}$$

Since $\widetilde{z}$ minimizes (IV.11) we have

$$\|\widetilde{D}^*\widetilde{z}\|_q^q \leq \|\widetilde{D}^*(z_0 + w)\|_q^q = \|\widetilde{D}_T^*z_0 + \widetilde{D}_T^*w\|_q^q + \|\widetilde{D}_{T^c}^*z_0 + \widetilde{D}_{T^c}^*w\|_q^q.$$

Moreover

$$\begin{aligned}\|\widetilde{D}^*\widetilde{z}\|_q^q &= \|\widetilde{D}^*(h+z_0)\|_q^q = \|\widetilde{D}_T^*h+\widetilde{D}_T^*z_0\|_q^q + \|\widetilde{D}_{T^c}^*h+\widetilde{D}_{T^c}^*z_0\|_q^q \\ &\geq \|\widetilde{D}_T^*z_0+\widetilde{D}_T^*w\|_q^q - \|\widetilde{D}_T^*h-\widetilde{D}_T^*w\|_q^q + \|\widetilde{D}_{T^c}^*h\|_q^q - \|\widetilde{D}_{T^c}^*z_0\|_q^q\end{aligned}$$

Combining the above two inequalities we get

$$\|\widetilde{D}_{T^c}^*h\|_q^q \leq \|\widetilde{D}_T^*h\|_q^q + 2\|\widetilde{D}_{T^c}^*z_0\|_q^q + \|\widetilde{D}^*w\|_q^q, \tag{IV.16}$$

Using the triangle inequality and (IV.16) we obtain the desired inequality:

$$\|D_{T^c}^*h\|_q^q \leq \|D_T^*h\|_q^q + \|D^*h-\widetilde{D}^*h\|_q^q + 2\|\widetilde{D}_{T^c}^*z_0\|_q^q + \|\widetilde{D}^*w\|_q^q. \tag{IV.17}$$

Step 3: Estimation of $\|D^*h\|_q$**.** Our ultimate goal is to estimate $\|h\|_2$. However, this can be done by first estimating $\|D^*h\|_q$ and thereby $\|D^*h\|_2$ and hence $\|h\|_2$ since D is a frame, by assumption. We do this, by combining the two inequalities in Steps 1 and 2, we get

$$\|D_T^*h\|_q^q \leq \frac{c}{1-c}\|D^*h-\widetilde{D}^*h\|_q^q + \frac{2c}{1-c}\|\widetilde{D}_{T^c}^*z_0\|_q^q + \frac{c}{1-c}\|\widetilde{D}^*w\|_q^q + \frac{1}{1-c}\|D^*\eta\|_q^q. \tag{IV.18}$$

By (IV.17) and (IV.18) we obtain

$$\begin{aligned}\|D^*h\|_q^q &= \|D_T^*h\|_q^q + \|D_{T^c}^*h\|_q^q \\ &\leq 2\|D_T^*h\|_q^q + \|D^*h-\widetilde{D}^*h\|_q^q + 2\|\widetilde{D}_{T^c}^*z_0\|_q^q + \|\widetilde{D}^*w\|_q^q \\ &\leq \frac{1+c}{1-c}\|D^*h-\widetilde{D}^*h\|_q^q + \frac{2+2c}{1-c}\|\widetilde{D}_{T^c}^*z_0\|_q^q + \frac{2}{1-c}\|D^*\eta\|_q^q + \frac{1+c}{1-c}\|\widetilde{D}^*w\|_q^q.\end{aligned} \tag{IV.19}$$

Step 4: Estimation of $\|h\|_2$**.** Rewriting the term $\widetilde{D}^*_{T^c}z_0$ in (IV.19) and using the fact that D is a frame and the inequality (II.3) we get

$$\begin{aligned}\|h\|_2 \leq & \frac{1}{\alpha}\|D^*h\|_2 \leq \frac{1}{\alpha}\|D^*h\|_q \\ \leq & Cn^{1/q-1/2}\|D^*-\widetilde{D}^*\|_{op}\|h\|_2 + C\|D^*\eta\|_q \\ & + 2^{1/q}C\left[\|\widetilde{D}^*z_0 - D^*z_0\|_q + \|D^*_{T^c}z_0\|_q\right] + C\|\widetilde{D}^*w\|_q,\end{aligned}$$

where

$$C = \frac{1}{5\alpha}\left(\frac{10}{1-c}\right)^{1/q}. \tag{IV.20}$$

This leads to the estimation of $\|h\|_2$ in terms of the perturbations

$$(1-\rho)\|h\|_2 \leq C\|D^*\eta\|_q + 2^{1/q}C\left[\|\widetilde{D}^*z_0 - D^*z_0\|_q + \|D^*_{T^c}z_0\|_q\right] + C\|\widetilde{D}^*w\|_q \tag{IV.21}$$

where $\rho := 2^{1/q}Cn^{1/q-1/2}\|D^*-\widetilde{D}^*\|_{op}$.

Step 5: Estimation of the perturbations. *1) Estimation of* $\|D^*\eta\|_q$. Using the fact that $\|\eta\|_2 \leq \dfrac{1}{\nu_A}\|Ah\|_2$, and

$$\|Ah\|_2 = \|A\tilde{z} - Az_0\|_2 \leq \|A\tilde{z} - y\|_2 + \|y - Bz_0\|_2 + \|Bz_0 - Az_0\|_2 \leq 2\varepsilon + \|(A-B)z_0\|_2,$$

we get

$$\|D^*\eta\|_q \leq n^{1/q-1/2}\|D^*\eta\|_2 \leq n^{1/q-1/2}\beta\|\eta\|_2 \leq n^{1/q-1/2}\frac{\beta}{\nu_A}(2\varepsilon + \|(A-B)z_0\|_2). \tag{IV.22}$$

2) Estimation of $\|\widetilde{D}^*w\|_q$. Using the upper frame bound β of D we get

$$
\begin{aligned}
\|\widetilde{D}^*w\|_q^q &\le \|\widetilde{D}^*w - D^*w\|_q^q + \|D^*w\|_q^q \\
&\le (n^{1/q-1/2}\|\widetilde{D}^*w - D^*w\|_2)^q + (n^{1/q-1/2}\|D^*w\|_2)^q \\
&\le n^{1-q/2}\|w\|_2^q(\|\widetilde{D}^* - D^*\|_{op}^q + \beta^q),
\end{aligned}
$$

from which we get (using (IV.15))

$$
\|\widetilde{D}^*w\|_q \le \frac{(2n)^{1/q-1/2}}{\nu_A}(\|\widetilde{D}^* - D^*\|_{op} + \beta)\|(B-A)z_0\|_2 \tag{IV.23}
$$

Step 6: Final estimate of $\|h\|_2$ Substitute (IV.22) and (IV.23) into (IV.21) and letting T be the index set corresponding to the s largest magnitude entries of D^*z_0, we get

$$
\begin{aligned}
\|h\|_2 \le \frac{2\beta}{5\nu_A}C_1 n^{1/q-1/2}\varepsilon + 2^{1/q}C_1\sigma_s(D^*z_0)_q + 2^{1/q}C_1 n^{1/q-1/2}\|D^* - \widetilde{D}^*\|_{op}\|z_0\|_2 \\
+ \frac{C_1}{\nu_A}n^{1/q-1/2}\left(\beta(1+2^{1/q}) + 2^{1/q}\|\widetilde{D}^* - D^*\|_{op}\right)\|A-B\|_{op}\|z_0\|_2
\end{aligned} \tag{IV.24}
$$

where

$$
C_1 = \frac{C}{1 - 2^{1/q}Cn^{1/q-1/2}\|D^* - \widetilde{D}^*\|_{op}} \tag{IV.25}
$$

is positive if $\|D^* - \widetilde{D}^*\|_{op} < 2^{-1/q}C^{-1}n^{1/2-1/q}$. □

Proof of Theorem IV.4.10

This proof is inspired by and follows closely the proof of Theorem 1.4 in [30]. Set $h = \widetilde{z} - z_0$.

Step 1: Consequence of the ℓ^q ***minimization.*** As in step 2 of the proof of Threom IV.4.9, let T be any index set such that $|T| \le s$, we get

$$\|\widetilde{D}^*_{T^c}h\|_q^q \leq \|\widetilde{D}^*_T h\|_q^q + 2\|\widetilde{D}^*_{T^c}z_0\|_q^q + \|\widetilde{D}^* w\|_q^q, \tag{IV.26}$$

where as before $Aw = (B-A)z_0$, $z_0 + w$ is feasible and

$$\|w\|_2 \leq \frac{1}{\nu_A}\|Aw\|_2 = \frac{1}{\nu_A}\|(B-A)z_0\|_2. \tag{IV.27}$$

As typically done in compressed sensing proofs using RIP, we divide the coordinates T^c into sets of size M (to be chosen later) in order of decreasing magnitude of $\widetilde{D}^*_{T^c}h$. Call these sets $T_1, T_2, \ldots, T_r$ and for simplicity set $T_{01} = T \cup T_1$. By construction:

$$\|\widetilde{D}^*_{T_{j+1}}h\|_\infty \leq \|\widetilde{D}^*_{T_j}h\|_1/M \leq M^{1-1/q}\|\widetilde{D}^*_{T_j}h\|_q/M, \quad j \geq 1$$

which yields

$$\|\widetilde{D}^*_{T_{j+1}}h\|_2^2 \leq M^{1-2/q}\|\widetilde{D}^*_{T_j}h\|_q^2. \tag{IV.28}$$

Using the triangle inequality, (II.3), (IV.26) and (IV.28), we have

$$\begin{aligned}
\sum_{j\geq 2}\|D^*_{T_j}h\|_2^q &\leq \sum_{j\geq 2}\left(\|D^*_{T_j}h - \widetilde{D}^*_{T_j}h\|_2 + M^{1/2-1/q}\|\widetilde{D}^*_{T_{j-1}}h\|_q\right)^q \\
&\leq \sum_{j\geq 2}\|D^*_{T_j}h - \widetilde{D}^*_{T_j}h\|_2^q + \sum_{j\geq 1} M^{q/2-1}\|\widetilde{D}^*_{T_j}h\|_q^q \\
&\leq r^{1-q/2}(\sum_{j\geq 2}\|D^*_{T_j}h - \widetilde{D}^*_{T_j}h\|_2^2)^{q/2} + \sum_{j\geq 1} M^{q/2-1}\|\widetilde{D}^*_{T_j}h\|_q^q \\
&= r^{1-q/2}\|D^*_{T_{01}^c}h - \widetilde{D}^*_{T_{01}^c}h\|_2^q + M^{q/2-1}\|\widetilde{D}^*_{T^c}h\|_q^q \\
&\leq r^{1-q/2}\|D^*_{T_{01}^c}h - \widetilde{D}^*_{T_{01}^c}h\|_2^q + M^{q/2-1}(\|\widetilde{D}^*_T h\|_q^q + 2\|\widetilde{D}^*_{T^c}z_0\|_q^q + \|\widetilde{D}^* w\|_q^q).
\end{aligned}$$

Taking the qth root of the previous inequality, writing $\widetilde{D}^*_T h = \widetilde{D}^*_T h - D^*_T h + D^*_T h$, and using the triangle inequality we get

$$\sum_{j\geq 2}\|D^*_{T_j}h\|_2 \leq \big(\sum_{j\geq 2}\|D^*_{T_j}h\|_2^q\big)^{1/q} = \rho(\|D^*_T h\|_2+\eta), \tag{IV.29}$$

where

$$\rho = 4^{1/q-1}(s/M)^{1/q-1/2}$$

and

$$\eta = (\frac{n}{s})^{1/q-1/2}\|D^*_{T^c_{01}}h - \widetilde{D}^*_{T^c_{01}}h\|_2 + \|\widetilde{D}^*_T h - D^*_T h\|_2 + s^{1/2-1/q}(2^{1/q}\|\widetilde{D}^*_{T^c}z_0\|_q + \|\widetilde{D}^* w\|_q). \tag{IV.30}$$

The term η can be made small by controlling $\|D^* - \widetilde{D}^*\|_{op}$, and w (through $\|A-B\|_{op}$) since the remaining term $\|\widetilde{D}^*_{T^c}z_0\|_q$ is small by assumption.

Step 2: The use of D-RIP. The inequality (IV.29) is exactly the same as the one in Lemma 2.2 of [30] except that the expressions for ρ and η are different since these expressions now contain terms that are due to perturbations of D and B. Thus, using Lemmas 2.4, 2.5 and 2.6 of [30], and the use of D-RIP combined with (IV.29) will give the following two inequalities

$$\sqrt{1-\delta_{s+M}}\|DD^*_{T_{01}}h\|_2 \leq \rho\sqrt{1+\delta_M}(\|h\|_2+\eta) + 2\varepsilon + \|(A-B)z_0\|_2, \tag{IV.31}$$

$$\sqrt{1-\frac{c_1}{2}-\rho^2-\rho^2 c_2}\|h\|_2 \leq \frac{1}{\sqrt{2c_1}}\|DD^*_{T_{01}}h\|_2 + \rho\eta\sqrt{1+\frac{1}{c_2}}, \tag{IV.32}$$

where we have used $\|Ah\|_2 \leq 2\varepsilon + \|(A-B)z_0\|_2$, instead of $\|Ah\|_2 \leq 2\varepsilon$ in Lemma 2.3 of [30].

Combining (IV.31) and (IV.32) to eliminate $\|DD^*_{T_{01}}h\|$ yields

$$\|h\|_2 \leq K_1(2\varepsilon + \|(A-B)z_0\|_2) + K_2\eta, \tag{IV.33}$$

where

$$K_1 = \frac{\sqrt{1-\delta_{s+M}}}{\sqrt{2c_1(1-\delta_{s+M})(1-\frac{c_1}{2}-\rho^2-\rho^2 c_2)}-\rho\sqrt{1+\delta_M}}, \tag{IV.34}$$

$$K_2 = \frac{\rho\sqrt{1+\delta_M}+\rho\sqrt{2c_1(1-\delta_{s+M})(1+1/c_2)}}{\sqrt{2c_1(1-\delta_{s+M})(1-\frac{c_1}{2}-\rho^2-\rho^2 c_2)}-\rho\sqrt{1+\delta_M}}, \tag{IV.35}$$

and the particular choice of the free parameters c_1, c_2, M making the expressions for K_1 and K_2 valid and positive will be chosen at the end of the proof.

Step 3: $\|h\|_2$ ***is small if*** $\|D^* - \widetilde{D}^*\|_{op}$ ***is small.*** Inequality (IV.33) is not the desired estimate of $\|h\|_2$ yet since h is still included in the term η. Therefore we need to estimate η. Obviously $(\frac{n}{s})^{1/q-1/2} \geq 1$, so

$$\begin{aligned}\eta &\leq \sqrt{2}(\frac{n}{s})^{1/q-1/2}\|D^*h-\widetilde{D}^*h\|_2 + s^{1/2-1/q}(2^{1/q}\|\widetilde{D}^*_{T^c}z_0\|_q + \|\widetilde{D}^*w\|_q) \\ &\leq \sqrt{2}(\frac{n}{s})^{1/q-1/2}\|D^*-\widetilde{D}^*\|_{op}\|h\|_2 + s^{1/2-1/q}(2^{1/q}\|\widetilde{D}^*_{T^c}z_0\|_q + \|\widetilde{D}^*w\|_q)\end{aligned} \tag{IV.36}$$

Substituting (IV.36) into (IV.33) and combining $\|h\|_2$ terms gives

$$(1-l)\|h\|_2 \leq K_1(2\varepsilon + \|(A-B)z_0\|_2) + K_2 s^{1/2-1/q}(2^{1/q}\|\widetilde{D}^*_{T^c}z_0\|_q + \|\widetilde{D}^*w\|_q) \tag{IV.37}$$

where

$$l = \sqrt{2}(\frac{n}{s})^{1/q-1/2}K_2\|D^*-\widetilde{D}^*\|_{op} \tag{IV.38}$$

Therefore (IV.37) gives an upperbound of $\|h\|_2$ if $\|D^*-\widetilde{D}^*\|_{op}$ is small enough such that $l < 1$.

Step 4: Estimation of perturbations. The estimation of $\|\widetilde{D}^*w\|_q$ is the same as (IV.23) in step 5 of of the proof of Theorem IV.4.9, except here $\beta = 1$:

$$\|\widetilde{D}^* w\|_q \leq \frac{(2n)^{1/q-1/2}}{\nu_A}(\|\widetilde{D}^* - D^*\|_{op} + 1)\|(B-A)z_0\|_2. \tag{IV.39}$$

For $\|\widetilde{D}^*_{T^c} z_0\|_q$ we have

$$\begin{aligned}\|\widetilde{D}^*_{T^c} z_0\|_q^q &\leq \|\widetilde{D}^*_{T^c} z_0 - D^*_{T^c} z_0\|_q^q + \|D^*_{T^c} z_0\|_q^q \\ &\leq n^{1-q/2}\|\widetilde{D}^* - D^*\|_{op}^q \|z_0\|_2^q + \|D^*_{T^c} z_0\|_q^q.\end{aligned}$$

Taking the qth root we get

$$\|\widetilde{D}^*_T z_0\|_q \leq (2n)^{1/q-1/2}\|\widetilde{D}^* - D^*\|_{op}\|z_0\|_2 + 2^{1/q-1}\|D^*_{T^c} z_0\|_q. \tag{IV.40}$$

Step 5: Final estimate of $\|h\|_2$**.** Substituting (IV.39) and (IV.40) into (IV.37) and letting T be the index set corresponding to the s largest magnitude entries of $D^* z_0$ yields

$$\begin{aligned}\|\widetilde{z} - z_0\|_2 \leq{} & \frac{2K_1}{1-l}\varepsilon + \frac{K_2}{1-l}(s/4)^{1/2-1/q}\|D^*_{T^c} z_0\|_q \\ & + \left(\frac{K_1}{1-l} + \frac{K_2}{\nu_A(1-l)}\left(\frac{2n}{s}\right)^{1/q-1/2}(1 + \|D^* - \widetilde{D}^*\|_{op})\right)\|B-A\|_{op}\|z_0\|_2 \\ & + \frac{\sqrt{2}K_2}{1-l}\left(\frac{4n}{s}\right)^{1/q-1/2}\|D^* - \widetilde{D}^*\|_{op}\|z_0\|_2\end{aligned} \tag{IV.41}$$

Step 6: The choice of the parameters for K_1 ***and*** K_2 ***in Step 2.*** It only remains to choose the parameters c_1, c_2 and M so that K_1 and K_2 are positive. The same as in [30], we choose $c_1 = 1, M = 6s$ and take c_2 arbitrarily small so that the denominator of K_1 and K_2 is positive if

$$\delta_{7s} < a(q) := \frac{6 - 3(2/3)^{2/q-2}}{6 - (2/3)^{2/q-2}}.$$

In this case,

$$K_1 = \frac{\sqrt{1-\delta_{7s}}}{\sqrt{2(1-\delta_{7s})(\frac{1}{2}-\frac{3}{8}(\frac{2}{3})^{2/q}(1+c_2))}-\frac{\sqrt{6}}{4}(\frac{2}{3})^{1/q}\sqrt{1+\delta_{7s}}}, \tag{IV.42}$$

$$K_2 = \frac{\frac{\sqrt{6}}{4}(\frac{2}{3})^{1/q}[\sqrt{1+\delta_{7s}}+\sqrt{2(1-\delta_{7s})(1+1/c_2)}\,]}{\sqrt{2(1-\delta_{7s})(\frac{1}{2}-\frac{3}{8}(\frac{2}{3})^{2/q}(1+c_2))}-\frac{\sqrt{6}}{4}(\frac{2}{3})^{1/q}\sqrt{1+\delta_{7s}}}. \tag{IV.43}$$

(choose c_2 so that K_1, K_2 are positive)

$a(1) = 0.6$ which coincides the result in [30]. Notice $a(q)$ tends to be 1 as $q \to 0$. For example, $a(q) = 0.84$ when $q = 1/2$. $\square$

CHAPTER V

THE KACZMARZ ALGORITHM WITH RANDOM MEASUREMENTS

The Kaczmarz algorithm is a classical iterative method for solving an overdetermined consistent linear system $\Phi x = y$. The algorithm is based on the mechanism of projection onto convex sets and also falls into the class of row-action methods. Within the spectrum of linear solvers, some key features of the Kaczmarz algorithm include its scalability and its simplicity; a single inner product is the dominant computation in each step of the algorithm. This has made the Kaczmarz algorithm a good candidate for high dimensional problems.

The Kaczmarz algorithm and its variants appear in a wide variety of settings. For example, it has been applied to computer tomography and image processing in [39, 40], and has been used for sparse signal recovery in compressed sensing in [41]. In signal processing, the closely related Rangan-Goyal algorithm is used for consistent reconstruction of quantized data, see [42, 43].

The main aim of this chapter is to study the issue of *almost sure convergence* for the Kaczmarz algorithm with random measurement vectors $\{\varphi_n\}_{n=1}^{\infty}$. We prove that the Kaczmarz algorithm *almost surely* converges exponentially fast and we provide quantitative bounds on the associated convergence rate.

The chapter is organized as follows: Section V.2 provides definitions and background properties of the random measurement vectors $\{\varphi_n\}_{n=1}^{\infty}$. Section V.3 gives basic formulas for the error $\|x - x_n\|$ in the Kaczmarz algorithm, and Section V.4 gives basic bounds on the moments $\mathbb{E}\|x - x_n\|^{2s}$ with $s > 0$. The main results appear in Section V.5 and Section V.6. The first main result, Theorem V.5.3 in Section V.5, provides sharp almost sure rates of exponential convergence for the Kaczmarz algorithm in the important case when the normalized measurement vectors $\varphi_n/\|\varphi_n\|$ are independent and uniformly distributed on $\mathbb{S}^{d-1}$ (for example, this applies to random vectors with i.i.d. Gaussian entries). Our next main results, Theorem V.6.2 and Theorem V.6.3 in Section V.6, provide quantitive bounds on the rate of almost sure exponential convergence for general classes of random measurement vectors.

V.1 Problem formulation

Given an overdetermined consistent linear system $\Phi x = y$, where $x \in \mathbb{R}^d$ is the signal we are trying to recover, and $y \in \mathbb{R}^N$ is the known linear measurement, the original Kaczmarz algorithm is used to approximately recover x from the linear measurements $\{y_n\}_{n=1}^N$.

We denote the rows of Φ by $\varphi_1^*, \varphi_2^*, \ldots, \varphi_N^*$. The original work of Kaczmarz [44] starts with an arbitrary initial estimate $x_0 \in \mathbb{R}^d$ and produces approximate solutions $x_n \in \mathbb{R}^d$ by the following iteration:

$$\forall\, n \geq 1, \quad x_n = x_{n-1} + \frac{y_n - \langle \varphi_i, x_{n-1} \rangle}{\|\varphi_n\|_2^2}\, \varphi_i, \tag{V.1}$$

where $i = n$ mod N. This method sweeps through the rows of Φ in a cyclic manner.

Geometrically, this is an iterative projection algorithm that updates the estimate $x_{n-1} \in \mathbb{R}^d$ by orthogonally projecting it onto the affine hyperplane

$$H_i = \{u \in \mathbb{R}^d : \langle u, \varphi_i \rangle = y_i\}.$$

The initial convergence analysis for this algorithm in [44] focuses on finite dimensional spaces, but there are also subsequent extensions to infinite dimensional spaces, e.g., in [45–47].

It is well known that the algorithm produces monotonically improving approximations as the iteration number i creases. Specifically, for any $x \in \mathbb{R}^d$ and $\{\varphi_n\}_{n=1}^N \subset \mathbb{R}^d$ and any initial estimate $x_0 \in \mathbb{R}^d$ the Kaczmarz algorithm satisfies

$$\|x - x_{n+1}\|_2 \leq \|x - x_n\|_2. \tag{V.2}$$

In fact, this basic fact will follow as a corollary to Proposition V.3.1.

Kaczmarz showed that iteratively cycling through the system produces estimates x_n that are guaranteed to converge to x:

$$\lim_{n\to\infty} \|x - x_n\|_2 = 0.$$

However, it can be difficult to quantify the associated rates of convergence. Geometric considerations imply that the specific rate at which the Kaczmarz algorithm converges depends strongly on the order in which measurements y_n are entered into the algorithm, and in certain circumstances the convergence can be quite slow. Motivated by this, Strohmer and Vershynin [48, 49] investigated a *randomized* version of the Kaczmarz algorithm where the new information (y_n, φ_n) processed at each step of the algorithm (V.4) is randomly selected from among the N measurements. They proved that this randomized approach achieves mean squared error with a rate that is quantifiable in terms of a particular matrix condition number $\kappa(\Phi)$.

Theorem V.1.1 (Randomized Kaczmarz algorithm [49])**.** *Let $\Phi x = y$ be a linear equation system, and x_0 be an arbitrary initial approximation, for $n = 1, 2, \ldots$, compute x_n as in* (V.1)*, where $i(n)$ is chosen from the set $\{1, 2, \ldots, N\}$ at random, with probability proportional to $\|a_{i(n)}\|_2^2$. Then x_n converges to x in expectation, with the average error*

$$\mathbb{E}\|x - x_n\|_2^2 \leq (1 - \kappa(\Phi)^{-2})^n \; \|x - x_0\|_2^2. \tag{V.3}$$

The theoretical and numerical analysis of the randomized Kaczmarz algorithm in [49] shows that this method converges exponentially fast and has features that are competitive with (and sometimes superior to) standard approaches such as the conjugate gradient method.

In addition to the analysis of convergence rates, there is recent work that highlights other favorable properties of the Kaczmarz algorithm. The work in [50] shows that the algorithm is robust against noise in the measurements y_n. There is work in [51] on accelerating the convergence of the Kaczmarz algorithm in high dimensions with help of the Johnson-Lindenstrauss Lemma. The discussion in [52–54] addresses choices of randomization for the algorithm.

V.1.1 Settings

In our setting, instead of having a matrix Φ, we have certain distributions, from which each vector of $\{\varphi_n\}_{n=1}^{\infty}$ is drawn. We wish to recover the signal x from the linear measurements $y_n = \langle x, \varphi_n \rangle, n \geq 1$.

The same formula from the Kaczmarz algorithm will be used:

$$\forall\, n \geq 1, \quad x_n = x_{n-1} + \frac{y_n - \langle \varphi_n, x_{n-1} \rangle}{\|\varphi_n\|_2^2}\, \varphi_n, \tag{V.4}$$

There is no need to cycle through $\{\varphi_n\}'s$ since we have an infinite number of them and each of them is chosen from certain distribution. The "infinite" here is a very loose concept, because we can have repeated measurements. For example, the randomized Kaczmarz algorithm by Strohmer and Vershynin is a special case of our settings. We will simply choose φ_n from a discrete distribution, with certain probability to choose certain row of the matrix Φ.

We would like to study the issue of *almost sure convergence* for the Kaczmarz algorithm with these random measurement vectors $\{\varphi_n\}_{n=1}^{\infty}$.

V.2 Random measurements

This section will discuss conditions on the random measurement vectors $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ which will be needed in our analysis of almost sure convergence in the Kaczmarz algorithm.

Suppose that the random measurement vectors $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ are used for the Kaczmarz algorithm (V.4). We always assume that each φ_n is almost surely nonzero, $\Pr[\,\|\varphi_n\| = 0\,] = 0$, to ensure that the Kaczmarz iteration (V.4) is well defined. Since most of our error analysis only involves the normalized random vectors $\varphi_n/\|\varphi_n\|_2$, the assumption that each φ_n is almost surely nonzero also guarantees that each $\varphi_n/\|\varphi_n\|_2$ is well defined.

Our general analysis of the Kaczmarz algorithm will require that the normalized random measurement vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ be independent but not necessarily identically distributed. Since it is common in practice to make assumptions directly on the measurement vectors $\{\varphi_n\}_{n=1}^{\infty}$, it is useful to note that independence of the measurement vectors $\{\varphi_n\}_{n=1}^{\infty}$ is a strictly stronger assumption than independence of the normalized measurement vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$. Our analysis will allow the possibility of non-independent $\{\varphi_n\}_{n=1}^{\infty}$, but will always require that $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ be independent.

Lemma V.2.1. *If the random vectors $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ are independent and almost surely nonzero, then the normalized random vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are also independent.*

As mentioned above, the converse of Lemma V.2.1 is not true.

Example V.2.2. Let θ_1, θ_2 be independent random variables that are uniform on $[0, 2\pi)$. Define the random vectors $\varphi_1 = (\cos\theta_1, \sin\theta_1)$ and φ_2 as follows

$$\varphi_2 = \begin{cases} (\cos\theta_2, \sin\theta_2), & \text{if } 0 \le \theta_1 < \pi, \\ 2(\cos\theta_2, \sin\theta_2), & \text{if } \pi \le \theta_1 < 2\pi. \end{cases}$$

Then $\varphi_1/\|\varphi_1\|_2$ and $\varphi_2/\|\varphi_2\|_2$ are independent, but φ_1, φ_2 are not independent.

Our analysis of almost sure convergence will involve the following frame-type assumptions on the normalized random measurement vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$.

Definition V.2.3 (Kaczmarz bound of order s)**.** Let $s>0$ be fixed. The unit-norm random vector $u \in \mathbb{R}^d$ has the *Kaczmarz bound* $0<\alpha<1$ *of order* s if

$$\forall x \in \mathbb{S}^{d-1}, \quad \left(\mathbb{E}\left(1-|\langle x,u\rangle|^2\right)^s\right)^{1/s} \leq \alpha. \tag{V.5}$$

If (V.5) holds with equality then we shall say that the Kaczmarz bound is *tight*.

Convergence rates in the Kaczmarz algorithm will depend on the specific value of the Kaczmarz bound $0<\alpha<1$. Qualitatively, if $u \in \mathbb{R}^d$ is a given random vector and $s>0$ is fixed, note that (V.5) holds for some $0<\alpha<1$ if and only if u is not concentrated on a subspace of $\mathbb{R}^d$ with positive codimension.

In the special case when $s=1$, Definition V.2.3 reduces to the notion of probabilistic frame and deserves further mention.

Definition V.2.4. The random vector $u \in \mathbb{R}^d$ has the *probabilistic lower frame bound* $\beta>0$ if

$$\forall x \in \mathbb{R}^d, \quad \mathbb{E}|\langle x,u\rangle|^2 \geq \beta\|x\|_2^2. \tag{V.6}$$

The random vector $u \in \mathbb{R}^d$ is a *tight probabilistic frame* if (V.6) holds with equality

$$\forall x \in \mathbb{R}^d, \quad \mathbb{E}|\langle x,u\rangle|^2 = \beta\|x\|_2^2. \tag{V.7}$$

If $u \in \mathbb{S}^{d-1}$ is a unit-norm tight probabilistic frame we shall simply say that u is *isotropic*.

Thus, a Kaczmarz bound $0<\alpha<1$ of order $s=1$ corresponds to a probabilistic frame bound $\beta = 1-\alpha$. A condition similar to (V.6) was used for the analysis of the Rangan-Goyal algorithm in [42], cf. [43]. Random vectors satisfying the probabilistic tight frame condition (V.7) are fully characterized in [55] and it is shown that if u is isotropic then the constant β in (V.7) must satisfy

$$\beta = \beta_d = 1/d.$$

Interested readers can find more properties of probabilistic frames in [55].

Example V.2.5. If $u \in \mathbb{R}^d$ is uniformly distributed on $\mathbb{S}^{d-1}$ then u is isotropic.

Example V.2.6. Let $\{f_n\}_{n=1}^N \subset \mathbb{R}^d$ be a deterministic unit-norm tight frame for $\mathbb{R}^d$, i.e.,

$$\forall x \in \mathbb{R}^d, \quad \|x\|_2^2 = \frac{d}{N}\sum_{n=1}^N |\langle x, f_n\rangle|^2.$$

If the discrete random vector $u \in \mathbb{R}^d$ is defined to be uniformly distributed on the set $\{f_n\}_{n=1}^N$, then u satisfies (V.7). For example, if $\{f_n\}_{n=1}^d \subset \mathbb{R}^d$ is an orthonormal basis for $\mathbb{R}^d$ and $u \in \mathbb{R}^d$ randomly selects an element of this basis, then the random vector u satisfies (V.7).

Example V.2.7. Let F be a full rank $N \times d$ matrix and let $\{f_n\}_{n=1}^N \subset \mathbb{R}^d$ be the rows of F. Let $u \in \mathbb{R}^d$ be the discrete random vector with the probability mass function

$$\forall\, 1 \le k \le N, \quad \Pr[u = f_k] = \|f_k\|_2^2 / \sum_{n=1}^N \|f_n\|_2^2.$$

It was shown in [49] that u has a probabilistic lower frame bound $\beta > 0$ that satisfies

$$\beta \ge \left(\frac{1}{\kappa(F)}\right)^2 = \frac{1}{\|F\|_{\mathrm{Fr}}^2 \|F^{-1}\|_2^2}. \tag{V.8}$$

For our analysis of almost sure convergence, it will be useful to have a version of Definition V.2.3 for the limiting case $s = 0$. The following standard lemma will be useful for this, for example, see page 71 of [56]. We also provide a proof in the Appendix.

Lemma V.2.8. *Let η be a random variable such that $\mathbb{E}|\eta|^s < \infty$ for some $s > 0$. Then*

$$\inf_{s>0}(\mathbb{E}|\eta|^s)^{1/s} = \lim_{s\to 0}(\mathbb{E}|\eta|^s)^{1/s} = \exp(\mathbb{E}\log|\eta|).$$

Corollary V.2.9. *If $u \in \mathbb{S}^d$ is a random unit-vector then*

$$\forall x \in \mathbb{S}^d, \quad \lim_{s\to 0}(\mathbb{E}(1-|\langle x,u\rangle|^2)^s)^{1/s} = \exp\left(\mathbb{E}[\log(1-|\langle x,u\rangle|^2)]\right). \tag{V.9}$$

In both Lemma V.2.8 and Corollary V.2.9, we interpret $\exp(-\infty) = 0$. Motivated by Corollary V.2.9, the following definition will naturally arise in our analysis of almost sure convergence in the Kaczmarz algorithm in Section V.6.2.

Definition V.2.10 (Logarithmic Kaczmarz bound)**.** The random unit-vector $u \in \mathbb{S}^{d-1}$ has a *logarithmic Kaczmarz bound* $0 < \rho < 1$ if

$$\forall x \in \mathbb{S}^{d-1}, \quad \exp\left(\mathbb{E}[\log(1-|\langle x,u\rangle|^2)]\right) \leq \rho. \tag{V.10}$$

We say that $u \in \mathbb{S}^{d-1}$ has a *tight logarithmic Kaczmarz bound* ρ if (V.10) holds with equality.

For perspective, $\mathbb{E}[\log(1-|\langle x,u\rangle|^2)]$ in (V.10) can be expressed as a perturbation of the familiar logarithmic potential [57] by

$$\forall x \in \mathbb{S}^{d-1}, \quad \mathbb{E}[\log(1-|\langle x,u\rangle|^2)] = 2\mathbb{E}[\log\|x-u\|_2] + \mathbb{E}[\log(1-4^{-1}\|x-u\|_2^2)].$$

Note that for $x,u \in \mathbb{S}^{d-1}$, $L(x,u) = \log(1-|\langle x,u\rangle|^2)$ is singular at both $u = x$ and $u = -x$.

Let F be a full rank $N \times d$ matrix and u be the discrete random vector defined in Example V.2.7. We can easily associate the Kaczmarz bound of u with the condition number of F, and consequently obtain the convergence rate in terms of this condition number (see Theorem V.4.1 and Theorem V.6.2). Therefore it is useful to relate the logarithmic Kaczmarz bound of u to the condition number of F as well. One trivial fact by Lemma V.2.8 (let $s = 1$) and Example V.2.7 is that $\rho \leq 1 - \kappa(F)^{-2}$, but an improved bound is not known yet and further investigation would be interesting.

Random vectors $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ with the following properties will play an important role in Section V.5. For convenience we collect these properties in the following definition:

Definition V.2.11. We shall say that the random vector $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ have the *normalized independence and uniformity (NIU) property* if each φ_n is almost surely nonzero and if the normalized vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent and uniformly distributed on $\mathbb{S}^{d-1}$.

Lemma V.2.1 and Example V.2.2 provide insight into the assumption in Definition V.2.11 that $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ be independent. The following examples provide some insight into the condition that each $\varphi_n/\|\varphi_n\|_2$ is uniformly distributed on $\mathbb{S}^{d-1}$.

Example V.2.12. Let $u \in \mathbb{R}^d$ be a uniform random vector on $\mathbb{S}^{d-1}$. We shall consider a random vector $\varphi \in \mathbb{R}^d$ to be *radial* if it is of the form $\varphi = ru$ where $r \in \mathbb{R}$ is a random variable that is independent of u. If the random vector $\varphi \in \mathbb{R}^d$ is radial and almost surely nonzero, then $\varphi/\|\varphi\|_2$ is uniform on $\mathbb{S}^{d-1}$. For example, if $\varphi \in \mathbb{R}^d$ is a random Gaussian vector with i.i.d. $N(0,1)$ entries then $\varphi/\|\varphi\|_2$ is uniformly distributed on $\mathbb{S}^{d-1}$.

However, if $\varphi/\|\varphi\|_2$ is uniform on $\mathbb{S}^{d-1}$, φ does not need to be radial.

Example V.2.13. Let θ be uniformly distributed on $[0, 2\pi)$. Define the random vector $\varphi \in \mathbb{R}^2$ by

$$\varphi = \begin{cases} (\cos\theta, \sin\theta), & \text{if } 0 \leq \theta < \pi, \\ 2(\cos\theta, \sin\theta), & \text{if } \pi \leq \theta < 2\pi. \end{cases}$$

Then $\varphi/\|\varphi\|_2$ is uniformly distributed on $\mathbb{S}^1$ but $\varphi \in \mathbb{R}^2$ is not radial.

V.3 Basic error formulas for the Kaczmarz algorithm

The following error formulas for the Kaczmarz algorithm will play an important role throughout this chapter.

Proposition V.3.1. *Suppose that $x \in \mathbb{R}^d$ and that the measurement vectors $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ are nonzero. Suppose that the measurements $y_n = \langle x, \varphi_n \rangle$, with $n \geq 1$, are used as input to the Kaczmarz algorithm with initial estimate $x_0 \in \mathbb{R}^d$.*

The error $z_n = x - x_n$ after the nth iteration of the Kaczmarz algorithm satisfies

$$\|z_n\|^2 = \|z_{n-1}\|_2^2 - \left| \left\langle z_{n-1}, \frac{\varphi_n}{\|\varphi_n\|_2} \right\rangle \right|^2 \tag{V.11}$$

and

$$\|z_n\|_2^2 = \|z_0\|_2^2 \prod_{k=1}^{n} \left(1 - \left| \left\langle \frac{z_{k-1}}{\|z_{k-1}\|}, \frac{\varphi_k}{\|\varphi_k\|} \right\rangle \right|^2 \right). \tag{V.12}$$

We adopt the convention that $z_{k-1}/\|z_{k-1}\|_2 = 0$ is the zero vector when $\|z_{k-1}\|_2 = 0$.

Proof. The defining iteration (V.4) can be written in terms of the error $z_n = x - x_n$ as

$$z_n = z_{n-1} - \left\langle z_{n-1}, \frac{\varphi_n}{\|\varphi_n\|} \right\rangle \frac{\varphi_n}{\|\varphi_n\|}.$$

Since φ_n is orthogonal to z_{n-1}, the equation (V.11) now follows

$$\|z_n\|_2^2 = \|z_n\|_2^2 - \left| \left\langle z_{n-1}, \frac{\varphi_n}{\|\varphi_n\|_2} \right\rangle \right|^2 = \|z_{n-1}\|_2^2 \left(1 - \left| \left\langle \frac{z_{n-1}}{\|z_{n-1}\|_2}, \frac{\varphi_n}{\|\varphi_n\|_2} \right\rangle \right|^2 \right) \tag{V.13}$$

A repeated application of (V.13) gives that for all $n \geq l$

$$\|z_n\|_2^2 = \|z_l\|_2^2 \prod_{k=l+1}^{n} \left(1 - \left| \left\langle \frac{z_{k-1}}{\|z_{k-1}\|_2}, \frac{\varphi_k}{\|\varphi_k\|_2} \right\rangle \right|^2 \right). \tag{V.14}$$

When $l = 0$ this yields the formula (V.12). $\square$

From Proposition V.3.1, we see that the monotonicity of the Kaczmarz algorithm in (V.2) is an immediate corollary of (V.11). Consequently, if $z_l = 0$ for some $l \geq 1$ then $z_j = 0$ for all $j \geq l$. So, if $z_l = 0$, the convention that $z_k/\|z_k\|_2 = 0$ for $k \geq l$ simply sets each term in the partial product in (V.14) to be one. While it is possible to have the desirable outcome of finite convergence to zero error $\|z_l\|_2 = 0$, this will generally not be the case for continuous random measurements. For example, if the normalized measurement vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are absolutely continuous with respect to the normalized surface measure on $\mathbb{S}^{d-1}$, then by (V.12) each error z_k is almost surely nonzero.

Corollary V.3.2. *Suppose the measurement vectors $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ are random vectors such that each φ_n is almost surely nonzero. Additionally suppose that $\{\varphi_n/\|\varphi_n\|\}_{n=1}^{\infty}$ are independent and that each $\varphi_n/\|\varphi_n\|_2$ is absolutely continuous with respect to the uniform measure on $\mathbb{S}^{d-1}$. If the initial error $z_0 = x - x_0$ in the Kaczmarz algorithm is nonzero, then for each $k \geq 1$, there holds* $\Pr[\|x - x_k\|_2 = 0] = \Pr[z_k = 0] = 0$.

V.4 Moment bounds in the Kaczmarz algorithm

The following moment bound first appeared in [58], and its proof is motivated by the work in [49] on mean squared error.

Theorem V.4.1. *[58] Let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be random vectors that are almost surely nonzero and such that $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent. Let $s > 0$ be fixed and assume that each $\varphi_n/\|\varphi_n\|$ has the common Kaczmarz bound $\alpha > 0$ of order s, as in* (V.5)*.*

The error after the nth iteration of the Kaczmarz algorithm satisfies

$$\mathbb{E}\|x - x_n\|_2^{2s} \leq \alpha^{ns}\|x - x_0\|_2^{2s}. \tag{V.15}$$

If additionally the Kaczmarz bound α is tight then

$$\mathbb{E}\|x - x_n\|_2^{2s} = \alpha^{ns}\|x - x_0\|_2^{2s}. \tag{V.16}$$

Proof. Let $z_n = x - x_n$. Note that $z_{n-1} = z_{n-1}(z_0, \varphi_1/\|\varphi_1\|_2, \cdots, \varphi_{n-1}/\|\varphi_{n-1}\|_2)$ is a function of the deterministic initial error $z_0 \in \mathbb{R}^d$ and the independent random vectors $\{\varphi_k/\|\varphi_k\|_2\}_{k=1}^{n-1}$. In particular, since $\{\varphi_k/\|\varphi_k\|_2\}_{k=1}^{\infty}$ are independent, the random vectors $z_{n-1}/\|z_{n-1}\|$ and $\varphi_n/\|\varphi_n\|$

are independent. Thus

$$
\begin{aligned}
\mathbb{E}\|z_n\|_2^{2s} &= \mathbb{E}\left(\|z_{n-1}\|_2^{2s}\left(1-\left|\left\langle \frac{z_{n-1}}{\|z_{n-1}\|_2}, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right|^2\right)^s\right) \\
&= \mathbb{E}\left(\mathbb{E}\left[\|z_{n-1}\|_2^{2s}\left(1-\left|\left\langle \frac{z_{n-1}}{\|z_{n-1}\|_2}, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right|^2\right)^s \,\middle|\, z_{n-1}\right]\right) \\
&= \mathbb{E}\left(\|z_{n-1}\|_2^{2s}\,\mathbb{E}\left[\left(1-\left|\left\langle \frac{z_{n-1}}{\|z_{n-1}\|_2}, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right|^2\right)^s \,\middle|\, z_{n-1}\right]\right) \\
&\leq \mathbb{E}\left(\|z_{n-1}\|_2^{2s}\alpha^s\right) \\
&= \alpha^s\,\mathbb{E}\|z_{n-1}\|_2^{2s}.
\end{aligned} \tag{V.17}
$$

Here, in the second equation, the outer expectation is taken over $\{\varphi_k/\|\varphi_k\|_2\}_{k=1}^{n-1}$, and the inner expectation is taken over φ_n. Therefore we can pull out the term $\|z_{n-1}\|_2^{2s}$ in the inner expectation, and hence the third equality holds. The inequality holds due to the common Kaczmarz bound as assumed.

Iterating (V.17) yields (V.15). A similar computation shows that if each $\varphi_n/\|\varphi_n\|_2$ has a tight Kaczmarz bound α, then (V.16) holds. $\square$

Taking $s=1$ in Theorem V.4.1 gives the mean squared error bound for the Kaczmarz algorithm as follows. Corollary V.4.2 is essentially the same as the mean squared error bounds in [49] but is expressed under a superficially more general model of randomization using probabilistic frames instead of the finite random vectors as in Example V.2.7.

Corollary V.4.2. *Let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be random vectors that are almost surely nonzero and such that $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent. If each $\varphi_n/\|\varphi_n\|_2$ has the common probabilistic lower frame bound $\beta > 0$ then the error after the nth iteration of the Kaczmarz algorithm satisfies*

$$
\mathbb{E}\|x-x_n\|_2^2 \leq (1-\beta)^n\|x-x_0\|_2^2. \tag{V.18}
$$

If additionally each $\varphi_n/\|\varphi_n\|_2$ *is isotropic* (V.7) *then*

$$\mathbb{E}\|x-x_n\|_2^2 = (1-d^{-1})^n\|x-x_0\|_2^2. \tag{V.19}$$

Similar to [49], Corollary V.4.2 yields the following examples. Versions of these examples appear in [49] under a slightly different statement of randomization, so we include them here to illustrate analogs for randomization using probabilistic frames, and for random measurements satisfying Definition V.2.11.

Example V.4.3. If $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ satisfy the properties of Definition V.2.11 then each $\varphi_n/\|\varphi_n\|_2$ is isotropic with tight probabilistic frame bound $\beta = 1/d$. Thus the mean squared error of the Kaczmarz algorithm for measurements with the properties of Definition V.2.11 satisfies

$$\mathbb{E}\|x-x_n\|_2^2 = (1-d^{-1})^n\|x-x_0\|_2^2.$$

Example V.4.4 (Computational Complexity). Let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be random vectors satisfying the properties of Definition V.2.11. Given $\varepsilon > 0$, let n_ε be the smallest number of iterations of the Kaczmarz algorithm needed to ensure the ε-precise mean squared error

$$\mathbb{E}\|x-x_{n_\varepsilon}\|_2^2 \leq \varepsilon^2\|x-x_0\|_2^2.$$

By (V.16), we seek the smallest integer n_ε such that $(1-\beta)^{n_\varepsilon} \leq \varepsilon^2$. Since $\beta = \beta_d = 1/d$, in high dimensions we have $\log(1-\beta) \approx -\beta = -1/d$ and

$$n_\varepsilon = \left\lceil \frac{2\log\varepsilon}{\log(1-d^{-1})} \right\rceil \approx 2d\,|\log\varepsilon|. \tag{V.20}$$

By (V.20), $\mathscr{O}(d)$ iterations suffice to ensure ε-precise mean squared error. Moreover, since each iteration of the Kaczmarz algorithm requires $\mathscr{O}(d)$ elementary operations, ε-precision is achieved with an overall quadratic complexity of $\mathscr{O}(d^2)$ operations.

Example V.4.5. Theorem V.4.1 together with Example V.2.7 recovers the mean squared error bound (V.3) from [49]. In particular, if the randomization from Example V.2.7 is used to solve a given $N \times d$ system $\Phi x = y$ then the Kaczmarz bound α of order $s = 1$ satisfies $\alpha \leq 1 - [\kappa(\Phi)]^{-2}$ so that $\mathbb{E}\|x - x_n\|_2^2 \leq \alpha^n \|x - x_0\|_2^2 \leq (1 - [\kappa(\Phi)]^{-2})^n \|x - x_0\|_2^2$.

V.5 Almost sure convergence for uniform random measurements on $\mathbb{S}^{d-1}$

As mentioned, (V.12) will play an important role in our analysis of almost sure convergence in the Kaczmarz algorithm. It will be convenient to introduce the following notation for the individual random variables in the random product (V.12):

$$\xi_k = \left(1 - \left|\left\langle \frac{z_{k-1}}{\|z_{k-1}\|_2}, \frac{\varphi_k}{\|\varphi_k\|_2} \right\rangle\right|^2\right). \tag{V.21}$$

Since the first step of the Kaczmarz algorithm requires an initial estimate $x_0 \in \mathbb{R}^d$, each random variable ξ_k is implicitly parametrized by the initial error $z_0 = x - x_0 \in \mathbb{R}^d$. When needed, we emphasize this dependence by writing $\xi_k = \xi_k(z_0)$.

With the notation (V.21), the error in the Kaczmarz algorithm satisfies

$$\|x - x_n\|_2^2 = \|x - x_0\|_2^2 \left(\prod_{k=1}^{n} \xi_k\right). \tag{V.22}$$

V.5.1 Independence of ξ_k's

In general, the random variables $\{\xi_k\}_{k=1}^{\infty}$ defined by (V.21) need not be independent, e.g., see Example V.6.1. However, in the special case when the random measurements $\{\varphi_n\}_{n=1}^{\infty}$ satisfy NIU (Definition V.2.11), it will follow that the random variables $\{\xi_n\}_{n=1}^{\infty}$ are independent and identically distributed. This will have pleasant consequences for the subsequent error analysis.

Lemma V.5.1. *Fix $z_0 \in \mathbb{R}^d$. Let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be random vectors that are almost surely nonzero and such that the normalized random measurement vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent and uniformly distributed on $\mathbb{S}^{d-1}$. Then the random variables $\{\xi_n\}_{n=1}^{\infty}$ defined by* (V.21) *are* *independent and identically distributed versions of the random variable*

$$\xi = 1 - |\langle e_1, u\rangle|^2, \tag{V.23}$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$ *and* $u \in \mathbb{R}^d$ *is a uniform random vector on* $\mathbb{S}^{d-1}$. *The random variable* ξ *does not depend on* z_0 *but does depend on the dimension* d.

Proof. Let $u_n = \varphi_n/\|\varphi_n\|_2$. The hypotheses on $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ mean that $\{u_n\}_{n=1}^{\infty}$ are independent random variables that are uniformly distributed on $\mathbb{S}^{d-1}$. Without loss of generality we assume that $z_0 \neq 0$. Moreover, as noted in the discussion following Proposition V.3.1, since each u_n is absolutely continuous, we have that $\Pr[z_k = 0] = 0$ for all k.

Note that the random vector

$$z_{n-1} = z_{n-1}(z_0, u_1, \cdots, u_{n-1})$$

is a function of the nonrandom initial error z_0 and the independent random vectors $\{u_k\}_{k=1}^{n-1}$. Thus, z_{n-1} and u_n are independent random vectors. This independence along with the rotational symmetry of u_n now implies that if $e_1 = (1, 0, \cdots, 0)$ then ξ_n has the same distribution as the random variable $\left(1 - |\langle e_1, u_n\rangle|^2\right)$. This shows that the random variables $\{\xi_n\}_{n=1}^{\infty}$ are identically distributed.

It remains to show that the random variables $\{\xi_n\}_{n=1}^{\infty}$ are independent. Let μ denote the normalized surface measure on $\mathbb{S}^{d-1}$. Let E_n denote the event that $\xi_n < \beta_n$, and let χ_{E_n} denote the associated indicator function. Note that $\chi_{E_n}(u_n,\cdots,u_1)$ is a function of $u_n,\cdots,u_1$.

$$
\begin{aligned}
\mathbb{E}[\chi_{E_n} \mid u_{n-1},\cdots,u_1] &= \Pr[\xi_n < \beta_n \mid \; u_{n-1},\cdots,u_1] \\
&= \Pr[\xi_n(u_n,\cdots,u_1) < \beta_n \mid u_{n-1},\cdots,u_1] \\
&= \Pr\left[1 - \left|\left\langle \frac{z_{n-1}(u_{n-1},\cdots,u_1)}{\|z_{n-1}(u_{n-1},\cdots,u_1)\|_2}, u_n\right\rangle\right|^2 < \beta_n \;\middle|\; u_{n-1},\cdots,u_1\right] \\
&= \Pr[1 - |\langle e_1, u_n\rangle|^2 < \beta_n] \\
&= \Pr[\xi_n < \beta_n].
\end{aligned}
$$

Indeed, the fourth equality holds because $z_{n-1}/\|z_{n-1}\|_2$ is independent of u_n, by rotation invariance of μ, we can replace $z_{n-1}/\|z_{n-1}\|_2$ by any vector with norm 1.

Thus

$$
\begin{aligned}
&\Pr[\xi_n < \beta_n, \xi_{n-1} < \beta_{n-1},\cdots,\xi_1 < \beta_1] = \mathbb{E}\left(\prod_{k=1}^{n} \chi_{E_k}(u_k,\cdots,u_1)\right) \\
&= \int_{(\mathbb{S}^{d-1})^n} \left(\prod_{k=1}^{n} \chi_{E_k}(u_k,\cdots,u_1)\right) d\mu(u_n)d\mu(u_{n-1})\cdots d\mu(u_1) \\
&= \int_{(\mathbb{S}^{d-1})^{(n-1)}} \mathbb{E}[\chi_{E_n} \mid u_{n-1},\cdots,u_1] \left(\prod_{k=1}^{n-1} \chi_{E_k}(u_k,\cdots,u_1)\right) d\mu(u_{n-1})\cdots d\mu(u_1) \\
&= \Pr[\xi_n < \beta]\Pr[\xi_{n-1} < \beta_{n-1},\cdots,\xi_1 < \beta_1].
\end{aligned}
$$

Iterating this argument shows that

$$
\Pr[\xi_n < \beta_n,\cdots,\xi_1 < \beta_1] = \prod_{k=1}^{n} \Pr[\xi_k < \beta_k].
$$

Thus, $\{\xi_k\}_{k=1}^n$ is independent for all $n \geq 1$, as required. $\square$

Lemma V.5.2. *Let $d \geq 2$ be an integer, and let ξ be the random variable given by* (V.23)*. Then*

$$\mathbb{E}(\log \xi) = \frac{\omega_{d-2}}{\omega_{d-1}} \int_0^{\pi} \sin^{d-2}\theta \, \log(\sin^2\theta) \, d\theta,$$

and

$$\mathbb{E}(\log \xi)^2 = \frac{\omega_{d-2}}{\omega_{d-1}} \int_0^{\pi} \sin^{d-2}\theta \, (\log(\sin^2\theta))^2 d\theta,$$

where $\omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$ is the surface area of $\mathbb{S}^d$.

Proof. Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$. By Lemma V.5.1

$$\begin{aligned}
\mathbb{E}(\log \xi) &= \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \log\left(1 - |\langle e_1, u\rangle|^2\right) du \\
&= \frac{\omega_{d-2}}{\omega_{d-1}} \int_{-1}^{1} (\sqrt{1-s^2})^{d-3} \log(1-s^2) ds \\
&= \frac{\omega_{d-2}}{\omega_{d-1}} \int_0^{\pi} \sin^{d-2}\theta \log \sin^2\theta d\theta.
\end{aligned}$$

Similarly,

$$\mathbb{E}(\log \xi)^2 = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} (\log(1 - |\langle e_1, u\rangle|^2))^2 du = \frac{\omega_{d-2}}{\omega_{d-1}} \int_0^{\pi} \sin^{d-2}\theta (\log(\sin^2\theta))^2 d\theta.$$

□

V.5.2 Almost sure exponential convergence rate

The independence of the random variables in $\{\xi_n\}_{n=1}^{\infty}$ in Lemma V.5.1 will allow us to apply classical tools such as the Strong Law of Large Numbers, the Central Limit Theorem, and the Law of the Iterated Logarithm to our analysis of almost sure convergence properties of the Kaczmarz algorithm. The following theorem first appeared in [58].

Theorem V.5.3. *[58] Let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be random vectors that are almost surely nonzero and such that the normalized random measurement vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent and uniformly distributed on $\mathbb{S}^{d-1}$. Let $R = \exp(-\mathbb{E}\log\xi)$ and $\sigma^2 = \mathbb{E}(\log\xi)^2 - (\mathbb{E}\log\xi)^2$ as computed in Lemma V.5.2. Then the error in the Kaczmarz algorithm satisfies*

$$\lim_{n\to\infty} \|x - x_n\|_2^{2/n} = R^{-1}, \quad \textit{almost surely,} \tag{V.24}$$

and

$$\forall\, t \in \mathbb{R}, \quad \lim_{n\to\infty} \Pr\left[R^n \|x - x_n\|_2^2 \geq \|x - x_0\|_2^2 e^{t\sqrt{n\sigma^2}}\right] = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du, \tag{V.25}$$

and

$$\limsup_{n\to\infty} \left(R^n \|x - x_n\|_2^2\right)^{\frac{1}{\sqrt{2\sigma^2 n \log(\log n)}}} = e, \quad \textit{almost surely,} \tag{V.26}$$

Proof. Let

$$S_n = \log\left(\prod_{k=1}^{n} \xi_k\right) = \sum_{k=1}^{n} \log(\xi_k). \tag{V.27}$$

By Lemma V.5.1 the $\{\xi_k\}_{k=1}^{\infty}$ are independent versions of the random variable ξ given by (V.23). By Lemma V.5.2, $\mathbb{E}(\log\xi) = \log(1/R)$ and $\mathrm{Var}(\log\xi) = \sigma^2$ are both finite.

Applying the Strong Law of Large Numbers to (V.27) yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log\xi_k = \mathbb{E}(\log\xi) = \log(1/R), \quad \text{a.s.} \tag{V.28}$$

Taking the exponential of (V.28) gives

$$\lim_{n\to\infty} \left(\prod_{k=1}^{n} \xi_n\right)^{\frac{1}{n}} = \exp(\mathbb{E}(\log\xi)) = R^{-1}, \quad \text{a.s.} \tag{V.29}$$

Equation (V.24) now follows from (V.22) and (V.29).

Applying the Central Limit Theorem to (V.27) gives

$$\forall\, t \in \mathbb{R}, \quad \lim_{n\to\infty} \Pr\left(\frac{\sum_{k=1}^{n} \log \xi_k - n\log(1/R)}{\sqrt{n\sigma^2}} \le t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du. \tag{V.30}$$

Exponentiating and reorganizing (V.30) gives

$$\forall\, t \in \mathbb{R}, \quad \lim_{n\to\infty} \Pr\left(\prod_{k=1}^{n} R\xi_k \ge e^{t\sqrt{n\sigma^2}}\right) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du. \tag{V.31}$$

Equation (V.25) now follows from (V.22) and (V.31).

To prove (V.26), apply the Law of the Iterated Logarithm to $\log(R\xi_n)$. Since $\mathbb{E}(\log(R\xi)) = \mathbb{E}(\log \xi + \log R) = 0$ and $\mathrm{Var}(\log(R\xi)) = \mathbb{E}(\log \xi + \log R)^2 = \mathbb{E}(\log \xi - \mathbb{E}(\log \xi))^2 = \sigma^2$, there holds

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \log(R\xi_k)}{\sqrt{2\sigma^2 n \log(\log n)}} = 1, \text{ a.s.}$$

which yields

$$\limsup_{n\to\infty} \left(\prod_{k=1}^{n} R\xi_k\right)^{\frac{1}{\sqrt{2\sigma^2 n \log(\log n)}}} = e, \text{ a.s.}$$

This implies (V.26). □

For a different perspective on Theorem V.5.3 we shall use following lemma. A proof is provided in the Appendix.

Lemma V.5.4. *Given $A \ge 1$ and a nonnegative sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, the following two statements are equivalent:*

(a) $\lim_{n\to\infty} (a_n)^{1/n} = 1/A$.

(b) $\forall\, 0 < r < A$, $\lim_{n\to\infty} r^n a_n = 0$ *and* $\forall\, A < r$, $\lim_{n\to\infty} r^n a_n = \infty$.

Thus, (V.24) in Theorem V.5.3 can be stated as follows.

Corollary V.5.5. *Let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be random vectors that are almost surely nonzero and such that the normalized random measurement vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent and uniformly distributed on $\mathbb{S}^{d-1}$. Let $1 < R$ be the constant defined in Theorem V.5.3.*

If $0 < r < R$ then

$$\lim_{n\to\infty} r^n \|x - x_n\|_2^2 = 0, \quad \textit{almost surely.} \tag{V.32}$$

If $r > R$ then

$$\lim_{n\to\infty} r^n \|x - x_n\|_2^2 = \infty, \quad \textit{almost surely.} \tag{V.33}$$

The boundary case $r = R$ in Corollary V.5.5 is addressed by (V.25) and (V.26). For example, taking $t = 0$ in (V.25) of Theorem V.5.3 shows that one does not have almost sure convergence of $R^n\|x - x_n\|_2^2$ to 0. Likewise, one does not have almost sure convergence of $R^n\|x - x_n\|_2^2$ to infinity either.

Example V.5.6. To compare the almost sure convergence rates in Theorem V.5.3 with the mean squared convergence rates in Corollary V.4.2, let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^2$ be independent random vectors that are uniformly distributed on $\mathbb{S}^1$. In dimension $d = 2$, we have that each φ_n is isotropic with $\beta = \beta_2 = 1/2$. Moreover, $\omega_1 = 2\pi$ and $\omega_0 = (2\sqrt{\pi})/\Gamma(1/2) = 2$, so that the constant R from Theorem V.5.3 satisfies

$$R = \exp\left(-\frac{1}{\pi}\int_0^{\pi} \log\sin^2\theta d\theta\right) = 4. \tag{V.34}$$

The computation of the integral in (V.34) follows from the fact that the Lobachevsky function

$$L(t) = -\int_0^t \log|2\sin\theta| d\theta = -t\log 2 - \frac{1}{2}\int_0^t \log\sin^2\theta d\theta$$

is π-periodic, e.g., see the appendix in [59]. So, $L(\pi) = L(0) = 0$ and this implies (V.34).

By (V.16), the mean squared error satisfies

$$\forall n \geq 1, \quad \mathbb{E}\|x - x_n\|_2^2 = (1/2)^n \|x - x_0\|^2.$$

By Corollary V.5.5, we have the following almost sure convergence:

$$\forall\, 0 < r < 4, \quad \lim_{n \to \infty} r^n \|x - x_n\|_2^2 = 0, \quad \text{almost surely.}$$

In particular, the mean squared error decreases at the rate $(1/2)^n$, whereas the squared error nearly decreases at the rate of $(1/4)^n$ in an almost sure sense.

V.6 Almost sure convergence for general random measurements

The results of Section V.5 shows that if the measurement vectors $\{\varphi_n\}_{n=1}^{\infty}$ satisfy the conditions of Definition V.2.11, then the random variables $\{\xi_n\}_{n=1}^{\infty}$ defined in (V.21) are independent and identically distributed, and moreover do not depend on the initial error z_0. This, in turn, made it possible to apply classical results on sums of i.i.d. random variables to the convergence analysis in Theorem V.5.3.

For general measurement vectors $\{\varphi_n\}_{n=1}^{\infty}$ without the properties in Definition V.2.11, it is possible for the random variables $\{\xi_n\}_{n=1}^{\infty}$ to be neither independent nor identically distributed (see Example V.6.1 below), and it is not possible to directly apply the classical convergence results used for Theorem V.5.3. In this section we address almost sure convergence of the Kaczmarz algorithm when a general collection of random measurements $\{\varphi_n\}_{n=1}^{\infty}$ is used.

Example V.6.1. Let $\varphi \in \mathbb{R}^2$ be a discrete random vector that satisfies

$$\Pr[\varphi = (1,0)] = 2/3 \quad \text{and} \quad \Pr[\varphi = (0,1)] = 1/3.$$

Let φ_1, φ_2 be independent versions of φ. We consider the random variables $\xi_1(z_0), \xi_2(z_0)$ that arise in the first two iterations of the Kaczmarz algorithm when $x = (\sqrt{3}/2, 1/2)$, $x_0 = (0,0)$, and the initial error $z_0 = x - x_0$ satisfies $z_0 = (\sqrt{3}/2, 1/2)$.

A direct computation shows that ξ_1 satisfies

$$\Pr[\xi_1 = 1/4] = 2/3 \quad \text{and} \quad \Pr[\xi_1 = 3/4] = 1/3.$$

Similarly, by considering a tree of probabilities, ξ_2 can be shown to satisfy

$$\Pr[\xi_2 = 1] = 5/9, \quad \text{and} \quad \Pr[\xi_2 = 0] = 4/9.$$

Moreover, it can be shown that $\Pr[\xi_1 = 3/4, \xi_2 = 1] = 1/9$. Thus ξ_1, ξ_2 are neither independent nor identically distributed.

V.6.1 With Kaczmarz bound

Theorem V.6.2. *[58] Let $\{\varphi_k\}_{k=1}^{\infty} \subset \mathbb{R}^d$ be random vectors that are almost surely nonzero and for which $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent. Let $s > 0$ be fixed and suppose that each $\varphi_n/\|\varphi_n\|_2$ has the common Kaczmarz bound $0 < \alpha < 1$ of order s. Then there exists a random variable X satisfying $\mathbb{E}|X|^s < \infty$ such that*

$$\lim_{n\to\infty} (1/\alpha)^n \|x - x_n\|_2^2 = X, \quad \textit{almost surely.} \tag{V.35}$$

Consequently,

$$\forall\, 0 < r < 1/\alpha, \quad \lim_{n\to\infty} r^n \|x - x_n\|_2^2 = 0, \quad \textit{almost surely.} \tag{V.36}$$

Proof. Let $Y_n = (1/\alpha)^{sn}\|x - x_n\|_2^{2s} = (1/\alpha)^{sn}\|z_n\|_2^{2s}$ and let $\mathscr{F}_n$ be the sigma algebra generated by the random vectors $\varphi_1/\|\varphi_1\|_2, \cdots, \varphi_n/\|\varphi_n\|_2$. It can be shown that Y_n is measureable with respect to $\mathscr{F}_n$. The same computations as in the proof of Theorem V.4.1 show that

$$\mathbb{E}[Y_n \mid \mathscr{F}_{n-1}] = \mathbb{E}[(1/\alpha)^{sn}\|z_n\|_2^{2s} \mid \mathscr{F}_{n-1}] \le (1/\alpha)^{s(n-1)}\|z_{n-1}\|_2^{2s}(1/\alpha)^s\alpha^s = Y_{n-1}.$$

Thus $\{(Y_n, \mathscr{F}_n)\}_{n=1}^{\infty}$ is a supermartingale. Moreover, by Theorem V.4.1, there holds

$$\forall n \ge 1, \quad \mathbb{E}[Y_n] \le \|z_0\|_2^{2s}.$$

An application of Doob's martingale convergence theorem (for example, see Theorem 1 on page 508 of [2]) to the submartingale $\{(-Y_n, \mathscr{F}_n)\}_{n=1}^{\infty}$ shows that the limit

$$\lim_{n\to\infty} Y_n = Y, \qquad \text{exists almost surely,}$$

and the limit satisfies $\mathbb{E}|Y| < \infty$. Thus,

$$\lim_{n\to\infty} (1/\alpha)^{sn} \|x - x_n\|_2^{2s} = Y, \qquad \text{almost surely.} \tag{V.37}$$

Letting $X = Y^{1/s}$, and taking the $1/s$ power of (V.37), we obtain (V.35)

$$\lim_{n\to\infty} (1/\alpha)^{n} \|x - x_n\|_2^{2} = X, \qquad \text{almost surely.}$$

This implies (V.36) and completes the proof. $\square$

The martingale convergence theorem is a natural tool for the proof of Theorem V.6.2. For comparison, Markov chain and martingale methods were previously applied to the error analysis of closely related algorithms such as the Rangan-Goyal algorithm in [42] and the Gibbs sampler in [60]. In the present setting, it is possible to give a direct alternative proof of the bound (V.36) in Theorem V.6.2 without appealing to martingale convergence in the following manner.

Alternative Proof of Equation (V.36). Fix $0 < r < 1/\alpha$ and let

$$P_n = r^n \prod_{k=1}^{n} \xi_k.$$

Recall that $P_n \geq 0$ and that $r^n\|x - x_n\|_2^2 = P_n\|x - x_0\|_2^2$. To prove (V.36), it suffices to show that

$$\forall \varepsilon > 0, \quad \lim_{N\to\infty} \Pr\left(\bigcup_{n=N}^{\infty} \{P_n > \varepsilon\} \right) = 0. \tag{V.38}$$

Let $\varepsilon > 0$ be fixed. A union bound together with Chebyshev's inequality implies that

$$\Pr\left(\bigcup_{n=N}^{\infty}\{P_n > \varepsilon\}\right) \leq \sum_{n=N}^{\infty} \Pr(P_n > \varepsilon) \leq \sum_{n=N}^{\infty} \frac{\mathbb{E}(P_n^s)}{\varepsilon^s}. \tag{V.39}$$

Theorem V.4.1 shows that

$$\mathbb{E}(P_n^s) \leq (r\alpha)^{sn}. \tag{V.40}$$

Combining (V.39) and (V.40), it follows that

$$\Pr\left(\cup_{n=N}^{\infty}\{P_n > \varepsilon\}\right) \leq \frac{1}{\varepsilon^s}\sum_{n=N}^{\infty}(r\alpha)^{sn} \leq \frac{(r\alpha)^{sN}}{\varepsilon^s(1-r^s\alpha^s)}.$$

Since $0 < r\alpha < 1$, it follows that (V.38) holds. This completes the proof. □

V.6.2 With logarithmic Kaczmarz bound

The next result improves the conclusion of Theorem V.6.2 by considering the limiting case when $s = 0$. Unlike Theorem V.6.2, the following theorem assumes that the $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are identically distributed.

Theorem V.6.3. *[58] Let $\{\varphi_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ be random vectors that are almost surely nonzero. Assume that the normalized vectors $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent and identically distributed versions of a random vector $u \in \mathbb{S}^{d-1}$ and assume that u has the logarithmic Kaczmarz bound $0 < \rho < 1$. Then the error in the Kaczmarz algorithm satisfies*

$$\forall\, 0 < r < 1/\rho, \quad \lim_{n\to\infty} r^n\|x - x_n\|_2^2 = 0, \quad \textit{almost surely.}$$

Proof. Fix $0 < r < 1/\rho$ and take α such that $\rho < \alpha < 1/r$. By Corollary V.2.9,

$$\forall x \in \mathbb{S}^{d-1}, \quad \inf_{s>0}(\mathbb{E}(1 - |\langle x, u\rangle|^2)^s)^{1/s} \leq \rho.$$

So, for every $x \in \mathbb{S}^{d-1}$ there exists $s_x > 0$ such that

$$(\mathbb{E}(1-|\langle x,u\rangle|^2)^{s_x})^{1/s_x} < \alpha.$$

It follows from the Lebesgue Dominated Convergence Theorem that

$$\forall x \in \mathbb{S}^{d-1}, \quad \lim_{\|y\|=1; y\to x} (\mathbb{E}(1-|\langle y,u\rangle|^2)^{s_x})^{1/s_x} = (\mathbb{E}(1-|\langle x,u\rangle|^2)^{s_x})^{1/s_x} < \alpha.$$

So, for every $x \in \mathbb{S}^{d-1}$, there exists an open neighborhood $U_x \subset \mathbb{S}^{d-1}$ of x such that

$$\forall y \in U_x, \quad (\mathbb{E}(1-|\langle y,u\rangle|^2)^{s_x})^{1/s_x} < \alpha.$$

Since $\mathbb{S}^{d-1}$ is compact and $\mathbb{S}^{d-1} \subset \cup_{x\in\mathbb{S}^{d-1}} U_x$, there exists a finite subcover $\{U_{x_j}\}_{j=1}^{J}$ of $\{U_x\}_{x\in\mathbb{S}^{d-1}}$. Letting $s^* = \min\{s_{x_j}\}_{j=1}^{J}$ and using Lyapunov's inequality (for example, see page 193 of [2]), we obtain

$$\forall x \in \mathbb{S}^{d-1}, \quad (\mathbb{E}(1-|\langle x,u\rangle|^2)^{s^*})^{1/s^*} < \alpha.$$

Since the $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent and identically distributed versions of the random vector u, each $\varphi_n/\|\varphi_n\|_2$ has the common Kaczmarz bound α of order $s^* > 0$. Since $r < 1/\alpha$ we conclude by Theorem V.6.2 that $\lim_{n\to\infty} r^n \|x - x_n\|_2^2 = 0$ almost surely. $\square$

Theorem V.6.3 provides stronger error bounds than Theorem V.6.2 since by Lemma V.2.8 and Corollary V.2.9, a logarithmic Kaczmarz bound ρ satisfies $\rho \leq \alpha = \alpha_s$ for each Kaczmarz bound α of order $s > 0$. In the special case when the $\{\varphi_n/\|\varphi_n\|_2\}_{n=1}^{\infty}$ are independent uniform random vectors on $\mathbb{S}^{d-1}$, Theorem V.6.3 recovers the sharp bound (V.32) of Corollary V.5.5. In particular, if $u = \varphi/\|\varphi\|_2$ is uniformly distributed on $\mathbb{S}^{d-1}$ then the logarithmic Kaczmarz bound ρ is tight and satisfies

$$\forall x \in \mathbb{S}^{d-1}, \quad \rho = \exp[\mathbb{E}\log(1-|\langle x,u\rangle|^2)] = \exp(\mathbb{E}(\log\xi)) = 1/R,$$

where R and $\mathbb{E}(\log\xi)$ are as in Lemma V.5.2 and Theorem V.5.3.

APPENDIX

V.7

Lemma V.7.1. *Let $x \geq 0$ be a bounded random variable, and let $\mathbb{E}(\log x)$ be finite, then*

$$\inf_{s>0}[\mathbb{E}(x^s)]^{1/s} = \lim_{s\to 0}[\mathbb{E}(x^s)]^{1/s} = \exp(\mathbb{E}(\log x))$$

Proof. The first equation is proven by Lyapunov's inequality (page 193 of [2]).

By L'Hospital's rule, it suffices down to show that

$$\frac{d}{ds}\int_\Omega x^s dx = \int_\Omega \frac{\partial}{\partial s} x^s dx. \tag{V.41}$$

This is not obvious because $\frac{\partial}{\partial s}x^s = x^s \log x$ has a singularity at $x = 0$.

Define $f(x,h) = x^h, E = \{x : 0 < x < \varepsilon\}, g(\varepsilon,h) = \int_{\Omega/E} f(x,h)dx$.

(1) Using mean value theorem,

$$\frac{\partial}{\partial h} g(\varepsilon,h) = \int_{\Omega/E} x^h \log x dx \tag{V.42}$$

(2) $\lim_{\varepsilon\to 0}\int_E x^h \log x dx = 0$.

This is true since we can write $\int_E x^h \log x dx$ as $\int_\Omega \chi_E x^h \log x dx$ and then use Lebesgue Dominated Convergence Theorem.

(3) The convergence in (3) is uniform in h, therefore together with (2) we also have

$$\lim_{\varepsilon\to 0}\int_0^h \frac{\partial}{\partial t} g(\varepsilon,t)dt = \int_0^h \int_\Omega x^t \log x dt$$

(4) Finally

$$\begin{aligned}
&g(\varepsilon,h)-g(\varepsilon,0)=\int_0^h \frac{\partial}{\partial t}g(\varepsilon,t)dt\\
\Rightarrow &\lim_{\varepsilon\to 0} g(\varepsilon,h)-\lim_{\varepsilon\to 0} g(\varepsilon,0)=\int_0^h\int_\Omega x^t\log x dt\\
\Rightarrow &\frac{d}{dh}\lim_{\varepsilon\to 0} g(\varepsilon,h)=\lim_{\varepsilon\to 0}\frac{\partial}{\partial h}g(\varepsilon,h)\\
\Rightarrow &\frac{d}{dh}\int_\Omega x^h dx=\int_\Omega x^h\log x dx,
\end{aligned}$$

where Fundamental Theorem of Calculus is used. □

V.8

Proof of Lemma V.5.4:

($\Rightarrow$) Assume $0<r<A$, then there exists c such that $1/A<c<1/r$, hence $(a_n)^{1/n}<c$ when n is sufficiently big. Therefore $r^n a_n \leq r^n c^n \to 0$. A similar argument applies to the case when $r>A$.

($\Leftarrow$) Suppose to the contrary that there exists $\varepsilon_0>0$ and a subsequence $\{n_k\}$ such that $|a_{n_k}^{1/n_k}-1/A|>\varepsilon_0$. Without loss of generality, we can assume there are infinitely many terms of $a_{n_k}^{1/n_k}$ that are bigger than $1/A$, so let us assume the subsequence $\{n_k\}$ satisfies $a_{n_k}^{1/n_k}-1/A>\varepsilon_0$. Pick $r=1/(\frac{1}{A}+\frac{\varepsilon_0}{2})<A$, then $r^{n_k}a_{n_k}>r^{n_k}(\frac{1}{A}+\varepsilon_0)^{n_k}\to\infty$, which is a contradiction to (b). □

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