Hyperbolic Structures on Groups

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In memory of my two lovely aunts : Ratna Kumaraswamy and Gayathri Srinivasan.
Much loved, gone too soon, never forgotten.

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CHAPTER 1

INTRODUCTION AND MAIN RESULTS

It is a common technique in geometric group theory to study groups via their action of some space. One standard way to do so is to convert a group G into a metric space by fixing a generating set X and endowing G with the corresponding word metric d_X; the group has a natural cobounded action on the corresponding Cayley graph $\Gamma(G,X)$. However, not all generating sets are equally good for this purpose : the most informative metric space is obtained when X is finite, while the space corresponding to $X = G$ forgets the structure of the group almost completely.

The first section of thesis (part of joint work with Carolyn Abbott and Denis Osin; (see [2]) focuses on formalizing this notion, so that generating sets can be ordered in a way according to the amount of information they provide about the group G. Indeed, we introduce an ordering on the generating sets of G as follows.

Let X, Y be two generating sets of a group G. We say that X is *dominated* by Y, written $X \preceq Y$, if the identity map on G induces a Lipschitz map between metric spaces $(G,d_Y) \to (G,d_X)$.

This is equivalent to the requirement that

$$\sup_{y\in Y} |y|_X < \infty.$$

The relation $\preceq$ is inclusion reversing and a preorder on the set of generating sets of G. It therefore induces an equivalence relation on the generating sets of G in the usual manner.

$$X \sim Y \quad \Leftrightarrow \quad X \preceq Y \text{ and } Y \preceq X.$$

The above condition is equivalent to the statement that the Cayley graphs $\Gamma(G,X)$ and $\Gamma(G,Y)$ are quasi-isometric. A quasi-isometry is a coarse version of an isometry for spaces. We denote the equivalence class of a generating set X by $[X]$ and the set of all such equivalence classes by $\mathcal{G}(G)$. $\mathcal{G}(G)$ is a poset under the order induced from above. Note also that under this equivalence, all finite generating sets are equivalent and the equivalence class of a finite generating set is the largest.

[2] also introduces the poset of *hyperbolic structures* on G, denoted $\mathcal{H}(G)$, which consists of equivalence classes of (possibly infinite) generating sets of G such that the corresponding Cayley graph is hyperbolic, endowed with the order from $\mathcal{G}(G)$. Since hyperbolicity is preserved under quasi-isometries, this notion is well-defined. Using arguments similar to those from the proof of the Milnor-Svarc Lemma, one can also define hyperbolic structures in terms of cobounded G-actions on hyperbolic spaces. This allows to work with either hyperbolic Cayley graphs or cobounded actions on hyperbolic spaces, according to our convenience.

We are especially interested in the subset $\mathcal{AH}(G) \subseteq \mathcal{H}(G)$ of *acylindrically hyperbolic structures* on G, i.e. hyperbolic structures $[X]$ such that the action of G on $\Gamma(G,X)$ is *acylindrical*.

Definition 1.0.1. An isometric action of a group G on a metric space (S,d) is *acylindrical* if for every $\varepsilon > 0$ there exist $R,N > 0$ such that for every two points $x,y \in S$ with $d(x,y) \geq R$, there are at most N elements $g \in G$ satisfying

$$d(x,gx) \leq \varepsilon \quad \textit{and} \quad d(y,gy) \leq \varepsilon.$$

Although it is not immediately obvious, the equivalence of generating sets preserves acylindricity, making this a well-defined notion as well.

An important class of groups for this thesis is the class of *acylindrically hyperbolic* groups. In order to define this class of groups, we need the following theorem by Osin.

Theorem 1.0.2. *[69, Theorem 1.1] Let G be a group acting acylindrically on a hyperbolic space. Then G satisfies exactly one of the following three conditions.*

(a) G has bounded orbits.

(b) G is virtually cyclic and contains a loxodromic element.

(c) G contains infinitely many independent loxodromic elements.

Cases (a) and (b) in the above theorem are called *elementary*, and Case (c) is called *non-elementary*.

Definition 1.0.3. A group G is called *acylindrically hyperbolic* if it admits a non-elementary, acylindrical action on a hyperbolic space.

The main goal of [2] was to initiate the study of the posets $\mathscr{H}(G)$ and $\mathscr{A}\mathscr{H}(G)$ for various groups G. My contribution to this body of work deals with the notion of *accessibility*.

Definition 1.0.4. A group G is said to be $\mathscr{H}$*-accessible* (respectively $\mathscr{A}\mathscr{H}$*-accessible*) if the poset $\mathscr{H}(G)$ (respectively $\mathscr{A}\mathscr{H}(G)$) contains the largest element.

Over the last few years, the class of acylindrically hyperbolic groups has received considerable attention. It is broad enough to include many examples of interest, e.g., non-elementary hyperbolic and relatively hyperbolic groups, all but finitely many mapping class groups of punctured closed surfaces, $Out(F_n)$ for $n \geq 2$, most 3-manifold groups, and finitely presented groups of deficiency at least 2. On the other hand, the existence of a non-elementary acylindrical action on a hyperbolic space is a rather strong assumption, which allows one to prove non-trivial results. In particular, acylindrically hyperbolic groups share many interesting properties with non-elementary hyperbolic and relatively hyperbolic groups. For details we refer to [30, 56, 69, 68] and references therein.

One can ask the question if every acylindrically hyperbolic group is $\mathscr{AH}$-accessible. While this is not true (a fact that follows from [2, Theorem 2.18]), I do prove that several well-known acylindrically hyperbolic groups are $\mathscr{AH}$-accessible.

Theorem 1.0.5. *The following acylindrically hyperbolic groups are $\mathscr{AH}$-accessible.*

(a) Finitely generated relatively hyperbolic groups whose parabolic subgroups are not acylindrically hyperbolic.

(b) Mapping class groups of punctured closed surfaces.

(c) Right-angled Artin groups.

(d) Fundamental groups of compact orientable 3*-manifolds with empty or toroidal boundary.*

The next part of this thesis answers the following question.

Problem 1.0.6. *Which groups admit acylindrical, non-elementary, cobounded actions on quasi-trees ?*

By a quasi-tree, I mean a connected graph quasi-isometric to a tree. Quasi-trees are a very specific subclass of hyperbolic spaces; indeed they are the “1- dimentional” hyperbolic spaces from the asymptotic point of view.

The motivation behind our question comes from the following observation. If instead of cobounded acylindrical actions we consider cobounded proper (i.e., geometric) ones, then there is a crucial difference between the groups acting on hyperbolic spaces and quasi-trees. Indeed a group G acts geometrically on a hyperbolic space if and only if G is a hyperbolic group. On the other hand, Stallings theorem on groups with infinitely many ends and Dunwoodys accessibility theorem implies that groups admitting geometric actions on quasi-trees are exactly virtually free groups. Yet another related observation is that acylindrical actions on unbounded locally finite graphs are necessarily proper. Thus if we restrict to quasi-trees of bounded valence in Question 4.1.1, we again obtain the class of virtually free groups (see Table 1.1). Other known examples of groups having non-elementary, acylindrical and cobounded actions on quasi-trees include groups associated with special cube complexes and right angled artin groups (see [10], [39], [47]).

	Hyperbolic spaces	**Quasi-trees**	**Locally finite quasi-trees**
Geometric action	Hyperbolic groups	Virtually free groups	Virtually free groups
Acylindrical, non-elementary, cobounded action	Acylindrically hyperbolic groups	**?**	Virtually free groups

Table 1.1

Thus one could expect that the restriction to a such a small subclass of hyperbolic spaces would yield a very specific subclass of the class of acylindrically hyperbolic groups. However, I prove that this does not happen. (See [8])

Theorem 1.0.7. *Every acylindrically hyperbolic group admits a non-elementary cobounded acylindrical action on a quasi-tree.*

In particular, this shows that every acylindrically hyperbolic group G has a structure $[X] \in \mathcal{AH}(G)$ such that the corresponding Cayley graph Γ is a (non-elementary) quasi-tree.) The proof of this result utilizes the notions of hyperbolically embedded subgroups and projection complexes. As an application, I also obtain some new results about hyperbolically embedded subgroups and quasi-convex subgroups of acylindrically hyperbolic groups.

Lastly, this thesis answers a particular question associated to *quasi-parabolic* hyperbolic structures (defined in the next section). Our understanding of quasi-parabolic structures on groups is far from being complete. The ultimate goal would be to obtain a classification of possible isomorphism types of $\mathcal{H}_{qp}(G)$. While achieving this goal does not seem realistic at the moment, there are some simpler questions which can be answered. One open questions posed in [2] is the following.

Problem 1.0.8. *Does there exist a group G such that $\mathcal{H}_{qp}(G)$ is finite and non-empty ?*

I prove that there an infinitely many groups which satisfy this property.

Theorem 1.0.9. *The lamplighter groups L_p have exactly two quasi-parabolic structures, when p is a prime.*

Organization of the thesis : The next few sections are dedicated to introducing several notions and definitions that will be used throughout the thesis. We will also record several useful results from [2]. The following chapters will then deal with the proofs of the main results of this thesis. Please note that Chapter 3 appears in [2], and is part of joint work with C. Abbott and D. Osin. Chapter 4 was first published in *Algebraic and Geometric Topology 17 (2017) 2145 - 2176*, published by Mathematical Sciences Publishers.

CHAPTER 2

PRELIMINARIES

2.1 Comparing Group Actions and Generating Sets

We begin with some standard terminology. Throughout this thesis, all group actions on metric spaces are isometric by default. Our standard notation for an action of a group G on a metric space S is $G \curvearrowright S$. Given a point $s \in S$ or a subset $X \subseteq S$ and an element $g \in G$, we denote by gs (respectively, gX) the image of s (respectively X) under the action of g. Given a group G acting on a space S and some $s \in S$, we also denote by Gs the G-orbit of s.

In order to avoid dealing with proper classes we fix a cardinal number $c \geq 2^{\aleph_0}$ and, henceforth, we assume that all metric spaces have cardinality at most c.

Definition 2.1.1. An action of a group G on a metric space S is said to be

(i) *proper* is for every bounded subset $B \subseteq S$ the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite;

(ii) *cobounded* if there exists a bounded subset $B \subseteq S$ such that $S = \bigcup_{g \in G} gB$;

(iii) *geometric* if it is proper and cobounded.

Given a metric space S, we denote by d_S the distance function on S unless another notation is introduced explicitly.

Definition 2.1.2. A map $f \colon R \to S$ between two metric spaces R and S is a *quasi-isometric embedding* if there is a constant C such that for all $x, y \in R$ we have

$$\frac{1}{C}\mathrm{d}_R(x,y) - C \leq \mathrm{d}_S(f(x), f(y)) \leq C\mathrm{d}_R(x,y) + C; \tag{2.1}$$

if, in addition, S is contained in the C–neighborhood of $f(R)$, f is called a *quasi-isometry*. Two metric spaces R and S are *quasi-isometric* if there is a quasi-isometry $R \to S$. It is well-known and easy to prove that quasi-isometry of metric spaces is an equivalence relation.

Definition 2.1.3. If a group G acts on metric spaces R and S, a map $f\colon R \to S$ is called *coarsely G-equivariant* if for every $r \in R$, we have

$$\sup_{g\in G} \mathrm{d}_S(f(gr), gf(r)) < \infty. \tag{2.2}$$

We now recall the definition of equivalent group actions introduced in [4].

Definition 2.1.4. Two actions $G \curvearrowright R$ and $G \curvearrowright S$ are *equivalent*, denoted $G \curvearrowright R \sim G \curvearrowright S$, if there exists a coarsely G-equivariant quasi-isometry $R \to S$. It is easy to prove (see [4]) that $\sim$ is indeed an equivalence relation.

We further develop another notion for comparing actions, especially when considering non-cobounded actions.

Definition 2.1.5. [2, Definition 31.] Let G be a group. We say that $G \curvearrowright R$ *dominates* $G \curvearrowright S$ and write $G \curvearrowright S \preceq G \curvearrowright R$ if there exist $r \in R$, $s \in S$, and a constant C such that

$$\mathrm{d}_S(s, gs) \leq C\mathrm{d}_R(r, gr) + C \tag{2.3}$$

for all $g \in G$.

Example 2.1.6. Assume that the action $G \curvearrowright S$ has bounded orbits. Then $G \curvearrowright S \preceq G \curvearrowright R$ for any other action of G on a metric space R.

Equivalently, we could define the relation $\preceq$ as follows.

Lemma 2.1.7. *[2, Lemma 3.3] $G \curvearrowright S \preceq G \curvearrowright R$ if and only if for any $r \in R$ and any $s \in S$ there exists a constant C such that (2.3) holds for all $g \in G$.*

Corollary 2.1.8. *[2, Corollary 3.4] The relation $\preceq$ is a preorder on the set of all G-actions on metric spaces.*

Proof. The relation $\preceq$ is obviously reflexive and is transitive by Lemma 2.1.7. $\square$

This allows us to introduce the following notion.

Definition 2.1.9. [2, Definition 3.5] We say that two actions of a group G on metric spaces R and S are *weakly equivalent* if $G \curvearrowright R \preceq G \curvearrowright S$ and $G \curvearrowright S \preceq G \curvearrowright R$. We use the notation $\sim_w$ for weak equivalence of group actions.

It is sometimes convenient to use the following alternative definition of weak equivalence.

Lemma 2.1.10. *[2, Lemma 3.6] Two actions $G \curvearrowright R$ and $G \curvearrowright S$ are weakly equivalent if and only if there exists a coarsely G–equivariant quasi-isometry from a G-orbit in R (endowed with the metric induced from R) to a G-orbit in S (endowed with the metric induced from S).*

Although the notions of equivalence and weak equivalence are not the same, they do coincide for cobounded actions.

Lemma 2.1.11. *[2, Lemma 3.8] Let $G \curvearrowright R$ and $G \curvearrowright S$ be two actions of a group G on metric spaces.*

(a) If $G \curvearrowright R \sim G \curvearrowright S$, then $G \curvearrowright R \sim_w G \curvearrowright S$.

(b) Suppose that the actions are cobounded and $G \curvearrowright R \sim_w G \curvearrowright S$. Then $G \curvearrowright R \sim G \curvearrowright S$.

We are now ready to formalize the notion of comparing two generating sets of a group.

Definition 2.1.12. [2, Definition 1.1] Let X, Y be two generating sets of a group G. We say that X is *dominated* by Y, written $X \preceq Y$, if the identity map on G induces a Lipschitz map between metric spaces $(G, \mathrm{d}_Y) \to (G, \mathrm{d}_X)$. This is obviously equivalent to the requirement that $\sup_{y \in Y} |y|_X < \infty$, where $|\cdot|_X = \mathrm{d}_X(1, \cdot)$ denotes the word length with respect to X. It is clear that $\preceq$ is a preorder on the set of generating sets of G and therefore it induces an equivalence relation in the standard way:

$$X \sim Y \quad \Leftrightarrow \quad X \preceq Y \text{ and } Y \preceq X.$$

We denote by $[X]$ the equivalence class of a generating set X and by $\mathcal{G}(G)$ the set of all equivalence classes of generating sets of G. The preorder $\preceq$ induces an order relation $\preccurlyeq$ on $\mathcal{G}(G)$ by the rule

$$[X] \preccurlyeq [Y] \quad \Leftrightarrow \quad X \preceq Y.$$

For example, all finite generating sets of a finitely generated group are equivalent and the corresponding equivalence class is the largest element of $\mathcal{G}(G)$; for every group G, $[G]$ is the smallest element of $\mathcal{G}(G)$. Note also that our order on $\mathcal{G}(G)$ is "inclusion reversing": if X and Y are generating sets of G such that $X \subseteq Y$, then $Y \preceq X$.

To define a hyperbolic structure on a group, we first recall the definition of a hyperbolic space. In this paper we employ the definition of hyperbolicity via the Rips condition.

Definition 2.1.13. A metric space S is called *δ-hyperbolic* if it is geodesic and for any geodesic triangle Δ in S, each side of Δ is contained in the union of the closed δ-neighborhoods of the other two sides.

Definition 2.1.14. [2, Definition 1.2] A *hyperbolic structure* on G is an equivalence class $[X] \in \mathcal{G}(G)$ such that $\Gamma(G, X)$ is hyperbolic. We denote the set of hyperbolic structures by $\mathcal{H}(G)$ and endow it with the order induced from $\mathcal{G}(G)$.

It is well-known that hyperbolicity of a space is a quasi-isometry invariant; thus the definition above is independent of the choice of a particular representative in the equivalence class $[X]$.

Recall that an isometric action of a group G on a metric space (S, d) is *acylindrical* [18] if for every constant ε there exist constants $R = R(\varepsilon)$ and $N = N(\varepsilon)$ such that for every $x, y \in S$ satisfying $\mathrm{d}(x,y) \geq R$, we have

$$\#\{g \in G \mid \mathrm{d}(x, gx) \leq \varepsilon,\ \mathrm{d}(y, gy) \leq \varepsilon\} \leq N.$$

This is a notion that dates back to Sela for groups acting on trees; the general definition is due to Bowditch. Groups acting acylindrically on hyperbolic spaces have received a lot of attention in the recent years. For a brief survey we refer to [69].

Definition 2.1.15. [2, Definition 1.2] The set of *acylindrically hyperbolic structures* on G, denoted $\mathscr{AH}(G)$, consists of hyperbolic structures $[X] \in \mathscr{H}(G)$ such that the action of G on the corresponding Cayley graph $\Gamma(G, X)$ is acylindrical.

It is easy to check that acylindricity is preserved under the equivalence of generating sets. Indeed, the map between Cayley graphs corresponding to equivalent generating sets is G-equivariant.

The last notion we introduce here is that of the Svarc-Milnor map, that allows us to interchangeably work with generating sets or cobounded group actions. Using the standard argument from the proof of the Svarc-Milnor Lemma, it is easy to show that elements of $\mathscr{H}(G)$ are in one-to-one correspondence with equivalence classes of cobounded actions of G on hyperbolic spaces considered up to the natural equivalence : two actions $G \curvearrowright S$ and $G \curvearrowright T$ are equivalent if there is a coarsely G-equivariant quasi-isometry $S \to T$. Indeed, we prove the following results in [2].

Lemma 2.1.16. *[2, Lemma 3.9] Let X, Y be generating sets of a group G. Then $G \curvearrowright \Gamma(G, X) \preceq G \curvearrowright \Gamma(G, Y)$ if and only if $X \preceq Y$. In particular, $G \curvearrowright \Gamma(G, X) \sim G \curvearrowright \Gamma(G, Y)$ if and only if $X \sim Y$.*

Lemma 2.1.17. *[2, Lemma 3.10] Let G be a group generated by a set X and acting on a metric space S. Suppose that for some $s \in S$, we have $\sup_{x \in X} \mathrm{d}_S(s, xs) < \infty$. Then the orbit* *map $g \mapsto gs$ is a Lipschitz map from (G, d_X) to S. In particular, if G is finitely generated, the orbit map is always Lipschitz.*

Lemma 2.1.18. *[2, Lemma 3.11] Let G be a group acting coboundedly on a geodesic metric space S. Let $B \subseteq S$ be a bounded subset such that $\bigcup_{g\in G} gB = S$. Let $D = \mathrm{diam}(B)$ and let b be any point of B. Then the group G is generated by the set*

$$X = \{g \in G \mid \mathrm{d}_S(b, gb) \leq 2D + 1\}$$

and the natural action of G of its Cayley graph $\Gamma(G,X)$ is equivalent to $G \curvearrowright S$.

In particular, given a group G, we let $\mathscr{A}_{cb}(G)$ denote the set of all equivalence classes of cobounded G-actions on geodesic metric spaces (of cardinality at most c). We define a relation $\preccurlyeq$ on $\mathscr{A}_{cb}(G)$ by

$$[G \curvearrowright R] \preccurlyeq [G \curvearrowright S] \ \Leftrightarrow \ G \curvearrowright R \preceq G \curvearrowright S.$$

Then we get the following results.

Proposition 2.1.19. *[2, Proposition 3.12] The map $\mathscr{G}(G) \to \mathscr{A}_{cb}(G)$ defined by $[X] \mapsto [G \curvearrowright \Gamma(G,X)]$ for every $[X] \in \mathscr{G}(G)$ is well-defined and is an isomorphism of posets.*

Proof. That the map is order-preserving and injective follows from Lemma 2.1.16. Surjectivity follows from Lemma 2.1.18. □

Definition 2.1.20 (Svarc-Milnor map)**.** [2, Definition 3.13] Given a group G, we denote by $\sigma\colon \mathscr{A}_{cb}(G) \to \mathscr{G}(G)$ the inverse of the isomorphism described in Proposition 2.1.19. We call σ the *Svarc-Milnor map*.

It follows from Proposition 2.1.19 that σ can be alternatively defined as an isomorphism of posets $\mathscr{A}_{cb}(G) \to \mathscr{G}(G)$ such that for every cobounded action $G \curvearrowright S$, we have

$$G \curvearrowright \Gamma(G,X) \sim G \curvearrowright S \tag{2.4}$$

for every $X \in \sigma([G \curvearrowright S])$.

In particular, the Svarc-Milnor map associates hyperbolic (respectively, acylindrically hyperbolic) structures on a group G to cobounded actions (respectively, cobounded acylindrical actions) of G on hyperbolic spaces. Indeed this follows from (2.4), the well-known fact that hyperbolicity of a geodesic space is a quasi-isometry invariant, and the fact that acylindricity of an action is preserved under the equivalence.

2.2 General classification of hyperbolic structures

We begin by recalling some standard facts about groups acting on hyperbolic spaces. For details the reader is referred to [37].

Given a hyperbolic space S, by ∂S we denote its Gromov boundary. In general, we do not assume that S is proper. Thus the boundary is defined as the set of equivalence classes of sequences convergent at infinity. More precisely, a sequence (x_n) of elements of S converges at infinity if $(x_i|x_j)_s \to \infty$ as $i, j \to \infty$ (this definition is clearly independent of the choice of s). Two such sequences (x_i) and (y_i) are equivalent if $(x_i|y_j)_s \to \infty$ as $i, j \to \infty$. If a is the equivalence class of (x_i), we say that the sequence x_i converges to a. This defines a natural topology on $S \cup \partial S$ with respect to which S is dense in $S \cup \partial S$.

From now on, let G denote a group acting (by isometries) on a hyperbolic space S. By $\Lambda(G)$ we denote the set of limit points of G on ∂S. That is,

$$\Lambda(G) = \partial S \cap \overline{Gs},$$

where $\overline{Gs}$ denotes the closure of a G-orbit in $S \cup \partial S$; it is easy to show that this definition is independent of the choice of $s \in S$. The action of G is called *elementary* if $|\Lambda(G)| \leq 2$ and *non-elementary* otherwise. The action of G on S naturally extends to a continuous action of G on ∂S.

Definition 2.2.1. An element $g \in G$ is called

(i) *elliptic* if $\langle g \rangle$ has bounded orbits;

(ii) *loxodromic* if the orbits of $\langle g \rangle$ are quasi-convex (equivalently, the translation number of g is positive);

(iii) *parabolic* otherwise.

Every loxodromic element $g \in G$ has exactly 2 fixed points $g^{\pm\infty}$ on ∂S, where $g^{+\infty}$ (respectively, $g^{-\infty}$) is the limit of the sequence $(g^n s)_{n\in\mathbb{N}}$ (respectively, $(g^{-n}s)_{n\in\mathbb{N}}$) for any fixed $s \in S$. We clearly have $\Lambda(\langle g \rangle) = \{g^{\pm\infty}\}$. Loxodromic elements $g, h \in G$ are called *independent* if the sets $\{g^{\pm\infty}\}$ and $\{h^{\pm\infty}\}$ are disjoint.

Definition 2.2.2. A *quasi-geodesic* is a quasi-isometric embedding of an interval (bounded or unbounded) $I \subseteq \mathbb{R}$ into a metric space X. Note that geodesics are $(1,0)$-quasi-geodesics. By slight abuse of notation, we may identify the map that defines a quasi-geodesic with its image in the space.

Every loxodromic element $g \in G$ preserves a bi-infinite quasi-geodesic l_g in S; adding $g^{\pm\infty}$ to l_g, we obtain a path in $S \cup \partial S$ that connects $g^{+\infty}$ to $g^{-\infty}$. Such a path is called a *quasi-geodesic axis* (or simply an axis) of g. Given any $s \in S$, we can always construct an axis of g that contains s: take the bi-infinite sequence $\ldots, g^{-2}s, g^{-1}s, s, gs, g^2s, \ldots$ and connect consecutive points by geodesics in S.

The following theorem summarizes the standard classification of groups acting on hyperbolic spaces due to Gromov [37, Section 8.2] (see also [40] for complete proofs in a more general context) and some results from [24, Propositions 3.1 and 3.2].

Theorem 2.2.3. *Let G be a group acting on a hyperbolic space S. Then exactly one of the following conditions holds.*

1) $|\Lambda(G)| = 0$*. Equivalently, G has bounded orbits. In this case the action of G is called* elliptic.

2) $|\Lambda(G)| = 1$*. Equivalently, G has unbounded orbits and contains no loxodromic elements. In this case the action of G is called* parabolic. *A parabolic action cannot be cobounded and the set of points of* ∂S *fixed by G coincides with* $\Lambda(G)$*.*

3) $|\Lambda(G)| = 2$*. Equivalently, G contains a loxodromic element and any two loxodromic elements have the same limit points on* ∂S*. In this case the action of G is called* lineal.

4) $|\Lambda(G)| = \infty$*. Then G always contains loxodromic elements. In turn, this case breaks into two subcases.*

  (a) G fixes a point of ∂S*. Equivalently, any two loxodromic elements of G have a common limit point on the boundary. In this case the action of G is called* quasi-parabolic. *Orbits of quasi-parabolic actions are always quasi-convex.*

  (b) G does not fix any point of ∂S*. Equivalently, G contains infinitely many independent loxodromic elements. In this case the action of G is said to be of* general type.

Parabolic and quasi-parabolic acylindrical actions do not exist. Moreover, we have the following [69].

Theorem 2.2.4. *Let G be a group acting acylindrically on a hyperbolic space. Then exactly one of the following three conditions holds.*

(a) The action is elliptic.

(b) The action is lineal and G is virtually cyclic.

(c) The action is of general type.

An acylindrical action is called *elementary* in cases (a) and (b), and *non-elementary* in case (c). In particular, being non-elementary is equivalent to being of general type for acylindrical actions.

Definition 2.2.5. A group G is said to be *acylindrically hyperbolic* if it admits a non-elementary, acylindrical action on a hyperbolic space.

Lemma 2.2.6. *[2, Lemma 4.4] Let $G \curvearrowright R$ and $G \curvearrowright S$ be equivalent actions of G on hyperbolic spaces. Then $G \curvearrowright R$ and $G \curvearrowright S$ have the same type.*

Proposition 2.2.7. *Let X and Y be equivalent generating sets of a group G. Then the following hold.*

(a) $\Gamma(G,X)$ *is hyperbolic if and only if* $\Gamma(G,Y)$ *is.*

(b) The action $G \curvearrowright \Gamma(G,X)$ is acylindrical if and only if $G \curvearrowright \Gamma(G,Y)$ is.

(c) The action $G \curvearrowright \Gamma(G,X)$ is elliptic (respectively lineal, quasi-parabolic, of general type) if and only if so is $G \curvearrowright \Gamma(G,Y)$.

Thus we obtain the following classification of hyperbolic structures, which follows almost immediately from the results stated above, and the fact that cobounded actions cannot be parabolic. The sets of elliptic, lineal, quasi-parabolic, and general type hyperbolic structures on G are denoted by $\mathcal{H}_e(G)$, $\mathcal{H}_\ell(G)$, $\mathcal{H}_{qp}(G)$, and $\mathcal{H}_{gt}(G)$ respectively. We use analogous notation for acylindrically hyperbolic structures.

Theorem 2.2.8. *[2, Theorem 4.6] For every group G, the following holds.*

(a)

$$\mathcal{H}(G) = \mathcal{H}_e(G) \sqcup \mathcal{H}_\ell(G) \sqcup \mathcal{H}_{qp}(G) \sqcup \mathcal{H}_{gt}(G)$$

and the subsets $\mathcal{H}_e(G) \sqcup \mathcal{H}_\ell(G)$ and $\mathcal{H}_e(G) \sqcup \mathcal{H}_\ell(G) \sqcup \mathcal{H}_{qp}(G)$ are initial segments of $\mathcal{H}(G)$.

(b) Either

$$\mathscr{AH}(G) = \mathscr{AH}_e(G) \sqcup \mathscr{AH}_\ell(G)$$

(if G is virtually cyclic) or

$$\mathscr{AH}(G) = \mathscr{AH}_e(G) \sqcup \mathscr{AH}_{gt}(G)$$

(if G is acylindrically hyperbolic).

2.3 Hyperbolically Embedded Subgroups

In this section, we recall the definition of the notion of a hyperbolically embedded collection of subgroups, introduced in [30]. We then define a strongly hyperbolically embedded collection of subgroups, which is a strengthening of the notion of a hyperbolically embedded collection of subgroups.

Suppose that we have a group G, a collection of subgroups $\{H_1, \dots, H_n\}$ of G, and a subset $X \subseteq G$ such that X together with the union of all H_i generate G. Let

$$\mathscr{H} = H_1 \sqcup H_2 \sqcup \ldots \sqcup H_n. \tag{2.5}$$

We think of X and $\mathscr{H}$ as abstract sets and consider the alphabet

$$\mathscr{A} = X \sqcup \mathscr{H} \tag{2.6}$$

together with the map $\mathscr{A} \to G$ induced by the obvious maps $X \to G$ and $H_i \to G$. By abuse of notation, we do not distinguish between subsets X and H_i of G and their preimages in $\mathscr{A}$. Note, however, the map $\mathscr{A} \to G$ is not necessarily injective. Indeed if X and a subgroup H_i (respectively, subgroups H_i and H_j for some $i \neq j$) intersect in G, then every element of $H_i \cap X \subseteq G$ (respectively, $H_i \cap H_j$) will have at least two preimages in $\mathscr{A}$: one in X and another in H_i (respectively, one in H_i and one in H_j) since we use disjoint unions in (2.5) and (2.6).

In these settings, we consider the Cayley graphs $\Gamma(G, X \sqcup \mathscr{H})$ and $\Gamma(H_i, H_i)$, and we naturally think of the latter as subgraphs of the former. For every $i \in \{1, \ldots, n\}$, we introduce a *relative metric* $\widehat{\mathrm{d}}_i \colon H_i \times H_i \to [0, +\infty]$ as follows: we say that a path p in $\Gamma(G, X \sqcup \mathscr{H})$ is *admissible* if it contains no edges of $\Gamma(H_i, H_i)$. Then $\widehat{\mathrm{d}}_i(h,k)$ is defined to be the length of a shortest admissible path in $\Gamma(G, X \sqcup \mathscr{H})$ that connects h to k. If no such a path exists, we set $\widehat{\mathrm{d}}_i(h,k) = \infty$. Clearly $\widehat{\mathrm{d}}_i$ satisfies the triangle inequality, where addition is extended to $[0, +\infty]$ in the natural way.

Remark 2.3.1. It is important that the union in the definition above is disjoint. This disjoint union leads to the following observation : for every $h \in H_i \cap H_j$, the alphabet $\mathscr{H}$ will have two letters representing h in G, one from H_i and another from H_j. It may also be the case that a letter from $\mathscr{H}$ and a letter from X represent the same element of the group G. In this situation, the corresponding Cayley graph $\Gamma(G, X \sqcup \mathscr{H})$ has bigons (or multiple edges in general) between the identity and the element, one corresponding to each of these letters.

It is convenient to extend the relative metric $\widehat{\mathrm{d}}_i$ to the whole group G by assuming

$$\widehat{\mathrm{d}}_i(f,g) \colon = \begin{cases} \widehat{\mathrm{d}}_i(f^{-1}g, 1), & \text{if } f^{-1}g \in H_i \\ \widehat{\mathrm{d}}_i(f,g) = \infty, & \text{otherwise.} \end{cases}$$

If the collection $\{H_1, \ldots, H_n\}$ consists of a single subgroup H, we use the notation $\widehat{\mathrm{d}}$ instead of $\widehat{\mathrm{d}}_i$.

Definition 2.3.2. A collection of subgroups $\{H_1, \ldots, H_n\}$ of G is *hyperbolically embedded in G with respect to a subset $X \subseteq G$*, denoted $\{H_1, \ldots, H_n\} \hookrightarrow_h (G, X)$, if the following conditions hold.

(a) The group G is generated by X together with the union of all H_i and the Cayley graph $\Gamma(G, X \sqcup \mathscr{H})$ is hyperbolic.

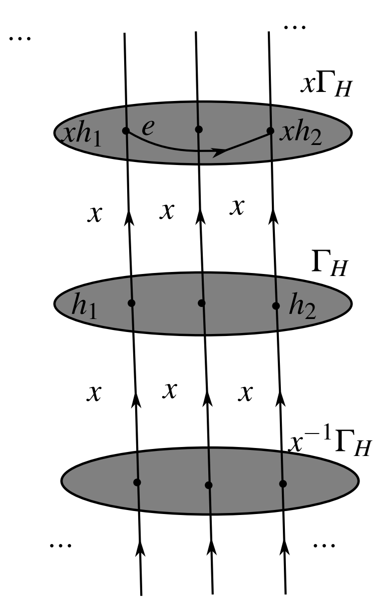

Figure 2.1: $H \times \mathbb{Z}$

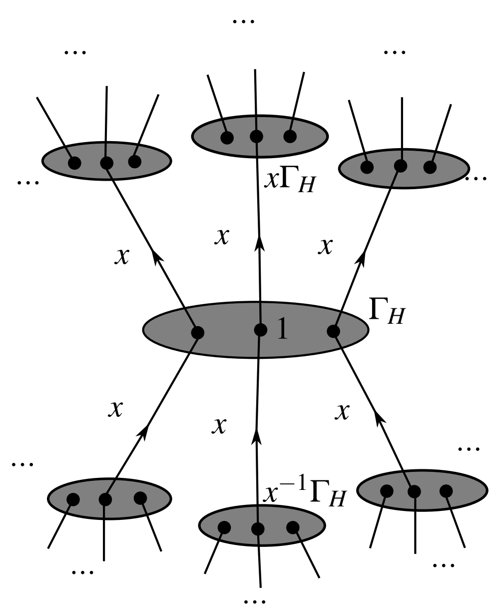

Figure 2.2: $H * \mathbb{Z}$

(b) For every i, the metric space $(H_i, \widehat{\mathrm{d}}_i)$ is proper, i.e., every ball (of finite radius) in H_i with respect to the metric $\widehat{\mathrm{d}}_i$ contains finitely many elements.

If, in addition, the action of G on $\Gamma(G, X \sqcup \mathscr{H})$ is acylindrical, we say that $\{H_1, \ldots, H_n\}$ is *strongly hyperbolically embedded in G with respect to X*.

Finally, we say that the collection of subgroups $\{H_1, \ldots, H_n\}$ is *hyperbolically embedded* in G and write $\{H_1, \ldots, H_n\} \hookrightarrow_h G$ if $\{H_1, \ldots, H_n\} \hookrightarrow_h (G, X)$ for some $X \subseteq G$.

Remark 2.3.3. Unlike the notion of a hyperbolically embedded subgroup, the notion of a strongly hyperbolically embedded subgroup depends on the choice of a generating set. In general, $\{H_1, \ldots, H_n\} \hookrightarrow_h (G, X)$ does not imply that $\{H_1, \ldots, H_n\}$ is strongly hyperbolically embedded in G with respect to X, but does imply that $\{H_1, \ldots, H_n\}$ is strongly hyperbolically embedded in G with respect to some other relative generating set Y containing X, see [69, Theorem 5.4] for details.

Since hyperbolically embedded subgroups and the metric $\widehat{\mathrm{d}}$ introduced above play a crucial role in this thesis, we consider two additional examples borrowed from [30].

Example 2.3.4. (a) Let $G = H \times \mathbb{Z}$, $X = \{x\}$, where x is a generator of $\mathbb{Z}$. Then $\Gamma(G, X \sqcup$

$H)$ is quasi-isometric to a line and hence it is hyperbolic. However the corresponding relative metric satisfies $\widehat{\mathrm{d}}(h_1,h_2) \leq 3$ for every $h_1,h_2 \in H$ (See Fig. 2.1). Indeed let Γ_H denote the Cayley graph $\Gamma(H,H)$. In the shifted copy $x\Gamma_H$ of Γ_H there is an edge (labeled by $h_1^{-1}h_2 \in H$) connecting h_1x to h_2x, so there is an admissible path of length 3 connecting h_1 to h_2. Thus if H is infinite, then $H \not\hookrightarrow_h (G,X)$.

(b) Let $G = H * \mathbb{Z}$, $X = \{x\}$, where x is a generator of $\mathbb{Z}$. In this case $\Gamma(G,X \sqcup H)$ is quasi-isometric to a tree (see Fig. 2.2) and $\widehat{\mathrm{d}}(h_1,h_2) = \infty$ unless $h_1 = h_2$. Thus $H \hookrightarrow_h (G,X)$. In fact, H is strongly hyperbolically embedded in G in this case.

The following result proved in [30] relates the notions of hyperbolically embedded collections of subgroups and relatively hyperbolic groups. (Readers unfamiliar with relative hyperbolicity can take this result as the definition of relatively hyperbolic groups.)

Theorem 2.3.5. *Let G be a group, $\{H_1,\dots,H_n\}$ a collection of subgroups of G. Then $\{H_1,\dots,H_n\} \hookrightarrow_h (G,X)$ for a finite $X \subseteq G$ if and only if G is hyperbolic relative to $\{H_1,\dots,H_n\}$.*

We will make use of several technical notions first introduced in [65, 63] for relatively hyperbolic groups and then generalized in the context of hyperbolically embedded subgroups in [30].

Definition 2.3.6. Let q be a path in the Cayley graph $\Gamma(G,X \sqcup \mathcal{H})$. A (non-trivial) subpath p of q is called an *H_i-subpath* if the label of p is a word in the alphabet H_i. An H_i-subpath p of q is an *H_i-component* if p is not contained in a longer H_i-subpath of q; if q is a loop, we require in addition that p is not contained in any longer H_i-subpath of a cyclic shift of q.

Two H_i-components p_1,p_2 of a path q in $\Gamma(G,X \sqcup \mathcal{H})$ are called *connected* if there exists a path c in $\Gamma(G,X \sqcup \mathcal{H})$ that connects some vertex of p_1 to some vertex of p_2, and the label of c is a word consisting only of letters from H_i. In algebraic terms this means that all vertices of p_1 and p_2 belong to the same left coset of H_i. Note also that we can always assume that c is an edge as every element of H_i is included in the set of generators. A component of a path p is called *isolated* in p if it is not connected to any other component of p.

The following result is a simplified version of [30, Proposition 4.13]. Given a path p in a metric space, we denote by p_- (respectively p_+) its initial (respectively, terminal) point.

Lemma 2.3.7. *Let G be a group and $\{H_1,\dots,H_n\}$ a fixed collection of subgroups in G. Let $X \subset G$ such that G is generated by X together with the union of all $\{H_1,\dots,H_n\}$. Then there exists a constant $C > 0$ such that for any n-gon p with geodesic sides in $\Gamma(G, X \sqcup \mathscr{H})$, any $\lambda \in \Lambda$, and any isolated H_λ component a of p, $\widehat{d_\lambda}(a_-, a_+) \leq Cn$.*

We also have the following results, which will be used in later sections.

Lemma 2.3.8 ([30], Corollary 4.27)**.** *Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, and X_1 and X_2 be relative generating sets. Suppose that $|X_1 \Delta X_2| < \infty$. Then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X_1)$ if and only if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X_2)$.*

Theorem 2.3.9 ([69], Theorem 5.4)**.** *Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a finite collection of subgroups of G, X a subset of G. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$. Then there exists $Y \subset G$ such that the following conditions hold.*

(a) $X \subset Y$

(b) $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, Y)$*. In particular, the Cayley graph* $\Gamma(G, Y \sqcup \mathscr{H})$ *is hyperbolic.*

(c) The action of G on $\Gamma(G, Y \sqcup \mathscr{H})$ is acylindrical.

We now recall some useful results and constructions from [4].

Definition 2.3.10. Let $H_1,\dots,H_n$ be subgroups of a group G and let X be a relative generating set for G with respect to $H_1,\dots,H_n$. Let

$$\iota_X : \mathscr{G}(H_1) \times \cdots \times \mathscr{G}(H_n) \to \mathscr{G}(G)$$

be the map defined by

$$\iota_X([Y_1],\ldots,[Y_n]) = \left[X \cup \left(\bigcup_{i=1}^{n} Y_i\right)\right]. \tag{2.7}$$

This map can be thought of as the analogue of the induced action map defined in [4] for equivalence classes of group actions on geodesic metric spaces. In general, very little can be said about this map. However, if the collection of subgroups is hyperbolically embedded, then the map behaves well. In the theorem below, we restate some of the results of [4] in this regard using terminology of [2].

Theorem 2.3.11. *Let G be a group, let $H_1,\ldots,H_n$ be subgroups of G, and let X be a relative generating set for G with respect to $H_1,\ldots,H_n$. Then the map ι_X defined by (2.7) is well-defined and order preserving. If, in addition, $\{H_1,\ldots,H_n\} \hookrightarrow_h (G,X)$, then ι_X sends $\mathcal{H}(H_1)\times\cdots\times\mathcal{H}(H_n)$ to $\mathcal{H}(G)$. In particular, ι_X is injective.*

We extend this result in [2] by showing that the induced action also preserves acylindricity.

Theorem 2.3.12. *[2, Theorem 5.20] Suppose that a collection of subgroups $\{H_1,\ldots,H_n\}$ is strongly hyperbolically embedded in a group G with respect to a relative generating set X. Then for every $A \in \mathcal{AH}(H_1)\times\cdots\times\mathcal{AH}(H_n)$, we have $\iota_X(A) \in \mathcal{AH}(G)$.*

2.4 Accessibility

The famous Stallings' theorem states that every finitely generated group with infinitely many ends splits as the fundamental group of a graph of groups with finite edge groups. This was a starting point of an accessibility theory developed by Dunwoody. A finitely generated group G is said to be *accessible* if the process of iterated nontrivial splittings of G over finite subgroups always terminates in a finite number of steps. Not every finitely generated group is accessible [32], but finitely presented groups are [31], as well as torsion free groups [51].

More generally, one can ask whether a given group has a maximal, in a certain precise sense, action on a tree satisfying various additional conditions on stabilizers (see, for example, [11, 75]). Yet another problem of similar flavor studied in the literature is whether a given group admits a maximal relatively hyperbolic structure [9]. It is natural to ask a similar question in our setting.

Definition 2.4.1. [2, Definition 2.17] We say that a group G is *$\mathcal{H}$-accessible* (respectively *$\mathcal{AH}$-accessible*) if $\mathcal{H}(G)$ (respectively $\mathcal{AH}(G)$) contains the largest element.

Definition 2.4.2. A group G is said to be *strongly $\mathcal{AH}$-accessible* if there exists an element $[X] \in \mathcal{AH}(G)$ such that for every acylindrical action (not necessarily cobounded) of G on any hyperbolic space S, we have $G \curvearrowright S \preceq G \curvearrowright \Gamma(G,X)$, with respect to the preorder on group actions from Definition 2.1.5. We say that the structure $[X]$ *realizes* the strong $\mathcal{AH}$-accessibility of G. Such a structure, if it exists, is obviously unique. In particular, a strongly $\mathcal{AH}$-accessible group is $\mathcal{AH}$-accessible.

Remark 2.4.3. Note that in the above definition, we do not restrict the cardinality of S. We consider actions of G on *all* hyperbolic metric spaces, not just those of bounded cardinality.

Example 2.4.4. Every hyperbolic group is strongly $\mathcal{AH}$-accessible. Indeed, if G is a hyperbolic group, then there exists a finite generating set X such that $[X] \in \mathcal{AH}(G)$. The result then follows from Lemma 2.1.17.

Example 2.4.5. Every group G which is not acylindrically hyperbolic is strongly $\mathcal{AH}$-accessible. Indeed, this is an immediate consequence of Theorem 2.2.4. If G is virtually cyclic, the result follows from Example 2.4.4. If every acylindrical action of G on a hyperbolic space is elliptic, then the trivial structure realizes the strong $\mathcal{AH}$-accessibility of G.

In particular, this example applies to groups with infinite amenable radicals (e.g. infinite center) (see [69, Corollary 7.2]) and to direct products of two infinite groups (see [69, Corollary 7.2]). In fact, if G is the direct product of two groups with infinite order elements, then the trivial structure realizes the strong $\mathscr{AH}$-accessibility of G (since G contains $\mathbb{Z}\times\mathbb{Z}$ as a subgroup, it is not virtually cyclic).

It is easy to find examples of groups which are not $\mathscr{H}$-accessible, e.g., the direct product $F_2 \times F_2$; however, this group is $\mathscr{AH}$-accessible, see Section 7.1. Finding $\mathscr{AH}$-inaccessible groups, especially finitely generated or finitely presented ones, is more difficult. However, for a lot of well-known classes of groups in geometric group theory, the question of accessibility can answered in the affirmative, as proved in the next chapter of this thesis.

CHAPTER 3

$\mathscr{AH}-$ ACCESSIBLE GROUPS

3.1 Sufficient condition for largest action

The goal of this chapter is to prove the following theorem.

Theorem 3.1.1. *The following groups are $\mathscr{AH}$-accessible.*

(a) Finitely generated relatively hyperbolic groups whose parabolic subgroups are not acylindrically hyperbolic.

(b) Mapping class groups of punctured closed surfaces.

(c) Right-angled Artin groups.

(d) Fundamental groups of compact orientable 3*-manifolds with empty or toroidal boundary.*

In fact, we will prove something stronger for each of the classes of groups mentioned in Theorem 3.1.1; namely that they are strongly $\mathscr{AH}-$accessible. This fact will be immediately clear from the proofs below. We would like to note that following an early draft of [2], parts (b) and (c) of the above theorem were independently and subsequently proven in [3], which additionally proves the $\mathscr{AH}$-accessibility of certain other groups using different methods. The special case of part (d) of the above theorem when the 3-manifold has no Nil or Sol in its prime decomposition is also proven in [3].

In order to prove that the groups listed in Theorem 3.1.1 have largest actions, we will use the following sufficient condition for largest actions, developed in [2]. Note that in the proposition, we use the order on group actions introduced in Definition 2.1.9. Here and in what follows, we always think of connected graphs as metric spaces with respect to the combinatorial metric.

Proposition 3.1.2. *Let G be a group acting cocompactly on a connected graph Δ and let $\mathscr{A}$ be a set of actions of G on metric spaces. Suppose that for every vertex $v \in V(\Delta)$ and every action $G \curvearrowright S \in \mathscr{A}$, the induced action of the stabilizer $Stab_G(v)$ on S has bounded orbits. Then $A \preceq G \curvearrowright \Delta$ for all $A \in \mathscr{A}$.*

As an immediate corollary of Proposition 3.1.2 and Proposition 2.1.19, we obtain the following.

Corollary 3.1.3. *[2, Corollary 4.14] Let G be a group acting cocompactly on a connected graph Δ and let $\mathscr{F} \subseteq \mathscr{G}(G)$ be any subset. Suppose that for every $v \in V(\Delta)$ and every $[X] \in \mathscr{F}$, the stabilizer $Stab_G(v)$ has bounded diameter with respect to d_X. Then $\sigma([G \curvearrowright \Delta])$ is an upper bound for $\mathscr{F}$ in $\mathscr{G}(G)$.*

3.2 Relatively hyperbolic groups

We start with the proof of Theorem 3.1.1 by dealing first with the case of relatively hyperbolic groups. Recall that a group G is *hyperbolic relative to subgroups* $H_1, \dots, H_n$ if $\{H_1, \dots, H_n\} \hookrightarrow_h (G, X)$ for some finite $X \subset G$ (see Theorem 2.3.5). The subgroups $H_1, \dots, H_n$ are called *peripheral subgroups* of G.

Theorem 3.2.1. *Let G be a relatively hyperbolic group with peripheral subgroups $H_1, \dots, H_n$. If each H_i is strongly $\mathscr{A}\mathscr{H}$-accessible, then G is strongly $\mathscr{A}\mathscr{H}$-accessible.*

Proof. Let X be a finite subset of G such that $\{H_1, \dots, H_n\} \hookrightarrow_h (G, X)$. Let $\mathscr{H} = \sqcup_{i=1}^n H_i$. By [69, Proposition 5.2], the action of G on $\Gamma(G, X \sqcup \mathscr{H})$ is acylindrical, i.e., $\{H_1, \dots, H_n\}$ is strongly hyperbolically embedded in G with respect to X.

Let $[Y_i]$ be the element in $\mathscr{A}\mathscr{H}(H_i)$ that realizes the strong $\mathscr{A}\mathscr{H}$-accessibility of H_i, for each $1 \leq i \leq n$. Then by Theorem 2.3.12, $[X \cup Y_1 \cup \dots \cup Y_n] \in \mathscr{A}\mathscr{H}(G)$. We will show that this element realizes the strong $\mathscr{A}\mathscr{H}$-accessibility of G.

Indeed, suppose that $G \curvearrowright Z$ is an acylindrical action of G on a hyperbolic space and let d_Z denote the metric on Z. Fix a base point $z \in Z$. Restricting the action of G on Z to each

H_i, we obtain an acylindrical action of each H_i on a hyperbolic space. But then

$$H_i \curvearrowright (Z, d_Z) \preceq H_i \curvearrowright \Gamma(H_i, Y_i)$$

for every $1 \le i \le n$. We can assume that there exists a constant C such that for every $i = 1, \ldots, n$, $d_Z(z, hz) \le C d_{Y_i}(1, h) + C$ for all $h \in H_i$. In particular, if $y \in Y_i$, then $d_Z(z, yz) \le 2C$.

Since X is finite, Lemma 2.1.17 applies and we conclude that

$$G \curvearrowright Z \preceq G \curvearrowright \Gamma(G, X \cup Y_1 \cup \ldots \cup Y_n).$$

□

Proof of Theorem 3.1.1(a). If G is a finitely generated relatively hyperbolic group, then it follows from [65, Theorem 1.1] that the collection of peripheral subgroups is finite. Let H be a peripheral subgroup of G, which by assumption, is not acylindrically hyperbolic. Example 2.4.5 applies, and we conclude that each peripheral subgroup is strongly $\mathscr{AH}$-accessible. The result follows from Theorem 3.2.1. □

3.3 Mapping class groups

We next deal with the case of mapping class groups of closed punctured surfaces, for which we will need several facts and definitions taken from [36], stated below. We refer the reader to [36] for proofs and details.

Definition 3.3.1 (Complex of curves)**.** A closed curve on S is called *essential* if it is not homotopic to a point or a puncture. The complex of curves associated to S is a graph defined as follows: vertices of the complex of curves are isotopy classes of essential, simple closed curves, and two vertices are joined by an edge if the curves have disjoint representatives on S. The complex of curves is denoted by $C(S)$.

We let g denote the genus of the surface S and p denote the number of punctures. We adopt the convention that if a is a vertex of $C(S)$, then by slight abuse of notation, we let a denote the associated curve, and let T_a be the Dehn twist about a. We now list some facts about Dehn twists and $C(S)$ which we will require for the proof.

(a) [36, Propositions 3.1 and 3.2] T_a is a non-trivial infinite order element of G.

(b) [36, Fact 3.6] $T_a = T_b$ if and only if $a = b$.

(c) [36, Fact 3.8] For any $f \in \mathrm{Mod}(S)$ and any isotopy class a of simple closed curves in S, we have that f commutes with T_a if and only if $f(a) = a$.

(d) [36, Fact 3.9] For any two isotopy classes a and b of simple closed curves in a surface S, we have that a and b are connected by an edge in $C(S)$ if and only if $T_aT_b = T_bT_a$.

(e) [36, Powers of Dehn twists, pg 75] For non-trivial Dehn twists T_a and T_b, and non-zero integers j,k, we have $T_a^j = T_b^k$ if and only if $a = b; j = k$.

(f) [36, Sec 1.3] G admits a cocompact, isometric action on $C(S)$. (This follows from the change of co-ordinates principle).

(g) [55, Theorem 1.1] $C(S)$ is a hyperbolic space. Except when S is a sphere with 3 or fewer punctures, C(S) has infinite diameter.

(h) [19, Theorem 1.3] If S is a surface satisfying $3g + p \geq 5$, then the action of $\mathrm{Mod}(S)$ on $C(S)$ is acylindrical.

Proof of Theorem 3.1.1(b). Let S be a compact, punctured surface without boundary, of genus g and with p punctures. We consider the following two cases.

Case 1. First assume that $3g + p < 5$. The mapping class groups for the cases $g = 0$ and $p = 0,1,2,3$ are finite and hence $\mathscr{AH}$-accessible. In the cases of $g = 0, p = 4$ (the four-punctured sphere) and $g = 1, p = 0,1$ (the torus and the once punctured torus), the mapping class groups are hyperbolic groups. Example 2.4.4 applies and we conclude that these groups are also $\mathscr{AH}$-accessible.

Case 2. We now assume that $3g+p \geq 5$. In this case, we will prove the result by using Corollary 3.1.3 applied to $\mathscr{AH}(G) \subset \mathscr{G}(G)$ for $G = \mathrm{Mod}(S)$.

By fact (f) above, G admits a cocompact (hence cobounded), isometric action on $C(S)$. By facts (g) and (h) above, $C(S)$ is an infinite diameter hyperbolic space and the action of $\mathrm{Mod}(S)$ on $C(S)$ is acylindrical.

To apply Corollary 3.1.3, we must consider stabilizers of vertices of $C(S)$. Let $H = Stab_G(a)$, where a is a vertex of $C(S)$. By fact (a) above, T_a is a non-trivial infinite order element of G. Further $T_a(a) = a$, so $T_a \in H$ by fact (c) above. For every element $f \in H$, using fact (c) again, we must have $fT_a = T_a f$ since $f(a) = a$. This implies that H has an infinite center and is thus not acylindrically hyperbolic (see Example 2.4.5).

Since $C(S)$ is connected and unbounded, there exists a vertex $b \neq a$ of $C(S)$ connected by an edge to a. Then T_b is a non-trivial infinite order element by fact (a), and by applying facts (d)and (c) above, $T_b \in H$. By using fact (e) above, one can easily show that $\langle T_a, T_b\rangle \cong \mathbb{Z}^2 \leq H$, so H is not virtually cyclic. By Theorem 2.2.4, every acylindrical action of H on a hyperbolic space is elliptic. In particular, for every acylindrical action of G on a hyperbolic space, the induced action of H is also acylindrical and thus elliptic. Applying Corollary 3.1.3, it follows that G is $\mathscr{AH}$-accessible with largest element $\sigma([G \curvearrowright C(S)])$. $\square$

Remark 3.3.2. The above proof also applies to the set $\mathscr{A}$ of acylindrical actions of $\mathrm{Mod}(S)$ on hyperbolic spaces. In this case, Proposition 3.1.2 applies, and we conclude that $\mathrm{Mod}(S)$ is strongly $\mathscr{AH}$-accessible. The same holds true for the cases of RAAGs and 3-manifolds discussed below.

3.4 Right Angled Artin groups

We now proceed to the case of right-angled Artin groups (RAAGs). We begin by defining a RAAG and its extension graph.

Definition 3.4.1. Given a finite graph Γ, the *right-angled Artin group* on Γ is the group defined by the presentation

$$A(\Gamma) = \langle V(\Gamma) \mid [a,b] = 1 \quad \forall \{a,b\} \in E(\Gamma)\rangle.$$

For example, the RAAG corresponding to a complete graph on n vertices is $\mathbb{Z}^n$.

Definition 3.4.2. The *extension graph* Γ^e corresponding to $A(\Gamma)$, introduced in [47], is a graph with vertex set

$$\{v^g \mid v \in V(\Gamma), g \in A(\Gamma)\}$$

and edges defined by the following rule: two distinct vertices u^g and v^h are joined by an edge if and only if they commute in $A(\Gamma)$.

There is a natural right-conjugation action of $A(\Gamma)$ on Γ^e given by $gv^h = v^{hg}$ for $v \in V(\Gamma)$ and $g,h \in A(\Gamma)$. Further, we may write

$$\Gamma^e = \bigcup_{g \in A(\Gamma)} g\Gamma,$$

where $g\Gamma$ denotes the graph Γ with its vertices replaced by the corresponding conjugates by g.

We will require the following theorems for the proof. For the proofs of these theorems and further details concerning RAAGs, we refer the reader to [47].

Theorem 3.4.3. *[48, Lemma 26] Let Γ be a finite connected graph. Then Γ^e is a quasi-tree.*

Theorem 3.4.4. *[47, Theorem 30] The action of $A(\Gamma)$ on Γ^e is acylindrical.*

We first prove the strong $\mathcal{AH}$-accessibility of RAAGs arising from finite connected graphs. We will then use this result to prove the $\mathcal{AH}$-accessibility of RAAGs arising from any finite graph.

Lemma 3.4.5. *Let Γ be a connected finite graph and $G = A(\Gamma)$. Then G is strongly $\mathscr{A}\mathscr{H}$-accessible.*

Proof. Let $V(\Gamma)$ denote the set of vertices of the graph Γ. If $|V(\Gamma)| = 1$, then $G \cong \mathbb{Z}$. Example 2.4.4 applies in this case and we conclude that G is strongly $\mathscr{A}\mathscr{H}$-accessible.

Thus we may assume that $|V(\Gamma)| \geq 2$. In this case, we will prove the result by using Proposition 3.1.2 applied to the set $\mathscr{A}$ of acylindrical actions of G on hyperbolic spaces. Observe that $G \curvearrowright \Gamma^e$ is cocompact and isometric. By Theorem 3.4.3 and 3.4.4, $G \curvearrowright \Gamma^e$ is acylindrical and Γ^e is a quasi-tree and hence a hyperbolic space. Thus $\sigma([G \curvearrowright \Gamma^e]) \in \mathscr{A}\mathscr{H}(G)$.

Since the action of G on Γ^e is by conjugation, stabilizers of vertices of the extension graph correspond to centralizers of conjugates of standard generators of G. So we must consider $H = C_G(a^g)$, where a represents a vertex of Γ, and g is any element of G.

Since Γ is connected and $|V(\Gamma)| \geq 2$, there exists a vertex $b \neq a$ such that b is connected to a in Γ^e, i.e. $[a,b] = 1$ in G. But then $[a^g, b^g] = 1$, so $b^g \in H$. It can be easily shown that $\langle a^g, b^g \rangle \cong \langle a, b \rangle \cong \mathbb{Z}^2 \leq H$, since the RAAG corresponding to a graph with 2 vertices and an edge connecting them is $\mathbb{Z}^2$. Thus H cannot be virtually cyclic.

Since the center of H contains the infinite cyclic group $\langle a^g \rangle$, H cannot be acylindrically hyperbolic by Example 2.4.5. Thus H cannot act non-elementarily and acylindrically on a hyperbolic space. By Theorem 2.2.4, for any acylindrical action of G on a hyperbolic space, the induced action of H is elliptic. Applying Proposition 3.1.2, we conclude that G is strongly $\mathscr{A}\mathscr{H}$-accessible. $\square$

Proof of Theorem 3.1.1(c). If Γ is connected, the result follows from Lemma 3.4.5. If Γ is a disconnected finite graph, then Γ has two or more connected components, say $\Gamma_1, \Gamma_2, ..., \Gamma_n$. Let $A(\Gamma_i)$ denote the RAAG associated to the connected subgraph Γ_i of Γ. It is easy to see that $G = A(\Gamma_1) * A(\Gamma_2) * ... * A(\Gamma_n)$, and so G is hyperbolic relative to the collection $\{A(\Gamma_i) \mid 1 \leq i \leq n\}$. By Lemma 3.4.5, each RAAG $A(\Gamma_i)$ is strongly $\mathscr{A}\mathscr{H}$-accessible. Using Theorem 3.2.1, we conclude that G is $\mathscr{A}\mathscr{H}$-accessible. $\square$

3.5 Fundamental groups of manifolds

Lastly, we consider the case of fundamental groups of compact, orientable 3-manifolds with empty or toroidal boundary. In order to prove the theorem, we will need the following results and definitions.

Definition 3.5.1. A 3-manifold N is said to be *irreducible* if every embedded S^2 bounds a 3-ball.

Definition 3.5.2. A 3-manifold N is said to be *atoroidal* if any map $T \to N$ which induces a monomorphism of fundamental groups can be homotoped into the boundary of N, i.e., N contains no essential tori.

Definition 3.5.3. A *Seifert fibered* manifold is a 3-manifold N together with a decomposition into disjoint simple closed curves (called *Seifert fibers*) such that each fiber has a tubular neighborhood that forms a standard fibered torus.

The *standard fibered torus* corresponding to a pair of coprime integers (a,b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $\frac{2\pi b}{a}$, equipped with natural fibering by circles.

Lemma 3.5.4. *[7, Lemma 1.5.1] Let N be a Seifert fibered manifold. If $\pi_1(N)$ is infinite, then it contains a normal, infinite cyclic subgroup.*

Definition 3.5.5. A compact 3-manifold is said to be *hyperbolic* if its interior admits a complete metric of constant negative curvature -1.

The following theorem was first announced by Waldhausen ([79]), and was proved independently by Jaco-Shalen ([43]) and Johannson ([44]).

Theorem 3.5.6. *[7, Theorem 1.6.1][JSJ decomposition Theorem] Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Then there exists a* *collection of disjointly embedded incompressible tori $T_1, T_2, ..., T_k$ such that each component of N cut along $T_1 \cup T_2 \cup ... \cup T_k$ is atoroidal or Seifert fibered. Furthermore, any such collection of tori with a minimal number of components is unique up to isotopy.*

The tori in the above theorem are referred to as *JSJ-tori*. If $T = \cup_{i=1}^{k} T_k$, the connected components of $N \backslash T$ are called *JSJ-components*. For details of atoroidal and Seifert fibered manifolds, we refer the reader to [7, Sections 1.5 and 1.6].

The following was proved by Perelman in his seminal papers (See [71, 72, 73]).

Theorem 3.5.7. *[7, Theorem 1.7.5][Hyperbolization Theorem] Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Suppose that N is not homeomorphic to $S^1 \times D^2$ (solid torus), $T^2 \times I$ (torus bundle), $K^2 \overset{\sim}{\times} I$ (twisted klein bottle bundle). If N is atoroidal and $\pi_1(N)$ is infinite, then N is a hyperbolic manifold.*

Remark 3.5.8. Note that the manifolds $S^1 \times D^2$, $T^2 \times I$, or $K^2 \tilde{\times} I$, although atoroidal, are also Seifert fibered manifolds, and are hence considered to be Seifert fibered JSJ components. Under this convention, the Hyperbolization theorem implies that JSJ components of N are either hyperbolic or Seifert fibered manifolds.

Definition 3.5.9. Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. We say N is a *graph manifold* if all its JSJ components are Seifert fibered manifolds.

The next result can be found in [7, Theorem 7.2.2]. This result follows easily from a combination theorem proved by Dahmani (see [29, Theorem 0.1]) or a more general combination theorem, later proved by the third author (see [67, Corollary 1.5]). The result has also been proved by Bigdely and Wise (see [15, Corollary E]).

Theorem 3.5.10. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Let $M_1, \ldots, M_k$ be the maximal graph manifold pieces of the JSJ-decomposition of N. Let $S_1, \ldots, S_l$ be the tori in the boundary of N that adjoin a hyperbolic* *piece and let $T_1, \ldots, T_m$ be the tori in the JSJ-decomposition of N that separate two (not necessarily distinct) hyperbolic components of the JSJ-decomposition. The fundamental group of N is hyperbolic relative to the set of peripheral subgroups*

$$\{H_i\} = \{\pi_1(M_p)\} \cup \{\pi_1(S_q)\} \cup \{\pi_1(T_r)\}.$$

The last theorem we mention here is a combination of [80, Lemma 2.4] and [58, Lemma 5.2]. Although [80, Lemma 2.4] was originally stated and proved for closed manifolds, the same proof also holds for manifolds with toroidal boundary. (See proofs of [80, Lemma 2.3 and Lemma 2.4]).

Theorem 3.5.11. *Let N be an orientable, irreducible 3-manifold with empty or toroidal boundary. Then either N has a finite-sheeted covering space that is a torus bundle over a circle or the action of $\pi_1(N)$ on the Bass-Serre tree associated to the JSJ decomposition of $\pi_1(N)$ is acylindrical.*

Proof of Theorem 3.1.1(d). We first observe that it suffices to prove the theorem for a compact, orientable, irreducible 3-manifold N with empty or toroidal boundary. Indeed, if N is not irreducible, we let $\widehat{N}$ denote the 3-manifold obtained from N by gluing 3-balls to all spherical components of ∂N. Then $\widehat{N}$ is irreducible, and $\pi_1(\widehat{N}) = \pi_1(N)$. Also observe that if $\pi_1(N)$ is finite, then it is $\mathscr{A}\mathscr{H}$-accessible by Example 2.4.4, so we may assume that $\pi_1(N)$ is infinite in what follows. We consider the following two cases.

Case 1. If there are no JSJ-tori, then it follows from Theorem 3.5.6 that N is either an atoroidal manifold or is Seifert fibered. If N is atoroidal and not homeomorphic to $S^1 \times D^2$, $T^2 \times I$, or $K^2 \tilde{\times} I$, then it follows from Theorem 3.5.7 that N is hyperbolic. Consequently, if N is closed, $\pi_1(N)$ is a hyperbolic group and hence $\mathscr{A}\mathscr{H}$-accessible by Example 2.4.4. If N has toroidal boundary, then $\pi_1(N)$ is hyperbolic relative to its peripheral subgroups, which are isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (see [35]). Applying Example 2.4.5, we get that $\mathbb{Z} \times \mathbb{Z}$ is strongly $\mathscr{A}\mathscr{H}$-accessible. By Theorem 3.2.1, we conclude that $\pi_1(N)$ is $\mathscr{A}\mathscr{H}$-accessible.

If N is Seifert fibered (recall that $S^1 \times D^2$, $T^2 \times I$, or $K^2 \tilde{\times} I$ are considered Seifert fibered JSJ components, as explained in Remark 3.5.8), then by Lemma 3.5.4, $\pi_1(N)$ has an infinite cyclic, normal subgroup. Since $\mathbb{Z}$ is not acylindrically hyperbolic, we can use [69, Corollary 1.5] to conclude that $\pi_1(N)$ is not acylindrically hyperbolic. Applying Theorem 2.2.4, we conclude that either $\pi_1(N)$ is virtually cyclic, or every acylindrical action of $\pi_1(N)$ on a hyperbolic space is elliptic. In the former situation, $\pi_1(N)$ is $\mathscr{AH}$-accessible by Example 2.4.4. In the latter case, $\pi_1(N)$ is obviously $\mathscr{AH}$-accessible.

Case 2. We now assume that N admits at least one JSJ torus, i.e., the JSJ decomposition of N is non-trivial. By the Seifert-van Kampen theorem, the JSJ decomposition of N induces a graph of groups decomposition of $\pi_1(N)$ whose vertex groups are the fundamental groups of the JSJ components, and the edge groups are the fundamental groups of the JSJ tori.

By Theorem 3.5.10, it suffices to prove the strong $\mathscr{AH}$-accessibility of each peripheral subgroup H_i provided by the theorem. Following the notation of Theorem 3.5.10, if $H_i = \pi_1(S_q)$ or $H_i = \pi_1(T_r)$, then $H_i \simeq \mathbb{Z} \times \mathbb{Z}$. By Example 2.4.5, such H_i are strongly $\mathscr{AH}$-accessible.

It thus remains to consider the graph manifolds M_p, which have at least one JSJ component. Using Theorem 3.5.11, either M_p has a finite-sheeted covering space that is a torus bundle over a circle or the action of $\pi_1(M_p)$ on the Bass-Serre tree associated to the JSJ decomposition of $\pi_1(M_p)$ is acylindrical. We denote this Bass-Serre tree by $\mathscr{T}_p$.

If M_p has a finite-sheeted covering space that is a torus bundle over a circle, then $\pi_1(M_p)$ is virtually polycyclic and is hence not acylindrically hyperbolic. Further, since we have at least one JSJ torus, $\mathbb{Z} \times \mathbb{Z} \leq \pi_1(M_p)$, which means that $\pi_1(M_p)$ is not virtually cyclic. By Theorem 2.2.4, every acylindrical action of $\pi_1(M_p)$ on a hyperbolic space is elliptic, allowing us to conclude that the trivial structure realizes the strong $\mathscr{AH}$-accessibility of $\pi_1(M_p)$.

If the action of $\pi_1(M_p)$ on the Bass-Serre tree $\mathscr{T}_p$ is acylindrical, then we will use

Proposition 3.1.2 applied to the set $\mathscr{A}$ of acylindrical actions of $\pi_1(M_p)$ on hyperbolic spaces in order to prove that $\pi_1(M_p)$ is strongly $\mathscr{A}\mathscr{H}$-accessible. Note that the action $\pi_1(M_p) \curvearrowright \mathscr{T}_p$ is cocompact and so $\sigma([\pi_1(M_p) \curvearrowright \mathscr{T}_p]) \in \mathscr{A}\mathscr{H}(\pi_1(M_p))$. Stabilizers of vertices for this action are isomorphic to the vertex groups, which are the fundamental groups of Seifert fibered components. Let M be a Seifert fibered component of M_p. Since we have at least one JSJ torus, $\mathbb{Z} \times \mathbb{Z} \leq \pi_1(M)$ and $\pi_1(M)$ is infinite. Arguing as in Case 1 by using Lemma 3.5.4, we can conclude that $\pi_1(M)$ is not acylindrically hyperbolic. Applying Theorem 2.2.4 allows us to conclude that the induced action of $\pi_1(M)$ in any acylindrical action of $\pi_1(M_p)$ on hyperbolic spaces is elliptic. Applying Proposition 3.1.2, we get that $\pi_1(M_p)$ is strongly $\mathscr{A}\mathscr{H}$-accessible. $\square$

CHAPTER 4

ACYLINDRICAL ACTIONS ON QUASI-TREES

4.1 Background

Recall that a group is called acylindrically hyperbolic if it admits a non-elementary, acylindrical action on a hyperbolic space. The main goal of this section is to answer the following.

Problem 4.1.1. *Which groups admit non-elementary cobounded acylindrical actions on quasi-trees?*

In this thesis, by a quasi-tree we mean a connected graph which is quasi-isometric to a tree. Quasi-trees form a very particular subclass of the class of all hyperbolic spaces. From the asymptotic point of view, quasi-trees are exactly "1-dimensional hyperbolic spaces". As explained in the introduction (see Table 1.1), one could expect that the answer to Question 4.1.1 would produce a proper subclass of the class of all acylindrically hyperbolic groups, which generalizes virtually free groups in the same sense as acylindrically hyperbolic groups generalize hyperbolic groups. Our main result shows that this does not happen.

Theorem 4.1.2. *Every acylindrically hyperbolic group admits a non-elementary cobounded acylindrical action on a quasi-tree.*

In other words, being acylindrically hyperbolic is equivalent to admitting a non-elementary acylindrical action on a quasi-tree. Although this result does not produce any new class of groups, it can be useful in the study of acylindrically hyperbolic groups and their subgroups. In this thesis, we concentrate on proving Theorem 4.1.2 and leave applications for future papers to explore (for some applications, see [56]).

It was known before that every acylindrically hyperbolic group admits a non-elementary cobounded action on a quasi-tree satisfying the so-called *weak proper discontinuity* property, which is weaker than acylindricity. Such a quasi-tree can be produced by using projection complexes introduced by Bestvina-Bromberg-Fujiwara in [12]. To the best of our knowledge, whether the corresponding action is acylindrical is an open question. The main idea of the proof of Theorem 4.1.2 is to combine the Bestvina-Bromberg-Fujiwara approach with an 'acylindrification' construction from [69] in order to make the action acylindrical. An essential role in this process is played by the notion of a hyperbolically embedded subgroup introduced in [30] - this fact is of independent interest since it provides a new setting for the application of the Bestvina-Bromberg-Fujiwara construction.

The above mentioned construction has been applied in the setting of geometrically separated subgroups (see [30, Section 4.5]). However, not every hyperbolically embedded subgroup $H \leq G$ arises from an action of G on a hyperbolic space in which H is geometrically separated. Nevertheless, it is possible to employ hyperbolically embedded subgroups in this construction, possibly with interesting applications. If fact, we prove much stronger results in terms of hyperbolically embedded subgroups (see Theorem 4.4.1) of which Theorem 4.1.2 is an easy consequence, and derive an application in this paper which is stated below (see Corollary 4.5.5).

Corollary 4.1.3. *Let G be a group. If $H \leq K \leq G$, H is countable and H is hyperbolically embedded in G, then H is hyperbolically embedded in K.*

We would like to note that the above result continues to hold even when we have a finite collection $\{H_1, H_2, ..., H_n\}$ of hyperbolically embedded subgroups in G such that $H_i \leq K$ for all $i = 1, 2, ..., n$. Interestingly, A.Sisto obtains a similar result in [77], Corollary 6.10. His result does not require H to be countable, but under the assumption that $H \cap K$ is a virtual retract of K, it states that $H \cap K \hookrightarrow_h K$. Although similar, these two theorems are distinct in the sense that neither follows from the other.

Another application of Theorem 4.4.1 is to the case of finitely generated subgroups, as stated below (see Corollary 4.5.8).

Corollary 4.1.4. *Let H be a finitely generated subgroup of an acylindrically hyperbolic group G. Then there exists a subset $X \subset G$ such that*

(a) $\Gamma(G,X)$ *is hyperbolic, and the action of* G *on* $\Gamma(G,X)$ *is non-elementary and acylindrical*

(b) H *is quasi-convex in* $\Gamma(G,X)$

The above result indicates that in order to develop a theory of quasi-convex subgroups in acylindrically hyperbolic groups, the notion of quasi-convexity is not sufficient, i.e., a stronger set of conditions is necessary in order to prove results similar to those known for quasi-convex subgroups in hyperbolic groups. For example, using Rips' construction from [74] and the above corollary, one can easily construct an example of an infinite, infinite index, normal subgroup in an acylindrically hyperbolic group, which is quasi-convex with respect to some non-elementary acylindrical action.

4.2 A silght modification to the relative metric

The aim of this section is to modify the relative metric on countable subgroups that are hyperbolically embedded, so that the resulting metric takes values only in $\mathbb{R}$, i.e., is finite valued. This will be of importance in section 4.4. The main result of this section is the following.

Theorem 4.2.1. *Let G be a group. Let $H < G$ be countable, such that $H \hookrightarrow_h G$. Then there exists a left-invariant metric $\widetilde{d}\colon H \times H \to \mathbb{R}$, such that*

(a) $\widetilde{d} \leq \widehat{d}$

(b) $\widetilde{d}$ *is proper, i.e., every ball of finite radius has finitely many elements.*

Proof. There exists a collection of finite, symmetric (closed under inverses) subsets $\{F_i\}$ of H such that $H = \bigcup_{i=1}^{\infty} F_i$ and $1 \subseteq F_1 \subseteq F_2 \subseteq ...$

Let $\widehat{d}$ be the relative metric on H. Let $H_0 = \{h \in H \mid \widehat{d}(1,h) < \infty\}$.

Define a function $w : H \to \mathbb{N}$ as

$$w(h) = \begin{cases} \widehat{d}(1,h) & \text{, if } h \in H_0 \\ \min\{i \mid h \in F_i\} & \text{, otherwise} \end{cases}$$

Since F_i's are symmetric, $w(h) = w(h^{-1})$ for all $h \in H$. Define a function l on H as follows- for every word $u = x_1 x_2 ... x_k$ in the elements of H, set

$$l(u) = \sum_{i=1}^{k} w(x_i).$$

Set a length function on H as

$$|g|_w = \min\{l(u) \mid \text{u is a word in the elements of } H \text{ that represents } g\},$$

for each g in H. We can now define a metric $d_w \colon H \times H \to \mathbb{N}$ as

$$d_w(g,h) = |g^{-1}h|_w.$$

It is easy to check that d_w is a (finite valued) well-defined metric. Since

$$d_w(ag,ah) = |(ag)^{-1}ah|_w = |g^{-1}a^{-1}ah|_w = |g^{-1}h|_w = d_w(g,h),$$

for all $a,g,h \in G$, the metric d_w is left-invariant. Further, it is easy to see that for all $h \in H$,

$$d_w(1,h) \leq w(h).$$

It remains to show that d_w is proper. Let $N \in \mathbb{N}$. Suppose $h \in H$ such that $w(h) \leq N$. If $h \in H_0$, then $\widehat{d}(1,h) \leq N$ which implies that there are finitely many choices for h, since $\widehat{d}$ is proper. If $h \notin H_0$, then $h \in F_i$ for some minimal i. But each F_i is a finite set, so there are finitely many choices for h. Thus $|\{h \in H \mid w(h) \leq N\}| < \infty$ for all $N \in \mathbb{N}$. This implies d_w is proper.

Indeed, if $y \neq 1$ is such that $|y|_w \leq n$, then there exists a word u, written without the identity element (which has weight zero), representing y in the alphabet H such that $u = x_1x_2...x_r$ and $\sum_{i=1}^{r} w(x_i) \leq n$. Since $w(x_i) \geq 1$ for every $x_i \neq 1$, $r \leq n$. Further, $w(x_i) \leq n$ for all i. Thus $x_i \in \{x \in H \mid w(x) \leq n\}$ for all i. So there only finitely many choices for each x_i, which implies there are finitely many choices for y. By definition, $d_w \leq \widehat{d}$. So we can set $\widetilde{d} = d_w$. □

4.3 The bottleneck property and a modified version of Bowditch's lemma

In this section, $\mathcal{N}_k(X)$ denotes the closed k-neighborhood of a set X in a metric space (S, d_S). i.e.

$$\mathcal{N}_k(X) = \{s \in S \mid \exists\, x \in X \text{ such that } d_S(s,x) \leq k\}.$$

In particular, $\mathcal{N}_k(x)$ denotes the closed k-neighborhood of a point x in a metric space.

The goal of this section is to prove the following theorem, which will be used in the proof of Theorem 4.1.2. Part (a) is a simplified form of a result taken from [46], which is in fact derived from a hyperbolicity criterion developed by Bowditch in [19].

Theorem 4.3.1. *Let Σ be a hyperbolic graph, and Δ be a graph obtained from Σ by adding edges.*

(a) [19] Suppose there exists $M > 0$ such that for all vertices $x, y \in \Sigma$ joined by an edge in Δ and for all geodesics p in Σ between x and y, all vertices of p lie in an M-neighborhood of x, i.e., $p \subseteq \mathcal{N}_M(x)$ in Δ. Then Δ is also hyperbolic, and there exists a constant k such that for all vertices $x,y \in \Sigma$, every geodesic q between x and y in Σ lies in a k-neighborhood in Δ of every geodesic in Δ between x and y.

(b) If, under the assumptions of (a), we additionally assume that Σ is a quasi-tree, then Δ is also a quasi-tree.

Definition 4.3.2. A graph Γ with the combinatorial metric d_Γ is said to be a *quasi-tree* if it is quasi-isometric to a tree T.

In order to prove Theorem 4.3.1, we will employ the following neccesary and sufficient condition for a geodesic metric space to be a quasi-tree, developed by Manning.

Theorem 4.3.3. *[53, Theorem 4.6, Bottleneck property] Let Y be a geodesic metric space. The following are equivalent.*

(a) Y is quasi-isometric to some simplicial tree Γ

(b) There is some $\mu > 0$ so that for all $x, y \in Y$, there is a midpoint $m = m(x,y)$ with $d(x,m) = d(y,m) = \frac{1}{2}d(x,y)$ and the property that any path from x to y must pass within less than μ of the point m.

We remark that if m is replaced with any point p on a geodesic between x and y, then the property that any path from x to y passes within less than μ of the point p still follows from (a), as proved below in Lemma 4.3.5. We will need the following lemma.

Lemma 4.3.4. *[22, Proposition 3.1]For all $\lambda \geq 1, C \geq 0, \delta \geq 0$, there exists an $R = R(\delta, \lambda, C)$ such that if X is a δ-hyperbolic space, γ is a (λ, C)-quasi-geodesic in X, and γ' is a geodesic segment with the same end points, then γ' and γ are Hausdorff distance less than R from each other.*

Lemma 4.3.5. *If Y is a quasi-tree, then there exists $\mu > 0$ such that for any point z on a geodesic connecting two points, any other path between the same end points passes within μ of z.*

Proof. Let T be a tree and $q\colon Y \to T$ be the (λ, C) quasi-isometry. Let d_Y and d_T denote the metrics in the spaces Y and T respectively. Note that since T is 0-hyperbolic, Y is δ-hyperbolic for some δ.

Let x, y be two points in Y, joined by a geodesic γ. Let z be any point of γ, and let α be another path from x to y. Let V denote the vertex set of α, ordered according to the

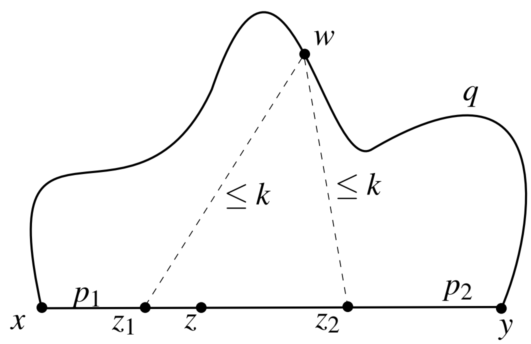

Figure 4.1: Corresponding to Lemma 4.3.6

geodesic γ. Take its image $q(V)$ and connect consecutive points by geodesics (of length at most $\lambda + C$) to get a path β in T from $q(x)$ to $q(y)$. Then the unique geodesic σ in T must be a subset of β. Since $q(V) \subset q \circ \alpha$, we get that any point of σ is at most $\lambda + C$ from $q \circ \alpha$. Also, $q \circ \gamma$ is a (λ, C)-quasi-isometric embedding of an interval, and hence a (λ, C)-quasi-geodesic. Thus, by Lemma 4.3.4 the distance from $q(z)$ to σ is less than $R = R(0, \lambda, C)$.

Let p be the point on σ closest to $q(z)$. There is a point $w \in Y$ on α such that $d(q(w), p) \leq \lambda + C$. Since $d(p, q(z)) < R$, we have $d(q(w), q(z)) \leq \lambda + C + R$. Thus

$$d(z, w) \leq \lambda^2 + 2\lambda C + R\lambda.$$

Thus α must pass within $\mu = \lambda^2 + 2\lambda C + R\lambda$ of the point z. $\square$

Lemma 4.3.6. *Let p,q be two paths in a metric space S between points x and y, such that p is a geodesic and $q \subseteq \mathcal{N}_k(p)$. Then $p \subseteq \mathcal{N}_{2k}(q)$.*

Proof. Let z be any point on p. Let p_1, p_2 denote the segments of the geodesic p with end points x, z and z, y respectively.

Define a function $f\colon q \to \mathbb{R}$ as $f(s) = d(s, p_1) - d(s, p_2)$. Then f is a continuous function. Further, $f(x) < 0$ and $f(y) > 0$. By the intermediate value theorem, there exists a point w on q such that $f(w) = 0$. Thus $d(w, p_1) = d(w, p_2)$ (see Fig.4.1). Let z_1

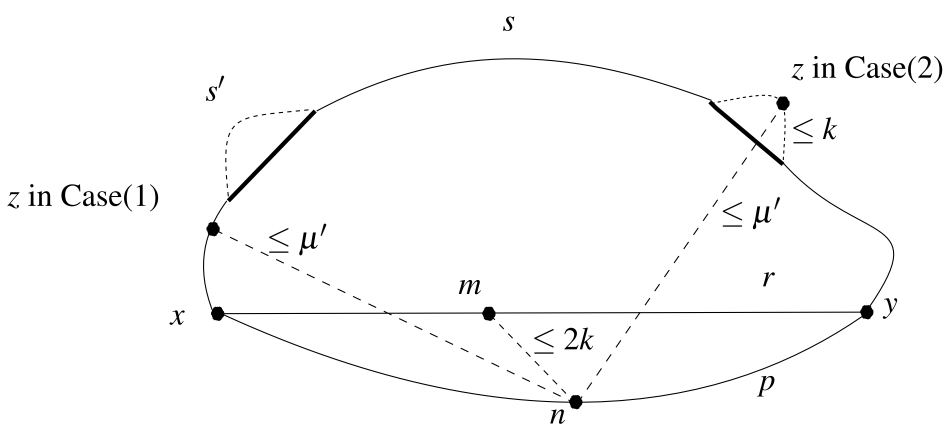

Figure 4.2: Corresponding to Theorem 4.3.1

(resp. z_2) be a point of p_1 (resp. p_2) such that $d(p_i,w) = d(z_i,w)$ for $i = 1,2$. Then $d(z_1,w) = d(z_2,w)$. By the hypothesis, $d(w,p) = \min\{d(w,p_1),d(w,p_2)\} \leq k$. So we get that $d(w,p_1) = d(w,p_2) \leq k$. Thus $d(z_1,z_2) \leq 2k$, which implies $d(z,w) \leq 2k$. □

Proof of Theorem 4.3.1. We proceed with the proof of part (b).

We prove that Δ is a quasi-tree by verifying the bottleneck property from Theorem 4.3.3. Let d_Σ (resp. d_Δ) denote the distance in the graph Σ (resp. Δ). Note that the vertex sets of the two graphs are equal.

Let x,y be two vertices. Let m be the midpoint of a geodesic r in Δ connecting them. Let s be any path from x to y in Δ. The path s consists of edges of two types

(i) edges of the graph Σ;

(ii) edges added in transforming Σ to Δ (marked as bold edges on Fig.4.2).

Let p be a geodesic in Σ between x and y. By Part (a), there exists k such that p is in the k-neighborhood of r in Δ. Applying Lemma 4.3.6 , we get a point n on p such that

$$d_\Delta(m,n) \leq 2k.$$

Let s' be the path in Σ between x and y, obtained from s by replacing every edge e of type (ii) by a geodesic path $t(e)$ in Σ between its end points (marked by dotted lines in Fig.4.2). Since Σ is a quasi-tree, by Lemma 4.3.5, there exists $\mu' > 0$ and a point z on s' such that

$$d_\Sigma(z,n) \leq \mu'.$$

Case 1: If z lies on an edge of s of type (i) , then

$$d_\Delta(z,m) \leq d_\Delta(z,n) + d_\Delta(n,m) \leq d_\Sigma(z,n) + d_\Delta(n,m) \leq \mu' + 2k.$$

Case 2: If z lies on a path $t(e)$ that replaced an edge e of type (ii), then by Part (a),

$$d_\Delta(e_-,m) \leq d_\Delta(e_-,z) + d_\Delta(z,n) + d_\Delta(n,m) \leq k + \mu' + 2k = \mu' + 3k.$$

Thus the bottleneck property holds for $\mu = \mu' + 3k > 0$. □

4.4 The main result

Our main result is the following theorem, from which Theorem 4.1.2 and other corollaries stated earlier can be easily derived (see Section 4.5).

Theorem 4.4.1. *Let $\{H_1, H_2, ..., H_n\}$ be a finite collection of countable subgroups of a group G such that $\{H_1, H_2, ..., H_n\} \hookrightarrow_h (G, Z)$ for some $Z \subset G$. Let K be a subgroup of G such that $H_i \leq K$ for all i. Then there exists a subset $Y \subset K$ such that:*

(a) $\{H_1, H_2, ..., H_n\} \hookrightarrow_h (K, Y)$

(b) $\Gamma(K, Y \sqcup \mathscr{H})$ *is a quasi-tree, where* $\mathscr{H} = \bigsqcup_{i=1}^n H_i$

(c) The action of K on $\Gamma(K, Y \sqcup \mathscr{H})$ is acylindrical

(d) $Z \cap K \subset Y$

4.4.1 Outline of the proof

Step 1: In order to prove Theorem 4.4.1, we first prove the following proposition. It is distinct from Theorem 4.4.1 since it does not require the action of K on the Cayley graph $\Gamma(K, X \sqcup \mathscr{H})$ to be acylindrical.

Proposition 4.4.2. *Let $\{H_1, H_2, ..., H_n\}$ be a finite collection of countable subgroups of a group G such that $\{H_1, H_2, ..., H_n\} \hookrightarrow_h G$ with respect to a relative generating set Z. Let K be a subgroup of G such that $H_i \leq K$ for all i. Then there exists $X \subset K$ such that*

(a) $\{H_1, H_2, ..., H_n\} \hookrightarrow_h (K, X)$

(b) $\Gamma(K, X \sqcup \mathscr{H})$ *is a quasi-tree, where* $\mathscr{H} = \bigsqcup_{1=1}^{n} H_i$

(c) $Z \cap K \subset X$

Step 2: Once we have proved Proposition 4.4.2, we will utilize an 'acylindrification' construction from [69] to make the action acylindrical, which will prove Theorem 4.4.1. The details of this step are as follows.

Proof. By Proposition 4.4.2, there exists $X \subseteq K$ such that

(a) $\{H_1, H_2, ..., H_n\} \hookrightarrow_h (K, X)$

(b) $\Gamma(K, X \sqcup \mathscr{H})$ is a quasi-tree

(c) $Z \cap K \subset X$

By applying Theorem 2.3.9 to the above, we get that there exists $Y \subset K$ such that

(a) $X \subseteq Y$

(b) $\{H_1, H_2, ..., H_n\} \hookrightarrow_h (K, Y)$. In particular, the Cayley Graph $\Gamma(K, Y \sqcup \mathscr{H})$ is hyperbolic

(c) The action of K on $\Gamma(K, Y \sqcup \mathscr{H})$ is acylindrical.

From the proof of Theorem 2.3.9 (see [69] for details), it is easy to see that the Cayley graph $\Gamma(G, Y \sqcup \mathscr{H})$ is obtained from $\Gamma(G, X \sqcup \mathscr{H})$ in a manner that satisfies the assumptions of Theorem 4.3.1, with $M = 1$. Thus by Theorem 4.3.1, $\Gamma(K, Y \sqcup \mathscr{H})$ is also a quasi-tree. Further

$$K \cap Z \subset X \subset Y.$$

Thus Y is the required relative generating set. □

We will thus now focus on proving Proposition 4.4.2. In order to prove this proposition, will use a construction introduced by Bestvina, Bromberg and Fujiwara in [12]. We describe the construction below and will retain the same terminology as introduced by the authors in [12].

4.4.2 The projection complex

Definition 4.4.3. Let $\mathbb{Y}$ be a set and $\xi > 0$ be a constant. Suppose that for each $Y \in \mathbb{Y}$ we have a function

$$d_Y^\pi \colon (\mathbb{Y} \backslash \{Y\} \times \mathbb{Y} \backslash \{Y\}) \to [0, \infty)$$

that satisfy the following axioms :

(A1) $d_Y^\pi(A,B) = d_Y^\pi(B,A)$ for all $Y \in \mathbb{Y}$ and all $A, B \in \mathbb{Y} \backslash \{Y\}$

(A2) $d_Y^\pi(A,B) + d_Y^\pi(B,C) \geq d_Y^\pi(A,C)$ for all $Y \in \mathbb{Y}$ and all $A, B, C \in \mathbb{Y} \backslash \{Y\}$

(A3) $\min\{d_Y^\pi(A,B), d_B^\pi(A,Y)\} < \xi$ for all distinct $Y, A, B \in \mathbb{Y}$

(A4) $\#\{Y \mid d_Y^\pi(A,B) \geq \xi\}$ is finite for all $A, B \in \mathbb{Y}$.

Let J be a positive constant. Then associated to this data we have the *projection complex* $P_J(\mathbb{Y})$, which is a graph constructed in the following manner : the set of vertices of $P_J(\mathbb{Y})$ is the set $\mathbb{Y}$. To specify the set of edges, one first defines a new function

$d_Y \colon (\mathbb{Y}\backslash\{Y\} \times \mathbb{Y}\backslash\{Y\}) \to [0,\infty)$, which can be thought of as a small perturbation of d^{π}_Y. The exact definition of d_Y can be found in [12]. An essential property of the new function is the following inequality, which is an immediate corollary of [12], Proposition 3.2.

For every $Y \in \mathbb{Y}$ and every $A, B \in \mathbb{Y}\backslash\{Y\}$, we have

$$|d^{\pi}_Y(A,B) - d_Y(A,B)| \leq 2\xi. \qquad (1)$$

The set of edge of the graph $P_J(\mathbb{Y})$ can now be described as follows : two vertices $A, B \in \mathbb{Y}$ are connected by an edge if and only if for every $Y \in \mathbb{Y}\backslash\{A,B\}$, $d_Y(A,B) \leq J$. This construction strongly depends on the constant J. Complexes corresponding to different J are not isometric in general.

We would like to mention that if $\mathbb{Y}$ is endowed with an action of a group G that preserves projections, i.e., $d^{\pi}_{g(Y)}(g(A), g(B)) = d^{\pi}_Y(A,B)$), then the action of G can be extended to an action on $P_J(\mathbb{Y})$. We also mention the following proposition, which has been proved under the assumptions of Definition 4.4.3.

Proposition 4.4.4 ([12], Theorem 3.16)**.** *For a sufficiently large $J > 0$, $P_J(\mathbb{Y})$ is connected and quasi-isometric to a tree.*

Definition 4.4.5. [Nearest point projection] In a metric space (S,d), given a set Y and a point $a \in S$, we define the nearest point projection as

$$\mathrm{proj}_Y(a) = \{y \in Y \mid d(Y,a) = d(y,a)\}.$$

If A, Y are two sets in S, then

$$\mathrm{proj}_Y(A) = \bigcup_{a \in A} \mathrm{proj}_Y(a).$$

We note that in our case, since elements of $\mathbb{Y}$ will come from a Cayley graph, which is a combinatorial graph, the nearest point projection will exist. This is because distances on a combinatorial graph take discrete values in $\mathbb{N} \cup \{0\}$. Since this set is bounded below, we cannot have an infinite strictly decreasing sequence of distances.

We make all geometric considerations in the Cayley graph $\Gamma(G, Z \sqcup \mathscr{H})$. Let $\mathrm{d}_{Z \sqcup \mathscr{H}}$ denote the metric on this graph. Since $\{H_1, H_2, ..., H_n\} \hookrightarrow_h G$ under the assumptions of Proposition 4.4.2, by Remark 4.26 of [30], $H_i \hookrightarrow_h G$ for all $i = 1, 2, ..., n$. By Theorem 4.2.1, we can define a finite valued, proper metric $\widetilde{d_i}$ on H_i, for all $i = 1, 2, ..., n$, satisfying

$$\widetilde{d_i}(x, y) \leq \widehat{d_i}(x, y) \text{ for all } x, y \in H_i \text{ and for all } i = 1, 2, ..., n \qquad (2)$$

We can extend both $\widehat{d_i}$ and $\widetilde{d_i}$ to all cosets gH_i of H_i by setting $\widetilde{d_i}(gx, gy) = \widetilde{d_i}(x, y)$ and $\widehat{d_i}(gx, gy) = \widehat{d_i}(x, y)$ for all $x, y \in H_i$. Let $\widehat{\mathrm{diam}}$ (resp. $\widetilde{\mathrm{diam}}$) denote the diameter of a subset of H_i or a coset of H_i with respect to the $\widehat{d_i}$ (resp. $\widetilde{d_i}$) metric.

Let

$$\mathbb{Y} = \{kH_i \mid k \in K, i = 1, 2, ..., n\}$$

be the set of cosets of all H_i in K. We think of cosets of H_i as a subset of vertices of $\Gamma(G, Z \sqcup \mathscr{H})$.

For each $Y \in \mathbb{Y}$, and $A, B \in \mathbb{Y} \backslash \{Y\}$, define

$$d_Y^{\pi}(A, B) = \widetilde{\mathrm{diam}}(\mathrm{proj}_Y(A) \cup \mathrm{proj}_Y(B)), \qquad (3)$$

where $\mathrm{proj}_Y(A)$ is defined as in Definition 4.4.5. The fact that (3) is well-defined will follow from Lemma 4.4.6 and Lemma 4.4.8, which are proved below. We will also proceed to verify the axioms $(A1) - (A4)$ of the Bestvina-Bromberg-Fujiwara construction in the above setting.

Lemma 4.4.6. *For any $Y \in \mathbb{Y}$ and any $x \in G$, $\widetilde{\mathrm{diam}}(\mathrm{proj}_Y(x)) \leq 3C$, where C is the constant as in Lemma 2.3.7. As a consequence, $\widetilde{\mathrm{diam}}(\mathrm{proj}_Y(x))$ is bounded.*

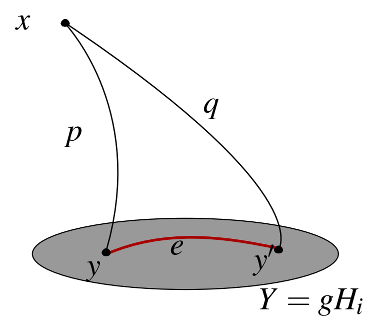

Figure 4.3: The bold red edge denotes a single edge labeled by an element of $\mathscr{H}$

Proof. By (2), it suffices to prove that $\widehat{\text{diam}}(\text{proj}_Y(x))$ is bounded. Let $y, y' \in \text{proj}_Y(x)$. Then $\text{d}_{Z\sqcup\mathscr{H}}(x,y) = \text{d}_{Z\sqcup\mathscr{H}}(x,y') = \text{d}_{Z\sqcup\mathscr{H}}(x,Y)$. Without loss of generality, $x \notin Y$, else the diameter is zero.

Let $Y = gH_i$. Let e denote the edge connecting y and y', and labeled by an element of H_i. Let p and q denote geodesics between the points x and y, and x and y' respectively. (see Fig.4.3)

Consider the geodesic triangle T with sides e, p, q. Since p and q are geodesics between the point x and Y, e is an isolated component in T, i.e., e cannot be connected to either p or q. Indeed if e is connected to, say, a component of p, then that would imply that e_+ and e_- are in Y, i.e., the geodesic p passes through a point of Y before y. But then y is not the nearest point from Y to x, which is a contradiction. By Lemma 2.3.7, $\widehat{d_i}(y,y') \leq 3C$. Hence

$$\widehat{\text{diam}}(\text{proj}_Y(x)) \leq 3C.$$

□

Remark 4.4.7. Observe that in the previous lemma, we proved the following fact : If x is a point in G and $y \in \text{proj}_Y(x)$, then every geodesic path p between x and y satisfies the property that no vertex of p, except for y, can belong to the coset Y. We will use this fact repeatedly in the following lemmas.

Lemma 4.4.8. *For every pair of distinct elements* $A, Y \in \mathbb{Y}$, $\widehat{\text{diam}}(\text{proj}_Y(A)) \leq 4C$, *where*

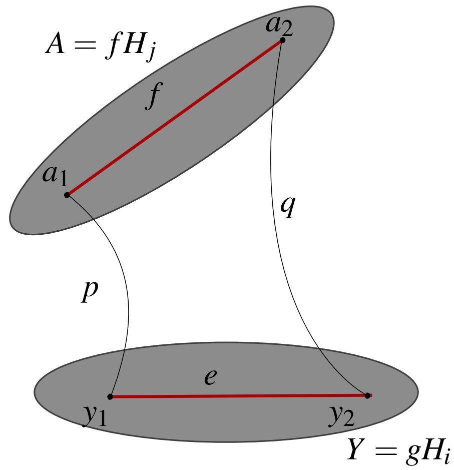

Figure 4.4: Lemma 4.4.8

C is the constant as in Lemma 2.3.7. As a consequence, $\widetilde{\text{diam}}(\text{proj}_Y(A))$ *is bounded.*

Proof. Let $Y = gH_i$ and $A = fH_j$. Let $y_1, y_2 \in \text{proj}_Y(A)$. Then there exist $a_1, a_2 \in A$ such that $\text{d}_{Z \sqcup \mathscr{H}}(a_1, y_1) = \text{d}_{Z \sqcup \mathscr{H}}(a_1, Y)$ and $\text{d}_{Z \sqcup \mathscr{H}}(a_2, y_2) = \text{d}_{Z \sqcup \mathscr{H}}(a_2, Y)$. Now y_1 and y_2 are connected by a single edge e, labeled by an element of H_i, and similarly, a_1 and a_2 are connected by an edge f, labeled by an element of H_j (see Fig.4.4). Let p and q denote geodesics that connect y_1, a_1 and y_2, a_2 respectively. We note that p and/or q may be trivial paths (consisting of a single point), but this does not alter the proof.

Consider e in the quadrilateral Q with sides p, f, q, e. By Remark 4.4.7, e cannot be connected to a component of p or q.

If $i = j$, then e cannot be connected to f since $A \neq Y$. If $i \neq j$, then obviously e and f cannot be connected. Thus e is isolated in this quadrilateral Q. By Lemma 2.3.7, $\widehat{d_i}(y_1, y_2) \leq 4C$. Thus

$$\widehat{\text{diam}}(\text{proj}_Y(A)) \leq 4C.$$

$\square$

Corollary 4.4.9. *The function d_Y^{π} defined by (3) is well-defined.*

Proof. Since the $\widetilde{d_i}$ metric takes finite values for all $i = 1, 2, \ldots, n$, using Lemma 4.4.8, we have that d_Y^{π} also takes only finite values. $\square$

Lemma 4.4.10. *The function d^{π}_Y defined by (3) satisfies conditions (A1) and (A2) in Definition 4.4.3.*

Proof. (A1) is obviously satisfied. For any $Y \in \mathbb{Y}$ and any $A,B,C \in \mathbb{Y}\backslash\{Y\}$, by the triangle inequality, we have that

$$d^{\pi}_Y(A,C) = \widetilde{\operatorname{diam}}(\operatorname{proj}_Y(A) \cup \operatorname{proj}_Y(C))$$

$$\leq \widetilde{\operatorname{diam}}(\operatorname{proj}_Y(A) \cup \operatorname{proj}_Y(B)) + \widetilde{\operatorname{diam}}(\operatorname{proj}_Y(B) \cup \operatorname{proj}_Y(C))$$

$$= d^{\pi}_Y(A,B) + d^{\pi}_Y(B,C).$$

Thus (A2) also holds. □

Lemma 4.4.11. *The function d^{π}_Y from (3) satisfies condition (A3) in Definition 4.4.3 for any $\xi > 14C$, where C is the constant from Lemma 2.3.7*

Proof. By (2), it suffices to prove that

$$\min\{\widehat{\operatorname{diam}}(\operatorname{proj}_Y(A) \cup \operatorname{proj}_Y(B)), \widehat{\operatorname{diam}}(\operatorname{proj}_B(A) \cup \operatorname{proj}_B(Y))\} < \xi.$$

Let $A,B \in \mathbb{Y}\backslash\{Y\}$ be distinct. Let $Y = gH_i$, $A = fH_j$ and $B = tH_k$. If $\widehat{\operatorname{diam}}(\operatorname{proj}_Y(A) \cup \operatorname{proj}_Y(B)) \leq 14C$, then we are done. So let

$$\widehat{\operatorname{diam}}(\operatorname{proj}_Y(A) \cup \operatorname{proj}_Y(B))) > 14C. \qquad (4)$$

Choose $a \in A, b \in B$, and $x,y \in Y$ such that $\mathrm{d}_{Z\sqcup\mathscr{H}}(A,Y) = \mathrm{d}_{Z\sqcup\mathscr{H}}(a,x)$ and $\mathrm{d}_{Z\sqcup\mathscr{H}}(B,Y) = \mathrm{d}_{Z\sqcup\mathscr{H}}(b,y)$. In particular,

$$x \in \operatorname{proj}_Y(A), y \in \operatorname{proj}_Y(B) \qquad (5)$$

and $b \in \operatorname{proj}_B(Y)$. Let p,q denote geodesics connecting a,x and b,y respectively. Let h_1

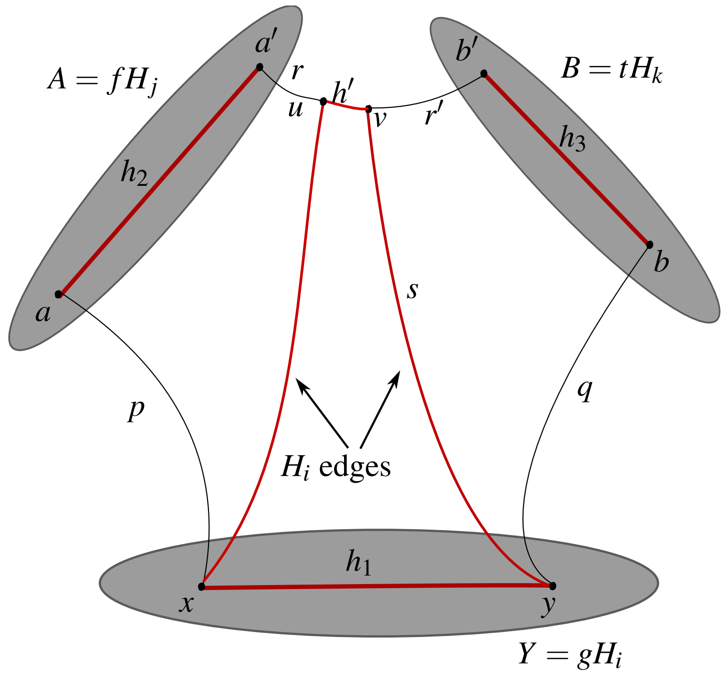

Figure 4.5: Condition $(A3)$

denote the edge connecting x and y, which is labeled by an element of H_i.

By (5), we have that

$$\widehat{\text{diam}}(\text{proj}_Y(A) \cup \text{proj}_Y(B)) \leq \widehat{\text{diam}}(\text{proj}_Y(A)) + \widehat{\text{diam}}(\text{proj}_Y(B)) + \widehat{d_i}(x,y).$$

Combining this with (4) and Lemma 4.4.8, we get

$$\widehat{d_i}(x,y) \geq \widehat{\text{diam}}(\text{proj}_Y(A) \cup \text{proj}_Y(B)) - \widehat{\text{diam}}(\text{proj}_Y(A)) - \widehat{\text{diam}}(\text{proj}_Y(B))$$

$$> 14C - 8C = 6C.$$

Choose any $a' \in A$ and $b' \in \text{proj}_B(a')$, i.e.,: $\text{d}_{Z \sqcup \mathscr{H}}(a', B) = \text{d}_{Z \sqcup \mathscr{H}}(a', b')$; (see Fig.4.5). (Note that if $a' = a$, the following arguments still hold). Let h_2 and h_3 denote the edges connecting a, a' and b, b'; which are labeled by elements of H_j and H_k respectively. Let r denote a geodesic connecting a' and b'. Consider the geodesic hexagon W with sides p, h_1, q, h_3, r, h_2. Then h_1 is not isolated in W, else by Lemma 2.3.7, $\widehat{d_i}(x,y) \leq 6C$, a contra-

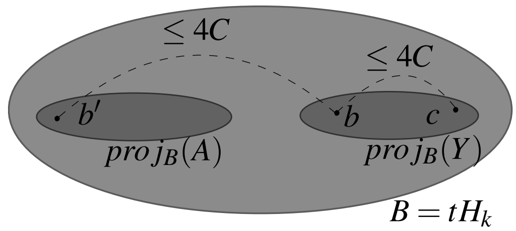

Figure 4.6: Estimating the distance between arbitrary points b and c of $\text{proj}_B(A)$ and $\text{proj}_B(Y)$ resp.

diction.

Thus h_1 is connected to another H_i-component in W. By Remark 4.4.7, h_1 cannot be connected to a component of p or q. Since A, B, Y are all distinct, h_1 cannot be connected to h_2 or h_3. So h_1 must be connected to an H_i-component on the geodesic r. Let this edge be h' with end points u and v as shown in Fig 4.5. Let s denote the edge (labeled by an element of H_i), that connects y, v. Let r' denote the segment of r that connects v to b'. Then r' is also a geodesic.

Consider the quadrilateral Q with sides r', h_3, q, s. By using arguments similar to those in the previous paragraph, h_3 cannot be connected to r', q or s. Thus h_3 is isolated in Q. By Lemma 2.3.7,

$$\widehat{d_k}(b, b') \leq 4C.$$

Since the above argument holds for any $a' \in A$ and for $b' \in \text{proj}_B(A)$, we have that $\widehat{d_k}(b, b') \leq 4C$. Using Lemma 4.4.8 (see Fig.4.6), we get that

$$\widehat{\text{diam}}(\text{proj}_B(Y) \cup \text{proj}_B(A)) \leq 4C + 4C = 8C < \xi.$$

□

Lemma 4.4.12. *The function d^{π}_Y defined by (3) satisfies condition (A4) in Definition 4.4.3, for $\xi > 14C$, where C is the constant from Lemma 2.3.7*

Proof. If $d^{\pi}_Y(A, B) \geq \xi$, then by (2), $\widehat{\text{diam}}(\text{proj}_Y(A) \cup \text{proj}_Y(B)) \geq d^{\pi}_Y(A, B) \geq \xi$. Thus it

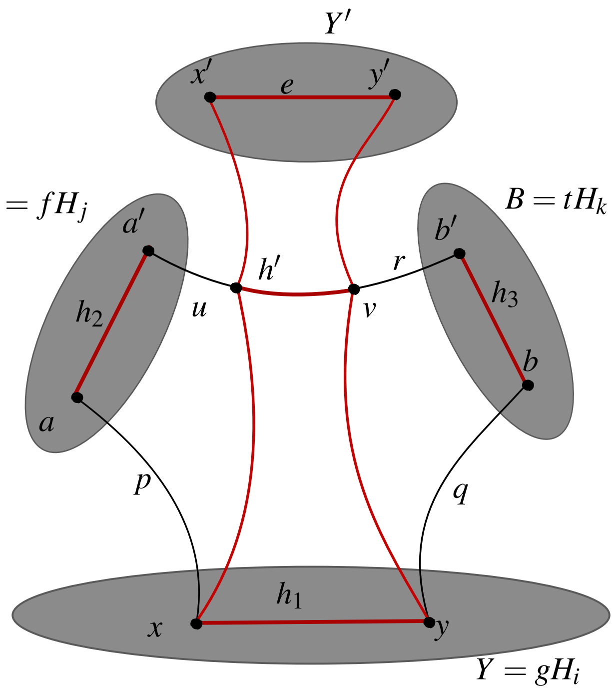

Figure 4.7: Condition $(A4)$

suffices to prove that the number of elements $Y \in \mathbb{Y}$ satisfying

$$\widehat{\operatorname{diam}}(\operatorname{proj}_Y(A) \cup \operatorname{proj}_Y(B)) \geq \xi \qquad (6)$$

is finite. Let $A, B \in \mathbb{Y}, A = fH_j$ and $B = tH_k$. Let $Y \in \mathbb{Y} \backslash \{A, B\}, Y = gH_i$. Let $a' \in A, b' \in \operatorname{proj}_B(a')$. By repeating the computations in Lemma 4.4.11, we can show that if Y is such that $\widehat{\operatorname{diam}}(\operatorname{proj}_Y(A) \cup \operatorname{proj}_Y(B)) \geq \xi$, then for any $a \in A, b \in B, x \in \operatorname{proj}_Y(a), y \in \operatorname{proj}_Y(b)$, we have that $\widehat{d_i}(x, y) > 6C$.

Let h_1 denote the edge connecting x, y, which is labeled by an element of H_i (see Fig.4.7). Let h_2 denote the edge connecting a, a', which is labeled by an element of H_j and h_3 denote the edge connecting b, b', which is labeled by an element of H_k. Let p be a geodesic between a, x, let q be a geodesic between b, y, and let r be a geodesic between a', b'. As argued in Lemma 4.4.11, we can show that h_1 cannot be isolated in the hexagon W with sides p, h_1, q, h_2, r, h_3 and must be connected to an H_i-component of r, say the edge h'.

We claim that the edge h' uniquely identifies Y. Indeed, let Y' be a member of $\mathbb{Y}$, with elements x', y' connected by an edge e (labeled by an element of the corresponding subgroup). Suppose that e is connected to h'. Then we must have that Y' is also a coset of H_i. But cosets of a subgroup are either disjoint or equal, so $Y = Y'$. Thus, the number of $Y \in \mathbb{Y}$ satisfying (6) is bounded by the number of distinct H_i-components of r, which is finite. $\square$

4.4.3 Choosing a relative generating set

We now have the necessary details to choose a relative generating set X which will satisfy conditions (a) and (b) of Proposition 4.4.2. This set will later be altered slightly to obtain another relative generating set which will satisfy all three conditions of Proposition 4.4.2. We will repeat arguments similar to those from pages 60-63 of [30].

Recall that $\mathscr{H} = \bigsqcup_{i=1}^{n} H_i$, and Z is the relative generating set such that $\{H_1, H_2, ..., H_n\} \hookrightarrow_h (G, Z)$. Let $P_J(\mathbb{Y})$ be the projection complex corresponding to the vertex set $\mathbb{Y}$ as specified in section 4.4.2 and the constant J is as in Proposition 4.4.4, i.e., $P_J(\mathbb{Y})$ is connected and a quasi-tree. Let d_P denote the combinatorial metric on$P_J(\mathbb{Y})$. Our definition of projections is K- equivariant and hence the action of K on $\mathbb{Y}$ extends to a cobounded action of K on $P_J(\mathbb{Y})$.

In what follows, by considering H_i to be vertices of the projection complex $P_J(\mathbb{Y})$, we denote by $star(H_i)$, the set

$$\{e \text{ is an edge in } P_J(\mathbb{Y}) \mid e \text{ connects } H_i \text{ to } kH_j, \text{ for some } k \in K \text{ and } 1 \leq j \leq n\}.$$

We choose the set X in the following manner. For all $i = 1, 2, ..., n$ and each edge e in $star(H_i)$ in $P_J(\mathbb{Y})$ that connects H_i to kH_j, choose all elements $x_e \in H_i k H_j$ such that

$$\mathrm{d}_{Z \sqcup \mathscr{H}}(1, x_e) = \mathrm{d}_{Z \sqcup \mathscr{H}}(1, H_i k H_j).$$

We say that all such x_e have *type* (i, j). Since $H_i \leq K$ for all i, $x_e \in K$. We observe the following:

(a) For each x_e of type (i,j) as above, there is an edge in $P_J(\mathbb{Y})$ connecting H_i and x_eH_j. Indeed if $x_e = h_1kh_2$, for $h_1 \in H_i, h_2 \in H_j$, then

$$d_P(H_i, x_eH_j) = d_P(H_i, h_1kh_2H_j) = d_P(H_i, h_1kH_j)$$

$$= d_P(h_1^{-1}H_i, kH_j) = d_P(H_i, kH_j) = 1.$$

(b) For each edge e connecting H_i and kH_j, there is a dual edge f connecting H_j and $k^{-1}H_i$. Thus for every element x_e of type (i,j), there is an element $x_f = (x_e)^{-1}$ of type (j,i). In particular, the set given by

$$X = \{x_e \neq 1 | e \in star(H_i), i = 1,2,...,n\} \qquad (7)$$

is symmetric, i.e., closed under taking inverses. Obviously, $X \subset K$.

(c) If $x_e \in X$ is of type (i,j), then x_e is not an element of H_i or H_j. Indeed if $x_e = h_1kh_2 \in H_i$ for some $h_1 \in H_i$ and some $h_2 \in H_j$, and x_e is an element of H_i or H_j, then $k = hf$ for some $h \in H_i$ and some $f \in H_j$. Consequently

$$\mathrm{d}_{Z\sqcup\mathscr{H}}(1, H_ikH_j) = \mathrm{d}_{Z\sqcup\mathscr{H}}(1, H_iH_j) = 0 = \mathrm{d}_{Z\sqcup\mathscr{H}}(1, x_e),$$

which implies $x_e = 1$, which is a contradiction to (7).

Lemma 4.4.13 (cf. Lemma 4.49 in [30])**.** *The subgroup K is generated by X together with the union of all H_i's. Further, the Cayley graph $\Gamma(K, X \sqcup \mathscr{H})$ is quasi-isometric to $P_J(\mathbb{Y})$, and hence a quasi-tree.*

Proof. Let $\Sigma = \{H_1, H_2, ..., H_n\} \subseteq \mathbb{Y}$. Let $\mathrm{diam}(\Sigma)$ denote the diameter of the set Σ in the combinatorial metric d_P. Since Σ is a finite set, $\mathrm{diam}(\Sigma)$ is finite. Define

$$\phi\colon K \to \mathbb{Y} \text{ as } \phi(k) = kH_1$$

By Property (a) above, if $x_e \in X$ is of type (i, j),

$$d_P(x_eH_1, H_1) \leq d_P(x_eH_1, x_eH_j) + d_P(x_eH_j, H_i) + d_P(H_i, H_1)$$

$$= d_P(H_1, H_j) + 1 + d_P(H_i, H_1) \leq 2\,\mathrm{diam}(\Sigma) + 1.$$

Further, for $h \in H_i$,

$$d_P(hH_1, H_1) \leq d_P(hH_1, hH_i) + d_P(hH_i, H_1)$$

$$= d_P(H_1, H_i) + d_P(H_i, H_1) \leq 2\,\mathrm{diam}(\Sigma).$$

Thus for all $g \in \langle X \cup H_1 \cup H_2 ... \cup H_n \rangle$, we have

$$d_P(\phi(1), \phi(g)) \leq (2\,\mathrm{diam}(\Sigma) + 1)|g|_{X \sqcup \mathscr{H}}, \qquad (8)$$

where $|g|_{X \sqcup \mathscr{H}}$ denotes the length of g in the generating set $X \cup H_1 \cup H_2 ... \cup H_n$. (We use this notation for the sake of uniformity).

Now let $g \in K$ and suppose $d_P(\phi(1), \phi(g)) = r$, i.e., $d_P(H_1, gH_1) = r$. If $r = 0$, then $H_1 = gH_1$, thus $g \in H_1$ and $|g|_{X \sqcup \mathscr{H}} \leq 1$. If $r > 0$, consider a geodesic p in $P_J(\mathbb{Y})$ connecting H_1 and gH_1. Let

$$v_0 = H_1 = g_0H_1(g_0 = 1), v_1 = g_1H_{\lambda_1}, v_2 = g_2H_{\lambda_2}, ..., v_{r-1} = g_{r-1}H_{\lambda_{r-1}}, v_r = gH_1(g_r = g)$$

be the sequence of vertices of p, for some $\lambda_j \in \{1, 2, ..., n\}$, and some $g_i \in K$ (see Fig.4.8).

Now $g_iH_{\lambda_i}$ is connected by a single edge to $g_{i+1}H_{\lambda_{i+1}}$. Thus $d_P(g_iH_{\lambda_i}, g_{i+1}H_{\lambda_{i+1}}) = 1$, which implies $d_P(H_{\lambda_i}, g_i^{-1}g_{i+1}H_{\lambda_{i+1}}) = 1$. Then there exists $x \in X$ such that

$$x \in H_{\lambda_i} g_i^{-1} g_{i+1} H_{\lambda_{i+1}}$$

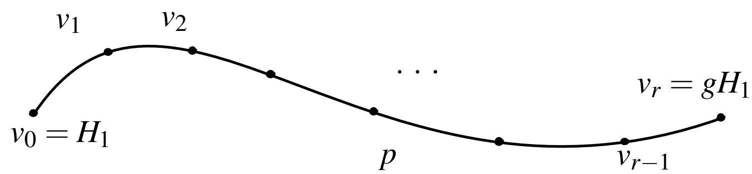

Figure 4.8: The geodesic p

and

$$\mathrm{d}_{Z\sqcup\mathcal{H}}(1,x) = \mathrm{d}_{Z\sqcup\mathcal{H}}(1, H_{\lambda_i} g_i^{-1} g_{i+1} H_{\lambda_{i+1}}).$$

Thus $x = hg_i^{-1}g_{i+1}k$ for some $h \in H_{\lambda_i}$ and some $k \in H_{\lambda_{i+1}}$ which implies $g_i^{-1}g_{i+1} = h^{-1}xk^{-1}$. So $|g_i^{-1}g_{i+1}|_{X\sqcup\mathcal{H}} \leq 3$, which implies

$$|g|_{X\sqcup\mathcal{H}} = \left|\prod_{i=1}^{r} g_{i-1}^{-1} g_i\right|_{X\sqcup\mathcal{H}} \leq \sum_{i=1}^{r} \left|g_{i-1}^{-1} g_i\right|_{X\sqcup\mathcal{H}} \leq 3r = 3d_P(\phi(1), \phi(g)) \qquad (9)$$

The above argument also provides a representation for every element $g \in K$ as a product of elements from $X \cup H_1 \cup H_2 ... \cup H_n$. Thus K is generated by the union of X and all H_i's. By (8) and (9), ϕ is a quasi-isometric embedding of $(K, |.|_{X\sqcup\mathcal{H}})$ into $(P_J(\mathbb{Y}), d_P)$ satisfying

$$\frac{1}{3}|g|_{X\sqcup\mathcal{H}} \leq d_P(\phi(1), \phi(g)) \leq (2\,\mathrm{diam}(\Sigma)+1)|g|_{X\sqcup\mathcal{H}}.$$

Since $\mathbb{Y}$ is contained in the closed diam(Σ)-neighborhood of $\phi(K)$, ϕ is a quasi-isometry. This implies that $\Gamma(K, X\sqcup\mathcal{H})$ is a quasi-tree. $\square$

Let $\widetilde{d_i}$ denote the modified relative metric on H_i associated with the Cayley graph $\Gamma(G, Z\sqcup\mathcal{H})$ from Theorem 4.2.1. Let $\widehat{d_i^X}$ denote the relative metric on H_i associated with the Cayley graph $\Gamma(K, X\sqcup\mathcal{H})$. We will now show that $\widehat{d_i^X}$ is proper for all $i = 1, 2, ..., n$. We will use the fact that $\widetilde{d_i}$ is proper and derive a relation between $\widetilde{d_i}$ and $\widehat{d_i^X}$.

Lemma 4.4.14 (cf. Lemma 4.50 in [30])**.** *There exists a constant α such that for any $Y \in \mathbb{Y}$ and any $x \in X\sqcup\mathcal{H}$, if*

$$\widetilde{\mathrm{diam}}(\mathrm{proj}_Y\{1,x\}) > \alpha,$$

then $x \in H_j$ and $Y = H_j$ for some j.

Proof. We prove the result for

$$\alpha = \max\{J + 2\xi, 6C\}.$$

Suppose that $\widetilde{\operatorname{diam}}(\operatorname{proj}_Y\{1,x\}) > \alpha$ and $x \in X$ has type (k,l), i.e., there exists an edge connecting H_k and gH_l in $P_J(\mathbb{Y})$, where $g \in K$. We consider three possible cases and arrive at a contradiction in each case.

Case 1: $H_k \neq Y \neq xH_l$. Then

$$\widetilde{\operatorname{diam}}(\operatorname{proj}_Y\{1,x\}) \leq d_Y^{\pi}(H_k, xH_l) \leq d_Y(H_k, xH_l) + 2\xi \leq J + 2\xi \leq \alpha,$$

using (1) and the fact that H_k and xH_l are connected by an edge in $P_J(\mathbb{Y})$, which is a contradiction.

Case 2: $H_k = Y$. Since $x \notin H_k$, let $y \in \operatorname{proj}_Y(x)$, i.e., $\mathrm{d}_{Z \sqcup \mathscr{H}}(x,y) = \mathrm{d}_{Z \sqcup \mathscr{H}}(x,H_k) = \mathrm{d}_{Z \sqcup \mathscr{H}}(x,Y)$.

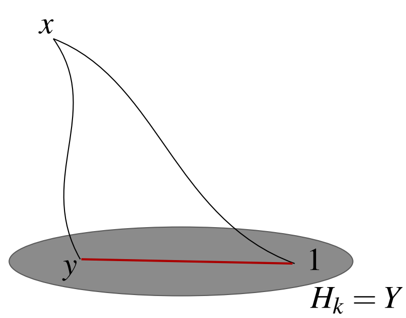

Figure 4.9: Case 2

By Lemma 4.4.6, if $\widehat{d}_k(1,y) \leq 3C$, then

$$\widehat{\operatorname{diam}}(\operatorname{proj}_Y\{1,x\}) \leq \widehat{\operatorname{diam}}(\operatorname{proj}_Y(1)) + \widehat{\operatorname{diam}}(\operatorname{proj}_Y(x)) + \widehat{d}_k(\operatorname{proj}_Y(1), \operatorname{proj}_Y(x))$$

$$\leq 0 + 3C + \widehat{d}_k(1,y) \leq 6C \leq \alpha.$$

Then by (2), we have

$$\widetilde{\operatorname{diam}}(\operatorname{proj}_Y\{1,x\}) \leq \alpha,$$

which is a contradiction. Thus $\widehat{d_k}(1,y) > 3C$. This implies that $1 \notin \operatorname{proj}_Y(x)$ (see Fig.4.9). By definition of the nearest point projection, $\mathrm{d}_{Z\sqcup\mathscr{H}}(1,x) > \mathrm{d}_{Z\sqcup\mathscr{H}}(y,x)$, which implies $\mathrm{d}_{Z\sqcup\mathscr{H}}(1,x) > \mathrm{d}_{Z\sqcup\mathscr{H}}(1,y^{-1}x)$. Since $y^{-1}x \in H_kgH_l$, we obtain $\mathrm{d}_{Z\sqcup\mathscr{H}}(1,x) > \mathrm{d}_{Z\sqcup\mathscr{H}}(1,H_kgH_l)$, which is a contradiction to the choice of x.

Case 3: $Y = xH_l, H_k \neq Y$. This case reduces to Case 2, since we can translate everything by x^{-1}.

Thus we must have $x \in H_j$ for some j. Suppose that $H_j \neq Y$. But then $\widetilde{\operatorname{diam}}(\operatorname{proj}_Y\{1,x\}) \leq \widetilde{\operatorname{diam}}(\operatorname{proj}_Y(H_j)) \leq 4C \leq \alpha$, by Lemma 4.4.8; which is a contradiction. □

Lemma 4.4.15 (cf. Lemma 4.45 in [30]). *If $H_i = fH_j$, then $H_i = H_j$ and $f \in H_i$. Consequently, if $gH_i = fH_j$, then $H_i = H_j$ and $g^{-1}f \in H_i$.*

Proof. If $H_i = fH_j$, then $1 = fk$ for some $k \in H_j$. Then $f = k^{-1} \in H_j$, which implies $H_i = H_j$. □

Lemma 4.4.16 (cf. Theorem 4.42 in [30]). *For all $i = 1,2,...,n$ and any $h \in H_i$, we have*

$$\alpha\widehat{d_i^X}(1,h) \geq \widetilde{d_i}(1,h),$$

where α is the constant from Lemma 4.4.14. Thus $\widehat{d_i^X}$ is proper.

Proof. Let $h \in H_i$ such that $\widehat{d_i^X}(1,h) = r$. Let e denote the H_i-edge in the Cayley graph $\Gamma(K, X \sqcup \mathscr{H})$ connecting h to 1, labeled by h^{-1}. Let p be an admissible path of length r in $\Gamma(K, X \sqcup \mathscr{H})$ connecting 1 and h. Then ep forms a cycle. Since p is admissible, e is isolated in this cycle.

Let $Lab(p) = x_1x_2...x_r$ for some $x_1, x_2, ..., x_r \in X \sqcup \mathscr{H}$. Let

$$v_0 = 1, v_1 = x_1, v_2 = x_1x_2, ..., v_r = x_1x_2...x_r = h.$$

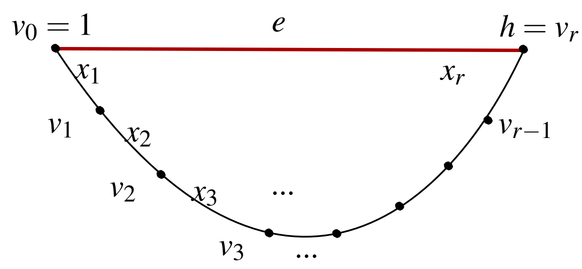

Figure 4.10: The cycle ep

Since these are also elements of G, for all $k = 1, 2, ..., r$ we have

$$\widetilde{\operatorname{diam}}(\operatorname{proj}_{H_i}\{v_{k-1}, v_k\}) = \widetilde{\operatorname{diam}}(\operatorname{proj}_{H_i}\{x_1x_2...x_{k-1}, x_1x_2...x_{k-1}x_k\})$$

$$= \widetilde{\operatorname{diam}}(\operatorname{proj}_Y\{1, x_k\}),$$

where $Y = (x_1x_2...x_{k-1})^{-1}H_i$.

If $\widetilde{\operatorname{diam}}(\operatorname{proj}_Y\{1, x_k\}) > \alpha$ for some k, then by Lemma 4.4.14, $x_k \in H_j$ and $Y = H_j$ for some j. By Lemma 4.4.15, $H_i = H_j$ and $x_1x_2...x_{k-1} \in H_j$. But then e is not isolated in the cycle ep, which is a contradiction.

Hence

$$\widetilde{\operatorname{diam}}(\operatorname{proj}_{H_i}\{v_{k-1}, v_k\}) \leq \alpha$$

for all $k = 1, 2, ..., r$, which implies

$$\widetilde{d}_i(1,h) \leq \widetilde{\operatorname{diam}}(\operatorname{proj}_{H_i}\{v_0, v_r\}) \leq \sum_{j=1}^{r} \widetilde{\operatorname{diam}}(\operatorname{proj}_{H_i}\{v_{j-1}, v_j\}) \leq r\alpha = \alpha \widehat{d_i^X}(1,h).$$

□

4.4.4 Proof of Proposition 4.4.2

The goal of this section is to alter our relative generating set X from Section 4.4.3, so that we obtain another relative generating set that satisfies all the conditions of Proposition

4.4.2. To do so, we need to establish a relation between the set X and the set Z. We will need the following obvious lemma.

Lemma 4.4.17. *Let X and Y be generating sets of G, and $sup_{x\in X}|x|_Y < \infty$ and $sup_{y\in Y}|y|_X < \infty$. Then $\Gamma(G,X)$ is quasi-isometric to $\Gamma(G,Y)$. In particular $\Gamma(G,X)$ is a quasi-tree if and only if $\Gamma(G,Y)$ is a quasi-tree.*

Remark 4.4.18. The above lemma implies that if we change a generating set by adding finitely many elements, then the property that the Cayley graph is a quasi-tree still holds.

We also need to note that from (1) in Definition 4.4.3, it easily follows that for each $Y \in \mathbb{Y}$ and every $A, B \in \mathbb{Y}\backslash\{Y\}$,

$$d_Y(A,B) \leq d_Y^{\pi}(A,B) + 2\xi. \qquad (10)$$

Lemma 4.4.19. *For a large enough J, the set X constructed in Section 4.4.3 satisfies the following property: If $z \in Z \cap K$ does not represent any element of H_i for all $i = 1,2,...,n$, then $z \in X$.*

Proof. Recall that $\mathrm{d}_{Z\sqcup\mathscr{H}}$ denotes the combinatorial metric on $\Gamma(G, Z\sqcup\mathscr{H})$. Let $z \in Z\cap K$ be as in the statement of the lemma. Then $z \in H_izH_i$ for all i and $1 \notin H_izH_i$. Thus

$$\mathrm{d}_{Z\sqcup\mathscr{H}}(1, H_izH_i) \geq 1 = \mathrm{d}_{Z\sqcup\mathscr{H}}(1,z) \geq \mathrm{d}_{Z\sqcup\mathscr{H}}(1, H_izH_i),$$

which implies

$$\mathrm{d}_{Z\sqcup\mathscr{H}}(1, H_izH_i) = \mathrm{d}_{Z\sqcup\mathscr{H}}(1,z) \text{ for all } i.$$

In order to prove $z \in X$, we must show that H_i and zH_i are connected by an edge in $P_J(\mathbb{Y})$. By Definition 4.4.3, this is true if

$$d_Y(H_i, zH_i) \leq J \text{ for all } Y \neq H_i, zH_i.$$

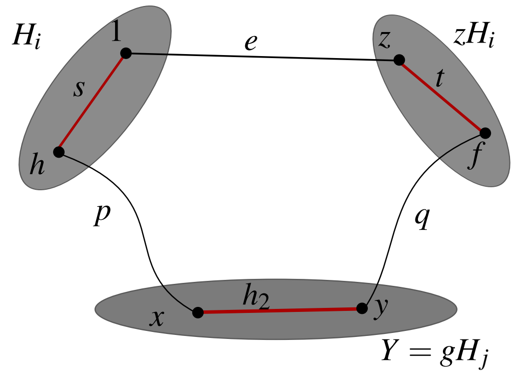

Figure 4.11: Dealing with elements of $Z \cap K$ that represent elements of $\mathscr{H}$

In view of (10), we will estimate $d^{\pi}_{Y}(H_i, zH_i)$.

Let $\mathrm{d}_{Z \sqcup \mathscr{H}}(h,x) = \mathrm{d}_{Z \sqcup \mathscr{H}}(H_i, Y)$ and $\mathrm{d}_{Z \sqcup \mathscr{H}}(f,y) = \mathrm{d}_{Z \sqcup \mathscr{H}}(zH_i, Y)$ for some $h \in H_i, f \in zH_i$ and for some $x, y \in Y = gH_j$. Let p be a geodesic connecting h and x; and q be a geodesic connecting y and f. Let h_2 denote the edge connecting x and y, labeled by an element of H_j. Similarly, let s,t denote the edges connecting $h, 1$ and z, f respectively, that are labeled by elements of H_i. Let e denote the edge connecting 1 and z, labeled by z. Consider the geodesic hexagon W with sides p, h_2, q, t, e, s (see Fig.4.11).

By using Remark 4.4.7 and the fact that $Y \neq H_i, zH_i$, we can show that h_2 cannot be connected to q , p, s or t. Since z does not represent any element of H_i for all i, h_2 cannot be connected to e. Thus, h_2 is isolated in W. By Lemma 2.3.7, $\widehat{d_j}(x,y) \leq 6C$. By Lemma 4.4.8,

$$d_Y(H_i, zH_i) \leq d^{\pi}_{Y}(H_i, zH_i) + 2\xi \leq 14C + 2\xi.$$

So we conclude that by taking the constant J to be sufficiently large so that Proposition 4.4.4 holds and J exceeds $14C + 2\xi$, we can ensure that $z \in X$ and the arguments of the previous section still hold. □

Lemma 4.4.20. *There are only finitely many elements of $Z \cap K$ that can represent an element of H_i for some $i \in \{1, 2, ..., n\}$.*

Proof. Let $z \in Z \cap K$ represent an element of H_i for some $i = 1, 2, ..., n$. Then in the Cayley

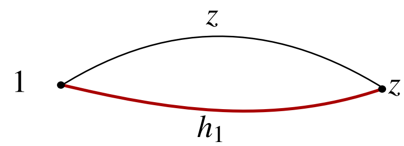

Figure 4.12: Bigons in the Cayley graph

graph $\Gamma(G, Z \sqcup \mathscr{H})$, we have a bigon between the elements 1 and h, where one edge is labeled by z, and the other edge is labeled by an element of H_i, say h_1 (see Rem. 2.3.1 and Fig.4.12).

This implies that $\widehat{d}_i(1,z) \leq 1$, so $\widetilde{d}_i(1,z) \leq 1$. But then $z \in \widetilde{B}_i(1,1)$, i.e., the ball of radius 1 in the subgroup H_i in the relative metric, centered at the identity. But this is a finite ball. Take

$$\rho = \left| \bigcup_{i=1}^{n} \widetilde{B}_i(1,1) \right|.$$

Then z has at most ρ choices, which is finite. $\square$

By Lemma 4.4.20 and by selecting the constant J as specified in Lemma 4.4.19, we conclude that the set X from Section 4.4.3 does not contain at most finitely many elements of $Z \cap K$. By adding these finitely many remaining elements of $Z \cap K$ to X, we obtain a new relative generating set X' such that $|X' \Delta X| < \infty$. By Proposition 2.3.8, $\{H_1, H_2, \ldots, H_n\} \hookrightarrow_h (K, X')$ and $Z \cap K \subset X'$. By Remark 4.4.18, $\Gamma(K, X' \sqcup \mathscr{H})$ is also a quasi-tree. Thus X' is the required set in the statement of Proposition 4.4.2, which completes the proof.

4.5 Applications

This section is dedicated to proving Theorem 4.1.2 and deriving other corollaries of Theorem 4.4.1. In order to prove Theorem 4.1.2, we first need to recall the following definitions.

Definition 4.5.1 (Loxodromic element)**.** Let G be a group acting on a hyperbolic space S. An element $g \in G$ is called *loxodromic* if the map $\mathbb{Z} \to S$ defined by $n \to g^n s$ is a quasi-isometric embedding for some (equivalently, any) $s \in S$.

Definition 4.5.2. [30][Elementary subgroup, Lemma 6.5] Let G be a group acting acylindrically on a hyperbolic space S, $g \in G$ a loxodromic element. Then g is contained in a unique maximal *elementary subgroup* $E(g)$ of G given by

$$E(g) = \{h \in G | d_{Hau}(l, h(l)) < \infty\},$$

where d_{Hau} denotes the Hausdorff distance and l is a quasi-geodesic axis of g in S.

Corollary 4.5.3. *A group G is acylindrically hyperbolic if and only if G has an acylindrical and non-elementary action on a quasi-tree.*

Proof. If G has an acylindrical and non-elementary action on a quasi-tree, by definition, G is acylindrically hyperbolic. Conversely, let G be acylindrically hyperbolic, with an acylindrical non-elementary action on a hyperbolic space X. Let g be a loxodromic element for this action. By Lemma 6.5 of [30] the elementary subgroup $E(g)$ is virtually cyclic and thus countable. By Theorem 6.8 of [30], $E(g)$ is hyperbolically embedded in G. Taking $K = G$ and $E(g)$ to be the hyperbolically embedded subgroup in the statement of Theorem 4.4.1 now gives us the result. Since $E(g)$ is non-degenerate, by [69, Lemma 5.12],, the resulting action of G on the associated Cayley graph $\Gamma(G, X \sqcup E(g))$ is also non-elementary. $\square$

Corollary 4.5.4. *For any acylindrically hyperbolic group G, $\mathcal{AH}(G)$ always contains a structure $[X]$ such that $\Gamma(G,X)$ is a non-elementary quasi-tree.*

The following corollary is an immediate consequence of Theorem 4.4.1.

Corollary 4.5.5. *Let $\{H_1, H_2, ..., H_n\}$ be a finite collection of countable subgroups of a group G such that $\{H1, H_2, ..., H_n\} \hookrightarrow_h G$. Let K be a subgroup of G. If $H_i \leq K$ for all $i = 1, 2, ..., n$, then $\{H_1, H_2, ..., H_n\} \hookrightarrow_h K$.*

Definition 4.5.6. Let (M, d) be a geodesic metric space, and $\varepsilon > 0$ a fixed constant. A subset $S \subset M$ is said to be *ε-coarsely connected* if there for any two points x, y in S, there exist points $x_0 = x, x_1, x_2, ..., x_{n-1}, x_n = y$ in S such that for all $i = 0, ..., n-1$,

$$d(x_i, x_{i+1}) \leq \varepsilon.$$

Further we say that S is *coarsely connected* if it is ε-coarsely connected for some $\varepsilon > 0$.

Recall that we denote the closed σ neighborhood of S by $S^{+\sigma}$.

Definition 4.5.7. Let (M, d) be a geodesic metric space, and $\sigma > 0$ a fixed constant. A subset $S \subset M$ is said to be *σ-quasi-convex* if for any two points x, y in S, any geodesic connecting x and y is contained in $S^{+\sigma}$. Further, we say that S is *quasi-convex* if it is σ-quasi-convex for some $\sigma > 0$.

Corollary 4.5.8. *Let H be a finitely generated subgroup of an acylindrically hyperbolic group G. Then there exists a subset $X \subset G$ such that*

(a) $\Gamma(G, X)$ is hyperbolic, and the action of G on $\Gamma(G, X)$ is non-elementary and acylindrical

(b) H is quasi-convex in $\Gamma(G, X)$

To prove the above corollary, we need the following two lemmas.

Lemma 4.5.9. *Let T be a tree, and let $Q \subset T$ be ε-coarsely connected. Then Q is ε-quasi-convex.*

Proof. Let $\varepsilon > 0$ be the constant from Definition 4.5.6. Let x, y be two points in Q, and p be any geodesic between them. Then there exist points $x_0 = x, x_1, x_2, ..., x_{n-1}, x_n = y$ in Q such that for all $i = 0, ..., n-1$, $d(x_i, x_{i+1}) \leq \varepsilon$. Let p_i denote the geodesic segments between x_i and $x_{i+1}$for all $i = 0, 1, ..., n-1$. Since T is a tree, we must have that

$$p \subseteq \bigcup_{i=0}^{n-1} p_i.$$

By definition, for all $i = 0, 1, ..., n-1$, $p_i \subseteq B(x_i, \varepsilon)$, the ball of radius ε centered at x_i. Since $x_i \in Q$ for all $i = 0, 1, ..., n-1$, we obtain

$$p_i \subseteq Q^{+\varepsilon}.$$

This implies $p \subseteq Q^{+\varepsilon}$. $\square$

Lemma 4.5.10. *Let Γ be a quasi-tree, and $S \subset \Gamma$ be coarsely connected. Then S is quasi-convex.*

Proof. Let T be a tree such that Γ is quasi-isometric to T. Let d_Γ and d_T denote distances in Γ and T respectively. Let $\delta > 0$ be the hyperbolicity constant of Γ. Let $q\colon T \to \Gamma$ be a (λ, C)-quasi-isometry. i.e.,

$$-C + \frac{1}{\lambda} d_T(a,b) \leq d_\Gamma(q(a), q(b)) \leq \lambda d_T(a,b) + C.$$

Let $\varepsilon > 0$ be the constant from Definition 4.5.6 for S. Set $Q = q^{-1}(S)$. Then $Q \subset T$. It is easy to check that Q is ρ-coarsely connected with constant $\rho = \lambda(\varepsilon + C)$. By Lemma 4.5.9, Q is ρ-quasi-convex.

Let x, y be two points in S, and p be a geodesic between them. Choose points a, b in Q such that $q(a) = x$ and $q(b) = y$. Let r denote the (unique) geodesic in T between a and b. Since Q is ρ-quasi-convex, we have

$$r \subseteq Q^{+\rho}.$$

Set $\sigma = \lambda\rho + C$. Then

$$q(r) \subseteq S^{+\sigma}.$$

Further $q \circ r$ is a quasi-geodesic between x and y. By Lemma 4.3.4, there exists a constant $R(= R(\lambda, C, \delta))$ such that $q(r)$ and p are Hausdorff distance less than R from each other. This implies that $p \subseteq S^{+(R+\sigma)}$. Thus S is quasi-convex. $\square$

Proof of Corollary 4.5.8. By Corollary 4.5.3, there exists a generating set X of G such that $\Gamma(G,X)$ is a quasi-tree (hence hyperbolic), and the action of G on $\Gamma(G,X)$ is acylindrical and non-elementary. Let d_X denote the metric on $\Gamma(G,X)$ induced by the generating set X. Let $H = \langle x_1, x_2, ..., x_n\rangle$. Set

$$\varepsilon = \max\{d_X(1, x_i^{\pm 1}) \mid i = 1, 2, ..., n\}.$$

We claim that H is coarsely connected with constant ε. Indeed if u, v are elements of H, then $u^{-1}v = \prod_{j=1}^{k} w_j$, where $w_j \in \{x_1^{\pm 1}, ..., x_n^{\pm 1}\}$. Set

$$z_0 = u, z_1 = uw_1, ..., z_{k-1} = uw_1w_2...w_{k-1}, z_k = v.$$

Clearly $z_i \in H$ for all $i = 0, 2, ..., k-1$. Further

$$d_X(z_i, z_{i+1}) = d_X(1, w_{i+1}) \leq \varepsilon$$

for all $i = 0, 1, 2, ..., k-1$. By Lemma 4.5.10, H is quasi-convex in $\Gamma(G,X)$. $\square$

CHAPTER 5

QUASI-PARABOLIC STRUCTURES

5.1 Structure of the Lamplighter groups

The goal of this chapter of the thesis is to answer the following open question from[2] regarding quasi-parabolic structures.

Problem 5.1.1. *Does there exist a group G such that $\mathscr{H}_{qp}(G)$ is non-empty and finite?*

We will answer the above question in the affirmative. Indeed, we will prove the following.

Theorem 5.1.2. *The lamplighter groups L_p have exactly two quasi-parabolic structures, when p is a prime.*

Theorem 5.1.2 will additionally enable us to give the complete $\mathscr{H}(G)$ structure for the Lamplighter groups when p is a prime. We begin by recalling relevant information about the Lamplighter groups.

The Lamplighter groups, denoted $L_n, n \geq 2$, are given by the presentation

$$\langle a, t \mid [a^{t^i}, a^{t^j}] = 1 \ \forall i, j \in \mathbb{Z},\, a^n = 1 \rangle,$$

where $x^y = y^{-1}xy$.

Equivalently, this group is the (restricted) wreath product $(\bigoplus_{\mathbb{Z}} \mathbb{Z}_n) \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ is taken to be t from the presentation given above. The "lamplighter" picture of elements of this group is the following : Take a bi-infinite road of light bulbs placed at integer points, each of which has n states corresponding to the elements of $\mathbb{Z}_n$, and a lamplighter indicating the particular bulb under consideration. The action of the group on this picture is such that t moves the cursor one position to the right, and powers of a change the state of the current bulb under consideration. Thus each element of L_n can be interpreted as a *configuration* of a finite collection of lit bulbs in some allowable states, with the lamplighter at a fixed integer. Algebraically, this means that if $x \in L_n$, then

$$x = t^m(t^{-i_1}a^{k_1}t^{i_1})(t^{-i_2}a^{k_2}t^{i_2})...(t^{-i_l}a^{k_l}t^{i_l}),$$

for some $m, i_1, i_2, ...i_l \in \mathbb{Z}$ and some $k_1, k_2, ..., k_l \in \{1, 2, ..., n-1\}$. This visualization can be very useful in making some arguments related to the Lamplighter groups. In this section, we prove some results about the structure of the lamplighter group.

Let $A = \bigoplus_{\mathbb{Z}} \mathbb{Z}_n$. This is the base group used to define the lamplighter group as a wreath product. Indeed,

$$A = \langle a^{t^i} \mid a^n = 1, [a^{t^i}, a^{t^j}] = 1 \, \forall i, j \in \mathbb{Z} \rangle,$$

, and its visualization in L_n is that of elements consisting of a finite number of illuminated lamps in some allowable states, while the lamplighter stands at index 0. i.e. If $b \in L_n$, then

$$b = (t^{-i_1}a^{k_1}t^{i_1})(t^{-i_2}a^{k_2}t^{i_2})...(t^{-i_l}a^{k_l}t^{i_l}),$$

for some $i_1, i_2, ...i_l \in \mathbb{Z}$ and some $k_1, k_2, ..., k_l \in \{1, 2, ..., n-1\}$. A is in fact, a characteristic subgroup of L_n (see [78]Lemma 4.2).

Remark 5.1.3. For $n \geq 2$, L_n has no general type structures. This follows from the fact that L_n is solvable, and thus cannot contain any non-abelian free subgroups. In particular, L_n is not acylindrically hyperbolic.

Although the lamplighter picture is very useful, we will use the description of elements of L_n in terms of group rings. Let $R = \mathbb{Z}_n[\mathbb{Z}]$ be the group ring, which consists of the following elements : all formal linear combinations of powers of t with coefficients $0, 1, ..., n-1$

(also called Laurent polynomials). For $b \in A$, $k \in \{0, 1, ...n-1\}$ and $m \in \mathbb{Z}$, define

$$kt^m.b = (b^k)^{t^m} = t^{-m} b^k t^m. \qquad (*)$$

Lemma 5.1.4. *$A \cong R$. In particular, A is a free left R-module , where the module multiplication is defined by extending (*) canonically to formal sums in R.*

Proof. We first prove that every $b \in A$ represents as $p(t).a$, where $p(t) \in R$. Suppose that

$$b = (t^{-i_1} a^{k_1} t^{i_1})(t^{-i_2} a^{k_2} t^{i_2})...(t^{-i_l} a^{k_l} t^{i_l}),$$

for some $i_1, i_2, ...i_l \in \mathbb{Z}$ and some $k_1, k_2, ..., k_l \in \{1, 2, ..., n-1\}$.

Set

$$p_b(t) = k_1 t^{-i_1} + k_2 t^{-i_2} + ...k_l t^{-i_l}.$$

Then

$$p_b(t).a = (t^{i_1} a^{k_1} t^{-i_1})(t^{i_2} a^{k_2} t^{-i_2})...(t^{i_l} a^{k_l} t^{-i_l}) = b.$$

It is easy to see that the map $b \mapsto p_b(t)$ is an injective and surjective homomorphism. Thus $A \cong R$.

Next observe that A is an abelian group (under component wise addition, though we will use the multiplicative notation), where each element is of order at most n. Further, the module multiplication is well defined and sends elements of A to A.

Let $x, y \in R$ and $b, c \in A$. Clearly, by definition, we have $(x+y).b = (x.b)(y.b)$. Further since shifting and adding configurations of lamps (with marker at zero) is equal to adding the configurations and then shifting, we also have $x.(bc) = (x.b)(x.c)$. Thus the module axioms are satisfied. It is also easy to see that since $A \cong R$, the set $B = \{a\} \subset A$ is a basis for A. Thus A is a free R-module. $\square$

Lemma 5.1.5. *Every element $g \in L_n$ represents as $t^m(p(t).a)$, for some $m \in \mathbb{Z}, p(t) \in R$.*

Proof. Recall that if $g \in L_n$, then

$$g = t^m (t^{-i_1} a^{k_1} t^{i_1})(t^{-i_2} a^{k_2} t^{i_2}) \ldots (t^{-i_l} a^{k_l} t^{i_l}),$$

for some $m, i_1, i_2, \ldots i_l \in \mathbb{Z}$ and some $k_1, k_2, \ldots, k_l \in \{1, 2, \ldots, n-1\}$. This yields the representation $g = t^m b$, where $b = (t^{-i_1} a^{k_1} t^{i_1})(t^{-i_2} a^{k_2} t^{i_2}) \ldots (t^{-i_l} a^{k_l} t^{i_l}) \in A$, and the result now follows from Lemma 5.1.4. $\square$

We will now recall the definition of the *Busemann pseudocharacter*. A map $q\colon G \to \mathbb{R}$ is a *quasi-character* (or *quasi-morphism*) if there exists a constant D such that

$$|q(gh) - q(g) - q(h)| \leq D$$

for all $g, h \in G$; one says that q has *defect at most* D. If, in addition, the restriction of q to every cyclic subgroup of G is a homomorphism, q is called a *pseudocharacter* (or *homogeneous quasi-morphism*). Every quasi-character $q\colon G \to \mathbb{R}$ gives rise to a pseudocharacter $p\colon G \to \mathbb{R}$ defined by

$$p(g) = \lim_{n \to \infty} \frac{q(g^n)}{n}$$

(the limit always exists); p is called the *homogenization of* q. It is straightforward to check that

$$|p(g) - q(g)| \leq D \tag{5.1}$$

for all $g \in G$ if q has defect at most D.

To every action of a group on a hyperbolic space fixing a point on the boundary, one can associate the so-called *Busemann pseudocharacter*. We briefly recall the construction and necessary properties here and refer to [37, Sec. 7.5.D] and [52, Sec. 4.1] for more details.

Definition 5.1.6. Let G be a group acting on a hyperbolic space S and fixing a point $\xi \in \partial S$. Fix any $s \in S$ and let $\mathbf{x} = (x_i)$ be any sequence of points of S converging to ξ. Then the function $q_{\mathbf{x}} \colon G \to \mathbb{R}$ defined by

$$q_{\mathbf{x}}(g) = \limsup_{n\to\infty} (\mathrm{d}_S(gs, x_n) - \mathrm{d}_S(s, x_n))$$

is a quasi-character. Its homogenization $p_{\mathbf{x}}$ is called the *Busemann pseudocharacter*. It is known that for any two sequences $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ converging to ξ, we have $\sup_{g\in G} |q_{\mathbf{x}}(g) - q_{\mathbf{y}}(g)| < \infty$ [52, Lemma 4.6]; in particular, this implies that $p_{\mathbf{x}} = p_{\mathbf{y}}$ and thus we can drop the subscript in $p_{\mathbf{x}}$. It is straightforward to verify that $g \in G$ acts loxodromically on S if and only if $p(g) \neq 0$; in particular, p is non-zero whenever $G \curvearrowright S$ is quasi-parabolic.

Lemma 5.1.7. *Let $[X]$ be a quasi-parabolic structure on $G = L_n$. Then t is a loxodromic element with respect to the action $G \curvearrowright \Gamma(G,X)$. Consequently, the action must fix either $t^{+\infty}$ or $t^{-\infty}$.*

Proof. Let p be the Busemann pseudocharacter associated to the action of G on $\Gamma(G,X)$. Let D be the defect of p. Assume, by contradiction, that t is not loxodromic with respect to this action. Then $p(t) = 0$.

Observe that since each $b \in A$ has finite order, it is not a loxodromic element and so $p(b) = 0$. Since p is a pseudocharacter, for any $x \in G$, $p(x^n) = np(x)$.

Consider for any $b \in A, k \in \mathbb{Z}$,

$$p(t^k b) = \lim_{i\to\infty} \frac{q((t^k b)^i)}{i} = \lim_{i\to\infty} \frac{q(t^{ik} b')}{i} \leq \lim_{x\to\infty} \frac{q(t^{ik}) + q(b') + D}{i} \leq \lim_{x\to\infty} \frac{q(t^{ik}) + 2D}{i} = p(t^k) = 0.$$

Similarly, we can show that $p(t^k b) \geq \lim_{x\to\infty} \dfrac{q(t^{ik}) - 2D}{i} = p(t^k) = 0$.

Thus $p(t^k b) = 0 \ \forall k \in \mathbb{Z}, b \in A$. But this contradicts the non-elementary structure of $[X]$. $\square$

Remark 5.1.8. The proof of Lemma 5.1.7 also shows that if we renormalize p so that $p(t) = 1$, then the Busemann pseudocharacter associated to any quasi-parabolic action of

L_n is the homomorphism that is the standard projection of elements of the group to $\mathbb{Z}$.

5.2 A necessary and sufficient condition for regular quasi-parabolic actions

In order to show that there are exactly two quasi-parabolic structures on L_p when p is a prime, we will need the following definitions and theorem taken from [24]. This theorem is a necessary and sufficient condition for the existence of regular quasi-parabolic structures on a group (defined below). Please note that the authors of [24] refer to quasi-parabolic actions as *focal* actions.

Definition 5.2.1. [24, Section 4] Let $G \curvearrowright S$ be a quasi-parabolic action, and $p \colon G \to \mathbb{R}$ the associated Busemann pseudocharacter. The action of G is said to be *regular* if p is a homomorphism.

Remark 5.2.2. It follows from Lemma 5.1.7 and Remark 5.1.8 that quasi-parabolic actions of $L_n, n \geq 2$ are always regular. Further, the lineal action associated to the Busemann pseudocharacter p from Lemma 5.1.7 is an orientable one; see [2, Lemma 4.15] and it is the only lineal structure on L_n.

Definition 5.2.3. [24, Section 4] Let H be a group and Q be a subset of H, and let α be an automorphism of H. We say that the action of α is *confining H into Q* (resp. *strictly confining*) if it satisfies the following conditions:

(a) $\alpha(Q)$ is contained (resp. strictly contained) in Q.

(b) $H = \bigcup_{n \geq 0} \alpha^{-n}(Q)$

(c) $\alpha^{n_0}(Q.Q) \subset Q$ for some non-negative integer n_0.

The next theorem is a particular case of [24, Theorem 4.1]. We state it in this particular way since we will apply the result in precisely the setting where $G = L_n$ and $A = \bigoplus_{\mathbb{Z}} \mathbb{Z}_n$ is the base of the semi-direct product structure for this group.

Although stated for a more general case, Theorem 4.1 of [24] can be restated as follows (See paragraph 4.A. of [24]) : Let X be a generating of a group G, and d_X be the associated word metric. Then (G, d_X) is regular focal if and only if G has a subgroup K and a decomposition $K = H \rtimes \langle \alpha \rangle$ and a subset $Q \subset H$ so that α is confining into Q and the inclusion map $(K, d_S) \to (G, d_X)$ is a quasi-isometry; where $S = \{Q, \alpha^{\pm 1}\}$. In the case when $G = H \rtimes \langle \alpha \rangle$, the statement reduces to the following.

Theorem 5.2.4. *Let $G = A \rtimes \langle t \rangle$ be a group and X a generating set of G. Then the following are equivalent.*

(1) $\Gamma(G, X)$ is hyperbolic and $G \curvearrowright \Gamma(G, X)$ is a regular quasi-parabolic action.

(2) There exists a subset $Q \subset A$ such that

(i) The action of t (or t^{-1}) on A is strictly confining into Q. (This action is by conjugation)

(ii) Setting $S = \{Q, t^{\pm 1}\}$, the inclusion map $i \colon (G, d_S) \hookrightarrow (G, d_X)$ is a quasi-isometry.

Moreover, if (2) holds, then the Busemann pseudocharacter in (1) is proportional to the obvious projection of G to $\mathbb{Z}$.

Remark 5.2.5. If $t^{-1}Qt = Q$, then (G, d_S) is quasi-isometric to a line. The above theorem implies that $[X] = [S] \in H_{qp}(G)$.

Since every quasi-parabolic action of L_n is regular, the above theorem reduced the problem of finding all quasi-parabolic structures on the group to finding all subsets of A that are strictly confining under the action of t or t^{-1}. Since $A \cong R$ by Lemma 5.1.4, we may reformulate the conditions of confining actions as follows. Note that condition (b) of Definition 5.2.3 ensures that $0 \in Q$ always.

Definition 5.2.6. Let $Q \subset R$. The action of t (resp. t^{-1}) on R is (strictly) confining into Q if

(a) tQ (resp. $t^{-1}Q$) is (strictly) contained in Q (Here we mean multiplication by t or t^{-1})

(b) For every $p(t) \in R$, there exists $n \geq 0$ such that $t^n p(t) \in Q$. (resp. $t^{-n} p(t) \in Q$)

(c) There exists a constant $n_0 \geq 0$ such that $t^{n_0}(Q+Q) \subset Q$. (resp. $t^{-n_0}(Q+Q) \subset Q$)

5.3 Groups with finitely many quasi-parabolic structures

We now turn to the proof of Theorem 5.1.2. For convenience, we set $G = L_p$, where p is a prime. Further, let $B_+ \subset R$ consist of all elements with non-negative powers on t, and $B_- \subset R$ consist of all elements with non-positive powers on t. Set

$$A_+ = \{t^{\pm 1}, B_+\}$$

and

$$A_- = \{t^{\pm 1}, B_-\}.$$

Our goal is to prove that $[A_+]$ and$[A_-]$ are the only two quasi-parabolic structures on G. We will prove this theorem in a series of smaller results.

Lemma 5.3.1. *The subset B_+ (respectively B_-) is strictly confining under the action of t (respectively t^{-1}).*

Proof. We prove the lemma for B_+; the proof for B_- has symmetric arguments. Clearly multiplication by t satisfies condition (a) of Definition 5.2.6. The action is strictly confining since $1 \in B_+$, but $1 \notin tB_+$. It is also easy to see that condition (b) of Definition 5.2.6 holds by taking n to be the absolute value of the smallest negative exponent on t in $p(t)$. Condition (c) holds for $n_0 = 0$ since B_+ is closed under addition. □

Corollary 5.3.2. *$[A-]$ and $[A_+]$ are quasi-parabolic structures on G.*

We now want to prove the following :

Proposition 5.3.3. *Let $Q \subset R$ be strictly confining under the action of t. Set $S = \{Q, t^{\pm 1}\}$. Then $[S] = [A_+]$.*

Indeed, if we can prove the above proposition, then by symmetric arguments, we can also show that if $Q \subset R$ strictly confining under the action of t^{-1}, then $[S] = [A_-]$. This will then imply that $[A_+]$ and$[A_-]$ are the only two quasi-parabolic structures on G, and it will remain to show that these structures are distinct. In order to prove the proposition, we will need the following lemmas.

In what follows, we fix $Q \subset R$ which is strictly confining under the action of t and set $S = \{Q, t^{\pm 1}\}$.

Lemma 5.3.4. *Suppose that $\{p_i(t) \mid i \in \Lambda\} \subset B_+$ is a collection of elements satisfying*

(i) $p_i(t) \notin Q$ for all $i \in \Lambda$, and

(ii) There exists a constant K such that $t^K p_i(t) \in Q$ for all $i \in \Lambda$.

Set $P = \{t^k p_i(t) \mid k \geq 0, i \in \Lambda\}$ and $Q' = Q \cup P$. Then Q' is also strictly confining with respect to the action of t. Further if $S' = \{Q', t^{\pm 1}\}$ then $[S] = [S']$.

Proof. Observe that $tQ \subset Q$ and $tt^k p_i(t) = t^{k+1} p_i(t)$ for all $i \in \Lambda, k \geq 0$. Thus

$$tQ' = t(Q \cup P) = tQ \cup tP \subset Q \cup P = Q'.$$

To see the strict containment, choose an element $p(t)$ from $\{p_i(t)\}$ such that $p(t)$ has minimal degree. Such an element exists because $\{p_i(t)\} \subset B_+$. We will show that $p(t) \notin Q'$. Suppose, by contradiction, that $p(t) \in tQ' = t(Q \cup P)$. If $p(t) \in tQ$, then $p(t) \in Q$ by condition (a) of Definition 5.2.6, which violates condition (i) of the statement. If $p(t) \in tP$, then $p(t)$ is not an element of minimal degree from $\{p_i(t)\}$, which contradicts the choice of $p(t)$. Thus $p(t) \notin tQ'$, and condition (a) of Definition 5.2.6 holds. Condition (b) obviously holds since $Q \subset Q'$.

It remains to show that condition (c) holds. Set $m_0 = K + n_0$, where n_0 is the constant that satisfies condition (c) for the action of t on Q.

Let $q_1(t), q_2(t) \in Q'$. Observe that $t^K q_i(t) \in Q$ for $i = 1,2$ because Q is confining under the action of t and by condition (ii) of the statement. Thus

$$t^{m_0}(q_1(t) + q_2(t)) = t^{n_0+K}(q_1(t) + q_2(t)) \in t^{n_0}(Q + Q) \in Q \subset Q'.$$

If $t^{m_0}(Q' + Q') = Q'$, then $t^{m_0+1}(Q' + Q') = tQ' \subset Q'$. Thus condition (c) is also satisfied.

Lastly, since $Q \subset Q'$, $\sup_{s \in S} |s|_{S'} = 1$. Conversely we have that

$$\sup_{s' \in S'} |s'|_S \leq \max\{1, K+1\} < \infty.$$

$\square$

Remark 5.3.5. If Λ is a finite index set, then condition (ii) of Lemma 5.3.4 is naturally satisfied. For each $i \in \Lambda$, there exists k_i such that $t^{k_i} p_i(t) \in Q$, by condition (b) of Definition 5.2.6. Taking $K = \max_{i \in \Lambda}\{k_i\}$ gives the required constant K for condition (ii).

We will first prove that $[S] \preceq [A_+]$. We will then use this fact to additionally prove that $[S] = [A_+]$.

Lemma 5.3.6. *Let $Q \subset R$ be strictly confining under the action of t. Then $[S] \preceq [A_+]$.*

Proof. First observe that by Remark 5.3.5, we may assume without loss of generality that $\{0, 1, ..., p-1\} \subset Q$. Consequently,

$$\{1, t, t^2, ..., t^i, ...\} \bigcup \{2, 2t, 2t^2, ...\} ... \bigcup \{(p-1), (p-1)t, (p-1)t^2, ...\} \subset Q.$$

i.e. Q contains all elements from B_+ that contain only one term.

Next, observe that for $0 \leq j < i$ and any $r, s \in \mathbb{Z}_p$, $rt^j + st^i \in Q + Q$. By using condition (c) of Definition 5.2.6, we get that $rt^{n_0+j} + st^{n_0+i} \in Q$. i.e. Q contains all element from B_+ with 2 terms and such that the smallest exponent on t is n_0.

Using Lemma 5.3.4, we may further assume that Q contains all elements from B_+ with exactly 2 terms. Indeed, this is a consequence of the conclusion of the previous paragraph and the observation that every element $p(t) \in B_+ \backslash Q$ with 2 terms satisfies $t^{n_0} p(t) \in Q$. By iterating the above steps finitely many times, we can conclude that Q contains all elements from B_+ with at most n_0 terms.

It suffices to prove the following claim : For any $n \geq n_0$, Q contains all elements from B_+ with n terms such that the smallest exponent on t is bigger than or equal to n_0. Indeed if the claim is proven, then by Lemma 5.3.4, we can conclude that Q contains all elements from B_+ since every element $p(t) \in B_+ \backslash Q$ with more than n_0 terms will satisfy $t^{n_0} p(t) \in Q$. Since Q contains all elements from B_+ with at most n_0 terms already, the result will follow.

To prove the claim, we will use induction on $n \geq n_0$. Since Q contains all elements from B_+ with at most n_0 terms, the base of the induction holds.

Assume that the claim is true for all integers n such that $n_0 \leq n \leq k$. i.e. Q contains all elements from B_+ with at most k terms such that the smallest exponent on t is n_0. We will show that Q contains all elements from B_+ with $k+1$ terms such that the smallest exponent on t is n_0.

Let $p(t) = r_1 t^{n_0+i_1} + r_2 t^{n_0+i_2} + \ldots r_{n_0} t^{i_{n_0}+n_0} + \ldots + r_{k+1} t^{n_0+i_{k+1}}$, where $0 \leq i_1 < i_2 < \ldots i_{n_0} < \ldots < i_{k+1}$ is a sequence of non-negative integers and $r_1, r_2, \ldots, r_{k+1}$ is a list of coefficients from $\mathbb{Z}_p$.

Then $r_1 t^{i_1} + r_2 t^{i_2} + \ldots + r_{n_0} t^{i_{n_0}} \in Q$ since this element has n_0 terms. We claim that $i_{n_0+1} \geq n_0$. Indeed if $i_{n_0+1} < n_0$, then we cannot choose n_0 non-negative integers strictly less than $n_0 - 1$. Thus $r_{n_0+1} t^{i_{n_0+1}} + \ldots + r_{k+1} t^{i_{k+1}}$ is an element such that the smallest exponent on t is bigger than or equal to n_0. This element has $k+1-n_0 \leq k$ terms. If $k+1-n_0 < n_0$, this element is in Q. If $k+1-n_0 > n_0$, then this element is in Q by the induction hypothesis.

Thus

$$r_1 t^{i_1} + r_2 t^{i_2} + \ldots + r_{k+1} t^{i_{k+1}} \in Q + Q.$$

Using condition (c), we get that

$$r_1 t^{n_0+i_1} + r_2 t^{n_0+i_2} + \ldots + r_{k+1} t^{n_0+i_{k+1}} \in Q.$$

This completes the proof of the lemma. $\square$

Lemma 5.3.7. *If* $\{rt^{-i} \mid i \geq 0,\, r \in \mathbb{Z}_p\} \subset Q$*, then* $[S] \preceq [A_-]$.

Proof. Arguing as in Lemma 5.3.6, by using induction on the number of terms in elements from B_-, we obtain that $r_1 t^{-i_1} + r_2 t^{-i_2} + r_3 t^{-i_3} + \ldots + r_n t^{-i_n} \in Q$, for any $n \geq 1$, any $0 \leq i_1 < i_2 < \ldots < i_n$, where $i_j \in \mathbb{Z}^+$ and any choice $r_1, r_2, \ldots r_n$ of coefficients from $\mathbb{Z}_p$.

$\square$

Corollary 5.3.8. *For any* $r \in \mathbb{Z}_p \backslash \{0\}$*,* Q *cannot contain* $\{rt^{-i} \mid i \geq 0\}$.

Proof. Since Q is strictly confining under the action of t, it defines a quasi-parabolic structure on G. By contradiction, suppose for some $r \in \mathbb{Z}_p$, $\{rt^{-i} \mid i \geq 0\} \subset Q$. Since p is a prime r generates $\mathbb{Z}_p$. By using condition (c) of Definition 5.2.6 repeatedly, we get that

$$\{zt^{-i} \mid i \geq 0,\, z \in \mathbb{Z}_p\} \subset Q.$$

The proof now follows from Lemma 5.3.6 and Lemma 5.3.7, which together imply that $[A_+], [A_-] \preceq Q$. Then any $p(t) \in R$ satisfies the condition $|p(t)|_S = 1$, by using condition (a) of Definition 5.2.6. By Lemma 5.1.5, every element $g \in G$ represents as $t^m((p(t))$, for some $m \in \mathbb{Z}, p(t) \in R$. Thus the map from $G \to \mathbb{Z}$ given by $t^k(p(t)) \mapsto k$ is surjective and satisfies

$$d_S(1, t^k p(t)) - 1 \leq |k| \leq d_S(1, t^k p(t)).$$

Thus Q must define a lineal structure on G, which is a contradiction. $\square$

Definition 5.3.9. For $p(t) \in R$, the *negative degree* of $p(t)$ is the smallest negative exponent on t that appears in $p(t)$. The *leading negative coefficient* is the coefficient of the term with the negative degree.

Example 5.3.10. For $G = \mathbb{Z}_4$, the negative degree of $p(t) = 2t^{-9} + 3t^{-6} + t^6 + 2t + 1$ is -9 and the leading negative coefficient is 2.

Proof of Proposition 5.3.3. By Lemma 5.3.6, $[S] \preceq [A_+]$. Thus, we may assume without loss of generality that $B_+ \subset Q$. We will now show that $[A_+] = [S]$.

First observe that if Q contains only finitely many more elements from R in addition to B_+, then we are done. So we need to consider the case when Q contains infinitely many elements from R in addition to B_+; let this collection be $\{p_j(t) \mid j \geq 1\}$.

If the negative degrees of these elements are bounded below, then again we have $[A_+] = [S]$. Indeed, if the lower bound on the negative degrees is $-k_0$, then

$$|p_j(t)|_{A_+} \leq t^{k_0} + 1$$

for all j, which implies that $[A_+] \preceq [S]$.

Thus, it remains to consider the case when the negative degrees of the elements $p_j(t)$ are not bounded below. Note that in this case we may add appropriate elements from B_+ to each $p_j(t)$ and use condition (c) of Definition 5.2.6 to obtain infinitely many elements from B_- in Q such that their negative degrees are not bounded below. So without loss of generality, we may assume that $\{p_j(t) \mid j \geq 1\} \subset B_-$.

Since $\mathbb{Z}_p$ is a finite set, there exists a coefficient $r \in \mathbb{Z}_p$ that occurs infinitely many times as the leading negative coefficient on elements with unbounded negative degrees. We fix this coefficient r and let $\{q_j(t)\} \subset \{p_j(t)\}$ be this collection of elements.

Assume that

$$q_j(t) = rt^{-j_1} + r_{j,2}t^{-j_2} + \ldots + r_{j,k_j}t^{-j_{k_j}}, \qquad (\star)$$

where $j_1 > j_2 > \ldots > j_{k_j} \geq 0$ and $r_{j,2}, r_{j,3}, \ldots, r_{j,k_j} \in \mathbb{Z}_p$ i.e. the negative degree of each $q_j(t)$ is $-j_1$. Then $j_1 \to \infty$.

For any element of the form $(\star)$, we will refer to the following process as *recovering* an element : Using condition (a), multiply $(\star)$ by t^{j_2} to obtain $rt^{-j_1+j_2}+b(t) \in Q$, where $b(t) \in B_+$. By adding an appropriate $b'(t)$, we get that $rt^{-j_1+j_2} \in Q+Q$. Using condition (c), we get $rt^{-j_1+j_2+n_0} \in Q$.

Case 1. The difference $j_1 - j_2$ takes arbitrarily large positive values as j varies. In this case, $-j_1+j_2+n_0$ takes arbitrary small negative values, and so we can recover

$$\{rt^{-i} \mid i \geq 0\} \in Q$$

by using condition (a) of Definition 5.2.6 in addition to the process of recovering elements described above. This is a contradiction by Corollary 5.3.8.

Case 2. The difference $j_1 - j_2$ is bounded for all j. i.e. $1 \leq j_1 - j_2 \leq K$. Choose any integer $M \geq 1$. Choose an element $q_j(t)$ which satisfies $-j_1 + M(p+1)n_0 < 0$. Such an element must exist since $j_1 \to \infty$ as $j \to \infty$.

Since Q is confining under the action of t, we may use condition (a) and multiply $q_j(t)$ by $t^{j_1-j_2}$ to obtain

$$rt^{-j_2} + r_{j,2}t^{j_1-2j_2} + \ldots + r_{j,k_j}t^{-j_{k_j}+j_1-j_2} \in Q.$$

Adding this new element to $q_j(t)$ we obtain the following element in $Q+Q$:

$$rt^{-j_1} + rr_{j,2}t^{-j_2} + \ldots . \qquad (\star\star)$$

By using condition (c), we get that

$$rt^{-j_1+n_0} + rr_{j,2}t^{-j_2'+n_0} + \ldots \in Q.$$

Since $o(r) = p$, we may repeat the above step $L \leq p$ times so that $r^L = r_{j,2}^{-1}$. At the end of L iterations, we obtain

$$rt^{-j_1+Ln_0}+r'_{j,x}t^{-j_x+Ln_0}+\ldots \in Q,$$

where $j_1-j_x>j_1-j_2$ and $-j_1+Ln_0<0$. i.e. the gap between the leading negative term and the next has increased.

By repeating this process M times, we can turn the M successive coefficients that follow the leading term to zero, allowing us to further increase the gap between the leading negative term and the next in each iteration. This allows us to recover rt^{-i} for larger values of i. Note that this process does not involve passing to an equivalent generating set; instead we are only using the conditions of Definition 5.2.6 to build elements in Q. By increasing M and choosing the appropriate $q_j(t)$, this allows us to eventually recover the entire set $\{rt^{-i} \mid i\geq 0\}\subset Q$ as in Case 1, which is a contradiction by Corollary 5.3.8. $\square$

Proof of Theorem 5.1.2. By Proposition 5.3.3, if $Q\subset R$ is strictly confining under the action of t, then $[S]=[A_+]$. By similar arguments, we can show that if $Q\subset R$ is strictly confining under the action of t^{-1}, then $[S]=[A_-]$. This yields two quasi-parabolic structures on G. It is easy to see that these two structures are incomparable since $\sup_{i\geq 1}|t^i|_{A_-}=+\infty$ and $\sup_{i\geq 1}|t^{-i}|_{A_+}=+\infty$; yielding precisely two quasi-parabolic structures on G. $\square$

Corollary 5.3.11. *The $\mathcal{H}(G)$ structure of $G=L_p$, where p is a prime is the following poset.*

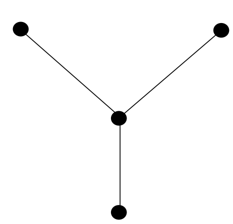

Figure 5.1: Poset of $\mathcal{H}(L_p)$

Proof. The proof follows from Theorem 5.1.2, Remark 5.2.2 and the fact that the Lamplighter groups have no general-type actions. Indeed, the Lamplighter groups are solvable and cannot contain non-abelian free subgroups. $\square$

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