On Extending Stallings 2-Cores of Diagram Groups to R. Thompson's Group T and Jones Subgroup of T

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Chapter 1

Introduction

In his seminal paper [19], Stallings introduced a notion of foldings of a graph, which were used to analyze and understand the structure of subgroups of free groups. This language was used to recast many theorems about free groups and their subgroups with straightforward proofs (e.g., see [15]). For example, Stallings foldings were used to create an algorithm for solving the membership problem for finitely generated subgroups of free groups. Specifically, given a set X of words over the generators of a free group, each can be drawn as a closed path that starts and ends at a distinguished vertex with edges labeled by the generators, and then adjacent edges with identical labels can be "folded together" (i.e., identified with each other). After removing any "hanging trees" (i.e., subgraphs which are trees and which intersect the rest of the graph at only one point) and folding as many times as possible, the resulting graph is called the Stallings core of $\langle X \rangle$. Moreover, each closed path that starts and ends at the distinguished vertex is labeled by reduced words in $\langle X \rangle$, and any such word corresponding to a path is said to be accepted by the core. It turns out that the subgroup of all elements accepted by the core is exactly $\langle X \rangle$, and a spanning tree can be used to determine a free set of generators.

In [7], Guba and Sapir extended the notion of Stallings foldings of edges to foldings of cells and created the notion of a Stallings 2-cores to diagram groups. They defined the Stallings 2-core of a subgroup H of a diagram group and gave a way to tell whether any element in the given diagram group is accepted by the 2-core or not. Then they showed that the subset of every element accepted by the 2-core is a subgroup that included H, but in general could be larger than H. This provides a partial solution to the membership problem and can be used in many ways. For example, if the 2-core of

H does not accept the generators of the diagram group, then H is a strict subgroup. In particular, Guba and Sapir showed that certain diagrams had what they called components, and that if the 2-core of H accepts an element, it must also accept all its components. Thus they made the conjecture around 1999 (see Section 2.4) that the subgroup of a diagram group G accepted by the 2-core of H is precisely the smallest subgroup of G containing H and closed with respect to components.

The proof of the conjecture remained completely open, however, until Golan showed in [6] that the conjecture holds specifically for Richard Thompson's group F, which can be described as a diagram group (see Section 2.2). In Chapter 3, we examine Guba and Sapir's conjecture in the case of free groups, and show that their construction completely generalizes Stallings' original construction by analyzing the Stallings 2-core of any subgroup of a particular and natural representation of a free group as a diagram group. We also show that given groups which satisfy their conjecture, closed subgroups and direct products of these groups also satisfy the conjecture. In particular, we use this to generalize Golan's result to the generalized Thompson groups F_n.

In Chapter 4, we show how to extend the construction of the Stallings 2-core to subgroups of Richard Thompsons's group T, and there we give a characterization of what a 2-core of a subgroup H accepts, showing that it is exactly the collection of all functions from T which are dyadically-piecewise-H. We use this to examine the 2-cores of many different subgroups of T, including the pointwise stabilizers of any finite subset U of $[0, 1]$. Along the way we also prove that T and V are quasi-residually finite, answering a question of Golan and Sapir from [5].

Finally, in Chapter 5 we present work which was joint with Yunxiang Ren. Specifically, we examine a subgroup of T which Jones discovered while investigating a reconstruction problem in subfactor theory. The subgroup in question was denoted $\vec{T}$, and among other things Jones proved that for certain representations of F and T, every link arises as the matrix coefficient with respect to the vacuum vector of some element of F and T, and every oriented link arises as the matrix coefficient of some element of $\vec{F}$ and $\vec{T}$.

He also showed how to test directly if an element f of F or T is in $\vec{F}$ or $\vec{T}$ respectively by constructing the Thompson graph of f and showing that f is in $\vec{F}$ or $\vec{T}$ respectively if and only if f has bipartite Thompson graph. Sapir and Golan studied $\vec{F}$ in [4] using this characterization. In particular, they proved $\vec{F}$ is isomorphic to F_3 and showed that $\vec{F}$ was precisely the subgroup of F of all elements that preserve the parity of the sums of digits of all dyadic rationals in $[0, 1]$ as binary words. They also proved that $\vec{F}$ was its own commensurator in F, and observed that this implies that the corresponding representation of F is irreducible.

In Chapter 5 we likewise explore the subgroup $\vec{T}$, Jones' subgroup of T. We present a finite set of generators for $\vec{T}$ in Theorem 5.2.1, describe the relationship of $\vec{T}$ to the stabilizer of the parity of the sums of digits of the dyadic rationals in Theorem 5.3.1, and show in Corollary 5.4.2 that $\vec{T}$ coincides with its commensurator in T, which implies that the corresponding unitary representation of T which Jones considered is irreducible. We also provide an explicit finite presentation for $\vec{T}$ as an abstract group. Moreover, we use the extended notion of the Stallings 2-core developed in Chapter 4 to further analyze $\vec{T}$. We prove that the 2-core of $\vec{T}$ is itself, which in turn gives a presentation of $\vec{T}$ as an annular diagram group that is curiously similar to one possible presentation of the generalized Thompson group T_3, even though we show that $\vec{T}$ and T_3 are not isomorphic, unlike the case of $\vec{F}$ and F_3.

Chapter 2

Background Information

2.1 Diagram Groups

A sufficient overview of diagram groups is given here, but for a more formal and detailed discussion, see [7]. Here we largely follow the language in [17].

Recall that a directed graph is a collection of vertices V and edges E, where each edge is an ordered pair of vertices. Given an edge $e = (v_1, v_2)$, v_1 is called the initial vertex and v_2 is called the terminal vertex. Moreover, there are maps ι and τ from E to V such that $\iota(e)$ is the initial vertex of e and $\tau(e)$ is the terminal vertex of e. A directed path of length n is a sequence of edges $e_1, \ldots, e_n$ such that $\tau(e_i) = \iota(e_{i+1})$. In this case, e_i is called the ith edge of the path, and a vertex is said to be on a path if it is either the initial or terminal vertex of some edge on the path. The path is said to be simple if each vertex on the path is the initial vertex of at most one edge and the terminal vertex of at most one edge. The path is called closed if the initial vertex of the first edge and the terminal vertex of the last edge are the same vertex. Moreover, the graph is said to be labeled by X if there is a map l from E to X. In this case, $l(e)$ is called the label of e.

Consider a semigroup presentation $\langle X|R \rangle$, and let u be a word over X. A diagram over the semigroup presentation is a type of directed graph labeled by X which can be constructed in one of the ways described in the following paragraphs. In particular, each diagram Δ also has two distinguished vertices, $\iota(\Delta)$ and $\tau(\Delta)$, called the initial and terminal vertex of Δ respectively, as well as two distinguished paths, top(Δ) and bot(Δ), called the top path and bottom path of Δ respectively.

The first type of diagram which we define is called a trivial diagram. For each

$x \in X$, the trivial diagram $\varepsilon(x)$ is a directed labeled graph consisting of a single directed edge labeled x and two distinct vertices. The initial vertex of $\varepsilon(x)$ is the initial vertex of the single edge in the diagram, while the terminal vertex of $\varepsilon(x)$ is the terminal of vertex of that same edge. The top path and bottom path of $\varepsilon(x)$ both consist of the single edge in the diagram.

The second type of diagram we can define is called a cell. For each $r \in R$, where r is a relation $x_1 \dots x_n = y_1 \dots y_m$ with $x_i, y_j \in X$, the cell Δ_r is the planar directed graph consisting of two directed simple paths, $\text{top}(\Delta_r)$ and $\text{bot}(\Delta_r)$, where $\text{top}(\Delta_r)$ has length n and the ith edge is labeled x_i for every i, and the $\text{bot}(\Delta_r)$ has length m and the jth edge is labeled y_j for every j. Furthermore, $\text{top}(\Delta_r)$ and $\text{bot}(\Delta_r)$ both start at the same vertex $\iota(\Delta_r)$ and end at the same vertex $\tau(\Delta_r)$, but share no other vertices in common.

All other diagrams can be obtained from trivial diagrams and cells by using the following operations of addition, multiplication, and inversion. Given two diagrams Δ_1 and Δ_2, $\Delta_1 + \Delta_2$ is the diagram Δ obtained by identifying $\tau(\Delta_1)$ and $\iota(\Delta_2)$, as shown in Figure 2.1. Then $\iota(\Delta) = \iota(\Delta_1)$, $\tau(\Delta) = \tau(\Delta_2)$, $\text{top}(\Delta)$ is the path $\text{top}(\Delta_1)$ followed by the path $\text{top}(\Delta_2)$ (this is well defined since the end of $\text{top}(\Delta_1)$ and the beginning of $\text{top}(\Delta_2)$ have been identified), and likewise $\text{bot}(\Delta)$ is the path $\text{bot}(\Delta_1)$ followed by $\text{bot}(\Delta_2)$.

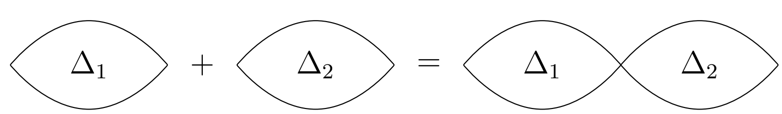


Figure 2.1: Addition of two diagrams Δ_1 and Δ_2 produces the diagram on the right, where $\tau(\Delta_1)$ and $\iota(\Delta_2)$ have been identified.

Given two diagrams Δ_1 and Δ_2 with $\text{bot}(\Delta_1)$ and $\text{top}(\Delta_2)$ sharing the same label, their multiplication $\Delta = \Delta_1 \cdot \Delta_2$ is defined by identifying $\text{bot}(\Delta_1)$ with $\text{top}(\Delta_2)$ as shown in Figure 2.2. Then $\text{top}(\Delta) = \text{top}(\Delta_1)$, $\text{bot}(\Delta) = \text{bot}(\Delta_2)$, $\iota(\Delta) = \iota(\Delta_1)$,

$\tau(\Delta) = \tau(\Delta_1)$.

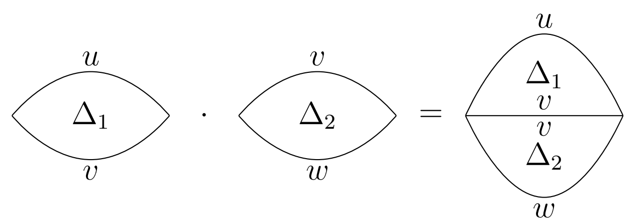


Figure 2.2: Multiplication of two diagrams Δ_1 and Δ_2 where $\mathrm{bot}(\Delta_1) = v = \mathrm{top}(\Delta_2)$ produces the diagram on the right, where $\mathrm{bot}(\Delta_1)$ and $\mathrm{top}(\Delta_2)$ have been identified.

Finally, given a diagram Δ, the diagram Δ^{-1} is the same diagram except that the top and bottom paths are swapped so that $\mathrm{top}(\Delta^{-1}) = \mathrm{bot}(\Delta)$ and $\mathrm{bot}(\Delta^{-1}) = \mathrm{top}(\Delta)$. The initial and terminal vertices remain unchanged: $\tau(\Delta^{-1}) = \tau(\Delta)$ and $\iota(\Delta^{-1}) = \iota(\Delta)$. Note that if Δ_r is a cell, Δ_r^{-1} is also called a cell, but no other diagrams are called cells.

A diagram Δ with $\mathrm{top}(\Delta)$ labeled by u and $\mathrm{bot}(\Delta)$ labeled by v is called a (u, v)-diagram over $\langle X|R\rangle$. For any diagram Δ, the diagram $\Delta \cdot \Delta^{-1}$ is called a dipole, and a diagram is said to be reduced if it contains no dipoles. A dipole $\Delta \cdot \Delta^{-1}$ can be removed from a diagram by identifying the paths along the top of the dipole and the bottom of the dipole together. The diagram group $D(\langle X|R\rangle, u)$ is the group of all reduced (u, u)-diagrams over $\langle X|R\rangle$ with multiplication of Δ_1 and Δ_2 being defined as the diagram obtained by removing all dipoles from $\Delta_1 \cdot \Delta_2$.

Another equivalent way to construct diagrams from trivial diagrams and cells is as follows. Given a word $w = x_1 \ldots x_n$ over X, $\varepsilon(w)$ is defined as $\varepsilon(x_1) + \ldots + \varepsilon(x_n)$. An atomic diagram is any diagram of the form $\varepsilon(u) + \Delta_r + \varepsilon(v)$ for words u, v over X and cell Δ_r corresponding to some relation in $r \in R$. Note that if Δ is an atomic diagram, then Δ^{-1} is also called an atomic diagram, since for any cell Δ_r, Δ_r^{-1} is also a cell. Then every diagram is simply a product of atomic diagrams.

Annular diagram groups can be defined similarly. Given a semigroup presentation

$\langle X|R\rangle$, every annular diagram Δ is a labeled directed graph with two distinguished paths $\mathrm{inn}(\Delta)$ and $\mathrm{out}(\Delta)$, called the inner path and outer path of Δ respectively, which are both simple closed directed paths with counterclockwise orientation. Moreover, $\mathrm{inn}(\Delta)$ must be inside the area bounded by $\mathrm{out}(\Delta)$, and all vertices and edges of Δ must be between $\mathrm{inn}(\Delta)$ and $\mathrm{out}(\Delta)$. Furthermore, both $\mathrm{inn}(\Delta)$ and $\mathrm{out}(\Delta)$ have distinguished vertices $\mathrm{i}(\Delta)$ and $\mathrm{o}(\Delta)$, called the inner and outer vertex of Δ respectively.

Annular diagrams can be constructed much like diagrams, by defining atomic annular diagrams and their multiplication. Specifically, for every atomic diagram Δ over $\langle X|R\rangle$ (in the sense of a diagram group), define the atomic annular diagram Δ_a to be the same directed labeled graph except with $\iota(\Delta)$ and $\tau(\Delta)$ identified. Both the inner vertex and the outer vertex of this annular diagram are the newly identified vertex $\iota(\Delta)$. The annular diagram has inner path $\mathrm{top}(\Delta)$ and outer path $\mathrm{bot}(\Delta)$, which are both now closed paths since their initial and terminal vertices were identified.

If $\mathrm{inn}(\Delta)$ has label u as read starting from $\mathrm{i}(\Delta)$ and $\mathrm{out}(\Delta)$ has label v as read starting from $\mathrm{o}(\Delta)$, then Δ is said to be annular diagram of type (u, v). The multiplication an annular diagram Δ_1 of type (u, v) and Δ_2 of type (v, w) is denoted $\Delta_1 \cdot \Delta_2$, and it is the annular diagram obtained by identifying $\mathrm{inn}(\Delta_2)$ and $\mathrm{i}(\Delta_2)$ with $\mathrm{out}(\Delta_1)$ and $\mathrm{o}(\Delta_1)$ respectively. The inner path of the new diagram is $\mathrm{inn}(\Delta_1)$ with distinguished vertex $\mathrm{i}(\Delta_1)$, and the outer path of the new diagram is $\mathrm{out}(\Delta_2)$ with distinguished vertex $\mathrm{i}(\Delta_2)$.

An annular diagram over $\langle X|R\rangle$, then, is some product of atomic annular diagrams over $\langle X|R\rangle$. Just as dipoles can be removed from diagrams, dipoles can be removed from annular diagrams, and doing so is called reducing the diagram. An annular diagram is called reduced if it contains no dipoles. The annular diagram group $D^a(\langle X|R\rangle, w)$ is the collection of all (w, w) reduced annular diagrams with the multiplication of Δ_1 and Δ_2 being the annular diagram obtained by removing all dipoles from $\Delta_1 \cdot \Delta_2$.

2.2 Thompson's Groups F, T and V

The three equivalent ways to define F, T, and V which we will use throughout the paper are defining the groups using functions, pairs of trees, and diagrams. A good survey of all three Thompson groups is given in [2], which uses both the function and the pairs of trees definitions of all three groups. For more details on F as a diagram group, T as an annular diagram group, and V as a braided diagram group, see [7]. In this paper, we choose to do function multiplication from left to right, and so to preserve the familiar relations of F, T, and V described in [2], we use the inverse generators as our canonical generators. For example, what we refer to as x_0 here is called x_0^{-1} in [2].

The first and most well-known definition of F is the following: the collection of all piecewise-linear orientation-preserving homeomorphism of the unit interval to itself, where there are only finitely breakpoints between each piece, each breakpoint is dyadic rational, i.e., of the form $a2^{-n}$ for some non-negative integers a and n, and the slope on each linear piece is an integer power of 2. More generally F_m is the collection of all piecewise-linear orientation-preserving homeomorphism of the unit interval to itself, where there are only finitely breakpoints between each piece, and each breakpoint is m-adic rational, i.e., of the form am^{-n} for some non-negative integers a and n, the slope on each linear piece is an integer power of m. Thus $F = F_2$. The group operation is composition, and a well-known generating set is $\{x_0, x_1\}$, where x_0 and x_1 are defined in Figure 2.3.

T can be defined similarly, by identifying the unit circle S_1 with $[0, 1]/\sim$ where 0 and 1 are identified with $0 \sim 1$. Generally, T_m is the collection of all piecewise-linear

$$x_0(t) = \begin{cases} 2t & t \in [0, \frac{1}{4}) \\ t + \frac{1}{4} & t \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{2}t + \frac{1}{2} & t \in [\frac{1}{2}, 1] \end{cases} \qquad x_1(t) = \begin{cases} t & t \in [0, \frac{1}{2}) \\ 2t - \frac{1}{2} & t \in [\frac{1}{2}, \frac{5}{8}) \\ t + \frac{1}{8} & t \in [\frac{5}{8}, \frac{3}{4}) \\ \frac{1}{2}t + \frac{1}{2} & t \in [\frac{3}{4}, 1] \end{cases}$$

$$c(t) = \begin{cases} \frac{1}{2}t + \frac{1}{2} & t \in [0, \frac{1}{2}) \\ t + \frac{1}{4} & t \in [\frac{1}{2}, \frac{3}{4}) \\ 2t - \frac{3}{2} & t \in [\frac{3}{4}, 1) \end{cases}$$

Figure 2.3: Two elements which generate F are x_0 and x_1 as defined above. Together with c, they generate T.

orientation-preserving homeomorphism of the unit circle to itself, where there are only finitely breakpoints between each piece, each breakpoint is m-adic rational, and the slope on each linear piece is an integer power of m. $T = T_2$, just as in the case of F, and notice that F can be viewed as $\mathrm{Stab}_T(0)$, the subgroup of all functions in T that stabilize 0. The function c defined in Figure 2.3 can be added to the generators of F to generate T.

To extend T to V, simply add the generator π_0 defined by

$$\pi_0 = \begin{cases} \frac{1}{2}x + \frac{1}{2} & 0 \leq x < \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \leq x < \frac{3}{4} \\ x & \frac{3}{4} \leq x < 1 \end{cases}$$

Notice that this function is no longer continuous, but it is right continuous. It turns out that V consists of all right continuous piecewise-linear bijections from $S_1 = [0, 1]/\sim$ to itself, where again there are only finitely many points of discontuity or breakpoints between each piece, each breakpoint is dyadic rational, and the slope on each linear piece is an integer power of 2. The groups V_m can be defined analogously, though we do not use them in this paper.

The next characterization of F utilized in this paper uses pairs of binary trees. A full binary tree is a rooted binary tree such that each vertex has 0 or 2 children. A caret is a binary tree where the root has two children which are both leaves. Then the elements of F are ordered pairs of full binary trees (R, S) such that R and S both have the same number of leaves, and there is an equivalence relation induced by the following equivalence: given such a pair (R, S), let R' be the tree obtained by attaching a caret at the ith leaf of R for some i counting leaves from left to right, and S' be the tree obtained by attaching a caret at the ith leaf of S for the same i, then (R', S') is equivalent to (R, S). This process of adding a caret to the same leaf in both R and S is called adding a dipole. Multiplication of (R_1, S_1) and (R_2, S_2) is done by adding dipoles to (R_1, S_1) and (R_2, S_2) until $S_1 = R_2$. Then $(R_1, S_1) \cdot (S_1, S_2) = (R_1, S_2)$.

Such a pair of trees (R, S) will be depicted in this paper by drawing the two trees side by side with an arrow from the left tree R to the right tree S labeled by the name of the element. For example, 2.4 shows two pairs of trees corresponding to x_0 and x_1, and such diagrams are called tree diagrams of x_0 and x_1.

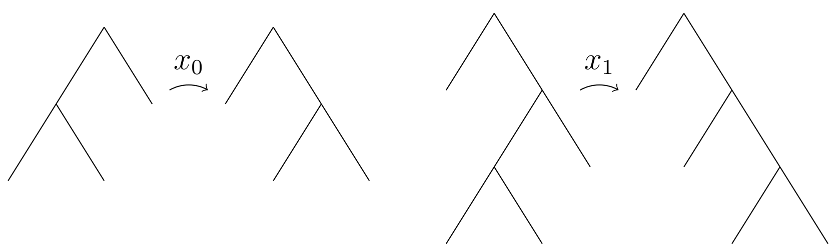


Figure 2.4: Pairs of trees corresponding to the standard two element generating set of F.

To see that this is still F, consider a pair of trees (R, S). Associate with the root of R and the root of S the interval $[0, 1]$. For each vertex in each tree associated with the interval $[a, b]$, associate its left child with $[a, \frac{a+b}{2}]$ and its right child with $[\frac{a+b}{2}, b]$ (if it has children). Then (R, S) corresponds with the function that linearly maps the interval associated with the ith leaf of R to the ith leaf of S for every S, and it is easy to see that this is indeed an element of F. In this way, the ith leaf of R is said to be identified with the ith leaf of S. For example, Figure 2.5 shows how an interval is associated with each vertex in a pair trees, and the graph of the corresponding function, which is x_0. For this reason, R and S may be referred to as the input tree and output tree respectively of (R, S).

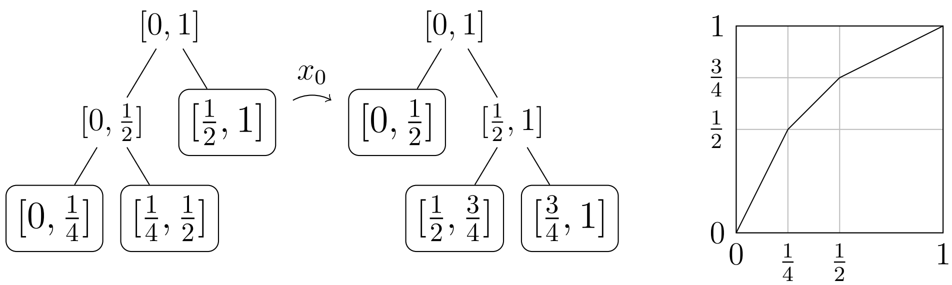


Figure 2.5: Going from a pair of trees to a function in F

Furthermore, removing or adding dipoles to a pair of trees does not affect the corresponding function, and it is easy to observe that the product of pairs of trees corresponds to the composition of the corresponding functions. To see that these groups are isomorphic, it only remains to check that x_0 and x_1 have corresponding pairs of trees, but these are shown in Figure 2.4.

Every element of T likewise corresponds to some (R, S, n), a pair of full binary trees R and S with the same number of leaves and additionally a number n which indicates which leaf of S the first leaf of R is identified with, where the rest of the leaves of R and S are identified cyclically. Often the k leaves of R and S are labeled by $1, \ldots, k$ to clarify these identifications, as shown in Figure 2.6, which shows the pair of trees corresponding to c. Likewise, a dipole can be added to (R, S, n) by adding a caret to a leaf in R and a leaf in S which are identified. In this case, the left child and right child of the caret in R and the left child and right child of the caret in S are identified respectively, and some (or all) of the leaves may need to be renumbered to reflect these new identifications. Likewise, n should be increased by one if the caret is added in S to the left of the leaf identified with the first leaf of R.

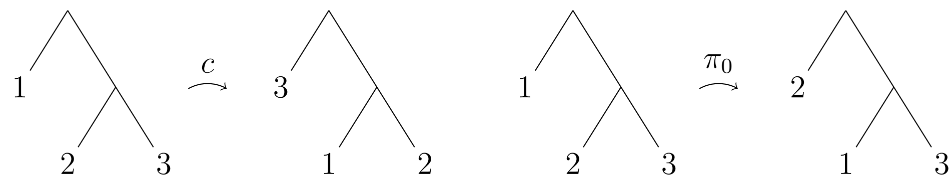


Figure 2.6: The pair of trees corresponding to c on left and π_0 on right, with the leaves numbered to show which leaves are identified.

The pair of trees (R, S, n) is likewise associated with a function by associating the root vertices of R and S with $[0, 1)$, and then if a vertex is associated with $[a, b)$, associate its left child with $[a, \frac{a+b}{2})$ and its right child with $[\frac{a+b}{2}, b)$. Then the corresponding function is the one that sends each interval of each leaf of R to the interval of the corresponding leaf in S linearly.

Extending this definition to V can be done by replacing n in (R, S, n) with σ, a permutation of $\{1, \ldots, n\}$. The ith leaf of R is then identified with the $\sigma(i)$th leave of S, and each such pair of trees corresponds with a function from V in exactly the same way as in T. The pair of trees diagram for π_0 is shown in Figure 2.6.

It is also possible to define F as the diagram group $D(\langle x | x = x^2 \rangle, x)$. Indeed, consider a pair of trees (R, S), and for every parent vertex in each tree with 2 children, replace it with a cell $x = x^2$ as shown in Figure 2.7. Then if R and S have n leaves, the result is two (x, x^n) diagrams Δ_R and Δ_S, so that $\Delta_R \cdot \Delta_S^{-1}$ is an (x, x) diagram, as shown in Figure 2.7. A dipole in a pair of trees clearly corresponds to a dipole in a diagram, and from this it is clear that multiplication also corresponds, since given two pairs of trees (R, S_1) and (S_1, S), the pair of trees (R, S) (possibly after removing dipoles) corresponds to the diagram formed from multiplying the diagrams for (R, S_1) and (S_1, S) and reducing. Finally, simply observe that the diagram Δ for any element of $D(\langle x | x = x^2 \rangle, x)$ has a longest directed path from $\iota(\Delta)$ to $\tau(\Delta)$, above which the cells are all of the form $x = x^2$ and below which the cells are all of the form $x^2 = x$.

Thus the portion of the diagram above this longest directed path corresponds to a tree and the portion of the diagram below this path corresponds to a tree, thereby giving a pair of trees.

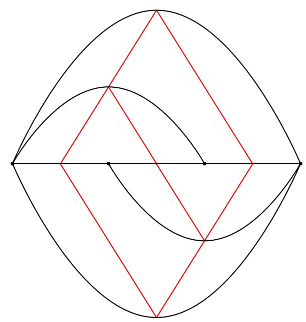


Figure 2.7: The diagram for x_0 as an element of $D(\langle x|x = x^2\rangle, x)$ is shown in black. Since every black edge is labeled x, the labels are omitted from the figure. The pair of trees (R, S) for x_0 are shown in red, with R on top and S on bottom and flipped vertically.

In much the same way, T is the annular diagram group $D^a(\langle x|x = x^2\rangle, x)$, and V can be described as the braided diagram group $D^b(\langle x|x = x^2\rangle, x)$, although we do not go into braided diagram groups any further and refer the reader to [7].

2.3 Addition and Normal Forms of Elements of F

A useful operation on F is the following addition operator [4]. If $f, g \in F$, then $f \oplus g$ is defined as

$$(f \oplus g)(t) = \begin{cases} \frac{f(2t)}{2} & t \in [0, \frac{1}{2}) \\ \frac{g(2t-1)+1}{2} & t \in [\frac{1}{2}, 1] \end{cases}$$

We can likewise demonstrate this operation on pairs of trees. If $f = (R_1, S_1)$ and $g = (R_2, S_2)$, then $f \oplus g = (R, S)$ where R is a tree where the left child of the root is a copy of R_1 and the right child of the root is a copy of R_2. S is likewise defined. In terms of diagrams, if Δ_1 and Δ_2 are two (x, x) diagrams and Δ is the cell corresponding to the relation $x = x^2$ with one edge on its top path and two edges on its bottom path, then $\Delta_1 \oplus \Delta_2 = \Delta \cdot (\Delta_1 + \Delta_2) \cdot \Delta^{-1}$.

Notice that although F is closed under addition, T is not. In fact, if $g \in T \setminus F$, then since g does not stabilize 0, $g \oplus f$ will not be a continuous function for any $f \in T$.

With this notation, notice that $x_1 = 1 \oplus x_0$. In general, another set of standard generators of F are given by $\{x_n\}_{n \in \mathbb{N}}$ where $x_n = 1 \oplus x_{n-1}$, and x_0 is as already defined. These generators are useful in that every element of F be written as a product of $x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} x_n^{-b_n} \ldots x_1^{-b_1} x_0^{-b_0}$ where n, a_i, and b_i are all non-negative integers [2]. Then if $f = x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} x_n^{-b_n} \ldots x_1^{-b_1} x_0^{-b_0}$, $x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n}$ is called the positive part of f, and $x_n^{-b_n} \ldots x_1^{-b_1} x_0^{-b_0}$ is called the negative part. If the negative part of f is trivial, then f is said to be a positive element, and likewise if the positive part is trivial, then f is a negative element.

Furthermore, given a pair of trees representation of $f = (R, S)$, n, a_i, and b_i can all be determined in the following way [2]: define the exponent of a vertex of a tree to be the maximal length of a path of left edges that end at the vertex and begins not on the right most path of the tree. Then $n + 1$ is the number of leaves in R and S, and a_i is the exponent of the ith vertex of R numbered from 0 to n, and b_i is the exponent of the ith vertex of S numbered similarly.

This naturally leads to the following observation, which will be useful in Chapter 5 in describing the elements of $\vec{T}$. Consider the tree S_n in Figure 2.8. If the output tree in a tree diagram for an element of F is S_n for some n, then the exponents b_i are all 0, and hence the element is positive. Likewise, if the input tree is S_n, then the exponents a_i are all 0, and hence the element is negative.

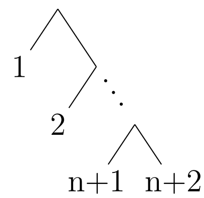


Figure 2.8: The tree S_n.

2.4 Stallings Cores and Stallings 2-Cores

In order to not confuse the free groups with the Thompson groups F_n, we will use $\mathcal{F}_n$ to refer to the free group on n generators.

Recall the Stalling's original construction of a core for subgroups of a free group [19, 15]. Let $\mathcal{F}_n$ have free generators $X = \{x_1, x_2, \ldots, x_n\}$, and let $H = \langle y_1, \ldots, y_n \rangle$. A directed graph in the sense of Serre is a directed graph together with an involution on edges. Specifically, given a set of vertices V, and a set of ordered pair of vertices called edges E labeled by a some set $L \cup L^{-1}$, the inverse edge of $e = (u, v)$ with label l is the edge $e^{-1} = (v, u)$ with label l^{-1}. E is assumed to be closed under taking inverses.

Then the Stallings core of H is a directed graph in the sense of Serre with edges labeled by elements of $X \cup X^{-1}$ constructed in the following way: begin with a single vertex, which we will refer to as the distinguished vertex. For each generator of H y_i, let w_i be the reduced word over $X \cup X^{-1}$ which equals y_i in $\mathcal{F}_n$, and then construct a directed path from the distinguished vertex to itself of the same length as w_i, where the kth edge along the path is labeled by the kth letter of w_i. Proceed to do the following foldings as many times as possible:

1. If two edges e_1 and e_2 share the same label and initial vertex, identify the edges together.

2. If two edges e_1 and e_2 share the same label and terminal vertex, identify the

edges together.

Note that when two edges are identified together, so are the corresponding initial or terminal vertices. Since their labels must already be the same, no labels are ever identified together.

A graph is said to have a hanging tree if it contains a tree as a subgraph and there is only one vertex v in the tree such that v is adjacent to some edge in the graph outside of the tree. Simplify the graph by removing any hanging trees, or in other words, remove all edges and vertices of the hanging tree except v from the graph.

Not only is the core of H a directed graph, but it can be viewed as an automaton over $X \cup X^{-1}$, which is said to accept a word w over $X \cup X^{-1}$ if there is a directed path on the core of H from the distinguished vertex to itself whose label reads w. It is not difficult to show that if a Stallings core is reduced, i.e., all possible foldings have occurred and all hanging trees have been removed, then w is accepted by the core implies that w is freely reduced.

Moreover, before the core is folded, it is clear that the core accepts exactly the language of all unreduced products of words over the generators of H, since any loop from the initial vertex to itself can only go through loops in their entirety or be forced to backtrack. The key to proving that the reduced core of H accepts exactly the words corresponding to reduced elements of H is then proving that folding two edges together does not change the language that the automaton accepts (see Lemma 3.4 of [15]).

Guba and Sapir extended these notions in [8] to diagram groups, where they approach diagrams as labeled directed 2-complexes, which are 2-complexes with labeled directed edges and where every cell has a top and bottom directed path, both of which starts at the same initial vertex and end at the same terminal vertex. A homomorphism of labeled directed 2-complexes is a map between 2-complexes which takes vertices to vertices, edges to edges of the same label, cells to cells, and respects the initial and terminal vertices of the edges and the top and bottom paths of each cell.

Now consider a diagram group $G = D(\langle X|R\rangle, w)$, and let $H = \langle X_H \rangle$ be a subgroup. The Stallings 2-core of H is constructed by the following steps. Consider the set of all diagrams Δ_x of elements $x \in X_H$, and identify all the top paths and bottom paths together to form a single 2-complex $C_0(H)$, and call this newly identified path the distinguished path of $C_0(H)$. Then do the following as many times as possible for all $u = v \in R$ and any pair of cells corresponding to $u = v$:

1. If the paths of both cells labeled by u are identified, identify the paths labeled by v.

2. If the paths of both cells labeled by v are identified, identify the paths labeled by u.

Two cells are said to be identified if their top and bottom paths have been identified. The resulting 2-complex $C(H)$ is called the Stallings 2-core of H. Notice that every identification in the above process is essentially a homomorphism of directed 2-complexes, and thus $C(H)$ contains a homomorphic image of the distinguished path in $C_0(H)$ which will be called the distinguished path of $C(H)$.

Just as with the Stallings core, the 2-core can be viewed as a 2-automaton, where an element $g \in G$ is said to be accepted by $C(H)$ if there is a homomorphism of labeled directed 2-complexes from the reduced diagram of g to the core of H which takes $\text{top}(g)$ and $\text{bot}(g)$ to the distinguished path of $C(H)$. For example, there is a clear homomorphism from each generator of H to $C_0(H)$ and hence also to $C(H)$. Furthermore, if there is a homomorphism of directed 2-complexes from a non-reduced diagram Δ to $C(H)$, then dipoles must be taken to dipoles, and so there is a homomorphism of directed 2-complexes from any diagram obtained from Δ by removing dipoles to $C(H)$. Thus if g and h are accepted by $C(H)$, then since $\text{bot}(g)$ and $\text{top}(h)$ are simply identified in the non-reduced product of g and h, the

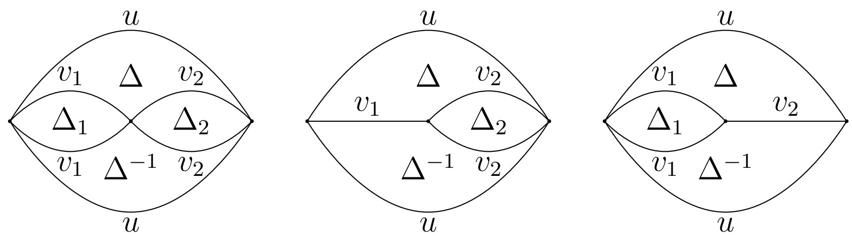


Figure 2.9: Shown are three (u, u) diagrams. The middle and right diagrams are called components of the left diagram and are accepted by any 2-core that accepts the diagram on the left since there is a clear homomorphism of directed 2-complexes from left diagram to the other two diagrams.

reduced product $g \cdot h$ is accepted by $C(H)$. Finally, if g is accepted by $C(H)$, then since the diagrams for g and g^{-1} are isomorphic, g^{-1} is also accepted by $C(H)$. This shows that $\{g \in G | C(H) \text{ accepts } g\}$ is a subgroup of G containing H. Define $\text{Core}(H) = \{g \in G | C(H) \text{ accepts } g\}$.

This should of course sound very similar to the language used to define the Stallings core of a subgroup of a free group, since the notion was motivated by Stallings' construction. In fact, we show in Section 3.1 that the Stallings 2-core of a subgroup of $\mathcal{F}_2$, represented in a natural way as a diagram group, also accepts exactly what the Stallings core of that same subgroup accepts.

To see that sometimes $\text{Core}(H)$ is strictly bigger than H, consider the following general example. Let $G = D(\langle X|R\rangle, u)$ be a diagram group such that there exists a (u, v_1v_2) diagram Δ, a (v_1, v_1) diagram Δ_1, and a (v_2, v_2) diagram Δ_2. Let H be the subgroup of G generated by $\Delta \cdot (\Delta_1 + \Delta_2) \cdot \Delta^{-1}$. Then $C(H)$ must accept both $\Delta \cdot (\varepsilon(v_1) + \Delta_2) \cdot \Delta^{-1}$ and $\Delta \cdot (\Delta_1 + \varepsilon(v_2)) \cdot \Delta^{-1}$ shown in Figure 2.9. However, these diagrams are often not in H.

Definition 2.4.1 (Guba and Sapir)**.** Any diagram of the form $\Delta \cdot (\Delta_1 + \Delta_2) \cdot \Delta^{-1}$, where Δ_1 is a (v_1, v_1) diagram and Δ_2 is a (v_2, v_2) diagram, is said to have components $\Delta \cdot (\varepsilon(v_1) + \Delta_2) \cdot \Delta^{-1}$ and $\Delta \cdot (\Delta_1 + \varepsilon(v_2)) \cdot \Delta^{-1}$, where $\cdot$ and $+$ are the standard multiplication and addition of diagrams.

A subgroup H is called closed under taking components if for any element h and component h' of h, $h \in H$ implies $h' \in H$. Let $\mathrm{Comp}(H)$ be the smallest subgroup of G containing H and closed under taking components. Then H is said to be closed if $H = \mathrm{Comp}(H)$. Guba and Sapir observed that $\mathrm{Comp}(H)$ is a subset of $\mathrm{Core}(H)$ and made the following conjecture around 1999, although it was never printed.

Conjecture 2.4.2. *Let H be a subgroup of a diagram group $D(\langle X|R\rangle, w)$. Then* $\mathrm{Comp}(H) = \mathrm{Core}(H)$.

Golan proved this conjecture true for the case of $D(\langle x|x = x^2\rangle, x) = F$ by showing that $\mathrm{Comp}(H)$ and $\mathrm{Core}(H)$ are both equal to a third subgroup which she called the subgroup of all elements of F which are dyadically-piecewise-H. In Chapter 3, we use Golan's result to construct more diagram groups where the conjecture holds, including the generalized Thompson groups F_n. Moreover, in Chapter 4, we generalize the construction of the 2-Core to T, and characterize $\mathrm{Core}(H)$ for any subgroup of H likewise as the subgroup of T of all functions that are dyadically-piecewise-T.

Chapter 3

Subgroups of Diagram Groups Accepted by Stallings 2-Cores

3.1 The Stallings 2-Core for Subgroups of Free Groups

Suppose $G = D(\langle X|R\rangle, u)$ is a diagram group. Originally, diagram groups were defined in terms of a string-rewriting system by Meakin and Sapir, and in this case it is more intuitive to say that a cell of the form $x = x$ is the same as its inverse, since they would both rewrite words in the same way. However, in [8], Guba and Sapir re-cast the definition of diagram groups into the language of directed 2-complexes. In particular, rather than thinking of R as a set of relations, it is thought of as a set of cells, and two different cells can have top paths and bottom paths which are respectively labeled identically. Put another way, R can be thought of as a multiset of relations, which could contain two different copies of the same relation. For example, if R contained two copies of $x = x^2$, each would correspond to a different cell. In this case, if the first of the two relations corresponds to the cell Δ_1 and the second to the cell Δ_2, then by definition of a dipole, $\Delta_1\Delta_2^{-1}$ is not a dipole (a dipole is only of the form $\Delta\Delta^{-1}$).

In particular, it is worth noting that in this paper we specifically allow for cells of the type $x = x$. Recall that for each relation r in R, there are two corresponding cells, Δ_r and Δ_r^{-1}. It is useful to distinguish between them by calling one of them, say Δ_r, positive and the other, Δ_r^{-1}, negative. Thus every cell is said to have positive or negative orientation, and its inverse has the opposite orientation, so that a cell and its inverse are never considered the same. Notably $\Delta_r^2 \neq \Delta_r\Delta_r^{-1}$, and so Δ_r^2 is never a dipole. As a result, $\mathcal{F}_1$ (recall that $\mathcal{F}_n$ is the free group of rank n) is isomorphic to the diagram group $D(\langle x|x = x\rangle, x)$. Note that if we wanted to avoid the relation $x = x$

and the ensuing discussion, we could use $D(\langle x, y, z | x = y, y = z, z = x\rangle, x)$, which is also isomorphic to $\mathcal{F}_1$. The former diagram group representation for $\mathcal{F}_1$, however, is more convenient.

Recall that given two diagram groups $G_i = D(\langle X_i | R_i\rangle, u_i)$ for $i = 1, 2$ with $X_1 \cap X_2 = \emptyset$ and a letter $a \notin X_1 \cup X_2$, the free product of G_1 and G_2, $G_1 * G_2$, is isomorphic to $D(\langle X|R\rangle, a)$ where $X = X_1 \cup X_2 \cup \{a\}$, $R = R_1 \cup R_2 \cup \{a = u_1, a = u_2\}$ [7]. For example, $\mathcal{F}_2$ is isomorphic to $\mathcal{F}_1 * \mathcal{F}_1 = D(\langle x, y, a | x = x, y = y, a = x, a = y\rangle, a)$.

The goal of this section is to extend the list of known diagram groups where Conjecture 2.4.2 holds. We begin this process by first observing through the following proposition that the Stallings core of a subgroup of $\mathcal{F}_2$ and the Stallings 2-core of the corresponding subgroup of $D(\langle x, y, a | x = x, y = y, a = x, a = y\rangle)$ both accept exactly the corresponding subgroups themselves.

Proposition 3.1.1. *Let $H = \langle x_1, \ldots, x_n\rangle$ be a subgroup of $\mathcal{F}_2$, represented as a diagram group by $D(\langle x, y, a | x = x, y = y, a = x, a = y\rangle)$. Then the 2-core of H accepts exactly H.*

Proof. Consider $\mathcal{F}_2 = \langle axa^{-1}, aya^{-1}\rangle$ as a subgroup of $\mathcal{F}_3 = \langle a, x, y\rangle$, then H can be viewed as a subgroup both of this representation of $\mathcal{F}_2$ and the aforementioned representation of $\mathcal{F}_2$ as a diagram group, which is generated by the diagrams in Figure 3.1. The goal is then to show that there is a correspondence between the core of H and the 2-core of H in this case, and that they accept exactly the same elements of $\mathcal{F}_2$ with respect to the isomorphism between these two representations of $\mathcal{F}_2$.

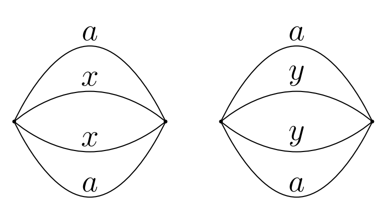


Figure 3.1: Two diagrams which generate $\mathcal{F}_2$, represented as the diagram group $D(\langle x, y, a | x = x, y = y, a = x, a = y\rangle, a)$.

Let us first show how to relate elements in the diagram group and $\{x^{\pm1}, y^{\pm1}, a^{\pm1}\}$. Consider the following correspondence between reduced (a, a) diagrams over $\mathcal{P} = \langle x, y, a | x = x, y = y, a = x, a = y\rangle$ and directed paths labeled by $\{x^{\pm1}, y^{\pm1}, a^{\pm1}\}$. Since every cell over $\mathcal{P}$ has only one top edge and one bottom edge, a diagram can be viewed as a sequence of cells corresponding to relations in $\mathcal{P}$. Then given a diagram that is a product of m cells $\Delta_1, \ldots, \Delta_m$, we correspond it to a directed path of edges $e_1, \ldots, e_m$ labeled by $\{x^{\pm1}, y^{\pm1}, a^{\pm1}\}$ where the label of e_i is determined as follows: if Δ_i is positively (respectively negatively) oriented and corresponds to the relation $x = x$, label the edge x (respectively x^{-1}). If Δ_i is positively (respectively negatively) oriented and corresponds to the relation $y = y$, label the edge y (respectively y^{-1}). If Δ_i is positively (respectively negatively) oriented and corresponds to the relation $a = y$ or $a = x$, label the edge a (respectively a^{-1}). In other words we are relating every 2-path over $\mathcal{P}$ with a directed labeled path in a very natural way, which gives a correspondence of elements in the diagram group with elements in $\langle axa^{-1}, aya^{-1}\rangle$.

This correspondence is also useful in looking at the 2-core of H as a subgroup of a diagram group and the core of H as a subgroup of $\langle axa^{-1}, aya^{-1}\rangle$. To make a clear distinction, in the rest of proof we will call the traditional Stallings core a 1-core.

To construct the 2-core of H, the first step is to identify the top and bottom paths of the diagrams corresponding to $x_1, \ldots, x_n$. In the 1-core, this corresponds to identifying the first and last vertex of each path corresponding to a diagram together. Observe that this correspondence also respects the process of folding: that is, if two cells are being folded as in the construction of the 2-core, and the corresponding to edges are folded as in the construction of the 1-core, the diagram and path still correspond. Furthermore, the distinguished vertex of the core clearly corresponds to the distinguished path of the 2-core.

Thus the 2-core and the 1-core are also in correspondence. Moreover if a diagram Δ is accepted by the 2-core, there is a homomorphism of directed 2-complexes from the diagram to the 2-core. But since the homomorphism preserves labels of cells, using the correspondence between diagrams and directed paths labeled by $\{x^{\pm 1}, y^{\pm 1}, a^{\pm 1}\}$ and distinguished vertex with distinguished path, this corresponds to being able to read the word corresponding to Δ as an element of $\mathcal{F}_2$ on a directed path that starts and ends at the distinguished vertex of the 1-core of H. Thus the corresponding element of $\mathcal{F}_2 = \langle axa^{-1}, aya^{-1}\rangle$ is also accepted by the 1-core of H. A similar argument gives the other direction. $\square$

Section 3.2 will show that a similar proposition holds for $\mathcal{F}_n$ by viewing it as a subgroup of $\mathcal{F}_2$. Thus the Stallings 2-core generalizes the notion of the Stallings core.

3.2 Direct Products and Subgroups

In another paper, Guba and Sapir show that not every subgroup of a diagram group is a diagram group [10]. However, it is easy to observe that for any subgroup H of a diagram group G, $\mathrm{Core}(H)$ is a diagram group.

Proposition 3.2.1. *If H is a closed subgroup of a diagram group $G = D(\mathcal{P}, w)$, i.e., $H = \mathrm{Core}(H)$, then H is a diagram group.*

Proof. The core of H consists of cells which are glued together, and this core can be used directly to form the presentation for a diagram group isomorphic to H. Give every edge e in the core of H a unique label, say x_e. Edges which are identified are given the same label since they are considered to be the same. Let R be the collection of all cells of the form $u = v$, where u is the new label of the top path of some cell in the core and v is the new label of the corresponding bottom path of that cell, and let X be the collection of all new labels of edges in the core. Let u be the new label of the distinguished path of the core. Then we claim that H is isomorphic to $H' = D(\langle X|R\rangle, u)$. If an edge has new label x_e, let $\phi(x_e)$ be the old label of the edge in the 2-core.

For example, we compute the core of F in chapter 4 (see Example 4.1.3) using tree diagram notation, and translating that back into the notation of diagram groups would give 4 cells, each of which had the type $x = x^2$ in $F = D(\langle x | x = x^2 \rangle, x)$. The diagram group F' constructed from the core of F would be $D(\langle a, b, c, d | a = bc, b = cd, c = dc, d = dd \rangle, a)$.

Then there is an obvious homomorphism $\Phi : H' \to H$ which simply takes diagrams over $\langle X|R \rangle$ to diagrams over $\mathcal{P}$ by replacing the label x_e of each edge in a diagram with the label $\phi(x_e)$. Clearly this homomorphism is surjective, since any diagram in H can be relabeled in the same way the core was relabeled. To see that it is injective, it suffices to show that only dipoles go to dipoles, and thus reduced diagrams go to reduced diagrams, implying that the kernel of the homomorphism is only the trivial diagram. Suppose Δ_1' and Δ_2' are two different cells over $\langle X|R \rangle$ such that $\Delta_1' \Delta_2'^{-1}$ is well-defined but not a dipole. Then it remains to check that Φ cannot send $\Delta_1' \Delta_2'^{-1}$ to a dipole. Suppose by way of contradiction that it does, that is, Δ_1' and Δ_2' are relabeled by Φ to be the same cell Δ. But Δ_1' and Δ_2' also correspond to cells Δ_1 and Δ_2 in the core of H, and the fact that the multiplication $\Delta_1' \Delta_2'^{-1}$ is well-defined means that $\mathrm{bot}(\Delta_1)$ and $\mathrm{bot}(\Delta_2^{-1})$ are identified. Thus Δ_1 and Δ_2 are identified by construction of the core, and so $\Delta_1' = \Delta_2'$. $\square$

This structure allows us to extend results about which diagram groups satisfy Conjecture 2.4.2 with the following proposition, for which we need some additional notation for clarity. Given a diagram group G, a closed subgroup H, and a subgroup K of both, let $\mathrm{Core}_G(K)$ denote the subgroup of G accepted by the core of K as a subgroup of the diagram group G, while $\mathrm{Core}_H(K)$ denotes the subgroup of H accepted by the core of K as a subgroup of the diagram group H. Likewise, let $\mathrm{Comp}_G(K)$ and $\mathrm{Comp}_H(K)$ be defined.

Proposition 3.2.2. *If G is a diagram group such that for any subgroup H of G, $\mathrm{Core}_G(H) = \mathrm{Comp}_G(H)$, then any for any closed subgroup H of G and any subgroup*

K *of* H*,* $\mathrm{Core}_H(K) = \mathrm{Comp}_H(K)$.

Proof. From the proof of Proposition 3.2.1, we can consider H as a diagram group with a natural homomorphism Φ from the diagrams of H as a diagram group to its diagrams as a subgroup of G. Our goal is to show that $\mathrm{Core}_G(K) = \mathrm{Core}_H(K)$. For any set of generators of K in the diagram group H, there are corresponding generators of K in the diagram group G whose diagrams are the same up to relabeling the edges via Φ. Thus consider the construction of the core of K in G and the core of K in H. After folding the distinguished paths of all generators of K together but before doing any other foldings, the core of K in H and the core of K in G are the same up to relabeling. Thus, it suffices to show that any folding that occurs in one also occurs in the other, and hence any diagram accepted by one will correspond to a diagram accepted by the other, and one of these diagrams is the image of the other under Φ.

Assume that for some number of foldings, the cores of K in H and G still coincide up to relabeling, and suppose $\Delta_{1,H}$ and $\Delta_{2,H}$ can be folded together in the core of K in H. Without loss of generality, they can be folded because $\mathrm{bot}(\Delta_{1,H}) = \mathrm{bot}(\Delta_{2,H})$ in the core of H, and $\mathrm{top}(\Delta_{1,H})$ and $\mathrm{top}(\Delta_{2,H})$ have the same label. Then if $\Delta_{1,G}$ and $\Delta_{2,G}$ are the corresponding cells in the core of K in G, since the cores coincide up to relabeling, we have that $\mathrm{bot}(\Delta_{1,G}) = \mathrm{bot}(\Delta_{2,G})$, and since the label of $\mathrm{top}(\Delta_{i,G})$ is the label of $\Phi(\mathrm{top}(\Delta_{i,H}))$, we must also have that $\mathrm{top}(\Delta_{1,G})$ and $\mathrm{top}(\Delta_{2,G})$ have the same label, and hence $\Delta_{1,G}$ and $\Delta_{2,G}$ can be folded in the core of K in G.

Likewise suppose that $\Delta_{1,G}$ and $\Delta_{2,G}$ can be folded in the core of K in G. Then without loss of generality, their bottom paths are identified in the core of K in G and their top paths have the same label. Let $\Delta_{1,H}$ and $\Delta_{2,H}$ be the corresponding cells in the core of K in H. Again since the cores coincide up to relabeling, their bottom paths must also be identified, and it suffices to show that their top paths have the same label.

Since $\Delta_{1,G}$ and $\Delta_{2,G}$ are cells in the core of K in G with their bottom paths identified, there is a diagram Δ_G accepted by the core of K in G containing $\Delta_{1,G}\Delta_{2,G}^{-1}$. Let Δ_H be the corresponding diagram in H. Then $\mathrm{Core}_G(K) = \mathrm{Comp}_G(K) \subset \mathrm{Comp}_G(H) = \mathrm{Core}_G(H)$, and so Δ_G is accepted by $\mathrm{Core}_G(H)$. In particular, $\Delta_{1,G}$ and $\Delta_{2,G}$ have homomorphic image in the core of H in G, and were thus folded together in the core of H in G. By the construction of H as a diagram group from its core in G, this implies that the tops of $\Delta_{1,H}$ and $\Delta_{2,H}$ must have the same label in H, exactly as desired.

Finally, observe that any diagram in the diagram group H with components also has components in G. Conversely, our above proof demonstrates that if a diagram $\Delta \cdot (\Delta_1 + \Delta_2) \cdot \Delta^{-1}$ from the diagram group G is in K as a subgroup of G, then the cells on the core of H in G corresponding to the cells of Δ and Δ^{-1} from the diagram would be folded, and hence the corresponding diagram in the diagram group H would also have components. Thus $\mathrm{Comp}_H(K) = \mathrm{Comp}_G(K) = \mathrm{Core}_G(K) = \mathrm{Core}_H(K)$.

□

For example, Golan and Sapir found particular representations of the generalized Thompson groups F_n in F [4], which we show are closed subgroups of F in the next proposition, and hence there exist diagram group representations of F_n that satisfy Conjecture 2.4.2.

Corollary 3.2.3. *There exists a representation of F_n as a diagram group for every $n \geq 2$ such that for any subgroup H of F_n,* $\mathrm{Core}(H) = \mathrm{Comp}(H)$.

Proof. By Lemma 5.10 of [4] and Theorem 5.11 of [4], the group $\vec{F_n}$ is isomorphic to F_{n+1} for $n \geq 2$. Fix n, and let S_i be the subset of all of finite dyadic rationals whose sums of digits are equivalent to i modulo n. Then $\vec{F_n} = \bigcap_{i=0}^{n-1} \mathrm{Stab}_F(S_i)$. In other words, $f \in \vec{F_n}$ if and only if for every i and every $s \in S_i$, $f(s) \in S_i$. Then if $f \in \mathrm{Core}(\vec{F_n})$, it is dyadically-piecewise-$\vec{F_n}$ by Theorem 5.6 of [6], and in particular $f(s) \in S_i$ for every i and $s \in S_i$. Hence $f \in \vec{F_N}$, and so $\mathrm{Core}(\vec{F_n}) = \vec{F_n}$. The proof is finished by Proposition 3.2.2. □

Next, recall that given two diagram groups $G_i = D(\langle X_i | R_i \rangle, u_i)$ for $i = 1, 2$ with $X_1 \cap X_2 = \emptyset$, the direct product $G_1 \times G_2$ is isomorphic to the diagram group $D(\langle X_1 \cup X_2 | R_1 \cup R_2 \rangle, u_1 u_2)$ [7], where every diagram in $G_1 \times G_2$ is simply a sum $\Delta_1 + \Delta_2$ for some $\Delta_i \in G_i$ and $i = 1, 2$.

Proposition 3.2.4. *If for $i = 1, 2$, G_i is a diagram group such that for any subgroup H_i of G_i we have* $\mathrm{Core}(H_i) = \mathrm{Comp}(H_i)$*, then for any finitely generated subgroup H of $G_1 \times G_2$, we also have* $\mathrm{Core}(H) = \mathrm{Comp}(H)$*.*

Proof. Let $H = \langle (x_1, y_1), \ldots, (x_n, y_n) \rangle$, and consider one of the generators of H, (x_i, y_i). Let Δ_1 be the diagram in G_1 corresponding to x_i and Δ_2 be the diagram in G_2 corresponding to y_i. Then (x_i, y_i) corresponds to the diagram $\Delta_1 + \Delta_2$. In particular, $\varepsilon(l(\mathrm{top}(\Delta_1))) + \Delta_2$ and $\Delta_1 + \varepsilon(l(\mathrm{top}(\Delta_2)))$ are components of (x_i, y_i), and hence accepted by the core. Thus $(1, y_i)$ and $(x_i, 1)$ are both accepted by the core of H. Let $H_1 = \langle x_1, \ldots, x_n \rangle$ and $H_2 = \langle y_1, \ldots, y_n \rangle$. Notice that in the core of H cells from G_1 and cells from G_2 have totally distinct labels and thus are never identified, so any folding done of cells with labels from G_1 are also folded in H_1, and cells from G_2 are folded in H_2. Thus if the core of H accepts (x, y) and Δ_1 is the diagram corresponding to x in G_1 and Δ_2 is the diagram corresponding to y in G_2, we have that x is also accepted by the core of H_1 and is thus in $\mathrm{Comp}(H_1)$, while y is accepted by the core of H_2 and is thus in $\mathrm{Comp}(H_2)$, implying that $\mathrm{Core}(H)$ is a subset of $\mathrm{Comp}(H_1) \times \mathrm{Comp}(H_2)$. It is clear that $\mathrm{Comp}(H_1) \times \mathrm{Comp}(H_2)$ is a contained in $\mathrm{Comp}(H)$, which is contained in $\mathrm{Core}(H)$ by Guba and Sapir. Thus $\mathrm{Core}(H) = \mathrm{Comp}(H) = \mathrm{Comp}(H_1) \times \mathrm{Comp}(H_2)$. □

Chapter 4

Extending the Stallings 2-Core Construction to Thompson's Group T

4.1 Generalized Definition of the Stallings 2-Core

There are many possible approaches to generalizing the Stallings 2-core construction to annular and braided diagram groups, and in this section we look at one way which works especially well for T. Specifically we translate the language of the Stallings 2-core construction for diagram groups into the pair of trees language used to describe F, since this language is the very natural for extending F to T and is easy to depict. Just as Golan proved in [6] that given a subgroup H of F, $\mathrm{Core}(H)$ is the subgroup of F consisting of functions that are dyadically-piecewise-H (see section 4.2), we show that given a subgroup H of T and the appropriate definition of $\mathrm{Core}(H)$, $\mathrm{Core}(H)$ is exactly the subgroup of T consisting of functions that are dyadically-piecewise-H. One inclusion here is the same as F, but the other direction is a rather different flavor.

It is also worth noting that trying to show that $\mathrm{Core}(H) = \mathrm{Comp}(H)$ in T is not exactly well-defined, since there is not a completely clear analog in annular diagram groups of components of an element in a diagram group. For example, consider that $x_0 \oplus x_0$ has components $1 \oplus x_0$ and $x_0 \oplus 1$. Indeed, it can be shown that all components in F arise from repeatedly combining 1 with other elements of F via $\oplus$, where those other elements fix 0 and 1, but no other elements of $[0, 1]$. However, as we have already observed, given $f, g \in T$, $f \oplus g \in T$ if and only if $f, g \in F$. The notion of being dyadically-piecewise-H does generalize well, and is our chosen method for characterizing the core of a subgroup H of T.

Definition 4.1.1. The core of a finitely generated subgroup $H = \langle x_1, \ldots, x_n \rangle \leq T$

is a labeled directed graph constructed in the following way. First, begin with all the reduced tree diagrams for $x_1, \ldots, x_n$ and identify all the roots of trees in these diagrams together. This vertex formed by identifying all the roots is called the root of the core. The label of each left edge is 0 and the label of each right edge is 1. Proceed by doing the following steps as many times as possible:

1. If two vertices are identified, identify their left children and left edges, along with their right children and right edges.

2. If two vertices have their left children and their right children identified respectively, then identify the vertices.

This process will stop since there are only finitely many edges and vertices. Let $C(H)$ denote the core of H.

Observe that the root of the core corresponds to the distinguished path of the core of a subgroup of a diagram group, and that the vertices correspond to edges and a vertex together with its left and right children correspond to cells. Thus the identification process used in creating the core also corresponds. We can also give an analogous way to define acceptance by a core.

Definition 4.1.2. An element $x \in T$ is said to be accepted by $C(H)$ if there is a homomorphism ϕ of labeled directed graphs from the reduced diagram of x to the core, where a graph homomorphism is a map such that vertices are sent to vertices, edges are sent to edges of the same label, and if u and v are vertices such that (u, v) is a directed edge e, then $(\phi(u), \phi(v))$ is the directed edge $\phi(e)$.

Checking whether such a homomorphism exists is practically accomplished by first mapping the roots of the reduced diagram of x to the root of $C(H)$, and then identifying edges and vertices from the diagram of x with edges and vertices of $C(H)$ using the same two steps outlined above repeatedly. Either a natural graph homomorphism to the core will arise from the identifications, or a contradiction (e.g., a vertex of in the diagram of x being identified with two different vertices in the core, or a vertex in x which has children is identified with a vertex in the core that has no children) is reached and the process cannot continue, implying that no such homomorphism exists. In particular, checking if x is accepted by $C(H)$ is decidable.

Example 4.1.3. For example, we can compute $C(F)$ by using the generators x_0 and x_1 of F. This example is depicted in Figure 4.1, where numbers are used for the vertices to signify that they are identified. The first step of constructing the core would be to label both roots of trees in the diagrams of x_0 and x_1 by 1, to show that they are all the same vertex on the core.

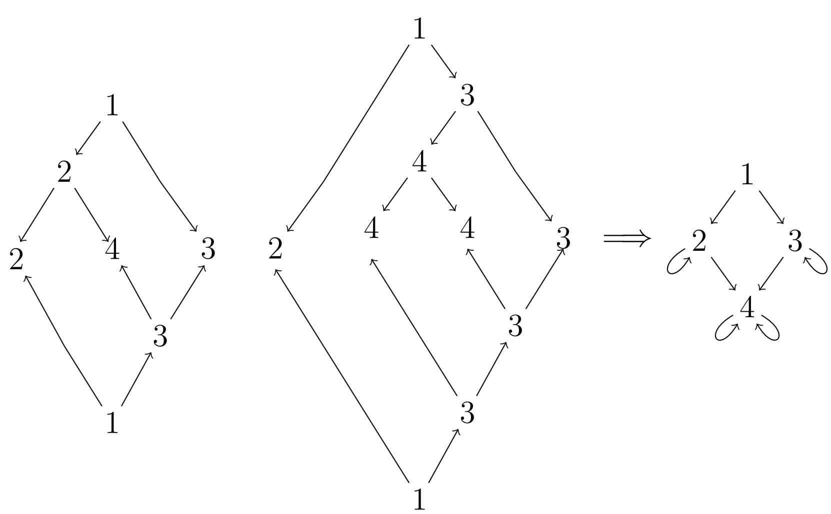


Figure 4.1: An intermediary step in constructing the core of $\langle x_0, x_1 \rangle$ is shown on the left, with The identified pairs of trees for x_0 on the left and x_1 in the middle. The final core is shown on the right.

Next identify all left children of vertex 1 as vertex 2, and identify all right children of vertex 1 as vertex 3. Notice that from the diagram of x_0, we see that vertex 2 has left child 2 and vertex 3 has right child 3, so we can label all corresponding children of vertices labeled 2 and 3 accordingly. Next, since it has not yet been identified with other vertices, let us label the right child of 2 by 4, and do so for all instances of the vertex 2 in the diagram. Then 4 is also in one instance the left child of 3, so update the core once again. Finally, all possible identifications and labels have been completed, so the core is complete, and it has 4 vertices and is shown on the right in the figure.

Let the subset of T accepted by the core of a finitely generated subgroup H be denoted by $\mathrm{Core}(H)$. Then the next proposition proves that $\mathrm{Core}(H)$ is also a subgroup of T that contains H, although it may be larger.

Lemma 4.1.4. $\mathrm{Core}(H)$ *is a subgroup that contains* H*.*

Proof. By construction, $C(H)$ accepts the generators of H, so it suffices to prove that $\mathrm{Core}(H)$ is a subgroup. Suppose $x \in \mathrm{Core}(H)$. It is clear that $x^{-1} \in \mathrm{Core}(H)$ since the diagrams for x and x^{-1} are isomorphic.

Suppose $x, y \in \mathrm{Core}(H)$. Then the unreduced tree diagram corresponding to xy is accepted by $C(H)$, since the only overlap between the two diagrams in the unreduced product diagram is the root, which is sent to the same place in the core. Furthermore, if a dipole is removed from a diagram accepted by the core, then the vertices which are identified by removing the dipole were also identified in the core, so the core accepts the reduced diagram. Thus $xy \in \mathrm{Core}(H)$. □

4.2 The Closure of a Subgroup of T

A subgroup $H \leq T$ is said to be closed if $\mathrm{Core}(H) = H$, in which case determining whether any element $g \in T$ is in H is very simple. In order to investigate which subgroups of T are closed, we need to first characterize $\mathrm{Core}(H)$, which is the goal of this section.

A function $f \in T$ is said to be dyadically-piecewise-H if there exists functions

$f_i \in H$ and dyadic rationals α_i for $i = 1, \ldots, n$ such that

$$f(t) = \begin{cases} f_1(t) & t \in [0, \alpha_1] \\ f_2(t) & t \in (\alpha_1, \alpha_2] \\ \vdots & \\ f_n(t) & t \in (\alpha_{n-1}, 1] \end{cases}$$

Thus f, although it may not be in H itself, is composed of pieces of functions from H. The subgroup of T of all elements which are dyadically-piecewise-H is written as $\mathrm{Piec}(H)$. For example, it is an observation that if $f \oplus g$ is in $\mathrm{Core}(H)$, then so is $1 \oplus g$, which is the identity on $[0, \frac{1}{2}]$ and $f \oplus g$ on $[\frac{1}{2}, 1]$. Nevertheless, the subgroup generated by $f \oplus g$ need not contain $1 \oplus g$, and so $\mathrm{Core}(H)$ may indeed contain more than H, just as was the case for F.

When proving that $\mathrm{Core}(H) = \mathrm{Comp}(H)$, Golan proved that $\mathrm{Core}(H) \subset \mathrm{Piec}(H)$ in the case of F [6], but it is also true for T with very little alteration. The proofs presented here for Lemmas 4.2.1 and 4.2.2 are the result of discussions between the author with Golan.

Since edges on trees and the core are labeled with 0 for left and 1 for right, paths and their labels will be referred to interchangeably. For every leaf in a pair of trees diagram of $x \in T$, let $t_+, t_- \in \{0, 1\}^*$ be the labels of the unique positive and negative paths to the leaf. Thus x is associated with $t_+ \to t_-$, since the final vertex on the path t_+ is associated with the final vertex on the path t_-. It also has the following equivalent meaning. Viewing x as a function on binary words, $t_+ \to t_-$ means that x applied to a binary word beginning with t_+ replaces the prefix with t_-. For example, x_0 contains $00 \to 0$, and rewrites binary words as follows, where α is any finite binary word:

$$00\alpha \to 0\alpha$$

$$01\alpha \to 10\alpha$$

$$1\alpha \to 11\alpha$$

This hearkens back to the original definition of diagram groups using rewriting systems.

Lemma 4.2.1. *Let $H = \langle x_1, \ldots, x_n \rangle \leq T$, and let u, v be a vertices of the diagrams for some x_i, x_j respectively. If u and v are identified in the core of H, then there exists a non-negative integer $l_{u,v}$ such that for any directed path with label t_u to u in x_i and any directed path with label t_v to v in x_j, as well as for any finite binary word α with $|\alpha| \geq l_{u,v}$, there exists a not necessarily reduced diagram for some $g \in H$ containing $t_u\alpha \to t_v\alpha$.*

Proof. We proceed by considering the construction of the core. At the beginning, only roots are identified, so either they are roots from different trees or $u = v$, and they are vertices from the same x_i. If they are roots, the only directed path from the root to the root is the empty path, and hence for $l = 0$ and any finite binary word α we can simply take g to be the unreduced trivial diagram of depth $|\alpha|$.

If $u = v$ and they are vertices in x_i, either they are leaves, in which case there are two directed paths to $u = v$ from the root, or they are not leaves in which case there is only one directed path to $u = v$ from the root. Observe that if there is $g \in H$ containing $t_u \to t_v$, then up to simply adding dipoles to g, g contains $t_u\alpha \to t_v\alpha$ for any finite binary word α. Thus if both paths are the same, g can once again be the trivial diagram, and if both paths are different, then g can be a diagram of x_i. Then $l = 0$ still works, and by the above observation, we are done with the base case of induction.

Now suppose that the result holds up to a certain point in the construction of the core, and then two vertices are identified in the core. Since there are two ways to identify vertices, we have two cases to consider.

First, assume that u and v are the left or right children of two identified vertices, w_u and w_v from x_i and x_j respectively. Since the cases are symmetric, assume they are both left children. If neither u nor v are leaves, then any directed path in x_i or x_j from the root to u or v go through w_u or w_v respectively. Thus, given any directed paths from the root to u or v with labels t_u and t_v respectively, we have $t_u = t_{w_u}0$ and $t_v = t_{w_v}0$ for some directed paths with label t_{w_u} and t_{w_v} to w_u and w_v. By induction there exists $l' = l_{w_u,w_v}$ such that if $|\alpha| \geq l'$, then there exists $g \in H$ containing $t_{w_u}\alpha \to t_{w_v}\alpha$. Thus if $|\alpha| \geq l'-1$, there exists $g \in H$ containing $t_u\alpha \to t_v\alpha$ since $t_u = t_{w_u}0$, and $t_v = t_{w_v}0$. Therefore $l_{u,v} = \max\{l'-1, 0\}$ suffices.

Suppose that u is a leaf and v is not a leaf. Up to taking x_i^{-1} instead of x_i, we may suppose that w_u is in the negative part of the diagram of x_i, hence x_i contains $t_{u,+} \to t_{w_u}0$ where $t_{u,+}$ is the unique positive path to u (recall that a path is positive if it is in the input tree for the pair of trees representation of g, and it is called negative if it is in the output tree). Then if t_u is a path in x_i to u, either t_u goes through w_u, in which case we can use the same argument as when u is not a leaf, or $t_u = t_{u,+}$. Thus x_i contains $t_{u,+} \to t_{w_u}0$, and by the previous case there is l_{w_u,w_v} such that for any finite binary word α of length at least l_{w_u,w_v}, there is a $g \in H$ containing $t_{w_u}\alpha \to t_{w_v}\alpha$, where $t_{w_v}0$ is any path in x_j to v which must go through w_v. Thus for any finite binary word α of length at least $l_{w_u,w_v} - 1$, $x_i \cdot g$ contains $t_{u,+}\alpha \to t_{w_v}0\alpha$. Since $t_u = t_{u,+}$, $t_v = t_{w_v}0$, $x_i \cdot g$ contains $t_u\alpha \to t_v\alpha$ as desired.

For the second case, suppose that the children of u and v have already been identified, and hence u and v are being identified. This time, as u and v have children, they are clearly not leaves. Let u_l, u_r, v_l, v_r be the left and right children of u and v

respectively. Then choose $l = \max\{l_{u_l,v_l}, l_{u_r,v_r}\} + 1$. Then if $|\alpha| \geq l$, we can write α as $0\alpha'$ or $1\alpha'$. Since both cases are similar, we will consider the former case. Then $|\alpha'| \geq l_{u_l,v_l}$, hence there exists $g \in H$ taking $t_{u_l}\alpha' \to t_{v_l}\alpha'$. Since $t_{u_l}\alpha' = t_u\alpha$ and $t_{v_l}\alpha' = t_v\alpha$, we are done. □

This lemma generalizes to any two directed paths in the core from the root to the same end point.

Lemma 4.2.2. *Let $H = \langle x_1, \ldots, x_n \rangle \leq T$, and let p and q be the labels of directed paths in $C(H)$ from the root to the same vertex. Then there exists a non-negative integer l such that for every $\alpha \in \{0,1\}^*$ with $|\alpha| \geq l$, there exists $g \in H$ containing $p\alpha \to q\alpha$.*

Proof. Observe that any directed path in the core can be decomposed into $p_1, \ldots, p_k$ where each p_i is the label of some directed path (not necessarily from the root) in some x_{j_i}, and the concatenation $p_1 \cdot p_2 \cdot \ldots \cdot p_k$ is well-defined and the original directed path.

Thus let $p = p_1, \ldots, p_k$ and $q = q_1, \ldots, q_l$ be such decompositions. We proceed by induction on $k+l$, with the base case of $k+l = 2$ being the previous proposition. Now suppose $m = k + l \geq 3$. Up to swapping the roles of p and q and taking the inverse of the element g that we find, we may assume that $k \geq 2$. Let $p_1 \in x_i$ and $p_2 \in x_j$, with u the last vertex of p_1 and v the first vertex of p_2. Since v is on a directed path in x_j, it is not a leaf, so there exists a unique path p_1' in x_j to v. Now, since p_1 and p_1' satisfy the conditions of the previous proposition, there exists l' such that for any $|\alpha'| \geq l$, there exists $g' \in H$ containing $t_u\alpha' \to t_v\alpha'$. Thus if $|\alpha| \geq l - |p_2 \ldots p_k|$, then g' contains $t_u p_2 \ldots p_k \alpha \to t_v p_2 \ldots p_k \alpha$. Since p_1' and p_2 are in the same tree, by induction there is a $g \in H$ containing $t_v p_2 \ldots p_k \alpha \to q_1 \ldots q_l \alpha$ if $|\alpha| \geq l''$ for some l''. Up to originally taking a larger quantity, we may assume $l' \geq l''$, and hence the element $g'g$ contains $p\alpha \to q\alpha$. □

In particular, this last proposition directly implies that every element of $\mathrm{Core}(H)$ is dyadically-piecewise-H. Notice that this implies that if $H \leq F$, then any element in $\mathrm{Core}(H)$ is dyadically-piecewise-H, and since every element of H stabilizes 0, so does every element in $\mathrm{Core}(H)$. Thus $\mathrm{Core}(H) \leq F$ and this extension of the Stallings 2-core to T coincides with its definition in F.

In our next step towards proving that $\mathrm{Core}(H) = \mathrm{Piec}(H)$, we need the following technical lemma.

Lemma 4.2.3. *For $i = 1, 2$, let h_i be the linear bijection from some standard dyadic interval $[\frac{a}{2^n}, \frac{a+1}{2^n}] \subset [0,1]$ to $[\frac{b_i}{2^{n_i}}, \frac{b_i+1}{2^{n_i}}] \subset [0,1]$ for some non-negative integers a, n, b_1, b_2, n_1, n_2. If there exists a dyadic rational $\gamma \in (\frac{a}{2^n}, \frac{a+1}{2^n})$ such that $h_1(\gamma) = h_2(\gamma)$, then $h_1(t) = h_2(t)$ for all $t \in [\frac{a}{2^n}, \frac{a+1}{2^n}]$.*

Proof. By definition, $h_i(t) = 2^{n-n_i}(t - \frac{a}{2^n}) + \frac{b_i}{2^{n_i}}$. Since $h_1(\gamma) = h_2(\gamma)$, we have the following:

$$\begin{aligned} 2^{n-n_1}(\gamma - \frac{a}{2^n}) + \frac{b_1}{2^{n_1}} &= 2^{n-n_2}(\gamma - \frac{a}{2^n}) + \frac{b_2}{2^{n_2}} \\ 2^{n_2}(2^n\gamma - a) + b_1 2^{n_2} &= 2^{n_1}(2^n\gamma - a) + b_2 2^{n_1} \\ b_1 2^{n_2} - b_2 2^{n_1} &= (2^n\gamma - a)(2^{n_1} - 2^{n_2}) \end{aligned}$$

Since $\gamma \in (\frac{a}{2^n}, \frac{a+1}{2^n})$ is a dyadic rational, $2^n\gamma - a$ is a dyadic rational in $(0,1)$. Since n_2, n_1, b_1, b_2 are non-negative integers, $b_1 2^{n_2} - b_2 2^{n_1}$ is an integer. Thus $(2^n\gamma - a)(2^{n_1} - 2^{n_2})$ must be an integer.

There are then two cases to consider, either $n_1 = n_2$ or $n_1 \neq n_2$. If $n_1 = n_2$, then since $h_1(\gamma) = h_2(\gamma)$, the above equation simplifies to $b_1 = b_2$, and thus h_1 and h_2 are the same linear function. If $n_1 \neq n_2$, then without loss of generality $n_1 > n_2$. Since

$2^n\gamma - a$ is a dyadic rational, let $2^n\gamma - a = \frac{c}{2^m}$ for some integers $m \geq 0$ and c. Then

$$\begin{aligned}
b_1 2^{n_2} - b_2 2^{n_1} &= (2^n\gamma - a)(2^{n_1} - 2^{n_2}) \\
\frac{b_1 2^{n_2} - b_2 2^{n_1}}{2^{n_1} - 2^{n_2}} &= \frac{c}{2^m} \\
\frac{2^m 2^{n_2}(b_1 - b_2 2^{n_1 - n_2})}{2^{n_2}(2^{n_1-n_2} - 1)} &= c \\
\frac{2^m(b_1 - b_2 2^{n_1-n_2})}{(2^{n_1-n_2} - 1)} &= c
\end{aligned}$$

Notice that 2^m and $2^{n_1-n_2} - 1$ are relatively prime, and thus $\dfrac{2^m(b_1 - b_2 2^{n_1-n_2})}{(2^{n_1-n_2} - 1)}$ is an integer if and only if $\dfrac{(b_1 - b_2 2^{n_1-n_2})}{(2^{n_1-n_2} - 1)}$ is an integer. But $\dfrac{c}{2^m} = \dfrac{(b_1 - b_2 2^{n_1-n_2})}{(2^{n_1-n_2} - 1)}$. Observe that $0 < \frac{c}{2^m} = 2^n\gamma - a < 1$ since $\gamma \in (\frac{a}{2^n}, \frac{a+1}{2^n})$, and thus $2^n\gamma - a = \frac{c}{2^m} \in (0, 1)$. Therefore $\dfrac{(b_1 - b_2 2^{n_1-n_2})}{(2^{n_1-n_2} - 1)}$ cannot be an integer, which is a contradiction since $c = \dfrac{(b_1 - b_2 2^{n_1-n_2})}{(2^{n_1-n_2} - 1)}$ is an integer. Thus $n_1 > n_2$ is impossible. $\square$

Lemma 4.2.4. *Let H be a subgroup of the Thompson group T, and let $g \in T$ be dyadically-piecewise-H. Then $g \in \mathrm{Core}(H)$.*

Proof. Since g is dyadically-piecewise-H,

$$g(t) = \begin{cases} h_1(t) & t \in [\alpha_0, \alpha_1] \\ h_2(t) & t \in [\alpha_1, \alpha_2] \\ \vdots & \\ h_m(t) & t \in [\alpha_{m-1}, \alpha_m] \end{cases}$$

where α_i are all dyadic rationals and $\alpha_0 = 0 \equiv 1 = \alpha_m$, and each $h_i \in H$. Since every such interval with dyadic rational endpoints can be subdivided into standard dyadic intervals of the form $[\frac{a}{2^n}, \frac{a+1}{2^n}]$ for some integers $n \geq 0$ and a, without loss of generality each $[\alpha_i, \alpha_{i+1}] = [\frac{a_i}{2^{n_i}}, \frac{a_i+1}{2^{n_i}}]$ for some integers $n_i \geq 0$ and a_i. We can furthermore assume g restricted to any particular $[\alpha_i, \alpha_{i+1}]$ is linear, up to simply dividing into more intervals.

With each standard dyadic interval $[\frac{a}{2^n}, \frac{a+1}{2^n}] \subset [0,1]$, there is a unique finite binary word u associated with that interval, where $0.u = \frac{a}{2^n}$ and $0.u111\ldots = \frac{a+1}{2^n}$. For example, associated with the interval $[\frac{1}{4}, \frac{2}{4}]$ is $u = 01$, noting that $0.01 = \frac{1}{4}$ in binary and $0.01111\ldots = 0.1 = \frac{1}{2}$ in binary.

Let u_i be the finite binary word associated with $[\alpha_{i-1}, \alpha_i] = [\frac{a_{i-1}}{2^{n_{i-1}}}, \frac{a_{i-1}+1}{2^{n_{i-1}}}]$. Recall that each u_i also corresponds to the label of a directed path on a binary tree consisting of choices of left indicated by 0 and right indicated by 1. Likewise, let v_i be the finite binary word associated with $g([\alpha_{i-1}, \alpha_i])$, which is also a standard dyadic interval. To show g is accepted by the core, it suffices to prove that for each u_i and v_i, both u_i and v_i are the labels of paths on the core of H starting from the root and ending at the same vertex on the core, as this will allow us to explicitly construct a homomorphism of labeled directed graphs from g to $C(H)$. For the rest of this proof, we will refer to u_i and v_i as both the labels of the directed paths and the directed paths themselves.

Since h_i linearly maps $[\alpha_{i-1}, \alpha_i]$ to $g([\alpha_{i-1}, \alpha_i])$, up to adding dipoles, the pairs of tree diagram for h_i must contain a path labeled by u_i from the root to a leaf on the input tree of h_i, and a path labeled by v_i from the root to a leaf on the output tree of h_i, such that the corresponding leaves are identified in the diagram. Thus in the reduced diagram for h_i, there exist paths u_i' and v_i' from the root to identified leaves in the domain and output trees respectively such that $u_i'\alpha = u_i$ and $v_i'\alpha = v_i$ for some possibly empty finite binary word α.

Then since each h_i is accepted by the core, u_i' and v_i' are paths on the core of H from the root to the same vertex. It suffices to prove that u_i and v_i are also paths on the core of H from the root to some vertices. Indeed, since u_i' and v_i' end at the same vertex on the core of H, and since $u_i = u_i'\alpha$, $v_i = v_i'\alpha$, u_i and v_i must end at the same place if the paths exist on the core of H. We can further simplify it to proving that

u_i is a path on the core, since if $u_i'\alpha$ is a path on the core and u_i' and v_i' end in the same place, then $v_i'\alpha$ must also be a path on the core.

Let u_i' be associated with the standard dyadic interval $[\alpha_{i-1}', \alpha_i']$. For the rest of the proof, it will be convenient to use a finite binary word interchangeably with the associated standard dyadic interval. For example, since u_i' is a prefix of u_i, treating them as intervals we have $u_i \subset u_i'$.

Suppose now that u_i ends with a 0. Then if v is any other binary word such that $v1^n = u_i 1^n$, we must have that u_i is a prefix of v. Said another way, $[\alpha_{i-1}, \alpha_i]$ is the largest standard dyadic interval with α_i as a right endpoint. Thus if $\alpha_i = \alpha_i'$, then $u_i' \subset u_i$, and since $u_i \subset u_i'$ as well, we have $u_i = u_i'$. Hence u_i would be a path on the core of H as desired.

Otherwise, if $\alpha_i \neq \alpha_i'$, then u_i is strictly contained in u_i'. Given any two dyadic intervals, their intersection is either empty, a single end point, or one of the two intervals. In particular, consider u_{i+1}', defined in the same way as u_i' (replace each instance of i in the above paragraphs defining u_i' with $i+1$ to define u_{i+1}'). Notice that h_i restricted to u_i' and h_{i+1} restricted to u_{i+1}' are linear maps, and since g is continuous, $h_{i+1}(\alpha_i) = h_i(\alpha_i)$. Since α_i is not the right end point of u_i', since $\alpha_i \in u_i' \cap u_{i+1}'$, and since u_{i+1}' contains elements to the right of α_i', α_i must also be interior to u_{i+1}'. Thus we can apply Lemma 4.2.3 to h_i and h_{i+1} restricted to $u_i' \cap u_{i+1}'$ to get that they must actually be the same linear function.

Since u_i as a binary word ends with a 0, we can write $u_i = u_i''0$. Then $u_{i+1} = u_i''10^n$ for some $n \geq 0$. If $n = 0$, then $[\alpha_{i-1}, \alpha_{i+1}]$ is a dyadic interval, and so we can replace the intervals $[\alpha_{i-1}, \alpha_i]$ and $[\alpha_i, \alpha_{i+1}]$ in the definition of g as dyadically-piecewise-H with the single interval $[\alpha_{i-1}, \alpha_{i+1}]$. By induction on m, where m is the number of functions required in the decomposition of g as dyadically-piecewise-H, we are done (if $m = 1$, the statement is obvious).

Thus suppose that $n > 0$. Then u_{i+1} ends with a 0, and we can use the same argument to show that either u_{i+1} is a path on the core of H or h_{i+1} and h_{i+2} are the same function. Continuing in this way, we will eventually stop, since either we will get a path on the core, or we will get to some k such that h_i, $h_{i+1}, \ldots, h_{i+k}$ are all the same function. If we get that u_{i+1} is a path on the core, then in particular $u_i''1$ is a path on the core. Since no vertex on the core has a right child and not a left, $u_i''0 = u_i$ is a path on the core as desired. In the second case, the endpoint that we stop at is $\alpha_{i+k} = 1$, since this is the furthest right we can go without wrapping back around. However, this cannot actually happen, since 1 is only the right end point of a standard dyadic interval if the corresponding binary word ends is a sequence of 1's. Each u_{i+j} in our sequence was shown to end with 0, since if it ended with a 1 we could have already concluded that u_i was a path on the core of H.

The final case to consider is when u_i ends with a 1. In this case, if v is a binary word corresponding to the same fraction as u_i, then u_i must be a prefix of v. Thus $[\alpha_{i-1}, \alpha_i]$ is the largest standard dyadic interval with α_{i-1} as a left endpoint, and we can use an analogous argument considering h_{i-1} and h_i together, rather than h_{i+1} and h_i. □

Taken together, Lemmas 4.2.2 and 4.2.4 directly prove the following theorem, which is the main goal of this section.

Theorem 4.2.5. *Let $H = \langle x_1, \ldots, x_n \rangle$ be a subgroup of T. Then $g \in T$ is accepted by the core of H if and only if g is dyadically-piecewise-H. That is,* $\mathrm{Core}(H) = \mathrm{Piec}(H)$.

In particular, the core of H does not depend upon the chosen set of generators of H. It also makes the proof of the following proposition quite simple.

Proposition 4.2.6. *Let $H \leq T$ and $g \in T$, then* $\mathrm{Piec}(gHg^{-1}) = g\,\mathrm{Piec}(H)g^{-1}$

Proof. Suppse $h \in \mathrm{Piec}(H)$. Then

$$h(t) = \begin{cases} h_1(t) & t \in [0, \alpha_1] \\ h_2(t) & t \in (\alpha_1, \alpha_2] \\ \vdots & \\ h_n(t) & t \in (\alpha_{n-1}, 1] \end{cases}$$

for some $h_1, \ldots, h_n \in H$ and dyadic rationals α_i.

Then we must also have

$$ghg^{-1}(t) = \begin{cases} gh_1g^{-1}(t) & t \in g^{-1}([0, \alpha_1]) \\ gh_2g^{-1}(t) & t \in g^{-1}((\alpha_1, \alpha_2]) \\ \vdots & \\ gh_ng^{-1}(t) & t \in g^{-1}((\alpha_{n-1}, 1]) \end{cases}$$

Since elements of T are bijections that take dyadic rationals to dyadic rationals, $g^{-1}([0, \alpha_1]), \ldots, g^{-1}((\alpha_{n-1}, 1])$ is still a dyadic partition of $[0, 1]$. Thus $ghg^{-1} \in \mathrm{Piec}(gHg^{-1})$, and so $g\,\mathrm{Piec}(H)g^{-1} \subset \mathrm{Piec}(gHg^{-1})$. The reverse inclusion is proved similarly. $\square$

4.3 Examples of Closed Subgroups

Our next goal is to explore some of the ramifications of Theorem 4.2.5 by looking at examples of closed subgroups of T.

Corollary 4.3.1. *If H is any subgroup of T, then $\mathrm{Core}(H)$ is closed.*

Proof. Since $\mathrm{Core}(H) = \mathrm{Piec}(H)$, and since $\mathrm{Piec}(\mathrm{Piec}(H)) = \mathrm{Piec}(H)$, $\mathrm{Core}(\mathrm{Core}(H)) = \mathrm{Core}(H)$. Thus $\mathrm{Core}(H)$ is closed. $\square$

Since a similar result is true in F, that is, the subgroup of F accepted by the core of a subgroup H of F is exactly $\mathrm{Piec}(H)$ [6], we also have the following immediate corollary.

Corollary 4.3.2. *If $H \leq F$ is closed, then H is also closed in T.*

Proof. If $h \in T$ is accepted by the core of H in T, then h is in $\mathrm{Piec}(H)$. Since every function in H fixes 0, so must h. Thus $h \in F$ and $h \in \mathrm{Piec}(H)$ by Theorem 5.6 of [6]. As $\mathrm{Piec}(H) = H$, $h \in H$. □

A natural question one might ask about T is whether any subgroup H such that $\mathrm{Core}(H) = T$ must itself be equal to T. We can show by example that this is not true, however.

Example 4.3.3. Let $H = \langle x_0, c_0 \rangle$, where x_0 and c_0 are the elements whose tree diagrams are depicted in Figure 4.2. Then $\mathrm{Core}(H) = T$, but $H \neq T$.

Proof. The fully simplified core of H is shown in Figure 4.2, as well as an intermediary step in computing the core. In short, vertices with the same number in the figure are identified in the process of constructing the core. For example, for the diagram of c_0 on the left of the figure, the first identification are the roots on top and bottom, which are both thus labeled 1. The left and right children of these two nodes are then identified, but we notice the the left child of the root of the top tree is the right child of the root of the bottom tree, thus we can label both left and right children as 2. Taking both diagrams for x_0 and c_0 together, we see that vertices 2 and 3 must be identified, as the children of 1 in both trees are identified. From this, the right child of 2 is 4 and the right child of 3 is 3 from x_0, and hence $4 = 3$ as well. We also see that the left child of 2 is 2, and thus vertices 1 and 2 both have 2 as the left and right children. Therefore vertices 1 and 2 are identified, and so the core of $\langle x_0, c_0 \rangle$ consists of a single vertex which is its own left and right children. This is exactly the core depicted on the right in the figure.

Observe that there is a graph homomorphism from any full binary tree to the core of $\langle x_0, c_0 \rangle$, given by simply sending every vertex of any tree to the vertex labeled 1. Since the single vertex in the core is its own children, any relation between vertices and edges is preserved by this homomorphism. Thus every element of T is accepted by this core, and we immediately get that $T \subset \text{Core}(H)$, and hence $\text{Core}(H) = T$.

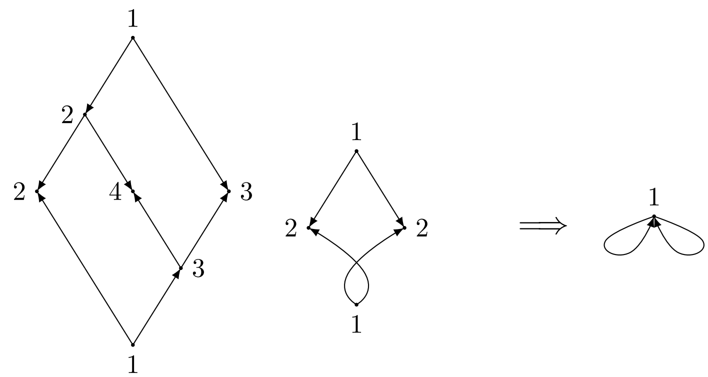


Figure 4.2: The core of $\langle x_0, c_0 \rangle$ on the right, computed from initial identifications done to the diagrams for x_0 (left) and c_0 (middle).

However, in order to show that $H \neq T$, it suffices to check that $H \cap F \neq F$. In order to do this, recall that $c_0 = x_0 c_1$, where $c_1^3 = 1$. Since $c_0^2 = 1$, every element of H can be written as $x_0^{n_0} c_0 x_0^{n_1} c_0 \ldots c_0 x_0^{n_k}$, for some $k, n_i \in \mathbb{Z}$, and $n_i \neq 0$ if $i \neq 0, k$.

Since $c_0 = x_0 c_1$, we have $x_0^{-1} c_0 = c_1$, and so $(x_0^{-1} c_0)^3 = 1$. Therefore we have

$$\begin{aligned}
c_0 x_0^{-1} c_0 &= x_0 x_0^{-1} (c_0 x_0^{-1} c_0) x_0^{-1} c_0 c_0 x_0 \\
&= x_0 (x_0^{-1} c_0)^3 c_0 x_0 \\
&= x_0 c_0 x_0
\end{aligned}$$

Thus, if n is a positive integer, we have

$$\begin{aligned} x_0^{-n} &= c_0(c_0x_0^{-1}c_0)^n c_0 \\ &= c_0(x_0c_0x_0)^n c_0 \end{aligned}$$

Therefore every element of H can be written as $x_0^{n_0}c_0x_0^{n_1}c_0\dots c_0x_0^{n_k}$, for some $k, n_i \in \mathbb{Z}$, $n_i \neq 0$ if $i \neq 0, k$, and $n_i \geq 0$, i.e., no negative powers of x_0 need be used. Furthermore, observe that since $(c_0x_0)^3 = 1$, we have $x_0c_0x_0c_0x_0 = c_0$. In particular, if $1 < i < k-1$, then if $n_i = 1$ we can replace the subword $x_0^{n_{i-1}-1}x_0c_0x_0^{n_i}c_0x_0x_0^{n_{i+1}-1}$ with simply $x^{n_{i-1}-1}c_0x_0^{n_{i+1}-1}$ and further reduce the word. Thus we may assume that $n_i \geq 2$ except possibly for $i = 0, 1, k-1, k$, and if $n_k \neq 0$, then $n_{k-1} \geq 2$ as well.

Finally, we will show if $h_1, h_2 \in H$ have the forms $h_1 = x_0^{n_0}c_0x_0^{n_1}c_0\dots c_0x_0^{n_k}$ and $h_2 = x_0^{m_0}c_0x_0^{m_1}c_0\dots c_0x_0^{m_l}$ as above, then $h_1(0) = h_2(0)$ if and only if $h_1h_2^{-1}$ is an integer power of x_0. Thus $H \cap F = \langle x_0 \rangle$. If $h_1h_2^{-1}$ is an integer power of x_0, then clearly $h_1(0) = h_2(0)$, since $x_0(0) = 0$.

For the other direction, it suffices to prove statement for $n_0 = 0$ and $m_0 = 0$. Indeed, if it is true for all such h_1 and h_2 with $n_0 = 0$ and $m_0 = 0$, then take h_1 and h_2 arbitrary. Then $h_1' = x_0^{-n_0}h_1$ and $h_2' = x_0^{-m_0}h_2$ have the desired form with $n_0 = 0 = m_0$. Since $x_0(0) = 0$, we still have that $h_1'(0) = h_2'(0)$, and thus $h_1'h_2'^{-1}$ is a power of x_0. Thus $x_0^{-n_0}h_1h_2^{-1}x_0^{-m_0}$ is a power of x_0, and so $h_1h_2^{-1}$ is as well, as desired.

We can view x_0 and c_0 as functions which rewrite binary words as described in the following table:

x_0	c_0
$00\alpha \to 0\alpha$	$0\alpha \to 1\alpha$
$01\alpha \to 10\alpha$	$1\alpha \to 0\alpha$
$1\alpha \to 11\alpha$	

where a binary string represents a dyadic rational. For example, $\frac{1}{2}$ is the binary number 0.1, hence corresponding to the string 1, $\frac{3}{8}$ is the binary number 0.011, and thus corresponds to the binary string 011. Since $x_0(\frac{1}{2}) = \frac{3}{4}$, we think of x_0 as rewriting 1 to 11.

Using this notation allows us to compactly compute that h_1 rewrites 0 in the following way (recall that functions are applied from left to right):

$$\begin{aligned}
h_1(0) &\to c_0x_0^{n_1}c_0 \ldots c_0x_0^{n_k}(0)\\
&\to x_0^{n_1}c_0 \ldots c_0x_0^{n_k}(1)\\
&\to c_0x_0^{n_2} \ldots c_0x_0^{n_k}(1^{n_1+1})\\
&\to x_0^{n_2}c_0 \ldots c_0x_0^{n_k}(01^{n_1})\\
&\to x_0^{n_2-1}c_0 \ldots c_0x_0^{n_k}(101^{n_1-1})\\
&\to c_0x_0^{n_3} \ldots c_0x_0^{n_k}(1^{n_2}01^{n_1-1})\\
&\to x_0^{n_3}c_0 \ldots c_0x_0^{n_k}(01^{n_2-1}01^{n_1-1})\\
&\to x_0^{n_3-1}c_0 \ldots c_0x_0^{n_k}(101^{n_2-2}01^{n_1-1})\\
&\to c_0x_0^{n_4} \ldots c_0x_0^{n_k}(1^{n_3}01^{n_2-2}01^{n_1-1})\\
&\vdots
\end{aligned}$$

$$\rightarrow c_0 x_0^{n_k}(1^{n_{k-1}}01^{n_{k-2}-2}0\ldots01^{n_2-2}01^{n_1-1})$$

$$\rightarrow 1^{n_k}01^{n_{k-1}-2}0\ldots01^{n_2-2}01^{n_1-1}$$

assuming $n_k > 0$. Note that we do use the fact that each $n_i \geq 2$ for $2 \leq i < k$, in order to guarantee that 1^{n_i-1} is not an empty string as we use the above rewriting rules. Similarly, if $n_k = 0$, then h_1 rewrites 0 to $01^{n_{k-1}-1}01^{n_{k-2}-2}0\ldots01^{n_2-2}01^{n_1-1}$.

Likewise h_2 rewrites 0 to $1^{m_l}01^{m_{l-1}-2}0\ldots01^{m_2-2}01^{m_1-1}$ if $m_l > 0$ and to $01^{m_l-1}01^{m_{l-1}-2}0\ldots01^{m_2-2}01^{m_1-1}$ if $m_l = 0$.

Thus if $h_1(0) = h_2(0)$, we must have $n_k = m_l$. Likewise, either h_1 and h_2 rewrite 0 to the exact same binary string, or one of them rewrites 0 to a binary string which ends with some number of 0s. In the first case, $h_1 = h_2$, and in the second case, without loss of generality, h_1 takes 0 to a longer binary string than h_2. In this case, we still must have $n_{k-i} = m_{l-i}$ for each $i < l$, and then $n_{k-i} = 2$ for each $i \geq l$, with the exception that $n_1 = 1$. It is possible either for $m_1 = n_{k-l+1}$ or $m_1 = 1$.

Thus $h_1h_2^{-1} = c_0x_0(c_0x_0^2)^n(c_0x_0)^{-\varepsilon}$ for $\varepsilon = 0$ or $\varepsilon = 1$ and $n \geq 0$ (since h_1 rewrites 0 to a longer string than h_2). If $\varepsilon = 0$, it is easy to verify that c_0x_0 rewrites 0 to 110, and that $c_0x_0^2$ rewrites 110α to 1100α. Thus if $\varepsilon = 0$, $h_1h_2^{-1}(0) \neq 0$, which is a contradiction. Thus $\varepsilon = 1$.

To finish the proof, we show that $c_0x_0(c_0x_0^2)^n(c_0x_0)^{-1} = x_0^{-n}$, and hence $h_1h_2^{-1}$ is a power of x_0, as desired. The statement is clearly true for $n = 0$, and for $n = 1$ the statement is $x_0^{-1} = c_0x_0c_0x_0^2x_0^{-1}c_0 = c_0x_0c_0x_0c_0$. This is equivalent to $1 = (c_0x_0)^3$, which is true since $c_0x_0 = c_1^{-1}$ is an element of order 3.

We proceed by induction on n, with $n > 1$:

$$
\begin{aligned}
c_0x_0(c_0x_0^2)^n(c_0x_0)^{-1} &= c_0x_0(c_0x_0^2)^{n-1}(c_0x_0)^{-1}(c_0x_0)(c_0x_0^2)(c_0x_0)^{-1} \\
&= x_0^{-(n-1)}x_0^{-1} \text{ (by induction)} \\
&= x_0^{-n}
\end{aligned}
$$

□

In the previous example, since $x_0c_1 = c_0$, we have $H = \langle x_0, c_1\rangle = \langle x_0, c_0\rangle$, where c_0 and c_1 are finite order elements. Thus it so happens that $\langle x_0, c_1\rangle \cap F = \langle x_0\rangle = \langle x_0, c_0\rangle \cap F$, and so one might ask whether a similar situation occurs for any subgroup H of F, and any finite order element $t \in T$. However, one of the defining relations of T is $c_1 = x_1x_0^{-1}c_1x_1$ [2]. From this it is clear that $x_0 \in \langle x_1, c_1\rangle$, hence we have the following example.

Example 4.3.4. $T = \langle x_1, c_1\rangle$, and thus $F \cap \langle x_1, c_1\rangle = F$, which is strictly larger than $\langle x_1\rangle$.

We can still make an effort to begin to classify the cores of subgroups of T of the form $\langle H, t\rangle$ for $H \leq F$ and $t \in T$ of finite order, however. Let us first suppose $H = 1$ is the trivial subgroup of F, the simplest case possible, and that $t = c_n$.

Corollary 4.3.5. $\mathrm{Core}(\langle c_n\rangle) = \langle c_n\rangle$ *for any* $n \geq 0$.

Proof. Since both trees in the reduced pair of trees representation of c_n are the same, any power of c_n simply permutes their leaves. In particular, c_n^k and c_n^j have no intersection as functions unless $k \equiv j \mod (n+2)$. Thus the only functions which are piecewise $\langle c_n\rangle$ are themselves in $\langle c_n\rangle$, and since $\mathrm{Core}(H) = \mathrm{Piec}(H)$, we thus have $\mathrm{Core}(\langle c_n\rangle) = \mathrm{Piec}(\langle c_n\rangle) = \langle c_n\rangle$. □

We can actually prove a similar statement about not only $\langle t \rangle$, but any finite subgroup of T.

Proposition 4.3.6. *If H is a finite subgroup of T, then* $\mathrm{Core}(H) = H$*.*

Proof. By Theorem 1.3 in [3], H is conjugate to any other finite subgroup of the same order. In particular, since c_n generates a finite subgroup of order $n+2$, H is either trivial or conjugate to some $\langle c_n \rangle$. If H is trivial, then its core consists of a single vertex with no children, and hence the core only accepts the trivial element. Otherwise, $\mathrm{Core}(H) = \mathrm{Piec}(H) = \mathrm{Piec}(\langle g c_n g^{-1} \rangle)$ for some $g \in T$, and so by Proposition 4.2.6, $\mathrm{Piec}(\langle g c_n g^{-1} \rangle) = g \,\mathrm{Piec}(c_n) g^{-1} = \langle g c_n g^{-1} \rangle = H$. Thus $\mathrm{Core}(H) = H$ as desired. □

We next turn our attention to a collection of subgroups of T which are all of quasi-finite index. A subgroup H of a group G is said to be of *quasi-finite index* if there are only finitely many subgroups of G containing H [5]. In particular, if H is a subgroup of T of quasi-finite index, then there are only finitely many options for $\mathrm{Core}(H)$, and hence if we can classify those options, we can easily test for which subgroup of T that $\mathrm{Core}(H)$ happens to be.

Fortunately, there is a nice class of such subgroups of T. Given any $\alpha \in [0, 1)$, the pointwise stabilizer of α, $\mathrm{PStab}_T(\alpha)$, is a maximal subgroup as the next lemma shows. Savchuk showed a similar result in [18] by proving that $\mathrm{PStab}_F(\alpha)$ is a maximal subgroup of F. In order to prove this result in T, we first prove the following technical lemma.

Lemma 4.3.7. *Let $\alpha, \beta, \gamma, \delta$ be dyadic rationals in $[0, 1)$ with $\alpha \neq \gamma$ and $\beta \neq \delta$. Then there exists an element $h \in T$ such that $h(\alpha) = \beta$ and $h(\gamma) = \delta$.*

Proof. Let k be the smallest positive integer such that $2^k(\delta - \beta)$, $2^k(\gamma - \alpha)$, and $2^k\alpha$ are integers, and let $\Delta_{\alpha,\gamma} \in [0, 1)$ such that $\Delta_{\alpha,\gamma} \equiv \gamma - \alpha \mod 1$. Likewise let $\Delta_{\beta,\delta} \in [0, 1)$ such that $\Delta_{\beta,\delta} \equiv \beta - \delta \mod 1$. Then there exists $k_1, \ldots, k_j$ such that

$2^{k_1} + \ldots + 2^{k_j} = 2^k \Delta_{\beta,\delta}$, and each k_i is an integer. Moreover, they can be chosen so that $j = 2^{k'+k}\Delta_{\alpha,\gamma}$ for some non-negative integer k', since $2^k\Delta_{\alpha,\gamma}$ is a positive integer, and j can always be increased by 1 by replacing 2^{k_j} with $2^{k_j-1} + 2^{k_j-1}$.

Our goal is to define h as a piecewise function, and we can now begin to do that. Since $2^k\alpha$ is an integer, $[\alpha, \alpha + \frac{1}{2^{k+k'}})$ is a standard interval. Thus we can define h on $[\alpha, \gamma]$ in the following way, recalling that $0 \sim 1$, so for example $[\frac{9}{8}, \frac{10}{8})$ is the same interval as $[\frac{1}{8}, \frac{2}{8})$.

$$h(t) = \begin{cases} 2^{k_1+k'}t + c_1 & t \in [\alpha, \alpha + \frac{1}{2^{k+k'}}) \\ 2^{k_2+k'}t + c_2 & t \in [\alpha + \frac{1}{2^{k+k'}}, \alpha + \frac{2}{2^{k+k'}}) \\ \vdots & \vdots \\ 2^{k_j+k'}t + c_j & t \in [\alpha + \frac{j-1}{2^{k+k'}}, \alpha + \frac{j}{2^{k+k'}}] \end{cases}$$

where c_1 is a dyadic rational chosen so that $h(\alpha) = \gamma$, and the remaining $c_2, \ldots, c_j$ are dyadic rationals chosen to make the function continuous. Thus we have $h(\gamma) = h(\alpha + \Delta_{\alpha,\gamma}) = h(\alpha + \frac{j}{2^{k+k'}})$, and since each interval has the same width of $\frac{1}{2^{k+k'}}$, we have $h(\alpha + \frac{j}{2^{k+k'}}) = h(\alpha) + \frac{2^{k_1+k'}}{2^{k+k'}} + \ldots + \frac{2^{k_j+k'}}{2^{k+k'}} = \beta + 2^{-k}(2^{k_1} + \ldots + 2^{k_j}) = \beta + \Delta_{\beta,\delta} = \delta$. Thus $h(\gamma) = \delta$ and $h(\alpha) = \beta$ as desired.

Repeat the process to define h on (γ, α) to get the desired element of T. □

Proposition 4.3.8. *For any $\alpha \in [0, 1)$, $PStab_T(\alpha)$ is a maximal subgroup of T.*

Proof. Let $f, g \in T \setminus \mathrm{PStab}_T(\alpha)$. If we show that $g \in \langle \mathrm{PStab}_T(\alpha), f\rangle$, then since g is arbitrary, $\langle \mathrm{PStab}_T(\alpha), f\rangle = T$, and hence $\mathrm{PStab}_T(\alpha)$ is a maximal subgroup. Let $\beta = f(\alpha)$ and $\gamma = g(\alpha)$. Since f and g are piecewise linear functions of the form $2^k x + m$ where m is a dyadic rational, we immediately get that $\beta - \alpha$ and $\gamma - \alpha$ are dyadic rational, and hence $\gamma - \beta$ is dyadic rational.

Since $f(\alpha) \neq \alpha \neq g(\alpha)$, we also have that $\alpha \neq \beta$ and $\alpha \neq \gamma$. Let $[a_\alpha, b_\alpha]$, $[a_\beta, b_\beta]$, and $[a_\gamma, b_\gamma]$ be standard dyadic intervals of the same width such that α, β, and γ are in their respective intervals, and such that $[a_\alpha, b_\alpha]$ is disjoint from both $[a_\beta, b_\beta]$ and $[a_\gamma, b_\gamma]$. Up to possibly choosing smaller intervals, we can assume that $[a_\beta, b_\beta] + \gamma - \beta = [a_\gamma, b_\gamma]$, since $\gamma - \beta$ is a dyadic rational.

By Lemma 4.3.7, there exists $h_1, h_2 \in T$ such that $h_1(b_\alpha) = b_\alpha$, $h_1(a_\beta) = a_\gamma$, $h_2(b_\alpha) = b_\alpha$, and $h_2(b_\beta) = b_\gamma$. Then define h by

$$h(t) = \begin{cases} t & t \in [a_\alpha, b_\alpha) \\ h_1(t) & t \in [b_\alpha, a_\beta) \\ t + \gamma - \beta & t \in [a_\beta, b_\beta) \\ h_2(t) & t \in [b_\beta, a_\alpha) \end{cases}$$

Thus $h(\alpha) = \alpha$ and $h(\beta) = \gamma$, and so $h \in \mathrm{PStab}_T(\alpha)$. Hence $fhg^{-1}(\alpha) = hg^{-1}(\beta) = g^{-1}(\gamma) = \alpha$, and so $fhg^{-1} \in \mathrm{PStab}_T(\alpha)$. In particular, $g \in \langle \mathrm{PStab}_T(\alpha), f \rangle$ as desired. □

As an immediate corollary, we get that for any $\alpha \in [0, 1)$ and for any $g \in T \setminus \mathrm{PStab}_T(\alpha)$, $\mathrm{Core}(\langle \mathrm{PStab}_T(\alpha), g \rangle) = \langle \mathrm{PStab}_T(\alpha), g \rangle = T$.

The next natural question is to extend the result to stabilizers of multiple points. It turns out that if U is any finite collection of points from $[0, 1)$, then $\mathrm{PStab}_T(U)$ also has quasi-finite index. Before proving this, we need a technical lemma similar to Lemma 4.2 from [2]. Elements $u_1, u_2, \ldots, u_n \in [0, 1)$ are said to be cyclically ordered if there exists j such that $u_j < u_{j+1} < \ldots u_n < u_1 < \ldots u_{j-1}$.

Lemma 4.3.9. *If $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are cyclically ordered elements of $[0, 1)/\sim$ with $v_i - u_i$ dyadic rational, then there exists $g \in T$ such that $g(u_i) = v_i$ for every $i = 1, \ldots, n$.*

Proof. First, assume $u_1 < u_2 < \ldots u_n$ and $v_1 < v_2 < \ldots v_n$ using the natural order on $[0, 1)$. Take dyadic rationals $a_{i,j}$ and $b_{i,j}$ such that $a_{i-1,2} < a_{i,1} < u_i < a_{i,2}$,

$b_{i-1,2} < b_{i,1} < v_i < b_{i,2}$, $a_{i,2} - a_{i,1} = b_{i,2} - b_{i,1}$, and $a_{i,2} - u_i = b_{i,2} - v_i$ for all i. Then by Lemma 4.2 of [2], there is an element $g \in F$ such that $g(a_{i,j}) = b_{i,j}$, and moreover, g can be taken to have slope 1 on $[a_{i,1}, a_{i,2}]$, hence $g(u_i) = v_i$.

If u_1 and v_1 are not the smallest elements of $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ respectively, then simply take dyadic rationals α, β so that $u'_1 - \alpha < u'_2 - \alpha < \dots < u'_n - \alpha$ and $v'_1 - \beta < v'_2 - \beta < \dots < v'_n - \beta$, where $u'_i - \alpha = u_i - \alpha$ or $u_i - \alpha + 1$ and $v'_i - \beta = v_i - \beta$ or $v_i - \beta + 1$, depending on which is in $[0, 1)$. Then we can use the result from the first case to get $g \in F$ such that $g(u'_i - \alpha) = v'_i - \beta$. Let $g_\alpha(t) = t - \alpha \mod 1$ and $g_\beta(t) = t + \beta \mod 1$. Then $g_\alpha g g_\beta(u_i) = g g_\beta(u'_i - \alpha) = g_\beta(v'_i - \beta) = v_i$, so $g_\alpha g g_\beta$ is the desired element of T. $\square$

Theorem 4.3.10. *Let U be a finite subset of $[0, 1)$. Then $PStab_T(U)$ is a subgroup of T of quasi-finite index.*

Proof. Enumerate U as $U = \{u_1, \dots, u_n\}$ where $u_1 < u_2 < \dots < u_n$, and let $g \in T \setminus \mathrm{PStab}_T(U)$. We proceed by induction on n. If $n = 1$, the statement is proven by the previous proposition. For $n > 1$, there are two cases to consider. Either there is some $u_i \in U$ such that the orbit of u_i under g is infinite, or the orbit of every u_i is finite under g. Let $U_g = \{u \in U | g(u) \neq u\}$.

In the first case, for each i such that u_i has finite orbit, there is some k_i such that $g^{k_i}(u_i) = u_i$. Thus if k is the product of all such k_i, then for every $u_i \in U$, $g^k(u_i)$ is either u_i or u_i has infinite orbit under g. It suffices to prove that $\langle \mathrm{PStab}_T(U), g^k \rangle$ is of quasi-finite index rather than $\langle \mathrm{PStab}_T(U), g \rangle$, since $\langle \mathrm{PStab}_T(U), g^k \rangle \leq \langle \mathrm{PStab}_T(U), g \rangle$. Thus, without loss of generality, for any $u \in U_g$, we have $g^k(u) = u$ if and only if $k = 0$. Moreover, U_g is not empty since at least one element of U has infinite order under g.

The remainder of this case follows an argument similar to that in the proof of Theorem 4.1 of [5]. There exists $k \in \mathbb{N}$ such that $g^k(U_g) \cap U$ is empty, since U is finite and the orbit of each element of U_g is infinite. Since U_g is non-empty by assumption, choose one element $u \in U_g$ and fix it for the rest of this case. By Lemma 4.3.9, there exists $h \in T$ that pointwise fixes $U \cup g^k(U_g \setminus \{u\})$ and does not fix $g^k(u)$. In particular, we may choose $h \in \text{PStab}_T(U)$. Then define $g_1 = g^k h g^{-k}$, and observe that g_1 pointwise fixes $U \setminus \{u\}$ and $g_1(u) \neq u$ by construction. It suffices now to prove that $\langle g_1, \text{PStab}_T(U)\rangle$ is of quasi-finite index, since it is a subgroup of $\langle g, \text{PStab}_T(U)\rangle$. In particular, we will show that $\langle g_1, \text{PStab}_T(U)\rangle = \text{PStab}_T(U \setminus \{u\})$, which is of quasi-finite index by induction.

Clearly $\langle g_1, \text{PStab}_T(U)\rangle \leq \text{PStab}_T(U \setminus \{u\})$. Let $f \in \text{PStab}_T(U \setminus \{u\})$. Without loss of generality, $f(u) \neq u$, since otherwise $f \in \text{PStab}_T(U)$ already. Since elements of T preserve the cyclic order of elements of $[0, 1)$, by Lemma 4.3.9, there exists $h_1 \in \text{PStab}_T(U)$ such that $fh_1(u) = g(u)$. In particular, $g^{-1}fh_1(u) = u$, and so $f \in \langle g_1, \text{PStab}_T(U)\rangle$ as desired.

Thus, in this case, $\langle g_1, \text{PStab}_T(U)\rangle = \text{PStab}_T(U \setminus \{v\})$ for some $v \in U$. As U is finite, there are only finitely many possibilities for subgroups containing $\langle g_1, \text{PStab}_T(U)\rangle$ by induction.

In the second case, the orbit of u_i under g is finite for every $u_i \in U$. Let $V = \{g^k(u_i) | k \in \mathbb{Z}, u_i \in U\}$. Then V is finite, and either $U \subsetneq V$, or $V = U$. If $U \subsetneq V$, then there exists $v \in V \setminus U$ and $u \in U$ such that $g(u) = v$. Choose $h \in \text{PStab}_T(U)$ so that v has infinite order under h. This is always possible, since there exists a dyadic rational $\frac{a}{2^m}$ such that $[\frac{a}{2^m}, \frac{a+1}{2^m}]$ contains v strictly inside and no elements of U, and then we can simply let h be defined as follows:

$$h(t) = \begin{cases} 2t - \frac{a}{2^m} & t \in [\frac{a}{2^m}, \frac{a+1/4}{2^m}) \\ t + \frac{1}{4} \cdot \frac{1}{2^m} & t \in [\frac{a+1/4}{2^m}, \frac{a+1/2}{2^m}) \\ \frac{1}{2}t + \frac{1}{2} \cdot \frac{a+1}{2^m} & t \in [\frac{a+1/2}{2^m}, \frac{a+1}{2^m}) \\ t & \text{otherwise} \end{cases}$$

Notice that h is a copy of x_0 shrunk down to the interval $[\frac{a}{2^m}, \frac{a+1}{2^m}]$ and the identity everywhere else. Thus v has infinite orbit under h, since all of $(0, 1)$ has infinite order under x_0, and h fixes every $u \in U$ since none of them are inside $[\frac{a}{2^m}, \frac{a+1}{2^m}]$. Consider the function ghg^{-1}, and compute $ghg^{-1}(u) = hg^{-1}(v) \neq u$. Since the orbit of v under h is infinite, $h^k g^{-1}(v) \neq u$ for any $k > 0$. In particular, u has an infinite orbit under ghg^{-1}, and so we are back in case 1, which is already solved.

Finally, it remains to check if $V = U$, that is, the orbit of every element of U is finite under g, and $g(u) \in U$ for every $u \in U$. By Lemma 4.3.9, there exists $h_1, \ldots, h_{n-1} \in T$ such that $h_j(u_i) = u_{i+j}$, where $u_{i+j} = u_{i+j-n}$ if $i + j > n$. Since g preserves cyclic order, there exists some k such that $g(u_i) = u_{i+k}$ for each i. Then $gh_k^{-1} \in \mathrm{PStab}_T(U)$, and so $\langle \mathrm{PStab}_T(U), g\rangle = \langle \mathrm{PStab}_T(U), h_k\rangle$. Thus there are at most n possibilities for $\langle \mathrm{PStab}_T(U), g\rangle$. In particular, $\langle \mathrm{PStab}_T(U)\rangle$ is determined by U and some cyclic permutation σ of $\{1, \ldots, n\}$, where $h_k(u_i) = u_{\sigma(i)}$.

It remains to see that each of these subgroups $\langle \mathrm{PStab}_T(U), h_k\rangle$ is of quasi-finite index. But the argument is basically the same as we have already seen, and we can show that every subgroup of T containing $\langle \mathrm{PStab}_T(U), h_k\rangle$ is of the form $\langle \mathrm{PStab}_T(V), h'_{k'}\rangle$ for some $V \subset U$ and $h'_{k'}$ which cyclically permutes the elements of V.

Indeed, let $g \in T \backslash \langle \mathrm{PStab}_T(U), h_k\rangle$. Either g permutes the elements of U, in which case $\langle \mathrm{PStab}_T(U), h_k, g\rangle = \langle \mathrm{PStab}_T(U), h_j\rangle$ for some j, or there exists $u \in U$ such that $g(u) \notin U$. In this case, we first restrict ourselves to considering $\langle \mathrm{PStab}_T(U), g\rangle$, which we have already seen is of quasi-finite index and of the form $\langle \mathrm{PStab}_T(V)\rangle$ for some $V \subset U$. Thus $\langle \mathrm{PStab}_T(U), g, h_k\rangle = \langle \mathrm{PStab}_T(V), h_k\rangle$, and either h_k permutes the elements of V or, as we showed in the beginning of the second main case of these proof, we can add a single element v to V so that $\langle \mathrm{PStab}_T(V), h_k\rangle = \langle \mathrm{PStab}_T(V')\rangle$, where $V' \subset V \cup \{v\}$. Without loss of generality, $v \in U \setminus V$, and so we are done. $\square$

If $\{u_n\}_{n\in\mathbb{N}}$ is an enumeration of all dyadic rationals, then $\bigcap_{n\in\mathbb{N}} \mathrm{PStab}_T(\{u_1, \ldots, u_n\}) = \{1\}$, and so T is quasi-residually finite, answering a question of Golan and Sapir from

[5]. Moreover, we can now classify $\mathrm{Core}(\langle \mathrm{PStab}_T(U), g\rangle)$ where U is a finite subset of $[0,1)$ and $g \in T$.

Corollary 4.3.11. *Let $U = \{u_1, \ldots, u_n\}$ be any finite subset of $[0,1)$, and $g \in T$. Then there exists a subset $V \subset U$ and a cyclic permutation of $\{1, \ldots, n\}$ such that for any $h \in T$ with $h(u_i) = u_{\sigma(i)}$ for each i, $\langle PStab_T(U), g\rangle = \langle PStab_T(V), h\rangle =$ $\mathrm{Core}(\langle PStab_T(V), h\rangle)$. Thus $\langle PStab_T(U), g\rangle$ is closed, and its core is one of only finitely many possibilities.*

Proof. If g is the trivial element, then $\mathrm{Core}(\mathrm{PStab}_T(U)) = \mathrm{PStab}_T(U)$, since any element of $\mathrm{Core}(\mathrm{PStab}_T(U))$ is dyadically-piecewise $\mathrm{PStab}_T(U)$, and in particular must fix every element of U.

For any other element g, by the proof of Theorem 4.3.10, there exists $V \subset U$ and a permutation σ of V such that $\langle \mathrm{PStab}_T(U), g\rangle = \langle \mathrm{PStab}_T(V), h\rangle$ for some subset $V \subset U$ and any element h such that $h(v) = \sigma(v)$ for every $v \in V$. The subgroup is determined uniquely by V and σ (or any other permutation $\sigma' \in \mathrm{Sym}(V)$ such that $\langle\sigma\rangle = \langle\sigma'\rangle$). To see that $\mathrm{Core}(\langle \mathrm{PStab}_T(V), h\rangle) = \langle \mathrm{PStab}_T(V), h\rangle$, observe that every element of $\langle \mathrm{PStab}_T(V)\rangle$ permutes the elements of V. For $f \in \mathrm{Core}(\langle \mathrm{PStab}_T(V), h\rangle)$, f is dyadically-piecewise $\langle \mathrm{PStab}_T(V), h\rangle$, and hence $f(v) \in V$ as well. In particular, for a fixed element $v_0 \in V$, there exists k such that $f(v_0) = \sigma^k(v_0)$. As $f \in T$, f preserves the cyclic order of V, and any cyclic permutation of V is determined uniquely by the permutation's action on any single element of V. Hence $f(v) = \sigma^k(v)$ for all other $v \in V$ as well. Thus $fh^{-k} \in \mathrm{PStab}_T(V)$, and so $f \in \langle \mathrm{PStab}_T(V), h\rangle$. □

For the sake of completeness, we also include a proof here that V is quasi-residually finite, again answering a question of Golan and Sapir.

Lemma 4.3.12. *Let U be a finite collection of dyadic rationals of S^1, and let $u, v \in S^1 \setminus U$. Then there exists $h \in H = PStab_V(U)$, the pointwise stabilizer of U in V, such that $h(u) = v$.*

Proof. Let α be any dyadic rational, and let $\phi_\alpha(f)(t) = f(t+\alpha) - \alpha$. ϕ_α is an inner automorphism of V that simply rotates the unit circle, and hence takes pointwise stabilizers to pointwise stabilizers. Thus without loss of generality, up to a rotation of S^1, we may assume that $0 \in U$. Thus neither u nor v are 0, and without loss of generality we may assume that $u < v$. Let $k \in \mathbb{N}$ be the smallest natural number such that every element of $U \cup \{u, v\}$ is a multiple of $2^{-(k-2)}$. Define $P = \{a2^{-k} \in S^1 | a \in \mathbb{Z}\}$. Clearly P contains $U \cup \{u, v\}$, and moreover contains at least two elements between u and v and at least one element larger than both u and v. Denote the elements of P as $p_1, \ldots, p_m$. Then there exists i, j such that $p_i = u$, $p_j = v$. Now we may define h as the function that linearly sends the following pieces of S^1 to each other (see Figure 4.3 for a graphical depiction of h):

$$\begin{aligned}
[0, p_{i-1}) &\rightarrow [0, p_{i-1}) \\
[p_{i-1}, p_{i+1}) &\rightarrow [p_{j-1}, p_{j+1}) \\
[p_{i+1}, p_{j-1}) &\rightarrow [p_{i+1}, p_{j-1}) \\
[p_{j-1}, p_{j+1}) &\rightarrow [p_{i-1}, p_{i+1}) \\
[p_{j+1}, 1) &\rightarrow [p_{j+1}, 1)
\end{aligned}$$

By the choice of k, we have that $i + 1 < j - 1$ since there are least two elements of P between $u = p_i$ and $v = p_j$, hence all the intervals are well-defined. Moreover, observe that by construction of P, the slope of h on each piece is 1. Finally, by the choice of k, we have that $p_{i-1}, p_{j-1}, p_i, p_j \notin U$, hence $h \in \mathrm{PStab}_V(U)$, and clearly $h(p_i) = p_j$, that is, $h(u) = v$.

$\square$

From this lemma, we derive a corollary, which will be the base case of induction in proving that V is quasi-residually finite.

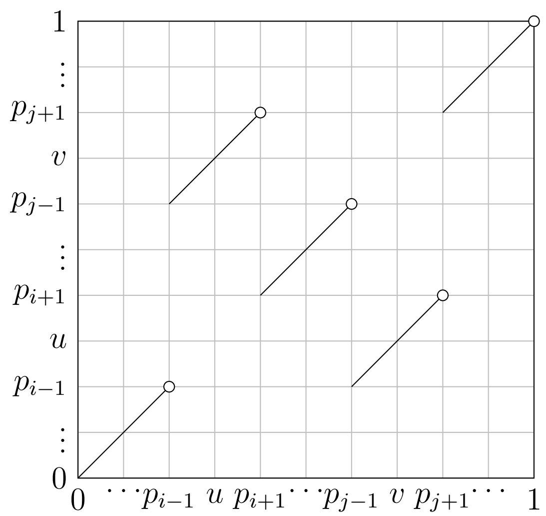


Figure 4.3: Function $h \in V$ that sends u to v and fixes everything else outside of small dyadic neighborhoods of u and v.

Corollary 4.3.13. *Let α be a dyadic rational. Then $Stab_V(\alpha)$ is a maximal subgroup.*

Proof. Let $H = \mathrm{Stab}_V(\alpha)$, and $f, g \in V \setminus H$. The goal is to show that $f \in \langle H, g\rangle$, hence for every $g \in V \setminus H$, we would have $V = \langle H, g\rangle$, proving directly that H is maximal.

Since $f, g \notin H$, $f(\alpha) \neq \alpha \neq g(\alpha)$. Thus by the Lemma 4.3.12, there exists $h \in H$ such that $h(f(\alpha)) = g(\alpha)$. Since $g^{-1}(h(f(\alpha))) = \alpha$, $g^{-1}hf \in H$, hence $f \in \langle H, g\rangle$. $\square$

Proposition 4.3.14. *V is quasi-residually finite.*

Proof. Our goal is to show that for every finite set of dyadic rationals U, $\mathrm{PStab}_V(U)$ is of quasi-finite index, by performing induction on $|U|$. By the Corollary 4.3.13, the base case is proven, so assume for every $|U| < n$, $\mathrm{PStab}_V(U)$ is of quasi-finite index, and let $|U| = n$, and $H = \mathrm{PStab}_V(U)$.

Let $g \in V \setminus H$, and let $U_g = \{u \in U | g(u) \neq u\}$. By assumption U_g is non-empty, and as in the proof of Theorem 4.3.10, we may assume up to taking a power of g that either the orbit of every $u \in U_g$ under the action of g is infinite, or that g permutes U. If g permutes U, then we can show just as in the proof of Theorem 4.3.10 that $\langle H, g\rangle$ is one of finitely many options, each of which is of finite index by induction.

Otherwise, using another technique from the previous proof, up to taking $g^{-k}hg^k$ for some $h \in H$ and $k \in \mathbb{N}$, we may assume that $|U_g| = 1$, i.e., $U_g = \{u\}$ for some u. Let $\hat{U} = U \setminus U_g$ and $\hat{H} = \mathrm{PStab}_V(\hat{U})$. To show that H is of quasi-finite index, it suffices to prove $\hat{H} \subset \langle H, g\rangle$.

Let $f \in \hat{H}$. If $f(u) = u$, there is nothing to prove, so assume $f(u) \neq u$. We also have $g(u) \neq u$, and hence by Lemma 4.3.12, there exists $h \in H$ such that $h(f(u)) = g(u)$. Thus $g^{-1}(h(f(u))) = u$, and since all elements involved stabilize $U \setminus \{u\}$, $g^{-1}hf \in H$, so $f \in \langle H, g\rangle$.

Thus there are only finitely many subgroups of V containing $\langle H, g\rangle$, and since g was unique only up to the element of U that it did not fix. Since U is finite, H is therefore of quasi-finite index. $\square$

Chapter 5

Jones' Subgroup of T

5.1 Thompson Graphs

Given any full binary tree with n leaves, the associated Thompson graph is defined by Jones [12] in the following way. Arrange all the leaves on a horizontal line, and call them $l_1, \ldots, l_n$. The vertices of the Thompson graph are points $v_1, \ldots, v_n$ on the same horizontal line, with v_1 to the left of l_1, and more generally v_i between l_{i-1} and l_i. For every left edge e of the tree, there is a unique pair of vertices v_i and v_j such that a path can be drawn connecting v_i and v_j which passes through the tree only at the edge e and stays above the horizontal line. Connect every such pair of vertices. An example Thompson graph is depicted in Figure 5.1.

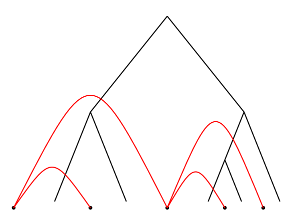


Figure 5.1: Example of a Thompson graph in red of a binary tree in black.

An alternative way to construct the Thompson graph was given by Golan and Sapir in [4], in which they recognize it as a subgraph of the diagram associated with a particular pair of trees. Recall that a tree can be associated with a diagram consisting of cells with one top edge and two bottom edges by replacing each caret in the tree with such a cell, where the top edge goes through the top vertex of the caret and the left child goes through the left bottom edge of the cell and the right child likewise goes through the right bottom edge of the cell (see Figure 2.7). Note that the edges in this diagram are considered oriented from left to right.

The Thompson graph can then be obtained from the diagram in the following way. Its vertices consist of all vertices of the diagram except the terminal vertex (the right-most vertex). For each internal vertex of the diagram (any vertex except the right-most or the left-most), the top-most incoming edge is also an edge of the Thompson graph, and these are the only edges in the Thompson graph.

Either way, we can now use the Thompson graph associated with a tree to define the Thompson graph associated with an element of T. Specifically, the Thompson graph associated to any element $f \in T$ with reduced pair of trees representation (R, S, n) is obtained by identifying the vertices of the Thompson graphs of R and S in the same way that their leaves are identified. An example is depicted in Figure 5.2, with the Thompson graph both depicted on the pair of trees and then simplified.

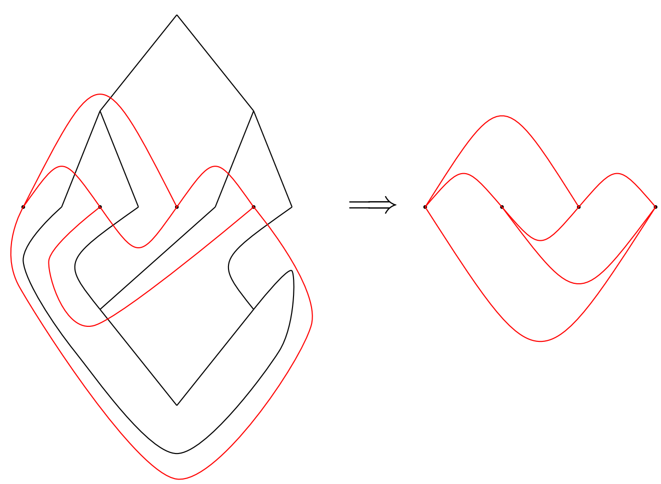


Figure 5.2: Thompson graph in red of an element of T depicted both on the pair of trees diagram and simplified.

The subgroup $\vec{T}$ of T then is the collection of all elements of T with bipartite Thompson graphs, and was first defined by Jones when he also defined $\vec{F}$ similarly. We will use the terms bipartite and 2-colorable interchangably. For more about $\vec{F}$, see [4], in which Golan and Sapir found explicit generators of $\vec{F}$ and discovered many other properties and characterizations of the subgroup.

5.2 Generators of $\vec{T}$

Denote by c_n the element depicted in Figure 5.3.

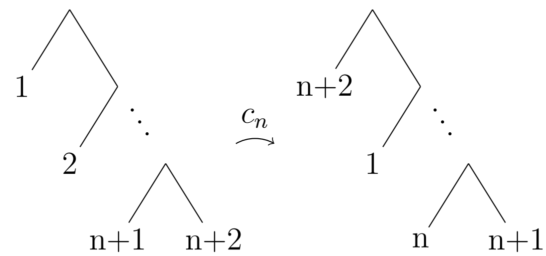


Figure 5.3: The element c_n from T.

Notice that $c = c_1$ is one of the standard generators of T, and c_n in general is an order $n+2$ element. Let $f_{\frac{1}{2}}$ denote the function given by $f_{\frac{1}{2}}(x) = x + \frac{1}{2} \mod 1$. Its pair of trees diagram is depicted in Figure 5.4, and $f_{\frac{1}{2}}^2 = 1$.

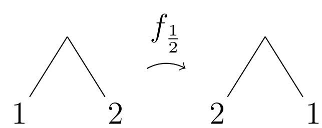


Figure 5.4: The element $f_{\frac{1}{2}}$.

By Theorem 1 of [4], we have $\vec{F} = \langle x_0x_1, x_1x_2, x_2x_3 \rangle$, where x_i are the generators defined in Section 2.2. The goal of this section is the following theorem, which adds just one element to the set of generators of $\vec{F}$ to get the generators of $\vec{T}$.

Theorem 5.2.1. $\vec{T} = \langle \vec{F}, f_{\frac{1}{2}} \rangle = \langle x_0x_1, x_1x_2, x_2x_3, f_{\frac{1}{2}} \rangle$

Before we prove the theorem, we need the following lemma to establish the structure of the tree diagrams of elements of $\vec{T}$.

Lemma 5.2.2. *Every element $f \in \vec{T}$ can be represented with a pair of trees (R, S, k) such that both trees have an even number of vertices, and such that there is a 2-coloring on the Thompson graphs of R and S where the colors on each vertex from left to right alternate.*

Proof. Let (R, S, k) be the reduced pair of trees representation for $f \in \vec{T}$. By definition of $\vec{T}$, the Thompson graph of f has a valid 2-coloring, which induces a 2-coloring on the vertices of the Thompson graph of R. Suppose that the Thompson graph of f contains two adjacent vertices m and $m+1$ with the same color, and consider inserting a dipole at vertex $m+1$, i.e., adding a caret to the $m+1$ vertices of both R and S. Adding a caret adds one left edge and right edge to R, and hence the corresponding Thompson graph of R is changed by adding a vertex in the middle of the caret, and connecting that vertex through the new left edge to the existing vertex on its left. A likewise addition is made to the Thompson graph of S, so that the new Thompson graphs of both R and S are obtained by adding a vertex in between vertices m and $m+1$, with an edge connecting the new vertex to m. Since m and $m+1$ have the same color, we may choose the other color for the new vertex, and the 2-coloring is still valid.

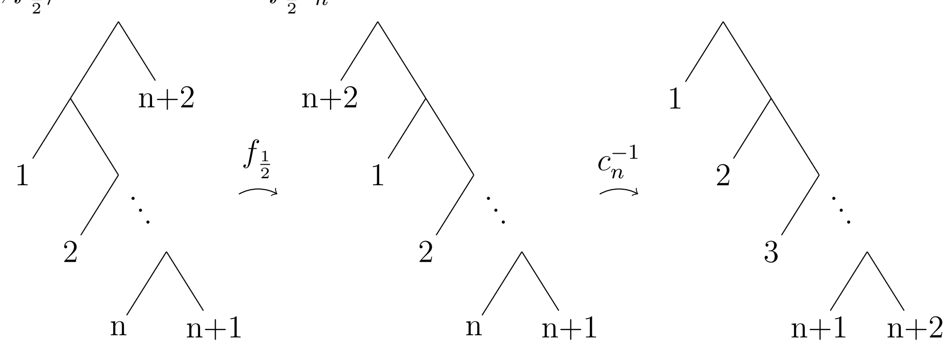
Continue adding dipoles in R and S in this way to add vertices to the Thompson graphs of R and S between all two vertices with the same color, ensuring that the colors alternate as desired. Likewise add a dipole at the last vertex of R and S if necessary to ensure that R and S has an even number of vertices. □

Proof of Theorem 5.2.1. Let $\vec{F}_+$ denote the positive elements of F that are in $\vec{F}$ and $\vec{F}_-$ denote the negative elements of F that are in $\vec{F}$, where positive and negative elements are as defined in Section 2.2. Let S_n be the tree depicted in Figure 2.8, and note that S_n is both the input and output tree for c_n in Figure 5.3. Then to prove the theorem we can use the previous lemma to first prove that elements of $\vec{T}$ can be written in the form $pc_{2n}^{m}q$, where $p \in \vec{F}_+$ and $q \in \vec{F}_-$, which is very similar to the form of elements for T given in Theorem 5.7 of [2]. Thus we will have proven that $\vec{T} = \langle \vec{F}_+, \vec{F}_-, \{f_{\frac{1}{2}}\}, \{c_{2n} | n \in \mathbb{N}\}\rangle$, and to finish the proof it will suffice to show that each of these generators is in $\langle x_0x_1, x_1x_2, x_2x_3, f_{\frac{1}{2}}\rangle$.

Let $f \in \vec{T}$, then by Lemma 5.2.2, there exists (R, S, k), a pair of trees representation of f, such that the Thompson graphs for R and S have $2n + 2$ vertices and are both 2-colorable with the colors on the vertices alternating. Consider the reduced pair of trees representation for c_{2n} denoted $(S_{2n}, S_{2n}, 2)$. It is easy to observe that the Thompson graph corresponding to S_{2n} is simply the path of length $2n+2$, and hence it is 2-colorable with the colors on the vertices alternating, just like the Thompson graphs of R and S.

In particular, $(R, S_{2n}, 1)$, (S_{2n}, S_{2n}, k), and $(S_{2n}, S, 1)$ are all bipartite elements of T since the 2-colorings of all of the trees are compatible. Furthermore, by the observation given in section 2.2, $(R, S_{2n}, 1)$ is in F_+ and $(S_{2n}, S, 1)$ is in F_-. We also have $(S_{2n}, S_{2n}, k) = c_{2n}^{k-1}$. Therefore $f \in \langle \vec{F}_+, \vec{F}_-, \{c_{2n} | n \in \mathbb{N}\}\rangle$, which is a subgroup of $\vec{T}$ since all generating elements have bipartite Thompson graphs and are in T.

Finally, since $\vec{F}$ is generated by $\{x_0x_1, x_1x_2, x_2x_3\}$, it suffices to show that $c_{2n} \in \langle \vec{F}, f_{\frac{1}{2}}\rangle$. Observe that $f_{\frac{1}{2}} c_n^{-1} \in F$:



Therefore, since $f_{\frac{1}{2}}$ and c_{2n} are bipartite, $f = f_{\frac{1}{2}} c_{2n}^{-1}$ is a bipartite element of $\vec{F}$. In particular, $c_{2n} = f_{\frac{1}{2}} f^{-1}$, so $c_{2n} \in \langle \vec{F}, f_{\frac{1}{2}}\rangle$. $\square$

5.3 $\vec{T}$ and Dyadic Parity

Every dyadic fraction has either an even or odd sum of binary digits, and we will refer to this parity as the dyadic parity of the number. For example, $\frac{7}{8} = 0.111$ in binary, and the sum of its digits is 3, so the fraction has odd dyadic parity. Observe that since every function $f \in T$ is piecewise of the form $ax + b$, where a is an integer power of 2 and b is dyadic, if f stabilizes odd dyadic parity, it also stabilizes even dyadic parity.

Since the generators of $\vec{F}$ preserve dyadic parity [4], and since $f_{\frac{1}{2}}$ switches dyadic parity, the subgroup $\vec{T}$ that they generate has the property that every element either exclusively preserves dyadic parity or exclusively switches dyadic parity. In fact, this property characterizes $\vec{T}$.

Theorem 5.3.1. *Let $f \in T$ either stabilize the dyadic parity of every dyadic rational or switch the dyadic parity of every dyadic rational. Then $f \in \vec{T}$.*

Proof. Let f have reduced pair of trees representation (R, S, k). Label left edges of R and S with 0 and right edges with 1, and label each leaf with the label of the path from the root of the tree to that leaf. Then each vertex v in the Thompson graph of R is to the left of some leaf a in R. If a has even dyadic parity then color the corresponding vertex in the Thompson graph "even", and otherwise color it "odd". Likewise the vertices in the Thompson graph of S can be colored.

If a_1 and a_2 are the labels of leaves in R and S respectively that are identified, then $f(a_1) = a_2$. Therefore, if f stabilizes the dyadic parity then a_1 and a_2 have the same parity. and if f switches the dyadic parity, then they have opposite parity. Thus, when the Thompson graphs of R and S are identified, up to swapping the colors, the colorings are identical.

It remains to show that for the Thompson graphs of R and S, this is indeed a valid 2-coloring, i.e., that no two adjacent vertices have the same color. Suppose two vertices v_1 and v_2 in the Thompson graph of a R are adjacent, and the leaves to their right are labeled by a_1 and a_2. Then either one of them is directly beneath a caret, or neither is. Both situations are depicted in Figure 5.5, and when one is directly beneath a caret, it is easily verified that a_1 and a_2 differ only by the last digit, and hence v_1 and v_2 have opposite colorings. In second case, the edge between v_1 and v_2 crosses a left edge e in R, which is labeled 0, and let k be the length of the path from the root to e, including e. Now the vertex in R labeled by a_1 is to the left of this edge in R, hence the kth digit of a_1 is 0, while the kth digit of a_2 is labeled 1. Furthermore, until the kth digits, a_1 and a_2 have the same digits. Finally, note that after the digits past the kth digits of a_1 and a_2 are all 0s, as that is the only way for these two vertices to be on either side of e. Indeed, if the path to either vertex ever went right, then the edge between the vertices in the Thompson graph would have to cross more than just the edge e of R. Thus the dyadic parity of a_1 and a_2 are exactly opposite, and the 2-coloring is valid. $\square$

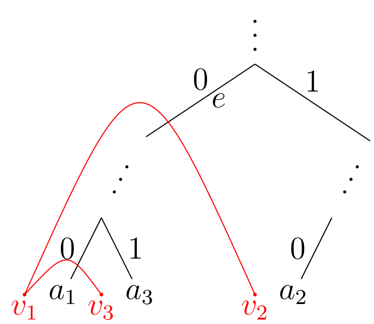


Figure 5.5: The two possible types of adjacencies in a Thompson graph, between v_1 and v_2 and v_1 and v_3. The leaves labeled a_1 and a_2 have different dyadic parity, as do the leaves labeled a_1 and a_3.

5.4 $\vec{T}$ Coincides With Its Commensurator in T

The following Lemma formalizes the statement that for any element $f \in T$, for t small, f either preserves the dyadic parity of all such t, or switches the dyadic parity. The proof simply relies on the fact that elements of T are piecewise linear with slopes equal to an integer power of 2.

Lemma 5.4.1. *Let $f \in T$, and let S_i denote the collection of dyadic rational numbers with sum $(i \mod 2)$. Then there exists m such that for all $t \in S_i$ with $t \leq 2^{-m}$ and $i = 0, 1$, either $f(t)$ is always in S_i or it is always in S_{1-i}.*

Proof. First, there exists m such that on $[0, 2^{-m}]$, f is given by $f(t) = 2^n t + \alpha$ for some $n \in \mathbb{Z}$ and α a dyadic rational. Let $t \in S_i$. Consider the length of α to be l, then m be chosen to be large enough that for $t \leq 2^{-m}$, $t \cdot 2^n$ can be written as a binary decimal of the form $0^l\beta$ for some finite binary string β depending on t, where $\beta \in S_i$, the same as t. Then $f(t) = 2^n t + \alpha$ as a binary string is simply $\alpha\beta$, hence if $\alpha \in S_0$, then $f(t)$ does not change the dyadic parity on $[0, 2^{-m}]$, and if $\alpha \in S_1$, then $f(t)$ does switch dyadic parity on $[0, 2^{-m}]$. □

With this technical fact, we can now prove that $\vec{T}$ coincides with its commensurator in T as a corollary of Theorem 5.3.1, where the commensurator of $\vec{T}$ is defined as $\{t \in T \mid t\vec{T}t^{-1} \cap \vec{T}$ has finite index in both $\vec{T}$ and $t\vec{T}t^{-1}\}$.

Corollary 5.4.2. *The commensurator of $\vec{T}$ in T is $\vec{T}$.*

Proof. Let $f \in T \setminus \vec{T}$. Then to check that f is not in the commensurator of $\vec{T}$, it suffices to show that there exists $g \in \vec{T}$ such that for large enough n, $(g^n)^f \notin \vec{T}$. Choose $g = (x_0 x_1)^{-1}$, and note that for any $t \in (0,1)$, $g^n(t)$ can be made arbitrarily small by taking n large enough.

Since $f \notin \vec{T}$, there exists $t_0 \in S_i$ and $t_1 \in S_j$ for some $i, j \in \{0, 1\}$ such that $f(t_0) \in S_i$ and $f(t_1) \notin S_j$. Let $t'_k = f(t_k)$ for $k = 1, 2$. Then consider $(f^{-1}g^n f)(t'_k) =$

$f(g^n(f^{-1}(t'_k))) = f(g^n(t_k))$. For large enough n, $g^n(t_i)$ is small enough that Lemma 1 applies to f. Now $t'_1 \in S_i$ and $t'_2 \notin S_j$ by assumption, noting that elements of T preserve dyadic rationals, so $t'_2 \in S_{1-j}$. Since $g \in \vec{F}$, $g^n(t_1) \in S_i$ and $g^n(t_2) \in S_j$.

By the previous lemma there are two cases. Suppose that for small enough values of t, f preserves dyadic parity. Then $f(g^n(f^{-1}(t'_1))) \in S_i$ with $t'_1 \in S_i$, and $f(g^n(f^{-1}(t'_2))) \in S_j$ with $t'_2 \notin S_j$, showing that $f^{-1}g^n f \notin \vec{T}$. The other case where f switches the dyadic parity is similar. $\square$

It was proven in [16] that the quasiregular representation of a subgroup is irreducible if the commensurator of the subgroup coincides with itself. Thus from Corollary 5.4.2, we have the following theorem:

Theorem 5.4.3. *The quasiregular representation $\ell^2(T/\vec{T})$ of T is irreducible.*

5.5 A Finite Presentation of $\vec{T}$

In this section, we determine first an infinite classical presentation for $\vec{T}$, and then deduce a finite presentation. Our infinite set of generators consists of all $x_{n-1}x_n = g_n$ and c_{2n}, for $n \in \mathbb{Z}$ and $n \geq 0$, using $c_0 = f_{\frac{1}{2}}$ for convenience. Note that each g_n is indeed in $\vec{F}$ by Lemma 4.5 of [4].

In the proof of Lemma 5.5.1, we will use the following relations of T from [2] which hold for any integers n, k such that $0 \leq k \leq n$:

$$x_k^{-1} x_n x_k = x_{n+1}, \qquad k < n; \tag{T1}$$

$$c_n = x_n c_{n+1}; \tag{T2}$$

$$c_n x_k = x_{k-1} c_{n+1}, \qquad 1 \leq k; \tag{T3}$$

$$c_n x_0 = c_{n+1}^2; \tag{T4}$$

$$c_n^{n+2} = 1; \tag{T5}$$

Lemma 5.5.1. *If n is a non-negative integer then*

$$g_k^{-1} g_n g_k = g_{n+2}, \quad 1 \leq k < n; \tag{5.1}$$

$$c_{2n}^{2n+2} = 1; \tag{5.2}$$

$$c_{2n} = g_{2n+1} c_{2n+2}; \tag{5.3}$$

$$c_{2n} g_k = g_{k-1} c_{2n+2}; \quad 1 < k < 2n+2 \tag{5.4}$$

$$c_{2n} g_1 = c_{2n+2}^3 \tag{5.5}$$

Proof.

$$\begin{aligned}
g_k^{-1} g_n g_k &= (x_k^{-1} x_{k-1}^{-1})(x_{n-1} x_n)(x_{k-1} x_k) \\
&= x_k^{-1} x_{k-1}^{-1} x_{n-1} (x_{k-1} x_{k-1}^{-1}) x_n x_{k-1} x_k \\
&\overset{T1}{=} x_k^{-1} x_n x_{n+1} x_k \\
&= x_k^{-1} x_n (x_k x_k^{-1}) x_{n+1} x_k \\
&\overset{T1}{=} x_{n+1} x_{n+2} \\
&= g_{n+2}
\end{aligned}$$

Thus (5.1) holds. (5.2) is the same as (T5).

$$c_{2n} \overset{T2}{=} x_{2n} c_{2n+1} \overset{T2}{=} x_{2n} x_{2n+1} c_{2n+2} = g_{2n+1} c_{2n+2}$$

Hence (5.3) holds. Relations (5.4) and (5.5) are proven similarly:

$$
\begin{aligned}
c_{2n}g_k &= c_{2n}x_{k-1}x_k \\
&\overset{T3}{=} x_{k-2}c_{2n+1}x_k \\
&\overset{T3}{=} x_{k-2}x_{k-1}c_{2n+2} \\
&= g_{k-1}c_{2n+2}
\end{aligned}
$$

$$
\begin{aligned}
c_{2n}g_1 &= c_{2n}x_0x_1 \\
&\overset{T4}{=} c_{2n+1}^2x_1 \\
&\overset{T3}{=} c_{2n+1}x_0c_{2n+2} \\
&\overset{T4}{=} c_{2n+2}^3
\end{aligned}
$$

□

These relations also determine directly the following relations.

Corollary 5.5.2. *Suppose n is a non-negative integer and $1 \leq m \leq 2n+1$, then*

$$c_{2n}^m = g_{2n+1-(m-1)}c_{2n+2}^m \tag{5.6}$$

$$c_{2n}^m = c_{2n+2}^{m+2}g_m^{-1} \tag{5.7}$$

Proof. The first relation is proven in the following way:

$$c_{2n}^m = c_{2n}^{m-1}c_{2n} \overset{(5.3)}{=} c_{2n}^{m-1}g_{2n+1}c_{2n+2} \overset{(5.4)}{=} g_{2n+1-(m-1)}c_{2n+2}^m$$

For the second relation, we instead prove that $c_{2n}^m g_m = c_{2n+2}^{m+2}$.

$$c_{2n}^m g_m \overset{(5.4)}{=} c_{2n}g_1c_{2n+2}^{m-1} \overset{(5.5)}{=} c_{2n+2}^{m+2}$$

□

Corollary 5.5.3. *Let k, n, and m be non-negative integers such that $k < 2n+2$ and $1 \leq m < 2n+2$. Then*

$$c_{2n}^{m} g_k = \begin{cases} g_{k-m} c_{2n+2}^{m} & k > m \\ c_{2n+2}^{m+2} & k = m \\ g_{2n+2+k-m} c_{2n+2}^{m+2} & k < m \end{cases}$$

Proof. If $k > m$, repeatedly apply (5.4) to get the first relation. If $k = m$, the relation is the same as (5.7). If $k < m$, then we use the same technique to get

$$c_{2n}^{m} g_k = c_{2n}^{m-k} c_{2n+2}^{k+2} \overset{(5.6)}{=} g_{2n+1-(m-k-1)} c_{2n+2}^{m-k+k+2} = g_{2n+2+k-m} c_{2n+2}^{m+2}$$

□

Corollary 5.5.4. *Suppose i, j, k, l are positive integers and $i < 2j+2, k < 2l+2$, then there exists m, n are positive integers with $m < 2n+2$, and p, q are positive elements in $\{g_t, t \geq 1\}$ such that*

$$c_{2j}^{i} c_{2l}^{k} = p c_{2n}^{m} q^{-1}$$

Proof. By using relations (5.6) and (5.7), we can increase j or i until $i = j$, up to multiplying by a postive element on the left or a negative element on the right. Since c_{2n} is an order $2n+2$ element, we may assume that $0 \leq m < 2n+2$. □

Now we determine an infinite presentation for the subgroup $\vec{T}$.

Theorem 5.5.5. *$\vec{T}$ is generated by $g_k, k \geq 1$ and $c_n, n \geq 0$ and is defined by the* *following relations:*

$$\begin{aligned}
g_k^{-1} g_n g_k &= g_{n+2} \qquad 1 \leq k < n \\
c_{2n}^{2n+2} &= 1 \\
c_{2n} &= g_{2n+1} c_{2n+2} \\
c_{2n} g_k &= g_{k-1} c_{2n+2} \qquad 1 < k < 2n+2 \\
c_{2n} g_1 &= c_{2n+2}^3
\end{aligned}$$

Before we proceed to the proof, we need some lemmas to establish the structure that these relations provide. For the rest of this section, we will refer to G as the group with presentation given in Theorem 5.5.5, and show that G is isomorphic to $\vec{T}$.

Lemma 5.5.6. *$\vec{F}$ is a subgroup of G.*

Proof. Since the relations of G also hold in $\vec{T}$, there is a natural homomorphism ϕ from G to $\vec{T}$, sending generators to generators. Since $\vec{F}$ has a finite presentation [7], and its relations are included in the relations of G (using g_k as its generators), there is also a homomorphism α from $\vec{F}$ to G that takes generators to generators. Thus $\alpha \circ \phi$ is a homomorphism taking g_k to g_k, and hence is the identity isomorphism of $\vec{F}$, implying that α is also an isomorphism, and that $\vec{F}$ is a subgroup of G. □

Lemma 5.5.7. *For every $g \in G$, we have*

$$g = p c_{2n}^m q^{-1}, \quad 0 \leq m < 2n+2,$$

where p, q are positive elements in $\vec{F} = \{g_k, k \geq 1\} \leq G$.

Proof. Let H be the subset of G consisting of all elements of the form $p c_{2n}^m q^{-1}$, with $m < 2n+2$, and p and q positive elements in $\vec{F}$. Since H contains the generators of G and is closed under taking inverses, it suffices to prove that H is closed under multiplication, and hence $H = G$. Let $p_1 c_{2n_1}^{m_1} q_1^{-1}$ and $p_2 c_{2n_2}^{m_2} q_2^{-1}$ be two elements of H, and note that the product of two positive elements of F is still a positive element of F [2], hence the same is true in $\vec{F}$. Moreover, $p_1 c_{2n_1}^{m_1} q_1^{-1} p_2 c_{2n_2}^{m_2} q_2^{-1} = p_1 c_{2n_1}^{m_1} p_3 q_3^{-1} c_{2n_2}^{m_2} q_2^{-1}$ for some positive elements p_3 and q_3, since $q_1^{-1} p_2$ is in $\vec{F}$.

First we show that $p_1 c_{2n_1}^{m_1} p_3$ can be written as $p_4 c_{2n_3}^{m_3}$ for some positive element p_4 and n_3, m_3 natural numbers. Let g_k to be the first letter in p_3. By Corollary 5.5.3, g_k can be moved to the left of $c_{2n_1}^{m_1}$, with possibly increasing n_1 or m_1. Continue in this way to move all of p_3 to the left of $c_{2n_1}^{m_1}$, to get the desired form $p_4 c_{2n_3}^{m_3}$, since the product of positive elements of $\vec{F}$ is still a positive element.

Similarly, since $(q_3^{-1} c_{2n_2}^{m_2} q_2^{-1})^{-1} = q_2 c_{2n_2}^{-m_2} q_3$, we can rewrite $q_3^{-1} c_{2n_2}^{m_2} q_2^{-1}$ as $c_{2n_4}^{m_4} q_4^{-1}$. Finally, $c_{2n_3}^{m_3} c_{2n_4}^{m_4}$ can be written as $p_5 c_{2n}^{m} q_5^{-1}$ by Corollary 5.5.4. Thus

$$\begin{aligned} p_1 c_{2n_1}^{m_1} q_1^{-1} p_2 c_{2n_2}^{m_2} q_2^{-1} &= p_1 c_{2n_1}^{m_1} p_3 q_3^{-1} c_{2n_2}^{m_2} q_2^{-1} = p_4 c_{2n_3}^{m_3} c_{2n_4}^{m_4} q_4^{-1} \\ &= p_4 p_5 c_{2n}^{m} q_4^{-1} q_5^{-1} = p c_{2n}^{m} q^{-1} \end{aligned}$$

□

Lemma 5.5.8. *If $\phi : G \to G/N$ is a proper quotient homomorphism, then ϕ restricted to $\vec{F}$ is a proper quotient of $\vec{F}$ as well.*

Proof. Let $g \in G$ such that $\phi(g) = 1$ and $g \neq 1$. It follows from Lemma 5.5.7 that g is of the form $g = p c_{2n}^{m} q^{-1}$, where $m < 2n + 2$ and p and q are positive elements in $\vec{F} \leq G$. Since $\phi(g) = 1$, we have $\phi(c_{2n}^{m}) = \phi(p^{-1} q)$, and since the order of c_{2n} is $2n + 2$, then $\phi((p^{-1} q)^{2n+2}) = \phi(c_{2n}^{2n+2}) = \phi(1) = 1$. We now have two cases: either $p = q$ or $p \neq q$. If $p \neq q$, then since $\vec{F}$ is torsion free, $(p^{-1} q)^{2n+2} \neq 1$, implying that ϕ restricted to $\vec{F}$ is a proper quotient of $\vec{F}$.

If $p = q$, since $g \neq 1$, it must be that $c_{2n}^m \neq 1$, and in particular $m > 0$. Then

$$\begin{aligned} 1 &= \phi(p^{-1}q) = \phi(c_{2n}^m) \\ &\overset{(5.6)}{=} \phi(g_{2n+1-(m-1)}c_{2n+2}^m) \end{aligned}$$

Note that $m < 2n+2$, so $2n+1-(m-1) > 0$. Then $\phi(g_{2n+1-(m-1)}^{2n+4}) = \phi(c_{2n+2}^{-(2n+4)m}) = 1$, but $g_{2n+1-(m-1)} \neq 1$. Thus once again, ϕ restricted to $\vec{F}$ is a proper quotient of $\vec{F}$.

□

Proof of Theorem 5.5.5. Let $\phi : G \to \vec{T}$ be the surjective homomorphism that sends generators to generators. By Lemma 5.5.6, ϕ restricted to $\vec{F}$ is an isomorphism. By Lemma 5.5.8, ϕ must then have trivial kernel. Thus ϕ is an isomorphism □

Now we can show that only finitely many of these relations are needed, and thus give a finite presentation.

Corollary 5.5.9. *$\vec{T}$ has a finite presentation with generators $\{g_1, g_2, g_3, c_0\}$ and re-*

lations

$$g_1^{-1}g_3g_1 = g_2^{-1}g_3g_2$$

$$g_1^{-3}g_2g_1^3 = g_3^{-1}g_1^{-2}g_2g_1^2g_3$$

$$g_1^{-2}g_3g_1^2 = g_2^{-1}g_1^{-1}g_3g_1g_2$$

$$g_1^{-2}g_3g_1^2 = g_3^{-1}g_1^{-1}g_3g_1g_3$$

$$g_1^{-2}g_2g_1^2 = g_2^{-1}g_1^{-1}g_2g_1g_2$$

$$g_1^{-2}g_2g_1^2 = g_3^{-1}g_1^{-1}g_2g_1g_3$$

$$c_2g_2 = g_1c_4$$

$$c_2g_3 = g_2c_4$$

$$c_4g_4 = g_3c_6$$

$$c_0g_1 = c_2^3$$

$$c_2g_1 = c_4^3$$

$$c_0^2 = 1$$

where the elements c_{2n} *and* g_n *are defined inductively by* $c_{2n} = g_{2n-1}^{-1}c_{2n-2}$ *and* $g_n = g_{n-2}^{-1}g_{n-1}g_{n-2}$.

Proof. Thorem 5.2.1 proves that the choice of generators is correct, so it remains to prove that the set of relations given in Lemma 5.5.1 follow, since these are shown to be the defining relations of $\vec{T}$ in Theorem 5.5.5. We will refer to these relations by their numbering given in Lemma 5.5.1.

The first six relations imply (5.1), since these are the relations for F_3 with generators g_1, g_2, and g_3 given in [7].

By definition of c_{2n} in this corollary, (5.3) is trivial. To prove (5.4), we do induction on both n for fixed base values of k, and then induct on k. For $k = 2$ and $n = 1$, $k = 3$ and $n = 1$, and $k = 4$ and $n = 2$, the corresponding relations are given in the finite presentation. For these fixed values of k, we induct on n then in the following way, referring to the induction step with abbreviation ind:

$$\begin{aligned}
c_{2n}g_k &\overset{(5.3)}{=} g_{2n-1}^{-1}c_{2n-2}g_k \\
&\overset{\text{ind}}{=} g_{2n-1}^{-1}g_1c_{2n} \\
&\overset{(5.1)}{=} g_1g_{2n+1}^{-1}c_{2n} \\
&\overset{(5.3)}{=} g_1c_{2n+2}
\end{aligned}$$

For the remaining cases, we induct on $k \geq 5$, assuming that for smaller values of k and all values of n that (5.4) holds. We will refer to using (5.4) when $k = 2$ as the base case, or base for short, and smaller values of k by ind, standing for proof by induction.

$$\begin{aligned}
c_{2n}g_k &\overset{\text{base}}{=} g_1^{-1}c_{2n-2}g_2g_k \\
&\overset{(5.1)}{=} g_1^{-1}c_{2n-2}g_{k-2}g_2 \\
&\overset{\text{ind}}{=} g_1^{-1}g_{k-3}c_{2n}g_2 \\
&= g_1^{-1}g_{k-3}g_1g_1^{-1}c_{2n}g_2 \\
&\overset{(5.1)}{=} g_{k-1}g_1^{-1}c_{2n}g_2 \\
&\overset{\text{base}}{=} g_{k-1}c_{2n+2}
\end{aligned}$$

Next we prove that (5.5) follows by induction on n, noting that the base cases of induction $n = 0$ and $n = 1$ are assumed in the finite relations. Similar to before, ind will stand for induction on n.

$$
\begin{aligned}
c_{2n}g_1 &\overset{(5.4)}{=} g_1^{-1}c_{2n-2}g_2g_1 \\
&\overset{(5.1)}{=} g_1^{-1}c_{2n-2}g_1g_4 \\
&\overset{\text{ind}}{=} g_1^{-1}c_{2n}^3g_4 \\
&\overset{(5.4)}{=} g_1^{-1}g_1c_{2n+2}^3 \\
&= c_{2n+2}^3
\end{aligned}
$$

We can now prove (5.2), the final set of relations.

$$
\begin{aligned}
c_{2n}^{2n+2} &= c_{2n}c_{2n}^{2n}c_{2n} \\
&\overset{(5.3)}{=} c_{2n}c_{2n}^{2n}g_{2n+1}c_{2n+2} \\
&\overset{(5.4)}{=} c_{2n}g_{2n+1-2n}c_{2n+2}^{2n}c_{2n+2} \\
&= c_{2n}g_1c_{2n+2}^{2n+1} \\
&\overset{(5.5)}{=} c_{2n+2}^3c_{2n_2}^{2n+1} \\
&= c_{2n+2}^{2n+4}
\end{aligned}
$$

Thus since c_0^2 is a relation, (5.2) follows by induction on n. $\square$

To understand more about the structure of $\vec{T}$, we can use the observation in Lemma 5.5.8 to show that every proper homomorphism of $\vec{T}$ factors through a certain homomorphism from $\vec{T}$ to the infinite dihedral group.

Corollary 5.5.10. *Any proper homomorphism of $\vec{T}$ factors through the homomorphism α from $\vec{T}$ to the infinite dihedral group, $\langle c_0, g_1 | c_0^2 = 1, g_1c_0g_1c_0 = 1\rangle$, where $\alpha(c_{2n}) = c_0$, $\alpha(g_{2n+1}) = g_1$, and $\alpha(g_{2n+2}) = 1$ for all $n \geq 0$.*

Proof. Let ϕ be a proper homomorphism of $\vec{T}$. Then by Lemma 5.5.8, ϕ restricted to $\vec{F}$ is also proper, and hence the image of $\vec{F}$ under ϕ is abelian [1, Theorem 4.13]. Therefore, since $g_k^{-1} g_n g_k = g_{n+2}$ in $\vec{T}$, $\phi(g_n) = \phi(g_{n+2})$, where $n > 1$. In all the equalities that follow, we will use $\phi(g_n) = \phi(g_{n+2})$ extensively, along with all other equalities that we prove hold true in the image of ϕ.

We also have

$$\phi(c_4 g_4) \overset{(5.4)}{=} \phi(g_1^{-1} c_2 g_2 g_4) = \phi(g_1^{-1} c_2 g_2^2)$$
$$\phi(c_4 g_4) \overset{(5.4)}{=} \phi(g_3 c_6) \overset{(5.4)}{=} \phi(g_3 g_1^{-2} c_2 g_2^2)$$

As a result, $\phi(g_3 g_1^{-1}) = 1$, so $\phi(g_3) = \phi(g_1)$, extending $\phi(g_n) = \phi(g_{n+2})$ to $n = 1$.

We also have

$$\phi(c_4) \overset{(5.3)}{=} \phi(g_5 c_6) = \phi(g_3 c_6) \overset{(5.4)}{=} \phi(g_3 g_1^{-1} c_4 g_2) = \phi(c_4 g_2)$$

This shows that $\phi(g_2) = 1$, and hence $\phi(g_{2n}) = 1$.

Next we show that $\phi(c_{2n+2}^2) = 1$ for any $n \geq 1$, noting that if $n = 0$ we already have $c_0^2 = 1$.

$$\phi(c_{2n} g_1) = \phi(c_{2n} g_3) \overset{(5.4)}{=} \phi(g_2 c_{2n+2}) = \phi(c_{2n+2})$$
$$\phi(c_{2n} g_1) \overset{(5.5)}{=} \phi(c_{2n+2}^3)$$

To extend this to $\phi(c_2^2) = 1$, we use that $\phi(c_2) \overset{(5.3)}{=} \phi(g_3 c_4) = \phi(g_1 c_4)$, hence $\phi(g_1) = \phi(c_2 c_4^{-1}) = \phi(c_2 c_4)$. Now $\phi(c_2 g_1) = \phi(c_2 g_3) \overset{(5.4)}{=} \phi(g_2 c_4) = \phi(c_4)$, so $\phi(g_1) = \phi(c_2^{-1} c_4)$. Combining these gives $\phi(c_2 c_4) = \phi(c_2^{-1} c_4)$, so $\phi(c_2^2) = 1$ as well, showing that $\phi(c_{2n}^2) = 1$ for any $n \geq 0$.

Thus $\phi(g_1)$ and $\phi(c_0)$ generate the image of ϕ, and we can use the relations shown to hold in $\phi(\vec{T})$ and the finite presentation given in Corollary 5.5.9 to verify that the relations in the infinite dihedral group hold, proving the corollary. We will call the generators $\phi(g_1)$ and $\phi(c_0)$ as simply g_1 and c_0. The first six relations in the finite presentation are all trivial. Since $\phi(c_2^2) = 1$, the relation $c_0 g_1 = c_2^3$ becomes $c_0 g_1 = c_2$. This follows naturally from the definition of $c_2 = g_1^{-1} c_0$ by inverting both sides, since c_0 and c_2 are their own inverses. Likewise, since $\phi(g_2) = 1$, $c_2 g_3 = g_2 c_4$ becomes $c_4 = c_2 g_1 = c_0 g_1 g_1$, which follows from the definition $c_4 = g_3^{-1} c_2 = g_1^{-1} c_2$ in the same way. The relation $c_2 g_2 = g_1 c_4$ then becomes $c_0 g_1 = g_1 c_0 g_1 g_1$, which simplifies to $c_0 = g_1 c_0 g_1$, which can be rewritten as one of the two relations in the presentation of the infinite dihedral group: $g_1 c_0 g_1 c_0 = 1$. The relation $c_0^2 = 1$ remains unchanged and is the other relation in the presentation of the infinite dihedral group. The remaining relations are easily verified to be unnecessary. □

5.6 An Annular Diagram Group Presentation for $\vec{T}$

The proof of the following proposition is basically identical to the proof of Proposition 3.2.1.

Proposition 5.6.1. *If $H < T$ is such that $H = \mathrm{Core}(H)$, then H is an annular diagram group.*

Proof. The core of H consists of carets with labels, which can be used directly to form the presentation for the annular diagram group. Let e be the name of the vertex in the core which was identified with all the roots of the generators of H. Suppose a caret in the core has label x on the root, y on the left child, and z on the right child. Then the corresponding rewriting is $x = yz$. Let S be the set of all distinct vertices in the core, and R be the set of all rewriting rules for every caret in the core. Then we claim that the collection of all annular (e, e) diagrams over the presentation $\langle S | R \rangle$ is an annular diagram group A isomorphic to H.

Indeed, let $\phi : A \to H$ be a homomorphism described as follows. For $\Delta \in A$ a reduced annular diagram, every cell in Δ has exactly one top edge and two bottom edges, and thus each cell can be identified with a caret as in Figure 2.7, which describes a similar situation for going between annular diagrams and tree diagrams for elements of T. Since each vertex in the core of H has at most two children, there are no two distinct rewriting rules $x = yz$ and $x = uv$. Moreover, since Δ is reduced, then since each unique pair of left and right children in the core of H have at most one parent, there are no dipoles created in this identification. As a result, replacing the cells of Δ with carets as described results in a tree diagram, in exactly the same way as annular diagrams in $D^a(\langle x|x = xx\rangle)$ correspond to tree diagrams. Call this tree diagram (R, S, n), and define $\phi(\Delta) = (R, S, n)$. Since every reduced representation of an element of H has a unique identification with the core of H consisting of carets labeled by the rewriting rules of A, it is clear that ϕ is surjective. Likewise, since no dipoles are created, no non-trivial diagram is sent to the identity, hence ϕ is injective. Finally, ϕ is a homomorphism since reduction of diagrams in A corresponds to removing dipoles in H, thus multiplication and reduction of two elements in A gives the same element of H. □

Proposition 5.6.2. $\mathrm{Core}(\vec{T}) = \vec{T}$, *and in particular* $\vec{T} = D^a(\langle e, f|e = ff, f = fe\rangle, e)$.

Proof. Let $f \in \mathrm{Core}(\vec{T})$. Then f is dyadically-piecewise-$\vec{T}$, and can be written as

$$f(t) = \begin{cases} f_1(t) & t \in [0, \alpha_1) \\ f_2(t) & t \in [\alpha_1, \alpha_2) \\ \vdots & \\ f_n(t) & t \in [\alpha_{n-1}, 1) \end{cases}$$

Now, each $f_i(t)$ either preserves or switches dyadic parity. Since f is continuous, $f_i(\alpha_i) = f_{i+1}(\alpha_i)$, where $f_{n+1} := f_1$. Thus if f_i preserves the dyadic parity of α_i, so does f_{i+1}. By induction, all f_i do the same, i.e., all preserve dyadic parity or all switch it. Thus f does the same, and hence $f \in \vec{T}$ by Theorem 5.3.1.

The particular presentation for $\vec{T}$ as an annular diagram group is a simple computation of the core of $\vec{T}$. $\square$

Notice that the presentation in the annular diagram group definition for $\vec{T}$ is a simple Tietze transformation away from $D^a(\langle f | f = f^3 \rangle, f^2)$. Compare this with the following presentation for T_3 as an annular diagram group: $D^a(\langle f | f = f^3 \rangle, f)$. Although they are quite similar, and although $\vec{F}$ is isomorphic to F_3, it turns out that $\vec{T}$ and T_3 are not isomorphic.

Proposition 5.6.3. *T_3 does not contain any element of order two, hence it is not isomorphic to $\vec{T}$.*

Proof. First, note that elements of T_3 have pairs of trees representations exactly like elements of T, except that the trees are ternary rather than binary, just as the functions have slopes integer powers of 3 and breakpoints at 3-adic rationals.

Now suppose that $f \in T_3$ has order 2. Let $f(0) = \alpha$. Then since $f^2 = 1$, $f^2(0) = f(\alpha) = 0$. Since f is continuous then, $f([0, \alpha]) = [\alpha, 1]$ and $f([\alpha, 1]) = [0, \alpha]$.

Now, let (R, S, k) be a pair of trees representation of f with each tree containing n leaves. Then since $f(0) = \alpha$, f sends the first leaf of R to a leaf of S corresponding to some interval that begins with α. Likewise, since $f(\alpha) = 0$, f sends some vertex in R whose interval begins with α to the first leaf of S. In particular, both R and S contain leaves whose intervals begin with α. Let $R_{-,\alpha}$ and $S_{-,\alpha}$ be the leaves of R and S respectively whose intervals combine to give $[0, \alpha]$. Similarly, let $R_{+,\alpha}$ and $S_{+,\alpha}$ be the remaining leaves in each tree.

There is a smallest full ternary tree that contains such a leaf whose interval begins with α, which is a subtree of R and S. Since adding vertices to the tree consists of giving one of the leaves three children, there is a net gain of 2 leaves, so the parity of the number of leaves of the tree doesn't change. Moreover, the intervals associated with the three children of a vertex partition the interval associated with that vertex, hence the parity of the number of leaves whose intervals partition $[0, \alpha]$ does not change. In other words, $|R_{-,\alpha}|$ and $|S_{-,\alpha}|$ have the same parity. But since the leaves in $R_{-,\alpha}$ are identified with the leaves in $S_{+,\alpha}$, $|R_{-,\alpha}|$ and $|S_{+,\alpha}|$ have the same parity. Thus $|S_{-,\alpha}| + |S_{-,\alpha}|$ is even. But by a similar argument, every full ternary tree has an odd number of leaves, since the smallest full ternary tree has 1 leaf, the root, and adding three children to a leaf changes the number of leaves by 2. Thus $|S_{-,\alpha}|+|S_{-,\alpha}|$ is odd, a contradiction. $\square$

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