TWO PROBLEMS IN COMPUTATIONAL MATHEMATICS: MULTIPLE ORTHOGONAL POLYNOMIALS AND GREEDY ENERGY POINTS

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This work is of course dedicated to Marta and Ariadna

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CHAPTER I

INTRODUCTION

In this thesis we investigate two subjects in asymptotic analysis. The first one focuses on the study of asymptotic properties of sequences of points called *greedy energy sequences*, obtained through an iterative algorithm that involves the minimization of certain energy functionals. The second subject concerns the study of asymptotic properties of sequences of *multiple orthogonal polynomials* in the complex plane. We give a detailed description of these two topics in what follows.

I.1 Greedy energy sequences

In order to define these sequences rigorously, we need to introduce a number of basic concepts and notations. Since some of the results in this part of the thesis are obtained in the context of locally compact metric spaces, we introduce these notions in this general setting.

Let X denote a locally compact metric space containing infinitely many points. A *kernel* in X is, by definition, a lower semicontinuous function $k : X \times X \to \mathbb{R} \cup \{+\infty\}$. It is called *positive* if $k(x, y) \geq 0$ for all $(x, y) \in X \times X$. The class of *M. Riesz kernels* in $X = \mathbb{R}^p$ is the most important one for our study, and it is defined as follows:

$$k_s(x, y) := \begin{cases} \log \frac{1}{|x-y|}, & \text{if } s = 0, \\ \frac{1}{|x-y|^s}, & \text{if } s > 0, \end{cases} \tag{1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^p$. The logarithmic kernel (case $s = 0$) plays a significant role in the asymptotic analysis of complex polynomials.

For a set $\omega_N = \{x_1, \dots, x_N\}$ of N ($N \geq 2$) points in X, not necessarily distinct, the *k-energy* of ω_N is defined as

$$E(\omega_N) := \sum_{1 \leq i \neq j \leq N} k(x_i, x_j) = \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} k(x_i, x_j). \tag{2}$$

If the kernel is *symmetric*, i.e. $k(x, y) = k(y, x)$ for all $x, y \in X$, we may also write

$$E(\omega_N) = 2 \sum_{1 \leq i < j \leq N} k(x_i, x_j).$$

We will use the notation $\operatorname{card}(\omega_N) = N$ to indicate that the set $\omega_N = \{x_1, \dots, x_N\}$ consists of N points, even if they are not distinct. If $k = k_s$ and $\omega_N = \{x_1, \dots, x_N\} \subset \mathbb{R}^p$, we will denote by $E_s(\omega_N)$ the Riesz s-energy of ω_N.

Definition I.1.1. *Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a symmetric kernel on a locally compact metric space X, and let $A \subset X$ be a compact set. We say that ω_N^* is an optimal N-point configuration for A if* $\operatorname{card}(\omega_N) = N$ *and*

$$E(\omega_N^*) = \inf\{E(\omega_N) : \omega_N \subset A,\ \operatorname{card}(\omega_N) = N\}.$$

For every N, the existence of optimal N-point configurations is guaranteed by the lower semicontinuity of k and the compactness of A. Of course, these configurations are not unique in general. Let us now define the notion of greedy energy sequences.

Definition I.1.2. *Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a symmetric kernel on a locally compact metric space X, and let $A \subset X$ be a compact set. A sequence $(a_n)_{n=1}^{\infty} \subset A$ is called a greedy k-energy sequence on A if it is generated in the following way:*

- *a_1 is selected arbitrarily on A.*

- *Assuming that $a_1, \ldots, a_n$ have been selected, a_{n+1} is chosen to satisfy*

$$\sum_{i=1}^{n} k(a_{n+1}, a_i) = \inf_{x \in A} \sum_{i=1}^{n} k(x, a_i), \tag{3}$$

  for every $n \geq 1$.

We remark that the choice of a_{n+1} is not unique in general. We will use the notation

$$\alpha_{N,k} := \{a_1, \ldots, a_N\}$$

for the set formed by the first N points of this sequence. In the context of Riesz kernels k_s, we write $\alpha_{N,s}$ instead of α_{N,k_s}.

We will later introduce in this thesis more general definitions of optimal configurations and greedy sequences, since we are also interested in analyzing their asymptotic behavior under the presence of an *external field*. One of the goals of this thesis is to find similarities and differences in the behavior of these two constructions.

It seems that A. Edrei was the first to introduce in [22] (see page 78 of that paper) the definition of configurations $\alpha_{N,0}$ in the complex plane (for the logarithmic kernel). However, in the literature these configurations are often called *Leja points* in recognition of F. Leja's article [38]. When the kernel employed is the Green function or the Newtonian kernel $1/|x-y|$ on the unit sphere S^2, the corresponding configurations $\alpha_{N,k}$ are also referred to as *Leja-Górski points* (see [31] and references therein). In [5], certain configurations known as *fast Leja points* were introduced, and an algorithm was presented to compute them. These configurations are defined over discretizations of planar sets and the kernel employed is the logarithmic kernel. In [17] a constrained energy problem for this kernel was considered and associated *constrained Leja points* were introduced.

If $A \subset \mathbb{C}$ is compact, an equivalent way to define the sequence of configurations $\alpha_{N,0}$ on A is to ask a_{n+1} to satisfy the property

$$\prod_{k=1}^{n} |a_{n+1} - a_k| = \max_{z \in A} \prod_{k=1}^{n} |z - a_k| =: M_n.$$

In particular, for every $n \geq 2$ the points a_n lie in the outer boundary of A, i.e. the boundary of the unbounded component of $\mathbb{C} \setminus A$. Edrei observed in [22] that if $(a_n)_{n=1}^{\infty} \subset A$ is an arbitrary greedy k_0-energy sequence on A, then

$$\lim_{n \to \infty} |V(a_1, \ldots, a_n)|^{2/n^2} = \operatorname{cap}_0(A), \tag{4}$$

$$\lim_{n \to \infty} M_n^{1/n} = \operatorname{cap}_0(A), \tag{5}$$

where $V(\zeta_1, \ldots, \zeta_n)$ denotes the Vandermonde determinant associated with $\zeta_1, \ldots, \zeta_n$, i.e.

$$V(\zeta_1, \ldots, \zeta_n) = \prod_{1 \leq i < j \leq n} (\zeta_j - \zeta_i),$$

and $\operatorname{cap}_0(A)$ denotes the logarithmic capacity of A, which is defined as

$$\operatorname{cap}_0(A) := e^{-\gamma(A)}, \tag{6}$$

$$\gamma(A) := \inf\{\int \int \log \frac{1}{|z-t|} \, d\mu(z) \, d\mu(t) : \mu \geq 0, \ \operatorname{supp}(\mu) \subset A, \ \|\mu\| = 1\}.[1]$$

The asymptotic formula (4) can be equivalently formulated as

$$\lim_{N \to \infty} \frac{E_0(\alpha_{N,0})}{N^2} = \gamma(A). \tag{7}$$

On the other hand, optimal configurations in the complex plane (i.e. when the total energy is minimized with respect to N variable points on compact sets $A \subset \mathbb{C}$,

[1] $\gamma(A)$ is known as the *Robin constant* of A.

see Definition I.1.1) associated with the logarithmic kernel are known in the literature as *Fekete points*. They can also be defined as those N-point configurations $\omega_N^* = \{z_1, \ldots, z_N\} \subset A$ that satisfy the property

$$|V(z_1, \ldots, z_N)| = \max_{\zeta_i \in A} |V(\zeta_1, \ldots, \zeta_N)|.$$

M. Fekete was the first to show in [25] that (7) also holds for any sequence of optimal N-point configurations ω_N^* on A.

Regarding the origin of Leja sequences in [22], let us explain the reason why these sequences were introduced. G. Pólya proved in [51] that if $E \subset \mathbb{C}$ is a compact set such that $\mathbb{C} \setminus E$ is connected, and $f(z)$ is the analytic continuation onto $\mathbb{C} \setminus E$ of the series expansion

$$\frac{b_0}{z} + \frac{b_1}{z^2} + \cdots + \frac{b_n}{z^{n+1}} + \cdots, \tag{8}$$

so that $\mathbb{C} \setminus E$ is the natural domain of $f(z)$, then

$$\limsup_{n \to \infty} |B_n|^{1/n^2} \leq \operatorname{cap}_0(E),$$

where

$$B_n := \begin{vmatrix} b_0 & b_1 & \cdots & b_{n-1} \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & b_n & \cdots & b_{2n-2} \end{vmatrix}.$$

Edrei used Leja sequences and applied their property (4) to show in [22] the following interesting converse result. If $E \subset \mathbb{C}$ is a compact set as before, then for every $\theta \in [0, 1]$ there exists a series expansion (8) representing an analytic function $f(z)$ on $\mathbb{C} \setminus E$ such that

$$\limsup_{n \to \infty} |B_n|^{1/n^2} = \theta \operatorname{cap}_0(E),$$

where $\mathbb{C} \setminus E$ is the natural domain of f.

There are several practical reasons for studying greedy energy sequences. First, these sequences are significantly easier to obtain numerically as compared to optimal configurations, since only one new particle (point) is generated at each step of the algorithm and all the previously defined particles are preserved (in the case of optimal configurations one generates N new particles at the Nth step and the previous ones are disregarded). This property makes greedy points useful, for instance, to sample the surface where they are generated, and to use them as nodes of a Newton interpolation scheme [53]. Greedy sequences also serve as a reference model to study the behavior of general sequences of particles. In addition, greedy sequences (especially Leja points) have been extensively used in numerical linear algebra [54, 14], numerical analysis [17, 39, 5] and approximation theory [22, 38, 7, 60].

Chapters II and III of this thesis are devoted to the study of greedy sequences. Chapter II contains results obtained using potential theoretic tools. These tools can be employed if there exists a positive measure supported on the set whose energy is finite (the notion of energy of a measure is defined in Section II.1). In the context of Riesz kernels, this situation corresponds to the case when $s < \dim_H(A)$, where $\dim_H(A)$ denotes the Hausdorff dimension of A. Chapter III describes those results obtained in the context of Riesz kernels under the assumption that $s \geq \dim_H(A)$ (the hyper-singular case).

The two main problems that we analyze regarding greedy energy points can be simply explained as follows. We investigate how the energy of these configurations behaves as the number of particles increases and tends to infinity (and obtain asymptotic formulas that are analogous to (4) and (5)). We also investigate how these configurations are asymptotically distributed. In so doing, we will show that greedy energy configurations are in many aspects similar to optimal configurations (especially in the context of potential theory). But we will also show that in other situations the behavior of greedy configurations differs significantly from that of optimal configurations.

The main results obtained in this thesis on asymptotic properties of greedy energy sequences can be outlined as follows:

- We show that for $s > 1$, greedy k_s-energy sequences on Jordan arcs or closed Jordan curves in $\mathbb{R}^p$ are not asymptotically s-energy minimizing (see Definition III.1.2 and Theorem III.2.5 for details). A similar result is proved for greedy best-packing configurations (see Definition III.2.7 and Theorem III.2.8).

- In fact, we show in Theorems III.2.5 and III.2.8 that for $s \in (1, \infty]$, *no* infinite sequence of points on Jordan arcs or curves in $\mathbb{R}^p$ can be asymptotically s-energy minimizing.

- We disprove a conjecture of L. Bos on the asymptotic distribution of greedy best-packing configurations (see Proposition III.2.9).

- It is shown that greedy k_d-energy sequences (case $s = d$) on the unit sphere $S^d \subset \mathbb{R}^{d+1}$ are asymptotically d-energy minimizing (see Theorem III.2.14). A similar result is proved for greedy k_1-energy sequences (case $s = 1$) on smooth Jordan arcs or curves in $\mathbb{R}^p$ (Theorem III.2.6). As an important consequence, we obtain that these sequences are asymptotically uniformly distributed (in both situations).

- It is shown that in terms of second-order asymptotics, greedy k_s-energy sequences and optimal configurations on the unit circle S^1 behave differently for $s \in (0, 1]$ (Propositions II.2.15 and III.2.3).

- In Chapters II and III, more general definitions of greedy energy sequences are introduced and their asymptotic properties are studied in the context of external fields (in Chapter II see e.g. Theorems II.2.5 and II.2.7) and weighted Riesz potentials (Chapter III).

- We provide several numerical computations that illustrate some of our results.

I.2 Multiple orthogonal polynomials

The origin of this subject is intimately related to the work of Charles Hermite on analytic number theory, and in particular to his proof in [33] of the transcendence of e. In this paper Hermite introduces the technique of simultaneous rational approximation of a system of analytic functions (in the case of [33] that system was formed by exponential functions). This important technique is now called *Hermite-Padé approximation.* If the functions to be approximated are Markov-type functions, i.e. functions of the form

$$\widehat{\mu}_i(z) = \int \frac{d\mu_i(x)}{z-x}, \qquad \operatorname{supp}(\mu_i) \subset \mathbb{R}, \tag{9}$$

where the measures μ_i are assumed to be finite and compactly supported, then the common denominator of the rational approximants is a polynomial that satisfies orthogonality conditions with respect to the measures μ_i.

More precisely, let μ_i, $1 \leq i \leq m$, be a system of non-trivial complex-valued measures that are compactly supported in the complex plane, and consider a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. Then, there exists a non-trivial polynomial $P_{\mathbf{n}}$ of degree at most $|\mathbf{n}| = n_1 + \cdots + n_m$, that satisfies the property

$$\int x^k\, P_{\mathbf{n}}(x)\, d\mu_i(x) = 0, \qquad 0 \leq k \leq n_i - 1, \quad 1 \leq i \leq m. \tag{10}$$

Such a polynomial is called a *multiple orthogonal polynomial* associated with $\mathbf{n}$ and

$(\mu_1, \ldots, \mu_m)$. Finding $P_{\mathbf{n}}$ reduces to the problem of solving a linear system of $|\mathbf{n}|$ homogeneous equations on $|\mathbf{n}| + 1$ unknowns, and therefore a non-trivial solution exists. $P_{\mathbf{n}}$ is of course not unique, but observe that there is only one monic polynomial of lowest degree that satisfies (10).

The asymptotic theory of multiple orthogonal polynomials studies the behavior of these polynomials as $|\mathbf{n}|$ approaches infinity. Different types of asymptotic properties can be analyzed, but in this thesis we investigate the *ratio* and *nth root* asymptotic behavior of certain sequences formed by such polynomials. Several obstacles must be overcomed before obtaining these asymptotic properties. One of them is to determine the exact degree of the polynomials considered (it is desirable that they have maximal degree). In order to solve this problem it is necessary to assume additional conditions on the orthogonality measures. It is also critical to determine the location of the zeros, since the asymptotic properties that we investigate must be analyzed in a region that excludes them.

The most important class of measure systems for which asymptotic properties of associated multiple orthogonal polynomials have been studied is the class of *Nikishin systems*. These systems were introduced by E.M. Nikishin in [48]. For the sake of simplicity, we explain how to construct such systems only in the case of two measures. Let σ_1 and σ_2 be measures supported on the real line, and assume that their supports are contained in disjoint compact intervals. Then the Nikishin system generated by (σ_1, σ_2) is the system (μ_1, μ_2) defined as

$$d\mu_1(x) := d\sigma_1(x), \qquad d\mu_2(x) := \widehat{\sigma}_2(x)\, d\sigma_1(x).$$

In this thesis we will consider a similar construction for measures supported on starlike sets in the complex plane. We would like to mention here that a large number of applications of multiple orthogonal polynomials associated with Nikishin systems have been found in diverse areas such as vector rational approximation [49, 48, 12], simultaneous quadrature formulas [26], analytic number theory [58], and more recently in integrable systems, random matrix theory, and brownian motions of non-intersecting paths [35, 18, 19].

The problem we investigate in this thesis is motivated by recent investigations in [3] on strong asymptotics of polynomials generated by a three-term higher-order recurrence of the form

$$zQ_n = Q_{n+1} + a_{n-p+1}Q_{n-p}, \qquad p \in \mathbb{N}, \quad n \geq p, \tag{11}$$

where the coefficients a_k are positive and satisfy the perturbation condition

$$\sum_{n=1}^{n} |a_n - a| < \infty. \tag{12}$$

It was shown in [2] that the positivity of the coefficients implies the fact that these polynomials are indeed multi-orthogonal with respect to a system of positive measures whose supports are compact and contained in the starlike set

$$S = \bigcup_{k=0}^{p} [0, \infty) \exp(2\pi i k/(p+1)).$$

Moreover, the orthogonality measures have a Nikishin-type structure.

The condition (12) allows the authors of [3] to prove a strong asymptotic formula of the form

$$\lim_{n\to\infty} \frac{Q_n(z)}{w_0^n(z)} = F_0(z).$$

This limit holds uniformly on compact subsets of the region $\Omega = \mathbb{C} \setminus S_0$, where

$$S_0 = \bigcup_{k=0}^{p} [0, \alpha] \exp(2\pi i k/(p+1)), \qquad \alpha = [(p+1)/p^{p/(p+1)}]a^{1/(p+1)},$$

F_0 is a certain function analytic in Ω, and w_0 is the unique branch of the algebraic equation

$$w^{p+1} - zw^p + a = 0$$

that satisfies $w_0(z) = z + O(1)$, $z \to \infty$, and has a holomorphic continuation onto Ω.

In this thesis we will start from rather weak assumptions on the orthogonality measures (instead of starting from assuming a condition on the recurrence coefficients such as (12)) to obtain ratio and nth root asymptotic formulas for the associated multiple orthogonal polynomials. Ratio asymptotics provides the limiting behavior (outside the support of the measures) for sequences of the form

$$\left\{ \frac{Q_{n+1}}{Q_n} \right\}_{n=1}^{\infty}, \tag{13}$$

and nth root asymptotics describes, in particular, the limiting distribution of the zeros of these polynomials. In the case of ratio asymptotics, we will in fact show the existence of different *periodic limits* for the sequence (13). The sequence of polynomials we investigate also satisfies a three-term recurrence relation of the form (11) with positive coefficients. The ratio asymptotic behavior of (13) will also allow us to prove that the sequence formed by the recurrence coefficients has different periodic limits. Therefore the situation we consider is different from that analyzed in [3]. Several relations between the limiting functions of the sequence (13) are obtained, as well as relations between the limiting values of the recurrence coefficients.

The main technique that we employ to obtain ratio asymptotics is to find a certain system of boundary value problems satisfied by the limiting functions of the sequence (13), and show that this system has a unique solution. In order to find the boundary value problems we will apply auxiliary results on ratio and relative asymptotics of *polynomials orthogonal with respect to varying measures.* To obtain nth root asymptotics we will use again techniques from logarithmic potential theory.

The following is an outline of the main results obtained in this thesis on properties of multiple orthogonal polynomials associated with measures supported on starlike sets (for a description of the measures of orthogonality and statement of the main results see Section IV.1):

- We prove that the multiple orthogonal polynomials have maximal degree and we describe the multiplicity and location of their zeros (see Proposition IV.1.1).
- It is shown that the multiple orthogonal polynomials satisfy a three-term recurrence relation of third order with positive recurrence coefficients (Proposition IV.1.2).
- The exact number of zeros of the *functions of second type* (see (176) for definition) is obtained, as well as their multiplicity and location (Proposition IV.1.3).
- An interlacing property of the zeros of the multiple orthogonal polynomials and the functions of second type is proved (see Theorem IV.1.4 and Proposition IV.1.5).
- Under mild conditions on the orthogonality measures, the ratio asymptotic behavior of the multiple orthogonal polynomials and the limiting behavior of the recurrence coefficients is described in Theorem IV.1.6. In particular, we show the existence of different periodic limits for the sequence of ratios of consecutive polynomials and the sequence of recurrence coefficients (see also Proposition IV.1.7 for relations between the limiting functions and the limiting values of the recurrence coefficients).
- We describe the limiting functions of the sequence of ratios of consecutive polynomials in terms of the branches of a three-sheeted compact Riemann surface of genus zero (Theorem IV.1.8).

- Under regularity assumptions on the measures of orthogonality (see Definition IV.1.10), we obtain the nth root asymptotic behavior of the multiple orthogonal polynomials, as well as the asymptotic distribution of their zeros. The limiting distribution of the zeros is described in terms of the solution to a vector equilibrium problem for logarithmic potentials (see Corollary IV.1.13).

- We also provide several numerical experiments that illustrate our results.

CHAPTER II

GREEDY ENERGY POINTS: THE POTENTIAL THEORETIC CASE

II.1 Introduction, background results and notation

Throughout this chapter, X will denote a locally compact metric space containing infinitely many points. If X is not compact, let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. Recall that $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ denotes a kernel in X. Kernels are always assumed to be symmetric.

Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, and let $\omega_N = \{x_1, \ldots, x_N\}$ be a configuration of N $(N \geq 2)$ points in X. In addition to the notion of k-energy (2) of ω_N, we define the *weighted energy* of ω_N as

$$E_f(\omega_N) := E(\omega_N) + 2(N-1) \sum_{i=1}^{N} f(x_i). \tag{14}$$

In potential theory, the function f is referred to as an *external field*. Recall that if $k = k_s$ and $\omega_N \subset \mathbb{R}^p$, then $E_s(\omega_N)$ denotes the Riesz s-energy of ω_N.

Definition II.1.1. *For a non-empty set $A \subset X$, the weighted N-point energy of A is given by*

$$\mathcal{E}_f(A, N) := \inf\{E_f(\omega_N) : \omega_N \subset A,\ \operatorname{card}(\omega_N) = N\}. \tag{15}$$

If $f \equiv 0$, we use instead the notation

$$\mathcal{E}(A, N) := \inf\{E(\omega_N) : \omega_N \subset A,\ \operatorname{card}(\omega_N) = N\}. \tag{16}$$

We say that $\omega_N^* \subset A$ is an optimal weighted N-point configuration on A if

$$E_f(\omega_N^*) = \mathcal{E}_f(A, N), \qquad \operatorname{card}(\omega_N^*) = N.$$

If A is compact, the existence of ω_N^* follows from the lower semicontinuity of k and f (see also Definition I.1.1 for the case $f \equiv 0$).

We will also use the notation

$$\mathcal{E}_s(A, N) := \inf\{E_s(\omega_N) : \omega_N \subset A,\ \operatorname{card}(\omega_N) = N\} \tag{17}$$

to denote the N-point Riesz s-energy of a compact set $A \subset \mathbb{R}^p$.

In order to state our results, we need to introduce the continuous analogues of the above notions. Given a non-empty set $A \subset X$, let $\mathcal{M}(A)$ denote the linear space of all real-valued Radon measures that are compactly supported on A, and let

$$\mathcal{M}^+(A) := \{\mu \in \mathcal{M}(A) : \mu \geq 0\}, \qquad \mathcal{M}_1(A) := \{\mu \in \mathcal{M}^+(A) : \mu(X) = 1\}. \tag{18}$$

Given a measure $\mu \in \mathcal{M}(X)$, the *energy* of μ is the double integral

$$I(\mu) := \int\int k(x, y)\, d\mu(x)\, d\mu(y), \tag{19}$$

whereas the function

$$U^\mu(x) := \int k(x, y)\, d\mu(y) \tag{20}$$

is called the *potential* of μ. The *weighted energy* of μ is defined by

$$I_f(\mu) := I(\mu) + 2\int f\, d\mu. \tag{21}$$

Since any lower semicontinuous function is bounded below on compact sets, the above integrals are well defined, although they may attain the value $+\infty$.

We shall use the notations $I_s(\mu)$, $I_{s,f}(\mu)$, and U_s^μ to denote, respectively, the energy (19), weighted energy (21), and potential (20) of a measure $\mu \in \mathcal{M}(\mathbb{R}^p)$ with respect to the Riesz s-kernel.

We say that k satisfies the *maximum principle* if for every measure $\mu \in \mathcal{M}_1(X)$,

$$\sup_{x \in \operatorname{supp}(\mu)} U^\mu(x) = \sup_{x \in X} U^\mu(x). \tag{22}$$

In $\mathbb{R}^p$, it is well known that Riesz kernels k_s satisfy the maximum principle for $s \in [p-2, p)$ (cf. [37, Theorem 1.10]).

The quantity $w(A) := \inf\{I(\mu) : \mu \in \mathcal{M}_1(A)\}$ is called the *Wiener energy* of A, and plays an important role in potential theory. The *capacity* of A is defined as $\operatorname{cap}(A) := w(A)^{-1}$ if k is positive, and otherwise, it is defined as $\operatorname{cap}(A) := \exp(-w(A))$. These notions generalize the concepts of Robin constant and logarithmic capacity of a compact set $A \subset \mathbb{C}$ (see (6)). A property is said to hold *quasi-everywhere* (q.e.), if the exceptional set (the set of all points where the property is not satisfied) has Wiener energy $+\infty$. In the context of Riesz kernels, we will use the symbols $w_s(A)$ and $\operatorname{cap}_s(A)$ to denote the Wiener s-energy and s-capacity of a set $A \subset \mathbb{R}^p$.

Given a net $\{\mu_\alpha\} \subset \mathcal{M}(A)$, we say that $\{\mu_\alpha\}$ converges in the *weak-star topology* to a measure $\mu \in \mathcal{M}(A)$ if

$$\lim_\alpha \int g \, d\mu_\alpha = \int g \, d\mu, \qquad \text{for all} \quad g \in C_c(A),$$

where $C_c(A)$ denotes the space of compactly supported continuous functions on A. We will use the notation

$$\mu_\alpha \xrightarrow{*} \mu$$

to denote the weak-star convergence of measures. If A is compact, we know by the Banach-Alaoglu theorem that $\mathcal{M}_1(A)$ equipped with the weak-star topology is compact.

If $w(A) < \infty$, a measure $\mu \in \mathcal{M}_1(A)$ satisfying the property $I(\mu) = w(A)$ is called an *equilibrium measure* for A. If A is compact, the existence of such a measure is guaranteed by the lower semicontinuity of k and the compactness of $\mathcal{M}_1(A)$ (cf. [28, Theorem 2.3]). However, uniqueness does not always hold in this case.

For Riesz kernels k_s in $\mathbb{R}^p$, the following are well known properties. Let $A \subset \mathbb{R}^p$ be a compact set, and assume that $0 \leq s < \dim_{\mathcal{H}}(A)$, where $\dim_{\mathcal{H}}(A)$ denotes the Hausdorff dimension of A. Then there exists only one measure $\lambda_{A,s} \in \mathcal{M}_1(A)$ such that $I_s(\lambda_{A,s}) = w_s(A)$, i.e. the equilibrium measure for A is unique. On the other hand, if $s \geq \dim_{\mathcal{H}}(A)$, then $I_s(\mu) = +\infty$ for all $\mu \in \mathcal{M}_1(A)$. We refer the reader to Theorems 8.5 and 8.9 in [45] for justifications of these facts.

The following result is central in this theory.

Theorem II.1.2 (Choquet [16]). *Let k be an arbitrary kernel on X and $A \subset X$ be a compact set. Then*

$$\lim_{N\to\infty} \frac{\mathcal{E}(A,N)}{N^2} = w(A), \tag{23}$$

where $\mathcal{E}(A,N)$ is defined by (16)*.*

The following is a variation of Theorem II.1.2.

Theorem II.1.3 (Farkas and Nagy [24]). *Assume that the kernel k is positive and is finite on the diagonal, i.e. $k(x,x) < +\infty$ for all $x \in X$. Then for arbitrary sets $A \subset X$,*

$$\lim_{N\to\infty} \frac{\mathcal{E}(A,N)}{N^2} = w(A).$$

We remark that Theorems II.1.2 and II.1.3 were proved in the context of locally compact Hausdorff spaces. Potential theory in these spaces was developed by Choquet

[15, 16], Fuglede [28] and Ohtsuka [50]. Recently Zorii [61, 62] has studied properties of potentials with external fields in this setting.

It was shown by Fuglede [28, Theorem 2.4] that if $A \subset X$ is compact and $\mu \in \mathcal{M}_1(A)$ is an equilibrium measure for A, then the inequality $U^\mu(x) \le w(A)$ is valid for all $x \in \operatorname{supp}(\mu)$. The *essential support* of μ is by definition the set

$$S_\mu^* := \{x \in A : U^\mu(x) \le w(A)\}. \tag{24}$$

Hence $\operatorname{supp}(\mu) \subset S_\mu^*$.

The following is a restricted version of Definition I.1.2.

Definition II.1.4. *Under the same assumptions as in Definition I.1.2, assume that $w(A) < \infty$, and let $\mu \in \mathcal{M}_1(A)$ be an equilibrium measure. A sequence $(a_n = a_{n,k,\mu})_{n=1}^\infty \subset A$ is called a greedy (k,μ)-energy sequence on A if it is generated in the following way:*

- *a_1 is selected arbitrarily on S_μ^*.*
- *Assuming that $a_1, \ldots, a_n$ have been selected, a_{n+1} is chosen to satisfy $a_{n+1} \in S_\mu^*$ and*
$$\sum_{i=1}^n k(a_{n+1}, a_i) = \inf_{x \in S_\mu^*} \sum_{i=1}^n k(x, a_i)$$
for every $n \ge 1$.

The set formed by the first N points of this sequence is denoted by $\alpha_{N,k,\mu}$.

In this chapter we are also interested in the so called *Gauss variational problem* in the presence of an external field f. In what follows we assume that $A \subset X$ is a closed set, and we will refer to A as the *conductor*. The Gauss variational problem asks for a solution to the minimization problem

$$V_f(A) := \inf_{\mu \in \mathcal{M}_f(A)} I_f(\mu), \tag{25}$$

where $\mathcal{M}_f(A)$ denotes the class of measures

$$\mathcal{M}_f(A) := \{\mu \in \mathcal{M}_1(A) : I(\mu) < +\infty, \int f \, d\mu < +\infty\}. \tag{26}$$

Throughout we will denote $V_f(A)$ simply as V_f. If $\mathcal{M}_f(A) = \emptyset$, then by definition $V_f = +\infty$. If $\mathcal{M}_f(A) \neq \emptyset$ and there exists a minimizing measure $\mu \in \mathcal{M}_f(A)$ satisfying $I_f(\mu) = V_f$, we call μ an *equilibrium measure in the presence of the external field* f. In this case we say that the Gauss variational problem is *solvable*, and observe that V_f is finite.

Sufficient conditions for the existence and uniqueness of solution for a similar variational problem were provided by N. Zorii [61, 62] in the more general context of locally compact Hausdorff spaces. She assumes that the kernel is positive if A is not compact, and allows measures to have non-compact support in this case. We remark that the theory of logarithmic potentials ($k = k_0$) with external fields in the complex plane is particularly rich in applications to physics and other branches of analysis. We will make use of this theory in Chapter IV, in order to obtain nth root asymptotics of multiple orthogonal polynomials. We refer the reader to [56] for details on this theory.

Let us introduce the notation

$$W_f(\mu) := V_f - \int f \, d\mu \tag{27}$$

for an equilibrium measure $\mu \in \mathcal{M}_f(A)$ in the presence of f. This value is finite. The *essential support of* μ *in the presence of* f is defined as

$$S^*_{f,\mu} := \{x \in A : U^\mu(x) + f(x) \leq W_f(\mu)\}. \tag{28}$$

If the Riesz kernel k_s is employed, we use the symbol $W_{s,f}(\mu)$ to denote the constant (27).

Using [28, Lemma 2.3.3] and the argument employed in [56] to prove parts (d) and (e) of Theorem I.1.3, it is easy to see that if $\mu \in \mathcal{M}_f(A)$ is an equilibrium measure in the presence of f, then

$$U^{\mu}(x) + f(x) \leq V_f - \int f\, d\mu \tag{29}$$

holds for all $x \in \operatorname{supp}(\mu)$ (i.e. $\operatorname{supp}(\mu) \subset S^*_{f,\mu}$) and

$$U^{\mu}(x) + f(x) \geq V_f - \int f\, d\mu \tag{30}$$

holds q.e. on A.

We are ready to introduce the following definition (compare with Definition II.1.4):

Definition II.1.5. *Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a kernel on a locally compact metric space X, $A \subset X$ be a closed set, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an external field. If X is not compact, we assume that f satisfies the following "growth" condition at infinity: for each compactly supported probability measure ν,*

$$\lim_{x\to\infty} (U^{\nu}(x) + f(x)) = +\infty, \tag{31}$$

(i.e. given $M > 0$, there exists a compact set $B \subset X$ such that $U^{\nu}(x) + f(x) > M$ for all $x \in X \setminus B$).

Assume that the Gauss variational problem is solvable and $\mu \in \mathcal{M}_f(A)$ is an equilibrium measure. A sequence $(a_n = a_{n,k,f,\mu})_{n=1}^{\infty} \subset A$ is called a weighted greedy (f, μ)-energy sequence on A if it is generated in the following way:

- *a_1 is selected arbitrarily on $S^*_{f,\mu}$.*

- *For every $n \geq 1$, assuming that $a_1, \ldots, a_n$ have been selected, a_{n+1} is chosen so that $a_{n+1} \in S^*_{f,\mu}$ and*

$$\sum_{i=1}^{n} k(a_{n+1}, a_i) + nf(a_{n+1}) = \inf_{x \in S^*_{f,\mu}} \Big\{ \sum_{i=1}^{n} k(x, a_i) + nf(x) \Big\}. \tag{32}$$

The set formed by the first N points of this sequence is denoted by $\alpha^f_{N,\mu}$. We also introduce the following associated function:

$$U^f_n(x) := \sum_{i=1}^{n-1} k(x, a_i) + (n-1)f(x), \qquad x \in A, \quad n \geq 2. \tag{33}$$

Remark II.1.6. *Condition* (31) *implies in particular that $S^*_{f,\mu}$ is compact. Consequently, for every $n \geq 1$, the existence of a_{n+1} is guaranteed by the lower semicontinuity of k and f. However, a_{n+1} may not be unique.*

In the context of Riesz kernels in $\mathbb{R}^p$, (31) *is one of the conditions that are usually required in order to prove the solvability of the Gauss variational problem* (*see [56]*)*. If $s = 0$,* (31) *is equivalent to the property*

$$\lim_{|x| \to \infty} (f(x) - \log |x|) = +\infty, \tag{34}$$

and if $s > 0$, then (31) *is equivalent to requiring that*

$$\lim_{|x| \to \infty} f(x) = +\infty. \tag{35}$$

In many practical circumstances it is not possible to determine the support or essential support of an equilibrium measure. For this reason it is of interest to introduce the following

Definition II.1.7. *Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a kernel on a locally compact metric space X, $A \subset X$ be a closed set, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an external field.*

In case it exists, a sequence $(a_n = a_{n,f})_{n=1}^{\infty} \subset A$ *is called a weighted greedy* f*-energy sequence on* A *if it is constructed inductively by selecting* a_1 *arbitrarily on* A *so that* $f(a_1) < +\infty$*, and* a_{n+1} *as in* (32) *but taking the infimum on* A*. We use the notation* α_N^f *to indicate the configuration formed by the first* N *points of this sequence.*

We will also consider more general constructions.

Definition II.1.8. *Let* $m \geq 2$ *be a fixed integer. Under the same assumptions of Definition II.1.5, suppose that the Gauss variational problem is solvable and* $\mu \in \mathcal{M}_f(A)$ *is an equilibrium measure in the presence of* f*. A sequence* $(a_n = a_{n,m,f,\mu})_{n=1}^{\infty} \subset A$ *is called a weighted greedy* (m, f, μ)*-energy sequence on* A *if it is generated inductively in the following way:*

- *The first* m *points* $a_1, \ldots, a_m$ *are selected so that* $\{a_1, \ldots, a_m\}$ *is an optimal weighted* m*-point configuration on* $S^*_{f,\mu}$*, i.e.*

$$E_f(\{a_1, \ldots, a_m\}) \leq E_f(\{x_1, \ldots, x_m\}) \tag{36}$$

  for all $(x_1, \ldots, x_m) \in S^*_{f,\mu} \times \cdots \times S^*_{f,\mu}$.

- *Assuming that* $a_1, \ldots, a_{mN}$ *have been selected, where* $N \geq 1$ *is an integer, the next set of* m *points* $\{a_{mN+1}, \ldots, a_{m(N+1)}\} \subset S^*_{f,\mu}$ *is chosen to minimize the energy functional*

$$U_{mN}^{(f,m)}(x_1, \ldots, x_m) := \sum_{i=1}^{m} \sum_{l=1}^{mN} k(x_i, a_l) + \sum_{1 \leq i < j \leq m} k(x_i, x_j) + ((N+1)m - 1) \sum_{i=1}^{m} f(x_i) \tag{37}$$

  on $S^*_{f,\mu} \times \cdots \times S^*_{f,\mu}$.

For every $N \geq 0$*, the subindices* $mN + 1, \ldots, m(N + 1)$ *are assigned to the points* $a_{mN+1}, \ldots, a_{m(N+1)}$ *in an arbitrary order. Let* $\alpha_{mN,\mu}^{(f,m)}$ *denote the configuration formed by the first* mN *points of this sequence.*

In analogy to Definition II.1.7, we also introduce the following

Definition II.1.9. *Under the same assumptions of Definition II.1.7, given an integer* $m \geq 2$*, a sequence* $(a_n = a_{n,m,f})_{n=1}^{\infty} \subset A$ *(in case it exists) is called a weighted greedy* (m, f)*-energy sequence on* A *if it is obtained inductively as in* (36) *and* (37) *but the minimization is taken on* A*. With* $\alpha_{mN}^{(f,m)}$ *we denote the configuration* $\{a_1, \dots, a_{mN}\}$.

II.2 Main results

II.2.1 Greedy energy sequences

Our first result on the asymptotic behavior of greedy sequences is the following:

Theorem II.2.1. *Let* $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ *be a kernel on a locally compact metric space* X *that satisfies the maximum principle. Assume that* $A \subset X$ *is a compact set and* $\{\alpha_{N,k}\}$ *is a greedy* k*-energy sequence on* A*. Then*

(i) the following limit holds:

$$\lim_{N \to \infty} \frac{E(\alpha_{N,k})}{N^2} = w(A); \tag{38}$$

(ii) if $w(A) < \infty$ *and the equilibrium measure* $\mu \in \mathcal{M}_1(A)$ *is unique, it follows that*

$$\frac{1}{N} \sum_{a \in \alpha_{N,k}} \delta_a \xrightarrow{\ *\ } \mu, \qquad N \to \infty, \tag{39}$$

where δ_a *is the unit Dirac measure concentrated at* a;

(iii) if $w(A) < \infty$*, there holds*

$$\lim_{n \to \infty} \frac{U_n(a_n)}{n} = w(A), \tag{40}$$

where a_n is the n-th element of the greedy k-energy sequence, and

$$U_n(x) := \sum_{j=1}^{n-1} k(x, a_j), \qquad n \geq 2.$$

Furthermore, if $w(A) < \infty$, the analogues of assertions $(i), (ii)$, and (iii) hold for any greedy (k, μ)-energy sequence on A without assuming the maximum principle.

Theorem II.2.1 generalizes a result for Riesz potentials due to Siciak [57, Lemma 3.1]. For sets of positive capacity, his result asserts that if $A \subset \mathbb{R}^p$ is a compact set, $p-2 \leq s < p$, $p \geq 2$, and $\{\alpha_{N,s}\}$ is a greedy k_s-energy sequence on A, then (40) holds for $k = k_s$.

As a consequence of Theorem II.2.1, we deduce the following corollaries. We denote the d-dimensional unit sphere in $\mathbb{R}^{d+1}$ by S^d.

Corollary II.2.2. *Let d be a positive integer and $s \in [0, d)$. If $\alpha_{N,s} \subset S^d$ is an arbitrary greedy k_s-energy sequence, then the asymptotic formula*[1]

$$\lim_{N\to\infty} \frac{E_s(\alpha_{N,s})}{N^2} = \begin{cases} \frac{\Gamma((d+1)/2)\,\Gamma(d-s)}{\Gamma((d-s+1)/2)\,\Gamma(d-s/2)}, & \text{if} \quad 0 < s < d, \\ \\ -\log(2) + \frac{1}{2}(\psi(d) - \psi(d/2)), & \text{if} \quad s = 0, \end{cases} \tag{41}$$

holds, where $\psi(x) := \Gamma'(x)/\Gamma(x)$ denotes the digamma function. In addition,

$$\frac{1}{N} \sum_{a \in \alpha_{N,s}} \delta_a \xrightarrow{*} \sigma_d, \qquad N \to \infty, \tag{42}$$

where σ_d is the normalized Lebesgue measure on S^d.

Figure 1 below shows the first 2000 points of a greedy k_1-energy sequence on the unit sphere S^2. Observe that these points are distributed in a uniform fashion, as is consistent with (42).

[1]We remark that for $d = 1$ and $s = 0$ we have $\mathcal{E}_0(S^1, N) = -N\log(N)$, $N \geq 2$, (cf. [10]).

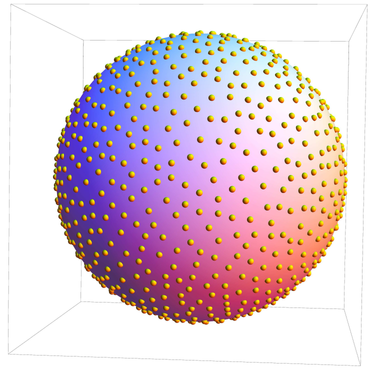

Figure 1: 2000 greedy energy points on S^2 for $s = 1$

We will also show in Chapter III that greedy k_d-energy sequences ($s = d$) on S^d satisfy (42), i.e. they are asymptotically uniformly distributed. However, it remains an open question to know if this property holds for $s > d$.

Corollary II.2.3. *Let $\alpha_{N,s}$ be any greedy k_s-energy sequence on $[-1, 1]$ for $s \in [0, 1)$. Then*

$$\lim_{N\to\infty} \frac{E_s(\alpha_{N,s})}{N^2} = \begin{cases} \frac{\sqrt{\pi}\,\Gamma(1+s/2)}{\cos(\pi s/2)\,\Gamma((1+s)/2)}, & \text{if} \quad 0 < s < 1, \\ \\ \log(2), & \text{if} \quad s = 0. \end{cases} \tag{43}$$

Furthermore,

$$\frac{1}{N} \sum_{a\in\alpha_{N,s}} \delta_a \xrightarrow{\ *\ } \frac{c_s}{(1-x^2)^{(1-s)/2}}\, dx, \qquad x \in [-1, 1], \quad N \to \infty, \tag{44}$$

where c_s is a normalizing constant.

Figures 23 – 6 below show the first 30 points of different greedy k_s-energy sequences on $[-1, 1]$. The values of s are indicated. In all examples, the first point is selected to be $a_1 = -1$. Observe that, as the parameter s increases, the points distribute themselves more uniformly on $[-1, 1]$. This phenomenon agrees with property (44). In fact, as a consequence of a more general result from Chapter III, we know that greedy k_1-energy sequences ($s = 1$) on $[-1, 1]$ are asymptotically uniformly distributed.

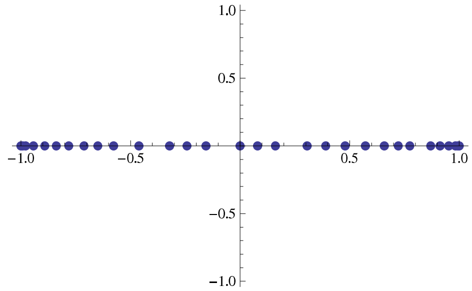

Figure 2: $s = 0$

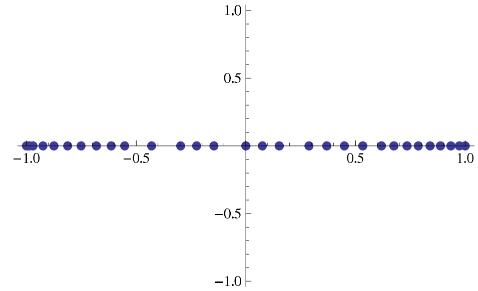

Figure 3: $s = 0.2$

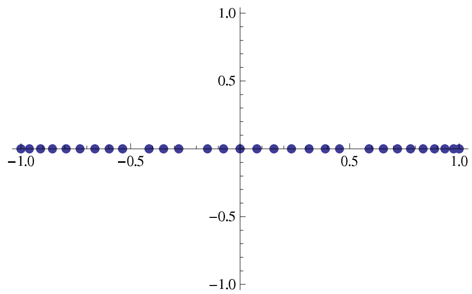

Figure 4: $s = 0.4$

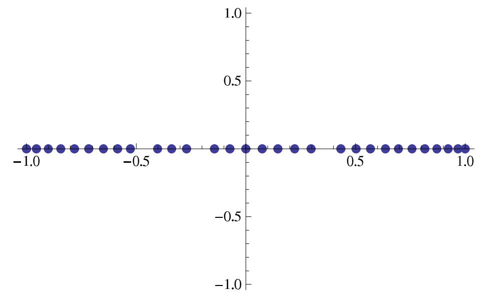

Figure 5: $s = 0.6$

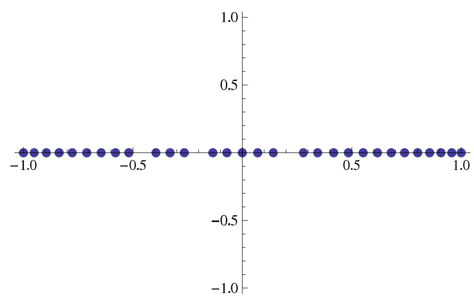

Figure 6: $s = 0.8$

II.2.2 Optimal weighted N-point configurations, weighted greedy energy sequences, and the Gauss variational problem in $\mathbb{R}^p$ for Riesz potentials

We present now the main results obtained in the context of potentials in the presence of external fields. The following is a generalization of Theorem II.1.2 to this setting.

Theorem II.2.4. *Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary kernel on a locally compact metric space X, $A \subset X$ be a compact conductor, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an external field. Assume that the Gauss variational problem is solvable. If $\{\omega_N^*\}$ is a sequence of optimal weighted N-point configurations on A, then*

$$\lim_{N\to\infty} \frac{E_f(\omega_N^*)}{N^2} = V_f. \tag{45}$$

Furthermore, if the Gauss variational problem has a unique solution $\mu \in \mathcal{M}_f(A)$, then

$$\frac{1}{N} \sum_{x\in\omega_N^*} \delta_x \xrightarrow{*} \mu, \qquad N \to \infty, \tag{46}$$

where δ_x is the unit Dirac measure concentrated at x.

As the proof of Theorem II.2.4 shows, without assuming the uniqueness of the equilibrium measure one can deduce that any convergent subsequence of $(1/N)\sum_{x\in\omega_N^*}\delta_x$ converges in the weak-star topology to an equilibrium measure. This observation is also applicable to all the results concerning greedy energy sequences.

The next result can be regarded as a generalization of Theorem II.2.1, but we remark that in Theorem II.2.1(i) we allow the possibility that $w(A) = +\infty$, whereas in Theorem II.2.5 the assumptions imply that $w(A) < +\infty$.

Theorem II.2.5. *Let $k : X \times X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary kernel on a locally compact metric space X, $A \subset X$ be a closed set, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an external field satisfying* (31) *in case that X is not compact. Assume that the Gauss variational problem is solvable and $\mu \in \mathcal{M}_f(A)$ is a solution. Let $\{\alpha_{N,\mu}^f\}$ be a weighted greedy (f, μ)-energy sequence on A. Then*

(i) the following limit holds:

$$\lim_{N\to\infty} \frac{E_f(\alpha_{N,\mu}^f)}{N^2} = V_f. \tag{47}$$

(ii) If the equilibrium measure $\mu \in \mathcal{M}_f(A)$ is unique, it follows that

$$\frac{1}{N} \sum_{a \in \alpha^f_{N,\mu}} \delta_a \xrightarrow{*} \mu, \qquad N \to \infty, \tag{48}$$

$$\lim_{n\to\infty} \frac{U^f_n(a_n)}{n} = V_f - \int f \, d\mu, \tag{49}$$

where a_n is the n-th element of the weighted greedy (f,μ)-energy sequence, and U^f_n is the function defined in (33).

The following result shows that if k is not allowed to take the value $+\infty$, then a certain relation can be established between conditions (47)–(49).

Proposition II.2.6. *Let $k : X \times X \to \mathbb{R}$ be a real-valued kernel on a locally compact metric space X, $A \subset X$ be a closed set, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an external field. Assume that the Gauss variational problem is solvable and $\mu \in \mathcal{M}_f(A)$ is a solution. Suppose that $\{b_n\}_{n=1}^{\infty} \subset S^*_{f,\mu}$ is a sequence of points such that*

$$\frac{1}{N} \sum_{n=1}^{N} \delta_{b_n} \xrightarrow{*} \mu, \qquad N \to \infty, \tag{50}$$

and set

$$T^f_n(x) := \sum_{i=1}^{n-1} k(x, b_i) + (n-1) f(x), \qquad x \in A, \quad n \geq 2.$$

If the following holds:

$$\lim_{n\to\infty} \frac{T^f_n(b_n)}{n} = V_f - \int f \, d\mu, \tag{51}$$

then

$$\lim_{N\to\infty} \frac{E_f(\{b_1, \ldots, b_N\})}{N^2} = V_f. \tag{52}$$

Theorem II.2.5 can be generalized for the class of greedy sequences introduced in Definition II.1.8.

Theorem II.2.7. *Let $m \geq 2$. Under the same assumptions of Theorem II.2.5, assume that $\{\alpha_{N,\mu}^{(f,m)}\}$ is a weighted greedy (m, f, μ)-energy sequence on A, where $\mu \in \mathcal{M}_f(A)$ is an equilibrium measure solving the Gauss variational problem. Then*

(i) the following limit holds:

$$\lim_{N\to\infty} \frac{E_f(\alpha_{mN,\mu}^{(f,m)})}{m^2N^2} = V_f. \tag{53}$$

(ii) If the equilibrium measure $\mu \in \mathcal{M}_f(A)$ is unique, it follows that

$$\frac{1}{mN} \sum_{a\in\alpha_{mN,\mu}^{(f,m)}} \delta_a \xrightarrow{*} \mu, \qquad N \to \infty, \tag{54}$$

$$\lim_{N\to\infty} \frac{U_{mN}^{(f,m)}(a_{mN+1}, \ldots, a_{m(N+1)})}{N} = m^2(V_f - \int f\, d\mu), \tag{55}$$

where a_i is the i-th element of the weighted greedy (m, f, μ)-energy sequence, and $U_{mN}^{(f,m)}$ is the function defined in (37)*.*

Remark II.2.8. *It is easy to see that* (54) *implies that*

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{a_i} \xrightarrow{*} \mu, \qquad N \to \infty.$$

All the results stated above, except Proposition II.2.6, are of course valid for Riesz kernels. We are also interested in obtaining asymptotic properties for greedy energy sequences of the type introduced in Definitions II.1.7 and II.1.9. These sequences have the advantage that their construction does not require the knowledge of the support of the equilibrium measure. We will show below that under natural assumptions on the external field f, these sequences can be constructed using Riesz potentials, and their asymptotic properties described.

Let $p \geq 2$ and consider the Riesz s-kernel k_s in $\mathbb{R}^p$ for $s \in (0, p)$. Assume that $A \subset \mathbb{R}^p$ is a closed set and f is an external field satisfying

$$\mathrm{cap}_s(\{x \in A : f(x) < +\infty\}) > 0. \tag{56}$$

If A is compact, no additional assumptions are needed. If A is not compact, we also assume that condition (35) holds.

Using the same arguments employed to prove Theorem I.1.3 in [56] (which concerns the case $p = 2$ and $s = 0$) and the fact that k_s is positive definite (see [37, Theorem 1.15]), it is not difficult to see that *the Gauss variational problem on A in the presence of f has a unique solution $\lambda = \lambda_{s,f} \in \mathcal{M}_f(A)$.* Furthermore, the inequality

$$U_s^\lambda(x) + f(x) \leq V_{s,f} - \int f \, d\lambda \tag{57}$$

is valid for all $x \in \mathrm{supp}(\lambda)$, where $V_{s,f} := I_{s,f}(\lambda)$ denotes the minimal energy constant (25), and

$$U_s^\lambda(x) + f(x) \geq V_{s,f} - \int f \, d\lambda \tag{58}$$

holds q.e. on A (relative to the s-capacity of sets).

We remark that if $p = 2$ and $s = 0$ then these properties hold if (35) is replaced by (34) (cf. [56]).

The following result holds.

Lemma II.2.9. *Let $p \geq 2$ and $p - 2 \leq s < p$. Suppose that $A \subset \mathbb{R}^p$ is closed and f satisfies* (56)*. If A is not compact, assume that* (35) *holds* (*or* (34) *in the case $p = 2$, $s = 0$*)*. Let $\lambda = \lambda_{s,f}$ be the equilibrium measure solving the Gauss variational problem on A in the presence of f. If $\{x_1, \ldots, x_n\} \subset \mathbb{R}^p$ is an arbitrary collection of points such that*

$$\sum_{i=1}^{n} \frac{1}{|x - x_i|^s} + nf(x) \geq M \qquad \text{for q.e.} \quad x \in \mathrm{supp}(\lambda), \tag{59}$$

then for all $x \in \mathbb{R}^p$,

$$\sum_{i=1}^{n} \frac{1}{|x - x_i|^s} \geq M - n(W_{s,f}(\lambda) - U_s^\lambda(x)), \tag{60}$$

where $W_{s,f}(\lambda)$ *is defined in* (27) *and* U_s^λ *is the potential associated to* λ*. Moreover,* (59) *implies that*

$$\sum_{i=1}^{n} \frac{1}{|x - x_i|^s} + nf(x) \geq M \qquad \text{for q.e.} \quad x \in A. \tag{61}$$

Remark II.2.10. *The case* $p = 2$, $s = 0$ *of Lemma II.2.9* (*the logarithmic kernel is employed in this case*) *is known as the generalized Bernstein-Walsh lemma and was proved by H. Mhaskar and E. Saff in [46].*

As a consequence of Lemma II.2.9, we obtain the following results.

Corollary II.2.11. *With the assumptions of Lemma II.2.9, let* $(a_n = a_{n,f})_{n=1}^\infty$ *be a weighted greedy* f*-energy sequence on* A *constructed using the Riesz kernel* k_s *for* $s \in [p-2, p)$*. Then this sequence is well-defined and* $a_n \in S^*_{f,\lambda}$ *for all* $n \geq 2$*, where* $S^*_{f,\lambda}$ *is the essential support* (28)*. Moreover, all the asymptotic properties in Theorem II.2.5 hold for this sequence* (*replacing* $\alpha^f_{N,\mu}$ *by* $\alpha^f_N = \{a_1, \ldots, a_N\}$ *and* μ *by* λ).

Corollary II.2.12. *Let* $m \geq 2$ *be an integer and assume that all the assumptions of Lemma II.2.9 hold. Let* $(a_n = a_{n,m,f})_{n=1}^\infty$ *be a weighted greedy* (m, f)*-energy sequence on* A *obtained using the Riesz kernel* k_s *for* $s \in [p-2, p)$*. Then this sequence is well-defined and* $a_n \in S^*_{f,\lambda}$ *for all* $n \geq 1$*. Furthermore, all the asymptotic properties in Theorem II.2.7 hold for this sequence* (*replacing* $\alpha^{(f,m)}_{mN,\mu}$ *by* $\alpha^{(f,m)}_{mN} = \{a_1, \ldots, a_N\}$ *and* μ *by* λ).

We remark that the problem of finding an explicit representation of the solution of a Gauss variational problem in $\mathbb{R}^p$ is a difficult task in general. However, there are certain assumptions on f that could alleviate the difficulty of this problem, as the following result shows in the case of Newtonian potentials.

Proposition II.2.13. *Let $p \geq 3$ and $s = p-2$. Assume that f is a radially symmetric function (i.e. $f(x) = f(|x|)$ for all $x \in \mathbb{R}^p$) satisfying* (35). *Assume further that, as a function of $\mathbb{R}_+$, f has an absolutely continuous derivative and obeys one of the following conditions*:

(i) $r^{p-1}f'(r)$ is increasing on $(0, \infty)$;

(ii) f is convex on $(0, \infty)$.

Let r_0 be the smallest number for which $f'(r) > 0$ for all $r > r_0$, and let R_0 be the smallest solution of $R_0^{p-1}f'(R_0) = p-2$ (it is easy to see that $r_0 < R_0$ and R_0 is finite). If $\lambda_{p-2,f}$ is the solution of the Gauss variational problem on $A = \mathbb{R}^p$ with f as the external field, then

$$\text{supp}(\lambda_{p-2,f}) = \{x \in \mathbb{R}^p : r_0 \leq |x| \leq R_0\},$$

and $\lambda_{p-2,f}$ is given by

$$d\lambda_{p-2,f}(x) = \frac{1}{p-2}(r^{p-1}f'(r))'\,dr\,d\sigma_{p-1}(\overline{x}), \qquad x = r\overline{x}, \quad r = |x|, \tag{62}$$

where $d\sigma_{p-1}$ denotes the normalized surface area measure of the unit sphere S^{p-1} ($\sigma_{p-1}(S^{p-1}) = 1$) in $\mathbb{R}^p$. Moreover,

$$W_{p-2,f}(\lambda_{p-2,f}) = \frac{1}{R_0^{p-2}} + f(R_0), \tag{63}$$

and

$$U_{p-2}^{\lambda_{p-2,f}}(x) = \begin{cases} 1/R_0^{p-2} + f(R_0) - f(r_0), & \textit{if} \quad |x| \leq r_0, \\ \\ 1/R_0^{p-2} + f(R_0) - f(x), & \textit{if} \quad r_0 < |x| < R_0, \\ \\ 1/|x|^{p-2}, & \textit{if} \quad |x| \geq R_0. \end{cases} \tag{64}$$

Remark II.2.14. *The case $p = 2$, $s = 0$ was analyzed by Mhaskar and Saff in [47] (see Example 3.2 of that paper).*

II.2.3 Second-order asymptotics on S^1 for greedy k_s-energy sequences

In this subsection we present a result that is in clear contrast with the previous ones. We have shown (see Theorems II.2.1 and II.1.2) that under certain conditions on A and k, the sequences

$$E(\alpha_{N,k})/N^2, \qquad\qquad E(\omega_N^*)/N^2,$$

have the same asymptotic behavior (ω_N^* denotes here an optimal N-point configuration on A, see Definition I.1.1). This property also holds in the external field case (see Theorems II.2.5 and II.2.4). However, the expression (66) below shows that *in terms of second-order asymptotics, greedy k_s-energy sequences and optimal N-point configurations on S^1 behave differently for $s \in (0,1)$.*

It is known that if $s \in (0,1)$, then the following limit holds (cf. [10]):

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(S^1, N) - I_s(\sigma)\, N^2}{N^{1+s}} = \frac{2\,\zeta(s)}{(2\pi)^s}, \tag{65}$$

where $\mathcal{E}_s(S^1, N)$ denotes (see (17)) the N-point minimal Riesz s-energy of S^1, σ is the normalized arclength measure on S^1, and $\zeta(s)$ is the analytic extension of the classical Riemann zeta function. The expression (65) is called a second-order asymptotic formula because it gives the second term in the asymptotic expansion of $\mathcal{E}_s(S^1, N)$, i.e. (65) can be written as

$$\mathcal{E}_s(S^1, N) = I_s(\sigma)\, N^2 + \frac{2\,\zeta(s)}{(2\pi)^s}\, N^{1+s} + o(N^{1+s}), \qquad N \to \infty.$$

Proposition II.2.15. *Let $s \in (0,1)$ and consider an arbitrary greedy k_s-energy sequence $\{\alpha_{N,s}\}_N$ on S^1. Then the following second-order asymptotics holds*:

$$\lim_{n\to\infty} \frac{E_s(\alpha_{3\cdot 2^n,s}) - I_s(\sigma)(3\cdot 2^n)^2}{(3\cdot 2^n)^{1+s}} = f(s)\frac{2\,\zeta(s)}{(2\pi)^s}, \tag{66}$$

where $f(s) = \frac{1}{2}(\frac{4}{3})^{1+s} + (\frac{1}{3})^{1+s} < 1$ for $s \in (0,1)$, $\zeta(s)$ is the analytic extension of the classical Riemann zeta function, and σ is the normalized arclength measure on S^1.

If $s \in (0,1)$, then $\zeta(s) < 0$, and therefore $f(s)\frac{2\zeta(s)}{(2\pi)^s} > \frac{2\zeta(s)}{(2\pi)^s}$. It is well known that on S^1, the minimal N-point Riesz s-energy $\mathcal{E}_s(S^1, N)$ is attained only by configurations consisting of N equally spaced points, and this property holds for every $s \geq 0$. We will show (see Lemma III.4.2 in Chapter III) that for such s, greedy configurations $\alpha_{2^n,s}$ on S^1 are formed by 2^n equally spaced points. Hence we obtain:

Corollary II.2.16. *For all $s \in (0,1)$ and for any greedy k_s-energy sequence $\{\alpha_{N,s}\}_N$ on S^1, the sequence*

$$\frac{E_s(\alpha_{N,s}) - I_s(\sigma)\, N^2}{N^{1+s}}$$

is not convergent.

II.3 Numerical experiments

In this section we provide some other numerical experiments. We illustrate in Figures 7–10 the first 200 points of four approximate greedy k_s-energy sequences on the unit square $[0,1]^2$ for four different values of s (for better visualization we have deleted the coordinate axes). The initial point is always selected to be the origin. The points in Figures 8–10 were obtained by minimizing over a discretization of $[0,1]^2$ formed by the set

$$\{(i/100, j/100) : 0 \leq i, j \leq 100\},$$

whereas, in the case of Figure 7, the points were obtained using a discretization of the boundary of $[0,1]^2$ consisting of 4000 equally spaced points. We remark that if $s = 0$, it follows from the maximum modulus principle that all greedy energy points will lie on the boundary of the square and thus only the boundary was discretized in this case.

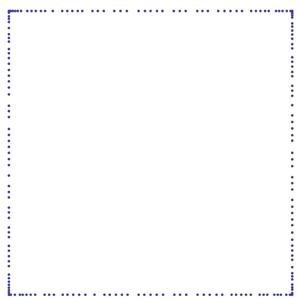

Figure 7: $s = 0$

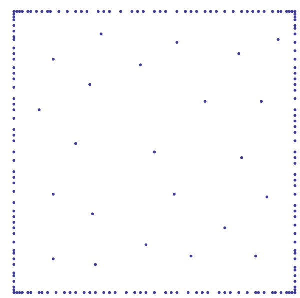

Figure 8: $s = 0.2$

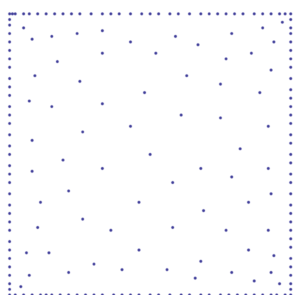

Figure 9: $s = 0.5$

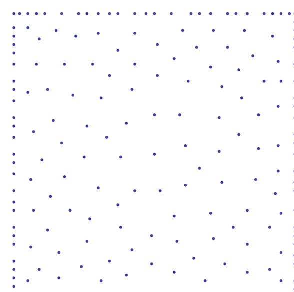

Figure 10: $s = 1$

The following figure shows the first 272 points of the same sequence illustrated in Figure 1. Observe that these points are already very well spread in the surface of the unit sphere.

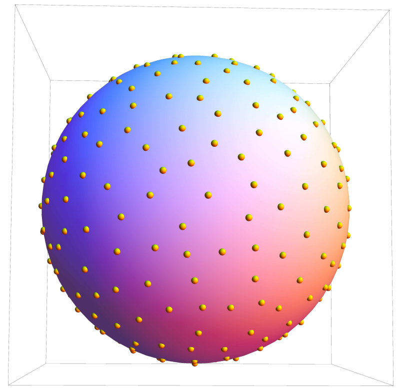

Figure 11: 272 greedy energy points on S^2 for $s = 1$

The configurations shown below in Figures 12–16 are obtained adding the next 20 points to the configurations shown in Figures 23–6. So the total number of points is 50. Recall that the first point is in all cases $a_1 = -1$.

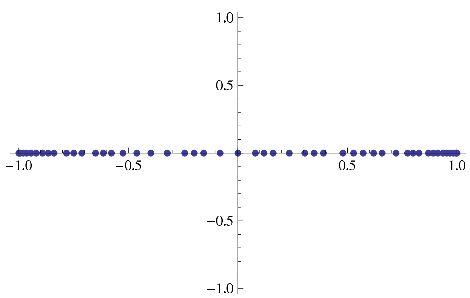

Figure 12: $s = 0$

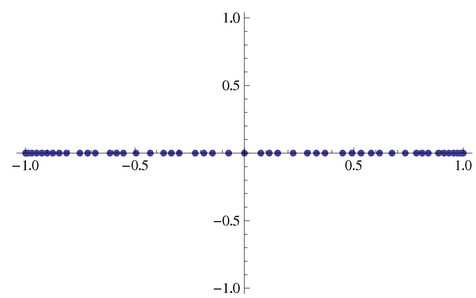

Figure 13: $s = 0.2$

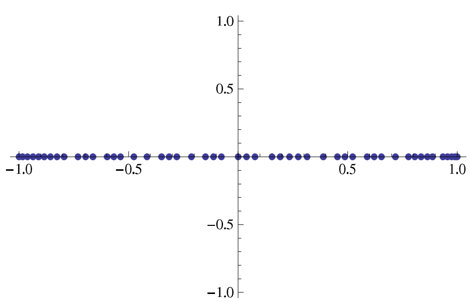

Figure 14: $s = 0.4$

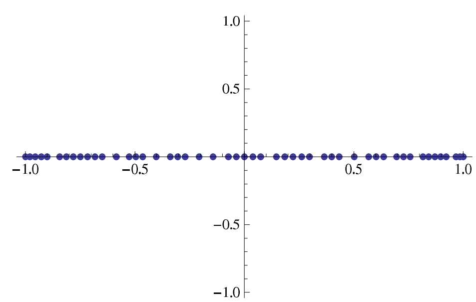

Figure 15: $s = 0.6$

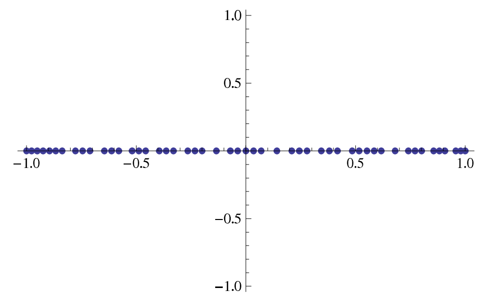

Figure 16: $s = 0.8$

We now present some plots of weighted greedy energy points. The following example shows the first 50 points of a weighted greedy f-energy sequence on $A = [-1, 1]$ (see Definition II.1.7) for the logarithmic kernel k_0 and the external field

$$f(x) = |x|, \qquad x \in [-1, 1]. \tag{67}$$

The first point selected for this sequence was $a_1 = -1$. Observe that the points are much more numerous near the origin, since f takes the lowest value there.

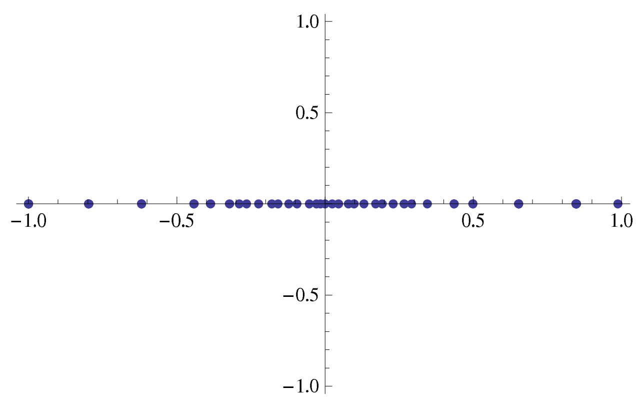

Figure 17: 50 weighted greedy f-energy points for the logarithmic kernel and the external field (67)

The next two examples are also weighted greedy f-energy sequences on $A = [-1, 1]$ for the logarithmic kernel k_0, but now the external field is

$$f(x) = -\log(w(x)), \qquad w(x) = (1-x)^{\lambda_1}(1+x)^{\lambda_2}, \qquad \lambda_1, \lambda_2 > 0. \tag{68}$$

The function w is called the Jacobi weight. It is known (cf. [56, page 241]) that in this case the equilibrium measure is

$$d\mu_{\lambda_1,\lambda_2}(x) = \frac{1}{\pi}\frac{(1+\lambda_1+\lambda_2)}{1-x^2}\sqrt{(x-a)(b-x)}, \qquad a \le x \le b,$$

with support $\operatorname{supp}(\mu_{\lambda_1,\lambda_2}) = [a,b]$, where

$$a = \theta_2^2 - \theta_1^2 - \sqrt{\Delta}, \qquad b = \theta_2^2 - \theta_1^2 + \sqrt{\Delta},$$

and

$$\theta_1 := \frac{\lambda_1}{1+\lambda_1+\lambda_2}, \quad \theta_2 := \frac{\lambda_2}{1+\lambda_1+\lambda_2}, \quad \Delta := [1-(\theta_1+\theta_2)^2][1-(\theta_1-\theta_2)^2].$$

The following example corresponds to the choice $\lambda_1 = 2, \lambda_2 = 1$. The point a_1 is the origin. In this case, $a \approx -0.83$ and $b \approx 0.45$.

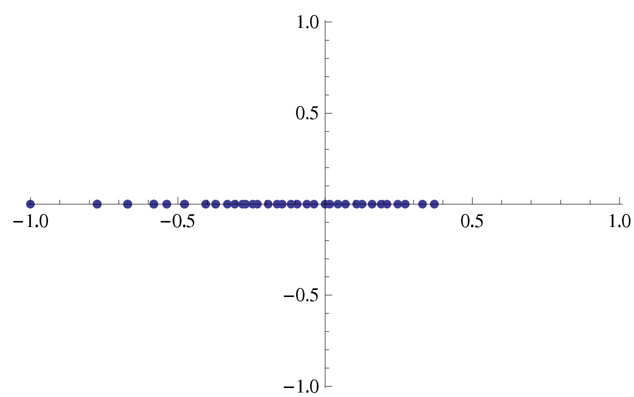

Figure 18: 50 weighted greedy f-energy points for the logarithmic kernel and the external field (68) with parameters $\lambda_1 = 2, \lambda_2 = 1$

In the following example we choose $\lambda_1 = 4, \lambda_2 = 1$, and again $a_1 = 0$. Observe that now all the points were pushed to the interval $[-1,0]$! Another interesting phenomenon can be observed, which is that many points are almost coincident. In this example, $a \approx -0.89$ and $b \approx 0.062$.

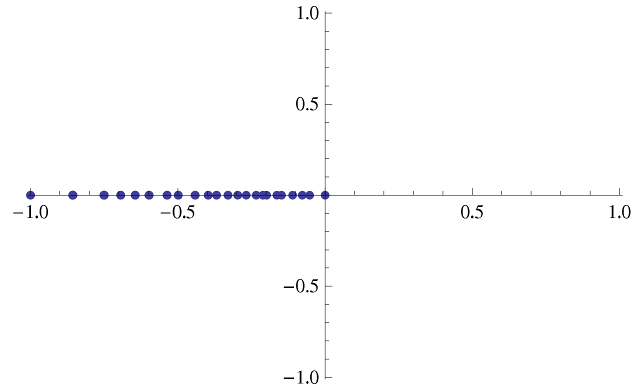

Figure 19: 50 weighted greedy f-energy points for the logarithmic kernel and the external field (68) with parameters $\lambda_1 = 4, \lambda_2 = 1$

In Figure 20 we show the first 200 points of a weighted greedy f-energy sequence on $A = [0,1]^2$. The initial point is the origin, $s = 0.8$ and the external field is $f(x,y) = x^2 + y^2$, $(x,y) \in A$.

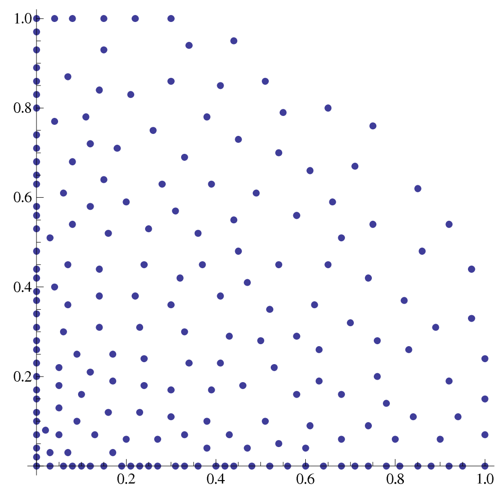

Figure 20: 200 weighted greedy f-energy points on $[0,1]^2$ for $s = 0.8$ and the external field $f(x,y) = x^2 + y^2$

II.4 Proofs

In this section we give the proofs of the results stated in Section II.2. Some auxiliary results are also contained in this section. Theorems II.2.1 and II.2.5 are proved using the same arguments, so we only give the proof of the latter result.

II.4.1 Proofs of results from Subsection II.2.1

Proof of Corollary II.2.2. It is well known (see for example [37]) that for any $s < d$ the equilibrium measure associated with the Riesz kernel k_s is unique and coincides with σ_d. Since $\operatorname{supp}(\sigma_d) = S^d$, any greedy k_s-energy sequence $\{\alpha_{N,s}\}_N \subset S^d$ is a greedy (k_s, σ_d)-energy sequence. Therefore by (38) we obtain that

$$\lim_{N\to\infty} \frac{E_s(\alpha_{N,s})}{N^2} = w_s(S^d) = I_s(\sigma_d).$$

The values on the right-hand side of (41) are the values of $I_s(\sigma_d)$. The case $s > 0$ follows from formula (1.2) of [36] and the case $s = 0$ from formula (2.26) of [11]. Finally (42) follows from (39). □

Proof of Corollary II.2.3. It is shown in [37] that for $s < 1$ the equilibrium measure associated with the Riesz kernel k_s is

$$\frac{c_s}{(1-x^2)^{(1-s)/2}}\, dx\,, \qquad x \in (-1, 1)\,,$$

and its energy is given by the value on the right-hand side of (43). Therefore, the results in Corollary II.2.3 follow from Theorem II.2.1. □

II.4.2 Proofs of results from Subsection II.2.2

Proof of Theorem II.2.4. Our first goal is to show that

$$\limsup_{N\to\infty} \frac{E_f(\omega_N^*)}{N^2} \le V_f. \tag{69}$$

Let $\nu \in \mathcal{M}_f(A)$ be arbitrary, and consider the measure $\lambda := \bigotimes_{j=1}^N \nu$ on the product space X^N. Define the function $h : X^N \to \mathbb{R} \cup \{+\infty\}$ by $h(x_1, \ldots, x_N) := E_f(\{x_1, \ldots, x_N\})$. Therefore, $E_f(\omega_N^*) \le h(x_1, \ldots, x_N)$ for all $(x_1, \ldots, x_N) \in A^N$. Integrating with respect to λ it follows that

$$E_f(\omega_N^*) \le \int_{A^N} h(x_1, \ldots, x_N)\, d\lambda(x_1, \ldots, x_N)$$

$$= \int_{A^N} \sum_{1\le i\ne j\le N} k(x_i, x_j)\, d\lambda(x_1, \ldots, x_N) + 2(N-1) \int_{A^N} \sum_{i=1}^N f(x_i)\, d\lambda(x_1, \ldots, x_N)$$

$$= \sum_{1\le i\ne j\le N} \int_{A^2} k(x_i, x_j)\, d\nu(x_i)\, d\nu(x_j) + 2(N-1) \sum_{i=1}^N \int_A f(x_i)\, d\nu(x_i)$$

$$= N(N-1)\Big(\int_{A^2} k(x,y)\, d\nu(x)\, d\nu(y) + 2\int_A f(x)\, d\nu(x) \Big) = N(N-1) I_f(\nu).$$

Taking the infimum over $\nu \in \mathcal{M}_f(A)$ we obtain that $E_f(\omega_N^*) \le N(N-1)V_f$, and therefore (69) holds.

Next we show that

$$V_f \le \liminf_{N\to\infty} \frac{E_f(\omega_N^*)}{N^2} \tag{70}$$

and at the same time we verify (46). Let $\omega_N^* = \{x_1, \ldots, x_N\}$ and define

$$\nu_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}.$$

Assume that $g_n : A \times A \to \mathbb{R}$ is a sequence of non-decreasing continuous functions that converges pointwise to k on A. We fix n. Then

$$\int\int g_n(x,y)\, d\nu_N(x)\, d\nu_N(y) + 2\int f\, d\nu_N \tag{71}$$

$$= \frac{1}{N^2}\Big(\sum_{i=1}^{N} g_n(x_i,x_i) + \sum_{1\le i\ne j\le N} g_n(x_i,x_j) + 2N\sum_{i=1}^{N} f(x_i)\Big)$$

$$\le \frac{1}{N^2}\Big(\sum_{i=1}^{N}(g_n(x_i,x_i)+2f(x_i)) + \sum_{1\le i\ne j\le N} k(x_i,x_j) + 2(N-1)\sum_{i=1}^{N} f(x_i)\Big)$$

$$= \frac{1}{N^2}\Big(\sum_{i=1}^{N}(g_n(x_i,x_i)+2f(x_i)) + E_f(\omega_N^*)\Big).$$

Let $C := \inf\{k(x,y) : (x,y) \in A^2\}$ and $D := \inf\{f(x) : x \in A\}$. Both C and D are finite since A is compact and k and f are lower semicontinuous. Using $E_f(\omega_N^*) \le N(N-1)V_f$ we obtain

$$ND \le \sum_{i=1}^{N} f(x_i) \le \frac{N}{2}(V_f - C). \tag{72}$$

By the compactness of A and the continuity of g_n, there exists a constant $M_n > 0$ such that

$$\sum_{i=1}^{N} |g_n(x_i,x_i)| \le N\, M_n.$$

In particular,

$$\frac{\sum_{i=1}^{N} g_n(x_i,x_i)}{N^2} \longrightarrow 0, \qquad N \longrightarrow \infty. \tag{73}$$

From (72) and (73) we conclude that

$$\frac{\sum_{i=1}^{N}(g_n(x_i,x_i)+2f(x_i))}{N^2} \longrightarrow 0, \qquad N \longrightarrow \infty. \tag{74}$$

Let $\nu \in \mathcal{M}_1(A)$ be a cluster point of the sequence $\{\nu_N\}$ in the weak-star topology. Then there exists a subsequence $\{\nu_N\}_{N\in\mathcal{N}}$ that converges weak-star to ν (cf. [28, Lemma 1.2.1]). Therefore

$$\int\int g_n(x,y)\,d\nu(x)\,d\nu(y) + 2\int f(x)\,d\nu(x) \tag{75}$$

$$\leq \liminf_{N\in\mathcal{N}} \Big(\int\int g_n(x,y)\,d\nu_N(x)\,d\nu_N(y) + 2\int f(x)\,d\nu_N(x) \Big).$$

Now we apply (75), (71), (74) and (69) to obtain

$$\int\int g_n(x,y)\,d\nu(x)\,d\nu(y) + 2\int f(x)\,d\nu(x) \leq V_f.$$

From the monotone convergence theorem we conclude that

$$I_f(\nu) = \lim_{n\to\infty} \int\int g_n(x,y)\,d\nu(x)\,d\nu(y) + 2\int f(x)\,d\nu(x) \leq V_f.$$

Therefore $\nu = \mu$, the equilibrium measure. Since μ is the only cluster point of $\{\nu_N\}$, (46) follows.

Using (71) we have

$$\int\int g_n(x,y)\,d\mu(x)\,d\mu(y) + 2\int f(x)\,d\mu(x)$$

$$\leq \liminf_{N\to\infty} \frac{1}{N^2}\Big(\sum_{i=1}^{N} (g_n(x_i,x_i) + 2f(x_i)) + E_f(\omega_N^*) \Big) = \liminf_{N\to\infty} \frac{1}{N^2} E_f(\omega_N^*),$$

from which (70) follows. Finally, (45) is a consequence of (70) and (69). □

Lemma II.4.1. *Let $k : X \times X \to \mathbb{R}\cup\{+\infty\}$ be a symmetric kernel on a locally compact metric space X, $A \subset X$ be a compact set, and $f : X \to \mathbb{R}\cup\{+\infty\}$ be an external field. Assume that the Gauss variational problem is solvable and $\mu \in \mathcal{M}_f(A)$ is a solution. Let $\{\tau_n\} \subset \mathcal{M}_1(S^*_{f,\mu})$ be a sequence of measures that converges to μ in* *the weak-star topology. Then*

$$\lim_{n\to\infty}\int f\,d\tau_n = \int f\,d\mu. \tag{76}$$

Proof. Since f and U^μ are lower semicontinuous we have

$$\int f\,d\mu \le \liminf_{n\to\infty}\int f\,d\tau_n,$$

$$\limsup_{n\to\infty}\int (W_f(\mu) - U^\mu)\,d\tau_n \le \int (W_f(\mu) - U^\mu)\,d\mu.$$

In addition, for $x \in S^*_{f,\mu}$ the inequality $f(x) \le W_f(\mu) - U^\mu(x)$ holds, and therefore

$$\limsup_{n\to\infty}\int f\,d\tau_n \le \limsup_{n\to\infty}\int (W_f(\mu) - U^\mu)\,d\tau_n.$$

By (29) and (30), $f = W_f(\mu) - U^\mu$ q.e. on S_μ, and since μ has finite energy this equality holds μ-a.e. Thus

$$\int f\,d\mu = \int (W_f(\mu) - U^\mu)\,d\mu,$$

and (76) follows. □

Proof of Theorem II.2.5. To prove this result we follow closely ideas from Chapter V of [56]. By definition,

$$U_n^f(a_n) \le U_n^f(x) \qquad \text{for all} \quad x \in S^*_{f,\mu}, \quad n \ge 2.$$

We have, for any $x \in S^*_{f,\mu}$,

$$E_f(\alpha^f_{N,\mu}) = 2\sum_{1\le i<j\le N} k(a_i, a_j) + 2(N-1)\sum_{i=1}^{N} f(a_i)$$

$$= 2\sum_{j=2}^{N}\Big(\sum_{i=1}^{j-1} k(a_i,a_j) + (j-1)f(a_j) + \sum_{i=1}^{j-1} f(a_i)\Big)$$

$$= 2\sum_{j=2}^{N}\Big(U_j^f(a_j) + \sum_{i=1}^{j-1} f(a_i)\Big) \le 2\sum_{j=2}^{N}\Big(U_j^f(x) + \sum_{i=1}^{j-1} f(a_i)\Big)$$

$$= 2\sum_{j=2}^{N}\sum_{i=1}^{j-1}\Big(k(x,a_i) + f(x) + f(a_i)\Big).$$

We now integrate with respect to μ to obtain

$$E_f(\alpha_{N,\mu}^f) \le 2\sum_{j=2}^{N}\sum_{i=1}^{j-1}\Big(U^\mu(a_i) + \int f\,d\mu + f(a_i)\Big).$$

Taking into account that $U^\mu(a_i) + f(a_i) \le W_f(\mu)$ for all i and $W_f(\mu) + \int f\,d\mu = V_f$, it follows that

$$E_f(\alpha_{N,\mu}^f) \le N(N-1)V_f. \tag{77}$$

Now, if $\{\omega_N^*\}$ is a sequence of optimal weighted N-point configurations on $S_{f,\mu}^*$, then $E_f(\omega_N^*) \le E_f(\alpha_{N,\mu}^f)$ for all N. Therefore (47) is a consequence of (77) and (45).

Throughout the rest of the proof we assume that the equilibrium measure $\mu \in \mathcal{M}_f(A)$ is unique. Consider the sequence of normalized counting measures

$$\nu_N := \frac{1}{N}\sum_{a\in\alpha_{N,\mu}^f} \delta_a.$$

As in the proof of Theorem II.2.4, we select a sequence $g_n : S_{f,\mu}^* \times S_{f,\mu}^* \to \mathbb{R}$ of non-decreasing continuous functions that converges pointwise to k on $S_{f,\mu}^*$. We have, as in (71),

$$\int\int g_n(x,y)\,d\nu_N(x)\,d\nu_N(y) + 2\,\frac{N-1}{N}\int f\,d\nu_N \le \frac{\sum_{i=1}^N g_n(a_i,a_i) + E_f(\alpha_{N,\mu}^f)}{N^2}.$$

Let $\{\nu_N\}_{N\in\mathcal{N}}$ be a subsequence that converges in the weak-star topology to a measure $\lambda \in \mathcal{M}_1(A)$. By the lower-semicontinuity of f,

$$\int f \, d\lambda \le \liminf_{N\in\mathcal{N}} \int f \, d\nu_N.$$

Thus from (73) and (47) we conclude that

$$\int\int g_n(x,y)\, d\lambda(x)\, d\lambda(y) + 2\int f\, d\lambda \le V_f.$$

Now we let $n \to \infty$ and obtain

$$I_f(\lambda) = \int\int k(x,y)\, d\lambda(x)\, d\lambda(y) + 2\int f\, d\lambda \le V_f.$$

It follows that $\lambda \in \mathcal{M}_f(A)$ and λ is an equilibrium measure. By hypothesis there is only one equilibrium measure, thus $\lambda = \mu$ and (48) is proved.

We next show (49). First,

$$\sum_{i=2}^{N} U_i^f(a_i) = \frac{1}{2}\, E_f(\alpha^f_{N,\mu}) - \sum_{i=1}^{N}(N-i) f(a_i). \tag{78}$$

By (48) and Lemma II.4.1,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(a_i) = \int f\, d\mu. \tag{79}$$

This implies that

$$\lim_{N\to\infty} \frac{2}{(N-1)N}\sum_{i=1}^{N}(N-i) f(a_i) = \int f\, d\mu. \tag{80}$$

Applying (47), (78), and (80), we obtain

$$\lim_{N\to\infty} \frac{2}{(N-1)N} \sum_{i=2}^{N} U_i^f(a_i) = V_f - \int f\, d\mu = W_f(\mu). \tag{81}$$

For every $n \geq 1$,

$$\frac{U_{n+1}^f(a_{n+1})}{n} = \inf_{x\in S_{f,\mu}^*} \left\{ \frac{1}{n} \sum_{i=1}^{n} k(x,a_i) + f(x) \right\}.$$

Integrating this expression with respect to μ it follows that

$$\frac{U_{n+1}^f(a_{n+1})}{n} \leq \frac{1}{n} \sum_{i=1}^{n} U^\mu(a_i) + \int f\, d\mu \leq W_f(\mu) + \int f\, d\mu - \frac{1}{n} \sum_{i=1}^{n} f(a_i). \tag{82}$$

Let

$$\rho_n := \int f\, d\mu - \frac{1}{n} \sum_{i=1}^{n} f(a_i), \qquad n \geq 1.$$

On the other hand, for every $n \geq 2$,

$$U_{n+1}^f(a_{n+1}) \geq U_n^f(a_n) + L, \tag{83}$$

where $L := \inf\{k(x,a) + f(x) : a, x \in S_{f,\mu}^*\}$. We may assume that $L \leq -1$.

Let $\epsilon \in (0,1)$. Assume that m is an integer such that

$$\frac{U_{m+1}^f(a_{m+1})}{m} < W_f(\mu) - \epsilon. \tag{84}$$

Applying (83) repeatedly we obtain for $(1+\epsilon/(3L))m \leq i \leq m$,

$$\frac{U_{i+1}^f(a_{i+1})}{m} \leq W_f(\mu) - \epsilon - \frac{(m-i)L}{m} \leq W_f(\mu) - \epsilon + \frac{\epsilon/3}{1+\epsilon/(3L)} \leq W_f(\mu) - \frac{\epsilon}{2},$$

and so

$$\frac{U_{i+1}^f(a_{i+1})}{i} \leq \frac{m}{i}(W_f(\mu) - \epsilon/2) \leq \frac{m}{i} W_f(\mu) - \frac{\epsilon}{2}.$$

Taking into account (82) and the previous inequality,

$$\frac{2}{(m+1)\,m}\sum_{i=1}^{m}U^{f}_{i+1}(a_{i+1}) \le \frac{2}{(m+1)\,m}\sum_{1\le i<(1+\epsilon/(3L))m} i(W_f(\mu)+\rho_i) \tag{85}$$

$$+\frac{2}{(m+1)\,m}\sum_{(1+\epsilon/(3L))m\le i\le m} m\,W_f(\mu) - \frac{\epsilon}{2}\frac{2}{(m+1)\,m}\sum_{(1+\epsilon/(3L))m\le i\le m} i.$$

Furthermore, it is easy to see that

$$-\frac{\epsilon}{2}\frac{2}{(m+1)\,m}\sum_{(1+\epsilon/(3L))m\le i\le m} i \le \frac{\epsilon^2}{6L(m+1)}\Big(1+2\,m+\frac{m\epsilon}{3L}\Big) \tag{86}$$

$$\le \frac{\epsilon^2(1+\epsilon/(3L))}{6L}.$$

By (79) we know that $\rho_n \longrightarrow 0$ as $n \longrightarrow \infty$, which implies that

$$\lim_{N\to\infty}\frac{2}{(N+1)\,N}\sum_{1\le i<(1+\epsilon/(3L))N} i\,\rho_i = 0. \tag{87}$$

If $W_f(\mu)\le 0$, then

$$\frac{2}{(m+1)\,m}\Big\{\sum_{1\le i<(1+\epsilon/(3L))m} i\,W_f(\mu) + \sum_{(1+\epsilon/(3L))m\le i\le m} m\,W_f(\mu)\Big\} \le W_f(\mu),$$

and hence it follows from (85) and (86) that

$$\frac{2}{m(m+1)}\sum_{i=1}^{m}U^{f}_{i+1}(a_{i+1}) \tag{88}$$

$$\le W_f(\mu) + \frac{\epsilon^2(1+3\epsilon/(3L))}{6L} + \frac{2}{(m+1)\,m}\sum_{1\le i<(1+\epsilon/(3L))m} i\,\rho_i.$$

Since the second term of the right-hand side of (88) is a negative constant, applying (87), (81), and (88), it follows that there are finitely many integers m satisfying (84). This together with (82) implies (49).

Now we assume that $W_f(\mu) > 0$. It is easy to verify that

$$\frac{2}{(m+1)m}\Big\{\sum_{1\le i<(1+\epsilon/(3L))m} i\,W_f(\mu) + \sum_{(1+\epsilon/(3L))m\le i\le m} m\,W_f(\mu)\Big\}$$

$$\le \Big(1 + \frac{2}{m+1} + \frac{\epsilon}{3L(m+1)} + \frac{\epsilon^2 m}{9(m+1)L^2}\Big) W_f(\mu),$$

and so, from (85) and (86), we deduce that

$$\frac{2}{(m+1)\,m}\sum_{i=1}^{m} U_{i+1}^f(a_{i+1}) \le \Big(1 + \frac{2}{m+1} + \frac{\epsilon}{3L(m+1)} + \frac{\epsilon^2 m}{9(m+1)L^2}\Big) W_f(\mu)$$

$$+\frac{\epsilon^2(1+\epsilon/(3L))}{6L} + \frac{2}{(m+1)\,m}\sum_{1\le i<(1+\epsilon/(3L))m} i\,\rho_i.$$

If we assume that there is an infinite sequence $\mathcal{N}$ of integers m satisfying (84), applying the last inequality and (87), we obtain

$$\limsup_{m\in\mathcal{N}} \frac{2}{(m+1)\,m}\sum_{i=1}^{m} U_{i+1}^f(a_{i+1}) \le W_f(\mu) + \frac{\epsilon^2\,W_f(\mu)}{9\,L^2} + \frac{\epsilon^2(1+\epsilon/(3L))}{6L}. \qquad (89)$$

We may assume without loss of generality that $L < -(1 + 2\,W_f(\mu))/3$. Then the right-hand side of (89) is a constant strictly less than $W_f(\mu)$, which contradicts (81). This concludes the proof of (49). □

Proof of Proposition II.2.6. We know that

$$E_f(\{b_1,\ldots,b_N\}) = 2\sum_{i=2}^{N} T_i^f(b_i) + 2\sum_{i=1}^{N}(N-i)f(b_i). \qquad (90)$$

Since k is real-valued and $\{b_n\}_{n=1}^{\infty} \subset S_{f,\mu}^*$, we have that $T_n^f(b_n) < +\infty$ for all n. From (51) it follows that

$$\lim_{N\to\infty} \frac{2}{N(N-1)}\sum_{i=2}^{N} T_i^f(b_i) = V_f - \int f\,d\mu, \qquad (91)$$

and applying (50) and Lemma II.4.1, we obtain

$$\lim_{N\to\infty} \frac{2}{N(N-1)} \sum_{i=1}^{N} (N-i)\, f(b_i) = \int f\, d\mu. \tag{92}$$

Therefore (52) is a consequence of (90)–(92). □

The proof of Theorem II.2.7 is similar to that of Theorem II.2.5, and consequently we only sketch it. The details are left to the reader.

Sketch of the proof of Theorem II.2.7. In order to prove (53), we use the fact that

$$E_f(\alpha^{(f,m)}_{mN,\mu}) = 2 \sum_{j=1}^{N-1} \left[U^{(f,m)}_{jm}(a_{jm+1}, \ldots, a_{(j+1)m}) + m \sum_{r=1}^{j-1} \sum_{l=1}^{m} f(a_{rm+l}) \right] + \phi_{m,N},$$

where

$$\phi_{m,N} = E(\{a_1, \ldots, a_m\}) + 2(mN-1) \sum_{i=1}^{m} f(a_i).$$

Using the definition of $\{a_{jm+1}, \ldots, a_{(j+1)m}\}$ and integrating the resulting inequality by $d\mu(x_{m+1}) \times \cdots \times d\mu(x_{mN})$ it follows that

$$E_f(\alpha^{(f,m)}_{mN,\mu}) \leq m^2 (N-1)(N-2) W_f(\mu) + m^2 (N+1) N \int f\, d\mu + o(N^2).$$

This inequality and (45) imply (53). The asymptotic expression (54) is an application of (53).

To prove (55) we use the inequalities

$$\frac{U^{(f,m)}_{mN}(a_{mN+1}, \ldots, a_{m(N+1)})}{mN} \leq m\, W_f(\mu) + \rho_{m,N},$$

$$U^{(f,m)}_{m(N+1)}(a_{m(N+1)+1}, \ldots, a_{m(N+2)}) \geq U^{(f,m)}_{mN}(a_{mN+1}, \ldots, a_{m(N+1)}) + m^2 L,$$

where

$$\rho_{m,N} = m\Big(\int f\,d\mu - \frac{1}{mN}\sum_{i=1}^{mN} f(a_i)\Big) + \frac{(m-1)}{2N} I(\mu) + \frac{(m-1)}{N}\int f\,d\mu$$

and $L = \inf\{k(x,a) + f(x) : a, x \in S^*_{f,\mu}\}$. The rest of the arguments in the proof of (55) are analogous to those used to justify (49). □

Lemma II.4.2. *Let $p \geq 2$ and $p-2 \leq s < p$. Assume that $A \subset \mathbb{R}^p$ is closed and f satisfies the conditions* (56) *and* (35) *(or* (34) *in the case $p = 2$, $s = 0$). Let $\lambda = \lambda_{s,f}$ be the equilibrium measure solving the Gauss variational problem on A in the presence of f. Then*

(i) for any measure $\nu \in \mathcal{M}_1(\mathbb{R}^p)$,

$$\text{“}\inf_{x\in S_\lambda}\text{”}\,(U_s^\nu(x) + f(x)) \leq W_{s,f}(\lambda), \tag{93}$$

where S_λ denotes the support of λ, and “inf” means that the infimum is taken quasi-everywhere.

(ii) If $\nu \in \mathcal{M}_1(A)$, then

$$\text{“}\sup_{x\in S_\nu}\text{”}\,(U_s^\nu(x) + f(x)) \geq W_{s,f}(\lambda), \tag{94}$$

where S_ν denotes the support of ν, and “sup” means that the supremum is taken quasi-everywhere.

(iii) Suppose that $\nu \in \mathcal{M}_1(A)$ has finite s-energy and there exists a constant M such that $U_s^\nu(x) + f(x) = M$ for q.e. $x \in S_\nu$ and $U_s^\nu(x) + f(x) \geq M$ for all $x \in A$. Then $\nu = \lambda$ and $M = W_{s,f}(\lambda)$.

Proof. The case $p = 2$, $s = 0$ of this result is part of Theorems I.3.1 and I.3.3 in [56]. We first justify (93). To the contrary, suppose that there exists a measure

$\nu \in \mathcal{M}_1(\mathbb{R}^p)$ and a constant $C > W_{s,f}(\lambda)$ such that

$$U_s^\nu(x) + f(x) \geq C \qquad \text{for q.e.} \quad x \in S_\lambda.$$

From (57) we obtain that

$$U_s^\lambda(x) + C - W_{s,f}(\lambda) \leq U_s^\nu(x) \qquad \text{for q.e.} \quad x \in S_\lambda. \tag{95}$$

Since $I_s(\lambda)$ is finite, S_λ is a compact set with positive s-capacity. Therefore, there exists a unique measure $\mu_\lambda \in \mathcal{M}_1(S_\lambda)$ such that $I_s(\mu_\lambda) = w_s(S_\lambda) > 0$. Since $U_s^{\mu_\lambda} \leq w_s(S_\lambda)$ on $\operatorname{supp}(\mu_\lambda)$, applying the first maximum principle ([37, Theorem 1.10]) it follows that $U_s^{\mu_\lambda} \leq w_s(S_\lambda)$ everywhere in $\mathbb{R}^p$. Using (58) we conclude that $U_s^{\mu_\lambda} = w_s(S_\lambda)$ q.e. on S_λ.

If we define now the measure $\eta := (C - W_{s,f}(\lambda))\, w_s(S_\lambda)^{-1}\mu_\lambda$, (95) yields

$$U_s^{\lambda+\eta}(x) \leq U_s^\nu(x) \qquad \text{for q.e.} \quad x \in S_\lambda. \tag{96}$$

Since λ and η have finite energy, this inequality holds $(\lambda + \eta)$-almost everywhere. Applying Theorem 1.27 (case $s = p - 2$) and Theorem 1.29 (case $p - 2 < s < p$) from [37] we obtain that the inequality (96) holds everywhere in $\mathbb{R}^p$. Finally, multiplying both sides by $|x|^s$ and letting $|x| \to \infty$ it follows that $C - W_{s,f}(\lambda) \leq 0$, which contradicts our initial assumption.

Now we prove (94). Let $L :=$ "$\sup$"$_{x\in S_\nu}(U_s^\nu(x) + f(x))$ and assume that L is finite. It follows from this assumption that ν has finite s-energy. Using (58) we have

$$U_s^\nu(x) + W_{s,f}(\lambda) - L \leq U_s^\lambda(x) \qquad \text{for q.e.} \quad x \in S_\nu. \tag{97}$$

The same argument employed above to prove part (i) shows that $W_{s,f}(\lambda) - L \leq 0$.

Finally, the assumptions of (iii) imply that

$$\text{“}\inf_{x\in A}\text{”}\,(U_s^\nu(x)+f(x)) = M = \text{“}\sup_{x\in S_\nu}\text{”}\,(U_s^\nu(x)+f(x)),$$

and consequently we obtain using (93) and (94) that $M = W_{s,f}(\lambda)$. Taking $C = M$ in (95) and $L = M$ in (97) we conclude that $U_s^\nu = U_s^\lambda$ everywhere in $\mathbb{R}^p$, which implies that $\lambda = \nu$ by Theorem 1.15 from [37]. □

Proof of Lemma II.2.9. From (59) and Lemma II.4.2(i) applied to the measure $\nu := (1/n)\sum_{i=1}^n \delta_{x_i}$ we obtain that $W_{s,f}(\lambda) \geq M/n$. Using (57) and (59) we have

$$U_s^\nu(x) + W_{s,f}(\lambda) - \frac{M}{n} \geq U_s^\lambda(x) \qquad \text{for q.e.} \quad x \in \operatorname{supp}(\lambda). \tag{98}$$

The same argument employed to prove Lemma II.4.2(i) shows that the inequality (98) is valid everywhere in $\mathbb{R}^p$, which is precisely (60). Finally, (61) is a consequence of (60) and (58). □

Proof of Corollary II.2.11. The fact that a_n is well-defined for all $n \geq 1$ follows from conditions (56) and (35) (or (34) in the case $p = 2$, $s = 0$). Applying Lemma $II.2.9$ to $\{x_1, \ldots, x_n\} := \{a_1, \ldots, a_n\}$ and

$$M := \sum_{i=1}^{n} \frac{1}{|a_{n+1} - a_i|^s} + n f(a_{n+1}),$$

it follows that $a_n \in S_{f,\lambda}^*$ for all $n \geq 2$. The case $p = 2$, $s = 0$ is justified in the same way. It is clear from the proof of Theorem $II.2.5$ that (47)–(49) are valid for the weighted greedy f-energy sequence $(a_n)_{n=1}^\infty$. □

Proof of Corollary II.2.12. For every $N \geq 0$, the existence of the minimizing configuration $\{a_{mN+1}, \ldots, a_{m(N+1)}\}$ is guaranteed by the conditions (56) and (35) (or (34) in the case $p = 2$, $s = 0$).

Next, we show that $\omega_N := \{a_{mN+1}, \ldots, a_{m(N+1)}\} \subset S^*_{f,\lambda}$ for every $N \geq 0$. It follows from the definition of ω_N that for each $i \in \{1, \ldots, m\}$, the inequality

$$\sum_{l=1}^{mN} \frac{1}{|a_{mN+i} - a_l|^s} + \sum_{j=1, j\neq i}^{m} \frac{1}{|a_{mN+i} - a_{mN+j}|^s} + ((N+1)m - 1)f(a_{mN+i}) \tag{99}$$

$$\leq \sum_{l=1}^{mN} \frac{1}{|x - a_l|^s} + \sum_{j=1, j\neq i}^{m} \frac{1}{|x - a_{mN+j}|^s} + ((N+1)m - 1)f(x)$$

holds for all $x \in A$. (If $N = 0$ then the first term on both sides of the inequality doesn't appear in the expression.) If we denote the left hand side of (99) by M, and apply Lemma $II.2.9$ to $\{x_1, \ldots, x_{(N+1)m-1}\} = \{a_l\}_{l=1}^{mN} \cup \{a_{mN+j}\}_{j=1, j\neq i}^{m}$, then (60) implies that $a_{mN+i} \in S^*_{f,\lambda}$.

It is clear that the sequence $(a_n)_{n\geq 1}$ is a weighted greedy (m, f, λ)-energy sequence and, therefore, all the assertions of Theorem II.2.7 are applicable to $(a_n)_{n\geq 1}$. □

Proof of Proposition II.2.13. It is easy to see that

$$\int_{S^{p-1}} \frac{1}{|r\overline{y} - x|^{p-2}} d\sigma_{p-1}(\overline{y}) = \begin{cases} 1/r^{p-2}, & \text{if } |x| \leq r, \\ \\ 1/|x|^{p-2}, & \text{if } |x| > r. \end{cases}$$

Let ν be the measure supported on $\{x \in \mathbb{R}^p : r_0 \leq |x| \leq R_0\}$ whose expression is given by the right-hand side of (62). From the definition of r_0 and R_0 it follows that ν is a probability measure and by simple computations we obtain that the potential U^ν_{p-2} coincides with the function on the right-hand side of (64). Therefore

$$U^\nu_{p-2}(x) + f(x) = \frac{1}{R_0^{p-2}} + f(R_0), \qquad r_0 \leq |x| \leq R_0. \tag{100}$$

Applying the definitions of r_0 and R_0 again, we get that $f(|x|) \geq f(r_0)$ if $|x| \leq r_0$ and $f(|x|) + 1/|x|^{p-2} \geq f(R_0) + 1/R_0^{p-2}$ if $|x| \geq R_0$ (regarding f as a function of $\mathbb{R}_+$). As a consequence

$$U_{p-2}^{\nu}(x) + f(x) \geq \frac{1}{R_0^{p-2}} + f(R_0) \tag{101}$$

for all $x \in \mathbb{R}^p$. Therefore, it follows from (100), (101), and Lemma II.4.2, that $\nu = \lambda_{p-2,f}$ and (63) holds. □

II.4.3 Proofs of results from Subsection II.2.3

Proof of Proposition II.2.15. We have

$$\frac{E_s(\alpha_{3\cdot 2^n,s}) - I_s(\sigma)(3\cdot 2^n)^2}{(3\cdot 2^n)^{1+s}} = \frac{1}{3^{1+s}} \frac{E_s(\alpha_{3\cdot 2^n,s}) - I_s(\sigma)(2^n)^2 - I_s(\sigma)2^{2n+3}}{(2^n)^{1+s}}\,. \tag{102}$$

As will be justified in Section III.4 (see Lemma III.4.3), the relation

$$E_s(\alpha_{3\cdot 2^n,s}) = \frac{1}{2}\,\mathcal{E}_s(S^1, 2^{n+2}) + \mathcal{E}_s(S^1, 2^n)$$

holds. Therefore, from (102), it follows that

$$\frac{E_s(\alpha_{3\cdot 2^n,s}) - I_s(\sigma)(3\cdot 2^n)^2}{(3\cdot 2^n)^{1+s}}$$

$$= \frac{1}{3^{1+s}}\left(\frac{\mathcal{E}_s(S^1, 2^n) - I_s(\sigma)(2^n)^2}{(2^n)^{1+s}} + \frac{4^{1+s}}{2}\frac{\mathcal{E}_s(S^1, 2^{n+2}) - I_s(\sigma)(2^{n+2})^2}{(2^{n+2})^{1+s}}\right).$$

Applying now (65) we get

$$\lim_{n\to\infty} \frac{E_s(\alpha_{3\cdot 2^n,s}) - I_s(\sigma)(3\cdot 2^n)^2}{(3\cdot 2^n)^{1+s}} = \left(\frac{1}{2}\left(\frac{4}{3}\right)^{1+s} + \left(\frac{1}{3}\right)^{1+s}\right)\frac{2\zeta(s)}{(2\pi)^s}\,.$$

Finally, it is easy to check that $f(s) = \frac{1}{2}(\frac{4}{3})^{1+s} + (\frac{1}{3})^{1+s} < 1$ for all $s \in (0,1)$. □

Proof of Corollary II.2.16. Since $\alpha_{2^n,s}$ consists of 2^n equally spaced points (see Lemma III.4.2 below), $E_s(\alpha_{2^n,s}) = \mathcal{E}_s(S^1, 2^n)$, and therefore

$$\lim_{n\to\infty} \frac{E_s(\alpha_{2^n,s}) - I_s(\sigma)2^{2n}}{2^{n(1+s)}} = \frac{2\zeta(s)}{(2\pi)^s},$$

but the subsequence $\{\alpha_{3\cdot 2^n,s}\}_n$ provides a different limit value, given by (66). □

CHAPTER III

GREEDY ENERGY POINTS: THE HYPER-SINGULAR CASE

III.1 Introduction, background results and notation

In Chapter II we investigated the asymptotic behavior of greedy energy sequences under conditions that allowed us to use potential theoretic methods. In the present chapter, we analyze greedy k_s-energy sequences on compact sets $A \subset \mathbb{R}^p$, assuming that $s \geq \dim_H(A)$. Therefore in this situation there exists no probability measure supported on A with finite Riesz s-energy, and other methods must be employed.

In this chapter we will also investigate greedy "best-packing" sequences, whose points are chosen to maximize the minimum distance to previously selected points. The definition of these sequences is introduced in Subsection III.2.2. In order to motivate our results, we will present in this section some background material.

In this chapter, A will denote a compact set in $\mathbb{R}^p$, and d will denote its Hausdorff dimension. For $s < d$, Theorem II.1.2 asserts that

$$\lim_{N\to\infty} \frac{E_s(\omega_{N,s}^*)}{N^2} = I_s(\lambda_{A,s}), \tag{103}$$

where $\{\omega_{N,s}^*\}$ denotes any sequence of optimal N-point configurations on A with respect to the Riesz s-kernel, and $\lambda_{A,s}$ denotes the corresponding equilibrium measure on A (see the paragraph preceding the statement of Theorem II.1.2). In addition (see [37] or Theorem II.2.4),

$$\frac{1}{N}\sum_{x\in\omega_{N,s}^*} \delta_x \xrightarrow{\ *\ } \lambda_{A,s}, \qquad N\to\infty,$$

where δ_x is the Dirac unit measure concentrated at x. If $s \geq d$, then Theorem II.1.2 tells us that

$$\lim_{N\to\infty} \frac{E_s(\omega^*_{N,s})}{N^2} = +\infty,$$

so the order of growth of $E_s(\omega^*_{N,s})$ is greater than N^2.

Throughout the rest of this chapter we denote by $\mathrm{Vol}(B^d)$ the volume of the unit ball B^d in $\mathbb{R}^d$, and $\mathcal{H}_d$ represents d-dimensional Hausdorff measure in $\mathbb{R}^p$ (normalized by the condition $\mathcal{H}_d([0,1]^d) = 1$, where $[0,1]^d$ denotes here the embedding of the d-dimensional unit cube in $\mathbb{R}^p$). Regarding the case $s \geq d$, in [32, 8] geometric measure theoretic tools were employed to obtain the following result:

Theorem III.1.1. *Let A be a compact subset of a d-dimensional C^1-manifold in $\mathbb{R}^p$. If $\{\omega^*_{N,d}\}$ is any sequence of optimal N-point configurations on A for $s = d$, then*

$$\lim_{N\to\infty} \frac{E_d(\omega^*_{N,d})}{N^2 \log N} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d(A)}. \tag{104}$$

Furthermore, if $\mathcal{H}_d(A) > 0$, any sequence $\{\widetilde{\omega}_N\}$ of configurations on A whose energies satisfy (104) *is uniformly distributed with respect to $\mathcal{H}_d$ in the sense that*

$$\frac{1}{N} \sum_{x\in\widetilde{\omega}_N} \delta_x \xrightarrow{\ *\ } \frac{\mathcal{H}_d|_A}{\mathcal{H}_d(A)}, \qquad N \to \infty. \tag{105}$$

Assume now that $A \subset \mathbb{R}^p$ is a d-rectifiable compact set, i.e. A is the image of a bounded set in $\mathbb{R}^d$ under a Lipschitz mapping. If $\{\omega^*_{N,s}\}$ is any sequence of optimal N-point configurations on A for $s > d$, then

$$\lim_{N\to\infty} \frac{E_s(\omega^*_{N,s})}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}}, \tag{106}$$

where $C_{s,d} > 0$ is a constant independent of A and p. In addition, if $\mathcal{H}_d(A) > 0$, any sequence of configurations on A whose energies satisfy (106) *is asymptotically* *uniformly distributed with respect to $\mathcal{H}_d$.*

We remark that the constant $C_{s,d}$ equals $2\,\zeta(s)$ when $d = 1$, where $\zeta(s)$ is the classical Riemann zeta function (cf. [44]). The value of $C_{s,d}$ for $d \geq 2$ is still unknown.

Definition III.1.2. *Let A be a compact set of Hausdorff dimension d. A sequence of point sets $\omega_N \subset A$, is said to be asymptotically s-energy minimizing on A, and we shall write $\{\omega_N\}_N \in$ AEM$(A;s)$, if it satisfies, with $\omega^*_{N,s}$ replaced by ω_N, the limit relation* (103)*,* (104) *or* (106)*, according to whether $s < d$, $s = d$, or $s > d$.*

As a particular consequence of Corollary II.2.11, we know that if $A \subset \mathbb{R}^p$ is compact, $s < d$ and $s \in [p-2, p)$, then any greedy k_s-energy sequence $\alpha_{N,s} \subset A$ is AEM$(A;s)$. One of the goals of this chapter is to determine whether or not greedy k_s-energy sequences are asymptotically s-energy minimizing for $s \geq d$. We will show examples where this property holds and other examples where it fails. See Section III.2 for details.

In Figures 21–22 below we show two examples of greedy k_s-energy sequences on $[0,1]^2$ for the values $s = 2$ and $s = 4$. As a particular consequence of our Theorem III.2.15, we know that greedy k_s-energy sequences on $[0,1]^2$ are asymptotically uniformly distributed for $s = 2$. But it remains an open question to know if this is also the case when $s > 2$ (on $[0,1]^2$ or S^2).

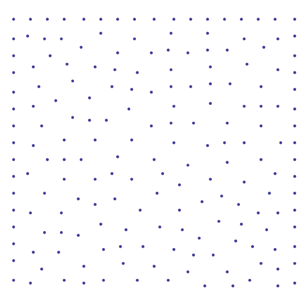

Figure 21: $s = 2$

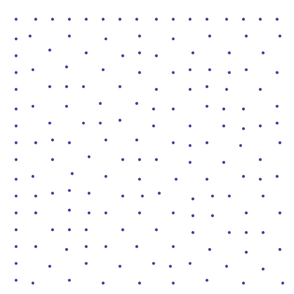

Figure 22: $s = 4$

In Section III.2 we state and discuss our main results. Their proofs are given in subsequent sections.

III.2 Main results

III.2.1 Greedy k_s-energy sequences on S^1

In this subsection we present some results about the asymptotic behavior of $E_s(\alpha_{N,s})$ for greedy k_s-energy sequences on S^1 when $s \geq 1$. As we shall see in Proposition III.2.2, greedy k_s-energy sequences on S^1 are not AEM($S^1; s$) for $s > 1$, which is perhaps a surprising result. We conclude that the behavior of $E_s(\alpha_{N,s})$ exhibits a transition at $s = 1$, the Hausdorff dimension of S^1, since as we saw in Chapter II greedy k_s-energy sequences are AEM($S^1; s$) for $s < 1$.

Remark III.2.1. *It follows from the geometric lemmas proved in Section III.4 that greedy k_s-energy sequences $\alpha_{N,s}$ on S^1 are independent of s, i.e. once the points $a_1, \ldots, a_n$ have been selected, the choice of a_{n+1} is independent of the value of s and depends only on the position of the first n points of the sequence.*

In [44, Theorem 3.1] it was proved that if Γ is a rectifiable Jordan arc, then for $s > 1$,

$$\lim_{N\to\infty} \frac{E_s(\omega^*_{N,s})}{N^{1+s}} = \frac{2\,\zeta(s)}{\mathcal{H}_1(\Gamma)^s}, \tag{107}$$

and if $s = 1$,

$$\lim_{N\to\infty} \frac{E_1(\omega^*_{N,1})}{N^2 \log N} = \frac{2}{\mathcal{H}_1(\Gamma)}, \tag{108}$$

where $\{\omega^*_{N,s}\}_N$ is any sequence of optimal N-point configurations with respect to the Riesz s-kernel.

We remind the reader that by $\mathcal{E}_s(S^1, N)$ we denote the N-point Riesz s-energy of S^1 (see (17)). As it was observed in Chapter II, optimal N-point configurations on S^1 consist precisely of N equally spaced points, and this property holds for all values of $s \in [0, \infty)$. From (107) we have

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(S^1, N)}{N^{1+s}} = \frac{2\,\zeta(s)}{(2\pi)^s}. \tag{109}$$

By Corollary II.2.2 and Theorem III.2.14 (see Subsection III.2.3) we know that if $s \in [0, d]$, then any greedy k_s-energy sequence $\{\alpha_{N,s}\}$ on S^d is AEM($S^d; s$). However the situation changes when $s > 1$ on S^1.

Proposition III.2.2. *For $s > 1$, any greedy k_s-energy sequence $\{\alpha_{N,s}\}_N$ on S^1 is not asymptotically s-energy minimizing. In fact, the subsequence $\alpha_{3\cdot 2^n,s}$ satisfies*

$$\lim_{n\to\infty} \frac{E_s(\alpha_{3\cdot 2^n,s})}{(3\cdot 2^n)^{1+s}} = f(s)\frac{2\,\zeta(s)}{(2\pi)^s},$$

where $f(s) = \frac{1}{2}(\frac{4}{3})^{1+s} + (\frac{1}{3})^{1+s} > 1$ for all $s > 1$.

As in the previous chapter, we want to describe the difference in terms of second-order asymptotics between greedy k_s-energy sequences and optimal N-point configurations when $s = 1$. The following formula holds (see [10]):

$$\lim_{N\to\infty} \frac{\mathcal{E}_1(S^1, N) - \frac{1}{\pi}N^2 \log N}{N^2} = \frac{1}{\pi}(\gamma - \log(\pi/2)), \tag{110}$$

where $\gamma = \lim_{M\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{M} - \log M)$ denotes the Euler-Mascheroni constant.

Proposition III.2.3. *For any greedy k_1-energy sequence $\{\alpha_{N,1}\}_N$ on S^1 we have*

$$\lim_{n\to\infty} \frac{E_1(\alpha_{3\cdot 2^n,1}) - \frac{1}{\pi}(3\cdot 2^n)^2 \log(3\cdot 2^n)}{(3\cdot 2^n)^2} = \frac{1}{\pi}(\gamma - \log(\pi/2) + \log(2^{\frac{16}{9}}/3)). \tag{111}$$

Since the first 2^m points of such sequences $\alpha_{N,1}$ are equally spaced on S^1 (see Lemma III.4.2), we obtain the following:

Corollary III.2.4. *For any greedy k_1-energy sequence $\{\alpha_{N,1}\}_N$ on S^1, the sequence*

$$\frac{E_1(\alpha_{N,1}) - \frac{1}{\pi}N^2 \log N}{N^2}$$

is not convergent.

III.2.2 k_s-Energy of sequences on Jordan arcs or curves in $\mathbb{R}^p$ for $s \geq 1$ and best-packing

Throughout this subsection, by a Jordan arc in $\mathbb{R}^p$ we understand a set homeomorphic to a closed segment. A closed Jordan curve refers to a set homeomorphic to a circle.

The main result in this subsection states that for $s > 1$ it is *not* possible to find *any* sequence of points on a Jordan arc or curve that is asymptotically s-energy minimizing.

Theorem III.2.5. *Let $\{x_k\}_{k=0}^{\infty} \subset \Gamma$ be an arbitrary sequence of distinct points, where Γ is a rectifiable Jordan arc or closed Jordan curve in $\mathbb{R}^p$. Set $\mathcal{X}_n := \{x_k\}_{k=0}^{n}$. Then $\{\mathcal{X}_n\}_n \notin \mathrm{AEM}(\Gamma; s)$ for all $s > 1$. In particular, $\{\alpha_{N,s}\} \notin \mathrm{AEM}(\Gamma; s)$ for any greedy k_s-energy sequence on Γ when $s > 1$.*

The next result shows that, in contrast to the case $s > 1$, for $s = 1$ greedy k_1-energy sequences on S^1 are $\mathrm{AEM}(S^1; 1)$. More generally, we shall prove this fact for *smooth* Jordan arcs or curves Γ by which we mean that the natural parametrization $\Phi : [0, L] \longrightarrow \Gamma$, where $L = \mathcal{H}_1(\Gamma)$, is of class C^1 and $\Phi'(t) \neq \mathbf{0}$ for all $t \in [0, L]$.

Theorem III.2.6. *Let $\Gamma \subset \mathbb{R}^p$ be a smooth Jordan arc or closed curve, and let $s = d = 1$. Then any greedy k_1-energy sequence $\{\alpha_{N,1}\}$ on Γ is $\mathrm{AEM}(\Gamma; 1)$, i.e.*

$$\lim_{N\to\infty} \frac{E_1(\alpha_{N,1})}{N^2 \log N} = \frac{2}{\mathcal{H}_1(\Gamma)}. \tag{112}$$

Furthermore,

$$\frac{1}{N}\sum_{a\in\alpha_{N,1}}\delta_a \xrightarrow{*} \frac{\mathcal{H}_1|_\Gamma}{\mathcal{H}_1(\Gamma)}, \qquad N\to\infty. \tag{113}$$

For the analogous result for greedy k_d-energy on the unit sphere $S^d \subset \mathbb{R}^{d+1}$, see Theorem III.2.14.

We next consider best-packing configurations. For a collection of N distinct points $\omega_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^p$, we set

$$\delta(\omega_N) := \min_{1\le i\ne j\le N} |x_i - x_j|,$$

and for an infinite set $A \subset \mathbb{R}^p$, we let

$$\delta_N(A) := \sup\{\delta(\omega_N) : \omega_N \subset A, \operatorname{card}(\omega_N) = N\}$$

be the *best-packing distance* of N-point configurations on A. In [9, Theorem 2.2] it is shown that if $A = \Gamma$ is a rectifiable Jordan curve or arc in $\mathbb{R}^p$, then

$$\lim_{N\to\infty} N\delta_N(\Gamma) = \mathcal{H}_1(\Gamma).$$

This fact leads us to the following.

Definition III.2.7. *Let $\Gamma \subset \mathbb{R}^p$ be a Jordan arc or curve, and let $\omega_N \subset \Gamma$ be a sequence of N-point configurations. We say that $\{\omega_N\} \in \mathrm{AEM}(\Gamma, \infty)$ if*

$$\lim_{N\to\infty} N\delta(\omega_N) = \mathcal{H}_1(\Gamma).$$

The following result is analogous to Theorem III.2.5 in the sense that it proves the impossibility of finding an infinite sequence on any rectifiable Jordan arc or curve that is $\mathrm{AEM}(\Gamma; \infty)$.

Theorem III.2.8. *Let $\Gamma \subset \mathbb{R}^p$ be a rectifiable Jordan arc or curve with length $L = \mathcal{H}_1(\Gamma)$, and let $\{x_k\}_{k=0}^{\infty} \subset \Gamma$ be an arbitrary infinite sequence such that $x_i \neq x_j$ if $i \neq j$. Set $\mathcal{X}_n := \{x_0, \ldots, x_n\}$. Then $\{\mathcal{X}_n\} \notin \mathrm{AEM}(\Gamma, \infty)$. In fact,*

$$\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n) \leq \frac{4+3\sqrt{2}}{4+4\sqrt{2}}\,L < L. \tag{114}$$

Moreover, if $c := \limsup_{n\to\infty} n\,\delta(\mathcal{X}_n) > \frac{2+\sqrt{2}}{4}L$, then

$$\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n) \leq \frac{L}{2} + \sqrt{c\,(L-c)} < c. \tag{115}$$

In particular, if $\limsup_{n\to\infty} n\,\delta(\mathcal{X}_n) = L$, then $\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n) \leq L/2$.

In analogy with finite s, we define *greedy best-packing configurations* on a compact set $A \subset \mathbb{R}^p$ by selecting $a_0 \in A$ and choosing $a_n \in A$ so that

$$\min_{0\leq i\leq n-1} |a_n - a_i| = \max_{x\in A} \min_{0\leq i\leq n-1} |x - a_i|.$$

Such points are referred to in [20] as *Leja-Bos* points. Theorem III.2.8 shows that such points are not asymptotically optimal on rectifiable Jordan arcs or curves.

In [20] there appears a conjecture attributed to L. Bos stating that if A is a compact domain of $\mathbb{C}$, every Leja-Bos sequence $\{a_n\}_{n=0}^{\infty}$ on A with $|a_0| = \max\{|x| : x \in A\}$ is asymptotically uniformly distributed. We show in the following result that this conjecture is false (see also Figure 33 in Section III.5).

Proposition III.2.9. *There exist greedy best-packing sequences on $[0,1]$ and $[0,1]^2$ that are not asymptotically uniformly distributed.*

It is not difficult to see, however, that greedy best-packing sequences are *dense* in the set A.

III.2.3 Weighted Riesz potentials

In this subsection we will consider the notion of weighted discrete Riesz energy introduced in [8]. We reproduce here the main definitions.

Definition III.2.10. *Let $A \subset \mathbb{R}^p$ be an infinite compact set whose d-dimensional Hausdorff measure $\mathcal{H}_d(A)$ is finite. A symmetric function $w : A \times A \longrightarrow [0, \infty]$ is called a* CPD*-weight function on $A \times A$ if*

- *w is continuous (as a function on $A \times A$) at $\mathcal{H}_d$-almost every point of the diagonal $D(A) := \{(x, x) : x \in A\}$,*
- *there is some neighborhood G of $D(A)$ (relative to $A \times A$) such that $\inf_G w > 0$, and*
- *w is bounded on any closed subset $B \subset A \times A$ such that $B \cap D(A) = \emptyset$.*

The term CPD stands for (almost) continuous and positive on the diagonal.

Definition III.2.11. *Let $s > 0$. Given a collection of N ($N \geq 2$) points $\omega_N := \{x_1, \dots, x_N\} \subset A$, the weighted Riesz s-energy of ω_N is defined by*

$$E_s^w(\omega_N) := \sum_{1 \leq i \neq j \leq N} \frac{w(x_i, x_j)}{|x_i - x_j|^s},$$

while the N-point weighted Riesz s-energy of A is given by

$$\mathcal{E}_s^w(A, N) := \inf\{E_s^w(\omega_N) : \omega_N \subset A,\ \operatorname{card}(\omega_N) = N\}.$$

The weighted Hausdorff measure $\mathcal{H}_d^{s,w}$ on Borel sets $B \subset A$ is defined by

$$\mathcal{H}_d^{s,w}(B) := \int_B (w(x, x))^{-d/s} d\mathcal{H}_d(x).$$

The following result about the asymptotic behavior of $\{\mathcal{E}_s^w(A,N)\}_N$ was obtained in [8].

Theorem III.2.12. *Let A be a compact subset of a d-dimensional C^1-manifold in $\mathbb{R}^p$ and assume that $w : A \times A \to [0,\infty]$ is a* CPD*-weight function on $A \times A$. Then*

$$\lim_{N\to\infty} \frac{\mathcal{E}_d^w(A,N)}{N^2 \log N} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d^{d,w}(A)}. \tag{116}$$

Furthermore, if $\mathcal{H}_d(A) > 0$ and $\{\widetilde{\omega}_N\}$ is a sequence of configurations on A satisfying (116)*, with $\mathcal{E}_d^w(A,N)$ replaced by $E_d^w(\widetilde{\omega}_N)$, then*

$$\frac{1}{N}\sum_{x\in\widetilde{\omega}_N}^{N} \delta_x \xrightarrow{\ *\ } \frac{\mathcal{H}_d^{d,w}|_A}{\mathcal{H}_d^{d,w}(A)}, \qquad N \to \infty. \tag{117}$$

Assume now that $A \subset \mathbb{R}^p$ is a d-rectifiable set. Then for $s > d$,

$$\lim_{N\to\infty} \frac{\mathcal{E}_s^w(A,N)}{N^{1+s/d}} = \frac{C_{s,d}}{[\mathcal{H}_d^{s,w}(A)]^{s/d}}, \tag{118}$$

where $C_{s,d}$ is the same positive constant that appears in Theorem III.1.1. In addition, if $\mathcal{H}_d(A) > 0$, any sequence $\{\widetilde{\omega}_N\}$ of configurations on A satisfying (118) *with $\mathcal{E}_s^w(A,N)$ replaced by $E_s^w(\widetilde{\omega}_N)$ also satisfies* (117)*.*

Definition III.2.13. *Let w be a lower semicontinuous CPD-weight function on $A\times A$. A sequence $(a_n)_{n=1}^\infty \subset A$ is called a greedy (w,s)-energy sequence on A if it is generated in the same way as generated in Definition I.1.2, with $k(x,y) := w(x,y)/|x-y|^s$.*

Our first result in this subsection concerns greedy (w,d)-energy points on the unit sphere $S^d \subset \mathbb{R}^{d+1}$ (compare with Proposition III.2.2 and Corollary II.2.2).

Theorem III.2.14. *Assume that $w : S^d \times S^d \to [0,\infty)$ is a continuous function such that $w(x,x) > 0$ for all $x \in S^d$. Let $\{\alpha_{N,d}^w\}_N$ be an arbitrary greedy (w,d)-energy* *sequence on S^d, $d \geq 1$. Then*

$$\lim_{N\to\infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d^{d,w}(S^d)}, \tag{119}$$

and therefore

$$\frac{1}{N} \sum_{a\in\alpha_{N,d}^w} \delta_a \xrightarrow{*} \frac{\mathcal{H}_d^{d,w}|_{S^d}}{\mathcal{H}_d^{d,w}(S^d)}, \qquad N \to \infty.$$

In particular, any greedy k_d-energy sequence $\{\alpha_{N,d}\}_N$ on S^d is AEM(S^d, d) *and satisfies* (42) *for $s = d$.*

In the following result we consider greedy (w, p)-energy sequences on sets in $\mathbb{R}^p$ with positive Lebesgue measure.

Theorem III.2.15. *Let $A \subset \mathbb{R}^p$ be a compact set such that $\mathcal{H}_p(A) > 0$, and let $\{\alpha_{N,p}^w\}_N$ be an arbitrary greedy (w, p)-energy sequence on A. Assume that $w : A\times A \to [0, \infty)$ is a continuous function such that $w(x, x) > 0$ for all $x \in A$. Then*

$$\lim_{N\to\infty} \frac{E_p^w(\alpha_{N,p}^w)}{N^2 \log N} = \frac{\mathrm{Vol}(B^p)}{\mathcal{H}_p^{p,w}(A)}, \tag{120}$$

and therefore

$$\frac{1}{N} \sum_{a\in\alpha_{N,p}^w} \delta_a \xrightarrow{*} \frac{\mathcal{H}_p^{p,w}|_A}{\mathcal{H}_p^{p,w}(A)}, \qquad N \to \infty. \tag{121}$$

In particular, any greedy k_p-energy sequence $\{\alpha_{N,p}\}_N$ on A is AEM$(A; p)$ *and is asymptotically uniformly distributed with respect to $\mathcal{H}_p$.*

In view of Proposition III.2.2, it is not in general possible to extend Theorem III.2.14 to $s > d$. However, for any compact set $A \subset \mathbb{R}^p$ with $\mathcal{H}_\delta(A) > 0$ (where $\delta > 0$ is arbitrary, not necessarily an integer), we can show that the order of growth of $E_s^w(\alpha_{N,s}^w)$ when $s > \delta$ ($s = \delta$) is at most $N^{1+s/\delta}$ ($N^2 \log N$). Let

$$\mathcal{H}_\delta^\infty(A) := \inf\{\sum_i (\mathrm{diam}\, G_i)^\delta : A \subset \bigcup_i G_i\}, \qquad \delta > 0.$$

Theorem III.2.16. *Let $0 < \delta \leq p$. Assume that $A \subset \mathbb{R}^p$ is a compact set such that $\mathcal{H}_\delta(A) > 0$. Let w be a bounded lower semicontinuous* CPD*-weight function on $A \times A$. Consider an arbitrary greedy (w, s)-energy sequence $\{\alpha^w_{N,s}\}_N \subset A$, for $s \geq \delta$. Then, for $N \geq 2$,*

$$E^w_s(\alpha^w_{N,s}) \leq \begin{cases} M_{s,\delta,A}\, \|w\|\, \mathcal{H}^\infty_\delta(A)^{-s/\delta} N^{1+s/\delta}, & \text{if } s > \delta, \\ \\ M_{\delta,A}\, \|w\|\, \mathcal{H}^\infty_\delta(A)^{-1} N^2 \log N, & \text{if } s = \delta, \end{cases}$$

where the constants $M_{s,\delta,A} > 0$ and $M_{\delta,A} > 0$ are independent of w and N, and $\|w\| := \sup\{w(x, y) : x, y \in A\}$.

Corollary III.2.17. *Let $A \subset \mathbb{R}^p$ be a d-rectifiable set. Suppose $s > d$ and w is a bounded lower semicontinuous* CPD*-weight function on $A \times A$. Consider an arbitrary greedy (w, s)-energy sequence $\{\alpha^w_{N,s}\}_N \subset A$. Then there are constants $C_1, C_2 > 0$ such that*

$$C_1\, N^{1+s/d} \leq E^w_s(\alpha^w_{N,s}) \leq C_2\, N^{1+s/d}. \tag{122}$$

If $s = d$ and A is assumed to be a compact subset of a d-dimensional C^1-manifold, then there are constants $C_3, C_4 > 0$ such that

$$C_3\, N^2 \log N \leq E^w_d(\alpha^w_{N,d}) \leq C_4\, N^2 \log N, \tag{123}$$

for any greedy (w, d)-energy sequence $\{\alpha^w_{N,d}\}_N \subset A$.

Corollary III.2.18. *Let $A \subset \mathbb{R}^p$ be a d-rectifiable set. Suppose $s > d$ and w is a bounded lower semicontinuous* CPD*-weight function on $A \times A$. Consider an arbitrary greedy (w, s)-energy sequence $\{a_n\}_{n=1}^\infty \subset A$. Then $\{a_n\}_{n=1}^\infty$ is dense in A. If $s = d$ and A is assumed to be a compact subset of a d-dimensional C^1-manifold, the same* *conclusion holds for any greedy (w, d)-energy sequence. Taking $w \equiv 1$ the result is applicable to greedy k_s-energy sequences.*

We can slightly improve the density result in certain cases like a real interval.

Proposition III.2.19. *Let $[a, b] \subset \mathbb{R}$ and $s > 1$. Assume that w is a bounded lower semicontinuous* CPD*-weight function on $[a, b] \times [a, b]$, and $(a_n)_{n=1}^{\infty}$ is a greedy (w, s)-energy sequence on $[a, b]$. If I is any closed subinterval of $[a, b]$, then*

$$\liminf_{N\to\infty} \frac{(\operatorname{card}\{1 \le n \le N : a_n \in I\})^{1+\frac{1}{s}}}{N} > 0. \tag{124}$$

III.3 Numerical experiments

In Theorem III.2.6, we proved that greedy k_1-energy sequences are asymptotically uniformly distributed on smooth Jordan arcs or closed Jordan curves (see (113)). In the case of an interval $[a, b] \subset \mathbb{R}$, this property can be formulated in an equivalent way as follows: If $(a_n)_{n=1}^{\infty}$ is an arbitrary greedy k_1-energy sequence on $[a, b]$, then

$$\lim_{N\to\infty} \frac{\operatorname{card}\{1 \le n \le N : a_n \in [c, d]\}}{N} = \frac{d-c}{b-a}, \qquad \text{for all} \quad [c, d] \subset [a, b]. \tag{125}$$

We do not know if this property also holds for greedy k_s-energy sequences in the case $s > 1$ (the best we can say so far is (124)). However, in view of the following numerical experiments we tend to believe that the answer is positive.

In all the examples below the points were generated in the interval $[-5, 5]$, and the first point is always selected to be $a_1 = -5$ (therefore the second and third points are always $a_2 = 5$ and $a_3 = 0$). The number of points in each example is indicated.

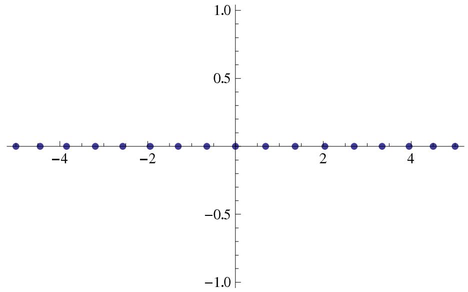

Figure 23: $s = 1,\ N = 17$

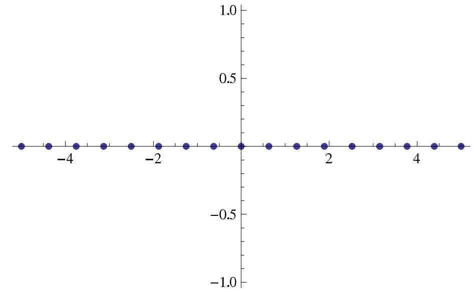

Figure 24: $s = 3,\ N = 17$

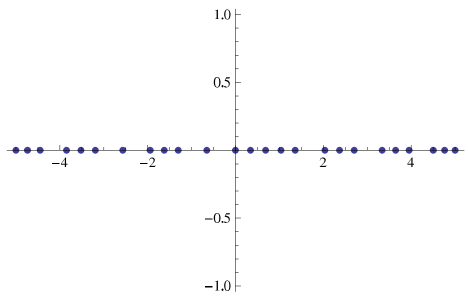

Figure 25: $s = 1,\ N = 25$

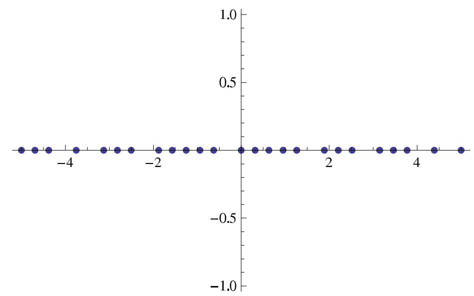

Figure 26: $s = 3,\ N = 25$

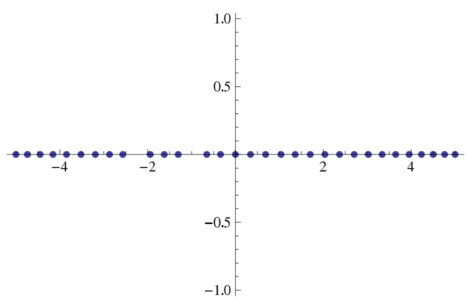

Figure 27: $s = 1,\ N = 31$

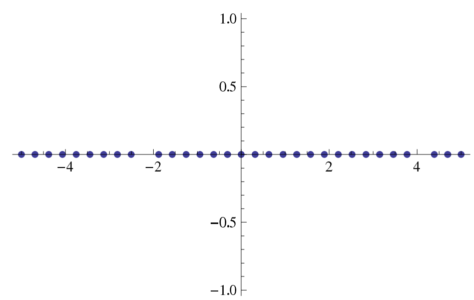

Figure 28: $s = 3,\ N = 31$

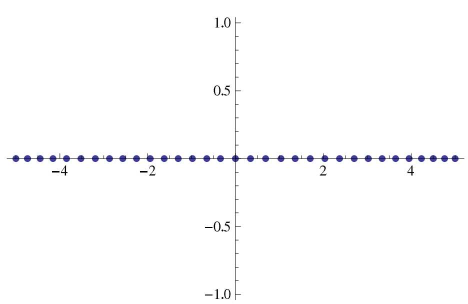

Figure 29: $s = 1,\ N = 33$

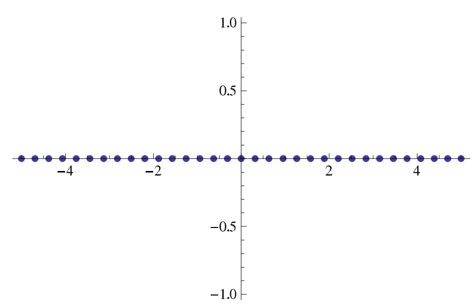

Figure 30: $s = 3,\ N = 33$

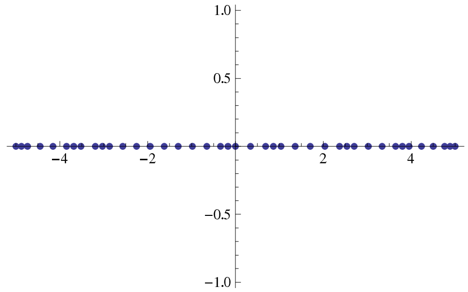

Figure 31: $s = 1$, $N = 41$

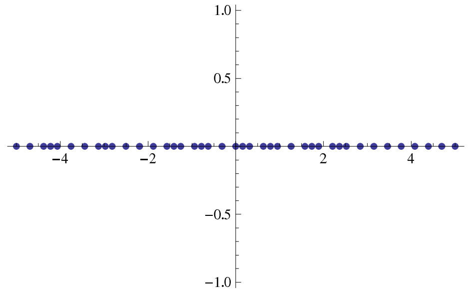

Figure 32: $s = 3$, $N = 41$

Observe that in the cases $N = 17$ and $N = 33$, the points are practically equally spaced! So the limit (125) should definitely hold for the subsequence $N = 2^n + 1$.

III.4 Proofs of results from Subsection III.2.1

In order to prove Proposition III.2.2 we need some auxiliary lemmas that give a geometric description of greedy k_s-energy sequences on S^1.

Lemma III.4.1. *Let $s \geq 0$ and consider two points $x_1, x_2 \in S^1$. Set*

$$f(x) := k_s(x, x_1) + k_s(x, x_2), \qquad x \in S^1,$$

where k_s is the Riesz s-kernel (1)*. Then on each arc determined by x_1 and x_2 the function f has only one minimum and it is attained at the midpoint of the arc.*

Proof. We write $x_1 = e^{i\lambda}$ and $x_2 = e^{i\phi}$, and without loss of generality we assume that $\lambda = 0$ and $\phi \in (0, 2\pi)$. We want to show that the function $g(\theta) := f(e^{i\theta})$ is strictly decreasing on $(0, \phi/2)$. Since $g(\theta)$ is symmetric on the interval $(0, \phi)$ with respect to the point $\phi/2$, the location and uniqueness of the minimum follows. Assume first that $s > 0$. We have that

$$g(\theta) = 2^{-\frac{s}{2}}\left[(1 - \cos(\phi - \theta))^{-\frac{s}{2}} + (1 - \cos\theta)^{-\frac{s}{2}}\right].$$

Thus

$$g'(\theta) = \left(\frac{s}{2}\right) 2^{-\frac{s}{2}} [\sin(\phi-\theta)(1-\cos(\phi-\theta))^{-\frac{s}{2}-1} - \sin(\theta)(1-\cos(\theta))^{-\frac{s}{2}-1}] .$$

Showing that $g'(\theta) < 0$ on $(0, \phi/2)$ is equivalent to

$$\frac{\sin(\phi-\theta)}{(1-\cos(\phi-\theta))^{\frac{s}{2}+1}} < \frac{\sin\theta}{(1-\cos\theta)^{\frac{s}{2}+1}}, \quad \theta \in (0, \phi/2) .$$

Since $\phi - \theta > \theta$, and the function $(\sin x)/(1-\cos x)^{\beta}$ is strictly decreasing on $(0, 2\pi)$ for $\beta > 1$, we obtain the desired result for $s > 0$.

If $s = 0$ we have

$$g(\theta) = -\log(2[\cos(\phi/2-\theta) - \cos(\phi/2)]) ,$$

and so the claim is also valid in this case. □

Lemma III.4.2. *Let $s \geq 0$ and assume that $(a_n)_{n=1}^{\infty}$ is an arbitrary greedy k_s-energy sequence on S^1. Then*

(i) for every positive integer m, the set $\alpha_{2^m,s}$ consists of 2^m equally spaced points, that is,

$$\alpha_{2^m,s} = \{a_1 e^{i\frac{2\pi n}{2^m}}\}_{n=1}^{2^m} ;$$

(ii) for every positive integer m, the set $\alpha_{3 \cdot 2^m}$ can be written as

$$\alpha_{3\cdot 2^m,s} = S_{2^{m+2}} \setminus S_{2^m}, \tag{126}$$

where $S_{2^{m+2}}$ and S_{2^m} are formed, respectively, by 2^{m+2} and 2^m equally spaced points, and $S_{2^m} \subset S_{2^{m+2}}$;

(iii) the choice of any point a_n is independent of s.

Proof. We first justify property (i). This property is well known for $s = 0$ (cf.[5]). The following argument applies to all values of $s \geq 0$. We proceed by induction on m. For $m = 1$ the result follows trivially. Assume now that the result is true for $m - 1$, i.e. given any greedy k_s-energy sequence $(b_n)_{n=1}^{\infty}$, the first 2^{m-1} points are equally spaced, and let us show that $\{a_n\}_{n=1}^{2^m}$ consists of 2^m equally spaced points. Consider the function

$$f_{2^{m-1}}(x) := \sum_{n=1}^{2^{m-1}} k_s(x, a_n), \qquad x \in S^1.$$

By hypothesis the points $a_1, \ldots, a_{2^{m-1}}$ are equally spaced. The symmetry of these points and Lemma $III.4.1$ allow us to conclude that $f_{2^{m-1}}$ attains its minimum at each midpoint of the 2^{m-1} arcs determined by $a_1, \ldots, a_{2^{m-1}}$, and only at these points. Thus,

$$a_{2^{m-1}+1} \in \{a_1 e^{i\frac{2\pi(2k-1)}{2^m}}\}_{k=1}^{2^{m-1}}. \tag{127}$$

Now we write

$$f_{2^{m-1}+1}(x) = \sum_{n=1}^{2^{m-1}+1} k_s(x, a_n) = f_{2^{m-1}}(x) + k_s(x, a_{2^{m-1}+1}).$$

The (only) point where the function $f_{2^{m-1}+1}$ attains its minimum is the point where $k_s(x, a_{2^{m-1}+1})$ attains its minimum, i.e. the point $-a_{2^{m-1}+1}$, since

$$\min_{x \in S^1} f_{2^{m-1}+1}(x) \geq \min_{x \in S^1} f_{2^{m-1}}(x) + \min_{x \in S^1} k_s(x, a_{2^{m-1}+1}),$$

and $f_{2^{m-1}}(x)$ and $k_s(x, a_{2^{m-1}+1})$ both attain their minimum at the same point. In general, by the symmetry of $\{a_n\}_{n=1}^{2^{m-1}}$, if we write

$$f_{2^{m-1}+l}(x) = f_{2^{m-1}}(x) + \sum_{k=1}^{l} k_s(x, a_{2^{m-1}+k}) \qquad l < 2^{m-1},$$

it follows that the point $a_{2^{m-1}+l+1}$ is a point where $\sum_{k=1}^{l} k_s(x, a_{2^{m-1}+k})$ attains its minimum. Therefore, the set $\{a_{2^{m-1}+k}\}_{k=1}^{2^{m-1}}$ is formed by the first 2^{m-1} points of some greedy k_s-energy sequence. By induction hypothesis, $\{a_{2^{m-1}+k}\}_{k=1}^{2^{m-1}}$ is formed by 2^{m-1} equally spaced points. From (127) we conclude that

$$\{a_n\}_{n=1}^{2^m} = \{a_n\}_{n=1}^{2^{m-1}} \cup \{a_{2^{m-1}+k}\}_{k=1}^{2^{m-1}}$$

is also formed by equally spaced points.

Properties (*ii*) and (*iii*) are immediate consequences of the above proof. $\square$

Since greedy k_s-energy sequences $\{\alpha_{N,s}\}$ on the unit circle S^1 are independent of s, we will denote them simply by α_N.

Lemma III.4.3. *Let $s \geq 0$. Given any greedy k_s-energy sequence $\{\alpha_N\}_N$ on S^1, the following relation holds for every $n \geq 1$:*

$$E_s(\alpha_{3\cdot 2^n}) = \frac{1}{2}\,\mathcal{E}_s(S^1, 2^{n+2}) + \mathcal{E}_s(S^1, 2^n). \tag{128}$$

Proof. If $\{x_k\}_{k=1}^N \subset S^1$ is an arbitrary collection of N equally spaced points, then using the simple equality $|e^{i\xi} - e^{i\theta}| = 2|\sin(\frac{\xi-\theta}{2})|$, we conclude that for $s > 0$,

$$\mathcal{E}_s(S^1, N) = E_s(\{x_k\}_{k=1}^N) = 2^{-s} N \sum_{n=1}^{N-1} \sin\left(\frac{\pi n}{N}\right)^{-s}. \tag{129}$$

Consider any greedy k_s-energy sequence $(\alpha_N)_{N=1}^\infty$ on S^1. We claim that

$$E_s(\alpha_{3\cdot 2^n}) = E_s(S_{2^{n+2}}) - 2^{n+1}\cdot 2^{-s} \sum_{k=1}^{2^{n+2}-1} \sin\left(\frac{\pi k}{2^{n+2}}\right)^{-s} + E_s(S_{2^n}),$$

where $\alpha_{3\cdot 2^n} = S_{2^{n+2}} \setminus S_{2^n}$ is as in (126). To see this, notice that $E_s(\alpha_{3\cdot 2^n})$ is obtained by removing twice from $E_s(S_{2^{n+2}})$ all terms $|e^{i\xi} - e^{i\theta}|^{-s}$ where either $e^{i\xi} \in S_{2^n}$ or $e^{i\theta} \in S_{2^n}$.

Since

$$E_s(S_{2^{n+2}}) = \mathcal{E}_s(S^1, 2^{n+2}), \qquad E_s(S_{2^n}) = \mathcal{E}_s(S^1, 2^n),$$

(128) follows by applying (129). The case $s = 0$ is proved similarly. □

Proof of Proposition III.2.2. Using (128) we obtain

$$\frac{E_s(\alpha_{3\cdot 2^n})}{3^{1+s}2^{n(1+s)}} = \frac{1}{3^{1+s}}\frac{1}{2}\frac{2^{(n+2)(1+s)}}{2^{n(1+s)}}\frac{\mathcal{E}_s(S^1, 2^{n+2})}{2^{(n+2)(1+s)}} + \frac{1}{3^{1+s}}\frac{\mathcal{E}_s(S^1, 2^n)}{2^{n(1+s)}}.$$

Simplifying the above expression and applying (109) we conclude that

$$\lim_{n\to\infty}\frac{E_s(\alpha_{3\cdot 2^n})}{(3\cdot 2^n)^{1+s}} = \left(\frac{1}{2}\left(\frac{4}{3}\right)^{1+s} + \left(\frac{1}{3}\right)^{1+s}\right)\frac{2\zeta(s)}{(2\pi)^s}.$$

It is straightforward to check that $f(s) = \frac{1}{2}\left(\frac{4}{3}\right)^{1+s} + \left(\frac{1}{3}\right)^{1+s} > 1$ for all $s > 1$. □

Proof of Proposition III.2.3. First observe that

$$\frac{E_1(\alpha_{3\cdot 2^n}) - \frac{1}{\pi}(3\cdot 2^n)^2\log(3\cdot 2^n)}{(3\cdot 2^n)^2}$$

$$= \frac{1}{9}\Big(\frac{(1/2)\,\mathcal{E}_1(S^1, 2^{n+2}) + \mathcal{E}_1(S^1, 2^n) - \frac{1}{\pi}(3\cdot 2^n)^2\log(3\cdot 2^n)}{2^{2n}}\Big).$$

We add and subtract $(1/\pi)2^{2n}\log(2^n)$ to obtain

$$\frac{E_1(\alpha_{3\cdot 2^n}) - \frac{1}{\pi}(3\cdot 2^n)^2\log(3\cdot 2^n)}{(3\cdot 2^n)^2} \tag{130}$$

$$= \frac{1}{9}\Big(\frac{\mathcal{E}_1(S^1, 2^n) - \frac{1}{\pi}2^{2n}\log(2^n)}{2^{2n}} + 16\frac{(1/2)\,\mathcal{E}_1(S^1, 2^{n+2}) - \frac{1}{\pi}\Lambda_n}{2^{2(n+2)}}\Big)$$

where $\Lambda_n = (3\cdot 2^n)^2\log(3\cdot 2^n) - 2^{2n}\log(2^n)$. Taking into account that

$$\Lambda_n = \frac{2^{2(n+2)}}{2}\log(2^{n+2}) + \log(3)(3\cdot 2^n)^2 - 8\log(4)2^{2n}$$

it follows that

$$16\frac{(1/2)\,\mathcal{E}_1(S^1,2^{n+2})-\frac{1}{\pi}\Lambda_n}{2^{2(n+2)}} \tag{131}$$

$$=8\frac{\mathcal{E}_1(S^1,2^{n+2})-\frac{1}{\pi}2^{2(n+2)}\log(2^{n+2})}{2^{2(n+2)}}+\frac{1}{\pi}(8\log(4)-9\log(3)).$$

Applying (110), (130) and (131) we conclude that

$$\lim_{n\to\infty}\frac{E_1(\alpha_{3\cdot 2^n})-\frac{1}{\pi}(3\cdot 2^n)^2\log(3\cdot 2^n)}{(3\cdot 2^n)^2}$$

$$=\frac{1}{\pi}(\gamma-\log(\pi/2))+\frac{1}{\pi}(\frac{8}{9}\log(4)-\log(3))=\frac{1}{\pi}(\gamma-\log(\pi/2)+\log(2^{\frac{16}{9}}/3)).$$

□

Proof of Corollary III.2.4. Since $E_1(\alpha_{2^n})=\mathcal{E}_1(S^1,2^n)$ for all n, the result follows from (110) and (111). □

III.5 Proofs of results from Subsection III.2.2

Proof of Theorem III.2.5. Assume first that Γ is a Jordan arc. If $x_1,x_2\in\Gamma$, we denote by (x_1,x_2) the subarc joining x_1 and x_2, and by $l(x_1,x_2)$ its length.

Let $\mathcal{X}_n:=\{x_{k,n}\}_{k=0}^{n}$ be a sequence of configurations on Γ, where we assume that the points $x_{k,n}$ are located in successive order. Set

$$d_{k,n}:=l(x_{k-1,n},x_{k,n}),\qquad k=1,\ldots,n. \tag{132}$$

In [44] the following result was proved:

Theorem III.5.1. *Let Γ be a rectifiable Jordan arc in $\mathbb{R}^p$. If $s>1$ and $\{\mathcal{X}_n\}_n\in$ AEM$(\Gamma;s)$, then*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left|d_{k,n}-\frac{L}{n}\right|=0,\qquad L:=\mathcal{H}_1(\Gamma). \tag{133}$$

We prove Theorem III.2.5 by contradiction. Let $\{x_k\}_{k=0}^{\infty} \subset \Gamma$ be an arbitrary sequence of distinct points and set $\mathcal{X}_n := \{x_k\}_{k=0}^{n}$. We will use the notation $\mathcal{X}_n = \{x_{0,n}, \ldots, x_{n,n}\}$. Assume that $\{\mathcal{X}_n\}_n \in \mathrm{AEM}(\Gamma; s)$. Let $\delta > 0$ and consider the sets

$$A_n^{\delta} := \{k :\in \{1, \ldots, n\} : \frac{L-\delta}{n} < d_{k,n} < \frac{L+\delta}{n}\}, \qquad B_n^{\delta} := \{1, \ldots, n\} \setminus A_n^{\delta}.$$

Let $\epsilon > 0$ be a fixed number. Then from (133) there exists $N = N(\epsilon) \in \mathbb{N}$ such that, if $n \geq N$,

$$\sum_{k=1}^{n} \left| d_{k,n} - \frac{L}{n} \right| \leq \epsilon. \tag{134}$$

If $k \in B_n^{\delta}$, then $|d_{k,n} - L/n| \geq \delta/n$, and from (134) it follows that

$$\mathrm{card}(B_n^{\delta}) \frac{\delta}{n} \leq \epsilon, \qquad n \geq N.$$

Therefore,

$$\mathrm{card}(A_n^{\delta}) = n - \mathrm{card}(B_n^{\delta}) \geq n\Big(1 - \frac{\epsilon}{\delta}\Big), \qquad n \geq N.$$

There are exactly n subarcs $(x_{k-1,n}, x_{k,n})$, and when we add the next $n/2$ points (we may assume that n is even) to the configuration $\mathcal{X}_n$, obviously at most $n/2$ of these new points will lie in the subarcs $(x_{k-1,n}, x_{k,n})$ where $k \in A_n^{\delta}$. Setting

$$C_n^{\delta} := \{k \in A_n^{\delta} : (x_{k-1,n}, x_{k,n}) \text{ does not contain a new point}\},$$

we have

$$\mathrm{card}(C_n^{\delta}) \geq n\Big(1 - \frac{\epsilon}{\delta}\Big) - \frac{n}{2} = n\Big(\frac{1}{2} - \frac{\epsilon}{\delta}\Big).$$

Now since the intervals $(x_{k-1,n}, x_{k,n})$ with $k \in C_n^{\delta}$ do not contain a new point, there are at least $\mathrm{card}(C_n^{\delta})$ values of k' in $\{1, \ldots, 3n/2\}$ such that $d_{k',3n/2} = d_{k,n}$ for some

$k \in C_n^{\delta}$. For these values of k' and the corresponding values of k, we have

$$\left| d_{k',3n/2} - \frac{L}{3n/2} \right| = \left| d_{k,n} - \frac{L}{n} + \frac{L}{3n} \right|.$$

Now we choose δ to be any fixed value less than $L/3$, say $\delta := L/6$. Then for $k \in C_n^{\delta}$,

$$\left| d_{k,n} - \frac{L}{n} + \frac{L}{3n} \right| \geq \left| \frac{L}{3n} - \left| \frac{L}{n} - d_{k,n} \right| \right| = \frac{L}{3n} - \left| \frac{L}{n} - d_{k,n} \right| > \frac{L}{3n} - \frac{L}{6n} = \frac{L}{6n}.$$

Finally,

$$\sum_{k'=1}^{3n/2} \left| d_{k',3n/2} - \frac{L}{3n/2} \right| \geq n \left(\frac{1}{2} - \frac{\epsilon}{\delta} \right) \frac{L}{6n} = \left(\frac{1}{2} - \frac{6\,\epsilon}{L} \right) \frac{L}{6}.$$

But the above estimate contradicts (134) since we can select ϵ sufficiently small so that

$$\left(\frac{1}{2} - \frac{6\,\epsilon}{L} \right) \frac{L}{6} > \epsilon.$$

If Γ is a closed Jordan curve, we select an orientation for it. Then the above reasoning used to prove the result in the case of Jordan arcs is also applicable. We only have to define $(x_{k-1,n}, x_{k,n})$ as the subarc joining $x_{k-1,n}$ and $x_{k,n}$ on which a particle moves from $x_{k-1,n}$ to $x_{k,n}$ following the orientation prescribed. The details of the argument are left to the reader. □

Proof of Theorem III.2.6. We first assume that Γ is a smooth Jordan arc of length L. We will reduce the problem of asymptotics of $\alpha_{N,1}$ on Γ to a weighted problem on $[0, L]$ and then apply Theorem III.2.15. Let $\Phi : [0, L] \longrightarrow \Gamma$ be the natural parametrization of Γ and define $w : [0, L] \times [0, L] \longrightarrow [0, \infty)$ by

$$w(x, y) := \frac{|x - y|}{|\Phi(x) - \Phi(y)|}. \tag{135}$$

Let $\Psi = \Phi^{-1}$ be the inverse function of Φ. If a_n is the n-th element of the greedy k_1-energy sequence on Γ, let $b_n := \Psi(a_n) \in [0, L]$ and $\beta_N := \{b_1, \dots, b_N\}$. Since for

$t = \Phi(x), x \in [0, L]$,

$$\inf_{t\in\Gamma} \sum_{i=1}^{n-1} \frac{1}{|t - a_i|} = \inf_{x\in[0,L]} \sum_{i=1}^{n-1} \frac{1}{|\Phi(x) - \Phi(b_i)|} = \inf_{x\in[0,L]} \sum_{i=1}^{n-1} \frac{w(x, b_i)}{|x - b_i|},$$

it follows that $\{\beta_N\}$ is a greedy $(w, 1)$-energy sequence on $[0, L]$ (see Definition III.2.13) associated with the weight function (135). Notice that

$$\mathcal{H}_1^{1,w}([0, L]) = \int_0^L w(x, x)^{-1}\, dx = \int_0^L |\Phi'(x)|\, dx = L.$$

Applying Theorem III.2.15 we obtain that

$$\lim_{N\to\infty} \frac{E_1(\alpha_{N,1})}{N^2 \log N} = \lim_{N\to\infty} \frac{E_1^w(\beta_N)}{N^2 \log N} = \frac{2}{\mathcal{H}_1^{1,w}([0, L])} = \frac{2}{L}.$$

If Γ is a smooth Jordan closed curve and $\Phi : [0, L] \longrightarrow \Gamma$ is the natural parametrization of Γ ($\Phi(0) = \Phi(L), \Phi'(0) = \Phi'(L)$), we set

$$\overline{w}(z, \xi) := \frac{|z - \xi|}{|\Phi(x) - \Phi(y)|}, \qquad z = e^{2\pi i x/L}, \xi = e^{2\pi i y/L}; \quad x, y \in [0, L],$$

and apply (with the aid of Theorem III.2.14) a similar argument as above on the unit circle S^1.

In both cases, (113) is a consequence of (112) and Theorem III.1.1. □

Proof of Theorem III.2.8. Let $p > 1$ be a rational number and let $n \in \mathbb{Z}_+$ be such that n/p is an integer. We denote the first $n + 1$ points of the sequence $\{x_k\}_{k=0}^{\infty}$ by $\mathcal{X}_n = \{x_{0,n}, \ldots, x_{n,n}\}$, where as in the proof of Theorem III.2.5 the points $x_{k,n}$ are located on Γ in successive order. There are exactly n subarcs $(x_{i,n}, x_{i+1,n})$. We add to $\mathcal{X}_n$ the next n/p points of the sequence $\{x_k\}$. Then there are at least $(p - 1)\, n/p$ subarcs $(x_{i,n}, x_{i+1,n})$ not containing a new point. These subarcs have length at least $\delta(\mathcal{X}_n)$. We select $(p - 1)\, n/p$ of those.

On the other hand, there are $2\,n/p$ subarcs $(x_{i,(p+1)n/p},\, x_{i+1,(p+1)n/p})$ remaining with length at least $\delta(\mathcal{X}_{(p+1)n/p})$. Consequently,

$$\frac{(p-1)\,n}{p}\,\delta(\mathcal{X}_n) + \frac{2\,n}{p}\,\delta(\mathcal{X}_{(p+1)n/p}) \le L. \tag{136}$$

Thus

$$\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n) \le \frac{p^2+p}{p^2+2p-1}\,L. \tag{137}$$

Letting $f(p)$ denote the right-hand side of (137), we see that for $p>1$ the function f attains its minimum when $p = 1+\sqrt{2}$, and $f(1+\sqrt{2}) = \frac{4+3\sqrt{2}}{4+4\sqrt{2}}\,L$, which establishes (114).

Let $\mathcal{X}_{n_k}$ be a subsequence of configurations such that $\lim_{k\to\infty} n_k\,\delta(\mathcal{X}_{n_k}) = c$. Notice that we cannot apply (136) directly because we cannot assume that n_k/p is an integer. Let $\lfloor x\rfloor$ denote the integral part of x and let $\{x\} := x - \lfloor x\rfloor$. Then we get

$$\left(n_k - \left\lfloor\frac{n_k}{p}\right\rfloor\right)\delta(\mathcal{X}_{n_k}) + 2\left\lfloor\frac{n_k}{p}\right\rfloor\delta(\mathcal{X}_{n_k+\lfloor n_k/p\rfloor}) \le L. \tag{138}$$

Since

$$\left|\left(n_k - \left\lfloor\frac{n_k}{p}\right\rfloor\right)\delta(\mathcal{X}_{n_k}) - \frac{(p-1)}{p}\,n_k\,\delta(\mathcal{X}_{n_k})\right| = \left\{\frac{n_k}{p}\right\}\delta(\mathcal{X}_{n_k}) \le \delta(\mathcal{X}_{n_k}),$$

it follows that

$$\lim_{k\to\infty}\left(n_k - \left\lfloor\frac{n_k}{p}\right\rfloor\right)\delta(\mathcal{X}_{n_k}) = \frac{(p-1)}{p}\,c. \tag{139}$$

Similarly,

$$\left|(p+1)\left\lfloor\frac{n_k}{p}\right\rfloor\delta(\mathcal{X}_{n_k+\lfloor n_k/p\rfloor}) - \left(n_k + \left\lfloor\frac{n_k}{p}\right\rfloor\right)\delta(\mathcal{X}_{n_k+\lfloor n_k/p\rfloor})\right| \le p\,\delta(\mathcal{X}_{n_k+\lfloor n_k/p\rfloor})$$

and thus

$$\liminf_{k\to\infty}\left(n_k+\left\lfloor\frac{n_k}{p}\right\rfloor\right)\delta(\mathcal{X}_{n_k+\lfloor n_k/p\rfloor})=\liminf_{k\to\infty}(p+1)\left\lfloor\frac{n_k}{p}\right\rfloor\delta(\mathcal{X}_{n_k+\lfloor n_k/p\rfloor}). \tag{140}$$

Since $\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n) \leq \liminf_{k\to\infty}(n_k+\lfloor n_k/p\rfloor)\,\delta(\mathcal{X}_{n_k+\lfloor n_k/p\rfloor})$, we obtain from (138)–(140) that

$$\frac{2}{p+1}\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n)\leq L-\frac{p-1}{p}\,c.$$

Therefore

$$\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n)\leq g(p):=\left(1+\frac{1}{p}\right)\frac{p\,(L-c)+c}{2}.$$

If $c=L$ we get immediately that $\liminf_{n\to\infty} n\,\delta(\mathcal{X}_n)\leq L/2$. The function g attains a minimum for $p=\sqrt{c/(L-c)}$ and takes the value $L/2+\sqrt{c\,(L-c)}$ at this point. This proves (115). □

Proof of Proposition III.2.9. Consider the sequence $\{a_n\}_{n=0}^{\infty}\subset[0,1]$ defined as follows:

- $a_0:=1, a_1:=0, a_2:=1/2.$
- Assuming that the first 2^n+1 points have been selected, let $a_{2^n+i}:=(2\,i-1)/2^{n+1}$, $1\leq i\leq 2^n$.

Obviously $\{a_n\}_{n=0}^{\infty}$ is a greedy best-packing sequence on $[0,1]$. However, the sequence of configurations $S_N:=\{a_n\}_{n=0}^{N}$ is not uniformly distributed since

$$\lim_{n\to\infty}\frac{\operatorname{card}(S_{3\cdot 2^{n-1}}\cap[0,1/2])}{3\cdot 2^{n-1}+1}=\lim_{n\to\infty}\frac{2^n+1}{3\cdot 2^{n-1}+1}=\frac{2}{3}\neq\frac{1}{2}.$$

Now we consider the sequence $\{b_n\}_{n=1}^{\infty}\subset[0,1]^2$ formed in the following way:

1) $b_1:=(1,1), b_2:=(0,0), b_3:=(0,1), b_4:=(1,0).$

2) Assume that the first $(2^{n-1}+1)^2$, $n\geq 1$, points have been selected.

2.1) We define the next $2^{2(n-1)}$ points as the centers of the $2^{2(n-1)}$ squares of area $2^{-2(n-1)}$ whose vertices are the first $(2^{n-1}+1)^2$ points $b_1, \ldots, b_{(2^{n-1}+1)^2}$. These $2^{2(n-1)}$ points are chosen in an arbitrary order.

2.2) Now we select the next $2^n(2^{n-1}+1)$ points to be the middle points of the edges of the $2^{2(n-1)}$ squares mentioned above. The first group of points that we add consists of those points with abscissa equal to 0. The second group is formed by those with abscissa equal to 2^{-n}. In general, the points from the i-th group have abscissa $(i-1)/2^n$. We add exactly 2^n+1 groups, and in each one of them, the points are selected in an arbitrary order.

Figure 33 illustrates the first 221 points of the sequence $\{b_n\}_{n=1}^{\infty}$.

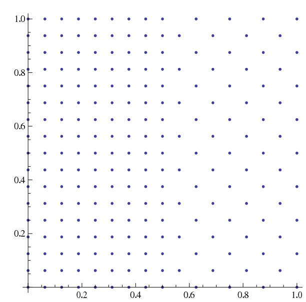

Figure 33: Greedy best-packing points for square: a counterexample to a conjecture of Bos

Using Voronoi cell decompositions one can show that $\{b_n\}_{n=1}^{\infty}$ is a greedy best-packing sequence on $[0,1]^2$. Indeed if we consider this Voronoi decomposition of $[0,1]^2$ corresponding to the points $\{b_i\}_1^N$, that is, $[0,1]^2 = \cup_{i=1}^N V_i$ where

$$V_i = \{x \in [0,1]^2 : |x - b_i| \leq |x - b_j| \text{ for all } j = 1, \ldots, N\},$$

then it is easy to see that each V_i is a convex polygon with 3, 4 or 5 sides and that b_{N+1} corresponds to a vertex of the V_i's that is of maximal distance from the points $\{b_i\}_{i=1}^N$.

To show that the sequence of configurations $T_N := \{b_i\}_{i=1}^N$ is not asymptotically uniformly distributed, we consider the subsequence of sets consisting of $N(n) = 3 \cdot 2^{2(n-1)} + 7 \cdot 2^{n-2} + 1$ points. We have that

$$\lim_{n\to\infty} \frac{\operatorname{card}(T_{N(n)} \cap [0,1/2] \times [0,1])}{N(n)} = \lim_{n\to\infty} \frac{(2^{n-1}+1)(2^n+1)}{N(n)} = \frac{2}{3} \neq \frac{1}{2}.$$

□

Using a similar argument it is possible to construct a greedy best-packing sequence on $[0,1]^p \subset \mathbb{R}^p$ that is not asymptotically uniformly distributed.

We remark that it is still plausible that for any infinite compact $A \subset \mathbb{R}^p$ there exists *at least one* greedy best-packing sequence that is asymptotically uniformly distributed on A.

III.6 Proofs of results from Subsection III.2.3

Proof of Theorem III.2.14. Given a point $x \in S^d$, we define $C(x,r) := \{y \in S^d : |y-x| \le r\}$. If σ_d denotes the normalized Lebesgue measure on S^d, then the following estimates hold (see formulas (3.7) and (3.4) in [36]):

$$\int_{S^d\setminus C(x,r)} \frac{1}{|x-y|^d} d\sigma_d(y) = \gamma_d \log\left(\frac{1}{r}\right) + \mathcal{O}(1), \quad r \to 0, \tag{141}$$

$$\sigma_d(C(x,r)) \le \frac{1}{d}\gamma_d r^d, \quad d \ge 2, \tag{142}$$

where

$$\gamma_d := \frac{\Gamma((d+1)/2)}{\Gamma(1/2)\Gamma(d/2)}. \tag{143}$$

If $d=1$, inequality (142) is not valid since $\sigma_1(C(x,r)) = \frac{2}{\pi}\arcsin(\frac{r}{2})$, but instead we have

$$\sigma_1(C(x,r)) = \gamma_1 r + \mathcal{O}(r^3), \quad r \to 0. \tag{144}$$

For $x \in S^d$ and $r > 0$,

$$\mathcal{H}_d^{d,w}(C(x,r)) = \int_{C(x,r)} w(y,y)^{-1} d\mathcal{H}_d(y) = \mathcal{H}_d(S^d) \int_{C(x,r)} w(y,y)^{-1} d\sigma_d(y).$$

Thus

$$\mathcal{H}_d^{d,w}(C(x,r)) \leq \frac{M\mathcal{H}_d(S^d)\,\gamma_d\, r^d}{d}, \quad d \geq 2, \tag{145}$$

$$\mathcal{H}_1^{1,w}(C(x,r)) \leq M\mathcal{H}_1(S^1)\gamma_1\, r + \mathcal{O}(r^3), \quad r \to 0, \tag{146}$$

where $M := \sup\{w(y,y)^{-1} : y \in S^d\}$.

Let $r \in (0,1)$ be fixed and set

$$D_i(r) := S^d \setminus C(a_i, rN^{-\frac{1}{d}}), \qquad D^N(r) := \bigcap_{i=1}^{N} D_i(r),$$

where a_i is the i-th element of the greedy (w,d)-energy sequence. From (145) and (146) we obtain that

$$\mathcal{H}_d^{d,w}(D^N(r)) \geq \mathcal{H}_d^{d,w}(S^d) - \frac{M\mathcal{H}_d(S^d)\gamma_d\, r^d}{d}, \quad d \geq 2, \tag{147}$$

$$\mathcal{H}_1^{1,w}(D^N(r)) \geq \mathcal{H}_1^{1,w}(S^1) - M\mathcal{H}_1(S^1)\gamma_1\, r + \mathcal{O}\Big(\frac{r^3}{N^2}\Big), \quad N \to \infty. \tag{148}$$

We may assume that the expressions in the right-hand side of the above inequalities are positive since we can take r sufficiently close to 0 and N sufficiently large (we will eventually let $r \to 0$ and $N \to \infty$).

Let $\epsilon > 0$. Since the function $w(x,y)/w(x,x)$ is uniformly continuous on $S^d \times S^d$, there exists $\delta > 0$ such that

$$\left|\frac{w(x,y)}{w(x,x)} - 1\right| < \epsilon, \qquad \text{for} \quad |x-y| < \delta.$$

Consider the function

$$U^w_{n,d}(x) := \sum_{i=1}^{n-1} \frac{w(x,a_i)}{|x-a_i|^d}, \quad x \in S^d, \quad n \geq 2. \tag{149}$$

From the definition of a greedy (w,d)-energy sequence we know that $U^w_{n,d}(a_n) \leq U^w_{n,d}(x)$ for all $x \in S^d$. Let $2 \leq n \leq N$ and assume that $r < \delta$. Then $C(a_i, rN^{-\frac{1}{d}}) \subset C(a_i, \delta)$ for all $1 \leq i \leq n-1$ and so

$$\int_{D^N(r)} U^w_{n,d}(x)\, d\mathcal{H}^{d,w}_d(x) \leq \sum_{i=1}^{n-1} \int_{D_i(r)} \frac{w(x,a_i)}{w(x,x)} \frac{d\mathcal{H}_d(x)}{|x-a_i|^d}$$

$$\leq \sum_{i=1}^{n-1} \left(\int_{C(a_i,\delta)\setminus C(a_i,rN^{-\frac{1}{d}})} \frac{1+\epsilon}{|x-a_i|^d}\, d\mathcal{H}_d(x) + \int_{S^d\setminus C(a_i,\delta)} \frac{w(x,a_i)}{w(x,x)} \frac{d\mathcal{H}_d(x)}{|x-a_i|^d}\right)$$

$$\leq (n-1)\Big((1+\epsilon)\mathcal{H}_d(S^d) \int_{S^d\setminus C(a_i,rN^{-\frac{1}{d}})} \frac{1}{|x-a_i|^d} d\sigma_d(x) + C(w,\delta)\Big),$$

where $C(w,\delta)$ is some constant depending on δ and w. Using (141) it follows that

$$\int_{D^N(r)} U^w_{n,d}(x)\, d\mathcal{H}^{d,w}_d(x) \leq (n-1)(1+\epsilon)\mathcal{H}_d(S^d)\Big(\frac{\gamma_d}{d}\log N - \gamma_d \log r + \mathcal{O}(1)\Big). \tag{150}$$

Therefore,

$$E^w_d(\alpha^w_{N,d}) = 2\sum_{n=2}^{N} U^w_{n,d}(a_n) \leq 2\sum_{n=2}^{N} \frac{1}{\mathcal{H}^{d,w}_d(D^N(r))} \int_{D^N(r)} U^w_{n,d}(x)\, d\mathcal{H}^{d,w}_d(x)$$

$$\leq \frac{N(N-1)}{\mathcal{H}^{d,w}_d(D^N(r))}(1+\epsilon)\mathcal{H}_d(S^d)\Big(\frac{\gamma_d}{d}\log N - \gamma_d \log r + \mathcal{O}(1)\Big).$$

Consequently, from (147) and (148) we get that for $d \geq 1$,

$$\limsup_{N\to\infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} \leq \frac{1}{\mathcal{H}_d^{d,w}(S^d) - \frac{M\mathcal{H}_d(S^d)\gamma_d r^d}{d}} (1+\epsilon)\mathcal{H}_d(S^d)\frac{\gamma_d}{d}.$$

After letting $r \to 0$ and $\epsilon \to 0$ we obtain that

$$\limsup_{N\to\infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} \leq \frac{\mathcal{H}_d(S^d)\,\gamma_d}{\mathcal{H}_d^{d,w}(S^d)\,d} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d^{d,w}(S^d)}.$$

Finally, since $\mathcal{E}_d^w(S^d, N) \leq E_d^w(\alpha_{N,d}^w)$ for all N, applying (116) it follows that

$$\lim_{N\to\infty} \frac{E_d^w(\alpha_{N,d}^w)}{N^2 \log N} = \frac{\mathrm{Vol}(B^d)}{\mathcal{H}_d^{d,w}(S^d)}.$$

The statement about the weak-star convergence of the normalized counting measure associated with $\alpha_{N,d}^w$ is also an application of Theorem III.2.12. □

Remark III.6.1. *It is not difficult to see that greedy k_s-energy sequences on $S^d \subset \mathbb{R}^{d+1}$ satisfy the following property for any $s \in [0,\infty)$. If $\{a_n\}_{n=1}^{\infty}$ denotes such a sequence, then for each integer $m \geq 1$, the choice of a_{2m} is unique and $a_{2m} = -a_{2m-1}$.*

It is also easily seen that on S^2 the configuration formed by the first six points of any greedy k_s-energy sequence does not depend on s and is a rotation of the configuration $\{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\}$.

Proof of Theorem III.2.15. If $R := \mathrm{diam}(A)$ is the diameter of A, $r < R$ and $x \in A$, then

$$\int_{A\setminus B(x,r)} \frac{1}{|x-y|^p}\,dy \leq \int_{B(x,R)\setminus B(x,r)} \frac{1}{|x-y|^p}\,dy = \mathcal{H}_{p-1}(S^{p-1})\log(R/r). \qquad (151)$$

Defining

$$D_i(r) := A \setminus B(a_i, rN^{-\frac{1}{p}}), \qquad D^N(r) := \bigcap_{i=1}^{N} D_i(r),$$

where a_i is the i-th element of the greedy (w,p)-energy sequence, the proof of Theorem III.2.14 is applicable here and yields the result. For instance, using (151) the expression similar to (150) is

$$\int_{D^N(r)} U_{n,p}^w(x)d\mathcal{H}_p^{p,w}(x) \leq (n-1)(1+\epsilon)\mathcal{H}_{p-1}(S^{p-1})\Big(\frac{1}{p}\log N - \log r + \mathcal{O}(1)\Big). \quad (152)$$

Since $\mathrm{Vol}(B^p) = p^{-1}\mathcal{H}_{p-1}(S^{p-1})$, (120) follows from (152) and Theorem III.2.12. The limit (121) is a consequence of (120) and Theorem III.2.12. □

Proof of Theorem III.2.16. We follow closely the argument on page 20 of [8]. The following result is known as Frostman's lemma (see [45]).

Lemma III.6.2. *Let $\delta > 0$ and A be a Borel set in $\mathbb{R}^p$. Then $\mathcal{H}_\delta(A) > 0$ if and only if there exists $\mu \in \mathcal{M}^+(A)$ such that $\mu(A) > 0$ and*

$$\mu(B(x,r)) \leq r^\delta, \qquad x \in \mathbb{R}^p, \quad r > 0, \quad (153)$$

where $B(x,r)$ denotes the open ball centered at x and radius r. Furthermore, one can select μ so that $\mu(A) \geq c_{p,\delta}\,\mathcal{H}_\delta^\infty(A)$, where $c_{p,\delta}$ is independent of A.

Let μ be a measure from Lemma III.6.2, and set $r_0 := (\mu(A)/\,2N)^{1/\delta}$. Define the sets

$$D_j := B(a_j, r_0), \qquad \mathcal{D}_N := A \setminus \bigcup_{j=1}^{N-1} D_j,$$

where a_j denotes the j-th element of the greedy (w,s)-energy sequence. Then, using (153),

$$\mu(\mathcal{D}_N) \geq \mu(A) - \sum_{j=1}^{N-1}\mu(D_j) \geq \mu(A) - (N-1)r_0^\delta > \frac{\mu(A)}{2} > 0. \quad (154)$$

Consider the function $U_{N,s}^w$ defined in (149). From (154) we obtain

$$U_{N,s}^w(a_N) \leq \frac{1}{\mu(\mathcal{D}_N)}\int_{\mathcal{D}_N} U_{N,s}^w(x)d\mu(x) \leq \frac{2}{\mu(A)}\sum_{j=1}^{N-1}\int_{\mathcal{D}_N}\frac{w(x,a_j)}{|x-a_j|^s}\,d\mu(x)$$

$$\leq \frac{2\|w\|}{\mu(A)} \sum_{j=1}^{N-1} \int_{A\setminus D_j} \frac{1}{|x-a_j|^s}\, d\mu(x),$$

where $\|w\| := \sup\{w(x,y) : x, y \in A\}$. Set $R := \operatorname{diam}(A)$. Then $\mu(A) \leq R^\delta$ by (153). If $y \in A$ and $r \in (0, R]$, then

$$\int_{A\setminus B(y,r)} \frac{1}{|x-y|^s}\, d\mu(x) \leq \int_0^{r^{-s}} \mu(\{x \in A : \frac{1}{|x-y|^s} > t\})dt$$

$$\leq \frac{\mu(A)}{R^s} + \int_{R^{-s}}^{r^{-s}} \mu(B(y, t^{-1/s}))dt \leq R^{\delta-s} + \int_{R^{-s}}^{r^{-s}} t^{-\delta/s} dt$$

$$\leq \begin{cases} R^{\delta-s} + \frac{s}{s-\delta} r^{\delta-s}, \text{ if } s > \delta, \\ \\ 1 + \delta \log\left(\frac{R}{r}\right), \text{ if } s = \delta. \end{cases}$$

Therefore, for $s > \delta$ we obtain

$$U^w_{N,s}(a_N) \leq \frac{2\|w\|}{\mu(A)}(N-1)\Big(R^{\delta-s} + \frac{s}{s-\delta} r_0^{1-s/\delta}\Big) \leq C_1\|w\|\Big(\frac{N}{\mu(A)}\Big)^{s/\delta}, \qquad (155)$$

where $C_1 > 0$ is a constant independent of N and w. If $s = \delta$, then

$$U^w_{N,\delta}(a_N) \leq \frac{2\|w\|}{\mu(A)}(N-1)\Big(1 + \delta\log\Big(\frac{R}{r_0}\Big)\Big) \leq C_2\|w\|\Big(\frac{N\log N}{\mu(A)}\Big), \qquad (156)$$

where $C_2 > 0$ is also independent of N and w. The sequence $\{U^w_{i,s}(a_i)\}_N$ is non-decreasing since

$$U^w_{i+1,s}(a_{i+1}) \geq U^w_{i,s}(a_i) + \frac{w(a_{i+1}, a_i)}{|a_{i+1} - a_i|^s}, \qquad i \geq 1.$$

Therefore, applying $\mu(A) \geq c_{p,\delta}\, \mathcal{H}^\infty_\delta(A)$ and (155)–(156), Theorem III.2.16 readily follows from

$$E^w_s(\alpha^w_{N,s}) = 2\sum_{i=2}^{N} U^w_{i,s}(a_i).$$

□

Proof of Corollary III.2.17. Since $E_s^w(\alpha_{N,s}^w) \geq \mathcal{E}_s^w(A, N)$ for every N and $s \geq d$, the lower bounds in (122) and (123) follow from (118) and (116), respectively. The upper bounds follow from Theorem III.2.16. □

Proof of Corollary III.2.18. Assume the existence of a point $a \in A$ and $\epsilon > 0$ such that $\{a_n\}_{n=1}^{\infty} \cap B(a, \epsilon) = \emptyset$. Let $\alpha_{N,s}^w = \{a_1, \ldots, a_N\}$. Then

$$E_s^w(\alpha_{N,s}^w) = 2 \sum_{1 \leq i < j \leq N} \frac{w(a_i, a_j)}{|a_i - a_j|^s} \leq 2 \sum_{j=2}^{N} \sum_{i=1}^{j-1} \frac{w(a_i, x)}{|a_i - x|^s},$$

where the last inequality is valid for any $x \in A$. In particular, taking $x = a$ we get

$$E_s^w(\alpha_{N,s}^w) \leq \frac{\|w\|}{\epsilon^s} N(N-1),$$

where $\|w\| = \sup\{w(x, y) : x, y \in A\}$. This inequality contradicts the first inequalities in (122) and (123). □

Proof of Proposition III.2.19. Assume that there exists a subinterval $I = [c, d] \subset [a, b]$ for which (124) is not satisfied. Let N_l be a subsequence such that

$$\lim_{l \to \infty} \frac{(\text{card}\{1 \leq n \leq N_l : a_n \in I\})^{1+\frac{1}{s}}}{N_l} = 0.$$

Select $\epsilon > 0$ sufficiently small so that $J = [c + \epsilon/2, d - \epsilon/2] \subset I$ is not empty. If we define $\nu_l := \text{card}\{1 \leq n \leq N_l : a_n \in J\}$, then there exists a subinterval of J of length at least $(d - c - \epsilon)/(\nu_l + 1)$ not containing any point from $\{a_n \in J : 1 \leq n \leq N_l\}$. Let x_l be the center of such a subinterval. We have, for $\alpha_{N_l,s}^w = \{a_1, \ldots, a_{N_l}\}$,

$$E_s^w(\alpha_{N_l,s}^w) = 2 \sum_{n=2}^{N_l} U_{n,s}^w(a_n) \leq 2 \sum_{n=2}^{N_l} U_{n,s}^w(x_l) = 2 \sum_{n=2}^{N_l} \sum_{i=1}^{n-1} \frac{w(x_l, a_i)}{|x_l - a_i|^s} \qquad (157)$$

$$\leq 2\|w\|\Big[\frac{N_l-1}{|x_l-a_1|^s}+\frac{N_l-2}{|x_l-a_2|^s}+\cdots+\frac{1}{|x_l-a_{N_l-1}|^s}\Big]=2\|w\|(S_{I,l}+T_{I,l}),$$

where $\|w\|=\sup\{w(x,y):x,y\in[a,b]\}$ and

$$S_{I,l}:=\sum_{a_i\in I,\,1\leq i\leq N_l-1}\frac{N_l-i}{|x_l-a_i|^s}\,,\qquad T_{I,l}:=\sum_{a_i\notin I,\,1\leq i\leq N_l-1}\frac{N_l-i}{|x_l-a_i|^s}.$$

For each $a_i\notin I$, $|a_i-x_l|\geq\epsilon/2$; hence

$$2\,T_{I,l}\leq(2/\epsilon)^s\,N_l^2. \tag{158}$$

If $a_i\in I, 1\leq i\leq N_l-1$, then $|a_i-x_l|\geq(d-c-\epsilon)/2(\nu_l+1)$. Therefore, if we define $\tau_l:=\mathrm{card}\{1\leq i\leq N_l-1:a_i\in I\}$, it follows that

$$2\,S_{I,l}\leq\frac{2^{s+1}}{(d-c-\epsilon)^s}\,(\nu_l+1)^s\,\tau_l\,N_l. \tag{159}$$

By hypothesis, $\tau_l^{1+s}/N_l^s\to 0$ as $l\to\infty$. We deduce from (157)–(159) that

$$\lim_{l\to\infty}\frac{E_s^w(\alpha_{N_l,s}^w)}{N_l^{1+s}}=0,$$

which contradicts the fact that

$$\liminf_{N\to\infty}\frac{E_s^w(\alpha_{N,s}^w)}{N^{1+s}}\geq\lim_{N\to\infty}\frac{\mathcal{E}_s^w([a,b],N)}{N^{1+s}}=\frac{2\zeta(s)}{\mathcal{H}_1^{s,w}([a,b])^s}>0.$$

□

CHAPTER IV

MULTIPLE ORTHOGONAL POLYNOMIALS ON STARLIKE SETS

IV.1 Introduction and statement of main results

In this chapter we present the results on the algebraic and asymptotic properties of multiple orthogonal polynomials associated with a system of two measures supported on starlike sets. The main results are described in this section. We start with the definition of the orthogonality measures and the associated polynomials.

Let

$$S_0 := \bigcup_{k=0}^{2} [0, \alpha] \exp(2\pi i k/3), \tag{160}$$

where $\alpha > 0$ is arbitrary and finite. Assume that s_1 is a complex-valued function defined on S_0 such that

$$s_1 \geq 0 \quad \text{on} \quad (0, \alpha), \qquad s_1 \in L^1(0, \alpha), \tag{161}$$

$$s_1(e^{\frac{2\pi i}{3}} z) = e^{\frac{4\pi i}{3}} s_1(z), \quad z \in S_0 \setminus \{0, \alpha, e^{\frac{2\pi i}{3}} \alpha, e^{\frac{4\pi i}{3}} \alpha\}. \tag{162}$$

Set

$$f(z) := z^2 \int_{-b}^{-a} \frac{s_2(t)}{z^3 - t^3} \, dt, \tag{163}$$

where s_2 is a real-valued function defined on $[-b, -a] \subset (-\infty, 0]$ that satisfies $s_2 \in L^1(-b, -a)$. We assume that $0 < a < b < \infty$. Notice that f satisfies

$$f(e^{\frac{2\pi i}{3}} z) = e^{\frac{4\pi i}{3}} f(z).$$

We assume of course that the measures $s_1(t)\,dt$ and $s_2(t)\,dt$ are non-trivial (i.e. their supports contain infinitely many points). We will also assume that

$$s_2 \geq 0 \qquad \text{on} \quad [-b, -a]. \tag{164}$$

We next construct the following weights

$$W_0(z) := s_1(z), \qquad z \in S_0, \tag{165}$$

$$W_1(z) := f(z)\, s_1(z), \qquad z \in S_0, \tag{166}$$

and define the sequence of *monic* polynomials $\{Q_n\}_{n=0}^{\infty}$ of lowest degree that satisfy the following conditions:

$$\deg Q_n \leq n, \tag{167}$$

$$\int_{S_0} Q_{2n}(t)\, t^k\, W_i(t)\, dt = 0\,, \qquad k = 0, \ldots, n-1, \quad i = 0, 1, \tag{168}$$

$$\int_{S_0} Q_{2n+1}(t)\, t^k\, W_0(t)\, dt = 0, \qquad k = 0, \ldots, n, \tag{169}$$

$$\int_{S_0} Q_{2n+1}(t)\, t^k\, W_1(t)\, dt = 0, \qquad k = 0, \ldots, n-1. \tag{170}$$

These are the polynomials whose algebraic and asymptotic properties we investigate. The first result concerns their degree and the location of their zeros.

Proposition IV.1.1. *The degree of each polynomial* Q_n *is maximal, i.e.* $\deg Q_n = n$. *Moreover, if* $n = 3j$, *then* Q_n *has exactly* j *simple zeros on the interval* $(0, \alpha)$. *If* $n = 3j + 1$, *then* Q_n *has a simple zero at the origin and* j *simple zeros on* $(0, \alpha)$. *Finally, if* $n = 3j + 2$, *then* Q_n *has a double zero at the origin and* j *simple zeros on* $(0, \alpha)$. *The remaining zeros of* Q_n *are simple, are located on the rays* $(0, \alpha)\exp(2\pi i/3)$, $(0, \alpha)\exp(4\pi i/3)$, *and are rotations of the zeros on* $(0, \alpha)$.

The proof of Proposition IV.1.1 is given in Section IV.2; it heavily relies on Lemma IV.2.4.

The following figures show the zeros of the polynomials $Q_n, 21 \leq n \leq 24$, associated with the following weights:

$$s_1 \equiv 1 \quad \text{on} \quad [0,5], \qquad s_2 \equiv 1 \quad \text{on} \quad [-2,-1]. \tag{171}$$

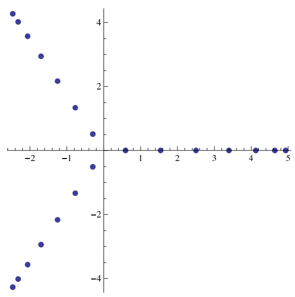

Figure 34: Zeros of Q_{21}

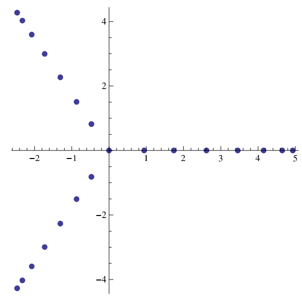

Figure 35: Zeros of Q_{22}

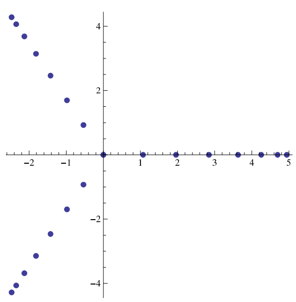

Figure 36: Zeros of Q_{23}

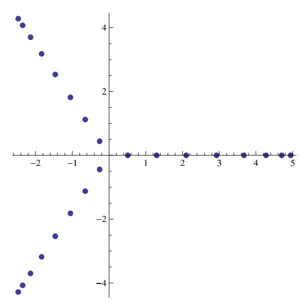

Figure 37: Zeros of Q_{24}

One of the most important properties of the polynomials Q_n is the fact that they satisfy a three-term recurrence relation of third order, as the following result shows.

Proposition IV.1.2. *The monic polynomials Q_n satisfy the following recurrence relation*

$$zQ_n = Q_{n+1} + a_n Q_{n-2}, \qquad n \geq 2, \quad a_n \in \mathbb{R}, \tag{172}$$

where

$$Q_j(z) = z^j, \quad j = 0, 1, 2. \tag{173}$$

The coefficients a_n are given by the formulas

$$a_{2n} = \frac{\int_0^\alpha t^n \, Q_{2n}(t) \, s_1(t) \, dt}{\int_0^\alpha t^{n-1} \, Q_{2n-2}(t) \, s_1(t) \, dt}, \tag{174}$$

$$a_{2n+1} = \frac{\int_0^\alpha t^n \, Q_{2n+1}(t) \, s_1(t) f(t) \, dt}{\int_0^\alpha t^{n-1} \, Q_{2n-1}(t) \, s_1(t) f(t) \, dt}. \tag{175}$$

Moreover, $a_n > 0$ for all $n \geq 2$.

Proposition IV.1.2 is proved in Section IV.2. One can show, using orthogonality properties of the polynomials Q_n with respect to varying measures (Proposition IV.3.6), that each integral in (174) and (175) is positive.

The following functions, called *functions of second type*, will play a crucial role in the asymptotic analysis of the polynomials Q_n. They are defined as follows:

$$\Psi_n(z) := \int_{S_0} \frac{Q_n(t)}{t - z} \, s_1(t) \, dt. \tag{176}$$

Observe that the functions Ψ_n satisfy the following immediate properties (see also Corollary IV.3.3):

$$\begin{cases} \Psi_n \in H(\overline{\mathbb{C}} \setminus S_0), \\ \\ \Psi_{2n}(z) = O(1/z^{n+1}), \qquad z \to \infty, \\ \\ \Psi_{2n+1}(z) = O(1/z^{n+2}), \quad z \to \infty. \end{cases} \tag{177}$$

(Throughout this chapter $H(\Omega)$ denotes the space of all holomorphic functions on an open set $\Omega \subset \overline{\mathbb{C}}$.) The functions Ψ_n also satisfy orthogonality conditions (see Propositions IV.2.2 and IV.3.4). It is important for our study to determine the exact number of zeros of each function Ψ_n outside the starlike set S_0, and their location. The following result gives the answers to these questions.

Proposition IV.1.3. *For each $j \in \{0,1,2,3,5\}$, the function Ψ_{6l+j} has exactly $3l$ simple zeros in $\mathbb{C} \setminus S_0$, of which l zeros are located in the interval $(-b,-a)$, and the remaining $2l$ zeros are rotations of these l zeros by angles of $2\pi/3$ and $4\pi/3$; Ψ_{6l+j} has no other zeros in $\mathbb{C} \setminus S_0$. The function Ψ_{6l+4} has exactly $3l+3$ simple zeros in $\mathbb{C} \setminus S_0$, of which $l+1$ zeros are located in the interval $(-b,-a)$, and the remaining $2l+2$ zeros are rotations of these $l+1$ zeros by angles of $2\pi/3$ and $4\pi/3$; Ψ_{6l+4} has no other zeros in $\mathbb{C} \setminus S_0$.*

This proposition is proved in Section IV.3, where other properties of the functions Ψ_n are described.

Notation: Let $Q_{n,2}$ denote the *monic* polynomial whose zeros coincide with the finite zeros of Ψ_n outside S_0, so that $\deg Q_{n,2} = 3l$ if $n = 6l+j$, $j \in \{0,1,2,3,5\}$, while $\deg Q_{n,2} = 3l+3$ if $n = 6l+4$.

The following result asserts that for consecutive values of n the zeros of the polynomials Q_n actually *interlace*, and the same is true for the zeros of $Q_{n,2}$. This property is relevant for analyzing the ratio asymptotic behavior of the sequences $\{Q_n\}_{n\geq 0}$ and $\{Q_{n,2}\}_{n\geq 0}$, since it implies, in particular, that the families of functions

$$\left\{\frac{Q_{n+1}}{Q_n}\right\}, \qquad \left\{\frac{Q_{n+1,2}}{Q_{n,2}}\right\},$$

are normal in the regions $\mathbb{C} \setminus S_0$ and $\mathbb{C} \setminus S_1$, respectively, where

$$S_1 := \bigcup_{k=0}^{2} [-b,-a] \exp(2\pi i k/3). \tag{178}$$

We have:

Theorem IV.1.4. *For every $n \geq 0$, the polynomials Q_n and Q_{n+1} do not have any common zeros in $S_0 \setminus \{0\}$. Moreover, there is exactly one zero of Q_{n+1} between two consecutive zeros of Q_n in $(0,\alpha)$. Conversely, there is exactly one zero of Q_n between* *two consecutive zeros of* Q_{n+1} *in* $(0, \alpha)$*. Therefore, the zeros of* Q_n *and* Q_{n+1} *interlace in* $S_0 \setminus \{0\}$*.*

Additionally, for every $n \geq 0$*, the functions* Ψ_n *and* Ψ_{n+1} *do not have any common zeros in* S_1*. There is exactly one zero of* Ψ_{n+1} *between two consecutive zeros of* Ψ_n *in* $(-b, -a)$*, and vice versa. Therefore, the zeros of* Ψ_n *and* Ψ_{n+1} *interlace in* S_1*.*

We can determine exactly how the zeros of Q_n interlace, thanks to the fact that the recurrence coefficients a_n are all positive.

Proposition IV.1.5. *Let the roots of the polynomials* $Q_{3k}, Q_{3k+1}, Q_{3k+2}$ *and* Q_{3k+3}*, in the interval* $(0, \alpha)$*, be denoted, respectively, as follows:*

$$x_1^{(3k)} < x_2^{(3k)} < x_3^{(3k)} < \cdots < x_{k-1}^{(3k)} < x_k^{(3k)},$$

$$x_1^{(3k+1)} < x_2^{(3k+1)} < x_3^{(3k+1)} < \cdots < x_{k-1}^{(3k+1)} < x_k^{(3k+1)},$$

$$x_1^{(3k+2)} < x_2^{(3k+2)} < x_3^{(3k+2)} < \cdots < x_{k-1}^{(3k+2)} < x_k^{(3k+2)},$$

$$x_1^{(3k+3)} < x_2^{(3k+3)} < x_3^{(3k+3)} < \cdots < x_k^{(3k+3)} < x_{k+1}^{(3k+3)}.$$

Then

$$x_1^{(3k)} < x_1^{(3k+1)} < x_2^{(3k)} < x_2^{(3k+1)} < \cdots < x_k^{(3k)} < x_k^{(3k+1)}, \tag{179}$$

$$x_1^{(3k+1)} < x_1^{(3k+2)} < x_2^{(3k+1)} < x_2^{(3k+2)} < \cdots < x_k^{(3k+1)} < x_k^{(3k+2)}, \tag{180}$$

$$x_1^{(3k+3)} < x_1^{(3k+2)} < x_2^{(3k+3)} < x_2^{(3k+2)} < \cdots < x_k^{(3k+2)} < x_{k+1}^{(3k+3)}. \tag{181}$$

Theorem IV.1.4 and Proposition IV.1.5 are proved in Section IV.4 (see also Proposition IV.4.1).

The following figures show the interlacing of the zeros of certain polynomials associated with the weights (171):

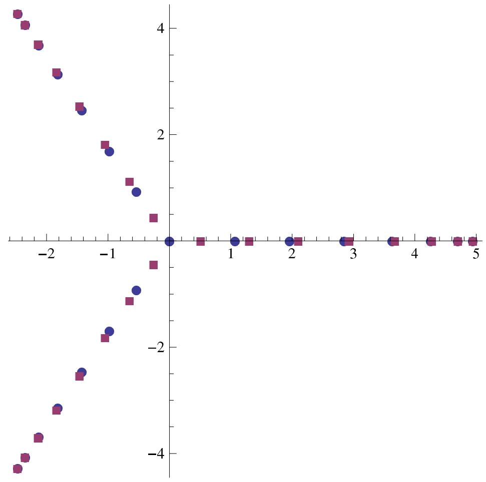

Figure 38: Zeros of Q_{23} (circles) and Q_{24} (squares)

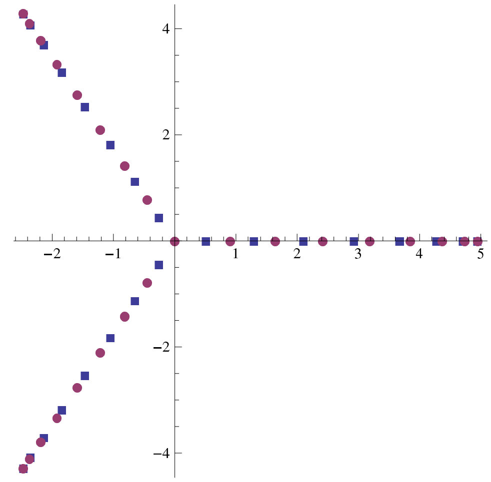

Figure 39: Zeros of Q_{24} (squares) and Q_{25} (circles)

We next describe the ratio asymptotics of the polynomials Q_n and $Q_{n,2}$, and the limiting behavior of the recurrence coefficients a_n. In order to state these results, we need to introduce the following polynomials:

$$P_{3k}(\tau) := Q_{3k}(\sqrt[3]{\tau}), \tag{182}$$

$$P_{3k+1}(\tau) := \frac{Q_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}, \tag{183}$$

$$P_{3k+2}(\tau) := \frac{Q_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}}, \tag{184}$$

$$P_{n,2}(\tau) := Q_{n,2}(\sqrt[3]{\tau}). \tag{185}$$

The fact that P_n and $P_{n,2}$ are indeed polynomials is a consequence of Propositions IV.1.1 and IV.1.3. Observe that the zeros of P_n and $P_{n,2}$ are contained in the interval $(0, \alpha^3)$ and $(-b^3, -a^3)$, respectively.

Theorem IV.1.6. *Assume that $s_1 > 0$ a.e. on $[0, \alpha]$ and $s_2 > 0$ a.e. on $[-b, -a]$. Then for each $i \in \{0, \ldots, 5\}$, the following limits hold:*

$$\lim_{k\to\infty} \frac{P_{6k+i+1}(z)}{P_{6k+i}(z)} = \widetilde{F}_1^{(i)}(z), \qquad z \in \mathbb{C} \setminus [0, \alpha^3], \tag{186}$$

$$\lim_{k\to\infty} \frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)} = \widetilde{F}_2^{(i)}(z), \qquad z \in \mathbb{C} \setminus [-a^3, -b^3], \tag{187}$$

where convergence is uniform on compact subsets of the indicated regions [1]. *Moreover (cf. (172)),*

$$\lim_{k\to\infty} a_{6k+i} = \begin{cases} -C_1^{(i)}, & \textit{for } i \in \{0,1,3,4\}, \\ \\ -C_0^{(i)}, & \textit{for } i \in \{2,5\}, \end{cases} \tag{188}$$

where

$$\widetilde{F}_1^{(i)}(z) = \begin{cases} 1 + C_1^{(i)}/z + O(1/z^2), & \textit{for } i \in \{0,1,3,4\}, \\ \\ z + C_0^{(i)} + O(1/z), & \textit{for } i \in \{2,5\}, \end{cases} \tag{189}$$

is the Laurent expansion at ∞ *of* $\widetilde{F}_1^{(i)}$. *Consequently,*

$$\lim_{k\to\infty} \frac{Q_{6k+i+1}(z)}{Q_{6k+i}(z)} = z\, \widetilde{F}_1^{(i)}(z^3), \qquad z \in \mathbb{C} \setminus S_0, \quad i \in \{0,1,3,4\}, \tag{190}$$

$$\lim_{k\to\infty} \frac{Q_{6k+i+1}(z)}{Q_{6k+i}(z)} = \frac{\widetilde{F}_1^{(i)}(z^3)}{z^2}, \qquad z \in \mathbb{C} \setminus S_0, \quad i \in \{2,5\}, \tag{191}$$

$$\lim_{k\to\infty} \frac{Q_{6k+i+1,2}(z)}{Q_{6k+i,2}(z)} = \widetilde{F}_2^{(i)}(z^3), \qquad z \in \mathbb{C} \setminus S_1, \quad i \in \{0,\ldots,5\}, \tag{192}$$

hold uniformly on compact subsets of the indicated regions.

As we remarked in the introduction of this thesis, the proof of the ratio asymptotic behavior of the polynomials Q_n and $Q_{n,2}$ relies on the application of results on ratio and relative asymptotics of polynomials orthogonal with respect to varying measures (see the discussion after Lemma IV.5.4). These auxiliary results from [6] allow us to find a system of boundary value equations satisfied by the limiting functions $\widetilde{F}_1^{(i)}, \widetilde{F}_2^{(i)}$ (see Proposition IV.5.5). The existence of the limits (186)–(187) then follows by

[1] If the degree of the numerator equals the degree of the denominator, then convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus [0, \alpha^3]$ or $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$.

proving that this system has a unique solution (Proposition IV.5.7). We do this by applying the maximum and minimum principle for harmonic functions.

We also describe in Proposition IV.5.8 the ratio asymptotic behavior of the functions of second type Ψ_n, as well as the ratio asymptotic behavior of the polynomials $p_n, p_{n,2}$ defined in (318) (these polynomials are "orthonormal versions" of the polynomials $P_n, P_{n,2}$ defined in (182)–(185), see Proposition IV.5.3) and their leading coefficients.

Several relations can be established among the limiting functions $\widetilde{F}_1^{(i)}, \widetilde{F}_2^{(i)}$, and the limiting values of the recurrence coefficients (see also the boundary value properties described in Proposition IV.5.5).

Let us define

$$a^{(i)} := \lim_{k\to\infty} a_{6k+i}, \qquad 0 \leq i \leq 5.$$

Proposition IV.1.7. *The following relations among the functions* $\widetilde{F}_j^{(i)}$ *are valid:*

$$\widetilde{F}_1^{(2)}(z) = z\widetilde{F}_1^{(0)}(z), \qquad \widetilde{F}_1^{(5)}(z) = z\widetilde{F}_1^{(3)}(z), \tag{193}$$

$$\widetilde{F}_1^{(0)}\widetilde{F}_1^{(1)} = \widetilde{F}_1^{(3)}\widetilde{F}_1^{(4)}, \qquad \widetilde{F}_1^{(1)}\widetilde{F}_1^{(2)} = \widetilde{F}_1^{(4)}\widetilde{F}_1^{(5)}, \qquad \widetilde{F}_1^{(2)}\widetilde{F}_1^{(3)} = \widetilde{F}_1^{(5)}\widetilde{F}_1^{(0)}, \tag{194}$$

$$\frac{1-\widetilde{F}_1^{(3)}}{1-\widetilde{F}_1^{(0)}} = \frac{a^{(3)}}{a^{(0)}}, \qquad \frac{1-\widetilde{F}_1^{(4)}}{1-\widetilde{F}_1^{(1)}} = \frac{a^{(4)}}{a^{(1)}}, \qquad \frac{z-\widetilde{F}_1^{(5)}(z)}{z-\widetilde{F}_1^{(2)}(z)} = \frac{a^{(5)}}{a^{(2)}}, \tag{195}$$

$$\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(2)}, \qquad \widetilde{F}_2^{(3)} = \widetilde{F}_2^{(5)}, \tag{196}$$

$$\widetilde{F}_2^{(0)}\widetilde{F}_2^{(1)} = \widetilde{F}_2^{(3)}\widetilde{F}_2^{(4)}, \qquad \widetilde{F}_2^{(1)}\widetilde{F}_2^{(2)} = \widetilde{F}_2^{(4)}\widetilde{F}_2^{(5)}, \qquad \widetilde{F}_2^{(2)}\widetilde{F}_2^{(3)} = \widetilde{F}_2^{(5)}\widetilde{F}_2^{(0)}. \tag{197}$$

Furthermore, the functions $\widetilde{F}_1^{(i)}$, $i \in \{0,\ldots,5\}$, *are all distinct, and the functions* $\widetilde{F}_2^{(i)}$, $i \in \{0,1,3,4\}$, *are also distinct.*

For every $i \in \{0,\ldots,5\}$, $a^{(i)} > 0$, *and the following relations hold:*

$$a^{(0)} = a^{(2)}, \qquad a^{(3)} = a^{(5)}, \qquad a^{(0)} + a^{(1)} = a^{(3)} + a^{(4)}. \tag{198}$$

The following inequalities also hold:

$$a^{(0)} \neq a^{(3)}, \quad a^{(0)} \neq a^{(4)}, \quad a^{(1)} \neq a^{(3)}, \quad a^{(1)} \neq a^{(4)}.$$

In fact, we will show that $a^{(4)} > a^{(1)}$, and therefore by (198) we also have $a^{(0)} > a^{(3)}$ (see Remark IV.6.2).

From (172) we immediately deduce that the following relations also hold everywhere in $\mathbb{C} \setminus S_0$:

$$\widetilde{F}_1^{(0)} \widetilde{F}_1^{(1)} (z - \widetilde{F}_1^{(2)}) = a^{(2)},$$

$$\widetilde{F}_1^{(1)} \widetilde{F}_1^{(2)} (1 - \widetilde{F}_1^{(3)}) = a^{(3)},$$

$$\widetilde{F}_1^{(2)} \widetilde{F}_1^{(3)} (1 - \widetilde{F}_1^{(4)}) = a^{(4)},$$

$$\widetilde{F}_1^{(3)} \widetilde{F}_1^{(4)} (z - \widetilde{F}_1^{(5)}) = a^{(5)},$$

$$\widetilde{F}_1^{(4)} \widetilde{F}_1^{(5)} (1 - \widetilde{F}_1^{(0)}) = a^{(0)},$$

$$\widetilde{F}_1^{(5)} \widetilde{F}_1^{(0)} (1 - \widetilde{F}_1^{(1)}) = a^{(1)}.$$

Theorem IV.1.6, Proposition IV.1.7, and other related results concerning ratio asymptotics of the polynomials Q_n and $Q_{n,2}$, are proved in Section IV.5.

Table 1 below lists the computed values of the recurrence coefficients $a_n, 2 \leq n \leq 24$, associated with the weights (171), while Table 2 lists the values of those coefficients associated with the weights

$$s_1 \equiv 1 \quad \text{on} \quad [0, 5], \qquad s_2 \equiv 1 \quad \text{on} \quad [-10, -1].$$

Observe that these numerical computations are consistent with the limiting relations (198).

We remark that at least one of the following inequalities must hold:

$$a^{(0)} \neq a^{(1)}, \qquad a^{(3)} \neq a^{(4)}, \tag{199}$$

otherwise all the limiting values $a^{(i)}$ would be equal, which is impossible. However, the numerical computations in Tables 1 and 2 suggest that both inequalities are true. So far we have not been able to show this.

Table 1:

n	a_n
2	31.250000000000000
3	13.117294027817388
4	27.061277400754041
5	6.9566203276935465
6	32.092059220810601
7	1.2666533338178369
8	30.232554389281338
9	9.4134893772834573
10	23.491822238001053
11	7.8798482592518220
12	31.448198155175568
13	0.9977706208003094
14	30.298124895839139
15	9.0993421406653429
16	23.195484548469524
17	8.1836828622050826
18	31.167379058897494
19	0.9455998438654098
20	30.418396962231367
21	8.9595044331899466
22	23.098955251172832
23	8.3152993124024974
24	31.044243836574903

Table 2:

n	a_n
2	31.250000000000000
3	23.654726542657228
4	16.523844885914200
5	17.731583489815896
6	26.357636064321322
7	10.512172941164216
8	22.629265657982933
9	21.483061273650794
10	14.316753958288949
11	18.833016666617695
12	25.360935576606374
13	9.8243828701623133
14	23.362195866879705
15	21.014090866438814
16	13.857149443187150
17	19.377667630986058
18	25.137061245771417
19	9.4515850248265041
20	23.841100160945267
21	20.743499158036020
22	13.712837073322134
23	19.610502893814671
24	25.051985211064199

Tables 3 and 4 below correspond, respectively, to the following weights:

$$s_1 \equiv 1 \quad \text{on} \quad [0,5], \qquad s_2 \equiv 1 \quad \text{on} \quad [-30,-1],$$

$$s_1 \equiv 1 \quad \text{on} \quad [0,5], \qquad s_2 \equiv 1 \quad \text{on} \quad [-100,-1].$$

The values displayed in these tables not only support the conjecture (199), but they also suggest that the following phenomenon holds: For a and α fixed,

$$a^{(2)} - a^{(3)} \longrightarrow 0, \qquad a^{(1)} - a^{(4)} \longrightarrow 0, \qquad \text{as} \quad b \longrightarrow \infty.$$

Table 3:

n	a_n
2	31.250000000000000
3	24.670637551289226
4	15.507933877282202
5	18.793727081605252
6	25.332457619748633
7	11.720413795766915
8	21.448846635763506
9	22.852085146053171
10	12.993134749701883
11	20.256032419069617
12	24.028086798792512
13	11.312952488783849
14	21.928402064229323
15	22.645750831199066
16	12.297891655914007
17	21.039607970438863
18	23.613071919035916
19	11.112849968988774
20	22.255444662619830
21	22.538797503708382
22	12.007749160328618
23	21.420855813072539
24	23.413918309071074

Table 4:

n	a_n
2	31.250000000000000
3	24.784783120101957
4	15.393788308469470
5	18.910289628010061
6	25.216064645109287
7	11.852299372605387
8	21.317094875233968
9	22.996155578103468
10	12.849288733098089
11	20.406089364171059
12	23.878501626014072
13	11.473406396504589
14	21.768253422855603
15	22.816679053194818
16	12.127380906657459
17	21.215636103754317
18	23.437898677590793
19	11.296133040848717
20	22.072665551683782
21	22.731954796025644
22	11.815214776104113
23	21.618523134985741
24	23.217532805177368

We next describe the limiting functions $\widetilde{F}_j^{(i)}$ in terms of the branches of a certain compact Riemann surface of genus zero.

Let $\Delta_1 := [0, \alpha^3]$ and $\Delta_2 := [-b^3, -a^3]$. Consider the three-sheeted compact Riemann surface

$$\mathcal{R} = \overline{\mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2}$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \qquad \mathcal{R}_1 := \overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2), \qquad \mathcal{R}_2 := \overline{\mathbb{C}} \setminus \Delta_2, \tag{200}$$

where the upper and lower banks of the cuts of two neighboring sheets are identified. Since $\mathcal{R}$ has genus zero (it is not difficult to show that the normal form of this surface is aa^{-1}, see also [23, Section I.2]), there exists a conformal representation ψ of $\mathcal{R}$ onto $\overline{\mathbb{C}}$ such that

$$\psi(z) = Az + O(1), \qquad z \to \infty^{(1)}, \quad A \neq 0, \tag{201}$$

$$\psi(z) = B/z + O(1/z^2), \qquad z \to \infty^{(2)}, \quad B \neq 0, \tag{202}$$

i.e. the divisor of ψ consists of a simple pole at $\infty^{(1)}$ and a simple zero at $\infty^{(2)}$ ($x^{(l)}$ denotes the point in the sheet l that projects onto $x \in \overline{\mathbb{C}}$). By Liouville's theorem, such conformal representation is uniquely determined up to a multiplicative constant.

We can certainly assume that the coefficient A in (201) is given by

$$A = -2/a^3, \tag{203}$$

and so we will assume throughout that ψ satisfies the three properties (201)–(203). Hence ψ *is uniquely determined*. Let

$$\psi = \{\psi_0, \psi_1, \psi_2\}$$

denote the branches of ψ.

Finally, given an arbitrary function $H(z)$ that has in a neighborhood of infinity a Laurent expansion of the form $H(z) = Cz^k + O(z^{k-1})$, $C \neq 0$, $k \in \mathbb{Z}$, we denote

$$\widetilde{H} := H/C.$$

Theorem IV.1.8. *The following representations are valid:*

$$\widetilde{F}_1^{(0)} = \frac{a^{(0)} - a^{(3)}}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}, \qquad \widetilde{F}_1^{(1)} = \frac{(a^{(4)} - a^{(1)})\,\widetilde{\psi}_0}{a^{(4)}\widetilde{\psi}_0 - a^{(1)}}, \qquad \widetilde{F}_1^{(2)}(z) = \frac{z(a^{(0)} - a^{(3)})}{a^{(0)}\widetilde{\psi}_0(z) - a^{(3)}},$$

$$\widetilde{F}_1^{(3)} = \frac{(a^{(0)} - a^{(3)})\,\widetilde{\psi}_0}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}, \qquad \widetilde{F}_1^{(4)} = \frac{a^{(4)} - a^{(1)}}{a^{(4)}\widetilde{\psi}_0 - a^{(1)}}, \qquad \widetilde{F}_1^{(5)}(z) = \frac{z(a^{(0)} - a^{(3)})\,\widetilde{\psi}_0(z)}{a^{(0)}\widetilde{\psi}_0(z) - a^{(3)}},$$

$$\widetilde{F}_2^{(0)}(z) = \widetilde{F}_2^{(2)}(z) = \frac{a^{(0)}(a^{(0)} - a^{(3)})\, z\,\widetilde{\psi}_0(z)\,\widetilde{\psi}_2(z)}{(a^{(0)} - a^{(3)}\omega_1^{(3)}\widetilde{\psi}_0(z)\,\widetilde{\psi}_2(z)/\omega_1^{(0)})(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})},$$

$$\widetilde{F}_2^{(3)}(z) = \widetilde{F}_2^{(5)}(z) = \frac{a^{(0)}(a^{(0)} - a^{(3)})\, z\,\widetilde{\psi}_0(z)}{(a^{(0)} - a^{(3)}\omega_1^{(3)}\widetilde{\psi}_0(z)\,\widetilde{\psi}_2(z)/\omega_1^{(0)})(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})},$$

$$\widetilde{F}_2^{(1)} = \frac{a^{(4)} - a^{(1)}}{\widetilde{\psi}_2(a^{(4)}\widetilde{\psi}_0 - a^{(1)})(\widetilde{\psi}_1 - (\omega_1^{(1)} - 1)/\omega_1^{(4)})},$$

$$\widetilde{F}_2^{(4)} = \frac{a^{(4)} - a^{(1)}}{(a^{(4)}\widetilde{\psi}_0 - a^{(1)})(\widetilde{\psi}_1 - (\omega_1^{(1)} - 1)/\omega_1^{(4)})}.$$

The constants $\omega_1^{(l)}$ are the reciprocals of the right-hand sides in the boundary value equations (363)–(365)*. They can be written in terms of the limiting values $a^{(i)}$ as follows:*

$$\omega_1^{(0)} = \omega_1^{(2)} = \frac{a^{(4)} - a^{(1)}}{a^{(0)}a^{(4)}},$$

$$\omega_1^{(3)} = \omega_1^{(5)} = \frac{a^{(0)}}{a^{(0)} - a^{(3)}},$$

$$\omega_1^{(1)} = \frac{a^{(4)}}{a^{(4)} - a^{(1)}}, \qquad \omega_1^{(4)} = \frac{a^{(0)} - a^{(3)}}{(a^{(0)})^2}.$$

Using Theorem 3.1 from [43], we can easily describe in the following result the cubic algebraic equation whose solutions are the branches of the conformal mapping ψ. The coefficients of this equation can be computed only in terms of the endpoints of the intervals Δ_1 and Δ_2.

Proposition IV.1.9. *Let*

$$\lambda := \frac{2b^3}{a^3} - 1, \qquad \mu := \frac{2\alpha^3}{a^3} + 1, \tag{204}$$

and let β and γ be the unique solutions of the algebraic system

$$\begin{cases} 2(\beta+\gamma)(3-\beta\gamma-\beta-\gamma)(3-\beta\gamma+\beta+\gamma)+(\lambda-\mu)(\beta-\gamma)^3=0, \\ (\lambda+\mu)^2(\beta-\gamma)^6 = 4(3+\beta\gamma)^3(1-\beta\gamma)(2+\beta+\gamma)(2-\beta-\gamma), \end{cases}$$

satisfying the conditions $-1<\gamma<\beta<1$. Then $w=\psi(z)$ is the solution of the cubic equation

$$w^3 + \left[\frac{2z}{a^3} + 1 + \frac{3+h+\Theta_2-\Theta_1}{H(\beta)}\right] w^2 \tag{205}$$

$$+\left[\frac{4z}{a^3 H(\beta)} + \frac{2}{H(\beta)} + \frac{2+2h+\Theta_2-3\Theta_1}{H(\beta)^2}\right] w - \frac{2\Theta_1}{H(\beta)^3} = 0,$$

where

$$H(z) = h + z + \frac{\Theta_1 z}{1-z} + \frac{\Theta_2 z}{1+z},$$

$$h = \frac{1}{4}(\beta+\gamma)\left(2\beta\gamma - \frac{(\beta-\gamma)^2}{1-\beta\gamma}\right),$$

$$\Theta_1 = \frac{1}{4}(1-c)(1-d)(1-\beta)(1-\gamma), \qquad \Theta_2 = \frac{1}{4}(1+c)(1+d)(1+\beta)(1+\gamma),$$

c and d are the solutions of equation

$$x^2 + (\beta+\gamma)\,x + \frac{(\beta-\gamma)^2}{1-\beta\gamma} - 3 = 0,$$

satisfying $c<-1, d>1$.

The proofs of Theorem IV.1.8 and Proposition IV.1.9 are given in Section IV.6.

We now describe the main results obtained on nth root asymptotics and zero asymptotic distribution for the polynomials Q_n and $Q_{n,2}$. First, we need to introduce certain definitions.

Definition IV.1.10. *Let μ be a positive, finite, compactly supported measure in the complex plane, where* $\operatorname{supp}(\mu)$ *contains infinitely many points. We say that μ is* ***regular*** *(in the sense of Stahl and Totik [59]) if*

$$\lim_{n\to\infty} \kappa_n^{1/n} = \frac{1}{\operatorname{cap}_0(\operatorname{supp}(\mu))},$$

where $\kappa_n > 0$ denotes the leading coefficient of the nth orthonormal polynomial associated with μ, and $\operatorname{cap}_0(\operatorname{supp}(\mu))$ *indicates the logarithmic capacity of* $\operatorname{supp}(\mu)$. *The class of regular measures is denoted by* **Reg**.

Given a compact set $E \subset \mathbb{C}$, recall that $\mathcal{M}_1(E)$ denotes the space of all probability Borel measures supported on E (see (18)). If P is a polynomial of degree n, we indicate by μ_P the associated normalized zero counting measure, i.e.

$$\mu_P := \frac{1}{n} \sum_{P(x)=0} \delta_x,$$

where δ_x is the Dirac measure with mass 1 at x (in the sum the zeros are repeated according to their multiplicity). If $\mu \in \mathcal{M}_1(E)$, let

$$V^{\mu}(z) = \int \log \frac{1}{|z-t|}\, d\mu(t)$$

denote in this chapter the logarithmic potential associated with μ. Finally, recall that if $\{\mu_n\} \subset \mathcal{M}_1(E)$ and $\mu \in \mathcal{M}_1(E)$, then the notation

$$\mu_n \xrightarrow{*} \mu$$

indicates the weak-star convergence of the sequence μ_n to μ, which means that for every continuous function f on E, the following holds:

$$\lim_{n\to\infty}\int_E f\,d\mu_n = \int_E f\,d\mu.$$

Let E_1, E_2 be compact subsets of $\mathbb{R}$, and let $M = [c_{j,k}], 1 \le j,k \le 2$ be a real, positive definite, symmetric matrix of order two. Given a vector measure $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathcal{M}_1(E_1) \times \mathcal{M}_1(E_2)$, we define the combined potential

$$W_j^{\boldsymbol{\mu}} := \sum_{k=1}^{2} c_{j,k} V^{\mu_k}, \qquad j = 1, 2,$$

and the constants

$$\omega_j^{\boldsymbol{\mu}} := \inf\{W_j^{\boldsymbol{\mu}}(x) : x \in E_j\}, \qquad j = 1, 2.$$

In [49, Chapter 5], a more general version of the following result is proved. We will make use of this result.

Lemma IV.1.11. *Assume that the compact sets E_1, E_2 are regular with respect to the Dirichlet problem, and let $M = [c_{j,k}], 1 \le j,k \le 2$ be a real, positive definite, symmetric matrix of order two. If $c_{j,k} \ge 0$ in case $E_j \cap E_k \ne \emptyset$, then there exists a unique vector measure $\overline{\boldsymbol{\mu}} = (\overline{\mu}_1, \overline{\mu}_2) \in \mathcal{M}_1(E_1) \times \mathcal{M}_1(E_2)$ such that*

$$W_j^{\overline{\boldsymbol{\mu}}}(x) = \omega_j^{\overline{\boldsymbol{\mu}}}, \qquad x \in \operatorname{supp}(\overline{\mu}_j), \quad j = 1, 2.$$

The matrix M is called the *interaction matrix*, $\overline{\boldsymbol{\mu}}$ is called the *vector equilibrium measure* determined by the matrix M on the system of compact sets (E_1, E_2), and $\omega_1^{\overline{\boldsymbol{\mu}}}, \omega_2^{\overline{\boldsymbol{\mu}}}$ are called the *equilibrium constants.*

Let λ_1 be the positive, rotationally invariant measure on S_0 whose restriction to the interval $[0, \alpha]$ coincides with the measure $s_1(x)\,dx$, and let λ_2 be the positive, rotationally invariant measure on S_1 whose restriction to the interval $[-b, -a]$ coincides with the measure $s_2(x)\,dx$.

The zero asymptotic distribution and nth root asymptotics of the polynomials P_n and $P_{n,2}$ can be described as follows:

Theorem IV.1.12. *Assume that the measures λ_1 and λ_2 are in the class* **Reg**, *and suppose that* $\operatorname{supp}(\lambda_1)$ *and* $\operatorname{supp}(\lambda_2)$ *are regular for the Dirichlet problem. Then*

$$\mu_{P_n} \xrightarrow{*} \overline{\mu}_1 \in \mathcal{M}_1(\Delta_1), \qquad \Delta_1 = [0, \alpha^3], \tag{206}$$

$$\mu_{P_{n,2}} \xrightarrow{*} \overline{\mu}_2 \in \mathcal{M}_1(\Delta_2), \qquad \Delta_2 = [-b^3, -a^3], \tag{207}$$

where $\overline{\boldsymbol{\mu}} = (\overline{\mu}_1, \overline{\mu}_2)$ is the vector equilibrium measure determined by the interaction matrix

$$\begin{bmatrix} 1 & -1/4 \\ & \\ -1/4 & 1/4 \end{bmatrix} \tag{208}$$

on the system of intervals (Δ_1, Δ_2). Therefore,

$$\lim_{n\to\infty} |P_n(z)|^{1/\lfloor n/3 \rfloor} = e^{-V^{\overline{\mu}_1}(z)}, \tag{209}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_1$, and

$$\lim_{n\to\infty} |P_{n,2}(z)|^{1/\lfloor n/6 \rfloor} = e^{-V^{\overline{\mu}_2}(z)}, \tag{210}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_2$. Moreover,

$$\lim_{k\to\infty} \left(\int_0^{\alpha^3} P_{6k+j}^2(\tau)\, d\nu_{6k+j}(\tau) \right)^{1/4k} = e^{-\omega_1^{\overline{\boldsymbol{\mu}}}}, \qquad \textit{for all} \quad j = 0, \ldots, 5, \tag{211}$$

$$\lim_{k\to\infty}\left(\int_{-b^3}^{-a^3} P_{6k+j,2}^2(\tau)\,d\nu_{6k+j,2}(\tau)\right)^{1/2k} = e^{-4\omega_2^{\overline{\boldsymbol{\mu}}}}, \qquad \text{for all} \quad j=0,\ldots,5, \tag{212}$$

where $(\omega_1^{\overline{\boldsymbol{\mu}}},\omega_2^{\overline{\boldsymbol{\mu}}})$ *is the corresponding vector of equilibrium constants, and the varying measures* $d\nu_{6k+j}$ *and* $d\nu_{6k+j,2}$ *are defined in* (320) *below.*

The next result follows immediately from the previous theorem.

Corollary IV.1.13. *Under the same assumptions of Theorem IV.1.12, let* $\overline{\boldsymbol{\mu}} = (\overline{\mu}_1,\overline{\mu}_2)$ *be the vector equilibrium measure determined by the interaction matrix* (208) *on the system of intervals* $[0,\alpha^3],[-b^3,-a^3]$*, and let* $(\omega_1^{\overline{\boldsymbol{\mu}}},\omega_2^{\overline{\boldsymbol{\mu}}})$ *be the corresponding vector of equilibrium constants. Consider the probability measures* $\vartheta_1\in\mathcal{M}_1([0,\alpha])$ *and* $\vartheta_2\in\mathcal{M}_1([-b,-a])$*, defined as follows:*

$$\vartheta_1(E) := \overline{\mu}_1(E^3),\ E\subset[0,\alpha],$$

$$\vartheta_2(E) := \overline{\mu}_2(E^3),\ E\subset[-b,-a],$$

where $E^3=\{x^3 : x\in E\}$*. If we denote by* Z_{Q_n} *the set of all roots of* Q_n *on* $(0,\alpha)$*, and by* $Z_{Q_{n,2}}$ *the set of all roots of* $Q_{n,2}$ *on* $(-b,-a)$*, then*

$$\frac{1}{n}\sum_{x\in Z_{Q_n}}\delta_x \overset{*}{\longrightarrow} \frac{1}{3}\vartheta_1,$$

$$\frac{1}{n}\sum_{x\in Z_{Q_{n,2}}}\delta_x \overset{*}{\longrightarrow} \frac{1}{6}\vartheta_2.$$

The limits

$$\lim_{n\to\infty}|Q_n(z)|^{1/n} = e^{-\frac{1}{3}V^{\overline{\mu}_1}(z^3)}, \qquad z\in\mathbb{C}\setminus S_0,$$

$$\lim_{n\to\infty}|Q_{n,2}(z)|^{1/n} = e^{-\frac{1}{6}V^{\overline{\mu}_2}(z^3)}, \qquad z\in\mathbb{C}\setminus S_1,$$

hold uniformly on compact subsets of the indicated regions. Finally, we have

$$\lim_{k\to\infty}\left(\int_0^{\alpha} Q_{3k}^2(t)\,\frac{s_1(t)}{Q_{3k,2}(t)}\,dt\right)^{1/k} = e^{-2\omega_1^{\overline{\mu}}},$$

$$\lim_{k\to\infty}\left(\int_0^{\alpha} Q_{3k+1}^2(t)\,\frac{t\,s_1(t)}{Q_{3k+1,2}(t)}\,dt\right)^{1/k} = e^{-2\omega_1^{\overline{\mu}}},$$

$$\lim_{k\to\infty}\left(\int_0^{\alpha} Q_{3k+2}^2(t)\,\frac{s_1(t)}{t\,Q_{3k+2,2}(t)}\,dt\right)^{1/k} = e^{-2\omega_1^{\overline{\mu}}},$$

$$\lim_{k\to\infty}\left(\int_{-b}^{-a} Q_{3k,2}^2(t)\,\frac{|th_{3k}(t)|}{|Q_{3k}(t)|}\,s_2(t)\,dt\right)^{1/k} = e^{-4\omega_2^{\overline{\mu}}},$$

$$\lim_{k\to\infty}\left(\int_{-b}^{-a} Q_{3k+1,2}^2(t)\,\frac{|h_{3k+1}(t)|}{|Q_{3k+1}(t)|}\,s_2(t)\,dt\right)^{1/k} = e^{-4\omega_2^{\overline{\mu}}},$$

$$\lim_{k\to\infty}\left(\int_{-b}^{-a} Q_{3k+2,2}^2(t)\,\frac{t^2|h_{3k+2}(t)|}{|Q_{3k+2}(t)|}\,s_2(t)\,dt\right)^{1/k} = e^{-4\omega_2^{\overline{\mu}}},$$

where the functions h_n are defined in (319) (*see also* (321)–(323)).

The following proposition provides a link between the results on ratio and nth root asymptotics.

Proposition IV.1.14. *Under the same assumptions of Theorem IV.1.6, the following relations hold:*

$$V^{\overline{\mu}_1}(z) = -\frac{1}{2}\sum_{i=0}^{5}\log|\widetilde{F}_1^{(i)}(z)|, \qquad z\in\mathbb{C}\setminus[0,\alpha^3], \tag{213}$$

$$V^{\overline{\mu}_2}(z) = -\sum_{i=0}^{5}\log|\widetilde{F}_2^{(i)}(z)|, \qquad z\in\mathbb{C}\setminus[-b^3,-a^3], \tag{214}$$

where $(\overline{\mu}_1,\overline{\mu}_2)$ is the vector equilibrium measure determined by the interaction matrix (208) *on the system of intervals* $[0,\alpha^3],[-b^3,-a^3]$.

Theorem IV.1.12, Corollary IV.1.13, Proposition IV.1.14, and other related results are proved in Section IV.7.

IV.2 The polynomials Q_n

Let

$$\Sigma_1 := \bigcup_{k=0}^{2} (-\infty, 0] \exp(2\pi i k/3). \tag{215}$$

We may assume that $s_2(x) = 0$ for all $x \in (-\infty, 0] \setminus [-b, -a]$, and we extend s_2 to Σ_1 through the symmetry property

$$s_2(e^{\frac{2\pi i}{3}} t) = e^{\frac{4\pi i}{3}} s_2(t), \quad t \in \Sigma_1. \tag{216}$$

Proposition IV.2.1. *The following holds:*

$$f(z) = \frac{1}{3} \int_{S_1} \frac{s_2(t)}{t-z} \, dt = \frac{z^2}{3} \int_{-b^3}^{-a^3} \frac{s_2(\sqrt[3]{\tau})}{(z^3 - \tau)\, \tau^{2/3}} \, d\tau, \qquad z \in \mathbb{C} \setminus S_1. \tag{217}$$

Therefore $f(z)/z^2$ is the Cauchy transform in z^3 of a weight supported on $[-b^3, -a^3]$.

Proof. Let

$$R_I := \{e^{\frac{\pi i}{3}} x : x \in [a, b]\}, \quad R_{II} := [-b, -a], \quad R_{III} := \{e^{\frac{5\pi i}{3}} x : x \in [a, b]\},$$

be the three rays forming the set S_1, and let $\gamma_I(t) = e^{\frac{\pi i}{3}} t$, $\gamma_{II}(t) = -t$, $\gamma_{III}(t) = e^{\frac{5\pi i}{3}} t$, $t \in [a, b]$ be the parametrizations of R_I, R_{II}, and R_{III}, respectively. We have

$$\int_{R_I} \frac{s_2(t)}{t-z} \, dt = \int_a^b \frac{s_2(e^{\frac{i\pi}{3}} t)\, e^{\frac{i\pi}{3}}}{e^{\frac{i\pi}{3}} t - z} \, dt = \int_a^b \frac{-s_2(-t)}{e^{\frac{i\pi}{3}} t - z} \, dt,$$

$$\int_{R_{II}} \frac{s_2(t)}{t-z} \, dt = \int_a^b \frac{-s_2(-t)}{-t-z} \, dt,$$

$$\int_{R_{III}} \frac{s_2(t)}{t-z} \, dt = \int_a^b \frac{s_2(e^{\frac{-i\pi}{3}} t)\, e^{-\frac{i\pi}{3}}}{e^{-\frac{i\pi}{3}} t - z} \, dt = \int_a^b \frac{-s_2(-t)}{e^{-\frac{i\pi}{3}} t - z} \, dt,$$

Therefore

$$\int_{S_1} \frac{s_2(t)}{t-z}\,dt = \int_a^b \Big(\frac{-1}{-t-z} + \frac{-1}{e^{\frac{i\pi}{3}}t-z} + \frac{-1}{e^{-\frac{i\pi}{3}}t-z}\Big) s_2(-t)\,dt. \tag{218}$$

The decomposition of $1/(t^3+z^3)$ in simple fractions is given by

$$\frac{1}{t^3+z^3} = \frac{1}{3z^2}\Big(\frac{-1}{-t-z} + \frac{-1}{e^{\frac{i\pi}{3}}t-z} + \frac{-1}{e^{-\frac{i\pi}{3}}t-z}\Big). \tag{219}$$

From (218) and (219) we obtain

$$\frac{1}{3}\int_{S_1} \frac{s_2(t)}{t-z}\,dt = z^2 \int_a^b \frac{s_2(-t)}{t^3+z^3}\,dt = z^2 \int_{-b}^{-a} \frac{s_2(t)}{z^3-t^3}\,dt.$$

The second equality in (217) follows after a simple change of variable. □

Proposition IV.2.2. *The functions Ψ_n satisfy the following orthogonality conditions:*

$$0 = \int_{S_1} t^\nu\, \Psi_{2n}(t)\, s_2(t)\,dt, \qquad \nu = 0,\ldots,n-1, \tag{220}$$

$$0 = \int_{S_1} t^\nu\, \Psi_{2n+1}(t)\, s_2(t)\,dt, \qquad \nu = 0,\ldots,n-1, \tag{221}$$

where S_1 is the starlike set (178).

Proof. We prove (220). The proof of (221) is identical. If $0 \le \nu \le n-1$, applying Fubini's theorem we have

$$\int_{S_1} t^\nu\, \Psi_{2n}(t)\, s_2(t)\,dt = \int_{S_1} t^\nu s_2(t) \int_{S_0} \frac{Q_{2n}(x)}{x-t}\, s_1(x)\,dx\,dt$$

$$= \int_{S_0} Q_{2n}(x)\, s_1(x) \int_{S_1} \frac{t^\nu - x^\nu + x^\nu}{x-t}\, s_2(t)\,dt$$

$$= \int_{S_0} Q_{2n}(x)\, p_\nu(x)\, s_1(x)\,dx - 3\int_{S_0} Q_{2n}(x)\, x^\nu f(x)\, s_1(x)\,dx,$$

where p_ν is a polynomial of degree at most $n-2$. Using (168), (220) follows. □

Proposition IV.2.3. *Let Q_n be the monic polynomial of smallest degree satisfying the conditions* (167)–(170). *If $d_n := \deg Q_n$, then*

$$Q_n(e^{\frac{2\pi i}{3}}z) = e^{\frac{2\pi i d_n}{3}}Q_n(z), \tag{222}$$

and

$$Q_n(z) = \overline{Q_n(\overline{z})}. \tag{223}$$

In particular, all the coefficients of Q_n are real. Furthermore, for each $0 \leq k \leq n-1$,

$$0 = \int_0^\alpha t^k Q_{2n}(t)(1 + e^{2\pi i(k+d_{2n})/3} + e^{4\pi i(k+d_{2n})/3})\, s_1(t)\, dt, \tag{224}$$

$$0 = \int_0^\alpha t^k Q_{2n}(t)(1 + e^{2\pi i(k+2+d_{2n})/3} + e^{4\pi i(k+2+d_{2n})/3})\, s_1(t) f(t)\, dt. \tag{225}$$

Similarly, for each $0 \leq k \leq n$,

$$0 = \int_0^\alpha t^k Q_{2n+1}(t)(1 + e^{2\pi i(k+d_{2n+1})/3} + e^{4\pi i(k+d_{2n+1})/3})\, s_1(t)\, dt, \tag{226}$$

and for $0 \leq k \leq n-1$,

$$0 = \int_0^\alpha t^k Q_{2n+1}(t)(1 + e^{2\pi i(k+2+d_{2n+1})/3} + e^{4\pi i(k+2+d_{2n+1})/3})\, s_1(t)\, f(t)\, dt. \tag{227}$$

Proof. If we define $P_n(t) := Q_n(e^{\frac{2\pi i}{3}}t)$ and perform the substitution $x = e^{\frac{2\pi i}{3}}t$, we obtain for any integer $k \geq 0$,

$$\int_{S_0} P_n(t)\, t^k\, W_0(t)\, dt = \int_{S_0} Q_n(e^{\frac{2\pi i}{3}}t)\, t^k\, s_1(t)\, dt$$

$$= \int_{S_0} e^{-\frac{2\pi i k}{3}} x^k\, Q_n(x)\, s_1(e^{\frac{4\pi i}{3}}x)\, e^{\frac{4\pi i}{3}}\, dx = e^{-\frac{2\pi i k}{3}} \int_{S_0} x^k\, Q_n(x)\, s_1(x)\, dx,$$

and similarly

$$\int_{S_0} P_n(t)\, t^k\, W_1(t)\, dt = \int_{S_0} e^{-\frac{2\pi ik}{3}} x^k\, Q_n(x) f(e^{\frac{4\pi i}{3}} x)\, s_1(e^{\frac{4\pi i}{3}} x)\, e^{\frac{4\pi i}{3}}\, dx$$

$$= e^{-\frac{2\pi i(k-1)}{3}} \int_{S_0} x^k\, Q_n(x)\, f(x)\, s_1(x)\, dx.$$

It follows that Q_n and P_n satisfy the same orthogonality conditions. Since they have the same degree,

$$P_n / e^{\frac{2\pi i d_n}{3}} = Q_n,$$

hence (222) holds.

Using the fact that s_1 and f are real-valued on $(0, \alpha)$ (see (217)), we get

$$\overline{s_1(e^{\frac{4\pi i}{3}} t)} = s_1(e^{\frac{2\pi i}{3}} t), \qquad t \in (0, \alpha),$$

$$\overline{s_1(e^{\frac{4\pi i}{3}} t)\, f(e^{\frac{4\pi i}{3}} t)} = s_1(e^{\frac{2\pi i}{3}} t)\, f(e^{\frac{2\pi i}{3}} t), \qquad t \in (0, \alpha).$$

Applying these relations it is immediate to see that

$$\int_{S_0} \overline{Q_n(\overline{t})}\, t^k\, s_1(t)\, dt = \overline{\int_{S_0} Q_n(t)\, t^k\, s_1(t)\, dt},$$

$$\int_{S_0} \overline{Q_n(\overline{t})}\, t^k\, f(t)\, s_1(t)\, dt = \overline{\int_{S_0} Q_n(t)\, t^k\, f(t)\, s_1(t)\, dt}.$$

Consequently $\overline{Q_n(\overline{t})}$ and $Q_n(t)$ are monic polynomials with the same degree and satisfying the same orthogonality relations, so (223) holds.

If we write the orthogonality relations (168) in terms of the interval $[0, \alpha]$, we get for $0 \le k \le n-1$ and $j \in \{0, 1\}$,

$$0 = \int_{S_0} t^k Q_{2n}(t) W_j(t)\, dt$$

$$= \int_0^\alpha t^k Q_{2n}(t) W_j(t)\, dt + \int_0^\alpha e^{\frac{2\pi i k}{3}} t^k Q_{2n}(e^{\frac{2\pi i}{3}} t) W_j(e^{\frac{2\pi i}{3}} t)\, e^{\frac{2\pi i}{3}}\, dt$$

$$+ \int_0^\alpha e^{\frac{4\pi i k}{3}} t^k Q_{2n}(e^{\frac{4\pi i}{3}} t) W_j(e^{\frac{4\pi i}{3}} t)\, e^{\frac{4\pi i}{3}}\, dt.$$

Since $W_0(e^{\frac{2\pi i}{3}} t) = e^{\frac{4\pi i}{3}} W_0(t)$ and $W_1(e^{\frac{2\pi i}{3}} t) = e^{\frac{2\pi i}{3}} W_1(t)$, using (222) we obtain (224) and (225). The proofs of (226) and (227) are analogous. □

Lemma IV.2.4. *Assume that $m \geq 1$ is an integer, and let P_1, P_2 be polynomials, not both identically equal to zero. If P_1 and P_2 have degree at most $m-1$, then the functions*

$$H_1(t) := P_1(t) + P_2(t) \sqrt[3]{t}\, f(\sqrt[3]{t}) \tag{228}$$

$$H_2(t) := P_1(t)\, t + P_2(t) \sqrt[3]{t}\, f(\sqrt[3]{t}) \tag{229}$$

have at most $2\,m-1$ zeros on $(0,\infty)$, counting multiplicities. Similarly, if P_1 has degree at most m and P_2 has degree at most $m-1$, then H_1 and H_2 have at most $2\,m$ zeros on $(0,\infty)$. If P_1 has degree at most $m-1$ and P_2 has degree at most m, then H_1 and H_2 also have at most $2\,m$ zeros on $(0,\infty)$.

Proof. Let σ be a finite positive measure with compact support $\operatorname{supp}(\sigma) \subset \mathbb{R}$, and let $\widehat{\sigma}$ denote its Cauchy transform, i.e.

$$\widehat{\sigma}(z) = \int \frac{d\sigma(x)}{z-x}.$$

Lemma 5 in [27] asserts that the system $\{1, \widehat{\sigma}\}$ forms an AT system on any closed interval $\Delta \subset \mathbb{R}$ disjoint from $\operatorname{Co}(\operatorname{supp}(\sigma))$ ($\operatorname{Co}(A)$ denotes the convex hull of A). This means that for any multi-index $(n_1, n_2) \in \mathbb{Z}_+^2$ and any pair of polynomials π_1, π_2 with $\deg \pi_1 \leq n_1 - 1$, $\deg \pi_2 \leq n_2 - 1$, not both identically equal to zero, the function

$$\pi_1 + \pi_2\, \widehat{\sigma}$$

has at most n_1+n_2-1 zeros on Δ, counting multiplicities. By Proposition IV.2.1 we have:

$$H_2(t)=t\Big(P_1(t)+\frac{P_2(t)}{3}\int_{-b^3}^{-a^3}\frac{s_2(\sqrt[3]{\tau})}{t-\tau}\frac{d\tau}{\tau^{2/3}}\Big),$$

so all the assertions concerning H_2 are valid.

Assume that there exist polynomials P_1,P_2 of degree at most $m-1$, not both identically equal to zero, such that the function H_1 in (228) has at least $2m$ zeros on $(0,\infty)$, counting multiplicities. If $P_2\equiv 0$ then we immediately reach a contradiction. So we assume that $P_2\not\equiv 0$. Pick $2m$ of these zeros and form a monic polynomial T_{2m} of degree $2m$ that vanishes at these points. The function H_1 can be analytically extended onto $\mathbb{C}\setminus[-b^3,-a^3]$, and in this region we have

$$\frac{H_1(z)}{T_{2m}(z)}=\frac{P_1(z)}{T_{2m}(z)}+\frac{zP_2(z)}{3T_{2m}(z)}\int_{-b^3}^{-a^3}\frac{s_2(\sqrt[3]{\tau})}{z-\tau}\frac{d\tau}{\tau^{2/3}}.$$

Observe that

$$\frac{H_1(z)}{T_{2m}(z)}=O\Big(\frac{1}{z^{m+1}}\Big),\qquad z\to\infty.$$

Let Γ be a simple closed curve surrounding $[-b^3,-a^3]$, so that the zeros of T_{2m} lie outside this curve. By Cauchy's theorem, Fubini's theorem and Cauchy integral formula, for any $0\le\nu\le m-1$ we have

$$0=\int_\Gamma z^\nu\frac{H_1(z)}{T_{2m}(z)}\,dz=\frac{1}{3}\int_{-b^3}^{-a^3}\frac{\tau^{\nu+1}P_2(\tau)\,s_2(\sqrt[3]{\tau})}{T_{2m}(\tau)\,\tau^{2/3}}\,d\tau,$$

and this contradicts the fact that $\deg P_2\le m-1$. Using the same argument one proves the case $\deg P_1\le m, \deg P_2\le m-1$.

In the remaining case we also use this argument by contradiction, but now we also divide H_1 by $\widehat{\sigma}$, where

$$d\sigma(\tau)=\frac{s_2(\sqrt[3]{\tau})\,d\tau}{3\,\tau^{2/3}},$$

and use the fact that

$$\frac{1}{\widehat{\sigma}(z)} = l(z) + \widehat{\mu}(z), \tag{230}$$

where $l(z)$ is a polynomial of degree one and μ is a measure of constant sign supported on $[-b^3, -a^3]$. A proof of (230) can be found in the appendix of [34]. □

Proof of Proposition IV.1.1. Assume first that $n = 3l$ and $d_{2n} = 3j$. Then (224) and (225) are equivalent to the following conditions:

$$\int_0^\alpha t^{3k} Q_{2n}(t)\, s_1(t)\, dt = 0, \qquad 0 \le k \le l-1, \tag{231}$$

$$\int_0^\alpha t^{3k} Q_{2n}(t)\, t\, f(t)\, s_1(t)\, dt = 0, \qquad 0 \le k \le l-1. \tag{232}$$

From (222) and the fact that $d_{2n} = 3j$, we deduce that

$$Q_{2n}(t) = a_0 + a_3\, t^3 + \cdots + a_{3j}\, t^{3j},$$

so $Q_{2n}(t) = \widetilde{Q}_{2n}(t^3)$ for some polynomial $\widetilde{Q}_{2n}$. Therefore (231) and (232) can be rewritten as follows:

$$\int_0^{\alpha^3} \tau^k\, \widetilde{Q}_{2n}(\tau)\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}} = 0, \qquad 0 \le k \le l-1, \tag{233}$$

$$\int_0^{\alpha^3} \tau^k\, \widetilde{Q}_{2n}(\tau)\, \sqrt[3]{\tau}\, f(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}} = 0, \qquad 0 \le k \le l-1. \tag{234}$$

Suppose that the polynomial $\widetilde{Q}_{2n}$ has $N < 2\,l$ sign changes on the interval $(0, \alpha^3)$. Let P_1 and P_2 be two polynomials of degree at most $l-1$, not both identically zero and with real coefficients, such that the function $H_1(t) = P_1(t) + P_2(t)\sqrt[3]{t}\, f(\sqrt[3]{t})$ has a zero at each point where $\widetilde{Q}_{2n}$ changes sign on $(0, \alpha^3)$, and a zero of order $2\,l-1-N$ at α^3. Finding P_1 and P_2 is equivalent to solving a homogeneous linear system with $2\,l-1$ equations and $2\,l$ unknowns, therefore a non-trivial solution exists. By Lemma

IV.2.4, the function H_1 has no zeros on $(0,\alpha^3]$ other than the $2\,l-1$ prescribed. Using (233) and (234) we have

$$\int_0^{\alpha^3} H_1(\tau)\,\widetilde{Q}_{2n}(\tau)\,s_1(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{2/3}}\,d\tau = 0.$$

But this contradicts the fact that $H_1\,\widetilde{Q}_{2n}$ is real-valued and has constant sign on $[0,\alpha^3]$. By applying (222) we conclude that Q_{2n} has exactly $2\,n$ simple zeros on S_0, $2\,n/3$ of them are located on $(0,\alpha)$ and the rest are obtained rotating the zeros on $(0,\alpha)$ by angles of $2\pi/3$ and $4\pi/3$.

Suppose now that $n = 3l$ and $d_{2n} = 3j+1$. We want to reach a contradiction. From (224) and (225) we have

$$0 = \int_0^{\alpha} t^{3k+2} Q_{2n}(t)\,s_1(t)\,dt, \qquad 0 \le k \le l-1,$$

$$0 = \int_0^{\alpha} t^{3k} Q_{2n}(t)\,f(t)\,s_1(t)\,dt, \qquad 0 \le k \le l-1.$$

The symmetry property (222) and $d_{2n} = 3j+1$ imply that Q_{2n} has the form

$$Q_{2n}(t) = b_1 t + b_4 t^4 + \cdots + b_{3j+1} t^{3j+1},$$

so $Q_{2n}(t) = t\,\widetilde{Q}_{2n}(t^3)$ for some polynomial $\widetilde{Q}_{2n}$ of degree j. Consequently, $\widetilde{Q}_{2n}$ satisfies the orthogonality conditions

$$0 = \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau)\,\tau\,s_1(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{2/3}}, \qquad 0 \le k \le l-1, \tag{235}$$

$$0 = \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau)\,\sqrt[3]{\tau}\,f(\sqrt[3]{\tau})\,s_1(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{2/3}}, \qquad 0 \le k \le l-1. \tag{236}$$

The polynomial $\widetilde{Q}_{2n}$ has $N \le j$ sign changes on $(0,\alpha)$. Notice that

$$d_{2n} = 3j+1 \le 2\,n = 6\,l \Rightarrow j + \frac{1}{3} \le 2\,l \Rightarrow j \le 2\,l-1.$$

We can find polynomials P_1 and P_2 of degree at most $l-1$ with real coefficients, not both identically zero, such that the function $H_2(t) = P_1(t)\,t + P_2(t)\sqrt[3]{t}\,f(\sqrt[3]{t})$ has a zero at each point where $\widetilde{Q}_{2n}$ changes sign on $(0, \alpha^3)$ and has a zero of order $2\,l-1-N$ at α^3. By Lemma IV.2.4, the function H_2 has no zeros on $(0, \alpha^3]$ other than the $2\,l-1$ prescribed. From (235) and (236) we obtain

$$0 = \int_0^{\alpha^3} H_2(\tau)\,\widetilde{Q}_{2n}(\tau)\,s_1(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{2/3}},$$

but this contradicts the fact that $\widetilde{Q}_{2n}$ is non-zero and the function $H_2\,\widetilde{Q}_{2n}$ is real-valued and has constant sign on $[0, \alpha^3]$. This contradiction shows that $d_{2n} = 3j+1$ is impossible if n is a multiple of 3.

If we assume that $n = 3\,l$ and $d_{2n} = 3j+2$, then (224) and (225) are equivalent to

$$0 = \int_0^{\alpha} t^{3k+1} Q_{2n}(t)\,s_1(t)\,dt, \qquad 0 \le k \le l-1,$$

$$0 = \int_0^{\alpha} t^{3k+2} Q_{2n}(t)\,f(t)\,s_1(t)\,dt, \qquad 0 \le k \le l-1.$$

In this case, there exists a polynomial $\widetilde{Q}_{2n}$ of degree j such that $Q_{2n} = t^2\widetilde{Q}_{2n}(t^3)$ and one obtains the orthogonality conditions

$$0 = \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau)\,s_1(\sqrt[3]{\tau})\,\sqrt[3]{\tau}\,d\tau, \qquad 0 \le k \le l-1, \tag{237}$$

$$0 = \int_0^{\alpha^3} \tau^k\,\widetilde{Q}_{2n}(\tau)\,\sqrt[3]{\tau}\,f(\sqrt[3]{\tau})\,s_1(\sqrt[3]{\tau})\,\sqrt[3]{\tau}\,d\tau, \qquad 0 \le k \le l-1. \tag{238}$$

The polynomial $\widetilde{Q}_{2n}$ has $N \le j$ sign changes on $(0, \alpha)$, and

$$d_{2n} = 3j + 2 \le 2\,n = 6\,l \Rightarrow j + \frac{2}{3} \le 2\,l \Rightarrow j \le 2\,l - 1.$$

Taking as a basis measure $s_1(\sqrt[3]{\tau})\,\sqrt[3]{\tau}\,d\tau$ and using that the function (228) has at most $2\,l-1$ zeros on $(0, \alpha^3]$ if the polynomial coefficients have degree at most $l-1$, we get a contradiction.

Let $n = 3l + 1$ and assume that $d_{2n} = 3j + 2$. We will show that $d_{2n} = 2n$. In this situation (237) and (238) are valid again. If we assume that the polynomial $\widetilde{Q}_{2n}$ has $N < 2l$ sign changes on the interval $(0, \alpha^3)$, then we obtain a contradiction as before.

If $n = 3l + 1$ and $d_{2n} = 3j$, then

$$\int_0^{\alpha} t^{3k} Q_{2n}(t)\, s_1(t)\, dt = 0, \qquad 0 \le k \le l,$$

$$\int_0^{\alpha} t^{3k+1} Q_{2n}(t)\, f(t)\, s_1(t)\, dt = 0, \qquad 0 \le k \le l-1.$$

Therefore

$$\int_0^{\alpha^3} \tau^k\, \widetilde{Q}_{2n}(\tau)\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}} = 0, \qquad 0 \le k \le l,$$

$$\int_0^{\alpha^3} \tau^k\, \widetilde{Q}_{2n}(\tau)\, \sqrt[3]{\tau}\, f(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}} = 0, \qquad 0 \le k \le l-1.$$

Since

$$d_{2n} = 3j \le 2n = 6l + 2 \Rightarrow j \le 2l + \frac{2}{3} \Rightarrow j \le 2l,$$

applying Lemma IV.2.4 we get a contradiction.

If $n = 3l + 1$ and $d_{2n} = 3j + 1$, then

$$0 = \int_0^{\alpha} t^{3k+2} Q_{2n}(t)\, s_1(t)\, dt, \qquad 0 \le k \le l-1,$$

$$0 = \int_0^{\alpha} t^{3k} Q_{2n}(t)\, f(t)\, s_1(t)\, dt, \qquad 0 \le k \le l.$$

So

$$0 = \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau)\, \tau\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}}, \qquad 0 \le k \le l-1, \tag{239}$$

$$0 = \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau)\, \sqrt[3]{\tau}\, f(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}}, \qquad 0 \le k \le l, \tag{240}$$

and using Lemma IV.2.4 and

$$d_{2n} = 3j + 1 \leq 2\,n = 6\,l + 2 \Rightarrow j \leq 2\,l + \frac{1}{3} \Rightarrow j \leq 2\,l,$$

we get a contradiction.

Let $n = 3l + 2$ and assume that $d_{2n} = 3j + 1$. We want to show that $d_{2n} = 2n$. In this case the relations (239) and (240) hold. If we assume that $\widetilde{Q}_{2n}$ has $N < 2\,l + 1$ sign changes on the interval $(0, \alpha^3)$, then we obtain a contradiction.

Let $n = 3l + 2$ and assume that $d_{2n} = 3j$. Then the relations (233) and (234) are both valid for $0 \leq k \leq l$. From $d_{2n} \leq 2n$ we deduce that $j \leq 2\,l + 1$. Applying Lemma IV.2.4 we reach a contradiction.

Let $n = 3l + 2$ and assume that $d_{2n} = 3j + 2$. Then (237) is valid for $0 \leq k \leq l$ and (238) holds. The inequality $d_{2n} \leq 2n$ implies that $j \leq 2l$, so Lemma IV.2.4 gives a contradiction.

The analysis for the polynomials Q_{2n+1} is similar. □

Corollary IV.2.5. *The polynomials Q_n and the functions Ψ_n satisfy*

$$Q_n(e^{\frac{2\pi i}{3}} z) = e^{\frac{2\pi i n}{3}} Q_n(z), \tag{241}$$

$$\Psi_n(e^{\frac{2\pi i}{3}} z) = e^{-\frac{2\pi i}{3}(1+2n)} \Psi_n(z), \tag{242}$$

for all $n \geq 0$.

Proof. (241) follows from (222) and $d_n = n$. Now,

$$\Psi_n(e^{\frac{2\pi i}{3}} z) = \int_{S_0} \frac{Q_n(t)\, s_1(t)}{t - e^{\frac{2\pi i}{3}} z}\, dt = \int_{S_0} \frac{e^{\frac{4\pi i}{3}}\, Q_n(t)\, s_1(t)}{e^{\frac{4\pi i}{3}} t - z}\, dt$$

$$= e^{-\frac{2\pi i}{3}(1+2n)} \int_{S_0} \frac{e^{\frac{4\pi i}{3}}\, Q_n(e^{\frac{4\pi i}{3}} t)\, s_1(e^{\frac{4\pi i}{3}} t)}{e^{\frac{4\pi i}{3}} t - z}\, dt = e^{-\frac{2\pi i}{3}(1+2n)} \Psi_n(z).$$

□

Remark IV.2.6. *The following example shows that the linear independence of two positive Borel measures supported on an interval is not sufficient to guarantee that the degrees of the associated multiple orthogonal polynomials are maximal. If we take the measures*

$$d\mu_1(x) = dx, \qquad d\mu_2(x) = (20x^3 - 30x^2 + 12x)\, dx, \qquad x \in [0,1],$$

then the polynomial $P(x) = x - 1/2$ *satisfies*

$$0 = \int_0^1 P(x)\, d\mu_1(x) = \int_0^1 P(x)\, d\mu_2(x),$$

but P *is not of degree two.*

A similar example can be constructed on a starlike set. If we let

$$m_1(t) := \begin{cases} 1 & \textit{if} \quad t \in [0,1], \\ e^{\frac{4\pi i}{3}} & \textit{if} \quad t \in \{e^{\frac{2\pi i}{3}} x : x \in (0,1]\}, \\ e^{\frac{2\pi i}{3}} & \textit{if} \quad t \in \{e^{\frac{4\pi i}{3}} x : x \in (0,1]\}, \end{cases}$$

and define

$$m_2(t) := 10 - 9t, \qquad t \in [0,1],$$

$$m_2(e^{\frac{2\pi i}{3}} t) := e^{\frac{2\pi i}{3}} (10 - 9t), \qquad t \in (0,1],$$

$$m_2(e^{\frac{4\pi i}{3}} t) := e^{\frac{4\pi i}{3}} (10 - 9t), \qquad t \in (0,1],$$

then m_1 *and* m_2 *satisfy*

$$m_1(e^{\frac{2\pi i}{3}} t) = e^{\frac{4\pi i}{3}} m_1(t), \qquad t \in \widetilde{S}_0 \setminus \{0\},$$

$$m_2(e^{\frac{2\pi i}{3}}t) = e^{\frac{2\pi i}{3}} m_2(t), \qquad t \in \widetilde{S}_0 \setminus \{0\},$$

where $\widetilde{S}_0 := \cup_{k=0}^{2}[0,1]\exp(2\pi i k/3)$. *For the polynomial* $P(t) = t^3 - 1/4$ *we have*

$$\int_{\widetilde{S}_0} P(t)\, t^j\, m_1(t)\, dt = 0, \qquad j = 0, 1,$$

$$\int_{\widetilde{S}_0} P(t)\, t^j\, m_2(t)\, dt = 0, \qquad j = 0, 1,$$

but $\deg P < 4$.

Lemma IV.2.7. *For any integer* $k \geq 0$*, the following holds:*

$$\int_{S_0} t^{3k} s_1(t)\, dt = 3\int_0^{\alpha} t^{3k} s_1(t)\, dt, \quad \int_{S_0} t^{3k+1} f(t)\, s_1(t)\, dt = 3\int_0^{\alpha} t^{3k+1} f(t)\, s_1(t)\, dt, \tag{243}$$

$$\int_{S_0} t^{3k+1}\, s_1(t)\, dt = 0, \qquad \int_{S_0} t^{3k+2} s_1(t)\, dt = 0. \tag{244}$$

Proof. Making use of (162),

$$\int_{S_0} t^{3k} s_1(t)\, dt = \int_0^{\alpha} t^{3k} s_1(t)\, dt + \int_0^{\alpha} t^{3k} s_1(e^{\frac{2\pi i}{3}}t)\, e^{\frac{2\pi i}{3}}\, dt + \int_0^{\alpha} t^{3k} s_1(e^{\frac{4\pi i}{3}}t)\, e^{\frac{4\pi i}{3}}\, dt$$

$$= 3\int_0^{\alpha} t^{3k} s_1(t)\, dt,$$

and similarly we get the other equality in (243). We have

$$\int_{S_0} t^{3k+1}\, s_1(t)\, dt = \int_0^{\alpha} t^{3k+1}\, s_1(t)(1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}})\, dt = 0,$$

$$\int_{S_0} t^{3k+2}\, s_1(t)\, dt = \int_0^{\alpha} t^{3k+2}\, s_1(t)(1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}})\, dt = 0.$$

□

Proof of Proposition IV.1.2. We first show that $Q_1(z) = z$ and $Q_2(z) = z^2$. Let us write $Q_1(z) = z + c_1$ and $Q_2(z) = z^2 + c_2 z + c_3$. Note that the integrals in (243) are non-zero because $s_1(x)\,dx$ is non-trivial and $f > 0$ on $(0, \alpha)$. Using (243) and (244) we have

$$0 = \int_{S_0} Q_1(t)\, s_1(t)\, dt \Rightarrow c_1 = 0,$$

$$0 = \int_{S_0} Q_2(t)\, s_1(t)\, dt \Rightarrow c_3 = 0,$$

$$0 = \int_{S_0} Q_2(t)\, f(t)\, s_1(t)\, dt \Rightarrow c_2 = 0.$$

If $n \geq 1$ and we write

$$zQ_{2n} = Q_{2n+1} + b_{2n}Q_{2n} + b_{2n-1}Q_{2n-1} + b_{2n-2}Q_{2n-2} + \cdots + b_1Q_1 + b_0Q_0, \tag{245}$$

let us show that

$$b_{2n-3} = b_{2n-4} = \cdots = b_1 = b_0 = 0, \tag{246}$$

and

$$b_{2n} = b_{2n-1} = 0. \tag{247}$$

We first prove (246) by induction. If $n = 1$ then there is nothing to prove. So we assume here that $n \geq 2$. If we integrate (245) with respect to $s_1(t)\,dt$, the integral on the left-hand side vanishes and on the right-hand side all integrals except the last one also vanish, hence

$$0 = b_0 \int_{S_0} s_1(t) \Rightarrow b_0 = 0.$$

To show that $b_1 = 0$ we now integrate (245) with respect to $f(t)\, s_1(t)\, dt$. Again the integral on the left-hand side vanishes and on the right-hand side all integrals vanish except

$$\int_{S_0} Q_1(t)\, f(t)\, s_1(t)\, dt = \int_{S_0} t\, f(t)\, s_1(t)\, dt \neq 0,$$

and it follows that $b_1 = 0$. We assume now that

$$0 = b_0 = b_1 = \cdots = b_{2k} = b_{2k+1} = 0$$

for some $k \leq n-3$, and let us prove that $b_{2k+2} = b_{2k+3} = 0$. We multiply (245) by z^{k+1} and apply the induction hypothesis to obtain

$$z^{k+2}Q_{2n} = z^{k+1}Q_{2n+1} + b_{2n}z^{k+1}Q_{2n} + \cdots + b_{2k+3}z^{k+1}Q_{2k+3} + b_{2k+2}z^{k+1}Q_{2k+2}. \quad (248)$$

Observe that

$$\int_{S_0} t^{k+1}Q_{2k+2}(t)\, s_1(t)\, dt \neq 0$$

because otherwise Q_{2k+2} and Q_{2k+3} would satisfy the same orthogonality relations, implying that these polynomials are equal, which is impossible. In addition, by (168) and (169) we know that

$$k+2 \leq n-1 \Rightarrow \int_{S_0} t^{k+2}Q_{2n}(t)\, s_1(t)\, dt = 0,$$

$$k+1 \leq j-1 \Rightarrow \int_{S_0} t^{k+1}Q_{2j}(t)\, s_1(t)\, dt = 0,$$
$$k+1 \leq j \Rightarrow \int_{S_0} t^{k+1}Q_{2j+1}(t)\, s_1(t)\, dt = 0,$$

and so

$$\int_{S_0} t^{k+1}Q_{2k+3}(t)\, s_1(t)\, dt = \cdots = \int_{S_0} t^{k+1}Q_{2n+1}(t)\, s_1(t)\, dt = 0,$$

therefore $b_{2k+2} = 0$.

To show that $b_{2k+3} = 0$ we now integrate (248) with respect to $f(t)\, s_1(t)\, dt$. Now,

$$\int_{S_0} t^{k+1}Q_{2k+3}(t)\, f(t)\, s_1(t)\, dt \neq 0,$$

because otherwise Q_{2k+3} and Q_{2k+4} satisfy the same orthogonality conditions, which is impossible by the maximality of the degrees. Since

$$k+2 \leq n-1 \Rightarrow \int_{S_0} t^{k+2} Q_{2n}(t)\, f(t)\, s_1(t)\, dt = 0,$$

$$k+1 \leq j-1 \Rightarrow \int_{S_0} t^{k+1} Q_{2j}(t)\, f(t)\, s_1(t)\, dt = 0,$$

$$k+1 \leq j-1 \Rightarrow \int_{S_0} t^{k+1} Q_{2j+1}(t)\, f(t)\, s_1(t)\, dt = 0,$$

implying that $b_{2k+3} = 0$.

Now we justify (247). Suppose that $2n = 3m + l$, where $l \in \{0, 1, 2\}$. Then we know by (241) that

$$Q_{2n}(t) = t^l \widetilde{Q}_{2n}(t),$$

where $\widetilde{Q}_{2n}$ is a monic polynomial of degree exactly m, and

$$Q_{2n+1}(t) = t^{l+1} \widetilde{Q}_{2n+1}(t),$$

where $\widetilde{Q}_{2n+1}$ is also a monic polynomial of degree exactly m. Therefore the polynomial $tQ_{2n}(t) - Q_{2n+1}(t)$ has degree at most $2n-2$. This implies (247).

Similarly one shows that for all $n \geq 1$,

$$zQ_{2n+1} = Q_{2n+2} + a_{2n+1} Q_{2n-1}, \qquad a_{2n+1} \in \mathbb{R}.$$

This completes the proof of (172).

Since

$$\int_{S_0} t^n Q_{2n}(t) s_1(t) dt = \int_{S_0} t^{n-1} Q_{2n+1}(t) s_1(t) dt + a_{2n} \int_{S_0} t^{n-1} Q_{2n-2}(t) s_1(t) dt,$$

and the first integral in the right-hand side vanishes, we get

$$a_{2n} = \frac{\int_{S_0} t^n Q_{2n}(t) s_1(t) dt}{\int_{S_0} t^{n-1} Q_{2n-2}(t) s_1(t) dt}.$$

We know by (224) that for every n,

$$\int_{S_0} t^n Q_{2n}(t) s_1(t) dt = 3 \int_0^\alpha t^n Q_{2n}(t) s_1(t) dt,$$

since $d_{2n} = 2n$. This shows (174), and similarly one proves (175).

The positivity of the recurrence coefficients is proved later in Proposition IV.3.8.

□

IV.3 The second type functions Ψ_n and associated polynomials $Q_{n,2}$

Proposition IV.3.1. *The following formula holds:*

$$\Psi_n(z) = \int_0^\alpha \left(\frac{1}{t-z} + \frac{e^{\frac{2\pi i n}{3}}}{e^{\frac{2\pi i}{3}} t - z} + \frac{e^{\frac{4\pi i n}{3}}}{e^{\frac{4\pi i}{3}} t - z} \right) Q_n(t)\, s_1(t)\, dt, \qquad z \notin S_0. \tag{249}$$

In particular, for $z \notin S_0$ and any integer $k \geq 0$,

$$\Psi_{3k}(z) = 3z^2 \int_0^\alpha \frac{Q_{3k}(t)\, s_1(t)}{t^3 - z^3}\, dt = z^2 \int_0^{\alpha^3} \frac{Q_{3k}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})}{\tau - z^3} \frac{d\tau}{\tau^{2/3}}, \tag{250}$$

$$\Psi_{3k+1}(z) = 3 \int_0^\alpha \frac{t^2\, Q_{3k+1}(t)\, s_1(t)}{t^3 - z^3}\, dt = \int_0^{\alpha^3} \frac{Q_{3k+1}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})}{\tau - z^3}\, d\tau, \tag{251}$$

$$\Psi_{3k+2}(z) = 3z \int_0^\alpha \frac{t\, Q_{3k+2}(t)\, s_1(t)}{t^3 - z^3}\, dt = z \int_0^{\alpha^3} \frac{Q_{3k+2}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})}{\tau - z^3} \frac{d\tau}{\tau^{1/3}}. \tag{252}$$

Proof. By definition,

$$\Psi_n(z) = \int_{S_0} \frac{Q_n(t)\, s_1(t)}{t - z}\, dt$$

$$= \int_0^{\alpha} \left[\frac{Q_n(t)\, s_1(t)}{t - z} + \frac{Q_n(e^{\frac{2\pi i}{3}} t)\, s_1(e^{\frac{2\pi i}{3}} t)}{e^{\frac{2\pi i}{3}} t - z}\, e^{\frac{2\pi i}{3}} + \frac{Q_n(e^{\frac{4\pi i}{3}} t)\, s_1(e^{\frac{4\pi i}{3}} t)}{e^{\frac{4\pi i}{3}} t - z}\, e^{\frac{4\pi i}{3}} \right] dt.$$

Applying the symmetry properties (162) and (241), we obtain (249). The formulas (250)–(252) follow immediately from (249). □

Proposition IV.3.2. *For any integer $l \geq 0$, the following orthogonality conditions hold:*

$$0 = \int_0^{\alpha^3} \tau^k\, Q_{6l}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}}, \qquad 0 \leq k \leq l - 1, \tag{253}$$

$$0 = \int_0^{\alpha^3} \tau^k\, Q_{6l+1}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, d\tau, \qquad 0 \leq k \leq l - 1, \tag{254}$$

$$0 = \int_0^{\alpha^3} \tau^k\, Q_{6l+2}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{1/3}}, \qquad 0 \leq k \leq l - 1, \tag{255}$$

$$0 = \int_0^{\alpha^3} \tau^k\, Q_{6l+3}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{2/3}}, \qquad 0 \leq k \leq l, \tag{256}$$

$$0 = \int_0^{\alpha^3} \tau^k\, Q_{6l+4}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, d\tau, \qquad 0 \leq k \leq l - 1, \tag{257}$$

$$0 = \int_0^{\alpha^3} \tau^k\, Q_{6l+5}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})\, \frac{d\tau}{\tau^{1/3}}, \qquad 0 \leq k \leq l. \tag{258}$$

Proof. It follows from Proposition IV.2.3 and (241) that for any integer $n \geq 0$, the following orthogonality properties hold:

$$0 = \int_0^{\alpha} t^j\, Q_{2n}(t)\, (1 + e^{2\pi i(j+2n)/3} + e^{4\pi i(j+2n)/3})\, s_1(t)\, dt, \qquad 0 \leq j \leq n - 1, \tag{259}$$

$$0 = \int_0^{\alpha} t^j\, Q_{2n+1}(t)\, (1 + e^{2\pi i(j+2n+1)/3} + e^{4\pi i(j+2n+1)/3})\, s_1(t)\, dt, \qquad 0 \leq j \leq n. \tag{260}$$

Taking $n = 3l$ in (259) and (260) we obtain

$$0 = \int_0^{\alpha} t^j\, Q_{6l}(t)\, (1 + e^{2\pi ij/3} + e^{4\pi ij/3})\, s_1(t)\, dt, \qquad 0 \leq j \leq 3l - 1,$$

$$0 = \int_0^{\alpha} t^j\, Q_{6l+1}(t)\, (1 + e^{2\pi i(j+1)/3} + e^{4\pi i(j+1)/3})\, s_1(t)\, dt, \qquad 0 \leq j \leq 3l.$$

Hence

$$0 = \int_0^\alpha t^{3k}\, Q_{6l}(t)\, s_1(t)\, dt, \qquad 0 \le k \le l-1,$$

$$0 = \int_0^\alpha t^{3k+2}\, Q_{6l+1}(t)\, s_1(t)\, dt, \qquad 0 \le k \le l-1,$$

and (253)–(254) follow after applying the transformation $\tau = t^3$. Similarly, replacing n by $3l+1$ and $3l+2$ in (259)–(260), we get

$$0 = \int_0^\alpha t^j\, Q_{6l+2}(t)\, (1 + e^{2\pi i(j+2)/3} + e^{4\pi i(j+2)/3})\, s_1(t)\, dt, \qquad 0 \le j \le 3l,$$

$$0 = \int_0^\alpha t^j\, Q_{6l+3}(t)\, (1 + e^{2\pi ij/3} + e^{4\pi ij/3})\, s_1(t)\, dt, \qquad 0 \le j \le 3l+1,$$

$$0 = \int_0^\alpha t^j\, Q_{6l+4}(t)\, (1 + e^{2\pi i(j+1)/3} + e^{4\pi i(j+1)/3})\, s_1(t)\, dt, \qquad 0 \le j \le 3l+1,$$

$$0 = \int_0^\alpha t^j\, Q_{6l+5}(t)\, (1 + e^{2\pi i(j+2)/3} + e^{4\pi i(j+2)/3})\, s_1(t)\, dt, \qquad 0 \le j \le 3l+2,$$

which imply (255)–(258). □

Corollary IV.3.3. *The following holds:*

$$\Psi_{6l}(z) = O\left(\frac{1}{z^{3l+1}}\right), \quad z \to \infty, \tag{261}$$

$$\Psi_{6l+1}(z) = O\left(\frac{1}{z^{3l+3}}\right), \quad z \to \infty, \tag{262}$$

$$\Psi_{6l+2}(z) = O\left(\frac{1}{z^{3l+2}}\right), \quad z \to \infty, \tag{263}$$

$$\Psi_{6l+3}(z) = O\left(\frac{1}{z^{3l+4}}\right), \quad z \to \infty, \tag{264}$$

$$\Psi_{6l+4}(z) = O\left(\frac{1}{z^{3l+3}}\right), \quad z \to \infty, \tag{265}$$

$$\Psi_{6l+5}(z) = O\left(\frac{1}{z^{3l+5}}\right), \quad z \to \infty. \tag{266}$$

Proof. By (177) we know that $\Psi_{2n}(z) = O(1/z^{n+1})$, which implies (261), (263), and (265). We can improve the estimate $\Psi_{2n+1}(z) = O(1/z^{n+2})$ given in (177). If we define the functions

$$G_{6l+1}(z) := \int_0^{\alpha^3} \frac{Q_{6l+1}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})}{\tau - z}\, d\tau,$$

$$G_{6l+3}(z) := \int_0^{\alpha^3} \frac{Q_{6l+3}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})}{\tau - z}\, \frac{d\tau}{\tau^{2/3}},$$

$$G_{6l+5}(z) := \int_0^{\alpha^3} \frac{Q_{6l+5}(\sqrt[3]{\tau})\, s_1(\sqrt[3]{\tau})}{\tau - z}\, \frac{d\tau}{\tau^{1/3}},$$

it follows from Proposition IV.3.2 that

$$G_{6l+1}(z) = O(1/z^{l+1}), \qquad z \to \infty,$$

$$G_{6l+3}(z) = O(1/z^{l+2}), \qquad z \to \infty,$$

$$G_{6l+5}(z) = O(1/z^{l+2}), \qquad z \to \infty,$$

therefore

$$\Psi_{6l+1}(z) = G_{6l+1}(z^3) = O(1/z^{3l+3}), \qquad z \to \infty,$$

$$\Psi_{6l+3}(z) = z^2\, G_{6l+3}(z^3) = O(1/z^{3l+4}), \qquad z \to \infty,$$

$$\Psi_{6l+5}(z) = z\, G_{6l+5}(z^3) = O(1/z^{3l+5}), \qquad z \to \infty.$$

□

It is convenient to rewrite the orthogonality conditions obtained in Proposition IV.2.2 in terms of the interval $(-b, -a)$.

Proposition IV.3.4. *The functions Ψ_n satisfy:*

$$0 = \int_{-b}^{-a} t^{\nu}\, \Psi_{2n}(t)\, (1 + e^{\frac{2\pi i}{3}(\nu - 4n - 1)} + e^{\frac{4\pi i}{3}(\nu - 4n - 1)})\, s_2(t)\, dt, \qquad \nu = 0, \ldots, n-1, \quad (267)$$

$$0 = \int_{-b}^{-a} t^{\nu}\, \Psi_{2n+1}(t)\, (1 + e^{\frac{2\pi i}{3}(\nu - n)} + e^{\frac{4\pi i}{3}(\nu - n)})\, s_2(t)\, dt, \qquad \nu = 0, \ldots, n-1. \quad (268)$$

In particular, for any integer $l \geq 0$,

$$0 = \int_{-b^3}^{-a^3} \tau^k \, \Psi_{6l}(\sqrt[3]{\tau}) \, s_2(\sqrt[3]{\tau}) \, \frac{d\tau}{\tau^{1/3}}, \qquad 0 \leq k \leq l-1, \tag{269}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, \Psi_{6l+1}(\sqrt[3]{\tau}) \, s_2(\sqrt[3]{\tau}) \, \frac{d\tau}{\tau^{2/3}}, \qquad 0 \leq k \leq l-1, \tag{270}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, \Psi_{6l+2}(\sqrt[3]{\tau}) \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l-1, \tag{271}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, \Psi_{6l+3}(\sqrt[3]{\tau}) \, s_2(\sqrt[3]{\tau}) \, \frac{d\tau}{\tau^{1/3}}, \qquad 0 \leq k \leq l-1, \tag{272}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, \Psi_{6l+4}(\sqrt[3]{\tau}) \, s_2(\sqrt[3]{\tau}) \, \frac{d\tau}{\tau^{2/3}}, \qquad 0 \leq k \leq l, \tag{273}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, \Psi_{6l+5}(\sqrt[3]{\tau}) \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l-1, \tag{274}$$

Proof. By (220), for $\nu = 0, \ldots, n-1$,

$$0 = \int_{S_1} t^\nu \, \Psi_{2n}(t) \, s_2(t) \, dt = \int_a^b t^\nu \, e^{\pi i(\nu+1)/3} \, \Psi_{2n}(e^{\frac{\pi i}{3}} t) \, s_2(e^{\frac{\pi i}{3}} t) \, dt$$

$$+ \int_a^b t^\nu (-1)^{\nu+1} \, \Psi_{2n}(-t) \, s_2(-t) \, dt + \int_a^b t^\nu e^{5\pi i(\nu+1)/3} \, \Psi_{2n}(e^{\frac{5\pi i}{3}} t) \, s_2(e^{\frac{5\pi i}{3}} t) \, dt.$$

By (242) and (216) we have

$$\Psi_{2n}(e^{\frac{\pi i}{3}} t) = e^{-4\pi i(1+4n)/3} \Psi_{2n}(-t), \qquad \Psi_{2n}(e^{\frac{5\pi i}{3}} t) = e^{-2\pi i(1+4n)/3} \Psi_{2n}(-t),$$

$$s_2(e^{\frac{\pi i}{3}} t) = e^{\frac{2\pi i}{3}} s_2(-t), \qquad s_2(e^{\frac{5\pi i}{3}} t) = e^{\frac{4\pi i}{3}} s_2(-t),$$

and (267) follows. The proof of (268) is similar and is left to the reader. The orthogonality conditions (269)–(274) follow immediately from (267)–(268). □

As a consequence of (269)–(274) we obtain

Corollary IV.3.5. *For each $j \in \{0, 1, 2, 3, 5\}$, the function Ψ_{6l+j} has at least l sign changes in the interval $(-b, -a)$, and the function Ψ_{6l+4} has at least $l+1$ sign changes* *in the interval* $(-b,-a)$. *Therefore the functions* $\Psi_{6l+j}, j \in \{0,1,2,3,5\}$ *have at least* $3l$ *zeros, counting multiplicities, in* $\mathbb{C}\setminus S_0$, *and* Ψ_{6l+4} *has at least* $3l+3$ *zeros, counting multiplicities, in* $\mathbb{C}\setminus S_0$.

Observe that the function Ψ_n satisfies the property

$$\Psi_n(\overline{z}) = -\overline{\Psi_n(z)}, \qquad z \in \mathbb{C}\setminus S_0,$$

hence, z is a zero of Ψ_n if and only if $\overline{z}$ is a zero of Ψ_n.

Let $j \in \{0,1,2,3,5\}$ and assume that $x_1,\ldots,x_l$ are l distinct zeros of Ψ_{6l+j} in $(-\infty,0)$. Then the points

$$e^{\frac{2\pi i}{3}}x_1,\ldots,e^{\frac{2\pi i}{3}}x_l, e^{\frac{4\pi i}{3}}x_1,\ldots,e^{\frac{4\pi i}{3}}x_l$$

are also zeros of Ψ_{6l+j}. Since

$$(z-x)(z-e^{\frac{2\pi i}{3}}x)(z-e^{\frac{4\pi i}{3}}x) = z^3 - x^3,$$

we have that

$$R_1(z) := \prod_{k=1}^{l}(z-x_k)\prod_{k=1}^{l}(z-e^{\frac{2\pi i}{3}}x_k)\prod_{k=1}^{l}(z-e^{\frac{4\pi i}{3}}x_k)$$

is a polynomial in z^3 with real coefficients. Assume further that Ψ_{6l+j} has more than $3l$ zeros in $\mathbb{C}\setminus S_0$, counting multiplicities. Then there exists a point $z_0 \in \mathbb{C}\setminus S_0$ such that the polynomial

$$R_2(z) := R_1(z)(z-z_0)(z-e^{\frac{2\pi i}{3}}z_0)(z-e^{\frac{4\pi i}{3}}z_0)$$

satisfies

$$\frac{\Psi_{6l+j}}{R_2} \in H(\overline{\mathbb{C}} \setminus S_0).$$

If $z_0 \in \mathbb{R}$ then R_2 is also a polynomial in z^3 with real coefficients. If $z_0 \notin \mathbb{R}$ then R_2 does not have real coefficients, but the polynomial

$$R_3(z) := R_1(z)(z - z_0)(z - e^{\frac{2\pi i}{3}} z_0)(z - e^{\frac{4\pi i}{3}} z_0)(z - \overline{z}_0)(z - e^{\frac{2\pi i}{3}} \overline{z}_0)(z - e^{\frac{4\pi i}{3}} \overline{z}_0)$$

is a polynomial in z^3 with real coefficients such that

$$\frac{\Psi_{6l+j}}{R_3} \in H(\overline{\mathbb{C}} \setminus S_0).$$

In any case, if we assume that $\Psi_{6l+j}, j \in \{0, 1, 2, 3, 5\}$, has more than $3l$ zeros in $\mathbb{C} \setminus S_0$, counting multiplicities, then we can find a polynomial R_{6l+j} with real coefficients and degree at least $3l + 3$ satisfying

$$R_{6l+j}(z) = R_{6l+j}(e^{\frac{2\pi i}{3}} z) = R_{6l+j}(e^{\frac{4\pi i}{3}} z), \qquad z \in \mathbb{C}, \tag{275}$$

$$\frac{\Psi_{6l+j}}{R_{6l+j}} \in H(\overline{\mathbb{C}} \setminus S_0). \tag{276}$$

Similarly, if we assume that Ψ_{6l+4} has more than $3l + 3$ zeros in $\mathbb{C} \setminus S_0$, counting multiplicities, then there exists a polynomial R_{6l+4} with real coefficients and degree at least $3l + 6$ such that

$$R_{6l+4}(z) = R_{6l+4}(e^{\frac{2\pi i}{3}} z) = R_{6l+4}(e^{\frac{4\pi i}{3}} z), \qquad z \in \mathbb{C}, \tag{277}$$

$$\frac{\Psi_{6l+4}}{R_{6l+4}} \in H(\overline{\mathbb{C}} \setminus S_0). \tag{278}$$

Proof of Proposition IV.1.3. Suppose that Ψ_{6l} has more than $3l$ zeros in $\mathbb{C} \setminus S_0$, counting multiplicities. Let R_{6l} be a polynomial with real coefficients and degree at least $3l+3$ satisfying (275) and (276). By (261) we have

$$\frac{\Psi_{6l}(z)}{R_{6l}(z)} = O\Big(\frac{1}{z^{6l+4}}\Big), \qquad z \to \infty.$$

Let Γ be a positively oriented, smooth Jordan curve surrounding S_0 such that the zeros of R_{6l} lie in the unbounded component of $\mathbb{C} \setminus \Gamma$. By Cauchy's theorem, formula (249), Fubini's theorem, and Cauchy's integral formula, for $\nu = 0, \ldots, 6l+2$,

$$0 = \int_\Gamma z^\nu \, \frac{\Psi_{6l}(z)}{R_{6l}(z)} \, dz$$

$$= \int_\Gamma \frac{z^\nu}{R_{6l}(z)} \frac{1}{2\pi i} \int_0^\alpha \Big(\frac{1}{t-z} + \frac{1}{e^{\frac{2\pi i}{3}}t - z} + \frac{1}{e^{\frac{4\pi i}{3}}t - z}\Big) Q_{6l}(t)\, s_1(t)\, dt\, dz$$

$$= -\int_0^\alpha t^\nu \Big[\frac{1}{R_{6l}(t)} + \frac{e^{2\pi i\,\nu/3}}{R_{6l}(e^{\frac{2\pi i}{3}}t)} + \frac{e^{4\pi i\,\nu/3}}{R_{6l}(e^{\frac{4\pi i}{3}}t)}\Big]\, Q_{6l}(t)\, s_1(t)\, dt,$$

and applying (275) we obtain

$$0 = \int_0^\alpha t^\nu (1 + e^{2\pi i\,\nu/3} + e^{4\pi i\,\nu/3})\, Q_{6l}(t)\, \frac{s_1(t)}{R_{6l}(t)}\, dt, \qquad 0 \le \nu \le 6l+2,$$

which implies

$$0 = \int_0^\alpha t^{3k}\, Q_{6l}(t)\, \frac{s_1(t)}{R_{6l}(t)}\, dt, \qquad 0 \le k \le 2l.$$

As a consequence, Q_{6l} has at least $2l+1$ sign changes in $(0,\alpha)$, which contradicts Proposition IV.1.1. This proves the claim in the case $n = 6l$. In the remaining cases we use the same argument. Indeed, we can select polynomials $R_{6l+j}, 1 \le j \le 5$ (recall that R_{6l+4} has degree at least $6l+4$) satisfying (275)–(278) and such that

$$\frac{\Psi_{6l+1}(z)}{R_{6l+1}(z)} = O\Big(\frac{1}{z^{6l+6}}\Big), \qquad z \to \infty,$$

$$\frac{\Psi_{6l+2}(z)}{R_{6l+2}(z)} = O\Big(\frac{1}{z^{6l+5}}\Big), \qquad z \to \infty,$$

$$\frac{\Psi_{6l+3}(z)}{R_{6l+3}(z)} = O\Big(\frac{1}{z^{6l+7}}\Big), \qquad z \to \infty,$$

$$\frac{\Psi_{6l+4}(z)}{R_{6l+4}(z)} = O\Big(\frac{1}{z^{6l+9}}\Big), \qquad z \to \infty,$$

$$\frac{\Psi_{6l+5}(z)}{R_{6l+5}(z)} = O\Big(\frac{1}{z^{6l+8}}\Big), \qquad z \to \infty.$$

The orthogonality conditions that we obtain for the polynomials $Q_{6l+j}, 1 \leq j \leq 5$, are

$$0 = \int_0^\alpha t^{3k+2}\, Q_{6l+1}(t)\, \frac{s_1(t)}{R_{6l+1}(t)}\, dt, \qquad 0 \leq k \leq 2l,$$

$$0 = \int_0^\alpha t^{3k+1}\, Q_{6l+2}(t)\, \frac{s_1(t)}{R_{6l+2}(t)}\, dt, \qquad 0 \leq k \leq 2l,$$

$$0 = \int_0^\alpha t^{3k}\, Q_{6l+3}(t)\, \frac{s_1(t)}{R_{6l+3}(t)}\, dt, \qquad 0 \leq k \leq 2l+1,$$

$$0 = \int_0^\alpha t^{3k+2}\, Q_{6l+4}(t)\, \frac{s_1(t)}{R_{6l+4}(t)}\, dt, \qquad 0 \leq k \leq 2l+1,$$

$$0 = \int_0^\alpha t^{3k+1}\, Q_{6l+5}(t)\, \frac{s_1(t)}{R_{6l+5}(t)}\, dt, \qquad 0 \leq k \leq 2l+1,$$

and they contradict the number of simple zeros that the polynomials $Q_{6l+j}, 1 \leq j \leq 5$, on the interval $(0, \alpha)$ (see Proposition IV.1.1). □

Recall that $Q_{n,2}$ denotes the monic polynomial whose zeros coincide with the zeros of Ψ_n outside S_0. So we have proved the following:

Proposition IV.3.6. *For each* $j \in \{0, 1, 2, 3, 5\}$, $\deg(Q_{6l+j,2}) = 3l$, *and* $\deg(Q_{6l+4,2}) = 3l + 3$. *Furthermore, the following orthogonality conditions with respect to varying measures hold:*

$$0 = \int_0^\alpha t^{3k}\, Q_{6l}(t)\, \frac{s_1(t)}{Q_{6l,2}(t)}\, dt, \qquad 0 \leq k \leq 2l-1, \tag{279}$$

$$0 = \int_0^\alpha t^{3k+2}\, Q_{6l+1}(t)\, \frac{s_1(t)}{Q_{6l+1,2}(t)}\, dt, \qquad 0 \leq k \leq 2l-1, \tag{280}$$

$$0 = \int_0^\alpha t^{3k+1} Q_{6l+2}(t) \frac{s_1(t)}{Q_{6l+2,2}(t)} dt, \qquad 0 \le k \le 2l-1, \tag{281}$$

$$0 = \int_0^\alpha t^{3k} Q_{6l+3}(t) \frac{s_1(t)}{Q_{6l+3,2}(t)} dt, \qquad 0 \le k \le 2l, \tag{282}$$

$$0 = \int_0^\alpha t^{3k+2} Q_{6l+4}(t) \frac{s_1(t)}{Q_{6l+4,2}(t)} dt, \qquad 0 \le k \le 2l, \tag{283}$$

$$0 = \int_0^\alpha t^{3k+1} Q_{6l+5}(t) \frac{s_1(t)}{Q_{6l+5,2}(t)} dt, \qquad 0 \le k \le 2l. \tag{284}$$

Proposition IV.3.7. *The following formulas are valid for any fixed $z \in \mathbb{C} \setminus S_0$. If q is a polynomial of degree at most $3k$, then*

$$\frac{q(z)\Psi_{3k}(z)}{Q_{3k,2}(z)} = \int_0^\alpha \frac{Q_{3k}(x)\, s_1(x)}{Q_{3k,2}(x)} \Big(\frac{q(x)}{x-z} + \frac{q(e^{\frac{2\pi i}{3}} x)}{e^{\frac{2\pi i}{3}} x - z} + \frac{q(e^{\frac{4\pi i}{3}} x)}{e^{\frac{4\pi i}{3}} x - z} \Big) dx. \tag{285}$$

If $\deg(q) \le 3k+2$, then

$$\frac{q(z)\Psi_{3k+1}(z)}{Q_{3k+1,2}(z)} = \int_0^\alpha \frac{Q_{3k+1}(x)\, s_1(x)}{Q_{3k+1,2}(x)} \Big(\frac{q(x)}{x-z} + \frac{e^{\frac{2\pi i}{3}} q(e^{\frac{2\pi i}{3}} x)}{e^{\frac{2\pi i}{3}} x - z} + \frac{e^{\frac{4\pi i}{3}} q(e^{\frac{4\pi i}{3}} x)}{e^{\frac{4\pi i}{3}} x - z} \Big) dx. \tag{286}$$

If $\deg(q) \le 3k+1$, then

$$\frac{q(z)\Psi_{3k+2}(z)}{Q_{3k+2,2}(z)} = \int_0^\alpha \frac{Q_{3k+2}(x)\, s_1(x)}{Q_{3k+2,2}(x)} \Big(\frac{q(x)}{x-z} + \frac{e^{\frac{4\pi i}{3}} q(e^{\frac{2\pi i}{3}} x)}{e^{\frac{2\pi i}{3}} x - z} + \frac{e^{\frac{2\pi i}{3}} q(e^{\frac{4\pi i}{3}} x)}{e^{\frac{4\pi i}{3}} x - z} \Big) dx. \tag{287}$$

In particular, we have

$$\frac{Q_{3k}(z)\Psi_{3k}(z)}{Q_{3k,2}(z)} = 3z^2 \int_0^\alpha \frac{Q_{3k}^2(x)}{Q_{3k,2}(x)} \frac{s_1(x)}{x^3 - z^3} dx, \tag{288}$$

$$\frac{Q_{3k+1}(z)\Psi_{3k+1}(z)}{Q_{3k+1,2}(z)} = 3z \int_0^\alpha \frac{Q_{3k+1}^2(x)}{Q_{3k+1,2}(x)} \frac{x\, s_1(x)}{x^3 - z^3} dx, \tag{289}$$

$$\frac{Q_{3k+2}(z)\Psi_{3k+2}(z)}{Q_{3k+2,2}(z)} = 3z^3 \int_0^\alpha \frac{Q_{3k+2}^2(x)}{Q_{3k+2,2}(x)} \frac{s_1(x)}{x(x^3 - z^3)} dx. \tag{290}$$

Proof. By (261) and Proposition IV.3.6, we know that if q is a polynomial of degree at most $6l$, then

$$\frac{q(z)\Psi_{6l}(z)}{Q_{6l,2}(z)} = O\Big(\frac{1}{z}\Big), \qquad z \to \infty, \tag{291}$$

and if q is a polynomial of degree at most $6l+3$, then by (264) and Proposition IV.3.6,

$$\frac{q(z)\Psi_{6l+3}(z)}{Q_{6l+3,2}(z)} = O\Big(\frac{1}{z}\Big), \qquad z \to \infty. \tag{292}$$

Let $z \in \mathbb{C} \setminus S_0$ and define a simple closed curve Γ surrounding S_0 so that z lies in the unbounded component of $\mathbb{C} \setminus \Gamma$. We also assume that Γ is oriented clockwise and the zeros of $Q_{6l,2}$ and $Q_{6l+3,2}$ lie in the unbounded component of $\mathbb{C} \setminus \Gamma$. If $\deg q \leq 6l$, by (291), Cauchy's theorem, (249), Fubini's theorem, and Cauchy's integral formula, we have

$$\frac{q(z)\Psi_{6l}(z)}{Q_{6l,2}(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{q(t)\Psi_{6l}(t)}{Q_{6l,2}(t)}\frac{dt}{t-z}$$

$$= \int_\Gamma \frac{q(t)}{Q_{6l,2}(t)(t-z)}\frac{1}{2\pi i}\int_0^\alpha \Big[\frac{1}{x-t} + \frac{1}{e^{\frac{2\pi i}{3}}x-t} + \frac{1}{e^{\frac{4\pi i}{3}}x-t}\Big] Q_{6l}(x)s_1(x)\,dx\,dt$$

$$= \int_0^\alpha Q_{6l}(x)\,s_1(x)\frac{1}{2\pi i}\int_\Gamma \frac{q(t)}{Q_{6l,2}(t)(t-z)}\Big[\frac{1}{x-t} + \frac{1}{e^{\frac{2\pi i}{3}}x-t} + \frac{1}{e^{\frac{4\pi i}{3}}x-t}\Big]dt\,dx$$

$$= \int_0^\alpha \frac{Q_{6l}(x)\,s_1(x)}{Q_{6l,2}(x)}\Big(\frac{q(x)}{x-z} + \frac{q(e^{\frac{2\pi i}{3}}x)}{e^{\frac{2\pi i}{3}}x-z} + \frac{q(e^{\frac{4\pi i}{3}}x)}{e^{\frac{4\pi i}{3}}x-z}\Big)dx,$$

where in the last equality we used that

$$Q_{6l,2}(t) = Q_{6l,2}(e^{\frac{2\pi i}{3}}t) = Q_{6l,2}(e^{\frac{4\pi i}{3}}t).$$

Analogously, if $\deg q \leq 6l+3$, applying (292) we obtain

$$\frac{q(z)\Psi_{6l+3}(z)}{Q_{6l+3,2}(z)} = \int_0^\alpha \frac{Q_{6l+3}(x)\,s_1(x)}{Q_{6l+3,2}(x)}\Big(\frac{q(x)}{x-z} + \frac{q(e^{\frac{2\pi i}{3}}x)}{e^{\frac{2\pi i}{3}}x-z} + \frac{q(e^{\frac{4\pi i}{3}}x)}{e^{\frac{4\pi i}{3}}x-z}\Big)dx.$$

Therefore (285) follows, since we checked that it is valid for $k = 2l$ and $k = 2l + 1$. The proofs of (286)–(287) are analogous.

To obtain (288) and (289), we replace q in formulas (285) and (286), by Q_{3k} and Q_{3k+1}, respectively. Formula (290) follows from (287) by taking $q(z) = Q_{3k+2}(z)/z$.

□

Proposition IV.3.8. *The recurrence coefficients* $\{a_n\}_{n=2}^{\infty}$ *that appear in* (172) *are all positive.*

Proof. We know by (174) that

$$a_{6l} = \frac{\int_0^\alpha t^{3l}\, Q_{6l}(t)\, s_1(t)\, dt}{\int_0^\alpha t^{3l-1}\, Q_{6l-2}(t)\, s_1(t)\, dt}.$$

Now we write

$$\int_0^\alpha t^{3l}\, Q_{6l}(t)\, s_1(t)\, dt = \int_0^\alpha t^{3l}\, Q_{6l}(t)\, Q_{6l,2}(t)\, \frac{s_1(t)}{Q_{6l,2}(t)}\, dt.$$

Since $\deg Q_{6l,2} = 3l$, by (279) we obtain that

$$\int_0^\alpha t^{3l}\, Q_{6l}(t)\, Q_{6l,2}(t)\, \frac{s_1(t)}{Q_{6l,2}(t)}\, dt = \int_0^\alpha Q_{6l}^2(t)\, \frac{s_1(t)}{Q_{6l,2}(t)}\, dt > 0.$$

If we write

$$\int_0^\alpha t^{3l-1}\, Q_{6l-2}(t)\, s_1(t)\, dt = \int_0^\alpha t^{3l-2}\, Q_{6l-2,2}(t)\, Q_{6l-2}(t) \frac{t\, s_1(t)}{Q_{6l-2,2}(t)}\, dt,$$

taking into account that $\deg\,(t^{3l-2} Q_{6l-2,2}) = 6l - 2$ and the orthogonality conditions (283), we conclude that

$$\int_0^\alpha t^{3l-2}\, Q_{6l-2,2}(t)\, Q_{6l-2}(t) \frac{t\, s_1(t)}{Q_{6l-2,2}(t)}\, dt = \int_0^\alpha Q_{6l-2}^2(t) \frac{t\, s_1(t)}{Q_{6l-2,2}(t)}\, dt > 0.$$

Therefore $a_{6l} > 0$. Since

$$a_{6l+2} = \frac{\int_0^\alpha t^{3l+1}\, Q_{6l+2}(t)\, s_1(t)\, dt}{\int_0^\alpha t^{3l}\, Q_{6l}(t)\, s_1(t)\, dt},$$

in order to show that $a_{6l+2} > 0$ we prove that the integral in the numerator is positive. We write

$$\int_0^\alpha t^{3l+1}\, Q_{6l+2}(t)\, s_1(t)\, dt = \int_0^\alpha t^{3l+2}\, Q_{6l+2,2}(t)\, Q_{6l+2}(t)\, \frac{s_1(t)}{t\, Q_{6l+2,2}(t)}\, dt,$$

and using (281) and the fact that $\deg\,(t^{3l+2} Q_{6l+2,2}) = 6l+2$, it follows that

$$\int_0^\alpha t^{3l+2}\, Q_{6l+2,2}(t)\, Q_{6l+2}(t)\, \frac{s_1(t)}{t\, Q_{6l+2,2}(t)}\, dt = \int_0^\alpha Q_{6l+2}^2(t)\, \frac{s_1(t)}{t\, Q_{6l+2,2}(t)}\, dt > 0.$$

Finally,

$$a_{6l+4} = \frac{\int_0^\alpha t^{3l+2}\, Q_{6l+4}(t)\, s_1(t)\, dt}{\int_0^\alpha t^{3l+1}\, Q_{6l+2}(t)\, s_1(t)\, dt} > 0,$$

since both integrals are positive (recall that $\int_0^\alpha t^{3l-1}\, Q_{6l-2}(t)\, s_1(t)\, dt > 0$).

It is easy to see that the functions Ψ_n satisfy the same recurrence relation satisfied by the polynomials Q_n. In particular,

$$t\, \Psi_{6l+1}(t) = \Psi_{6l+2}(t) + a_{6l+1} \Psi_{6l-1}(t). \tag{293}$$

From (267) we have that

$$0 = \int_{-b}^{-a} t^{3k+2}\, \Psi_{6l+2}(t)\, s_2(t)\, dt = 0, \qquad 0 \le k \le l-1,$$

so if we multiply (293) by t^{3l-1} and integrate we obtain that

$$\int_{-b}^{-a} t^{3l}\, \Psi_{6l+1}(t)\, s_2(t)\, dt = a_{6l+1} \int_{-b}^{-a} t^{3l-1}\, \Psi_{6l-1}(t)\, s_2(t)\, dt. \tag{294}$$

We claim that both integrals in (294) are positive. From (288)–(290) we have

$$\frac{\Psi_{3k}(z)}{Q_{3k,2}(z)} = \frac{3z^2}{Q_{3k}(z)} \int_0^\alpha \frac{Q_{3k}^2(x)}{Q_{3k,2}(x)} \frac{s_1(x)}{x^3 - z^3}\, dx,$$

$$\frac{\Psi_{3k+1}(z)}{Q_{3k+1,2}(z)} = \frac{3z}{Q_{3k+1}(z)} \int_0^\alpha \frac{Q_{3k+1}^2(x)}{Q_{3k+1,2}(x)} \frac{x\, s_1(x)}{x^3 - z^3}\, dx,$$

$$\frac{\Psi_{3k+2}(z)}{Q_{3k+2,2}(z)} = \frac{3z^3}{Q_{3k+2}(z)} \int_0^\alpha \frac{Q_{3k+2}^2(x)}{Q_{3k+2,2}(x)} \frac{s_1(x)}{x(x^3 - z^3)}\, dx.$$

Therefore, if $z = t < 0$, then

$$\operatorname{sign}\Big(\frac{\Psi_{3k}(t)}{Q_{3k,2}(t)}\Big) = (-1)^{3k}, \tag{295}$$

$$\operatorname{sign}\Big(\frac{\Psi_{3k+1}(t)}{Q_{3k+1,2}(t)}\Big) = (-1)^{3k}, \tag{296}$$

$$\operatorname{sign}\Big(\frac{\Psi_{3k+2}(t)}{Q_{3k+2,2}(t)}\Big) = (-1)^{3k+1}. \tag{297}$$

Observe that since $\deg Q_{6l+1,2} = 3l$ and $\deg Q_{6l-1,2} = 3l-3$, by the orthogonality conditions satisfied by Ψ_{6l+1} and Ψ_{6l-1} and (295)–(297), we obtain that

$$\int_{-b}^{-a} t^{3l}\, \Psi_{6l+1}(t)\, s_2(t)\, dt = \int_{-b}^{-a} Q_{6l+1,2}(t)\Psi_{6l+1}(t)\, s_2(t)\, dt$$

$$= \int_{-b}^{-a} Q_{6l+1,2}^2(t) \frac{\Psi_{6l+1}(t)}{Q_{6l+1,2}(t)}\, s_2(t)\, dt > 0,$$

$$\int_{-b}^{-a} t^{3l-1}\, \Psi_{6l-1}(t)\, s_2(t)\, dt = \int_{-b}^{-a} t^{3l-3}\Psi_{6l-1}(t)\, t^2\, s_2(t)\, dt$$

$$= \int_{-b}^{-a} Q_{6l-1,2}(t)\Psi_{6l-1}(t)\, t^2\, s_2(t)\, dt = \int_{-b}^{-a} Q_{6l-1,2}^2(t) \frac{\Psi_{6l-1}(t)}{Q_{6l-1,2}(t)}\, t^2\, s_2(t)\, dt > 0.$$

Thus from (294) we get that $a_{6l+1} > 0$. Reasoning as before, from

$$t^{3l+1}\Psi_{6l+3}(t) = t^{3l}\Psi_{6l+4}(t) + a_{6l+3}\, t^{3l}\, \Psi_{6l+1}(t),$$

we have

$$a_{6l+3} = \frac{\int_{-b}^{-a} t^{3l+1}\Psi_{6l+3}(t)\, s_2(t)\, dt}{\int_{-b}^{-a} t^{3l}\Psi_{6l+1}(t)\, s_2(t)\, dt},$$

since $\int_{-b}^{-a} t^{3l}\Psi_{6l+4}(t)\, s_2(t)\, dt = 0$. Using the orthogonality conditions satisfied by Ψ_{6l+3}, the fact that $\deg Q_{6l+3,2} = 3l$, and (295), we obtain

$$\int_{-b}^{-a} t^{3l+1}\Psi_{6l+3}(t)\, s_2(t)\, dt = \int_{-b}^{-a} t^{3l}\Psi_{6l+3}(t)\, t\, s_2(t)\, dt$$

$$= \int_{-b}^{-a} Q_{6l+3,2}(t)\Psi_{6l+3}(t)\, t\, s_2(t)\, dt = \int_{-b}^{-a} Q^2_{6l+3,2}(t)\frac{\Psi_{6l+3}(t)}{Q_{6l+3,2}(t)}\, t\, s_2(t)\, dt > 0,$$

and so $a_{6l+3} > 0$. Finally, from

$$t^{3l+2}\Psi_{6l+5}(t) = t^{3l+1}\Psi_{6l+6}(t) + a_{6l+5}\, t^{3l+1}\Psi_{6l+3}(t),$$

$$\int_{-b}^{-a} t^{3l+1}\Psi_{6l+6}(t)\, s_2(t)\, dt = 0,$$

we have

$$a_{6l+5} = \frac{\int_{-b}^{-a} t^{3l+2}\Psi_{6l+5}(t)\, s_2(t)\, dt}{\int_{-b}^{-a} t^{3l+1}\Psi_{6l+3}(t)\, s_2(t)\, dt} > 0,$$

since both integrals are positive. □

IV.4 Interlacing properties of the zeros of Q_n and Ψ_n

Proposition IV.4.1. *Let $A, B \in \mathbb{R}$ be two constants such that $|A| + |B| > 0$. Let*

$$Y_n(z) := A\, z\Psi_n(z) + B\,\Psi_{n+1}(z), \tag{298}$$

$$T_n(z) := A\, zQ_n(z) + B\, Q_{n+1}(z). \tag{299}$$

Then, for every $n \geq 0$, the function Y_n has only simple zeros on $(-\infty, 0)$. Similarly, for every $n \geq 0$, the polynomial T_n has only simple zeros on $(0, \alpha)$.

Proof. From (269)–(274) it follows that

$$0=\int_{-b^3}^{-a^3}\tau^k Y_{6l}(\sqrt[3]{\tau})\,s_2(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{2/3}},\qquad 0\le k\le l-1,$$

$$0=\int_{-b^3}^{-a^3}\tau^k Y_{6l+1}(\sqrt[3]{\tau})\,s_2(\sqrt[3]{\tau})\,d\tau,\qquad 0\le k\le l-2,$$

$$0=\int_{-b^3}^{-a^3}\tau^k Y_{6l+2}(\sqrt[3]{\tau})\,s_2(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{1/3}},\qquad 0\le k\le l-1,$$

$$0=\int_{-b^3}^{-a^3}\tau^k Y_{6l+3}(\sqrt[3]{\tau})\,s_2(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{2/3}},\qquad 0\le k\le l-1,$$

$$0=\int_{-b^3}^{-a^3}\tau^k Y_{6l+4}(\sqrt[3]{\tau})\,s_2(\sqrt[3]{\tau})\,d\tau,\qquad 0\le k\le l-1,$$

$$0=\int_{-b^3}^{-a^3}\tau^k Y_{6l+5}(\sqrt[3]{\tau})\,s_2(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{1/3}},\qquad 0\le k\le l-1.$$

These orthogonality conditions show that for each $j\in\{0,2,3,4,5\}$, the function Y_{6l+j} has at least l sign change knots in $(-\infty,0)$, and the function Y_{6l+1} has at least $l-1$ sign change knots in $(-\infty,0)$. From (242) it follows that for every n,

$$Y_n(e^{\frac{2\pi i}{3}}z)=C_n\,Y_n(z),$$

where C_n denotes a constant. Therefore the functions $Y_{6l+j}, j\in\{0,2,3,4,5\}$, have at least $3l$ zeros in $\Sigma_1\setminus\{0\}$, and Y_{6l+1} has at least $3l-3$ zeros in $\Sigma_1\setminus\{0\}$. For each $0\le j\le 5$, let R_{6l+j} denote the monic polynomial whose zeros coincide with the zeros of Y_{6l+j} on $\Sigma_1\setminus\{0\}$. Then R_{6l+j} satisfies (275), $Y_{6l+j}/R_{6l+j}\in H(\overline{\mathbb{C}}\setminus S_0)$, and using (261)–(274) we have

$$\frac{Y_{6l}(z)}{R_{6l}(z)}=O\Big(\frac{1}{z^{6l}}\Big),\qquad z\to\infty,$$

$$\frac{Y_{6l+1}(z)}{R_{6l+1}(z)}=O\Big(\frac{1}{z^{6l-1}}\Big),\qquad z\to\infty,$$

$$\frac{Y_{6l+2}(z)}{R_{6l+2}(z)}=O\Big(\frac{1}{z^{6l+1}}\Big),\qquad z\to\infty,$$

$$\frac{Y_{6l+3}(z)}{R_{6l+3}(z)} = O\left(\frac{1}{z^{6l+3}}\right), \qquad z \to \infty,$$

$$\frac{Y_{6l+4}(z)}{R_{6l+4}(z)} = O\left(\frac{1}{z^{6l+2}}\right), \qquad z \to \infty,$$

$$\frac{Y_{6l+5}(z)}{R_{6l+5}(z)} = O\left(\frac{1}{z^{6l+4}}\right), \qquad z \to \infty.$$

As before, we let Γ denote a closed curve surrounding S_0, such that the zeros of the polynomials R_{6l+j} lie in the unbounded component of $\mathbb{C} \setminus S_0$. Using Cauchy's theorem, (249), Fubini's theorem, and Cauchy's integral formula, we obtain that for $\nu = 0, \dots, 6l-2$,

$$0 = \int_\Gamma z^\nu \, \frac{Y_{6l}(z)}{R_{6l}(z)} \, dz = \int_0^\alpha x^\nu \, T_{6l}(x) \, (1 + e^{2\pi i(\nu+1)/3} + e^{4\pi i(\nu+1)/3}) \frac{s_1(x)}{R_{6l}(x)} \, dx.$$

Similarly, we have that

$$0 = \int_0^\alpha x^\nu \, T_{6l+1}(x) \, (1 + e^{2\pi i(\nu+2)/3} + e^{4\pi i(\nu+2)/3}) \frac{s_1(x)}{R_{6l+1}(x)} \, dx, \quad 0 \le \nu \le 6l-3,$$

$$0 = \int_0^\alpha x^\nu \, T_{6l+2}(x) \, (1 + e^{2\pi i\nu/3} + e^{4\pi i\nu/3}) \frac{s_1(x)}{R_{6l+2}(x)} \, dx, \quad 0 \le \nu \le 6l-1,$$

$$0 = \int_0^\alpha x^\nu \, T_{6l+3}(x) \, (1 + e^{2\pi i(\nu+1)/3} + e^{4\pi i(\nu+1)/3}) \frac{s_1(x)}{R_{6l+3}(x)} \, dx, \quad 0 \le \nu \le 6l+1,$$

$$0 = \int_0^\alpha x^\nu \, T_{6l+4}(x) \, (1 + e^{2\pi i(\nu+2)/3} + e^{4\pi i(\nu+2)/3}) \frac{s_1(x)}{R_{6l+4}(x)} \, dx, \quad 0 \le \nu \le 6l,$$

$$0 = \int_0^\alpha x^\nu \, T_{6l+5}(x) \, (1 + e^{2\pi i\nu/3} + e^{4\pi i\nu/3}) \frac{s_1(x)}{R_{6l+5}(x)} \, dx, \quad 0 \le \nu \le 6l+2,$$

and so

$$0 = \int_0^\alpha x^{3k+2} \, T_{6l}(x) \frac{s_1(x)}{R_{6l}(x)} \, dx, \qquad 0 \le k \le 2l-2, \tag{300}$$

$$0 = \int_0^\alpha x^{3k+1} \, T_{6l+1}(x) \frac{s_1(x)}{R_{6l+1}(x)} \, dx, \qquad 0 \le k \le 2l-2, \tag{301}$$

$$0 = \int_0^\alpha x^{3k} \, T_{6l+2}(x) \frac{s_1(x)}{R_{6l+2}(x)} \, dx, \qquad 0 \le k \le 2l-1, \tag{302}$$

$$0 = \int_0^\alpha x^{3k+2}\, T_{6l+3}(x) \frac{s_1(x)}{R_{6l+3}(x)}\, dx, \qquad 0 \le k \le 2l-1, \tag{303}$$

$$0 = \int_0^\alpha x^{3k+1}\, T_{6l+4}(x) \frac{s_1(x)}{R_{6l+4}(x)}\, dx, \qquad 0 \le k \le 2l-1, \tag{304}$$

$$0 = \int_0^\alpha x^{3k}\, T_{6l+5}(x) \frac{s_1(x)}{R_{6l+5}(x)}\, dx, \qquad 0 \le k \le 2l. \tag{305}$$

The orthogonality conditions (300) imply that the polynomial T_{6l} has at least $2l-1$ sign change knots in $(0,\alpha)$. Taking into account that

$$T_{6l}(e^{\frac{2\pi i}{3}} z) = e^{\frac{2\pi i}{3}} T_{6l}(z),$$

we see that any sign change knot of T_{6l} in $(0,\alpha)$ (or even in $(0,\infty)$) must be a simple zero, because otherwise T_{6l} would have at least $6l+3$ zeros, contradicting the fact that $\deg(T_{6l}) \le 6l+1$. Moreover, T_{6l} cannot have any zero of multiplicity ≥ 2 in $(0,\infty)$, because then one also obtains that T_{6l} would have at least $6l+3$ zeros. Therefore we conclude that all the zeros of T_{6l} in $(0,\infty)$ are simple. Similarly, using (301)–(305) one argues that the polynomials $T_{6l+j}, 1 \le j \le 5$, must have only simple zeros in $(0,\infty)$.

Now we show that the functions Y_n have only simple zeros in $(-\infty,0)$. We already know that Y_{6l} has at least l sign change knots in $(-\infty,0)$. If we assume that one of these sign change knots is a zero of multiplicity ≥ 3, then R_{6l} would have degree at least $3l+6$, and so we would have

$$\frac{Y_{6l}(z)}{R_{6l}(z)} = O\Big(\frac{1}{z^{6l+6}}\Big), \qquad z \to \infty.$$

Reasoning as above, we derive that (300) would be valid for $0 \le k \le 2l$, which implies that T_{6l} has at least $6l+3$ zeros, which is a contradiction. Therefore all the sign change knots of Y_{6l} in $(-\infty,0)$ must be simple zeros. Furthermore, if Y_{6l} has a zero of multiplicity ≥ 2 in $(-\infty,0)$, we can also take R_{6l} to be of degree at least

$3l + 6$, and we will arrive to a contradiction. Similarly we see that all the zeros of $Y_{6l+j}, 1 \leq j \leq 5$, contained in $(-\infty, 0)$, must be simple. □

Proof of Theorem IV.1.4. Let $x \in (0, \alpha)$ and assume that $Q_n(x) = Q_{n+1}(x) = 0$. Then x is a simple zero of Q_n and Q_{n+1}. Therefore, $Q_n'(x) \neq 0$ and $Q_{n+1}'(x) \neq 0$. Take $A = 1$ and $B = -xQ_n'(x)/Q_{n+1}'(x)$. For this choice of A and B, we have that the polynomial T_n defined by (299) satisfies

$$T_n(x) = T_n'(x) = 0,$$

contradicting Proposition IV.4.1. This shows that Q_n and Q_{n+1} do not have common zeros in $(0, \alpha)$.

Let $x \in (0, \alpha)$ be arbitrary but fixed. Taking $A = Q_{n+1}(x)/x$ and $B = -Q_n(x)$, we have that $|A| + |B| > 0$. For this choice of A and B we have $T_n(x) = 0$ trivially, therefore we must have $T_n'(x) \neq 0$, and so

$$L_n(x) := \frac{Q_{n+1}(x)\, Q_n(x)}{x} + Q_{n+1}(x)\, Q_n'(x) - Q_n(x)\, Q_{n+1}'(x) \neq 0,$$

and this is valid for every $x \in (0, \alpha)$. In particular, the sign of L_n is constant on $(0, \alpha)$. Without loss of generality we assume that $L_n > 0$ on $(0, \alpha)$. If x_1, x_2 are two consecutive zeros of Q_n in $(0, \alpha)$, since

$$L_n(x_1) = Q_{n+1}(x_1)\, Q_n'(x_1) > 0,$$

$$L_n(x_2) = Q_{n+1}(x_2)\, Q_n'(x_2) > 0,$$

and the sign of Q_n' changes at these two points, by Bolzano's theorem we find that there must be an intermediate zero of Q_{n+1}. Analogously, one shows that between two consecutive zeros of Q_{n+1} on $(0, \alpha)$ there is one of Q_n. By counting the zeros of

Q_n and Q_{n+1}, it is easy to see that between two consecutive zeros of Q_n on $(0, \alpha)$, there is exactly one intermediate zero of Q_{n+1}, and viceversa.

The same argument proves the interlacing property of the zeros of Ψ_n and Ψ_{n+1}.

□

Proof of Proposition IV.1.5. If we write

$$Q_{3k-2}(z) = b_1^{(3k-2)} z + b_4^{(3k-2)} z^4 + \cdots + z^{3k-2},$$

$$Q_{3k}(z) = b_0^{(3k)} + b_3^{(3k)} z^3 + \cdots + z^{3k},$$

$$Q_{3k+1}(z) = b_1^{(3k+1)} z + b_4^{(3k+1)} z^4 + \cdots + z^{3k+1},$$

by the recurrence relation we obtain

$$b_0^{(3k)} - b_1^{(3k+1)} = a_{3k}\, b_1^{(3k-2)}. \tag{306}$$

From Vieta formulas we derive that

$$b_0^{(3k)} = (-1)^{3k} (x_1^{(3k)} \cdots x_k^{(3k)})^3,$$

$$b_1^{(3k+1)} = (-1)^{3k} (x_1^{(3k+1)} \cdots x_k^{(3k+1)})^3,$$

and similarly $b_1^{(3k-2)}$ equals $(-1)^{3k-1}$ times the product of all non-zero roots of Q_{3k-2}. Using (306), Proposition IV.3.8, and the fact the product of all non-zero roots of Q_{3k-2} is positive, we obtain that

$$(x_1^{(3k)} \cdots x_k^{(3k)})^3 < (x_1^{(3k+1)} \cdots x_k^{(3k+1)})^3.$$

This inequality and Theorem IV.1.4 imply (179). Similarly, if we write

$$Q_{3k-1}(z) = b_2^{(3k-1)} z^2 + b_5^{(3k-1)} z^5 + \cdots + z^{3k-1},$$

$$Q_{3k+2}(z) = b_2^{(3k+2)} z^2 + b_5^{(3k+2)} z^5 + \cdots + z^{3k+2},$$

we have

$$b_1^{(3k+1)} - b_2^{(3k+2)} = a_{3k+1}\, b_2^{(3k-1)},$$

$$b_2^{(3k+2)} = (-1)^{3k} (x_1^{(3k+2)} \cdots x_k^{(3k+2)})^3,$$

and $b_2^{(3k-1)}$ equals $(-1)^{3k+1}$ times the product of all nonzero roots of Q_{3k-1}. Hence

$$(x_1^{(3k+1)} \cdots x_k^{(3k+1)})^3 < (x_1^{(3k+2)} \cdots x_k^{(3k+2)})^3,$$

which implies (180) by Theorem IV.1.4. The property (181) follows directly from Theorem IV.1.4. □

Remark IV.4.2. *For every $n \geq 0$, the polynomials Q_n and Q_{n+3} do not have any common zeros in $S_0 \setminus \{0\}$, and their zeros also interlace. Similarly, the functions Ψ_n and Ψ_{n+3} do not have common zeros in S_1 and they interlace. This follows from the fact that if A, B are real constants so that $|A| + |B| > 0$, then the functions $AQ_n + BQ_{n+3}$ and $A\Psi_n + B\Psi_{n+3}$ have only simple zeros on $(0, \alpha)$ and $(-\infty, 0)$, respectively.*

IV.5 Ratio asymptotics of the polynomials Q_n and $Q_{n,2}$

Let

$$H_n := \frac{Q_n \Psi_n}{Q_{n,2}}. \tag{307}$$

Notice that H_n is a real-valued function with constant sign in $(-\infty, 0)$.

Proposition IV.5.1. *Let $l \geq 0$ be an arbitrary integer. Then the following orthogonality conditions hold:*

$$0 = \int_{-b^3}^{-a^3} \tau^k \, Q_{6l,2}(\sqrt[3]{\tau}) \frac{|H_{6l}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau} \; Q_{6l}(\sqrt[3]{\tau})|} \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l-1. \tag{308}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, Q_{6l+1,2}(\sqrt[3]{\tau}) \frac{|H_{6l+1}(\sqrt[3]{\tau})|}{|\tau^{2/3} \, Q_{6l+1}(\sqrt[3]{\tau})|} \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l-1. \tag{309}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, Q_{6l+2,2}(\sqrt[3]{\tau}) \frac{|H_{6l+2}(\sqrt[3]{\tau})|}{|Q_{6l+2}(\sqrt[3]{\tau})|} \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l-1. \tag{310}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, Q_{6l+3,2}(\sqrt[3]{\tau}) \frac{|H_{6l+3}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau} \; Q_{6l+3}(\sqrt[3]{\tau})|} \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l-1. \tag{311}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, Q_{6l+4,2}(\sqrt[3]{\tau}) \frac{|H_{6l+4}(\sqrt[3]{\tau})|}{|\tau^{2/3} \, Q_{6l+4}(\sqrt[3]{\tau})|} \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l. \tag{312}$$

$$0 = \int_{-b^3}^{-a^3} \tau^k \, Q_{6l+5,2}(\sqrt[3]{\tau}) \frac{|H_{6l+5}(\sqrt[3]{\tau})|}{|Q_{6l+5}(\sqrt[3]{\tau})|} \, s_2(\sqrt[3]{\tau}) \, d\tau, \qquad 0 \leq k \leq l-1. \tag{313}$$

For each $j \in \{0, 1, 2, 3, 5\}$, $Q_{6l+j,2}(\sqrt[3]{\tau})$ is a polynomial in τ of degree l, and $Q_{6l+4,2}(\sqrt[3]{\tau})$ is a polynomial in τ of degree $l+1$.

Proof. The orthogonality conditions (308)–(313) follow immediately from (269)–(274). The claims concerning the degree of the polynomials $Q_{n,2}(\sqrt[3]{\tau})$ are a consequence of Proposition IV.1.3. □

Proposition IV.5.2. *Let $k \geq 0$ be an arbitrary integer. Then the following orthogonality conditions hold:*

$$0 = \int_0^{\alpha^3} \tau^j \, Q_{3k}(\sqrt[3]{\tau}) \frac{s_1(\sqrt[3]{\tau})}{Q_{3k,2}(\sqrt[3]{\tau})} \frac{d\tau}{\tau^{2/3}}, \qquad 0 \leq j \leq k-1. \tag{314}$$

$$0 = \int_0^{\alpha^3} \tau^j \, \frac{Q_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}} \frac{s_1(\sqrt[3]{\tau})}{Q_{3k+1,2}(\sqrt[3]{\tau})} \sqrt[3]{\tau} \, d\tau, \qquad 0 \leq j \leq k-1. \tag{315}$$

$$0 = \int_0^{\alpha^3} \tau^j \, \frac{Q_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}} \frac{s_1(\sqrt[3]{\tau})}{Q_{3k+2,2}(\sqrt[3]{\tau})} \sqrt[3]{\tau} \, d\tau, \qquad 0 \leq j \leq k-1. \tag{316}$$

For each $k \geq 0$, the expressions

$$Q_{3k}(\sqrt[3]{\tau}), \qquad \frac{Q_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}, \qquad \frac{Q_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}},$$

denote polynomials in τ of degree k.

Proof. The orthogonality conditions follow immediately from (279)–(284). □

For each integer $j \geq 0$, let

$$K_{3j} := \Big(\int_0^{\alpha^3} P_{3j}^2(\tau)\frac{s_1(\sqrt[3]{\tau})}{P_{3j,2}(\tau)}\frac{d\tau}{\tau^{2/3}}\Big)^{-1/2},$$

$$K_{3j+1} := \Big(\int_0^{\alpha^3} P_{3j+1}^2(\tau)\frac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+1,2}(\tau)}\,d\tau\Big)^{-1/2},$$

$$K_{3j+2} := \Big(\int_0^{\alpha^3} P_{3j+2}^2(\tau)\frac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+2,2}(\tau)}\,d\tau\Big)^{-1/2},$$

where the polynomials P_n and $P_{n,2}$ are defined in (182)–(185). Similarly, we define for each integer $j \geq 0$ the following constants

$$K_{3j,2} := \Big(\int_{-b^3}^{-a^3} P_{3j,2}^2(\tau)\,\frac{|H_{3j}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau}\,P_{3j}(\tau)|}\,s_2(\sqrt[3]{\tau})\,d\tau\Big)^{-1/2},$$

$$K_{3j+1,2} := \Big(\int_{-b^3}^{-a^3} P_{3j+1,2}^2(\tau)\,\frac{|H_{3j+1}(\sqrt[3]{\tau})|}{|\tau P_{3j+1}(\tau)|}\,s_2(\sqrt[3]{\tau})\,d\tau\Big)^{-1/2},$$

$$K_{3j+2,2} := \Big(\int_{-b^3}^{-a^3} P_{3j+2,2}^2(\tau)\,\frac{|H_{3j+2}(\sqrt[3]{\tau})|}{|\tau^{2/3}P_{3j+2}(\tau)|}\,s_2(\sqrt[3]{\tau})\,d\tau\Big)^{-1/2}.$$

We need to introduce more notations. Let

$$\kappa_n := K_n, \qquad \kappa_{n,2} := \frac{K_{n,2}}{K_n}, \tag{317}$$

consider the polynomials

$$p_n := \kappa_n P_n, \qquad p_{n,2} := \kappa_{n,2} P_{n,2}, \tag{318}$$

and the functions

$$h_n := K_n^2 H_n. \tag{319}$$

Finally, we introduce the following positive varying measures:

$$\begin{aligned}
d\nu_{3j}(\tau) &:= \tfrac{s_1(\sqrt[3]{\tau})}{P_{3j,2}(\tau)} \tfrac{d\tau}{\tau^{2/3}},\\
d\nu_{3j+1}(\tau) &:= \tfrac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+1,2}(\tau)}\, d\tau,\\
d\nu_{3j+2}(\tau) &:= \tfrac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+2,2}(\tau)}\, d\tau,\\
d\nu_{3j,2}(\tau) &:= \tfrac{|h_{3j}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau}\, P_{3j}(\tau)|}\, s_2(\sqrt[3]{\tau})\, d\tau,\\
d\nu_{3j+1,2}(\tau) &:= \tfrac{|h_{3j+1}(\sqrt[3]{\tau})|}{|\tau P_{3j+1}(\tau)|}\, s_2(\sqrt[3]{\tau})\, d\tau,\\
d\nu_{3j+2,2}(\tau) &:= \tfrac{|h_{3j+2}(\sqrt[3]{\tau})|}{|\tau^{2/3} P_{3j+2}(\tau)|}\, s_2(\sqrt[3]{\tau})\, d\tau.
\end{aligned} \tag{320}$$

Proposition IV.5.3. *The polynomials p_n and $p_{n,2}$ are orthonormal polynomials with respect to the measures $d\nu_n$ and $d\nu_{n,2}$, respectively. That is, for every $n \geq 0$,*

$$\int_0^{\alpha^3} p_n^2(\tau)\, d\nu_n(\tau) = 1,$$

$$\int_{-b^3}^{-a^3} p_{n,2}^2(\tau)\, d\nu_{n,2}(\tau) = 1,$$

and

$$\int_0^{\alpha^3} \tau^j p_n(\tau)\, d\nu_n(\tau) = 0, \qquad \textit{for all} \quad j < \deg p_n,$$

$$\int_{-b^3}^{-a^3} \tau^j p_{n,2}(\tau)\, d\nu_{n,2}(\tau) = 0, \qquad \textit{for all} \quad j < \deg p_{n,2}.$$

Proof. It follows immediately from Propositions IV.5.1 and IV.5.2. □

Using (288)–(290), it is easy to check that the functions h_n have the following representations:

$$h_{3k}(z) = z^2 \int_0^{\alpha^3} \frac{p_{3k}^2(\tau)}{\tau - z^3}\, d\nu_{3k}(\tau), \tag{321}$$

$$h_{3k+1}(z) = z \int_0^{\alpha^3} \frac{p_{3k+1}^2(\tau)}{\tau - z^3}\, d\nu_{3k+1}(\tau), \tag{322}$$

$$h_{3k+2}(z) = z^3 \int_0^{\alpha^3} \frac{p_{3k+2}^2(\tau)}{\tau - z^3}\, d\nu_{3k+2}(\tau). \tag{323}$$

Lemma IV.5.4. *Assume that $s_1(x) > 0$ a.e. on $[0, \alpha]$, and $s_2(x) > 0$ a.e. on $[-b, -a]$. If f is continuous on $[0, \alpha^3]$, then*

$$\lim_{n\to\infty} \int_0^{\alpha^3} f(\tau)\, p_n^2(\tau)\, d\nu_n(\tau) = \frac{1}{\pi} \int_0^{\alpha^3} f(\tau) \frac{d\tau}{\sqrt{(\alpha^3 - \tau)\tau}}. \tag{324}$$

Similarly, if g is continuous on $[-b^3, -a^3]$, then

$$\lim_{n\to\infty} \int_{-b^3}^{-a^3} g(\tau)\, p_{n,2}^2(\tau)\, d\nu_{n,2}(\tau) = \frac{1}{\pi} \int_{-b^3}^{-a^3} g(\tau) \frac{d\tau}{\sqrt{(-a^3 - \tau)(\tau + b^3)}}. \tag{325}$$

In particular, the following limits hold uniformly on closed subsets of $\overline{\mathbb{C}} \setminus S_0$:

$$\lim_{k\to\infty} h_{3k}(z) = -\frac{z^2}{\sqrt{(z^3 - \alpha^3)z^3}}, \tag{326}$$

$$\lim_{k\to\infty} h_{3k+1}(z) = -\frac{z}{\sqrt{(z^3 - \alpha^3)z^3}}, \tag{327}$$

$$\lim_{k\to\infty} h_{3k+2}(z) = -\frac{z^3}{\sqrt{(z^3 - \alpha^3)z^3}}, \tag{328}$$

where the branch of the square root is taken so that $\sqrt{x} > 0$ for $x > 0$.

Proof. Let us define the measures

$$d\mu_{3k}(\tau) = \frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}}\, d\tau, \qquad d\mu_{3k+1}(\tau) = d\mu_{3k+2}(\tau) = s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}\, d\tau,$$

According to Definition 2 in [6], for each $i \in \{0,1,2\}$ and $k \in \mathbb{Z}$, we know that the system $(\{d\mu_{3l+i}\}, \{P_{3l+i,2}\}, k)_{l\geq 1}$ is strongly admissible on $[0,\alpha^3]$. Then by Corollary 3 in [6], we obtain that

$$\lim_{l\to\infty} \int_0^{\alpha^3} f(\tau)\, p_{3l+i}^2(\tau)\, \frac{d\mu_{3l+i}(\tau)}{P_{3l+i,2}(\tau)} = \frac{1}{\pi} \int_0^{\alpha^3} f(\tau) \frac{d\tau}{\sqrt{(\alpha^3-\tau)\tau}},$$

for every f continuous on $[0,\alpha^3]$. Since $d\nu_{3l+i}(\tau) = d\mu_{3l+i}(\tau)/P_{3l+i,2}(\tau)$, (324) follows. The asymptotic formulas (326)–(328) are a consequence of (324) and (321)–(323).

Similarly, if we define the measures

$$dm_{3k}(\tau) = \frac{|h_{3k}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau}|}\, s_2(\sqrt[3]{\tau})\, d\tau,$$

$$dm_{3k+1}(\tau) = \frac{|h_{3k+1}(\sqrt[3]{\tau})|}{|\tau|}\, s_2(\sqrt[3]{\tau})\, d\tau,$$

$$dm_{3k+2}(\tau) = \frac{|h_{3k+2}(\sqrt[3]{\tau})|}{|\tau^{2/3}|}\, s_2(\sqrt[3]{\tau})\, d\tau,$$

then for each $i \in \{0,1,2\}$ and each $k \in \mathbb{Z}$, the system $(\{dm_{3l+i}\}, \{|P_{3l+i}|, k\})$ is strongly admissible on $[-b^3, -a^3]$, and (325) follows as before. $\square$

For each $i \in \{0,\ldots,5\}$, we consider the families of rational functions

$$\left\{\frac{P_{6k+i+1}(z)}{P_{6k+i}(z)}\right\}_k, \qquad \left\{\frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)}\right\}_k.$$

By Theorem IV.1.4, the first family is uniformly bounded on compact subsets of $\mathbb{C}\setminus[0,\alpha^3]$, and the second family is uniformly bounded on compact subsets of $\mathbb{C}\setminus[-b^3,-a^3]$. Therefore, by Montel's Theorem we can extract convergent subsequences from each family. Let $\Lambda \subset \mathbb{N}$ be a sequence of integers so that for each $i \in \{0,\ldots,5\}$,

$$\lim_{k\in\Lambda} \frac{P_{6k+i+1}(z)}{P_{6k+i}(z)} = \widetilde{F}_1^{(i)}(z), \qquad z \in \mathbb{C}\setminus[0,\alpha^3], \tag{329}$$

$$\lim_{k\in\Lambda}\frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)}=\widetilde{F}_2^{(i)}(z),\qquad z\in\mathbb{C}\setminus[-a^3,-b^3], \tag{330}$$

where the limits hold uniformly on compact subsets of the indicated regions. Our goal is to show that we obtain the same limiting functions $\widetilde{F}_j^{(i)}$, no matter which convergent subsequences we choose.

Since the zeros of the polynomials P_n are all contained in $[0,\alpha^3]$ and they interlace, from (329) we derive that for each $i\in\{0,\ldots,5\}$, the functions $\widetilde{F}_1^{(i)}, 1/\widetilde{F}_1^{(i)}$ are analytic in $\mathbb{C}\setminus[0,\alpha^3]$. Moreover, since $\deg(P_{3k})=\deg(P_{3k+1})=\deg(P_{3k+2})$ and $\deg(P_{3k+3})=\deg(P_{3k+2})+1$, we know that if $i\in\{0,1,3,4\}$, then $\widetilde{F}_1^{(i)}$ is analytic at infinity and $\widetilde{F}_1^{(i)}(\infty)=1$, whereas the functions $\widetilde{F}_1^{(2)},\widetilde{F}_1^{(5)}$ have a simple pole at infinity and

$$\widetilde{F}_1^{(2)}(z)=z+O(1),\qquad z\to\infty,$$

$$\widetilde{F}_1^{(5)}(z)=z+O(1),\qquad z\to\infty.$$

Similarly, for each $i\in\{0,\ldots,5\}$, the functions $\widetilde{F}_2^{(i)}, 1/\widetilde{F}_2^{(i)}$ are analytic in the region $\mathbb{C}\setminus[-b^3,-a^3]$. Since $\deg(P_{6k+i,2})=k$ for $i\in\{0,1,2,3,5\}$ and $\deg(P_{6k+4,2})=k+1$, we have that for $i\in\{0,1,2\}$, the functions $\widetilde{F}_2^{(i)}$ are analytic at infinity and $\widetilde{F}_2^{(i)}(\infty)=1$, whereas

$$\widetilde{F}_2^{(3)}(z)=z+O(1),\qquad z\to\infty,$$

$$\widetilde{F}_2^{(4)}(z)=1/z+O(1/z^2),\qquad z\to\infty,$$

$$\widetilde{F}_2^{(5)}(z)=z+O(1),\qquad z\to\infty.$$

Given a Borel measurable function $w\geq 0$ defined on the interval $[c,d]$ that satisfies the Szegő condition

$$\frac{\log w(t)}{\sqrt{(d-t)(t-c)}}\in L^1(dt),$$

the function

$$S(w;z) := \exp\left\{\frac{d-c}{4\pi}\sqrt{\left(\frac{2z-c-d}{d-c}\right)^2-1}\int_c^d \frac{\log w(t)}{t-z}\frac{dt}{\sqrt{(d-t)(t-c)}}\right\}$$

is called the Szegő function on $\overline{\mathbb{C}}\backslash[c,d]$ associated with w. If we introduce the notation

$$D(f;z) = \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log f(\theta)\,d\theta\right\},$$

then $S(w;z)$ can be written as

$$S(w;z) = D(\widetilde{w}; 1/\psi_{[c,d]}(z)),$$

where

$$\widetilde{w}(\theta) := w\left(\frac{d-c}{2}\cos\theta + \frac{c+d}{2}\right), \qquad \theta\in[0,2\pi],$$

and $\psi_{[c,d]}$ is the conformal mapping of $\overline{\mathbb{C}}\setminus[c,d]$ onto $\{|z|>1\}$ satisfying that $\psi(\infty) = \infty$ and $\psi'(\infty)>0$, i.e.

$$\psi_{[c,d]}(z) = \frac{2z-c-d}{d-c} + \sqrt{\left(\frac{2z-c-d}{d-c}\right)^2-1}.$$

In particular, if w is continuous at $x\in[c,d]$ and $w(x)>0$, then the limit

$$\lim_{z\to x}|S(w;z)|^2 = \frac{1}{w(x)}$$

holds. We will indicate this below by writing $|S(w;x)|^2\,w(x) = 1$.

Throughout this section we are always assuming that $s_1 > 0$ a.e. on $[0,\alpha]$, and $s_2 > 0$ a.e. on $[-b,-a]$. By (314)–(315) we have

$$0 = \int_0^{\alpha^3}\tau^j P_{6k}(\tau)\,d\nu_{6k}(\tau), \qquad 0\le j\le 2k-1,$$

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+1}(\tau)\, g_{6k}(\tau)\, d\nu_{6k}(\tau), \qquad 0 \le j \le 2k-1,$$

where $g_{6k}(\tau) := \tau P_{6k,2}(\tau)/P_{6k+1,2}(\tau)$. Using (330),

$$\lim_{k\in\Lambda} g_{6k}(\tau) = \frac{\tau}{\widetilde{F}_2^{(0)}(\tau)},$$

uniformly on $[0,\alpha^3]$. Since $\deg(P_{6k}) = \deg(P_{6k+1})$, using Theorem 2 in [6] (result on relative asymptotics of polynomials orthogonal with respect to varying measures), we obtain that

$$\lim_{k\in\Lambda} \frac{P_{6k+1}(z)}{P_{6k}(z)} = \frac{S_1^{(0)}(z)}{S_1^{(0)}(\infty)} = \widetilde{F}_1^{(0)}(z), \qquad z \in \overline{\mathbb{C}} \setminus [0,\alpha^3], \tag{331}$$

uniformly on compact subsets of the indicated region, where $S_1^{(0)}$ is the Szegő function on $\overline{\mathbb{C}} \setminus [0,\alpha^3]$ associated with the weight $\tau/\widetilde{F}_2^{(0)}(\tau)$, $\tau \in [0,\alpha^3]$. Therefore $S_1^{(0)}$ satisfies the following boundary value condition,

$$|S_1^{(0)}(\tau)|^2 \frac{\tau}{\widetilde{F}_2^{(0)}(\tau)} = 1, \qquad \tau \in (0,\alpha^3]. \tag{332}$$

Similarly, by (315)–(316) we have

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+1}(\tau)\, d\nu_{6k+1}(\tau), \qquad 0 \le j \le 2k-1,$$

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+2}(\tau)\, g_{6k+1}(\tau)\, d\nu_{6k+1}(\tau), \qquad 0 \le j \le 2k-1,$$

where $g_{6k+1}(\tau) := P_{6k+1,2}(\tau)/P_{6k+2,2}(\tau)$, and so applying the same argument we obtain that

$$\lim_{k\in\Lambda} \frac{P_{6k+2}(z)}{P_{6k+1}(z)} = \frac{S_1^{(1)}(z)}{S_1^{(1)}(\infty)} = \widetilde{F}_1^{(1)}(z), \qquad z \in \overline{\mathbb{C}} \setminus [0,\alpha^3], \tag{333}$$

uniformly on compact subsets of the indicated region, where $S_1^{(1)}$ is the Szegő function on $\overline{\mathbb{C}} \setminus [0, \alpha^3]$ associated with the weight $1/\widetilde{F}_2^{(1)}(\tau), \tau \in [0, \alpha^3]$. Therefore

$$|S_1^{(1)}(\tau)|^2 \frac{1}{\widetilde{F}_2^{(1)}(\tau)} = 1, \qquad \tau \in [0, \alpha^3]. \tag{334}$$

By Proposition IV.5.2 we know that

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+2}(\tau)\, d\nu_{6k+2}(\tau), \qquad 0 \le j \le 2k-1,$$

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+3}(\tau)\, g_{6k+2}(\tau)\, d\nu_{6k+2}(\tau), \qquad 0 \le j \le 2k,$$

where $g_{6k+2}(\tau) := P_{6k+2,2}(\tau)/(\tau P_{6k+3,2}(\tau))$. Let P_{6k+2}^* be the monic polynomial of degree $2k$ orthogonal with respect to the measure $d\nu_{6k+3}(\tau) = g_{6k+2}(\tau)\, d\nu_{6k+2}(\tau)$. Since $\deg(P_{6k+2}^*) = \deg(P_{6k+2})$, by Theorem 2 in [6] we have

$$\lim_{k \in \Lambda} \frac{P_{6k+2}^*(z)}{P_{6k+2}(z)} = \frac{S_1^{(2)}(z)}{S_1^{(2)}(\infty)}, \tag{335}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [0, \alpha^3]$, where $S_1^{(2)}$ is the Szegő function on $\overline{\mathbb{C}} \setminus [0, \alpha^3]$ with respect to the weight $1/(\tau \widetilde{F}_2^{(2)}(\tau)), \tau \in [0, \alpha^3]$. Therefore,

$$|S_1^{(2)}(\tau)|^2 \frac{1}{\tau \widetilde{F}_2^{(2)}(\tau)} = 1, \qquad \tau \in (0, \alpha^3]. \tag{336}$$

Let ϕ_1 denote the conformal mapping of $\mathbb{C} \setminus [0, \alpha^3]$ onto the exterior of the unit circle and satisfies $\phi_1(\infty) = \infty$ and $\phi_1'(\infty) > 0$. Then, by Theorem 1 in [6] (result on ratio asymptotics of polynomials orthogonal with respect to varying measures) we have

$$\lim_{k \in \Lambda} \frac{P_{6k+3}(z)}{P_{6k+2}^*(z)} = \frac{\phi_1(z)}{\phi_1'(\infty)}, \tag{337}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \alpha^3]$. Therefore by (335) and (337) we have

$$\lim_{k \in \Lambda} \frac{P_{3k+3}(z)}{P_{6k+2}(z)} = \frac{S_1^{(2)}(z)}{S_1^{(2)}(\infty)} \frac{\phi_1(z)}{\phi_1'(\infty)} = \widetilde{F}_1^{(2)}(z), \tag{338}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \alpha^3]$.

The same arguments used before show that

$$\lim_{k \in \Lambda} \frac{P_{6k+4}(z)}{P_{6k+3}(z)} = \frac{S_1^{(3)}(z)}{S_1^{(3)}(\infty)} = \widetilde{F}_1^{(3)}(z), \qquad \text{uniformly on} \quad \overline{\mathbb{C}} \setminus [0, \alpha^3], \tag{339}$$

$$\lim_{k \in \Lambda} \frac{P_{6k+5}(z)}{P_{6k+4}(z)} = \frac{S_1^{(4)}(z)}{S_1^{(4)}(\infty)} = \widetilde{F}_1^{(4)}(z), \qquad \text{uniformly on} \quad \overline{\mathbb{C}} \setminus [0, \alpha^3], \tag{340}$$

$$\lim_{k \in \Lambda} \frac{P_{6k+6}(z)}{P_{6k+5}(z)} = \frac{S_1^{(5)}(z)}{S_1^{(5)}(\infty)} \frac{\phi_1(z)}{\phi_1'(\infty)} = \widetilde{F}_1^{(5)}(z), \qquad \text{uniformly on} \quad \mathbb{C} \setminus [0, \alpha^3], \tag{341}$$

where $S_1^{(3)}, S_1^{(4)}, S_1^{(5)}$ are the Szegő functions on $\overline{\mathbb{C}} \setminus [0, \alpha^3]$ associated with the weights $\tau / \widetilde{F}_2^{(3)}(\tau), 1/\widetilde{F}_2^{(4)}(\tau), 1/(\tau \widetilde{F}_2^{(5)}(\tau))$, respectively. Therefore

$$|S_1^{(3)}(\tau)|^2 \frac{\tau}{\widetilde{F}_2^{(3)}(\tau)} = 1, \qquad \tau \in (0, \alpha^3], \tag{342}$$

$$|S_1^{(4)}(\tau)|^2 \frac{1}{\widetilde{F}_2^{(4)}(\tau)} = 1, \qquad \tau \in [0, \alpha^3], \tag{343}$$

$$|S_1^{(5)}(\tau)|^2 \frac{1}{\tau \widetilde{F}_2^{(5)}(\tau)} = 1, \qquad \tau \in (0, \alpha^3]. \tag{344}$$

Now we will derive other relations between the functions $\widetilde{F}_1^{(i)}$ and $\widetilde{F}_2^{(i)}$ which are valid on $[-b^3, -a^3]$. From (308)–(309) we have

$$0 = \int_{-b^3}^{-a^3} \tau^j P_{6k,2}(\tau) \, d\nu_{6k,2}(\tau), \qquad 0 \le j \le k-1,$$

$$0 = \int_{-b^3}^{-a^3} \tau^j P_{6k+1,2}(\tau) \, g_{6k,2}(\tau) \, d\nu_{6k,2}(\tau), \qquad 0 \le j \le k-1,$$

where

$$g_{6k,2}(\tau) := \frac{|h^{(1)}_{6k+1}(\sqrt[3]{\tau})|}{|\tau^{2/3}\, h^{(1)}_{6k}(\sqrt[3]{\tau})|}\, \frac{|P_{6k}(\tau)|}{|P_{6k+1}(\tau)|}.$$

Using Lemma IV.5.4 and (329),

$$\lim_{k\in\Lambda} g_{6k,2}(\tau) = \frac{1}{|\tau \widetilde{F}^{(0)}_1(\tau)|},$$

uniformly on $[-b^3, -a^3]$. Again, using the fact that $\deg(P_{6k,2}) = \deg(P_{6k+1,2})$, by Theorem 2 in [6] we obtain that

$$\lim_{k\in\Lambda} \frac{P_{6k+1,2}(z)}{P_{6k,2}(z)} = \frac{S^{(0)}_2(z)}{S^{(0)}_2(\infty)} = \widetilde{F}^{(0)}_2(z), \tag{345}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$, where $S^{(0)}_2$ is the Szegő function on $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$ associated with the weight $1/|\tau\, \widetilde{F}^{(0)}_1(\tau)|$, and so

$$|S^{(0)}_2(\tau)|^2 \frac{1}{|\tau\, \widetilde{F}^{(0)}_1(\tau)|} = 1, \quad \tau \in [-b^3, -a^3]. \tag{346}$$

Similarly, we have that the limits

$$\lim_{k\in\Lambda} \frac{P_{6k+2,2}(z)}{P_{6k+1,2}(z)} = \frac{S^{(1)}_2(z)}{S^{(1)}_2(\infty)} = \widetilde{F}^{(1)}_2(z), \tag{347}$$

$$\lim_{k\in\Lambda} \frac{P_{6k+3,2}(z)}{P_{6k+2,2}(z)} = \frac{S^{(2)}_2(z)}{S^{(2)}_2(\infty)} = \widetilde{F}^{(2)}_2(z), \tag{348}$$

hold uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$, where $S^{(1)}_2, S^{(2)}_2$ are the Szegő functions associated with the weights $|\tau|/|\widetilde{F}^{(1)}_1(\tau)|, 1/|\widetilde{F}^{(2)}_1(\tau)|$, respectively. Therefore

$$|S^{(1)}_2(\tau)|^2\, \frac{|\tau|}{|\widetilde{F}^{(1)}_1(\tau)|} = 1, \quad \tau \in [-b^3, -a^3], \tag{349}$$

$$|S^{(2)}_2(\tau)|^2\, \frac{1}{|\widetilde{F}^{(2)}_1(\tau)|} = 1, \quad \tau \in [-b^3, -a^3]. \tag{350}$$

Let ϕ_2 be the conformal mapping of $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$ onto the exterior of the unit circle, and satisfies the conditions $\phi_2(\infty) = \infty$ and $\phi_2'(\infty) > 0$. As a result of Theorems 1 and 2 in [6], we also obtain that the limits

$$\lim_{k\in\Lambda} \frac{P_{6k+4,2}(z)}{P_{6k+3,2}(z)} = \frac{S_2^{(3)}(z)}{S_2^{(3)}(\infty)} \frac{\phi_2(z)}{\phi_2'(\infty)} = \widetilde{F}_2^{(3)}(z), \tag{351}$$

$$\lim_{k\in\Lambda} \frac{P_{6k+5,2}(z)}{P_{6k+4,2}(z)} = \frac{S_2^{(4)}(\infty)}{S_2^{(4)}(z)} \frac{\phi_2'(\infty)}{\phi_2(z)} = \widetilde{F}_2^{(4)}(z), \tag{352}$$

$$\lim_{k\in\Lambda} \frac{P_{6k+4,2}(z)}{P_{6k+3,2}(z)} = \frac{S_2^{(5)}(z)}{S_2^{(5)}(\infty)} \frac{\phi_2(z)}{\phi_2'(\infty)} = \widetilde{F}_2^{(5)}(z), \tag{353}$$

hold uniformly on compact subsets of $\mathbb{C} \setminus [-b^3, -a^3]$, where $S_2^{(3)}, S_2^{(4)}$, and $S_2^{(5)}$ are the Szegő functions on $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$ with respect to the weights

$$1/|\tau \widetilde{F}_1^{(3)}(\tau)|, \qquad |\widetilde{F}_1^{(4)}(\tau)|/|\tau|, \qquad 1/|\widetilde{F}_1^{(5)}(\tau)|, \qquad \tau \in [-b^3, -a^3],$$

respectively. Therefore we have

$$|S_2^{(3)}(\tau)|^2 \frac{1}{|\tau\, \widetilde{F}_1^{(3)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3], \tag{354}$$

$$|S_2^{(4)}(\tau)|^2 \frac{|\widetilde{F}_1^{(4)}(\tau)|}{|\tau|} = 1, \quad \tau \in [-b^3, -a^3], \tag{355}$$

$$|S_2^{(5)}(\tau)|^2 \frac{1}{|\widetilde{F}_1^{(5)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3]. \tag{356}$$

Proposition IV.5.5. *There exist positive constants $c_k^{(l)}$, $1 \le k \le 2$, $0 \le l \le 5$, such that the functions $F_k^{(l)} := c_k^{(l)} \widetilde{F}_k^{(l)}$ satisfy the following boundary value conditions:*

$$|F_1^{(l)}(\tau)|^2 \frac{\tau}{F_2^{(l)}(\tau)} = 1, \quad \tau \in (0, \alpha^3], \quad l = 0, 3, \tag{357}$$

$$|F_1^{(l)}(\tau)|^2 \frac{1}{F_2^{(l)}(\tau)} = 1, \quad \tau \in [0, \alpha^3], \quad l = 1, 4, \tag{358}$$

$$|F_1^{(l)}(\tau)|^2 \frac{1}{\tau \, F_2^{(l)}(\tau)} = 1, \quad \tau \in (0, \alpha^3], \quad l = 2, 5, \tag{359}$$

$$|F_2^{(l)}(\tau)|^2 \frac{1}{|\tau \, F_1^{(l)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3], \quad l = 0, 3, \tag{360}$$

$$|F_2^{(l)}(\tau)|^2 \frac{|\tau|}{|F_1^{(l)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3], \quad l = 1, 4, \tag{361}$$

$$|F_2^{(l)}(\tau)|^2 \frac{1}{|F_1^{(l)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3], \quad l = 2, 5. \tag{362}$$

Proof. It follows from the relations (331)–(334), (336), and (338)–(344), that there exist positive constants $\omega_1^{(l)}$ such that

$$|\widetilde{F}_1^{(l)}(\tau)|^2 \frac{\tau}{\widetilde{F}_2^{(l)}(\tau)} = \frac{1}{\omega_1^{(l)}}, \quad \tau \in (0, \alpha^3], \quad l = 0, 3, \tag{363}$$

$$|\widetilde{F}_1^{(l)}(\tau)|^2 \frac{1}{\widetilde{F}_2^{(l)}(\tau)} = \frac{1}{\omega_1^{(l)}}, \quad \tau \in [0, \alpha^3], \quad l = 1, 4, \tag{364}$$

$$|\widetilde{F}_1^{(l)}(\tau)|^2 \frac{1}{\tau \, \widetilde{F}_2^{(l)}(\tau)} = \frac{1}{\omega_1^{(l)}}, \quad \tau \in (0, \alpha^3], \quad l = 2, 5, \tag{365}$$

where

$$\omega_1^{(l)} = (S_1^{(l)}(\infty))^2, \quad \text{for} \quad l = 0, 1, 3, 4, \tag{366}$$

$$\omega_1^{(l)} = (S_1^{(l)}(\infty) \, \phi_1'(\infty))^2, \quad \text{for} \quad l = 2, 5. \tag{367}$$

Similarly, from (345)–(356) we obtain that there exist positive constants $\omega_2^{(l)}$ such that

$$|\widetilde{F}_2^{(l)}(\tau)|^2 \frac{1}{|\tau \, \widetilde{F}_1^{(l)}(\tau)|} = \frac{1}{\omega_2^{(l)}}, \quad \tau \in [-b^3, -a^3], \quad l = 0, 3, \tag{368}$$

$$|\widetilde{F}_2^{(l)}(\tau)|^2 \frac{|\tau|}{|\widetilde{F}_1^{(l)}(\tau)|} = \frac{1}{\omega_2^{(l)}}, \quad \tau \in [-b^3, -a^3], \quad l = 1, 4, \tag{369}$$

$$|\widetilde{F}_2^{(l)}(\tau)|^2 \frac{1}{|\widetilde{F}_1^{(l)}(\tau)|} = \frac{1}{\omega_2^{(l)}}, \quad \tau \in [-b^3, -a^3], \quad l = 2, 5, \tag{370}$$

where

$$\omega_2^{(l)} = (S_2^{(l)}(\infty))^2, \quad \text{for} \quad l = 0, 1, 2, \tag{371}$$

$$\omega_2^{(l)} = (S_2^{(l)}(\infty)\, \phi_2'(\infty))^2, \quad \text{for} \quad l = 3, 5, \tag{372}$$

$$\omega_2^{(4)} = 1/(S_2^{(4)}(\infty)\, \phi_2'(\infty))^2. \tag{373}$$

Therefore, finding the positive constants $c_k^{(l)}$ reduces to solving the equations

$$\frac{(c_1^{(l)})^2}{c_2^{(l)}\omega_1^{(l)}} = 1 = \frac{(c_2^{(l)})^2}{c_1^{(l)}\omega_2^{(l)}}, \qquad l = 0, \ldots, 5.$$

If we take logarithms we transform these equations into the linear system

$$\begin{cases} 2\log c_1^{(l)} - \log c_2^{(l)} = \log \omega_1^{(l)}, \\ \\ -\log c_1^{(l)} + \log c_2^{(l)} = \log \omega_2^{(l)}, \end{cases}$$

in the unknowns $\log c_1^{(l)}, \log c_2^{(l)}$, which has a unique solution. □

In order to prove the uniqueness of the limiting functions $\widetilde{F}_j^{(i)}$, we need to use Lemma IV.5.6 below. More general versions of this result can be found in [4] (see Lemma 4.1) and [1] (see Proposition 1.1), so we omit the proof.

Let us first introduce some notations. Assume that Δ_1, Δ_2 are disjoint compact intervals in $\mathbb{R}$, and let $C(\Delta_i)$ denote the space of all real-valued continuous functions on Δ_i. We write $\mathbf{u} = (u_1, u_2)^t \in C$ if $u_1 \in C(\Delta_2)$, and $u_2 \in C(\Delta_1)$. Given $u_1 \in C(\Delta_2)$, let $T_{2,1}(u_1)$ denote the harmonic function in $\overline{\mathbb{C}} \setminus \Delta_2$ that solves the Dirichlet problem with boundary conditions

$$T_{2,1}(u_1)(x) = u_1(x), \quad x \in \Delta_2,$$

and given $u_2 \in C(\Delta_1)$, let $T_{1,2}(u_2)$ denote the harmonic function in $\overline{\mathbb{C}} \setminus \Delta_1$ that solves the Dirichlet problem with boundary conditions

$$T_{1,2}(u_2)(x) = u_2(x), \quad x \in \Delta_1.$$

Consider the linear operator $T : C \longrightarrow C$ defined as follows

$$T = \begin{bmatrix} 0 & T_{1,2} \\ T_{2,1} & 0 \end{bmatrix},$$

and $I : C \longrightarrow C$ the identity operator. The auxiliary result is the following

Lemma IV.5.6. *If* $\mathbf{u} \in C$ *and* $(2I - T)(\mathbf{u}) = \mathbf{0}$*, then* $\mathbf{u} = \mathbf{0}$.

Now we prove that the limiting functions do not depend on the subsequence $\Lambda \subset \mathbb{N}$ selected for which (329) and (330) hold.

Proposition IV.5.7. *The limiting functions* $\widetilde{F}_j^{(i)}$ *are unique for every* $j \in \{1, 2\}$ *and* $i \in \{0, \ldots, 5\}$.

Proof. For each fixed $i \in \{0, \ldots, 5\}$, by Proposition IV.5.5 the functions $\log |F_1^{(i)}|$ and $\log |F_2^{(i)}|$ satisfy the system

$$\begin{cases} 2 \log |F_1^{(i)}(\tau)| - \log |F_2^{(i)}(\tau)| = \log |f_i(\tau)|, \quad \tau \in (0, \alpha^3], \\ \\ - \log |F_1^{(i)}(\tau)| + 2 \log |F_2^{(i)}(\tau)| = \log |g_i(\tau)|, \quad \tau \in [-b^3, -a^3], \end{cases} \tag{374}$$

where the functions $f_i(\tau), g_i(\tau)$ equal $1/\tau, 1$, or τ, depending on the value of i (f_i and g_i are not equal). Assume that the functions $\widetilde{G}_1^{(i)}, \widetilde{G}_2^{(i)}$ satisfy

$$\lim_{k \in \Lambda'} \frac{P_{6k+i+1}(z)}{P_{6k+i}(z)} = \widetilde{G}_1^{(i)}(z), \qquad z \in \mathbb{C} \setminus [0, \alpha^3],$$

$$\lim_{k\in\Lambda'} \frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)} = \widetilde{G}_2^{(i)}(z), \qquad z \in \mathbb{C} \setminus [-a^3, -b^3],$$

for some other subsequence $\Lambda' \subset \mathbb{N}$, where the limits hold uniformly on compact subsets of the indicated regions. Then as before we can find positive constants $d_1^{(i)}, d_2^{(i)}$ so that the functions $G_j^{(i)} := d_j^{(i)} \widetilde{G}_j^{(i)}$ satisfy the same system (374).

If we define the functions

$$u_1 := \log |F_1^{(i)}| - \log |G_1^{(i)}|, \qquad u_2 := \log |F_2^{(i)}| - \log |G_2^{(i)}|,$$

then observe that u_1 is harmonic in $\overline{\mathbb{C}} \setminus [0, \alpha^3]$, u_2 is harmonic in $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$ (the possible singularities at infinity of the functions $\log |F_j^{(i)}|, \log |G_j^{(i)}|$ cancel out by subtraction), and they are also bounded in the corresponding regions. Moreover we have

$$\begin{cases} 2u_1(\tau) - u_2(\tau) = 0, \quad \tau \in (0, \alpha^3], \\ \\ -u_1(\tau) + 2u_2(\tau) = 0, \quad \tau \in [-b^3, -a^3]. \end{cases}$$

Let $\Delta_1 := [0, \alpha^3]$, $\Delta_2 := [-b^3, -a^3]$. From the first equation and the generalized minimum (maximum) principle for superharmonic (subharmonic) functions, we obtain that $2u_1 - T_{1,2}(u_2) = 0$ on $\overline{\mathbb{C}} \setminus \Delta_1$. Similarly $2u_2 - T_{2,1}(u_1) = 0$ on $\overline{\mathbb{C}} \setminus \Delta_2$. In particular,

$$\begin{cases} 2u_1(\tau) - T_{1,2}(u_2)(\tau) = 0, \quad \tau \in \Delta_2, \\ \\ -T_{2,1}(u_1)(\tau) + 2u_2(\tau) = 0, \quad \tau \in \Delta_1, \end{cases}$$

so by Lemma IV.5.6 we get that $u_1 = 0$ on Δ_2, and $u_2 = 0$ on Δ_1. Therefore $T_{1,2}(u_2) = 0$ on $\overline{\mathbb{C}} \setminus \Delta_1$ and $T_{2,1}(u_1) = 0$ on $\overline{\mathbb{C}} \setminus \Delta_2$. This implies that u_1 and u_2 are identically zero.

From $|F_j^{(i)}| = |G_j^{(i)}|$ it easily follows that $c_j^i = d_j^i$ and $\widetilde{F}_j^{(i)} = \widetilde{G}_j^{(i)}$. □

Proof of Theorem IV.1.6. The existence of the limits (186) and (187) follows from Proposition IV.5.7. Notice that the polynomials P_n satisfy the recurrence relations

$$P_{3k}(z) = P_{3k+1}(z) + a_{3k}P_{3k-2}(z),$$

$$P_{3k+1}(z) = P_{3k+2}(z) + a_{3k+1}P_{3k-1}(z),$$

$$zP_{3k+2}(z) = P_{3k+3}(z) + a_{3k+2}P_{3k}(z),$$

and so we have

$$a_{6k+i} = \frac{P_{6k+i}(z)}{P_{6k+i-2}(z)} - \frac{P_{6k+i+1}(z)}{P_{6k+i-2}(z)}, \qquad i \in \{0,1,3,4\},$$

$$a_{6k+i} = \frac{zP_{6k+i}(z)}{P_{6k+i-2}(z)} - \frac{P_{6k+i+1}(z)}{P_{6k+i-2}(z)}, \qquad i \in \{2,5\}.$$

By (186) we obtain the existence of the limits

$$\lim_{k\to\infty} a_{6k+i} = \widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(1-\widetilde{F}_1^{(i)}(z)), \qquad i \in \{0,1,3,4\}, \tag{375}$$

$$\lim_{k\to\infty} a_{6k+i} = \widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(z-\widetilde{F}_1^{(i)}(z)), \qquad i \in \{2,5\}, \tag{376}$$

where the relations are valid for every $z \in \mathbb{C} \setminus [0,\alpha^3]$, and we identify $\widetilde{F}_1^{(-2)} = \widetilde{F}_1^{(4)}$, $\widetilde{F}_1^{(-1)} = \widetilde{F}_1^{(5)}$.

If $i \in \{0,1,3,4\}$ then

$$\widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(1-\widetilde{F}_1^{(i)}(z)) = -c_1^{(i)} + O(1/z), \qquad z\to\infty,$$

and for $i \in \{2,5\}$,

$$\widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(z-\widetilde{F}_1^{(i)}(z)) = -c_0^{(i)} + O(1/z), \qquad z\to\infty,$$

and so (188) follows from (375)–(376). Using the definition of the polynomials P_n, (190)–(192) follow directly from (186)–(187). $\square$

Proposition IV.5.8. *Assume that the hypotheses of Theorem IV.1.6 hold. Then the polynomials* $p_n, p_{n,2}$ *defined in* (318) *satisfy for each* $i \in \{0, \ldots, 5\}$:

$$\lim_{k\to\infty} \frac{p_{6k+i+1}(z)}{p_{6k+i}(z)} = \kappa_1^{(i)} \widetilde{F}_1^{(i)}(z), \qquad z \in \mathbb{C} \setminus [0, \alpha^3], \tag{377}$$

$$\lim_{k\to\infty} \frac{p_{6k+i+1,2}(z)}{p_{6k+i,2}(z)} = \kappa_2^{(i)} \widetilde{F}_2^{(i)}(z), \qquad z \in \mathbb{C} \setminus [-b^3, -a^3], \tag{378}$$

uniformly on compact subsets of the indicated regions, where

$$\kappa_j^{(i)} = \sqrt{\omega_j^{(i)}}, \qquad j = 1, 2,$$

and the constants $\omega_j^{(i)}$ *are defined in* (366)–(367) *and* (371)–(373)*. Consequently, for the leading coefficients* $\kappa_n, \kappa_{n,2}$ *defined in* (317) *we have:*

$$\lim_{k\to\infty} \frac{\kappa_{6k+i+1}}{\kappa_{6k+i}} = \kappa_1^{(i)}, \tag{379}$$

$$\lim_{k\to\infty} \frac{\kappa_{6k+i+1,2}}{\kappa_{6k+i,2}} = \kappa_2^{(i)}. \tag{380}$$

In addition, the following limits hold uniformly on compact subsets of $\mathbb{C} \setminus (S_0 \cup S_1)$:

$$\lim_{k\to\infty} \frac{\Psi_{6k+i+1}(z)}{\Psi_{6k+i}(z)} = \frac{1}{\omega_1^{(i)}} \frac{\widetilde{F}_2^{(i)}(z^3)}{z^2\, \widetilde{F}_1^{(i)}(z^3)}, \qquad i = 0, 3, \tag{381}$$

$$\lim_{k\to\infty} \frac{\Psi_{6k+i+1}(z)}{\Psi_{6k+i}(z)} = \frac{1}{\omega_1^{(i)}} \frac{z\widetilde{F}_2^{(i)}(z^3)}{\widetilde{F}_1^{(i)}(z^3)}, \qquad i = 1, 2, 4, 5. \tag{382}$$

Proof. Using the same argument employed before and Theorems 1 and 2 from [6], we obtain

$$\lim_{k\to\infty} \frac{p_{6k+i+1}(z)}{p_{6k+i}(z)} = S_1^{(i)}(z), \qquad z \in \mathbb{C} \setminus [0, \alpha^3], \quad i = 0, 1, 3, 4,$$

$$\lim_{k\to\infty} \frac{p_{6k+i+1}(z)}{p_{6k+i}(z)} = S_1^{(i)}(z)\,\phi_1(z), \qquad z \in \mathbb{C}\setminus[0,\alpha^3], \quad i = 2,5,$$

$$\lim_{k\to\infty} \frac{p_{6k+i+1,2}(z)}{p_{6k+i,2}(z)} = S_2^{(i)}(z), \qquad z \in \mathbb{C}\setminus[-b^3,-a^3], \quad i = 0,1,2,$$

$$\lim_{k\to\infty} \frac{p_{6k+i+1,2}(z)}{p_{6k+i,2}(z)} = S_2^{(i)}(z)\,\phi_2(z), \qquad z \in \mathbb{C}\setminus[-b^3,-a^3], \quad i = 3,5,$$

$$\lim_{k\to\infty} \frac{p_{6k+5,2}(z)}{p_{6k+4,2}(z)} = (S_2^{(4)}(z)\,\phi_2(z))^{-1}, \qquad z \in \mathbb{C}\setminus[-b^3,-a^3],$$

so (377) and (378) follow. (379) and (380) are immediate consequences of (377) and (378).

Observe that by (307) we can write

$$\frac{\Psi_{n+1}}{\Psi_n} = \frac{\kappa_n^2}{\kappa_{n+1}^2}\frac{h_{n+1}}{h_n}\frac{Q_n}{Q_{n+1}}\frac{Q_{n+1,2}}{Q_{n,2}},$$

so if we apply (379)–(380) together with Lemma IV.5.4 and Theorem IV.1.6, we obtain (381)–(382). □

Recall the definition

$$a^{(i)} := \lim_{k\to\infty} a_{6k+i}, \qquad 0 \le i \le 5.$$

Proof of Proposition IV.1.7. We first show that $a^{(i)} > 0$ for all i. If we assume that $a^{(0)} = 0$, then (375) implies that $\widetilde{F}_1^{(0)} \equiv 1$. Now using (357) we obtain that $\widetilde{F}_2^{(0)}(z) = z$ for all $z \in \mathbb{C}\setminus[-b^3,-a^3]$, contradicting the fact that $\widetilde{F}_2^{(0)}(\infty) = 1$. If we assume that $a^{(1)} = 0$, then again by (375) we get $\widetilde{F}_1^{(1)} \equiv 1$, and so by (358) we have $\widetilde{F}_2^{(1)} \equiv 1$, contradicting (361). If $a^{(2)} = 0$, then from (376) it follows that $\widetilde{F}_1^{(2)}(z) = z$ for all $z \in \mathbb{C}\setminus[0,\alpha^3]$, and so (359) implies that $\widetilde{F}_2^{(1)}(z) = z$, which is impossible. Similar arguments show that $a^{(i)} > 0$ for $i \in \{3,4,5\}$.

We prove now simultaneously that $\widetilde{F}_1^{(2)}(z) = z\,\widetilde{F}_1^{(0)}(z)$ and $\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(2)}$. Let

$$u_1(z) := \log|F_1^{(2)}(z)| - \log|z\,F_1^{(0)}(z)|, \qquad u_2(z) := \log|F_2^{(2)}(z)| - \log|F_2^{(0)}(z)|.$$

Then u_1 is harmonic in $\overline{\mathbb{C}}\setminus[0,\alpha^3]$ and u_2 is harmonic in $\overline{\mathbb{C}}\setminus[-b^3,-a^3]$. By (360) and (362) we see that u_2 is also bounded on $\overline{\mathbb{C}}\setminus[-b^3,-a^3]$. To show that u_1 is bounded on $\overline{\mathbb{C}}\setminus[0,\alpha^3]$ it suffices to show that it is bounded near the origin.

Taking into account that $F_1^{(0)}(z) = C\,S_1^{(0)}(z)$ and $F_1^{(2)}(z) = D\,S_1^{(2)}(z)\,\phi_1(z)$ (C and D are constants), and the definitions of the functions $S_1^{(0)}$ and $S_1^{(2)}$, the boundedness of u_1 near the origin is equivalent to the boundedness of the expression

$$\frac{1}{2\pi}\int_0^{2\pi}\Re\left[\frac{e^{i\theta}+1/\phi_1(z)}{e^{i\theta}-1/\phi_1(z)}\right]\log(1+\cos\theta)\,d\theta - \log|z|, \qquad z\notin[0,\alpha^3],$$

near the origin. If we apply the substitution $w = 1/\phi_1(z)$, this in turn is equivalent to the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}\Re\left[\frac{e^{i\theta}+w}{e^{i\theta}-w}\right]\log|1+e^{i\theta}|\,d\theta - \log|1+w|, \qquad |w|<1,$$

is bounded near -1. But in fact we have

$$\frac{1}{2\pi}\int_0^{2\pi}\Re\left[\frac{e^{i\theta}+w}{e^{i\theta}-w}\right]\log|1+e^{i\theta}|\,d\theta = \log|1+w|, \qquad |w|<1.$$

Now Proposition IV.5.5 implies that

$$2u_1(\tau) - u_2(\tau) = 0, \qquad \tau\in(0,\alpha^3],$$

$$-u_1(\tau) + 2u_2(\tau) = 0, \qquad \tau\in[-b^3,-a^3].$$

The same argument used in the proof of Proposition IV.5.7 shows that u_1 and u_2 are identically zero, and so $\widetilde{F}_1^{(2)}(z) = z\widetilde{F}_1^{(0)}(z)$ and $\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(2)}$. Similarly one proves that $\widetilde{F}_1^{(5)}(z) = z\widetilde{F}_1^{(3)}(z)$ and $\widetilde{F}_2^{(5)} = \widetilde{F}_2^{(3)}$.

From (193), (188), and (189), it follows that $a^{(0)} = a^{(2)}$ and $a^{(3)} = a^{(5)}$. We have by (375)–(376) that

$$\widetilde{F}_1^{(0)}(z)\widetilde{F}_1^{(1)}(z)(z - \widetilde{F}_1^{(2)}) = a^{(2)},$$

$$\widetilde{F}_1^{(4)}(z)\widetilde{F}_1^{(5)}(z)(1 - \widetilde{F}_1^{(0)}) = a^{(0)}.$$

So if we apply that $a^{(0)} = a^{(2)}$ and $\widetilde{F}_1^{(2)}(z) = z\widetilde{F}_1^{(0)}(z)$, dividing one equation by the other we get that $z\widetilde{F}_1^{(0)}\widetilde{F}_1^{(1)} = \widetilde{F}_1^{(4)}\widetilde{F}_1^{(5)}$, which is equivalent to $\widetilde{F}_1^{(1)}\widetilde{F}_1^{(2)} = \widetilde{F}_1^{(4)}\widetilde{F}_1^{(5)}$. The other two relations in (194) follow immediately using this equality and (193).

The relations in (197) are an easy consequence of (194) and (357)–(359). Now, (195) is obtained by dividing appropriate relations from (375)–(376) and taking into account (194). The equality $a^{(0)} + a^{(1)} = a^{(3)} + a^{(4)}$ follows by identifying the Laurent expansions at infinity of $\widetilde{F}_1^{(0)}\widetilde{F}_1^{(1)}$ and $\widetilde{F}_1^{(3)}\widetilde{F}_1^{(4)}$.

We next show that the functions $\widetilde{F}_1^{(i)}$, $i \in \{0, \ldots, 5\}$, are all distinct. If $i \in \{0, 1, 3, 4\}$, then evidently $\widetilde{F}_1^{(i)} \neq \widetilde{F}_1^{(2)}$ and $\widetilde{F}_1^{(i)} \neq \widetilde{F}_1^{(5)}$. If $\widetilde{F}_1^{(0)} = \widetilde{F}_1^{(1)}$, then (363) and (364) imply that

$$\frac{\widetilde{F}_2^{(1)}(\tau)}{\widetilde{F}_2^{(0)}(\tau)} = \frac{\omega_1^{(1)}}{\omega_1^{(0)}}\frac{1}{\tau}, \qquad \tau \in (0, \alpha^3],$$

which is contradictory since $1/\widetilde{F}_2^{(0)}$ is holomorphic outside $[-b^3, -a^3]$. The same argument proves that $\widetilde{F}_1^{(0)} \neq \widetilde{F}_1^{(4)}, \widetilde{F}_1^{(1)} \neq \widetilde{F}_1^{(3)}$, and $\widetilde{F}_1^{(3)} \neq \widetilde{F}_1^{(4)}$. If $\widetilde{F}_1^{(0)} = \widetilde{F}_1^{(3)}$, then from (363) we obtain that $\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(3)}$, which is impossible since $\widetilde{F}_2^{(0)}$ is analytic at infinity and $\widetilde{F}_2^{(3)}$ is not. Similarly (using now (364) and (365)) we see that $\widetilde{F}_1^{(1)} \neq \widetilde{F}_1^{(4)}$ and $\widetilde{F}_1^{(2)} \neq \widetilde{F}_1^{(5)}$.

Now we show that the functions $\widetilde{F}_2^{(i)}$, $i \in \{0, 1, 3, 4\}$, are all different. If we assume that $\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(1)}$, then (368)–(369) imply that

$$\frac{|\widetilde{F}_1^{(1)}(\tau)|}{|\widetilde{F}_1^{(0)}(\tau)|} = \frac{\omega_2^{(1)}}{\omega_2^{(0)}}\tau^2, \qquad \tau \in [-b^3, -a^3].$$

Since $\widetilde{F}_1^{(0)}$ and $\widetilde{F}_1^{(1)}$ are real-valued on $[-b^3, -a^3]$, it follows that $\widetilde{F}_1^{(1)}(z) = z^2 \widetilde{F}_1^{(0)}(z)$, which is impossible. The other cases hold trivially just by looking at the Laurent expansion at infinity.

By (195) we obtain that $a^{(0)} \neq a^{(3)}$ and $a^{(1)} \neq a^{(4)}$ (otherwise $\widetilde{F}_1^{(0)} = \widetilde{F}_1^{(3)}$ or $\widetilde{F}_1^{(1)} = \widetilde{F}_1^{(4)}$). Now we show that $a^{(1)} \neq a^{(3)}$. Applying (375) for $i = 0$ and the relation $\widetilde{F}_1^{(1)} \widetilde{F}_1^{(2)} = \widetilde{F}_1^{(4)} \widetilde{F}_1^{(5)}$, we get

$$\widetilde{F}_1^{(1)} \widetilde{F}_1^{(2)} (1 - \widetilde{F}_1^{(0)}) = a^{(0)}. \tag{383}$$

Using (383) and the relation (375) for $i = 4$, we obtain

$$\widetilde{F}_1^{(1)} (1 - \widetilde{F}_1^{(0)}) = \frac{a^{(0)}}{a^{(4)}} \widetilde{F}_1^{(3)} (1 - \widetilde{F}_1^{(4)}). \tag{384}$$

Applying the first two equations from (195), we derive that

$$\widetilde{F}_1^{(1)} (1 - \widetilde{F}_1^{(0)}) = \frac{a^{(3)}}{a^{(1)}} (1 - \widetilde{F}_1^{(1)})(\widetilde{F}_1^{(0)} - 1) + \frac{a^{(0)}}{a^{(1)}} (1 - \widetilde{F}_1^{(1)}). \tag{385}$$

If we assume now that $a^{(1)} = a^{(3)}$, then (385) yields

$$\frac{1 - \widetilde{F}_1^{(0)}}{1 - \widetilde{F}_1^{(1)}} = \frac{a^{(0)}}{a^{(1)}}. \tag{386}$$

But from (375) we know that

$$\frac{(1 - \widetilde{F}_1^{(0)}) \widetilde{F}_1^{(4)}}{(1 - \widetilde{F}_1^{(1)}) \widetilde{F}_1^{(0)}} = \frac{a^{(0)}}{a^{(1)}}, \tag{387}$$

so (386) and (387) imply that $\widetilde{F}_1^{(4)} = \widetilde{F}_1^{(0)}$, which is contradictory. Therefore $a^{(1)} \neq a^{(3)}$, and so by (198) we also obtain that $a^{(0)} \neq a^{(4)}$. $\square$

Corollary IV.5.9. *The following relations hold:*

$$\omega_1^{(0)}\omega_1^{(1)} = \omega_1^{(3)}\omega_1^{(4)}, \quad \omega_1^{(1)}\omega_1^{(2)} = \omega_1^{(4)}\omega_1^{(5)}, \quad \omega_1^{(2)}\omega_1^{(3)} = \omega_1^{(5)}\omega_1^{(0)},$$

$$\omega_1^{(0)} = \omega_1^{(2)}, \qquad \omega_1^{(3)} = \omega_1^{(5)},$$

$$\omega_2^{(0)}\omega_2^{(1)} = \omega_2^{(3)}\omega_2^{(4)}, \quad \omega_2^{(1)}\omega_2^{(2)} = \omega_2^{(4)}\omega_2^{(5)}, \quad \omega_2^{(2)}\omega_2^{(3)} = \omega_2^{(5)}\omega_2^{(0)},$$

$$\omega_2^{(0)} = \omega_2^{(2)}, \qquad \omega_2^{(3)} = \omega_2^{(5)}.$$

Proof. All these relations follow immediately from the relations established in Proposition IV.1.7 and the boundary value equations (363)–(365) and (368)–(370) (multiply or divide appropriately these equations). □

IV.6 The Riemann surface representation of the limiting functions $\widetilde{F}_j^{(i)}$

We will give now the proof of Theorem IV.1.8. Before doing so, we need some definitions and comments. Let

$$G_1^{(i,j)} := \frac{F_1^{(i)}}{F_1^{(j)}}, \qquad G_2^{(i,j)} := \frac{F_2^{(i)}}{F_2^{(j)}}, \qquad 0 \leq i, j \leq 5.$$

Recall that we chose the conformal representation ψ of $\mathcal{R}$ onto $\overline{\mathbb{C}}$ so that it satisfies the conditions (201)–(203). As a consequence, we have $\psi(z) = \overline{\psi(\overline{z})}$. To see this, observe that ψ and $\overline{\psi(\overline{z})}$ have the same divisor, and therefore $\psi(z) = C\overline{\psi(\overline{z})}$, for some constant C. Using the fact that the coefficient A in (201) is real, we get that $C = 1$. The symmetry property $\psi(z) = \overline{\psi(\overline{z})}$ implies in particular that

$$\psi_k : \overline{\mathbb{R}} \setminus (\Delta_k \cup \Delta_{k+1}) \longrightarrow \overline{\mathbb{R}}, \qquad k = 0, 1, 2, \qquad \Delta_0 = \Delta_3 = \emptyset,$$

and

$$\psi_k(x_\pm) = \overline{\psi_k(x_\mp)} = \overline{\psi_{k+1}(x_\pm)}, \qquad x \in \Delta_{k+1}. \tag{388}$$

In addition, all the coefficients in the Laurent expansion at infinity of the branches ψ_k are real numbers. Given a function F with Laurent expansion at infinity

$$F(z) = C\, z^k + O(z^{k-1}), \qquad C \in \mathbb{R} \setminus \{0\}, \quad k \in \mathbb{Z},$$

we use the symbol $\operatorname{sign}(F(\infty))$ to denote the sign of C (i.e. $\operatorname{sign}(F(\infty)) = 1$ if $C > 0$ and $\operatorname{sign}(F(\infty)) = -1$ if $C < 0$).

The function $\psi_0\, \psi_1\, \psi_2$ is analytic and bounded on $\overline{\mathbb{C}}$ (when multiplying two consecutive branches, the singularities on the common slit cancel out by the Schwarz reflection principle), so by Liouville's theorem this function is constant. Let us denote this constant by C (from now on we will reserve in this section the letter C for this constant). So we have

$$(\psi_0\, \psi_1\, \psi_2)(z) \equiv C, \qquad (\widetilde{\psi}_0\, \widetilde{\psi}_1\, \widetilde{\psi}_2)(z) \equiv 1, \qquad z \in \overline{\mathbb{C}}. \tag{389}$$

Proposition IV.6.1. *The following relations hold:*

$$G_1^{(0,3)}(z) = \frac{\operatorname{sign}((\psi_1\psi_2)(\infty))\, (\psi_1\psi_2)(z)}{|C|^{2/3}}, \tag{390}$$

$$G_2^{(0,3)}(z) = \frac{\operatorname{sign}(\psi_2(\infty))\, \psi_2(z)}{|C|^{1/3}}. \tag{391}$$

Proof. By (357) and (360) we have

$$|G_1^{(0,3)}(\tau)|^2 \frac{1}{G_2^{(0,3)}(\tau)} = 1, \qquad \tau \in (0, \alpha^3], \tag{392}$$

$$|G_2^{(0,3)}(\tau)|^2 \frac{1}{|G_1^{(0,3)}(\tau)|} = 1, \qquad \tau \in [-b^3, -a^3]. \tag{393}$$

Observe also that the functions $G_1^{(0,3)}$ and $G_2^{(0,3)}$ are bounded on $\overline{\mathbb{C}} \setminus \Delta_1$ and $\overline{\mathbb{C}} \setminus \Delta_2$, respectively. Moreover,

$$G_1^{(0,3)}(z) = D + O(1/z), \qquad z \to \infty,$$

$$G_2^{(0,3)}(z) = E/z + O(1/z^2), \qquad z \to \infty.$$

Let us call v_1 and v_2 the functions on the right hand side of (390) and (391), respectively. The function v_2 is positive on $\Delta_1 = [0, \alpha^3]$ since $\operatorname{sign}(v_2(\infty)) = 1$. Using (388) and (389), we have that for any $x \in (0, \alpha^3)$,

$$\frac{|v_1(x_\pm)|^2}{v_2(x)} = \frac{|\psi_1(x_\pm)|^2\, \psi_2(x)^2}{\operatorname{sign}(\psi_2(\infty))\, \psi_2(x)} = \frac{|\psi_1(x_\pm)||\psi_1(x_\pm)||\psi_2(x)|}{|C|}$$

$$= \frac{|\psi_0(x_\mp)||\psi_1(x_\pm)||\psi_2(x)|}{|C|} = \frac{|\overline{\psi_0(x_\pm)}||\psi_1(x_\pm)||\psi_2(x)|}{|C|} = 1,$$

i.e. v_1 and v_2 satisfy (392) on $(0, \alpha^3)$. On the other hand, for $x \in (-b^3, -a^3)$,

$$\frac{|v_2(x_\pm)|^2}{|v_1(x)|} = \frac{|\psi_2(x_\pm)|}{|\psi_1(x_\pm)|} = 1,$$

so v_1 and v_2 also satisfy (393) on $(-b^3, -a^3)$.

Finally, the same argument used to prove Proposition IV.5.7 yields the validity of the relations (390) and (391). □

Proof of Theorem IV.1.8. By Proposition IV.6.1 we have

$$\frac{\widetilde{F}_1^{(0)}}{\widetilde{F}_1^{(3)}} = \widetilde{\psi}_1\, \widetilde{\psi}_2 = 1/\widetilde{\psi}_0, \tag{394}$$

$$\frac{\widetilde{F}_2^{(0)}}{\widetilde{F}_2^{(3)}} = \widetilde{\psi}_2. \tag{395}$$

From the first relation in (195) and (394), simple algebraic manipulations show that

$$\widetilde{F}_1^{(0)} = \frac{a^{(0)} - a^{(3)}}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}, \qquad \widetilde{F}_1^{(3)} = \frac{(a^{(0)} - a^{(3)})\,\widetilde{\psi}_0}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}.$$

The representations of $\widetilde{F}_1^{(2)}$ and $\widetilde{F}_1^{(2)}$ in Theorem IV.1.8 follow immediately from the relations $\widetilde{F}_1^{(2)}(z) = z\widetilde{F}_1^{(0)}(z)$ and $\widetilde{F}_1^{(5)}(z) = z\widetilde{F}_1^{(3)}(z)$.

Since $\widetilde{F}_1^{(0)}\widetilde{F}_1^{(1)} = \widetilde{F}_1^{(3)}\widetilde{F}_1^{(4)}$, from (394) we have $\widetilde{F}_1^{(1)}/\widetilde{F}_1^{(4)} = \widetilde{\psi}_0$. Using this relation and (195) we obtain

$$\widetilde{F}_1^{(1)} = \frac{(a^{(4)} - a^{(1)})\,\widetilde{\psi}_0}{a^{(4)}\widetilde{\psi}_0 - a^{(1)}}, \qquad \widetilde{F}_1^{(4)} = \frac{a^{(4)} - a^{(1)}}{a^{(4)}\widetilde{\psi}_0 - a^{(1)}}.$$

From the definition of the functions Ψ_n and Proposition IV.1.2 it follows that these functions satisfy the same recurrence relation satisfied by the polynomials Q_n, i.e.

$$z\Psi_n(z) = \Psi_{n+1} + a_n\Psi_{n-2}, \qquad n \geq 2. \tag{396}$$

Therefore, if we define the functions

$$U^{(i)}(z) := \lim_{k\to\infty} \frac{\Psi_{6k+i+1}(z)}{\Psi_{6k+i}(z)}, \qquad z \in \mathbb{C} \setminus (S_0 \cup S_1), \qquad 0 \leq i \leq 5,$$

(by Proposition IV.5.8 we know that such limits exist) then we know by (396) that

$$a^{(i)} = U^{(i-2)}(z)U^{(i-1)}(z)(z - U^{(i)}(z)), \qquad 0 \leq i \leq 5,$$

where we understand that $U^{(-2)} = U^{(4)}, U^{(-1)} = U^{(5)}$. In particular, applying (381) and (382) we obtain for $i = 0, 1, 4, 5$, that

$$a^{(0)} = \frac{1}{\omega_1^{(4)}\omega_1^{(5)}} \frac{\widetilde{F}_2^{(5)}(z)}{\widetilde{F}_1^{(5)}(z)} \frac{\widetilde{F}_2^{(4)}(z)}{\widetilde{F}_1^{(4)}(z)} \Big(z - \frac{\widetilde{F}_2^{(0)}(z)}{\omega_1^{(0)}\widetilde{F}_1^{(0)}(z)}\Big), \tag{397}$$

$$a^{(1)} = \frac{1}{\omega_1^{(0)}\omega_1^{(5)}} \frac{\widetilde{F}_2^{(0)}(z)}{\widetilde{F}_1^{(0)}(z)} \frac{\widetilde{F}_2^{(5)}(z)}{\widetilde{F}_1^{(5)}(z)} \Big(1 - \frac{\widetilde{F}_2^{(1)}(z)}{\omega_1^{(1)}\widetilde{F}_1^{(1)}(z)}\Big), \tag{398}$$

$$a^{(4)} = \frac{1}{\omega_1^{(2)}\omega_1^{(3)}} \frac{\widetilde{F}_2^{(2)}(z)}{\widetilde{F}_1^{(2)}(z)} \frac{\widetilde{F}_2^{(3)}(z)}{\widetilde{F}_1^{(3)}(z)} \Big(1 - \frac{\widetilde{F}_2^{(4)}(z)}{\omega_1^{(4)}\widetilde{F}_1^{(4)}(z)}\Big), \tag{399}$$

$$a^{(5)} = \frac{1}{\omega_1^{(3)}\omega_1^{(4)}} \frac{\widetilde{F}_2^{(3)}(z)}{\widetilde{F}_1^{(3)}(z)} \frac{\widetilde{F}_2^{(4)}(z)}{\widetilde{F}_1^{(4)}(z)} \Big(z - \frac{\widetilde{F}_2^{(5)}(z)}{\omega_1^{(5)}\widetilde{F}_1^{(5)}(z)}\Big), \tag{400}$$

where these relations are valid for every $z \in \mathbb{C} \setminus ([-b^3, -a^3] \cup [0, \alpha^3])$. If we apply the relations $a^{(3)} = a^{(5)}$, $\widetilde{F}_1^{(5)} = z\widetilde{F}_1^{(5)}$, $\widetilde{F}_2^{(5)} = \widetilde{F}_2^{(3)}$, from (397) and (400) we obtain

$$z\frac{a^{(0)}}{a^{(3)}}\Big(1 - \frac{1}{\omega_1^{(5)}} \frac{\widetilde{F}_2^{(3)}(z)}{\widetilde{F}_1^{(5)}(z)}\Big) = \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\Big(z - \frac{\widetilde{F}_2^{(0)}(z)}{\omega_1^{(0)}\widetilde{F}_1^{(0)}(z)}\Big)$$

Using (395) we get

$$z\Big(\frac{a^{(0)}}{a^{(3)}} - \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\Big) = \Big(\frac{za^{(0)}}{a^{(3)}\widetilde{F}_1^{(5)}(z)} - \frac{\omega_1^{(3)}\widetilde{\psi}_2(z)}{\omega_1^{(0)}\widetilde{F}_1^{(0)}(z)}\Big)\frac{\widetilde{F}_2^{(3)}(z)}{\omega_1^{(5)}}.$$

If we substitute in this expression the functions $\widetilde{F}_1^{(0)}, \widetilde{F}_1^{(5)}$ by their representations in terms of the branches $\widetilde{\psi}_k$, we obtain

$$z\Big(\frac{a^{(0)}}{a^{(3)}} - \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\Big) = \frac{(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})}{(a^{(0)} - a^{(3)})}\Big(\frac{a^{(0)}}{a^{(3)}\widetilde{\psi}_0(z)} - \frac{\omega_1^{(3)}\widetilde{\psi}_2(z)}{\omega_1^{(0)}}\Big)\frac{\widetilde{F}_2^{(3)}(z)}{\omega_1^{(5)}}$$

The factors in the right hand side of this equation never vanish on $\mathbb{C} \setminus ([0, \alpha^3] \cup [-b^3, -a^3])$, and so we can write

$$\widetilde{F}_2^{(3)}(z) = \frac{z\Big(\frac{a^{(0)}}{a^{(3)}} - \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\Big)\omega_1^{(5)}(a^{(0)} - a^{(3)})}{(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})\Big(\frac{a^{(0)}}{a^{(3)}\widetilde{\psi}_0(z)} - \frac{\omega_1^{(3)}\widetilde{\psi}_2(z)}{\omega_1^{(0)}}\Big)}.$$

If we move z to the left hand side and evaluate both sides at infinity we obtain the relation

$$\omega_1^{(5)}\Big(\frac{a^{(0)}}{a^{(3)}} - \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\Big) = \frac{a^{(0)}}{a^{(3)}}, \tag{401}$$

and so the Riemann surface representation for the function $\widetilde{F}_2^{(3)}$ that we give in Theorem IV.1.8 follows. This also proves the representation for the functions $\widetilde{F}_2^{(5)}, \widetilde{F}_2^{(0)}$, and $\widetilde{F}_2^{(2)}$.

From (398) and (399) we derive the relation

$$\frac{a^{(1)}}{a^{(4)}}\Big(1-\frac{\widetilde{F}_2^{(4)}}{\omega_1^{(4)}\widetilde{F}_1^{(4)}}\Big)=\frac{\omega_1^{(2)}\omega_1^{(3)}}{\omega_1^{(0)}\omega_1^{(5)}}\Big(1-\frac{\widetilde{F}_2^{(1)}}{\omega_1^{(1)}\widetilde{F}_1^{(1)}}\Big).$$

From Corollary IV.5.9 we know that $\omega_1^{(2)}\omega_1^{(3)}=\omega_1^{(5)}\omega_1^{(0)}$. Since $\widetilde{F}_2^{(4)}/\widetilde{F}_2^{(1)}=\widetilde{F}_2^{(0)}/\widetilde{F}_2^{(3)}=\widetilde{\psi}_2$ and $\widetilde{F}_1^{(4)}/\widetilde{F}_1^{(1)}=1/\widetilde{\psi}_0=\widetilde{\psi}_1\widetilde{\psi}_2$, we get

$$\frac{a^{(1)}}{a^{(4)}}-1=\frac{\widetilde{F}_2^{(4)}}{\widetilde{F}_1^{(4)}}\Big(\frac{a^{(1)}}{a^{(4)}\omega_1^{(4)}}-\frac{\widetilde{\psi}_1}{\omega_1^{(1)}}\Big) \tag{402}$$

Evaluating at infinity we obtain the relation

$$\frac{a^{(1)}}{a^{(4)}}-1=-\frac{1}{\omega_1^{(1)}},$$

and so

$$\omega_1^{(1)}=\frac{a^{(4)}}{a^{(4)}-a^{(1)}}. \tag{403}$$

From (402) we can write

$$\widetilde{F}_2^{(4)}=\frac{\widetilde{F}_1^{(4)}}{(\widetilde{\psi}_1-(\omega_1^{(1)}-1)/\omega_1^{(4)})}.$$

So the Riemann surface representation of $\widetilde{F}_2^{(4)}$ follows from that of $\widetilde{F}_1^{(4)}$ and the representation of $\widetilde{F}_2^{(1)}$ follows from the relation $\widetilde{F}_2^{(4)}=\widetilde{\psi}_2\widetilde{F}_2^{(1)}$.

Now from (401) and Corollary IV.5.9 we get

$$\omega_1^{(3)}=\omega_1^{(5)}=\frac{a^{(0)}}{a^{(0)}-a^{(3)}}. \tag{404}$$

If we evaluate both sides of the equation (400) at infinity we obtain

$$a^{(5)} = a^{(3)} = \frac{1}{\omega_1^{(3)}\omega_1^{(4)}}(1 - 1/\omega_1^{(3)}),$$

and so (404) gives

$$\omega_1^{(4)} = \frac{a^{(0)} - a^{(3)}}{(a^{(0)})^2}.$$

Finally, from Corollary IV.5.9 and the above computations we deduce that

$$\omega_1^{(0)} = \omega_1^{(2)} = \frac{a^{(4)} - a^{(1)}}{a^{(0)}a^{(4)}}.$$

□

Remark IV.6.2. *Observe that since* $\omega_1^{(1)} > 0$*, it follows from* (403) *that* $a^{(4)} > a^{(1)}$*, and so from* (198) *we have* $a^{(0)} > a^{(3)}$*.*

Proof of Proposition IV.1.9. It is straightforward to check that the function

$$\chi(z) = \psi\Big(-\frac{a^3}{2}(1+z)\Big) - \psi(\infty^{(0)}), \qquad \infty^{(0)} \in \mathcal{R},$$

is a conformal representation of the Riemann surface $\mathcal{S}$ constructed as $\mathcal{R}$ (200) but formed by the sheets

$$\mathcal{S}_0 := \overline{\mathbb{C}} \setminus [-\mu, -1], \qquad \mathcal{S}_1 := \overline{\mathbb{C}} \setminus ([-\mu, -1] \cup [1, \lambda]), \qquad \mathcal{S}_2 := \overline{\mathbb{C}} \setminus [1, \lambda],$$

where λ and μ are defined in (204). χ also satisfies

$$\chi(z) = z + O(1), \qquad z \to \infty^{(1)},$$

and has a simple zero at $\infty^{(0)} \in \mathcal{S}$. Observe that

$$\chi(\infty^{(2)}) = -\psi(\infty^{(0)}). \tag{405}$$

(The reader is cautioned that in (405), $\infty^{(2)} \in \mathcal{S}$ and $\infty^{(0)} \in \mathcal{R}$).

χ and $\mathcal{S}$ are the type of conformal mappings and Riemann surfaces considered in [43]. It follows from [43, Theorem 3.1] that

$$\chi(\infty^{(2)}) = \frac{2}{H(\beta)},$$

where H and β are described in the statement of the Proposition we are proving. So $\chi(z) = \psi(-a^3(1+z)/2) + 2/H(\beta)$. It also follows from [43, Theorem 3.1] that the function $w = H(\beta)\chi(z) - 1$ is the solution of the algebraic equation

$$w^3 - (H(\beta)z + \Theta_1 - \Theta_2 - h)w^2 - (1 + \Theta_1 + \Theta_2)w + H(\beta)z - h = 0,$$

where Θ_1, Θ_2, and h are the constants described in the statement of this Proposition. Simple computations and a change of variable yield immediately that $w = \psi(z)$ is the solution of the equation (205). □

IV.7 The nth root asymptotics and zero asymptotic distribution of the polynomials Q_n and $Q_{n,2}$

We start this section with the following basic result from [56]:

Lemma IV.7.1. *Let $E \subset \mathbb{C}$ be a compact set with positive logarithmic capacity which is regular with respect to the Dirichlet problem, and ϕ a continuous function on E.*

Then there exists a unique $\widetilde{\mu} \in \mathcal{M}_1(E)$ *and a constant* w *such that*

$$V^{\widetilde{\mu}}(z) + \phi(z) \begin{cases} \leq w, & z \in \operatorname{supp}(\widetilde{\mu}), \\ \geq w, & z \in E. \end{cases}$$

The measure $\widetilde{\mu}$ is precisely the solution of the Gauss variational problem on E (for the logarithmic potential) in the presence of the external field ϕ, and of course

$$w = \int\int \log \frac{1}{|z-t|} d\widetilde{\mu}(z)\, d\widetilde{\mu}(t) + \int \phi(z)\, d\widetilde{\mu}(z).$$

So we call $\widetilde{\mu}$ the *equilibrium measure* in the presence of the external field ϕ on E and w the *equilibrium constant.* We already know (see (57) and (58)) that if the compact set E is not regular with respect to the Dirichlet problem, then the second inequality holds except on a set $e \subset E$ with zero logarithmic capacity. When E is regular, it is well known (see [56, Theorem I.4.8]) that the continuity of ϕ implies that the second inequality holds for all points in E.

Recall that if P is a polynomial of degree n, we indicate by μ_P the associated normalized zero counting measure.

The following result will also be needed. The proof is a combination of the arguments employed in [13], [29] and [59].

Lemma IV.7.2. *Let* σ *be a positive Borel measure in the class* **Reg** *such that* $\operatorname{supp}(\sigma)$ *is regular for the Dirichlet problem. Suppose that* $\{\phi_n\}, n \in \Lambda \subset \mathbb{N}$*, is a sequence of positive continuous functions defined on* $\operatorname{supp}(\sigma)$ *such that*

$$\lim_{n\in\Lambda} \frac{1}{2n} \log \frac{1}{\phi_n(x)} = \phi(x), \qquad \phi \in C(\operatorname{supp}(\sigma)), \tag{406}$$

uniformly on $\mathrm{supp}(\sigma)$*. By* $\{q_n\}_{n\in\Lambda}$ *denote a sequence of monic polynomials such that* $\deg q_n = n$ *and*

$$\int x^k\, q_n(x)\, \phi_n(x)\, d\sigma(x) = 0, \qquad k = 0, \ldots, n-1.$$

Then

$$\mu_{q_n} \xrightarrow{*} \widetilde{\mu}, \tag{407}$$

and

$$\lim_{n\in\Lambda} \Big(\int |q_n(x)|^2 \phi_n(x)\, d\sigma(x) \Big)^{1/2n} = e^{-\omega}, \tag{408}$$

where $\widetilde{\mu}$ *and* ω *are the equilibrium measure and equilibrium constant in the presence of the external field* ϕ *on* $\mathrm{supp}(\sigma)$*.*

Proof. Let $E := \mathrm{supp}(\sigma)$. From (406) and Lemma IV.7.1, it follows that for any $\epsilon > 0$ there exists l_0 such that for all $l \geq l_0, l \in \Lambda$, and $z \in \mathrm{supp}(\widetilde{\mu}) \subset E$,

$$\frac{1}{l} \log \frac{|p_l(z)|}{\|p_l \phi_l^{1/2}\|_E} \leq \frac{1}{2l} \log \frac{1}{|\phi_l(z)|} \leq \phi(z) + \varepsilon \leq w - V^{\widetilde{\mu}}(z) + \varepsilon,$$

where $\{p_l\}, l \in \Lambda$, is any sequence of monic polynomials such that $\deg p_l = l$ (there is no possibility of confusion with the sequence p_n defined in (318)), and $\|\cdot\|_E$ denotes the supremum norm on E. Hence,

$$u_l(z) := V^{\widetilde{\mu}}(z) + \frac{1}{l} \log \frac{|p_l(z)|}{\|p_l \phi_l^{1/2}\|_E} \leq w + \epsilon, \quad z \in \mathrm{supp}(\widetilde{\mu}), \quad l \geq l_0.$$

Since u_l is subharmonic in $\overline{\mathbb{C}} \setminus \mathrm{supp}(\widetilde{\mu})$, by the continuity and maximum principles, we have

$$u_l(z) \leq w + \epsilon, \quad z \in \overline{\mathbb{C}}, \quad l \geq l_0.$$

In particular,

$$u_l(\infty) = \frac{1}{l} \log \frac{1}{\|p_l \phi_l^{1/2}\|_E} \leq w + \varepsilon.$$

The last two relations imply

$$\limsup_{l \in \Lambda} \left(\frac{|p_l(z)|}{\|p_l \phi_l^{1/2}\|_E} \right)^{1/l} \leq \exp\left(w - V^{\widetilde{\mu}}(z)\right), \tag{409}$$

uniformly on compact subsets of $\mathbb{C}$, and

$$\liminf_{l \in \Lambda} \|p_l \phi_l^{1/2}\|_E^{1/l} \geq \exp\left(-w\right). \tag{410}$$

In particular, these relations hold for the sequence of polynomials $\{q_l\}, l \in \Lambda$.

Let t_l be the weighted Fekete polynomial of degree l for the weight $e^{-\phi}$ on $E = \mathrm{supp}(\sigma)$ (see [56, page 150] for definition) and $|\sigma|$ be the total variation of σ, i.e. $|\sigma| = \sigma(E)$. From the extremal property in the L^2 norm of q_l, we have

$$\|q_l \phi_l^{1/2}\|_2 := \left(\int |q_l(x)|^2 \phi_l(x) d\sigma(x) \right)^{1/2} \leq \|t_l \phi_l^{1/2}\|_2 \leq |\sigma|^{1/2} \|t_l \phi_l^{1/2}\|_E \leq$$

$$|\sigma|^{1/2} \|t_l e^{-l\phi}\|_E \|\phi_l^{1/2} e^{l\phi}\|_E.$$

Then, using (406) and [56, Theorem III.1.9], we obtain that

$$\limsup_{l \in \Lambda} \|q_l \phi_l^{1/2}\|_2^{1/l} \leq e^{-w}. \tag{411}$$

Since $\mathrm{supp}(\sigma)$ is regular with respect to the Dirichlet problem, Theorem 3.2.3 vi) in [59] yields

$$\limsup_{l \in \Lambda} \left(\frac{\|q_l \phi_l^{1/2}\|_E}{\|q_l \phi_l^{1/2}\|_2} \right)^{1/l} \leq 1,$$

which combined with (410) (with $p_l = q_l$) and (411) implies

$$\lim_{l\in\Lambda}\left(\frac{\|q_l\phi_l^{1/2}\|_E}{\|q_l\phi_l^{1/2}\|_2}\right)^{1/l} = 1. \tag{412}$$

Thus, we obtain (408) since (410), (411), and (412) give

$$\limsup_{l\in\Lambda}\|q_l\phi_l^{1/2}\|_E^{1/l} = \limsup_{l\in\Lambda}\|q_l\phi_l^{1/2}\|_2^{1/l} = e^{-w}. \tag{413}$$

All the zeros of q_l lie in $\mathrm{Co}(\mathrm{supp}(\sigma)) \subset \mathbb{R}$. The unit ball in the weak star topology of measures is compact. Take any subsequence of indices $\Lambda' \subset \Lambda$ such that

$$\mu_{q_l} \overset{*}{\longrightarrow} \mu_{\Lambda'}, \qquad l \in \Lambda',$$

for some probability measure $\mu_{\Lambda'}$. Then,

$$\lim_{l\in\Lambda'}\frac{1}{l}\log|q_l(z)| = -\lim_{n\in\Lambda'}\int \log\frac{1}{|z-x|}\mu_{q_l}(x) = -V^{\mu_{\Lambda'}}(z),$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathrm{Co}(\mathrm{supp}(\sigma))$. This, together with (408) and (409) (applied to $\{q_l\}, l \in \Lambda'$), implies

$$(V^{\widetilde{\mu}} - V^{\mu_{\Lambda'}})(z) \le 0, \qquad z \in \overline{\mathbb{C}}\setminus\mathrm{Co}(\mathrm{supp}(\sigma)).$$

Since $V^{\widetilde{\mu}} - V^{\mu_{\Lambda'}}$ is subharmonic in $\overline{\mathbb{C}}\setminus\mathrm{supp}(\widetilde{\mu})$ and $(V^{\widetilde{\mu}} - V^{\mu_{\Lambda'}})(\infty) = 0$, from the maximum principle, it follows that $V^{\widetilde{\mu}} \equiv V^{\mu_{\Lambda'}}$ in $\mathbb{C}\setminus\mathrm{Co}(\mathrm{supp}(\sigma))$ and thus $\mu_{\Lambda'} = \widetilde{\mu}$. Consequently, (407) holds. □

Let λ_1 be the positive, rotationally invariant measure on S_0 whose restriction to the interval $[0,\alpha]$ coincides with the measure $s_1(x)\,dx$, and let λ_2 be the positive, rotationally invariant measure on S_1 whose restriction to the interval $[-b,-a]$ coincides with the measure $s_2(x)\,dx$. We also need the following auxiliary result:

Lemma IV.7.3. *Suppose that* $\lambda_1, \lambda_2 \in \mathbf{Reg}$. *Then the measures*

$$\frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}}\,d\tau, \qquad s_1(\sqrt[3]{\tau})\,\sqrt[3]{\tau}\,d\tau, \qquad \tau \in [0, \alpha^3], \tag{414}$$

$$s_2(\sqrt[3]{\tau})\,d\tau, \quad \frac{s_2(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}\,d\tau, \quad \frac{s_2(\sqrt[3]{\tau})}{\tau^{2/3}}\,d\tau, \quad \tau \in [-b^3, -a^3], \tag{415}$$

are also regular.

Proof. Let π_n be the nth monic orthogonal polynomial associated with λ_1, i.e. π_n is the monic polynomial of degree n that satisfies

$$\int_{S_0} \pi_n(t)\,\overline{t^k}\,d\lambda_1(t) = 0, \qquad 0 \le k \le n-1. \tag{416}$$

The regularity of λ_1 is equivalent to the property

$$\lim_{n\to\infty} \|\pi_n\|_2^{1/n} = \operatorname{cap}_0(\operatorname{supp}(\lambda_1)),$$

where $\|\pi_n\|_2$ denotes the L^2 norm of π_n with respect to λ_1, and recall that $\operatorname{cap}_0(A)$ denotes the logarithmic capacity of A. It is immediate to check that

$$\pi_n(e^{\frac{2\pi i}{3}} z) = e^{\frac{2\pi i n}{3}}\,\pi_n(z), \tag{417}$$

and so using this property and (416) we get

$$0 = \int_0^\alpha t^{3l}\,\pi_{3k}(t)\,s_1(t)\,dt = \int_0^{\alpha^3} \tau^l\,\pi_{3k}(\sqrt[3]{\tau})\,s_1(\sqrt[3]{\tau})\,\frac{d\tau}{\tau^{2/3}}, \qquad 0 \le l \le k-1.$$

Similarly we have

$$0 = \int_0^{\alpha} t^{3l+1}\pi_{3k+1}(t)\, s_1(t)\, dt = \int_0^{\alpha^3} \tau^l \frac{\pi_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}\, s_1(\sqrt[3]{\tau})\, d\tau, \qquad 0 \le l \le k-1,$$

$$0 = \int_0^{\alpha} t^{3l+2}\pi_{3k+2}(t)\, s_1(t)\, dt = \int_0^{\alpha^3} \tau^l \frac{\pi_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}}\, s_1(\sqrt[3]{\tau})\, \tau^{2/3} d\tau, \qquad 0 \le l \le k-1.$$

Therefore the polynomials

$$\pi_{3k}(\sqrt[3]{\tau}), \qquad \frac{\pi_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}, \qquad \frac{\pi_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}},$$

are the monic orthogonal polynomials of degree k, respectively, associated with the measures

$$\frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}}\, d\tau, \qquad s_1(\sqrt[3]{\tau})\, d\tau, \qquad s_1(\sqrt[3]{\tau})\, \tau^{2/3}\, d\tau. \tag{418}$$

It also follows that

$$\int_{S_0} |\pi_{3k}(t)|^2\, d\lambda_1(t) = \int_0^{\alpha^3} (\pi_{3k}(\sqrt[3]{\tau}))^2\, \frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}}\, d\tau,$$

$$\int_{S_0} |\pi_{3k+1}(t)|^2\, d\lambda_1(t) = \int_0^{\alpha^3} \left(\frac{\pi_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}\right)^2 s_1(\sqrt[3]{\tau})\, d\tau,$$

$$\int_{S_0} |\pi_{3k+2}(t)|^2\, d\lambda_1(t) = \int_0^{\alpha^3} \left(\frac{\pi_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}}\right)^2 s_1(\sqrt[3]{\tau})\, \tau^{2/3}\, d\tau.$$

So taking into account (see [52, Theorem 5.2.5]) that

$$\mathrm{cap}_0(\mathrm{supp}(\lambda_1)) = \mathrm{cap}_0(\mathrm{supp}(\rho))^{1/3},$$

where ρ denotes any of the three measures in (418), the regularity of λ_1 implies the regularity of the three measures in (418).

Let l_n denote the nth monic orthogonal polynomial associated with the measure $d\rho_1(\tau) := s_1(\sqrt[3]{\tau})\, \sqrt[3]{\tau}\, d\tau$, and let T_n be the nth Chebyshev polynomial (see [52], page 155) for the set $E := \operatorname{supp}(\rho_1)$. By the L^2 extremal property of orthogonal polynomials, we have

$$\Big(\int l_n^2(\tau)\, d\rho_1(\tau)\Big)^{1/2} \leq \Big(\int T_n^2(\tau)\, d\rho_1(\tau)\Big)^{1/2} \leq \|T_n\|_E\, \rho_1(E)^{1/2},$$

where $\|T_n\|_E$ denotes the supremum norm of T_n on E, and so by [52, Corollary 5.5.5] we obtain

$$\limsup_{n\to\infty} \|l_n\|_2^{1/n} \leq \lim_{n\to\infty} \|T_n\|_E^{1/n} = \operatorname{cap}_0(\operatorname{supp}(\rho_1)). \tag{419}$$

On the other hand, if we call $\widetilde{l}_n$ the nth monic orthogonal polynomial associated with the measure $d\rho_2(\tau) := s_1(\sqrt[3]{\tau})\, \tau^{2/3}\, d\tau$, we have

$$\Big(\int \widetilde{l}_n^2(\tau) d\rho_2(\tau)\Big)^{1/2} \leq \alpha^{1/2}\Big(\int l_n^2(\tau)\, d\rho_1(\tau)\Big)^{1/2},$$

and so using the regularity of ρ_2 and (419) we obtain that ρ_1 is also regular. This proves that the measures in (414) are regular. Similar arguments show that the measures in (415) are also regular. □

Proof of Theorem IV.1.12. Let $j \in \{0, \ldots, 5\}$ be fixed, and assume that for some subsequence $\Lambda \subset \mathbb{N}$ we have that

$$\mu_{P_{6k+j}} \xrightarrow{*} \mu_1 \in \mathcal{M}_1(\Delta_1), \tag{420}$$

$$\mu_{P_{6k+j,2}} \xrightarrow{*} \mu_2 \in \mathcal{M}_1(\Delta_2). \tag{421}$$

It follows from (420) and (421) that

$$\lim_{k\in\Lambda} \frac{1}{2k} \log |P_{6k+j}(z)| = -V^{\mu_1}(z), \qquad z \in \mathbb{C}\setminus\Delta_1, \tag{422}$$

$$\lim_{k\in\Lambda} \frac{1}{4k} \log |P_{6k+j,2}(z)| = -\frac{1}{4}V^{\mu_2}(z), \qquad z \in \mathbb{C}\setminus\Delta_2, \tag{423}$$

uniformly on compact subsets of the indicated regions.

We know by Proposition IV.5.2 that there exists a fixed measure $d\rho$ supported on Δ_1 ($d\rho$ is one of the measures in (414)) such that

$$0 = \int_{\Delta_1} \tau^j \, P_{6k+j}(\tau) \, \frac{d\rho(\tau)}{P_{6k+j,2}(\tau)}, \qquad 0 \le j < \deg(P_{6k+j}), \tag{424}$$

where $\deg(P_{6k+j}) = 2k$ if $j \le 2$ and $\deg(P_{6k+j}) = 2k+1$ if $j \ge 3$. We know by Lemma IV.7.3 that the measure $d\rho$ is regular. If we apply Lemma IV.7.2 (taking $d\sigma = d\rho$, $\phi_{2k} = 1/P_{6k+j,2}$ and $\phi = -(1/4)V^{\mu_2}$), we obtain from (423) and (424) that μ_1 is the unique solution of the extremal problem

$$V^{\mu_1}(\tau) - \frac{1}{4} V^{\mu_2}(\tau) \begin{cases} = w_1, & \tau \in \operatorname{supp}(\mu_1), \\ \\ \ge w_1, & \tau \in \Delta_1, \end{cases} \tag{425}$$

and

$$\lim_{k \in \Lambda} \left(\int_{\Delta_1} P^2_{6k+j}(\tau) \, d\nu_{6k+j}(\tau) \right)^{1/4k} = e^{-\omega_1}, \tag{426}$$

where the measure $d\nu_{6k+j}$ is defined in (320).

By Proposition IV.5.1, there exists a fixed measure $d\eta$ ($d\eta$ is one of the measures in (415)) supported on Δ_2 such that

$$0 = \int_{\Delta_2} \tau^j \, P_{6k+j,2}(\tau) \, \frac{|h_{6k+j}(\sqrt[3]{\tau})|}{|P_{6k+j}(\tau)|} \, d\eta(\tau), \qquad 0 \le j < \deg(P_{6k+j,2}), \tag{427}$$

where $\deg(P_{6k+j}, 2) = k$ if $j \ne 4$ and $\deg(P_{6k+j,2}) = k+1$ if $j = 4$. The function h_{6k+j} is defined in (319). We also know by Lemma IV.7.3 that $d\eta$ is regular. Taking into account the representations (321)–(323) and the fact that p_n is orthonormal with respect to $d\nu_n$ (see (318) and Proposition IV.5.3), it follows that there exist positive constants C_1, C_2 such that

$$C_1 \leq |h_{6k+j}(\sqrt[3]{\tau})| \leq C_2, \qquad \text{for all} \quad \tau \in \Delta_2.$$

So applying Lemma IV.7.2 (now take $d\sigma = d\eta$, $\phi_k(\tau) = |h_{6k+j}(\sqrt[3]{\tau})|/|P_{6k+j}(\tau)|$ and $\phi = -V^{\mu_1}$), we get from (427) and (422) that μ_2 satisfies

$$V^{\mu_2}(\tau) - V^{\mu_1}(\tau) \begin{cases} = w_2, & \tau \in \operatorname{supp}(\mu_2), \\ \\ \geq w_2, & \tau \in \Delta_2, \end{cases} \tag{428}$$

and

$$\lim_{k\in\Lambda} \left(\int_{\Delta_2} P^2_{6k+j,2}(\tau)\, d\nu_{6k+j,2}(\tau) \right)^{1/2k} = e^{-\omega_2}, \tag{429}$$

where the measure $d\nu_{6k+j,2}$ is defined in (320).

Therefore by (425) and (428), the vector measure (μ_1, μ_2) solves the potential equilibrium problem determined by the interaction matrix (208) on the intervals Δ_1, Δ_2. By Lemma IV.1.11 this solution is unique, so (206) and (207) follow. (426) and (429) imply (211) and (212). Finally, (209) and (210) are an immediate consequence of (206) and (207). □

Proof of Proposition IV.1.14. By Theorem IV.1.6 we know that the following limit holds:

$$\lim_{k\to\infty} \frac{Q_{6(k+1)}(z)}{Q_{6k}(z)} = \prod_{i=0}^{5} \widetilde{F}_1^{(i)}(z^3), \qquad z \in \mathbb{C} \setminus S_0.$$

Therefore we obtain that

$$\lim_{k\to\infty} |Q_{6k}(z)|^{1/k} = \prod_{i=0}^{5} |\widetilde{F}_1^{(i)}(z^3)|, \qquad z \in \mathbb{C} \setminus S_0,$$

and by Corollary IV.1.13 it follows that

$$e^{-\frac{1}{3}V^{\overline{\mu}}(z^3)} = \prod_{i=0}^{5} |\widetilde{F}_1^{(i)}(z^3)|^{1/6} \qquad z \in \mathbb{C} \setminus S_0.$$

So (213) is proved. The same argument proves (214). $\square$

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