Hypercomplex-Transfinite Numbers: A Theory of Chaos, Totality and Modality

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DEDICATION

In loving memory of my parents,

Mom and Dad

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LIST OF FIGURES

TABLE OF CONTENTS

INTRODUCTION

A wide range of startling discoveries in the last decades of the twentieth century gave birth to a revolutionary new science, devoted to the exploration of the strange but ubiquitous effects of "organized chaos." Chaos theory challenges traditional thought by undermining the classical conception of hard linear determinism, replacing it with an unstable species of nonlinear dynamics.[1] In the early 1980s, personal computers made it possible to discover the hidden beauty of fractal geometry, made evident simply by running recursive algorithms on complex numbers. Within physical contexts, chaotic effects have been discovered within the orbital paths of planets and galaxies, the currents of oceans and atmosphere, the contours of landmasses, Earth's magnetic field, which fluctuates in response to the flow of Earth's iron magma core, the clouds of Jupiter, certain chemical reactions, and the sporadic dripping of a water faucet. Yet the influence of chaos is scarcely confined to inanimate matter, but thrives in the shape, growth and function of plants and biological systems. The structures of brain, heart, lungs and intestines reflect the recursive architecture of fractal geometry. Even more, chaos plays an integral role in outcomes of animal consciousness. Not only are aggregate features of animal populations chaotic, a feature causally interlocked with other chaotic parameters of environment, such as food chain and weather, but even some of the most individualistic animal behavior, such as the pecking of a bird, for instance, have been statistically demonstrated as chaotic, when compared to the behavior of wider populations, such as flock, herd, school, etc. As for humans,

[1] "The most passionate advocates of the new science go so far as to say that twentieth-century science will be remembered for just three things: relativity, quantum mechanics, and chaos. Chaos, they contend, has become the centuries' third great revolution in the physical sciences. Like the first two revolutions, chaos cuts away at the tenets of Newton's physics. As one physicist put it: 'Relativity eliminated the Newtonian illusion of absolute space and time; quantum theory eliminated the Newtonian dream of a controllable measurement process; and chaos eliminates the Laplacian fantasy of deterministic predictability.' Of the three, the revolution in chaos applies to the universe we see and touch, to objects at human scale." James Gleick, *Chaos: The Making of a New Science* (New York: Penguin, 1987), 5–6.

neuroscience has demonstrated that human brainwave activity, mood swings and many additional features of the mind are governed by chaos theory, as are certain aspects of human personality and behavior. Further still, chaosticians have concluded that the collective output of human behavior produces chaotic results in the stock market and other economic indexes, indexes previously described as random.

Once the various areas of chaos research are set alongside one another, chaos theory is seen as supplanting or, in some sense, infringing into the modal categories of necessity, determinism, free will and contingency. With such extreme diversity, chaos theory has generated tremendous interest, coming to be considered by many as constituting a universal science. One is naturally led by the wide-ranging effects of chaos to consider what phenomena in nature, if any, might be demonstrated as non-chaotic. But how is chaos theory to be interpreted? After its bewildering yet beautiful features are contemplated, what does it all mean? This question is problematic in that the *meaning* of chaos might not be ascertainable until its *scope* is first clarified. It may be that chaos theory ultimately falls back into the lap of the determinist, though with a different brand of determinism than before. Or else it may be that chaos theory implies such a strange admixture of determinism and indeterminism, of necessity and contingency as to dissolve the familiar understanding of these concepts. In this event, traditional categories of modality would be regarded as naïve, presenting the challenge as to how these fundamental terms should be re-conceptualized.

A chief interest of the present study is the tremendous reach of chaos theory, as exhibited throughout fractal geometry, physical sciences, psychology, and probability. The spirit of this study is to carry forward in the advocacy of chaos theory, probing the extent to which its borders might be extended, and inquiring into what it lacks from constituting a universal science. The present discussion considers how chaos theory might become positioned to subsume modal categories of necessity, determinism, free will and contingency. The strategy is to: (1) propose a theory of transfinite probability as a science of contingency, (2) propose a theory of hypercomplex-transfinite (HT) numbers as a science of necessity, (3) demonstrate that these two theories are isomorphic to one another, and (4) produce an algebraic synthesis of these theories by means of HT number theory and HT modality, thereby producing a synthesis of necessity and contingency. Because the logical structure of the Mandelbrot fractal is the same as the logical structure of HT number theory, chaos theory has the same logical power as HT theory, and can thus be interpreted as deeply involved in the synthesis accomplished through HT modality. The conclusion to be drawn is that, if chaos theory reaches so widely as to provide for the embrace of the polar extremes of necessity and contingency, then recognition must be given to the dialectical synthesis, and thus erasure, of these modal categories.

Chapters 1–3 present the background and context for the philosophical concerns motivating the present study. Chapter 1 describes the classical dichotomy of Being vs. Nonbeing, and then describes how Neoplatonists of the via negativa envision a synthesis of these modal opposites. Chapter 2 describes the twentieth century crisis in mathematical foundations, the field of study most widely regarded as the locus classicus of logical necessity. Chapter 3 describes modal problems in broad outline, giving special emphasis to questions of contingency. Chapter 4 proposes a theory of transfinite probability, contrasting it with the standard mathematical theory of classical probability. Chapter 5 proposes a unified theory of hypercomplex-transfinite (HT) numbers. This system gives expression to an HT ramified zigzag theory. Chapter 6 shows that, not only does HT ramified zigzag solve the set theoretic paradoxes and Gödel's incompleteness theorem, but it effects a synthesis of HT numbers and transfinite probability within the rubric of

HT modality, thus accomplishing the fundamental objective of the present study. Chapter 7 reflects on how the material proposed in chapters 4–6 relates to the issues discussed in chapters 1–3, and reorients questions of modality in terms of the philosophical-mathematical program proposed here. Interestingly, HT theory implies that P = NP. As generally conceived, the present discussion can be understood as offering a Hegelian response to Kant’s antinomies, and also as offering a Neoplatonist synthesis of Being and Nonbeing. The distinctive quality of the present study is the pursuit of these philosophical objectives in terms of the philosophy of mathematics.

CHAPTER 1

BEING, NONBEING AND CONTRARIETY

The chief concern of the Parmenidean tradition is to identify and defend an absolutist conception of *being* or *substance,* both of which translate the Greek word *ousia.* This metaphysical program has been attempted in a number of various ways. Aristotle is an exemplary member of this tradition. He believed that absolute being is to be found within empirical reality. Although his views on this were subject to revision, his initial claim was that particular physical objects, such as a horse or a man, are what he referred to as *primary substances*. According to this theory, primary substances are absolute in the Parmenidean sense because they do not have contraries, as is also true of secondary substances, such as species of primary substances. Aristotle wrote:

> Another characteristic of substances is that there is nothing contrary to them. For what would be contrary to a primary substance? For example, there is nothing contrary to an individual man, nor yet is there anything contrary to man or to animal.[2]

An individual man is a primary substance for Aristotle because there is no such thing as an anti-man that, when brought near to man, makes him cease to be or disappear. In this sense, a primary substance has no opposite or contrary. But although primary substances have no contraries, they are able to receive contraries. For instance, a man is able to receive contraries of health or sickness, or of tallness or shortness. Additionally, primary substances can be abstracted from so as to produce a concept or form. This form is spoken of as the species of the primary substance,

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[2] Aristotle, *Categories* 5, 3b24–28; Jonathan Barnes, ed., *The Complete Works of Aristotle*, 2 vols. (Princeton: Princeton University Press, 1984), 1: 7.

such as the species of horse or the species of man, what Aristotle refers to as secondary substances. Insofar as scientific knowledge seeks to understand general or universal characteristics and not particular individuals, it studies forms or species. But for Aristotle, the forms of these secondary substances are always grounded in and derived from primary substances.

Based on this discussion, it can be understood that, like Plato, Aristotle believed in forms, though Aristotle understood forms as empirical rather than transcendent. But despite their differences, the common feature of the Parmenidean tradition is to identify a specific feature of reality that is claimed to be absolute, and then to deny that that absolute can be subject to contrariety. Should contrariety prevail on what is regarded as *being* or *substance*, then Heraclitean flux would erode the Parmenidean absolute. Nonbeing poses a special concern within this context because it poses the threat of being construed as a contrary to being. Thus providing an account of nonbeing such that it offers no contrariety to being has been a fundamental concern of the Parmenidean tradition. When Parmenides denied nonbeing, his claim was primarily logical. When Aristotle denied nonbeing, his claim was primarily empirical. When Plato denied or otherwise contemplated nonbeing, the issue was primarily metaphysical. But despite these various claims, absolutist sanctions against contrariety have suffered remarkable defeat. This drama has played out in logic, mathematics, the empirical sciences, and metaphysics. The objective of the discussion in this chapter is to consider the degree to which contrariety and nonbeing threaten or repudiate the Parmenidean tradition, particularly within the context of Plato's classical theory of forms. The chief interest is to explain how the problem of contrariety became especially acute within Platonism, causing the emergence of the via negativa from the via positiva.

Platonism and the **Classical Conception of *Being***

Although Plato constantly revised his theory of forms, at least three basic claims represent what can be described as his classical theory of forms. These claims can be identified as the ontological thesis, the metaphysical thesis and the epistemological thesis.

1. Ontological thesis: Forms or ideals exist as transcendent absolutes, constituting the essence and substance (or ultimate *being*) of reality. They are intelligible, discrete realities that always exist, never coming into being or passing away. As such, they have the highest rank of existence among all existing things. Their existence is fully actual and objective, independent of the human mind.
2. Metaphysical thesis: Although the forms are transcendent and separate unto themselves within the divine realm, they are the archetypes or prototypes that give form to physical reality. Either the forms exert a presence within physical reality, or else physical reality copies or shares in the forms (the specific relation is not clear). The forms superintend the physical world, such that the physical, changing, lesser reality is governed by the higher, intelligible, absolute reality.
3. Epistemological thesis: True and ultimate knowledge pertains only to the forms. That is, knowledge is achieved by the pure, rationalistic activity of the mind as it grasps the intelligible essence of the forms, which is eternal and absolute. True and stable knowledge does not pertain to empirical observations of the senses which observe transient physical images.

In the following passage from the *Phaedo,* Socrates describes the forms of Beauty, the Great and the Good. His view of causality is that empirical reality copies various forms: big things are made big by the form of the Big, small things are made small by the form of the Small, etc. Here he claims that beautiful things are made beautiful by the form of Beauty.

> I assume the existence of a Beautiful, itself by itself, of a Good and a Great and all the rest. If you grant me these and agree that they exist, I hope to show you the cause as a result…. I think that, if there is anything beautiful besides the Beautiful itself, it is beautiful for no other reason than that it shares in that Beautiful, and I say so with everything…. I simply, naively and perhaps foolishly cling to this, that nothing else makes it beautiful other than the presence of, or the sharing in, or however you may describe its relationship to that Beautiful we mentioned, for I will not insist on the precise nature of the relationship, but that all beautiful things are beautiful by the Beautiful. That, I think, is the safest answer I can give myself or anyone else. And if I stick to this I think I shall never fall into error. This is the safe answer for me or anyone else to give, namely, that it is through Beauty that beautiful things are made beautiful.[3]

In the following passage in *Symposium,* Socrates quotes Diotima as she instructed him on Love and Beauty in his youth. This passage describes the same theory of causality as above, again with reference to the form of Beauty. Socrates quotes Diotima speaking to him as follows:

> You see, the man who has been thus far guided in matters of love, who has beheld beautiful things in the right order and correctly, is coming now to the goal of loving. All of a sudden he will catch sight of something wonderfully beautiful in its nature. That, Socrates, is the reason for all his earlier labors: First, it always *is* and neither comes to be nor passes away, neither waxes nor wanes. Second, it is not beautiful this way and ugly that way, nor beautiful at one time and ugly at another, nor beautiful in relation to one thing and ugly in relation to another, nor is it beautiful here but ugly there, as it would be if it were beautiful for some people and ugly for others…. It is not anywhere in another thing, as in an animal, or in earth, or in heaven, or in anything else, but itself by itself with itself, it is always one in form; and all the other beautiful things share in that, in such a way that when those other things come to be or pass away, this does not become the least bit smaller or greater nor suffer any change. This is what it is to go aright, or be led by another, into the mystery of Love: one goes always upwards for the sake of this Beauty, starting out from beautiful things and using them like rising stairs.[4]

As in the *Phaedo* passage, Socrates speaks of the form of Beauty as having magisterial autonomy, "itself by itself with itself." But in the present passage, he places much greater

[3] Plato, *Phaedo* 100b–e; John M. Cooper, ed., *Plato: Complete Works* (Indianapolis: Hackett Publishing, 1997).

[4] Plato, *Symposium* 210e–211c.

emphasis on the absolute existence and universal character of the form. As seen here, the influence or scope of a form is not qualified, as if only in heaven or in earth or in some other thing, but “always one in form,” with unqualified attribution.

Augustine is often described as synthesizing Platonism with Christianity. The Judeo-Christian tradition describes God as unchanging in the same way as Plato described the forms (“I the Lord change not,” and “Yesterday and today the same, and forever”). In terms of onto-theology (that is, a theology grounded in Absolute Being), Augustine described God as having absolute Being and sovereignty over the physical world in the same way as Plato described the forms as transcending the world of flux.

Augustine claims that even condemned souls maintain their own just rank within the hierarchical order of beings created by God. This order extends through intermediate steps from one degree of praise to the next, such that no level of existence should be missing from the others. As Augustine writes:

> For even though our souls are decayed with sin, they are better and more sublime than they would be if they were transformed into visible light. And you see that even souls that are addicted to the bodily senses give God great praise for the grandeur of light. Therefore, don’t let the fact that sinful souls are condemned lead you to say in your heart that it would be better if they did not exist. For they are condemned only in comparison with what they would have been if they had refused to sin. Nonetheless, God their Creator deserves the most noble praise that human beings can offer him, not only because he places them in a just order when they sin, but also because he created them in such a way that even the filth of sin could in no way make them inferior to corporeal light, for which he is nonetheless praised.
>
> If you saw that the earth had been made but not the heavens, then you would have a legitimate complaint, for you could say that the earth ought to have been made like the heavens that you can imagine. But since you see that the pattern to which you wanted the earth to conform has indeed been made (but is called “the heavens” and not “the earth”), I’m sure that you would not begrudge the fact that the inferior thing has also been made, and that the earth exists, since you are not deprived of the better thing. And there is so great a variety of parts in the earth that we cannot conceive of any earthly form that God has not created…. By intermediate steps one passes gradually from the most fertile and pleasant land to

> the briniest and most barren, so that you would not dream of disparaging any of them except in comparison with a better. Thus you ascend through every degree of praise, so that even when you come to the very best land, you would not want it to exist without the others. And how great a distance there is between the whole earth and the heavens! God has numbered them all.

Furthermore, Augustine claims that the human soul is naturally connected to the divine reasons on which it depends, allowing it to infer through right reason certain things that must exist, though not able to infer all things that do exist.

> Therefore, it is possible for something to exist in the universe that you do not conceive with your reason, but it is not possible for something that you conceive by right reason not to exist. For you cannot conceive anything better in creation that has slipped the mind of the Creator. Indeed, the human soul is naturally connected with the divine reasons on which it depends…. If, therefore, it knows by right reason that God ought to have made something, let it believe that God has in fact done so, even if it does not see the thing among those that God has made.

Augustine says that unhappiness can be avoided simply by cherishing one's will to exist, for the will to exist moves one closer to him who exists in the highest degree.

> So if you will to escape from unhappiness, cherish your will to exist. For if you will more and more to exist, you will approach him who exists in the highest degree. And give thanks that you exist now, for even though you are inferior to those who are happy, you are superior to things that do not have even the will to be happy – and many such things are praised even by those who are unhappy…. Nonetheless, all things that exist deserve praise simply in virtue of the fact that they exist, for they are good simply in virtue of the fact that they exist.

Augustine admonishes others to love things which exist eternally, rather than temporal things that are set on the road toward nonexistence.

> The more you love existence, the more you will desire eternal life, and so the more you will long to be refashioned so that your affections are no longer temporal, branded upon you by the love of temporal things that are nothing before they exist, and then, once they do exist, flee from existence until they exist no more. How can you expect such things to endure, when for them to begin to exist is to set out on the road to nonexistence? … Someone who loves existence approves of such things insofar as they exist and loves that which always exists. If once he used to waiver in the love of temporal things, he now grows firm in the love of the eternal. Once he wallowed in the love of fleeting things, but he will stand steadfast in the love of what is permanent. Then he will obtain the very

> existence that he willed when he was afraid not to exist but could not stand upright because he was entangled in the love of fleeting things…. Your will to exist is like a first step. "If you go on from there to set your sights more and more on existence, you will rise to him who exists in the highest degree.[5]

Augustine claims that it is absurd to say one would rather not exist than to be unhappy, for not to exist is nothing, and so no one can be right in choosing not to exist. "How can we concur with one who is choosing nothing? We desire peace, and this has the character of being." Augustine concludes, "Therefore, just as no one can desire not to exist, no one ought to be ungrateful to the goodness of the Creator for the fact that he exists."[6]

A related question to this discussion is that, if all existence traces to God, then is not God the author of evil? Augustine's interlocutor Evodius asks how we perform evil acts, apart from learning of evil. Augustine responds by saying:

> Perhaps because they turn away from learning and become strangers to it. But whether that is the correct explanation or not, one thing is certainly clear: since learning is good, and the word 'learning' is correctly applied only when we come to know something, we simply cannot come to know evil things. If we could, then they would be part of learning, and so learning would not be a good thing. But it is a good thing, as you said yourself. Therefore, we do not come to know evil things, and there is no point in your asking from whom we learn to do evil things. Or else we do come to know them, but only as things to be avoided, not as things to be done. It follows that doing evil is nothing but turning away from learning…. So someone who wants to know the cause of our learning something really wants to know the cause of our doing good. So let's have no more of your wanting to hunt down this mysterious evil teacher. If he is evil, he is no teacher; and if he is a teacher, he is not evil.[7]

In denying the possibility of an evil teacher, Augustine is defending a Parmenidean view of both Being and Truth, such that neither has a contrary, as if there should be an anti-Being or an anti-Truth. Augustine had converted to Christianity from Manichaeism, and so was vigilant not to

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[5] Augustine, *On Free Choice of the Will*, trans. Thomas Williams (Indianapolis: Hackett Publishing, 1993), Book 3, Chap. 5.

[6] Ibid., Book 3, Chap. 8.

[7] Ibid., Book 1, Chap. 1.

claim that Good and Evil were coequal contraries. Augustine and others of the Parmenidean tradition thus describe evil as merely a privation or absence of Good. According to Manichaeism, a belief that flourished in the fourth century AD, there are two equal and yet opposing gods, one the god of Good and the other the god of Evil. But inasmuch as these gods are equal, they have the same metaphysical standing as the other, and thus there is no real basis by which one is superior or inferior to the other. More specifically, Mani, the founder of Manichaeism, offered the formulation that each of these rival gods constitutes a separate *substance,* thus conceiving of *being* in terms of contrariety. This contrariety was regarded not only as a theological heresy from within monotheistic religions but also a metaphysical heresy from within Parmenidean metaphysics.

Although not a Platonist, Descartes considered the relations of being and nonbeing and of truth and falsehood in similar fashion as did Plato and Augustine. Descartes considered his faculty of judging to be fallible because humans are situated midway between “the supreme being” and “non-being,” and thus concluded that “error as such is not something real that depends upon God, but rather is merely a defect.” As he explains:

> On looking for the cause of these errors, I find that I possess not only a real and positive idea of God, or a being who is supremely perfect, but also what may be described as a negative idea of nothingness, or of that which is farthest removed from all perfection. I realize that I am, as it were, something intermediate between God and nothingness, or between supreme being and non-being: my nature is such that in so far as I was created by the supreme being, there is nothing in me to enable me to go wrong or lead me astray; but in so far as I participate in nothingness or non-being, that is, in so far as I am not myself the supreme being and am lacking in countless respects, it is no wonder that I make mistakes. I understand, then, that error as such is not something real which depends on God, but merely a defect. Hence my going wrong does not require me to have a faculty specially bestowed on me by God; it simply happens as a result of the fact that the faculty of true judgement which I have from God is in my case not infinite.

> But this is still not entirely satisfactory. For error is not a pure negation, but rather a privation or lack of some knowledge which somehow should be in me.[8]

Descartes rejects the view that error or falsehood is *real*, as if truth were opposed by pure negation which depends on God for its existence, as if divine concurrence could be given to what is opposed to truth. For Descartes, false reason might agree with what is, but right reason can never oppose absolute truth. The falsity that denies truth is not something that has *being*, but merely what is a privation of something else that has being. This construction of nonbeing is the classic understanding of the Parmenidean tradition, whether with respect to the nonbeing of falsehood or the nonbeing of evil. Within the Parmenidean tradition, neither falsehood nor evil possess their own nature, for such natures would constitute substances that present contrariety to truth and goodness. Contrariety such as this is thought heretical and impossible by those who follow Parmenides.

Critique of the Classical Theory of *Being*

The Greek mathematician Thales proved the following theorem. Consider a circle with diameter AB, center C and point D on the circle. Theorem: The triangle ABD is a right triangle. This theorem offers a basis for universal, necessary knowledge, giving knowledge of any possible triangle considered, though only one triangle is considered. Plato's view highlights the fact that the theorem does not pertain to a particular depiction, for any depiction inevitably deviates from a perfect triangle. The theorem pertains rather to geometrical objects that are grasped only through the mind. Therefore, let the triangle of this theorem be identified as Plato's form of triangle.

[8] *The Philosophical Writings of Descartes,* trans. John Cottingham, Robert Stoothoff and Dugald Murdoch, 2 vols. (Cambridge: Cambridge University Press, 1984), 2:38.

But the above discussion produces an inconsistency for classical Platonism. A triangle must be scalene, isosceles, or equilateral. The Platonic form of the theorem must be equally applicable to all of these possibilities, for the theorem is true for any triangle whatsoever. But yet a Platonic form is to have not only the definitional "essence" of what it is to be something, but must also have the actual *substance* or *being* of the universal archetype. But the triangle of Thales' theorem cannot be delimited as strictly scalene, isosceles and equilateral at the same time, since it must be a universal that covers all possible cases. Thus the Platonic form of the present discussion cannot be sustained due to a fundamental modal distinction; the form cannot satisfy the demands of actuality and possibility simultaneously, and thus the classical theory of forms is shown to be naïve. Forms cannot be said to exist for any mathematical entity whatsoever without qualification. It was seemingly with this problem in mind that Plato revised his theory in *Republic* so as to describe forms in hierarchal relationships to one another, allowing for some forms to be nested within other forms. The form of a triangle, it seems, does not exist absolutely unto itself but is built up from other forms. A scalene triangle, for example, can be built up by the forms of three, line and unequal. An isosceles triangle can be built up by the forms of one, unequal, two, equal and line, etc. But such interdependency of the forms drastically alters the theory of forms from which they were formerly described as autonomous and absolute. Heraclitean flux is threatening the Parmenidean character of the forms.

The sort of problem described in the preceding discussion gives rise to distinctions of realism, conceptualism and nominalism. Because many believe that Platonic realism reflects an excessive metaphysical outlook, much of western thought has opted to defend the more modest views of conceptualism and nominalism. Hume and Kant were leading proponents of conceptualism within modern philosophy. But here too, the Parmenidean tradition of no contrariety continues to exert a profound influence.

Kant is a leading proponent of conceptualism. In espousing the Parmenidean logic of rejecting contrariety, he insists that there can be no antithetic within pure reason.

> It is grievous, indeed, and disheartening, that there should be any such thing as an antithetic of pure reason, and that reason, which is the highest tribunal for all conflicts, should thus be at variance with itself.[9]

But further, Kant spells out the specific conditions for which he deemed such an antithetic impossible. He writes, "There would indeed be a real conflict, if pure reason had anything to say on the negative side which amounted to a positive ground for its negative contentions."[10] But after rejecting any possibility for such a conflict, he concludes:

> There is thus no real antithetic of pure reason…. This is a comforting consideration, and affords reason fresh courage; for upon what could it rely, if, while it alone is called upon to remove all errors, it should yet be at variance with itself, and without hope of peace and quiet possession.… There is, therefore, properly speaking, no polemic in the field of pure reason.[11]

Yet Kant also diagnoses human reason as involved in speculations that illegitimately exceed the proper limits of reason, and as thus falling into various errors which need correction. As he writes:

> The root of these disturbances, which lies deep in the nature of human reason, must be removed. But how can we do so, unless we give it freedom, nay, nourishment, to send out shoots so that it may discover itself to our eyes, and that it may then be entirely destroyed? We must, therefore, bethink ourselves of objections which have never yet occurred to any opponent, and indeed lend him our weapons, and grant him the most favourable position which he could possibly

[9] Immanuel Kant, *Critique of Pure Reason,* trans. Norman Kemp Smith (New York: St. Martin's Press, 1929), A740/B768.

[10] Ibid., A741/B769.

[11] Ibid., A743/B771.

> desire. We have nothing to fear in all this, but much to hope for; namely, that we may gain for ourselves a possession which can never again be contested.[12]

On Kant's view, mathematics is especially immune from antithetic. As he writes: "In mathematics this subreption is impossible; and it is there, therefore, that apagogical proofs have their true place."[13] This is to say that an apogogical or indirect proof in mathematics can never result in an antithetic or contrariety.

On Kant's view, logic and mathematics never allow antithetic whatsoever, although science can temporarily fall into a state of contrariety, and even perhaps of contradiction, but only due to the uncertain nature of competing hypotheses. In the end, however, the ultimate and ideal state of the physical sciences is such as to be immune from opposition or polemic. Like the physical sciences, philosophy exhibits antithetic within its impure state, and so Kant's admonition is to nourish philosophical polemics so that the offending errors can be fully exposed, only then to be "entirely destroyed," leaving uncontested truth as a permanent possession. In the end, he insists that all the employments of pure reason are free from antithetic or contrariety. This indicates that Kant subscribes to the Parmenidean conception of absolute Being, whether conceived of in metaphysical, empirical or mathematical terms.

The long struggle to prove Euclid's fifth postulate as a theorem, rather than to assume it, ultimately was resolved through means of an apogogical or indirect proof. Because the axiom could not be proven true by direct methods, mathematicians began to assume that it was false, so as to then derive a contradiction. Yet just as the fifth postulate resisted direct proof, it also resisted proof by contradiction. To the amazement of those involved, it eventually became clear that Euclid's fifth postulate gives rise to an antithetic – it can be either true or false, and a

[12] Ibid., A778/B806.

[13] Ibid., A792/B820.

consistent geometry results either case. Significantly, this example provides an instantiation of the specific conditions which Kant claimed to be impossible. Not only is there a simple antithetic with respect to the fifth postulate, but non-Euclidean geometries are able to provide a positive ground for their negative claim which is antithetic or contrary to Euclidean geometry.

The discovery of non-Euclidean geometries is a much discussed topic. But consider whether the discovery actually presents a contradiction. (The preceding discussion of "antithetic" is quite different from an antinomy or contradiction. "Antithetic," as used by Kant, refers merely to contrariety or opposition.) The critical point to understand here is that contradictions occur in the context of particular assumptions, and thus when a contradiction is discovered, consideration is given as to which assumption (or assumptions) is to be blamed and thereupon abandoned. So then, are Euclidean and non-Euclidean geometries mutually contradictory? Yes, but only insofar as mathematical truth is thought to be absolute. Once belief is abandoned that mathematical truths can have no contraries, an assumption aptly described as the Parmenidean-Kantian assumption, then the contradiction disappears. Now rather than speaking of one absolute and universal body of mathematical truths, opposing models and systems have come to be recognized throughout mathematics, each having their own truths and counter-truths. Truth, rather than being absolute and universal, is now interpreted as delimited to particular contexts. But of course such findings are not limited to mathematics. Niels Bohr emphasized the interplay of truth and counter-truth within quantum physics, particularly with respect to the uncertainty principle. As he famously stated: "The opposite of a correct statement is a false statement. The opposite of a profound truth may well be another profound truth." Kant's prohibitions against antithetic are thus no longer recognized within mathematics or the physical sciences.[14]

[14] The argument here is not intended to challenge Kant's philosophy of transcendental idealism, but only to show that his prohibition against mathematical and scientific contrariety is incorrect.

As touching on the debate of realism, conceptualism and nominalism, the negative effects described here equally challenge absolutist views of realism and conceptualism, though the question arises as to how these implications might affect non-absolutists of the via negativa. It thus remains to consider the question of contrariety within the metaphysics of Neoplatonism.

The Via Negativa, Nonbeing and Contrariety

Plato's classical theory of forms underwent constant revision. But after the revisions culminate in the via negativa in the later dialogues, it seems in retrospect that the sweep of revisions was moving in that direction even from the early dialogues (though note that it is a minority view that the dialogues culminate in the via negativa). The most rudimentary version of the classical theory of forms seems to be found in *Symposium,* and then the theory is slightly revised in moving to the *Phaedo*.[15] In *Symposium,* the form of Beauty is described in terms of ultimate autonomy, "itself by itself with itself." The scope of the form is universal and without qualification. In the *Phaedo,* the form of Beauty is still described in terms of autonomy ("itself by itself"), though the autonomy of other forms is less than absolute. Such is the case with the Big and Small, or more particularly the Odd and the Even, where neither is fully independent of the other; rather, each element of the pair stands in a correlative relation with its complement (its opposite). The *Phaedo* stresses that forms are always the same and are not subject to change.

The theory of forms undergoes significant changes in *Republic* Book 6. The Analogy of the Divided Line describes forms as standing in the same place as mere hypotheses, the difference

[15] Although the theory of forms progresses in moving from *Symposium* to *Phaedo,* another progression moves in the opposite direction, moving from *Phaedo* to *Symposium*. The *Phaedo* describes a strong almost irreconcilable opposition between the physical and spiritual, describing the body as the prison of the soul; the body drags the soul down, making it drunk and dizzy, thereby hindering the soul's pursuit of the divine and intelligible. But in *Symposium,* the physical and spiritual are alike in that both seek Beauty *erotically.* And so in this one context, the physical and spiritual overcome their alienation such that each is the complement of the other. Their mutual interaction is especially evident in the Ascent passage.

based on their mode of presentations in moving up or down the divided line. Just as shadows and reflections gain intelligibility when seen in relation to their corresponding physical objects, so physical objects are mere shadows that are explained in terms of mathematical precision. The progression in moving from physical objects to mathematical clarity is the critical step up the divided line from the visible to the intelligible. But the mathematical intelligibility gained herein is one based merely on hypotheses, and thus the analogy of shadowy images continues to hold force even within the intelligible realm. Just as physical objects lack clarity with respect to mathematics, so too mathematical hypotheses lack intelligibility with respect to a secure mathematical foundation. It is the view of Plato (and so too of many modern mathematicians), that mathematical hypotheses lack intelligibility in the absence of an unhypothetical first principle. Thus the many and varied claims of mathematical hypotheses are positioned high on the divided line with respect to visible objects, but low with respect to the unhypothetical first principle, which stands supreme atop the divided line. Hypotheses had been part of the upward force that pulls the soul upward from the visible realm into the intelligible, but once the soul apprehends what is intelligible, hypotheses become a hindrance to the ascent toward the intelligible by taking on the character of shadowy images. According to Plato, a hypothesis is a concocted reality a mathematician might contrive as a mere thought experiment. Plato claims that the pursuit for intelligibility must press beyond the hypothetical to what is fundamentally unhypothetical. He claims further that this final step must be made without the aid of hypotheses at all, and thus, insofar as hypotheses pertain in some important sense to what is visible, the final ascent of reason takes place within a realm that is wholly invisible and purely intelligible. According to the logic of the divided line, the unhypothetical first principle stands in the same relation to hypotheses as hypotheses relate to physical objects, which in turn is the same relation in which physical objects relate to images (such as shadows and reflections). The path upward proceeds from hypothesis to hypothesis, without any knowledge of why the hypotheses are as they are, until the unhypothetical fir principle is attained. At the top of the divided line, the power of dialectic operates purely within the realm of the intelligible, without any benefit of hypotheses or visible objects whatsoever. Once the "unhypothetical first principle of everything" is grasped, Plato says the path of reason reverses itself, now looking down on the path just ascended. Upon taking the downward path, what was formerly hypothetical is now understood in light of the unhypothetical as determinate forms. Thus the path downward proceeds "to a conclusion without making use of anything visible at all," that is, no hypotheses or physical objects, "but moving on from forms to forms, and ending in forms" *(Republic* 511b).

The Analogy of the Divided Line offers a remarkable presentation of forms. Although intended to be fully determinate and necessary, forms first appear in *Republic* Book 6 in the guise of contingent hypotheses. The intelligibility of a form is not autonomous from within the form itself, but is completely heteronomous with respect to the unhypothetical first principle. Except for the intelligibility conferred on a form by the unhypothetical first principle, a form is otherwise indistinguishable from a mere hypothesis. This is a radical departure from a form as classically conceived, existing "itself by itself with itself." Additionally, because forms are arranged according to nested hierarchies, the range and effect of the forms are delimited, just as certain hypotheses are valid only within contexts of other supporting hypotheses. This interdependence diminishes the absolute quality of forms even further. As presented in *Republic*, the theory of forms is very much a work in progress, though it has already been revised beyond the point of *Symposium* or *Phaedo*.

Before continuing on to consider how Plato revises the theory of forms further in *Parmenides* and *Sophist,* it will be helpful to consider a similar theory of being and nonbeing as offered by Augustine. Like Plato, Augustine's writings reflect a significant amount of development with respect to the classical theory of *being.* Augustine's earlier uncritical writings on the subject are expressed in *On Free Choice of the Will,* but his more developed critical views are apparent in *Confessions.* Within this latter context, Augustine conceived of God in Parmenidean terms as described in the theory of divine simplicity. This theory expresses the view that God is a perfect absolute being who is ontologically self-sufficient, not having proper parts that are, in some sense, logically prior to the divine being. The theory of divine simplicity holds that God is immaterial, immutable, timeless, impassible and independent. Augustine, Anselm, Aquinas, Maimonides and other monotheists embraced the theory, though Muslim theologians tended to reject it. The God of divine simplicity has more affiliation with Aristotle's Unmoved Mover and "Nous nousing Nous" rather than answering to the description of a God who is a loving, compassionate father. According to this theory, God relates to his internal attributes necessarily but to external objects contingently. But if God's substance is necessary and yet God is also the creator of Adam contingently, then how is the act of creating contingent things to be understood as a necessary or intrinsic attribute of God? This is the modal divide through which Augustine was trying to relate to God in *Confessions*. Based on the context of divine simplicity, Augustine developed the view that God stands eternally, but humanity is broken up in time. Augustine interprets time as tending toward nonbeing, which places time in sharp distinction from eternity. Augustine writes as follows:

> But the two times, past and future, how can they *be,* since the past is no more and the future is not yet? On the other hand, if the present were always present and never flowed away into the past, it would not be time at all, but eternity. But if the present is only time, because it flows away into the past, how can we say that it

> *is?* For it is only because it will cease to be. Thus we can affirm that time *is* only in that it tends toward not-being.[16]

Based on the predicament of having his life grounded in not-being, Augustine wrote, "My life is but a scattering," and then added, "I am divided up in time, whose order I do not know, and my thoughts and the deepest places of my soul are torn with every kind of tumult."[17]

Here in *Confessions*, the distinction between time and eternity implies that humanity is radically separated from God. Augustine describes all humanity as divided up and scattered in the nonbeing of time, so as (except for divine grace) to be metaphysically cut off from God.[18] In *On Free Choice of the Will,* humanity is separated from God only through intermediate steps of being, according to which humans have a lesser and derived status of being, though one which nonetheless is completely connected with God, who exists in the highest degree. The relation of humanity to God in *On Free Choice of the Will* is autological. This autological relation is the defining positive relation of the via positiva. In *Confessions,* the relation of humanity to God is heterological. This heterological relation is the defining negative relation of the via negativa.

The via negativa is strikingly evident within Plato's *Parmenides* and Sophist. The *Parmenides* is perhaps the most puzzling and controversial dialogue in all of Plato. The problem centers on why Plato is attacking – or even ruining – his own theory. The controversy seems to result from the intent of many interpreters to resist the tremendous criticism directed against the theory, as if the appropriate response is to somehow retreat back to the classical naive theory,

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[16] Augustine, *Confessions,* Book 11, Chap. 14, in Steven M. Cahn, ed., *Classics of Western Philosophy,* 7th ed. (Indianapolis: Hackett Publishing, 2006), 377–78.

[17] Ibid., Book 11, Chap. 29; Cahn, 384.

[18] G. J. P. O'Daly writes: "Perhaps uniquely among ancient Platonists, Augustine does not attempt to understand time with reference to its supposed *paradeigma* or model, eternity. Elsewhere, indeed, he will refer to time as a 'trace' or 'copy'of eternity, but in *Conf. 11* it is rather the total contrast between God's transcendence of time and our anguished sense of dispersion and fragmentation in time that he wishes to emphasize." Quoted in *Augustine: Confessions,* ed. James J. O'Donnell, 3 vols. (Oxford: Clarendon Press, 1992), III: 278.

whereas Plato's intent seems to be to push ahead toward critically revising the theory. Because of the difficulty of interpreting this dialogue, the following discussion will analyze the text closely in certain key respects. The view advanced here is that the dramatic narrative and logical argument of the dialogue converge so as to reach the same conclusion. This conclusion is extremely clear within the dramatic context, though it is tremendously obscure in logical terms and in departing from and repudiating essential features of the classical theory.

The dialogue depicts Parmenides (about 65) and Zeno (about 40) visiting Athens, where they engage in discussion with the young Socrates (about 20). The dialogue begins with Zeno reading aloud from his book, and then Socrates responding to and critiquing what Zeno has said. Socrates then presents the classical theory of forms (as described in Plato's middle dialogues), whereupon Parmenides subjects the theory to a number of criticisms. The first part of the dialogue closes with Parmenides and Socrates discussing how the theory of forms might remain viable. The second part of the dialogue consists of Parmenides presenting eight deductions. It is clear that these deductions destabilize Platonic forms, but beyond that the nature of the deductions or how to offer an appropriate response to them is highly contested.

At the beginning of the dialogue, Socrates partially critiques Zeno's claim that things cannot be many, for if they were many they would be both like and unlike, which is impossible, "because unlike things can't be like or like things unlike." In partially rejecting this claim, Socrates asserts that contrariety is possible for sensible things but not for intelligible forms. As he says to Zeno:

> But tell me this: don't you acknowledge that there is a form, itself by itself, of likeness, and another form, opposite to this, which is what unlike is? Don't you and I and the other things we call 'many' get a share of those two entities? And don't things that get a share of likeness come to be like in that way and to the extent that they get a share, whereas things that get a share of unlikeness come to be unlike, and things that get a share of both come to be both? And even if all

> things get a share of both, though they are opposites, and by partaking of them are both like and unlike themselves, what's astonishing about that? (*Parmenides* 129a)

Then in five successive statements, Socrates considers a thesis that he believes would be astonishing, if true. Such repetition as this by Plato is not superfluous, but, much rather, it is this astonishing claim that seems to prefigure the essential point of the deductions. This thesis is that forms are subject to contrariety. As Socrates says in the first instance, "If someone showed that the likes themselves come to be unlike or the unlikes like – that, I think, would be a marvel" (129b). Second, "But if he should demonstrate this thing itself, what one is, to be many, or, conversely, the many to be one – at this I'll be astonished" (129b). Third, "And it's the same with all the others: if he could show that the kinds and forms themselves have in themselves these opposite properties, that would call for astonishment" (129c). And then in the clearest passage, in which the contrariety of forms is mentioned a fourth and fifth time, Socrates states:

> But if someone first distinguishes as separate the forms, themselves by themselves, of the things I was talking about a moment ago – for example, likeness and unlikeness, multitude and oneness, rest and motion, and everything of that sort – and then shows that in themselves they can mix together and separate, I for my part," he said, "would be utterly amazed, Zeno. I think these issues have been handled with great vigor in your book, but I would, as I say, be much more impressed if someone were able to display that this same difficulty which you and Parmenides went through in the case of visible things, also similarly entwined in multifarious ways in the forms themselves – in things that are grasped by reasoning. (129d–130a)

After Socrates finishes speaking with Zeno, Parmenides asks him, "Is there a form, itself by itself, of just, and beautiful, and good, and everything of that sort?" Socrates answered yes (130b). But this question makes clear that, in addition to the contrariety of forms, the present discussion is also emphasizing a form as "itself by itself." This is the third time this language has been used in the dialogue. This language is often construed as stressing the separate nature of forms from the sensible objects for which they are archetypes, which is certainly valid. However, this language also emphasizes the nature of a form as autonomous unto itself. But if the astonishing conclusion arises that forms have contraries, then the equally astonishing conclusion would result that forms stand in heteronomous relation to other forms, rather than as autonomous and absolute. Thus in advancing the astonishing claim that forms are subject to contrariety, Parmenides is also critiquing the autonomy of the forms as standing unto themselves. Critiquing the autonomy of forms is a necessary step in showing them to have contrariety. Parmenides refers to the autonomy of a form as being "marked off" unto itself.

After discussing a number of objections to the theory of forms, Parmenides once again calls attention to the autonomous nature of forms: "And yet, Socrates," said Parmenides, "the forms inevitably involve these objections and a host of others besides – if there are those characters or things, and a person is to mark off each form as 'something itself' (135a). Parmenides makes clear that the existence of a form and its solitary character as marked off "by itself" are two different things. But he also makes clear that his intention is not to ruin the theory, even though it stands in opposition to his own. As he states:

> If someone, having an eye on all the difficulties we have just brought up and others of the same sort, won't allow that there are forms for things and won't mark off a form for each one, he won't have anywhere to turn his thought, since he doesn't allow that for each thing there is a character that is always the same. In this way he will destroy the power of dialectic entirely. (135b–c)

After Socrates concedes to Parmenides that he has no strategy by which to repair the theory of forms, Parmenides says, "Socrates, that's because you are trying to mark off something beautiful, and just, and good, and each one of the forms, too soon," he said, "before you have been properly trained" (135c). Parmenides states that he was impressed by the way Socrates responded to Zeno's reading. He states:

> You didn't allow him to remain among visible things and observe their wandering between opposites. You asked him to observe it instead among those things that one might above all grasp by means of reason and might think to be forms. (135e)

As enumerated here, this is the sixth time the dialogue has referred to the contrariety of forms. Socrates responds to Parmenides by saying, " I did that because I think that here, among visible things, it's not at all hard to show that things are both like and unlike and anything else you please" (135e).

Parmenides then explains to Socrates his method of training, which is necessary in order "to achieve a full view of the truth." He says one must consider a hypothesis not only in terms of its affirmation but also its negation; and further, a hypothesis must be considered not only in terms of its internal consequences, but also for its external consequences, both how these external consequences relate to what is hypothesized itself, and to how the various consequences relate among themselves. This method consists of a bivalent alternative considered on three levels, yielding a schema of 2^3 categories of analysis. This is the logical basis of the eight deductions. After describing how this method applies generally to the many or the like, Parmenides states as follows:

> And the same method applies to unlike, to motion, to rest, to generation and destruction, and to being itself and not-being. And, in a word, concerning whatever you might ever hypothesize as being or as not being or as having any other property, you must examine the consequences for the thing you hypothesize in relation to itself and in relation to each one of the others, whichever you select, and in relation to several of them and to all of them in the same way; and, in turn, you must examine the others, both in relation to themselves and in relation to whatever other thing you select on each occasion, whether what you hypothesize you hypothesize as being or as not being. (136a–c)

Socrates says the task Parmenides describes seems scarcely manageable, but that he does not yet fully understand it, and so he asks Parmenides to go through the exercise for him. Parmenides and Zeno both say that this is a big assignment (namely, working through the eight deductions), and Zeno says it will not be easy for Parmenides, advanced in age, to accomplish. Parmenides says the task is strenuous and causes him anxiety, but that he will undertake it for the sake of his hearers. Note that Socrates had said earlier it was easy for him to answer Zeno in showing that sensible objects are contrary to one another, but that he would be impressed and astonished if someone could show that this contrariety applies to forms. What Socrates did was easy, but Plato's readers are to understand that what Parmenides is setting out to accomplish in the eight deductions is strenuous, hard and astonishing.

In discussing the one and many, the like and unlike, motion and rest, generation and corruption, past, present and future time, being and nonbeing, Parmenides is critiquing within the deductions the very forms that are of fundamental importance to Plato's theory. In urging that a particular hypothesis be analyzed as to its consequences on "the others," Parmenides is proposing that forms be analyzed in the deductions by means of other forms. This syntactical analysis of forms envisions the same sort of logical grammar as found within Aristotle's categories or even in Kant's categories, though Plato never succeeds in completing the sort of logical system he imagines. But then in a remarkable turn of events, Parmenides offers to subject himself and his own theory to the same criticism as the theory of forms: "Is it all right with you if I begin with myself and my own hypothesis? Shall I hypothesize about the one itself and consider what the consequences must be if it is one or if it is not one" (137b)? Parmenides claims that absolute being is monistic. Plato claims that absolute being is pluralistic, at least insofar as his classical theory describes each of the forms as absolute. Thus in the eight deductions, Plato is not only subjecting his own theory to criticism, but he portrays Parmenides as doing the same thing. The effect is to suggest that the two opposing theories of the absolute are themselves contraries of one another.

The logical structure of the eight deductions takes the following scheme. The antecedent of each of the first four is: “if the one is;” the consequents are as follows:

(D1) then the one is not *F* and not con-*F* (in relation to itself and in relation to the others).

(D2) then the one is *F* and con-*F* (in relation to itself and in relation to the others).

(D3) then the others are *F* and con-*F* (in relation to themselves and in relation to the one).

(D4) then the others are not *F* and not con-*F* (in relation to themselves and in relation to the one).

The remaining four deductions each begins with the antecedent: “if the one is not;” the consequents are as follows:

(D5) then the one is *F* and con-*F* (in relation to itself and in relation to the others).

(D6) then the one is not *F* and not con-*F* (in relation to itself and in relation to the others).

(D7) then the others are *F* and con-*F* (in relation to themselves and in relation to the one).

(D8) then the others are not *F* and not con-*F* (in relation to themselves and in relation to the one).

According to most interpreters, D1 and D2 taken together entail that the one is contradictory, and thus does not exist. Similarly, the same conclusion follows for D3 and D4 when taken together. Although note that D1–D4 all claim individually that the one does exist. Contrary to this, D5 and D6 taken together entail that the one does exist, as also D7 and D8 when taken together. Although note that D5–D8 all claim individually that the one does not exist.

What is to be made of this nested series of contradictions? Most interpreters regard the eight deductions as implying local contradictions and also a master contradiction. The predominant view is that there is no constructive way forward. As described thus far, the effect is Pyrrhonian, in that the deductions, equally valid as taken in isolation, work together to mutually block one another from making headway. On this view, this is the end of Plato’s rationalism and the beginning of his mysticism. But it seems clear that more can be said than this. What about the claim advanced within the dialogue that forms are to be treated as contraries?

Samuel C. Rickless argues that the "radical purity" of forms ("**RP**" is the assumption that no form can have contrary properties) is a background assumption of D1 and D4. D1 and D2 entail together that the one cannot exist, though this entailment is predicated on the assumption of **RP**. But since D5 and D6 entail that the one exists, then **RP** must be rejected. According to Rickless, rather than entailing a grand contradiction, the deductions entail both that the one exists and that RP is false.[19] But how is this possible? This interpretation directly repudiates Plato's classical theory. If forms can be treated as contraries, as Rickless proposes, that would be astonishing. And yet that is precisely the astonishment that the drama of the dialogue has prefigured. But yet something is missing. How can the theory of forms be transformed by contrariety merely by saying so? The theory must clearly be reworked and redeveloped in technical detail.

In order to transform the theory of forms so that it admits of contrariety, some sort of theoretical basis needs to be supplied that does not belong to the classical theory. Toward this end, note that Plato provides a supplementary discussion to the first two deductions. In this appendix, he describes the forms as acting at "a definite time," but yet he also speaks of their separating and combining as if occurring outside or beyond time, in what Plato calls the "instant," which is said to be "in no time at all." To frame this discussion within a contemporary context, it is as if Plato is attempting to resolve the contradictions of the deductions by construing them in terms of a superposed cat-state within quantum mechanics, as if the logic of superposition can be interpreted by means of imaginary time. Plato writes as follows:

> Let's speak of it yet a third time. If the one is as we have described it – being both one and many and neither one nor many, and partaking of time – must it not,

[19] Samuel C. Rickless, *Plato's Forms in Transition: A Reading of the Parmenides* (Cambridge: Cambridge University Press, 2007), 136–37; 211.

> because it is one, sometimes partake of being, and in turn because it is not, sometimes not partake of being? So it partakes at one time, and doesn't partake at another; for only in this way could it both partake and not partake of the same thing. Isn't there, then, a definite time when it gets a share of being and when it parts from it? Or how can it at one time have and at another time not have the same thing, if it never gets and releases it? The one, as it seems, when it gets and releases being, comes to be and ceases to be. And since it is one and many and comes to be and ceases to be, doesn't its being many cease to be whenever it comes to be one, and doesn't its being one cease to be whenever it comes to be many? Whenever it comes to be one and many, must it not separate and combine? Furthermore, whenever it comes to be like and unlike, must it not be made like and unlike? And whenever it comes to be greater and less and equal, must it not increase and decrease and be made equal? And whenever, being in motion, it comes to a rest, and whenever, being at rest, it changes to moving, it must itself, presumably, be in no time at all. For it does not change while it is at rest or in motion, or while it is in time. Is there, then, this queer thing in which it might be, just when it changes? The instant. The instant seems to signify something such that changing occurs from it to each of two states. For a thing doesn't change from rest while rest continues, or from motion while motion continues. Rather, this queer creature, the instant, lurks between motion and rest – being in no time at all – and to it and from it the moving thing changes to resting and the resting thing changes to moving. And the one, if in fact it both rests and moves, could change to each state – for only in this way could it do both. But in changing, it changes at an instant, and when it changes, it would be in no time at all, and just then it would be neither in motion nor at rest.
>
> Is it so with the other changes too? Whenever the one changes from being to ceasing-to-be, or from not-being to coming-to-be, isn't it then between certain states of motion and rest? And then it neither is nor is not, and neither comes to be nor ceases to be? Indeed, according to the same argument, when it goes from one to many and from many to one, it is neither one nor many, and neither separates nor combines. And when it goes from like to unlike and from unlike to like, it is neither like nor unlike, nor is it being made like or unlike. And when it goes from small to large and to equal and vice versa, it is neither small nor large nor equal; nor would it be increasing or decreasing or being made equal. The one, if it is, could undergo all that. (155e–157b, abridged)

This passage appears to provide the theoretical basis necessary to resolve the contradictory nature of the eight deductions, and also to satisfy the dramatic prefiguring within the dialogue that Parmenides is to astonish Socrates by demonstrating the contrariety of the forms.[20] On this

[20] "Plato takes it to be imposible for something that is not in time to be both F and not F. For the only way he can conceive of something's being both F and not F is for it to be F at one time and not F at another. The result of this is that the conjunction of RP with the hypothesis that the one is leads to absurdity. Hence, if it can be shown that the one is, then it will thereby be shown that RP is false"(Rickless 211).

reading, Parmenides has saved the forms, and has thereby preserved the power of dialectic and of reasoning itself. But in order to do so, he had to reject the claim of Plato's classical theory that forms are autonomous and absolute. In Parmenides' view as expressed here, Socrates was in error by marking off the forms "too soon" as unto themselves. The correct view, according to the revision of the theory, is to regard forms as heteronomous contraries, existing unto one another rather than unto themselves.

The contrariety of forms described here recalls the reciprocal nature of odd and even and of big and small in *Phaedo,* though the effect here is stunningly different. The pairing of odd and even in *Phaedo* suggests a lack of autonomy for each individual form, but yet the pair offers heteronomous support to each other to make up for the autonomy that was individually lost. Then in *Republic,* forms are stripped of their autonomy even more fully, but then made whole by the extrinsic guarantee of the unhypothetical first principle. So now in *Parmenides,* forms are paired off with their contraries, though without the stabilizing guarantee of the unhypothetical first principle. This contrariety renders the forms extremely unstable, and so, if not absolute, it is unclear why they can behave as forms at all. Heraclitean flux is poised to destroy the Parmenidean absolute.

The reading of the *Parmenides* advanced here makes it closely parallel to the problem in *Republic.* In moving up the analogy of the divided line, Plato advances from the visible to the intelligible. According to the classical theory, visible things are subject to contrariety whereas intelligible forms are not. The intelligible portion of the divided line ideally consists of forms, though Plato says mathematicians deal much more with mere hypotheses than with forms, and that the logical import of the forms on mathematics cannot be determined apart from the unhypothetical first principle, which is unknown. The mere hypotheses in the divided line are akin to the conflicting hypotheses in the eight deductions. The interpretation advanced here within the deductions can thus be extended to the divided line, allowing the contrariety of mathematical hypotheses in the divided line to be construed within terms of imaginary time, as provided for in the deductions. That is, Plato proposes a logical theory of imaginary time in *Parmenides* that becomes a mathematical theory of imaginary time when applied to *Republic.* In this combined theory of forms it becomes possible to extend the analogy of the divided line so as to claim that intelligible forms are subjected to contrariety in imaginary time, just as visible things are subjected to contrariety in real time. If then an unhypothetical first principle can be found that explains the contrariety of forms, it would solve the combined problem of both *Republic* and *Parmenides.* As cast within the drama of the *Parmenides,* the unhypothetical first principle of *Republic* has become the shared philosophical aspiration of both Parmenides and Plato, thus collapsing the distinctions of their respective theories. *Parmenides* adds logical support to *Republic,* while *Republic* adds mathematical support to *Parmenides.* The one of *Parmenides* can now be conceptualized as the unhypothetical first principle of *Republic.*

In this revised critical theory, the Parmenidean one or unhypothetical first principle is the supreme form above all others. Should all forms give way to contrariety but the one, then even so the Parmenidean absolute is not lost. It seems that Plato’s ultimate philosophical program involves the willingness to subject all forms to contrariety, except the form of the Good, which he is intent on describing in immediate association with the unhypothetical first principle. In this scenario, the revised critical theory would continue to resist the Manichaean heresy.

Plato’s *Sophist* seems to belong midway between *Republic* and *Parmenides.* The *Sophist* describes a discussion between Socrates, the Eleatic visitor, Theodorus, Theaetetus, and the young Socrates (namesake of the philosopher Socrates). Socrates asked the Eleatic visitor about the difference between statesmen, philosophers and sophists. His interest was to determine whether their natures are separate or combined. The conversation then gravitated to the sophist. After providing six different accounts of the sophist, the visitor's descriptions were still incomplete. In making his seventh attempt, the visitor realized that the proclivity of the sophist to say what is not as if it is makes it especially difficult to specify his true character. But the visitor is intent on hunting down his prey: "the beast is complex and can't be caught with one hand," he says. The drama of the dialogue involves the escalating concessions the visitor must make to his Eleatic philosophy in order to track down the sophist, and then the various implications that follow. If the visitor can offer an accurate account of deceit, falsehood and nonbeing, then he will be able to describe the sophist in exact detail.

The Eleatic visitor explains to the young Theaetetus that giving an adequate account of falsehood is problematic. He states:

> This is a very difficult investigation we're engaged in. This appearing, and this seeming but not being, and this saying things but not true things—all these issues are full of confusion, just as they always have been. It's extremely hard, Theaetetus, to say what form of speech we should use to say that there really is such a thing as false saying or believing, and moreover to utter this without being caught in a verbal confusion…. This form of speech of ours involves the rash assumption that that which is not is, since otherwise falsity wouldn't come into being. But when we were boys, my boy, the great Parmenides testified to us from start to finish, speaking in both prose and poetic rhythms, that *Never shall this force itself on us, that that which is not may be.* (236d–237a)

The visitor then asks, "Do we dare to utter the sound *that which in no way is?"* Or if we say *that which is not,* what can that name be applied to? What would be indicated to someone who wanted to find out about it? Or is there a *something* that these words can be applied to? His conclusion is, "This much is obvious to us, that *that which is not can't be applied to any of those which are"* (237b–c).

The visitor then expresses concern as to the grammatical distinction between *that which is not* and *those things which are not,* as if nonbeing should be differentiated by number. According to the strict doctrine of the Eleatic school, it is not correct to attach *that which is* to that which is not. The visitor says, "Do you understand, then, that it's impossible to say, speak, or think *that which is not* itself correctly by itself. It's unthinkable, unsayable, unutterable, and unformulable in speech (238b–c)." But then in complete candor, the visitor turns his logic against himself:

> My good young friend, don't you notice on the basis of the things we said that *that which is not* even confuses the person who's refuting it in just this way, that whenever someone tries to refute it, he's forced to say mutually contrary things about it? … I was the one who made the statement that *that which is not* should not share either in *one* or in *plurality.* But even so I've continued after all that to speak of it as *one,* since I say *that which is not.* … And again a little earlier I said that it *is* unutterable, unsayable, and inexpressible in speech. … So in trying to attach *being* to it wasn't I saying things that were the contrary of what I'd said before? … And in attaching *that which,* wasn't I speaking of it as *one?* … And also in speaking of it as something inexpressible in speech, unsayable, and unutterable, I was speaking of it as one thing. … Then what would somebody say about me? He'd find that the refutation of that which is not has been defeating me for a long time. (238d–239b)

The visitor then asks Theaetetus to say something correct about nothing without attaching *being, one,* or *plurality* to it. Theaetetus acknowledges he cannot do so. But this concession leaves the sophist beyond the grammatical reach of the visitor. The visitor thus says, "Until we meet someone who can do it let's say that the sophist has stopped at nothing. He's escaped down into inaccessible confusion" (239c).

After further discussion, the visitor asks Theaetetus if they should abandon their hunt for the sophist. Theaetetus says no. The visitor then asks Theaetetus not to think him a patricide; "We're going to have to subject father Parmenides' saying to further examination, and insist by brute force both that *that which is not* somehow is, and then again that *that which is* somehow is not"

(241d). The visitor states further, “We’ll be trying to refute what Parmenides said—*if* we can do it” (242b).

The visitor then describes different metaphysical outlooks of Greek mythology. Some myths say everything is derived from a primordial triad in which the original beings were sometimes at war with each other and sometimes friendly, so as to marry and have offspring. Other myths say that everything comes from a primordial dyad. The Eleatic myth claims that everything comes from one. But then the visitor describes a hybrid view:

> Later on, some Ionian and Sicilian muses both had the idea that it was safer to weave the two views together. They say that *that which is* is both many and one, and is bound by both hatred and friendship. According to the terser of these views, in being taken apart they’re brought together. The more relaxed muses, though, allow things to be free from that condition sometimes. They say that all that there is alternates, and that sometimes it’s one and friendly under Aphrodite’s influence, but at other times it’s many and at war with itself because of some kind of strife. It’s hard to say whether any one of these thinkers has told us the truth or not, and it wouldn’t be appropriate for us to be critical of such renowned and venerable men. (242c–243a) (The two muses/men referred to here are Heraclitus and Empedocles.)

The visitor says he has always been clear on what is meant by *is not*, though now he is confused by it, and so too, he has always thought himself clear on *what is,* but may now be confused on that as well. “But maybe we’re in the same state [of confusion] about both” (243c). The visitor and Theaetetus thus discuss the question of being as applied to the metaphysics of primordial dyads such as hot and cold.

The visitor suggests the question of *being* is like a battle between gods and giants. One group drags everything down to earth from divine intelligibility, and then despises anyone who speaks of *being* apart from a body. The visitor describes the other group as defending their position “from somewhere up out of sight,” and insist that “true being is certain nonbodily forms that can be thought about.” The visitor adds, “They take the bodies of the other group, and also what they call the truth, and they break them up verbally into little bits and call them a process of coming-to-be instead of being." The visitor concludes by saying, "There's a never-ending battle going on constantly between them about this issue" (246a–c).

The visitor and Theaetetus then critique the traditional view of *being* as described in Plato's classical theory (and as frequently taken up in the doctrine of divine simplicity in medieval thought). The visitor says:

> But for heaven's sake, are we going to be convinced that it's true that change, life, soul, and intelligence are not present in *that which wholly is,* and that it neither lives nor thinks, but stays changeless, solemn and holy, without any understanding? (248e–249a)

The visitor and Theaetetus then agree that *that which wholly is* must have attributes of intelligence, life, soul and change. The visitor states, "Then both *that which changes* and also *change* have to be admitted as being" (249b). They then try to describe their view in the same way they had been seeking descriptions of primordial metaphysics (250a). They agree that change and rest are completely contrary to each other, and agree that both equally are and that each equally is. So they agree that *that which is* is a third thing beyond change and rest; that it is not both of them together but something beyond them. "Therefore by its own nature *that which is* doesn't either rest or change." Yet if something does not change, how can it not be resting? And if something is not resting, how can it not be changing? The discussion thus seems to have led to an impossible conclusion (250c–d). The visitor recalls that they were initially confused as to what is meant by *what is not,* but now they are in just as much confusion about *that which is.* The visitor concludes:

> Then we've now given a complete statement of our confusion. But there's now hope, precisely because both *that which is* and *that which is not* are involved in equal confusion. That is, in so far as one of them is clarified, either brightly or dimly, the other will be too. (250e–251a)

The visitor then asks Theaetetus to consider the question of blending. To what degree do things have the capacity of associating or blending with others? They consider three options. The first is that there is no blending at all. In that case, change and rest will not have any share in being. As the visitor says, "It seems that agreeing to that destroys everything right away, both for the people who make everything change, for the ones who make everything an unchanging unit, and for the ones who say that beings are forms that always stay the same and in the same state. All of these people apply *being.* Some do it when they say that things really are changing, and others do it when they say that tings really are at rest." (252a)

The second option is to allow everything to blend with everything else.

> Also there are people who put everything together at one time and divide them at another. Some put them together into one and divide them into indefinitely many, and others divide them into a finite number of elements and put them back together out of them. None of these people, regardless of whether they take this to happen in stages or continuously, would be saying anything if there isn't any blending. (252b)

Theaetetus objects to this second alternative on the following ground: "Because if change and rest belonged to each other then change would be completely at rest and conversely rest itself would be changing." The visitor agrees, stating, "But I suppose it's ruled out by very strict necessity that change should be at rest and that rest should change" (252b). Having eliminated the first two alternatives, they agree on the third: "So the third option is the only one left…. Certainly one of the following things has to be the case: either everything is willing to blend, or nothing is, or some things are and some are not" (252d–e). The visitor then states:

> Since some things will blend and some won't, they'll be a good deal like letters of the alphabet. Some of them fit together with each other and some don't…. More than other letters the vowels run through all of them like a bond, linking them together, so that without a vowel no one of the others can fit with another (253a).

The visitor concludes the discussion on blending by saying:

> Well then, we've agreed that kinds mix with each other in the same way. So if someone's going to show us correctly which kinds harmonize with which and which kinds exclude each other, doesn't he have to have some kind of knowledge as he proceeds through the discussion? And in addition doesn't he have to know whether there are any kinds that run through all of them and link them together to make them capable of blending, and also, when there are divisions, whether certain kinds running through wholes are always the cause of the division? (253b–c)

Theaetetus responds, "Of course that requires knowledge—probably just about the most important kind. The visitor is surprised to ask, "Maybe we've found the philosopher even though we were looking for the sophist" (253c)? The discussion began by considering how the sophist mixes statements and appearances of being with nonbeing. But now it is the philosopher who is interested in how various kinds mix with others.

The dialogue then achieves a critical development in the following passages. The visitor states:

> We'll find that the philosopher will always be in a location like this if we look for him. He's hard to see clearly too, but not in the same way as the sophist…. The sophist runs off into the darkness of *that which is not,* which he's had practice dealing with, and he's hard to see because the place is so dark.… But the philosopher always uses reasoning to stay near the form, *being.* He isn't at all easy to see because that area is so bright and the eyes of most people's souls can't bear to look at what's divine. (254a–b)

Shortly after, the visitor and Theaetetus gain awareness of the being of *"that which is not*—which we were looking for because of the sophist" (258b). The visitor then considers the *that which is not* as follows:

> Then does it have just as much being as any of the others, as you said it did? Should we work up the courage now to say that *that which is not* definitely is something that has its own nature? Should we say that just as *the large* was large, *the beautiful* was beautiful, *the not large* was not large, and *the not beautiful* was not beautiful, in the same way *that which is not* also was and is not being, and is one form among the many *that are?* Do we, Theaetetus, still have any doubts about that? (258b–c)

With the agreement of Theaetetus, the visitor concludes as follows:

> You know our disbelief in Parmenides has gone even farther than his prohibition…. We've pushed our investigation ahead and shown him something even beyond what he prohibited us from even thinking about….Because he says, remember, *Never shall it force itself on us, that that which is not may be….* But we've not only shown that *those which are not* are. We've also caused what turns out to be the form of *that which is not* to appear. Since we showed that the nature of *the different* is, chopped up among all beings in relation to each other, we dared to say that *that which is not* really is just this, namely, each part of the nature of the different that's set over against *that which is.* (258c–e)

The visitor states that speech is one of the things that are and that to be deprived of speech would be to be deprived of philosophy. The visitor then considers whether nonbeing blends with belief and speech, which is clearly the case, due to the existence of false belief and false speech. The visitor then says, "When we've seen that clearly we can show that falsity is, and when we've shown that we can tie the sophist up in it, if we can keep hold of him" (261a).

Now that the dialogue has presented an account of the form of nonbeing, an important twist has occurred within the narrative. The Eleatic visitor and Theaetetus have been on the hunt throughout the dialogue to ascertain the exact character of the sophist, who mixes falsehood with truth, appearance with reality, and nonbeing with being. Their initial endeavor was hindered because of the visitor's commitment to his Eleatic persuasion; the visitor was attempting to describe falsehood and nonbeing while he himself remained tethered to the form of being. Thus anything the visitor could express grammatically about nonbeing was shallow and superficial, whereas the sophist was exploiting the vagaries of nonbeing at depth. So, in service of learning the truth about the sophist, the visitor abandoned his Eleatic strictures in order to reach the realm of the sophist's inaccessible confusion. But once willing to engage the question of nonbeing and to grant the existence of the form of nonbeing itself, whom does the visitor find first except the philosopher? It turns out that, like the sophist, the philosopher too is interested in mixing various kinds together. And now, in order to discover and exhibit the falsehood of the sophist, the visitor's philosophical intent is to blend truth with falsehood, reality with appearance, and being with nonbeing. Within the precincts of the form of nonbeing, the philosopher is now taking up the tools and engaging in the very expertise of the sophist. The visitor and Theaetetus have believed throughout their discussion that philosophers and sophists are entirely unlike. Yet contrary to their intent, their inquiry has now shown them to be alike. More specifically, the philosopher and sophist are opposites that are alike. As stated by Pogo, "We have met the enemy and he is us."

The polar opposition between philosophers and sophists is stated most obviously in the passage where they are described as *hard to see* (254a–b). Sophists are hard to see because they skulk about in the darkness of nonbeing and falsehood. Philosophers are hard to see because they approach the dazzling radiance of being and truth. Philosophers are lovers of wisdom, which they acknowledge not to possess, while sophists are teachers of wisdom, which they assuredly claim to possess. Sophists have mass appeal and thus exert significant influence in the polis. Philosophers do not have mass appeal, and thus do not exert influence in the polis. The influence of the sophist and the lack of influence of the philosopher both accrue to the injury of the polis. If the features of the philosopher and sophist could somehow be combined, allowing the positive features of both to emerge, then this unification would prove a great benefit to society. To this end, Plato wishes to show that philosopher and sophist are not merely polar opposites, as to remain separate, but dialectical opposites, as to be combined. Plato thus wishes to suggest that, in their most salient features, philosophers and sophists are both like and unlike. They are systematically opposite with respect to being and nonbeing, truth and falsehood, light and darkness. Based on this systematic opposition which separates them, Plato hopes to find some sort of dialectical synthesis by which they can be combined.

Just as the *Sophist* presents philosophers and sophists as dialectical opposites, so too the dialogue asserts the same opposition applies to being and nonbeing. The dialectical opposition of being and nonbeing is evident in the first instance in that the form of being is a source of light and truth to the philosopher, whereas the form of nonbeing is the source of darkness and falsehood to the sophist. In the analogy of the divided line, the visible and intelligible are not only contraries of one another but they comprise a dialectical system. Physical objects are visible but not intelligible, while forms are intelligible but not visible. Physical objects require forms to fulfill their lack of intelligibility, though forms have no such need of physical objects. Forms and physical objects are systematically like and unlike, making each the complement of the other. This dialectical opposition is advanced in *Republic* as representing the totality of perceptible things, whether visible or intelligible. However, the analogy of the divided line is remarkable in that Plato presents it as bearing only on epistemological concerns, making no reference to ontology. This is a strange omission. The ascent up the divided line is driven solely by the desire for greater intelligibility, there being no mention of increasing levels of existence. But as reflecting the architectonic plan Plato employs between one dialogue and another, the ontological dimension that is missing from *Republic* is supplied by *Sophist.* Physical objects in *Republic* lack intelligibility because, as explained in *Sophist,* they are closely associated with the form of nonbeing. Likewise, forms in *Republic* have a great deal of intelligibility because, as explained in *Sophist,* they are closely associated with the form of being. *Republic* provides a dramatic representation of the divided line by means of the allegory of the cave. *Sophist* provides a dramatic representation of the dialectic of being and nonbeing by means of the opposition between philosopher and sophist.

But if the dialectical synthesis of philosopher and sophist is intended to provide effective political leaders for the polis, what is the significance of the dialectical synthesis of being and nonbeing? From the dialectical point of view, this synthesis is the metaphysical basis for the world of becoming and change. That which comes to be does so because it has a share in being and in nonbeing. Likewise, that which changes has a share both in *that which is* and *that which is not.*

In order to develop the contrariety of being and nonbeing in *Sophist,* Plato had to clarify some basic grammatical misunderstandings on the part of Parmenides and the Eleatic school. Plato made explicit that predication is not limited to the mere identity of the subject, as thought by Parmenides, as if the predicate can express only what is autological to the subject. In opposition to Parmenides, Plato asserted that the predicate can express what is heterological to the subject. Thus it is not a contradiction to say that one has being and many have being, but the many have a being that the one has not. The pseudo-contradictions that worried the Eleatic school depended on interpreting predicates as absolute and without qualification. The preeminent stature of Parmenides does not owe to his view of logical grammar but to his prioritization of the absolute. If the absolute is not conveyed in Parmenidean grammar, then much of western philosophy has sought to find the Parmenidean absolute elsewhere. It is perfectly grammatical, then, to express *not* in relation to things that are.

Beyond clarifying the relation of subject and predicate, Plato also developed the schema of genus and species, which he describes in terms of his theory of division and collection, found in some of his later dialogues (such as *Sophist, Parmenides, Statesman, Phaedrus* and *Philebus).* Not only do different species exist within the genus but the genus itself has its own existence. The difference between *being* and *nonbeing* can thus be described as different species or *modes* within the genus of existence itself. Expressed more commonly, *being* and *nonbeing* are different modalities within the genus of modality. The contrariety of being and nonbeing becomes evident once they are analyzed as having common membership within the same genus or kind.

- First, being and nonbeing are like in that they are both species of the genus of modality. But they are unlike in that being is the positive species and nonbeing the negative. Stated differently, being is autological to the genus and nonbeing is heterological to it.
- Second, the visitor and Theaetetus deny the claim that everything in nature blends with everything else. On their limited initial view, some things in nature blend and others do not, and those things that blend do so according to their genus or kind.
- Third, the modal blending of being and nonbeing is possible even within the narrow limitations agreed upon by the visitor and Theaetetus. This blending is made possible because being and nonbeing belong to the same genus, but also in that they are opposite to one another within that genus. According to the logic of the dialogue, being and nonbeing are like, in that they belong to the same kind, but unlike in that one is positive and the other negative. The opposition of being and nonbeing within the same kind makes them suitable for blending, and thus the basis for becoming and change.
- Fourth, the modal blending of being and nonbeing implies that they are contraries of one another. Because the *Sophist* describes being and nonbeing as forms, this dialogue advances the claim that contrariety exists among the forms, and thus the forms are not strictly autonomous, selfsame and absolute.

The contrariety of being and nonbeing within *Sophist* prefigures the contrariety of forms as presented in *Parmenides.* The *Sophist* thus stands in relation to the *Parmenides* in the same relation as the One to the Many. This is a provocative image in that the problem of the One and the Many is a fundamental issue in these dialogues. For this and other reasons (discussed below), the logical progression between the dialogues seems to move from *Sophist* to *Parmenides*. (In terms of Socrates' age, the chronology of the dialogues advances from *Parmenides* to *Sophist.)*

Beyond the grammatical clarification of subject and predicate and the development of the relation of genus and species, there are two great philosophical theses advanced in Plato's *Sophist.* First, an account is given of the form of nonbeing, such that it is claimed to have its own existence and nature, having the same rank as such forms as the beautiful and the large. Second, the form of being and the form of nonbeing are shown to exist within the same scheme of contrariety as asserted of forms more generally in Plato's *Parmenides,* including the forms of being and nonbeing themselves.

In the discussion above, it was claimed that the theory of forms needed innovation if the *Parmenides* is to revise the theory to allow for the contrariety of forms. The discussion there pointed out the significance of Plato's description of the contrariety of forms as set within a context of "in no time at all." The account advanced here in *Sophist* give support to such a construction, since the existence of the genus transcends the existence of the various species within the genus. The genus maintains a stable existence, even though the species within the genus may be subject to contrariety. Relative to the species, the genus can be described as "in no time at all, though the species experience contrariety in real time."[21] Plato is clearly envisioning a common solution to the problem of the contrariety of forms in both *Sophist* and *Parmenides*,

[21] Indeed, some or all of the species could go extinct as a result of any manner of contrariety, though the genus could still be said to exist, though as an empty or null genus.

though the solution is expressed differently within the two dialogues. The account of genus in *Sophist* is more conceptual in that it is framed as part of a logical hierarchy. The account of "in no time at all" in *Parmenides* is more dynamical in that it superintends actual physical processes. In combining these two descriptions, the expression emerges of a "genus" that exists "in no time at all." This combined construction provides theoretical justification to the claim that forms can exist in a mutual state of contrariety while maintaining a hierarchal state of constant existence. This is the achievement of the *Sophist* and *Parmenides* as taken together.

Even further, this dual solution of *Sophist* and *Parmenides* with respect to imaginary time is also a precise instantiation of the hierarchal relation of forms as described in *Republic.* The theory of imaginary time in *Parmenides* has already been extended to *Republic*, and the preceding paragraph showed that a synthesis can be achieved between the epistemology of *Republic* with the ontology of *Sophist.* It thus becomes possible to take these three dialogues together as advancing a combined revision of the theory of forms; *Sophist* offers a theory of subject-predicate, *Parmenides* offers a theory of logic, *Republic* offers a theory of mathematics, and *Sophist* offers a theory of modality. *Parmenides* offers a theory of imaginary time that can be directly extended to *Sophist* and *Republic.* All of these theories are expressed in relation to forms. In Plato's estimation, these theories are not only mutually consistent but interwoven within his systematic revision of the theory of forms.

As an aside, it is significant to note that the historical figure Parmenides as described by the Eleatic visitor in *Sophist* is extremely different from the character Parmenides as depicted in *Parmenides.* The representation in *Sophist* is much closer to what is known of him from other sources, and so the eponymous character of the *Parmenides* was probably an invention of Plato, after Plato had presented him more faithfully in *Sophist.* As attested both in other sources and in

Sophist, Parmenides did not allow the assertion of *that which is not*. If the Eleatic visitor committed patricide in *Sophist* by admitting of *that which is not,* then the Parmenides of *Parmenides* committed suicide by allowing the same admission.

A fundamental concern of Plato's *Sophist* is to establish the analogy between the contrariety of being and nonbeing and the contrariety of philosopher and sophist. However, the contrariety of being and nonbeing is explicitly denied by the Eleatic visitor, and so this point requires special attention. Upon close reading of *Sophist,* the visitor seems to be unaware of wider considerations that bear on the discussion which are of deep significance to Socrates and Plato. The visitor states, "It seems that when we say *that which is not,* we don't say something contrary to *that which is,* but only something different from it" (257b). The visitor then makes the same point in greater detail:

> Nobody can say that this *that which is not,* which we've made to appear and now dare to say is, is the contrary of *that which is.* We've said good-bye long ago to any contrary of *that which is,* and to whether it is or not, and also to whether or not an account can be given of it. With regard to *that which is not,* which we've said is, let someone refute us and persuade us that we've made a mistake—or else, so long as he can't do that, he should say just what we say. (258e–259a).

The Eleatic visitor is clearly saying what is at odds with the present analysis. What is the meaning of this? Does Plato intend the Eleatic visitor to speak for him, or does he intend his readers to critique and see past the visitor's limitations? As the visitor says here, "Let someone refute us and persuade us that we've made a mistake."

Why does the Eleatic visitor fail to reach the same conclusion as the present discussion? The primary reason is that the visitor fails to recognize the sophist as like himself, thereby failing to recognize the contrariety between philosopher and sophist. For the visitor, the sophist is wholly other – a strange alien to be hunted down like a wild beast. But when the visitor has to begin to adopt the methods of the sophist in order to track him down, he fails to recognize that, in mixing being with nonbeing, he is practicing the very expertise of the sophist. Plato develops this analogy in detail, not only to apply to the philosopher and sophist generally but even more specifically to the visitor himself and the sophist. The visitor describes the angler as one who hunts for fish, and then says: “Well then, let’s use that model to try and find the sophist, and see what he is” (221c). The visitor describes the sophist as one who hunts other people (as paying customers). The visitor and Theaetetus then agree that the sophist has an expertise in hunting, like the angler. But Theaetetus seeks clarification on the sophist’s expertise in hunting. The visitor answers, “For heaven’s sake, don’t we recognize that the one man belongs to the same kind as the other?” Theaetetus asks, “Which men?” “The angler and the sophist.” “In what way?” The visitor answers, “To me they both clearly appear to be hunters.” Theaetetus asks, “We said which kind of hunting the angler does. What kind does the sophist do” (221d–e)? The visitor can draw analogies as to how the sophist is like a hunter and he can specify the different kinds of hunting that differentiate the angler from the sophist, but he cannot see how the analogy extends to himself to show that he is hunting the sophist. When Theaetetus asks “which men” are alike, the visitor sees the likeness between the angler and the sophist. But he might just as well have recognized himself as like the sophist. Certain things are readily apparent “to me,” he says. But the visitor’s perspective is egocentric. His analogies show him what others are like, yet he fails to recognize himself within the same analogies. He failed to recognize the likeness he shared with the sophist either with respect to hunting or with respect to mixing being and nonbeing. Because the visitor is unable to recognize the sophist as his dialectical opposite, he fails to recognize the contrariety between philosophers and sophists. And because he fails to recognize this contrariety, he also fails to recognize the contrariety between being and nonbeing.

But in addition to his ethical failure to recognize the sophist as like himself, the visitor also followed a logical path of egocentrism. He argued for the existence of the form of nonbeing by saying, "that *that which is not* really is just this, namely, each part of the nature of the different that's set over against *that which is"* (258e). Although the visitor argued for the ontological independence of the form of nonbeing, his theoretical justification for the form subjugated it as merely "different" from that which is. He never considered, for instance, that being is different from nonbeing, nor did he consider the sense in which being and nonbeing are the same. For this reason, he failed to allow the forms of being and nonbeing to encounter one another in mutual likeness and difference as dialectical opposites. The visitor was willing to adapt his logical outlook in order to catch hold of the sophist, but he failed to develop his logic adequately so as to understand the reciprocal relation he shared with the sophist. The visitor said, "The beast is complex and can't be caught with one hand" (226a). Yet he failed to gain an evenhanded understanding of the sophist. Accordingly, his development of logic falls short of the dialectical logic envisioned by Plato in which the relation of philosopher and sophist and of being and nonbeing is expressed in terms of contrariety. In this regard, Plato presents the visitor as an exemplar of the via positiva, pushing this tradition as far as possible in the direction of the negative while remaining short of the via negativa. The theoretical justification for the form of nonbeing as offered by the visitor is not to be confused with the theoretical justification given by Parmenides in the eight deductions. The visitor described the form of nonbeing as a half-truth. The complete truth, as suggested by Plato, involves blending opposites through a logic of contrariety.

The visitor discusses with Theaetetus the problem of failing to recognize how certain forms relate to others. One should not think that the same form is different in separate cases, or fail to recognize that different forms belong to one another because they are combined under a common form. As he says:

> Aren't we going to say that it takes expertise in dialectic to divide things by kinds and not to think that the same form is a different one or that a different form is the same? … So if a person can do that, he'll be capable of adequately discriminating a single form spread out all through a lot of other things, each of which stands separate from the others. In addition he can discriminate forms that are different from each other but are included within a single form that's outside them, or a single form that's connected as a unit throughout many wholes, or many forms that are completely separate from others. That's what it is to know how to discriminate by kinds how things can associate and how they can't…. And you'll assign this dialectical activity only to someone who has a pure and just love of wisdom. (253d–e)

The visitor claims that the ability to discriminate forms properly depends on a pure and just love of wisdom. Plato is demonstrating to his careful reader that the visitor's antipathy for the sophist has clouded his ability to reason properly. Plato intends the visitor to be a case study who shows the truth more than stating it, and who also states the truth more than attaining to it. Without realizing it, the visitor accurately describes the conditions of his own failure.

The sophist is not the only person who mixes reality with appearance. At the beginning of the *Sophist*, gods are described as disguising themselves as humans. In the discussion there, Socrates quotes from Homer in alluding to a passage where Odysseus disguised himself in order to observe the deeds of just and unjust people. The difficulty of distinguishing gods, philosophers, statesmen and sophists is the topic at the beginning of the dialogue. As Socrates says:

> Certainly the genuine philosophers who "haunt our cities"—by contrast to the fake ones—take on all sorts of different appearances just because of other people's ignorance…. Some people think they're worthless and others think they're worth everything in the world. Sometimes they take on the appearance of statesmen, and sometimes of sophists. Sometimes, too, they might give the impression that they're completely insane. (217c–d)

It is not only the sophist, but gods, heroes and philosophers who take on different appearances, as if the *that which is not* should conceal the *that which is*. But for Socrates, these changing appearances reflect the blended essences of the individuals involved – that is, the essences of different types of individuals do not cut clean. On this account, the mixing of being and nonbeing is not nearly so nefarious as what the visitor supposes.

When Theodorus first brought the Eleatic visitor to visit with Socrates and others, Socrates asks Theodorus the following:

> Did they (those from Elea) think that sophists, statesmen, and philosophers make up one kind of thing or two? Or did they divide them up into three kinds corresponding to the three names and attach one name to each of them? (217a)

Socrates believes that, in one way or another, the natures of statesmen, philosophers and sophists are somehow *blended.* In this and a number of other dialogues, Socrates believes the nature of the philosopher is not autonomous and autological, and so he continually seeks to find reciprocal and complementary (heterological) counterparts from within others. The unity of opposites between the philosopher and the sophist that emerges from the discussion between the Eleatic visitor and Theaetetus did not come as a surprise to Socrates, but was the very sort of explanation for which he posed the question at the beginning of the dialogue. The next step for Socrates is to consider how the combined nature of the philosopher and sophist might blend with the nature of the statesman.

The Eleatic visitor believes that some things blend and others do not. Most assuredly, he believes that forms do not blend. But in addition to forms, the visitor is convinced that his nature does not blend with the nature of the sophist. Given the view of Empedocles that nature combines and separates according to love and strife, Plato regards the visitor's alienation from the sophist as eristic, making the visitor's judgment impure and thus obscuring his ability to reason properly about the forms.

In considering the blending natures of statesmen, philosophers and sophists, Plato is considering the problem of how the philosopher is to gain influence within the polis. This aspiration does not reflect personal ambition on the part of the philosopher but the desire for wise and just governance for society. One of Plato's primary objectives is to ascertain what mixture is necessary among individuals and society in order to sustain a just polis. The synthesis of philosopher and sophist is simply a means toward achieving a synthesis of the philosopher and statesman. The fact that Socrates was condemned to death in his public trial shows what little political standing he had in the assembly. Similarly, Athens suffered defeat in the Peloponnesian War largely due to the infamous influence of Alcibiades, the chief advocate for the disastrous naval expedition to Sicily. From Plato's point of view, if Socrates and Alcibiades had developed a more constructive relationship (than what is described in *Symposium* and *Alcibiades),* then perhaps both Socrates and Athens might have been saved. But if a suitable dialectical relationship could not be achieved between Socrates and Alcibiades, then what other blends might be more efficacious? Plato's first rudimentary effort in this regard was the philosopher-king of *Republic,* a theory based on the myth of metals. Plato continued to pursue this objective through *Sophist, Statesman* and *Laws,* among other dialogues. But beyond the limited concern of political theory, the question of metaphysical, cosmological, psychological, epistemological, political and ethical blending is one that runs throughout the Platonic corpus. The rational basis for considering the question of blending is the dialectical unity of opposites. The blending of opposites is thus part of Plato's large-scale program, and so the attempted blending of philosopher and sophist and of being and nonbeing in *Sophist* is in no way incidental or peripheral to Plato's overall project. (However, it seems that Plato's attempted blending of philosopher-king and of philosopher-sophist-statesman were more philosophical exercises than real proposals. The sophist had dropped out of Plato's conception of the philosopher ruler in *Laws*, though the blending of light and darkness remained.)

Once Plato's philosophical program as described here is applied to the theory of forms, it becomes evident that there is far too much contrariety than what can be sustained from within the classical theory. Consequently, the classical theory must be abandoned, though Plato leaves it on display for those who do not see the need to move beyond it. Accordingly, to follow Plato into his revised critical theory of forms is to follow him into contrariety, negation and the via negativa. At the heart of this revised theory is the theory of contrariety and modality as found in *Republic, Sophist* and *Parmenides.* This revised critical theory is the basis of the study that follows.

CHAPTER 2

THE CRISIS IN MATHEMATICAL FOUNDATIONS

Upon completion of the second volume of his magnum opus in 1902, Gottlob Frege believed he had successfully reduced the theorems of arithmetic to first-order logic. But upon sending the volume to the printer, he received a letter from Bertrand Russell, congratulating him on his first volume, but also conveying the now famous Russell paradox. Frege was quick to realize the calamitous nature of this logical contradiction and immediately prepared an appendix for the second volume in which he announced the downfall of his theory. He began the appendix as follows:

> A scientist can hardly meet with anything more undesirable than to have the foundations give way just as the work is finished. In this position I was put by a letter from Mr. Bertrand Russell, as the work was nearly through the press.

Frege stated in the appendix that the consequence of the Russell paradox was not merely the ruin of his own theory but the seeming insurmountable difficulty it presents to any logical foundation whatsoever. In Frege's assessment, the paradox ran deep. In a letter to his wife, Russell described Frege's response by saying, "Arithmetic totters." And so began the crisis in mathematical foundations.

Frege and Russell founded the logicist school of mathematics. This tradition believed that mathematics generally and set theory specifically can be reduced to pure logic. After the onset of the set theoretic paradoxes, this view was cast into significant doubt. In *The Principles of*

Mathematics, Russell believed that "the contradiction" (his term for the Russell paradox) was caused by allowing a set or class to be a member of itself. He proposed a type theory that distinguished a set from its members. He wrote, "It is the distinction of logical types that is the key to the whole mystery."[22] Before the book was printed, however, he added Appendix B, which presented a different version of the theory, one that was inconsistent with the first. The appendix only "tentatively" asserts the second version of the theory, stating "it requires, in all probability, to be transformed into some subtler shape before it can answer all difficulties." By 1905 Russell had abandoned type theory, proposing three different theories: (1) the zigzag theory, (2) the limitation of size theory, and (3) the no-classes theory. He advanced these three without adopting any one of them. Then, in a brief note of February, 1906, he indicated he felt "hardly any doubt that the no-classes theory affords the complete solution of all the difficulties." Nonetheless, by 1908 in *Principia Mathematica* he had returned to the theory of types. Russell's theoretical inconstancy during this time period was an accurate reflection of his statement of 1904, "Everything now is chaos."[23]

In the early years of the foundational crisis, Frege and Russell believed in the universal applicability of logic. As described in the axiom of comprehension: for any definite property *p*, there exists a set *P* for which *p* is the defining property. On this view, the logical import of the universal quantifier is unbounded. This view contrasts sharply with algebraic logicists who, following DeMorgan, understand logical implication in terms of a "universe of discourse" in which algebraic operations are valid within certain limited sets, depending on where these sets exist in the cumulative hierarchy. In turning to his theory of types, Russell abandoned his earlier

[22] Bertrand Russell, *The Principles of Mathematics* (Cambridge: Cambridge University Press, 1903), §104.

[23] Russell, "ZN" unpublished manuscript, (Hamilton, Canada: McMaster University Archives), folio 760. The title page is missing from these worksheets; "FN" might have meant "Fundamental Notions." Russell wrote on fol. 762, "If this fails, arithmetic totters."

view of the universal scope of propositions. His revised type theory refers to a "range of significance" such that propositions are valid only within limited ranges, though Russell continued to treat sets of numbers as having universal scope.

Frege eventually abandoned belief that arithmetic can be reduced to logic. He concluded that any adequate foundation for arithmetic would have to be obtained through geometry. Similarly, Whitehead and Russell abandoned hope for the logical perfectibility of set theory. They wrote in 1925:

> Infallibility is never attainable, and therefore some element of doubt should always attach to every axiom and to all its consequences. In formal logic, the element of doubt is less than in most sciences, but it is not absent, as appears from the fact that the paradoxes followed from premises which were not previously known to require limitations.[24]

Russell wrote candidly of the disillusionment he suffered as a result of the unresolved crisis in mathematical foundations. He is often quoted as follows:

> I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than those that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable.[25]

L.E.J. Brouwer was the founder of the intuitionist or constructivist school of thought. In 1908 he wrote *The Untrustworthiness of the Principles of Logic* in which he argued that the

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[24] A. N. Whitehead and Bertrand Russell, *Principia Mathematica*, 2nd ed., 3 vols. (Cambridge: Cambridge University Press, 1925), 1:59.

[25] Bertrand Russell, *Portraits from Memory*, 1958; quoted in Morris Kline, *Mathematics: The Loss of Certainty* (Oxford: Oxford University Press, 1980), 229–30.

validity of mathematical logic is confined to finite terms and cannot be extended to infinite sets. Russell's paradox was predicated on the concept of universality, and so Brouwer regarded infinity and universality as the basis of the contradiction. Brouwer claimed that the law of the excluded middle cannot be sustained in infinite contexts, and that it was for this reason that logic had failed or become untrustworthy within the foundational crisis. He advanced the highly restrictive claim that mathematics must be limited to considerations of strict finite construction, never allowing an inference that cannot be completely grasped by finite human intuition. Because Brouwer's proposals were inadequate to reconstruct much of commonly accepted mathematics, his view remained a minority opinion, though his critique has been regarded as an important part of the foundational debate.

Errett Bishop, a member of the constructivist school, expresses the outlook of constructivism in the following quotation:

> Mathematics belongs to man, not to God. We are not interested in properties of the positive integers that have no descriptive meaning for finite man. When a man proves a positive integer to exist, he should show how to find it. If God has mathematics of his own that needs to be done, let him do it himself.[26]

David Hilbert founded what became known as the formalist school of mathematics. Giving an address in 1925, he described his program by saying, "The goal of my theory is to establish once and for all the certitude of mathematical methods.[27]" In the same speech, he complained of mathematical foundations as follows:

> Admittedly, the present state of affairs where we run up against the paradoxes is intolerable. Just think, the definitions and deductive methods which everyone learns, teaches and uses in mathematics, the paragon of truth and certitude, lead to

[26] *Foundations of Constructive Analysis* (New York: Academic Press, 1967), 2.

[27] David Hilbert, "On the Infinite," in Paul Benacerraf and Hilary Putnam, eds., *Philosophy of Mathematics* (London: Cambridge University Press, 1985), 135.

> absurdities! If mathematical thinking is defective, where are we to find truth and certitude?[28]

Hilbert sought to construct an absolute proof theory that would secure the legitimacy of mathematical reasoning that had been called into question. Mathematical proofs could do no better than achieve relative consistency, i.e., demonstrating a system consistent only insofar as another system is thought consistent. Hilbert realized that at some point, a mathematical system needed to step forward to provide a final basis of absolute consistency on which could hang the whole of mathematics. Otherwise, all mathematical proofs relied on a vicious circle of borrowed credibility. Unless this liability were corrected, mathematical reasoning could constitute nothing more than one big assumption. So what Hilbert wanted was an unhypothetical first principle that would guarantee all mathematical knowledge in rigorous terms.

Where would Hilbert turn to find the basis for an ultimate absolute within mathematics? In *The Foundations of Geometry* he had proved that Euclidean geometry is consistent if the real numbers are consistent. Thus his search for the final authority in mathematics led him to number theory, which he called "that purest and simplest offspring of the human mind." He proposed to demonstrate that, without appeal to any external evidence, number theory could prove its own validity. (If the proof relied on external argumentation, it would not be absolute, but relative.) Hilbert's concern to prove the absolute consistency of number theory extended to infinite as well as finite sets. Hilbert was an admirer of Cantor and so emphatically stated, "No one will drive us from the paradise which Cantor has created for us."[29] He even went so far as to say:

> What we have twice experienced, once with the paradoxes of the infinitesimal calculus and once with the paradoxes of set theory, will not be experienced a third time, nor ever again.[30]

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[28] Ibid., 141.
[29] Ibid.

[30] Ibid., 150.

Hilbert formulated four questions that constituted what has come to be known as "Hilbert's program." If all four questions could be proved in the affirmative, then number theory would have demonstrated its own veracity. Kurt Gödel definitively answered Hilbert's questions, though not in the way Hilbert expected. The success of Hilbert's program depended on each of his four yes-or-no questions being answered in the affirmative. Gödel resolved Hilbert's fourth question in the affirmative in his doctoral dissertation of 1929, but proved subsequently that the first three questions were answered in the negative, thus destroying Hilbert's program. What is more, Gödel stunned the mathematical community with what are known as his two incompleteness theorems of 1930 and 1931.

Gödel's first incompleteness theorem proved that any system of number theory must necessarily be incomplete. The proof is essentially a mathematical variation of the liar paradox. Gödel adapted the Liar within terms of first-order logic so as to say, "This statement is not a theorem." Gödel showed that this statement is expressible as a well-formed formula of first-order logic and that, further, it necessarily attaches to any logical system sufficient to generate a number theory. Thus no theory of number can shake free of this so-called "Gödel statement." If the Gödel statement were proved as a theorem it would imply a self-contradiction, since the statement says that it cannot be proven. But conversely, if the statement is left unproven, as it must be, then it is clearly true in saying that it is not a theorem, and thus it represents a true statement that cannot be proven. And thus, if every number theory has a truth that cannot be proven, then every number theory is therein incomplete. That is, there is a semantic truth for every number system that cannot be proven syntactically.

Gödel's second incompleteness theorem is often referred to as his "great theorem," and it is this latter proof that directly demolished Hilbert's program. To understand this theorem, consider the following: A witness in court swears to tell the truth, the whole truth and nothing but the truth. Though this oath does not guarantee a witness will tell the truth, it is considered meaningful for an honest person to take such an oath. Honesty should be willing to commend itself, and so a truthful person should be willing to make a self-declaration of truthfulness. In the same way, Hilbert believed that number theory ought to be able to make a self-declaration of its own truthfulness, or consistency. Because number theory was believed by everyone to be consistent, Hilbert proposed that its autonomous self-declaration of consistency would not only be meaningful, but would provide the breakthrough necessary for an absolute consistency proof. Not only did Gödel show this impossible, but true to the paradoxical nature of the Liar and his first incompleteness theorem, he demonstrated that if number theory were to prove its own consistency, it would thereby force its own inconsistency! It was this remarkable paradox that destroyed Hilbert's program. As Hermann Weyl emphatically stated, "No Hilbert will be able to assure us of consistency forever."[31]

Gödel's incompleteness theorems hit against two different targets, one internal and the other external. Internally, his theorems apply to first-order logic itself. Externally, they apply to number theory. What is more, these theorems apply to all branches of mathematics having at least the same power as elementary arithmetic. Even as Hilbert's program offered the hope of extending a single logical guarantee throughout the whole of mathematics, even so Gödel's incompleteness theorems disseminate incompleteness and doubt. Gödel's first incompleteness theorem proves that mathematical logic and number theory are incomplete. His second incompleteness theorem proves that first-order logic and number theory cannot prove their own

[31] Hermann Weyl, *Philosophy of Mathematics and Natural Science* (Princeton: Princeton University Press, 1949), 235.

consistency. Not only is mathematics unable to testify to the whole truth (first incompleteness theorem), it is not known to speak nothing but the truth (second incompleteness theorem).

Gödel's work has had a major influence on the development of mathematical thought. Many fundamental beliefs have had to be reexamined because of it, such as the doctrine of decidability. This notion meant that, in the final analysis, there were no gaps or hidden truths in mathematics, but that every well-formed mathematical question was fully accessible to investigation, leading to a conclusive answer, whether true or false. Hilbert articulated this belief as follows:

> As an example of the way in which fundamental questions can be treated I would like to choose the thesis that every mathematical problem can be solved. We are all convinced of that. After all, one of the things that attract us most when we apply ourselves to a mathematical problem is precisely that within us we always hear the call: here is the problem, search for the solution; you can find it by pure thought, for in mathematics there is no *ignorabimus* [we shall not know].[32]

After Gödel's incompleteness proofs were first published, additional general conditions of Gödelian incompleteness have become familiar, such as non-computability, partial decidability, and undecidability.

W. V. Quine wrote in 1961, with reference to the discovery of the Russell paradox, "I remarked earlier that the discovery of antinomy is a crisis in the evolution of thought. In general set theory the crisis began sixty years ago and is not yet over." Quine goes on to speak of Gödel's theorems as follows:

> Gödel's discovery is not an antinomy but a veridical paradox. That there can be no sound and complete deductive systematization of elementary number theory, much less of pure mathematics generally, is true. It is decidedly paradoxical, in the sense that it upsets crucial preconceptions. We used to think that mathematical truth consisted in provability.

[32] Kline, *Mathematics: The Loss of Certainty*, 259. A different translation appears in the full text: Hilbert, "On the Infinite," 150.

> Like any veridical paradox, this is one we can get used to, thereby gradually sapping its quality of paradox. But this one takes some sapping. And mathematical logicians are at it, most assiduously.[33]

Descartes had likened the teachings of philosophy to “towering and magnificent palaces with no better foundation than sand and mud,” and believed they could be bolstered by the foundations of mathematics, which he described as “strong and secure.”[34] But mathematical foundations of the twentieth century turned out very different from what Descartes had imagined. Zermelo-Fraenkel set theory emerged as the most widely adopted foundational account, but what philosophical tradition does ZF support? It turns out that ZF was developed with more of a view of avoiding paradoxes than of defending any philosophical program. As Davis and Hersh write:

> The work on this program played a major role in the development of logic. But it was a failure in terms of its original intention. By the time set theory had been patched up to exclude the paradoxes, it was a complicated structure which one could hardly identify with logic in the philosophical sense of ‘the rules for correct reasoning.’ So it became untenable to argue that mathematics is nothing but logic—that mathematics is one vast tautology.[35]

Davis and Hersh comment further on the philosophical implications of the foundational crisis as follows:

> Foundationalism, i.e., attempts to establish a basis for mathematical indubitability, has dominated the philosophy of mathematics in the twentieth century. . . . Formalism, intuitionism and logicism, each left its trace in the form of a certain mathematical research program that ultimately made its own contribution to the body of mathematics itself. As *philosophical* programs, as attempts to establish a secure foundation for mathematical knowledge, all have run their course and petered out or dried up. Yet there remains, as a residue, an unstated consensus that the philosophy of mathematics *is* research on the foundations of mathematics.[36]

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[33] W. V. Quine, *The Ways of Paradox and Other Essays* (Cambridge: Harvard University Press, 1976), 16–17.

[34] Descartes came to believe that mathematical foundations required grounding in his *Meditations.*

[35] Phillip J. Davis and Reuben Hersh, *The Mathematical Experience* (Boston: Houghton Mifflin, 1981), 333.

[36] Ibid., 345–46.

Morris Kline makes this same point even stronger. He describes the quest for an infallible body of reasoning as a grand illusion. He writes:

> The current predicament of mathematics is that there is not one but many mathematics and that for numerous reasons each fails to satisfy the members of the opposing schools. It is now apparent that the concept of a universally accepted infallible body of reasoning—the majestic mathematics of 1800 and the pride of man—is a grand illusion. Uncertainty and doubt concerning the future of mathematics have replaced the certainties and complacency of the past. The disagreements about the foundations of the "most certain" science are both surprising and, to put it mildly, disconcerting. The present state of mathematics is a mockery of the hitherto deep-rooted and widely reputed truth and logical perfection of mathematics.[37]

Imre Lakatos proposed a "philosophy of dubitability" in which he radically departed from the traditional understanding of mathematics, which he called "Euclidianism." His view of "quasi-empiricism" argued that the deductive methods used in mathematical discovery are intrinsically conjectural and fallible, inevitably giving place to uncertainty. He argued that mathematics is far too complex to be formally deduced from a priori axioms. Like the natural sciences, they must be inductively formulated through informal and a posteriori criticisms. In attempting to deductively construct their systems, mathematicians inevitably find themselves adding "first principles" after the fact, which purportedly should have been declared from the start. These "foundational" elements, however, could never have been anticipated until their consequences had come to imply their prior existence. The need for this midcourse adjustment can never be overcome in mathematics because it is a discipline faced with the potential of infinite growth. This effect, Lakatos insists, represents a crushing blow to deductivism, which claims to set forth all its axioms from the start. Despite the loss of predictability and certitude, he argued mathematicians must allow for induction, gathering their needed axioms along the way. He disparaged mathematicians who allow themselves to fall into inductivism without

[37] Kline, *Mathematics: The Loss of Certainty*, 6.

acknowledging it: "It is intriguing how mathematical logicians who are so squeamish about rigour, and who set out to achieve absolute certainty, can slip into the morass of inductivism."[38]

Lakatos wrote: "For more than 2000 years there has been an argument between dogmatists and sceptics. In this great debate, mathematics has been the proud fortress of dogmatism…. A challenge is now overdue." He showed how mathematical proofs fail to be irrefutable by becoming an ordeal of examples and counter-examples in which uncertainty gives way to certainty and then again to uncertainty. In his most famous work, *Proofs and Refutations,* he wrote: " mathematics does not grow through a monotonous increase of the number of indubitably established theorems, but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations." He argues the growth of mathematics "cannot be properly understood without understanding the method of proofs and refutations, without adopting a fallibilist approach."[39]

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[38] Imre Lakatos, *Mathematics, Science and Epistemology,* eds. John Worrall and Gregory Currie (Cambridge: Cambridge University Press, 1978), 2:17. Lakatos goes on to write, "Russell was probably the first modern logician to claim that the evidence for mathematics and logic may be 'inductive.'" Lakatos, 25. In 1924, Russell had written: "When pure mathematics is organized as a deductive system - *i.e.,* as the set of all those propositions that can be deduced from an assigned set of premises - it becomes obvious that, if we are to believe in the truth of pure mathematics, it cannot be solely because we believe in the truth of the set of premises. Some of the premises are much less obvious than some of their consequences, and are believed chiefly because of their consequences. This will be found to be always the case when a science is arranged as a deductive system." Bertrand Russell, *Logic and Knowledge,* ed.,R. C. Marsh (London: George Allen and Unwin, 1956), 325.

[39] Lakatos, *Proofs and Refutations,* John Worrall and Elie Zahar, eds. (New York: Cambridge University Press, 1976), 4–5, 140. Lakatos provides a number of quotes from leading mathematicians who have adopted a fallibilist perspective. He refers to Carnap, who in 1930 "still thought that 'any uncertainty in the foundations of the "most certain of all the sciences" is extremely disconcerting,'" but by 1958 was referring to "the impossibility of absolute certainty." In 1953, J. B. Rosser wrote: "if a system of logic is adequate for even a reasonable facsimile of present-day mathematics, then there can be no adequate assurance that it is free from contradiction. Failure to derive the known paradoxes is very negative assurance at best and may merely indicate lack of skill on our part." In 1963, Haskell Curry stated, "The search for absolute certainty was evidently a principal motivation for both Brouwer and Hilbert. But does mathematics need absolute certainty for its justification?" In 1939, Alonzo Church stated "there is no convincing basis for a belief in the consistency either of Russell's or Zermelo's system, even as probable." Lakatos says of Gödel, that in 1944 he "stressed that under the influence of modern criticism of its foundation, mathematics has already lost a good deal of its 'absolute certainty' and that in the future, by the appearance of further axioms of set theory, it will be increasingly fallible." Lakatos writes: "It will take more than the paradoxes and Gödel's results to prompt philosophers to take the empirical aspects of mathematics seriously, and to elaborate a philosophy of critical fallibilism, which takes inspiration not from the so-called foundations but from the *growth* of

Quine is also an advocate for an empirical understanding of logic and mathematics. On his view, foundations are more of an afterthought. He writes:

> Foundation ceases to be the apt metaphor; it is as if a frail foundation were supported by suspension from a sturdy superstructure. . . . We may look upon set theory, or its notation, as just a conveniently restricted vocabulary in which to formulate a general axiom system for classical mathematics—let the sets fall where they may.[40]

The quest for mathematical foundations attempts to determine whether or not mathematics is a fully objective science, such as reflecting necessary truths, or else whether it might only embody expressions of human language and creativity. Herman Weyl offers a striking statement as to this latter possibility. He writes:

> The question of the foundations and the ultimate meaning of mathematics remains open; we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization.[41]

Morris Kline describes the migration from the view of mathematics as metaphysical reality to the view of mathematics as human creation:

> Creations of the early 19th century, strange geometries and strange algebras, forced mathematicians, reluctantly and grudgingly, to realize that mathematics proper and the mathematical laws of science were not truths. They found, for example, that several differing geometries fit spatial experience equally well. All could not be truths. Apparently mathematical design was not inherent in nature, or if it was, man's mathematics was not necessarily the account of that design. The key to reality had been lost.[42]

William James advocated the emerging view of mathematics as human language. He wrote:

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mathematical knowledge." *Mathematics, Science and Epistemology,* vol. 2, 24-27, 42; see also, Lakatos, *Problems in the Philosophy of Mathematics*, vol. 1, (New York: North-Holland Publishing, 1972).

[40] Quine, *The Ways of Paradox and Other Essays*, 31– 32.

[41] Kline, *Mathematics: The Loss of Certainty*, 6.

[42] Ibid., 4.

> When the first mathematical, logical and natural uniformities, the first *laws,* were discovered, men were so carried away by the clearness, beauty and simplification that resulted, that they believed themselves to have deciphered authentically the eternal thoughts of the Almighty. His mind also thundered and reverberated in syllogisms. He also thought in conic sections, squares and roots and ratios, and geometrized like Euclid. He made Kepler's laws for the planets to follow; he made velocity increase proportionally to the time in falling bodies; he made the law of the sines for light to obey when refracted; he established the classes, orders, families and genera of plants and animals, and fixed the distances between them. He thought the archetypes of all things, and devised their variations; and when we rediscover any one of these his wondrous institutions, we seize his mind in its very literal intention.
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> But as the sciences have developed farther, the notion has gained ground that most, perhaps all, of our laws are only approximations. The laws themselves, moreover, have grown so numerous that there is no counting them; and so many rival formulations are proposed in all the branches of science that investigators have become accustomed to the notion that no theory is absolutely a transcript of reality, but that any one of them may from some point of view be useful. Their great use is to summarize old facts and to lead to new ones. They are only a man-made language, a conceptual shorthand, as someone calls them, in which we write our reports of nature; and languages, as is well known, tolerate much choice of expression and many dialects.
>
> The human arbitrariness has driven divine necessity from scientific logic.[43]

Mathematicians increasingly became freed up from constraints of mathematics as truth, and were thus free to produce mathematics as art. *Pure* mathematicians reveled in the desire to push back the frontiers of the bizarre and counter-intuitive. G. H. Hardy epitomized this spirit by writing in 1940:

> I have never done anything 'useful.' No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. . . Judged by all practical standards, the value of my mathematical life is nil; and outside mathematics it is trivial anyhow. I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. And that I have created something is undeniable: the question is about its value. The case for my life . . . is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.[44]

[43] William James, "What Pragmatism Means," delivered 1906–1907.

[44] Davis and Hersh, *The Mathematical Experience,* 85–86.

Contemporary mathematicians are of a split mind as to how real they believe mathematics to be. On the one hand they acknowledge the traditional imperative of grounding mathematics in some sort of objective context, and yet on the other they view themselves as creating mathematics as art in such a way as to resist or even defy absolutist treatment. The following statement of Nicholas Bourbaki (a collective pen name for an influential group of mathematicians during the 1950s and 60s) is often quoted as an example of the ambivalence mathematicians have on the subject.

> On foundations we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say, "Mathematics is just a combination of meaningless symbols," and then we bring out Chapters 1 and 2 on set theory. Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling that each mathematician has that he is working with something real. This sensation is probably an illusion, but is very convenient. That is Bourbaki's attitude toward foundations.[45]

Bertrand Russell described a conversation he had with Ludwig Wittgenstein, during the time he was helping Wittgenstein publish *Tractatus Logico-Philosophicus.* Russell was disturbed by Wittgenstein's rejection of any totalizing or universalizing claims of logic. Russell relates how he took out a sheet of paper and put three blobs of ink on it. "I besought him to admit that, since there were these three blobs, there must be at least three things in the world; but he refused resolutely." Russell explained Wittgenstein's refusal by adding, "He would admit that there were three blobs on the page, because that was a finite assertion, but he would not admit that anything

[45] J. A. Dieudonné, "The Work of Nicholas Bourbaki," *American Mathematical Monthly* 77 (1970): 145. Davis and Hersh describe the same ambivalence on the part of many mathematicians. They write: "Most writers on the subject seem to agree that the typical working mathematician is a Platonist on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all…. The typical mathematician is both a Platonist and a formalist—a secret Platonist with a formalist mask that he puts on when the occasion calls for it." *The Mathematical Experience,* 321–22.

at all could be said about the world as a whole."[46] The set theoretic paradoxes all involve, in one respect or another, reference to the *universal set*, or at least an unqualified appeal to "all sets." This universal conception is at the heart of the Russell paradox, the Cantor paradox, and the Burali-Forti paradox. Russell was intent on preserving the universal application of logic (as allowed for by the axiom of comprehension), even though he eventually granted the possibility that logical propositions might not be consistent in universal contexts, though he always insisted that numbers should be so. His diagnosis of the paradoxes was that they were all self-referential, and he hoped to avoid the paradoxes by eliminating this self-reference. But Wittgenstein's view was that the paradoxes were too large. As he wrote, *"The limits of my language* mean the limits of my world." [47]

What conclusions are to be drawn regarding the crisis in mathematical foundations? William James described the false ideal of mathematical and scientific perfection by saying that human arbitrariness has driven out divine necessity. As he and a number of other critics would have it, mathematics is nothing more than an extension of human language. As Wittgenstein wrote on a number of occasions, "Mathematics is a motley." But just as advocates have gone too far in describing mathematics as overly absolute and indubitable, so too detractors have gone too far in deriding it in deflationary terms. Galileo claimed that the book of nature is written in the language of mathematics. Although this language is clearly given to more freedom of expression than many had thought, cosmologists have nonetheless determined that the age of the universe is about 13.798 billion years old. Mathematics seems to conform to nature with too great a precision than to be thought arbitrary on a human level. The problem is that mathematics has too

[46] Ray Monk, *Ludwig Wittgenstein: The Duty of Genius* (New York: The Free Press, 1990), 182.

[47] Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, trans. D. F. Pears and B. F. McGuinness (New York: Routledge and Kegan Paul, 1974), 5.6.

much contrariety to belong in Platonic heaven, and yet is too objective and unimaginably complex to be likened to human arbitrariness. How real, then, is mathematics?

The discovery of fractal geometry within number systems presents a stunning counterpoint to those who have measured the integrity of mathematical logic on a linear scale. Because chaos theory is ubiquitous throughout nature, it may be that the crisis in mathematical foundations results from linear logic attempting to capture what is significantly alien to it. The present study proposes that the logic of chaos theory be adapted and applied toward the development of a nonlinear mathematical foundation. Such a foundation should allow for varying regimes of contrariety to coexist within a non-absolutist superstructure. That is to say, the present study advances the claim that the nature of mathematics may belong neither to divine necessity nor to human arbitrariness but, like all else of nature, to the mystifying reality of organized chaos. Mathematical foundations grounded in chaos theory would possess sufficient objectivity and beauty to counter the claim that mathematics is nothing but a motley artifact of social construction. So too, such foundations would render the mathematical superstructure unstable in such a way as to make it seem to totter from within the narrow scope of linear foundations. This proposal would make mathematical reality akin to the rest of nature, and would serve to advance chaos theory as a universal science.

CHAPTER 3

CONTINGENCY, POSSIBILITY AND CHAOS

Two of the most important schools of thought in probability theory are those of frequentists and Bayesians. The frequentist view is a physicalist or objective account that interprets probability as the relative frequency of outcomes measured or abstracted over long or infinite trials, especially on random outcomes, such as games of chance. The randomness in these cases results from the physical nature of the games or systems that produce neutral or fair results. A differing physicalist account is the propensity view that analyzes empirical outcomes of scientific interest where the outcomes exert a particular propensity, whether random or not (cf. Karl Popper). The Bayesian view considers the degree of confidence a person may have in a given circumstance, focusing only on the data at hand. This is a subjectivist account that deals more with propositional logic than repeated random outcomes. There are four main interpretations of Bayesian probability: (1) the classical Bayesian interpretation (Laplace); (2) the subjectivist interpretation (de Finetti and Savage); (3) the epistemic or inductive interpretation (Ramsey and Cox); and (4) the logical interpretation (Keynes and Carnap).

Leonard Savage, a leading Bayesian, wrote in 1954 that the foundations of statistics rest in turn on the foundations of probability, which as he described, are "as controversial a subject as one could name." He went on to write:

> It is unanimously agreed that statistics depends somehow on probability. But, as to what probability is and how it is connected with statistics, there has seldom

> been such complete disagreement and breakdown of communication since the Tower of Babel…. There must be dozens of different interpretations of probability defended by living authorities, and some authorities hold that several different interpretations may be useful, that is, that the concept of probability may have different meaningful senses in different contexts.[48]

Yet the divisions within probability theory have certainly proliferated since the time Savage made this statement. According to Jim Berger and I.J. Good, a combinatorial analysis of recognized Bayesian views produces an upper limit of 47,632 different positions. However, as Savage indicated, probability can be interpreted as having different senses at the same time, and so it does not follow that frequentist and Bayesian thought contradict each other but only that they analyze probabilities differently.

In 1964, Paul Cootner wrote *The Random Character of Stock Market Prices*. Cootner interpreted stock market data as conforming randomly to the standard bell curve. This idea was popularized by Burton Malkiel in his 1973 book, *A Random Walk Down Wall Street.* Malkiel simulated stock market data by adjusting a hypothetical stock up or down each day by half a point based on flipping a coin. Malkiel then presented his simulated data to a stock analyst, who urged him to buy the stock in question. Malkiel concluded that, because a stock market analyst was unable to differentiate a random simulated stock from real stock market data, the stock market can be interpreted as random. Martin Weber, Andrew Lo and Craig MacKinlay have all argued against the random walk interpretation, claiming that detectable trends within the data allow for a modicum of predictability. Lo and MacKinlay wrote *A Non-Random Walk Down Wall Street* in 2002.

Benoit Mandelbrot wrote “A Multifractal Walk Down Wall Street” in 1999. He writes, “The geometry that describes the shape of coastlines and the patterns of galaxies also elucidates how

[48] Leonard J. Savage, *The Foundations of Statistics* (New York: J. Wiley & Sons, 1954), 1–2.

stock prices soar and plummet."[49] Mandelbrot criticizes the classical financial models used throughout the twentieth century. He cites two main premises of standard portfolio theory with which he disagrees: that market events are independent of each other, and that the distribution of market activities conform to the standard bell curve. Mandelbrot argues that when the entire range of market behavior is taken into account, it is the multifractal and not bell curve that best describes market behavior. While he grants that standard portfolio theory is accurate about ninety-five percent of the time, it fails to address the precipitous extremes of sudden market volatility. Just because the weather might be moderate ninety-five percent of the time, a mariner would never ignore the possibility of a typhoon. Unlike models based on the bell curve, chaos theory pertains to both the regularities and irregularities of the market. Though multifractals "do not purport to predict the future with certainty," Mandelbrot writes, "they do create a more realistic picture of market risks." As he explains in greater detail:

> The mathematics underlying portfolio theory handles extreme situations with benign neglect: it regards large market shifts as too unlikely to matter or as impossible to take into account. . . . Typhoons are, in effect, defined out of existence. . . . According to portfolio theory, the probability of these large fluctuations would be a few millionths of a millionth of a millionth of a millionth. (The fluctuations are greater than 10 standard deviations.) But in fact, one observes spikes on a regular basis—as often as every month—and their probability amounts to a few hundredths. Granted, the bell curve is often described as normal—or, more precisely, as the normal distribution. But should financial markets then be described as abnormal? Of course not—they are what they are, and it is portfolio theory that is flawed.[50]

The principle of "self-affinity" is the means by which the stock market is seen as chaotic. As Mandelbrot explains:

> A fractal is a geometric shape that can be separated into parts, each of which is a reduced-scale version of the whole. In finance, this concept is not a rootless abstraction but a theoretical reformulation of a down-to-earth bit of market

[49] Benoit Mandelbrot, "A Multifractal Walk Down Wall Street," *Scientific American* (February, 1999): 70.

[50] Ibid.

> folklore—namely, that movements of a stock or currency all look alike when a market chart is enlarged or reduced so that it fits the same time and price scale. An observer then cannot tell which of the data concern prices that change from week to week, day to day or hour to hour. This quality defines the charts as fractal curves and makes available many powerful tools of mathematical and computer analysis.[51]

The issue between the random walk interpretation and the multifractal walk interpretation as concerning economic data seems to hang on the difference between a frequentist abstract interpretation versus that of a propensity interpretation, the latter being attuned to differentiate between data that is driven by real world pressures and data that is purely random. This point is particularly relevant in that Paul Cootner's work of 1964, which stressed the random nature of the stock market, was significantly influenced by Louis Bachelier's book of 1900, The Theory of Speculation. Cootner read Bachelier as promoting a random interpretation of economic data, though Bachelier's interpretation was actually modeled in terms of the stochastic dynamics of Brownian motion.

A major concern of the present study is to suggest that the foundational crisis of probability theory might be resolved by ongoing developments of chaos theory. If probability theorists have mistakenly described as random that which is described more accurately as chaotic, then there can be little wonder that they have met with so much controversy. The random walk probabilist who forces the bell curve on multifractal data is in much the same situation as the pre-Keplerian astronomer who wearily sought in the circle what belonged to the ellipse. But insofar as understanding of planetary motion has been revised from the circle to the ellipse, and then from the ellipse to a relativistic version of the ellipse, and then again from a relativistic version to a version both relativistic and chaotic, the question arises for empirical scientists generally as to how variegated and subtle is the path of discovery that leads from one distorted view of reality to another, until finally giving place to chaos. It is a striking fact that empirical scientists are now looking at realms previously considered as inhabited by randomness and are now finding evidence of chaos. As chaos theory continues to supplant the place of randomness, the question arises as to whether the

[51] Ibid., 71.

purely random actually exists in nature, or whether it is a figment of the imagination. It may be that the concept of the purely random relates to chaos theory in the same way that Newtonian physics relates to relativity.

But to press beyond differentiating randomness and chaos theory, what can be said about the logical foundations of probability theory directly, without specific interest as to the determinacy or randomness of the events under investigation? Of primary significance here is Hume's critique of causation and induction. Because Hume's critique and Kant's response to it are central to the present study, it will be described here in some detail. Central to Hume's critique is the distinction known as Hume's Fork. This fork classifies all the endeavors of human reason as either part of "relations of ideas" or "matters of fact." Relations of ideas are necessary formal truths which bear on such issues as mathematics, though do not say anything about the real world. Matters of fact relate to reasoning concerning the empirical world, which can be known (at least in part) inferentially through cause and effect. Hume claimed that these categories exhaust all human reasoning and are mutually exclusive of one another.

In his *Enquiry Concerning Human Understanding,* Hume critiques causality and induction in Section IV, Part II. He begins by asking two questions that summarize the discussion from Part I, and then a third question that introduces the problem to be considered in Part II. First, what is the nature of all our reasoning concerning matters of fact? Answer: the relation of cause and effect. Second, what is the foundation of our reasoning concerning cause and effect? Answer: experience. Third, what is the foundation of all conclusions from experience? The negative answer given here is that such reasoning lacks rational justification. (Hume's positive answer is given in Section V.)

Experience teaches that certain types of causes have always been followed by certain types of effects, and thus the expectation is acquired that, for future events, these same causes will be followed by the same effects. But Hume argues that these propositions of experience and expectation cannot be bound to each other with logical certainty. He explains that Nature hides her secrets from us so that we are never allowed to see the hidden forces at work in natural processes. Thus when we see a set of appearances in the future that matches a set of appearances in the past, we have no way of knowing rationally that the underlying dynamics continue to be at work in the different situations. Despite the lack of logical connection between experience and expectations, we commonly act as if these two are connected. Hume writes, "Now this is a process of the mind or thought, of which I would willingly know the foundation." Hume's intent is to deny that this foundation can be demonstrated with logical certainty. Thus Hume's Fork is defended inasmuch that contingent matters of fact cannot be analyzed in terms of logical necessity.

Hume realizes that many individuals will believe that he is underestimating the reasonableness of expectation, and so he writes, "But if you insist that the inference is made by a chain of reasoning, I desire you to produce that reasoning." (By "reasoning" Hume means an account that is logically certain.) He grants that inasmuch as his line of questioning is yet new, it may be doubtful whether an initial failure would actually imply that the specified result does not exist. "For this reason, he writes, "it may be requisite to venture upon a more difficult task; and enumerating all the branches of human knowledge, endeavour to show that none of them can afford such an argument." In order to encompass all these branches of human reason, Hume reasserts the primacy of his division between relations of ideas and matters of fact, holding that they exhaust all the branches of human reason.

Hume explains that the expectation of future events cannot be united to past experience through logical demonstration since there is no logical certainty that the future will resemble the past. He writes: "That there are no demonstrative arguments in the case seems evident; since it implies no contradiction that the course of nature may change, and that an object, seemingly like those which we have experienced, may be attended with different or contrary effects." Since then the desired assumption does not belong among the relations of ideas, perhaps it is a matter of fact. Hume then states that all matters of fact or of real existence are known through the relation of cause and effect; that this relation is itself known entirely through experience; and that "all our experimental conclusions proceed upon the supposition that the future will be conformable to the past." He thus concludes: "To endeavour, therefore, the proof of this last supposition by probable arguments, or arguments regarding existence, must be evidently going in a circle, and taking that for granted, which is the very point in question." In explicit terms, it seems from past experience and from experimental sciences that the claim "the future will resemble the past" has always been correct. But this observation from experience cannot be used to prove the general conclusion that "the future will always resemble the past" without begging the question. Thus inductive reasoning about matters of fact or real existence cannot be rationally justified.

Hume's purpose is certainly not to advocate a reckless abandon in regard to the operations of nature, for as he insists, "none but a fool or madman will ever pretend to dispute the authority of experience, or to reject that great guide of human life…." But, as he continues, this fact does not supplant the philosophical curiosity to "examine the principle of human nature, which gives this mighty authority to experience, and makes us draw advantage from that similarity which nature has placed among different objects." As a thoroughgoing empiricist Hume is seeking to foil high rationalism in its attempt to provide a deductive account of necessary causal connections. This is his target when he states, "I cannot find, I cannot imagine any such reasoning. But I keep my mind still open to instruction, if any one will vouchsafe to bestow it on me." Then, turning against his fellow empiricists Hume reasserts that no amount of experience can infer logical connection between sensible qualities that are conjoined with what is taken to be their secret hidden powers. Hume concludes that philosophical reason cannot improve on what nature teaches by experience to stupid peasants, infants and brute beasts.

Hume's critique invites comparison to the science of induction as described within the formal probability calculus.[52] Hume claims that the future cannot be known to resemble the past with logical certainty, unless one argues in a circle. Yet the calculus of classical probability assumes the axiom of total probability, thereby assuming the homogeneity of all past, present and future events to be considered. How does this calculus compare to Hume's critique? Insofar as the probability calculus is entirely formal, Hume would not find it objectionable, but would classify it within the relations of ideas. But does not this calculus pertain to real world phenomena? This is the difference between frequentist and propensity theories. Frequentists describe probabilities merely in terms of relative frequencies, though propensity theory describes probabilities in terms of underlying causes. There is a complicated set of considerations here. The present discussion seeks to argue both for and against Hume, and likewise for frequentist and propensity theories. (Critiquing and engaging Bayesian probability is more complicated, and so will not be undertaken here.)

The discussion of transfinite probability proposed below in chapter 4 is expressed in terms completely favorable to frequentist thought. To this extent, the present discussion is affirming of frequentism. But insofar as transfinite probability rejects the axiom of unitary total probability, the present discussion rejects the modern probability calculus held in common by classical expressions of frequentist, propensity and Bayesian thought (although any of these theories can be expressed in unclassical terms). However, the rejection of unitary total probability is consistent with Hume's insistence that the future cannot be known (in logical terms) to resemble the past.

[52] A. N. Kolmogorov axiomatized probability in 1933, This formal system is critiques below in chapter 4.

But there is an interesting twist here. Hume claims that Nature hides her secrets, so that empirical investigation is never allowed to witness Nature's hidden powers of causality. But for Hume, this claim must be taken as a matter of fact (since relations of ideas have no relevance for the real world), and so the claim must be taken as contingent. In that case, it is possible that Nature might at one time disclose secrets formerly kept hidden. If Hume's future does not necessarily resemble his past, then a secret power of Nature's constancy hidden in his past and present might become revealed in his future. Such a power of Nature's constancy might have been discovered in the last two decades of the twentieth century by chaos theory. In arguing that Nature keeps her powers hidden, Hume realized he must enumerate the various branches of learning, showing that Nature's secrets are not disclosed by any of the empirical sciences. But Hume's future knows of sciences he knew not of. Because the effects of chaos are so ubiquitous and consistent throughout nature, the hypothesis has become possible that chaos theory is now revealing an intimation of Nature's former secrets. This revelation might indicate that Nature's secret of cause and effect operates consistently and uniformly through chaos theory. If this hypothesis can be adopted as a working hypothesis or even confirmed theoretically (perhaps within a quasi-empirical context), then a principle of induction would result such that the future could be regarded as resembling the past.

There is another implication here for Hume. The empirical discovery of the new science of chaos theory offers a uniform and consistent presentation of cause and effect as exemplified in fractal geometry, both in terms of number theory and of random or near-random data pertaining to the stock market. Chaos theory thus seems to be so ubiquitous as to subsume fundamental categories of modality, including necessity, determinism, free will and contingency. But such a broad modal reach implies the collapse of Hume's Fork and Hume's modal theory. Hume claims relations of ideas and matters of fact are mutually exclusive of each other, this exclusivity owing to the extreme difference between necessity and contingency. Hume grants that mathematics and number theory are necessary truths reflective of relations of ideas. But insofar as chaos theory is regarded as inclusive of both necessary and contingent realities, it spans and thereby unifies a modal divide that Hume thought to be absolute and insurmountable.

The claim was made above that, in denying the axiom of total probability, the present study is consistent with Hume's critique of induction that the future cannot be known to resemble the past. But then because of the ubiquitous effects of chaos theory, the proposal was advance that chaos theory might reveal the secrets of nature regarding cause and effect, thereby giving support to the inductive belief in the resemblance of past and future. In taking these two claims together, the present study agrees with Hume analytically that the future cannot be known to resemble the past, but yet offers support to the claim that such resemblance might be inferred synthetically. As based on other proposals offered within the present study, such a synthetic judgment would be a posteriori and would not be absolute.

Not only Hume, but the vast majority of western philosophy has regarded the modal divide between necessity and contingency as absolute and insurmountable. Kant's fourth antinomy engages this modality in an interesting manner. The antinomy considers whether the universe was created by a necessary being or whether it emerged contingently as a matter of chance. Kant believed that these opposing views are equally valid, and that this controversy arises because the question seeks to transcend the limits of possible experience. Kant claimed that speculative questions such as this (and also the question of determinism vs. free will) fall out of bounds of the proper scope of reason, and thus his antinomies are intended to chasten reason so as to conform to the critical metaphysics of his transcendental idealism. But because chaos theory has now been discovered to apply to number algorithms and probabilistic data, then empirical evidence now suggests that chaos applies equally well to both necessary and contingent realities. If there is an absolute and insurmountable divide between necessity and contingency, then this divide is unknown to chaos theory. The metaphysical polar opposites of Kant's fourth antinomy are thus subsumed within the modal expanse of chaos.

For Kant, the antinomies chasten[53] reason by delimiting the scope of reason, though not its absolutist character. On his view, if absolute knowledge is not possible with universal scope, then absolute knowledge must remain possible in describing the necessary conditions of possible experience. Contrary to Kant, Neoplatonists of the via negativa frame the antinomies quite differently, interpreting them as reflecting the need for transcendental synthesis. From this point of view, Kant's antinomies do not result from the quantitative mistake of extending reason beyond its proper limits but from the qualitative mistake of treating reason as absolute. From such a perspective, members of the via negativa are in a position to offer a Hegelian response to Kant's antinomies. The dialectical outlook of the via negativa seeks to combine being with nonbeing and necessity with contingency.

Because number and probability both suffer from a foundational crisis, the mathematical conceptions of necessity and contingency remain theoretically obscure. Yet because these modalities are opposing species within the genus of modal logic, their combined predicament is a

[53] Jeffrey Tlumak describes Kant's metaphysics as "suitably chastened," and describes the delimited nature of Kant's theory as follows: "The unbridled yea-sayers are the rational dogmatists, who are convinced that with proper method pure reason can know all sorts of substantive truths about ourselves, the world, God, and the relations among the three. Theoretical reason, whose interest is knowledge, discovers that there *are* several significant structural facts about the world that we can know with certainty, but precisely given the conditions for the possibility of such knowledge, theoretical reason has strictly definable *limits* too, limits which require that such knowledge apply only to objects of possible experience. Its natural urges to act unanchored in possible experience have to be curbed to avoid all sorts of illusions. But this limiting of theoretical reason uniquely enables practical reason to function defensibly; reason in its practical employment is interested in how we should live, including morally and politically. Such practical thinking presupposes free will, which properly delimited theory can neither establish nor rule out." *Classical Modern Philosophy: A Contemporary Introduction* (New York: Routledge, 2007), 245.

means of progress. There is now hope, precisely because both necessity and contingency are involved in equal confusion, and both have affinities with chaos theory. In so far as one of them is clarified, then perhaps the other will be too. The discussion below in chapters 4–6 seeks to provide a dialectical synthesis of necessity and contingency as expressed within set theory and probability. This synthesis rejects the belief that number and probability are absolute and autonomous but exist in a relation of contrariety so as to be combined within a unified theory of modality. This proposal reflects the understanding of Neoplatonists of the via negativa as described above in chapter 1.

CHAPTER 4

TRANSFINITE PROBABILITY

In reminiscence of Hilbert's hotel (an imaginary hotel with an infinite number of rooms), consider a similar thought experiment in which a casino has a countably infinite number of roulette tables. Consider that the casino management performs the following experiment.

1. The casino management wishes to calculate the odds for individual gamblers who wager not simply on a single roulette outcome but on an entire countably infinite sequence or *string* of roulette outcomes. Accordingly, the casino adopts procedures for selecting, recording and placing bets on these infinite *target strings* prior to spinning the roulette wheels. Such target roulette strings might be determined randomly or else in terms of decimal expansions of particular irrational numbers, such as the decimal expansion of π, the square root of prime numbers, etc.
2. The infinite number of roulette wheels on the casino floor are spun an infinite number of times. Each roulette table generates an infinite string of actual outcomes.
3. The actual strings of roulette outcomes on the casino floor are compared to one particular target string and to the probability space of the entire casino. The casino then calculates the odds and establishes the amount of payouts for these infinite wagers.

The fundamental question arising from this experiment is: what is the probability that the infinite number of roulette wheels, each of which produces an infinite string of outcomes, will produce at least one exact match to the infinite target string, beginning with the first result and continuing to infinity? This question will be referred to as *the casino question*.

The casino question is interesting in that it yields as many as three arguable solutions. Each of these solutions will be considered here. As for the first, one which represents a pre-critical or layperson's approach to probability, the casino question can be analyzed by scaling the question down from two dimensions of infinity to one; that is, where an infinite number of roulette wheels are compared to finite target strings (as opposed to infinite target strings). On this basis, let the first target in the target string be 27. Of the infinite roulette wheels, how many will match 27 on the first spin? Since the chance of any one roulette wheel producing a 27 is 1 out of 38, the answer is one thirty-eighth of infinity, a value equal to infinity. Similarly, there will be an infinite number of roulette wheels matching the target string in the second, third and any *n*th position. But if the first, second and third positions of the target string each has an infinite number of roulette wheels corresponding to it, how many roulette wheels will correspond to the sequence of all three targeted outcomes? As before, the answer is an infinite number. And so the generalization seems to follow that for any finite number of targeted outcomes, there will be an infinite number of roulette wheels corresponding exactly to the target string. If then the logic of finite strings can be extended to strings of infinite length, there should be an infinite number of roulette wheels whose outcomes correspond to a target string of infinite length. Thus according to this reasoning, the answer to the casino question is probability 1. That is, an infinite number of roulette wheels that are spun an infinite number of times will inevitably produce a match with a specified target string of infinite length. This view agrees with naive intuition that infinity implies totality, that infinite activity exhausts all possibilities.

The second solution, given by classical probability[54] (as developed by Kolmogorov), maintains that the answer to the casino question is probability 0. The countably infinite strings of actual roulette outcomes on the casino floor relate to the total probability space of the casino in the same way as a countably infinite number of isolated real numbers relate to the real number interval [0, 1]. Because such a countable infinity of real points has zero measure in the real interval [0, 1] (according to the measure theoretic terms of classical probability), classical probability treats these isolated points as vanishing magnitudes. In the same way, classical probability interprets the countable infinity of roulette outcome strings on the casino floor as lacking measure theoretic space and thus, as vanishing magnitudes, having probability 0. Thus if an infinite number of countably infinite roulette outcome strings (all having probability 0) are compared to a specific target string of countably infinite outcomes, the probability of finding an exact match from among such vanishing magnitudes is 0.

The third solution, non-classical in nature, gives rise to a theory of transfinite probability. This demonstration can be accomplished by patterning the casino question after Cantor's diagonalization method of 1891, a method which proves that the real numbers are nondenumerable (or uncountable). Because Cantor's transfinite theory is central to the present discussion, it will be described here in some detail. It is interesting to note that Cantor's original diagonal argument did not pertain strictly to the real numbers themselves. Cantor first proved that the real numbers are uncountable in 1874, but this proof came under the attack of Leopold Kronecker, a finitist mathematician who rejected the view that irrational numbers have decimal expansions extending to actual infinity. In response to Kronecker, Cantor sought a proof for the

[54] The term "classical probability" is used both in a philosophical, interpretive sense and in a mathematical, metric sense. Throughout the present study, the term is used in the mathematical sense to refer to A. N. Kolmogorov's formal axiomatization of probability theory in terms of measure theory. This classical theory of mathematical probability is not intended as for or against any of the opposing philosophical interpretations of probability.

existence of uncountable sets as such, "one which is independent of the consideration of irrational numbers," as Cantor wrote. Nevertheless, Cantor's diagonal proof easily extends to irrational numbers as canonically understood, and thus serves to legitimate Cantor's earlier proof. But it was the diagonalization and related techniques concerning the power set which Cantor first articulated in 1891 that provided the original impetus for his development of an infinite hierarchy of transfinite numbers. Cantor wrote of his diagonal proof as follows:

> This proof seems remarkable not only because of its great simplicity, but also because the principle which it follows can be extended directly to the general theorem, that the powers of well-defined sets have no maximum or, what is the same, that in place of any given set *L* another set *M* can be placed which is of greater power than *L*.[55]

Cantor's presentation of the diagonalization procedure considers a set in which elements $E = (x_1, x_2, \ldots, x_n, \ldots)$, where each $x_n = m$ or w. Cantor then proved that the set of all such elements was uncountable. He began by producing a countably infinite table of elements E_μ in terms of the following array in which $a_{\mu, \nu}$ is either m or w:

$$
\begin{aligned}
&E_1 = (a_{11}, a_{12}, \ldots, a_{1\nu}, \ldots) \\
&E_2 = (a_{21}, a_{22}, \ldots, a_{2\nu}, \ldots) \\
&\vdots \\
&E\mu = (a_{\mu 1}, a_{\mu 2}, \ldots, a_{\mu\nu}, \ldots) \\
&\vdots
\end{aligned}
$$

Then Cantor defined a new series E_0 of $b_1, b_2, \ldots b_\nu, \ldots$ where E_0 was to be defined as a diagonal series on the array as having coordinates $a_{11}, a_{22}, \ldots, a_{\mu\nu}, \ldots$. Each value $b_{\mu\nu}$ of E_0 was to be m or w, determined so as not to equal the values of E_μ at $a_{\mu\nu}$. Thus for whatever E_μ might be considered, $E_0 \neq E_\mu$ at the νth coordinate. As a consequence, the diagonalization series E_0 proves that the set of all possible series of E_μ is uncountable. After proving this, Cantor immediately

[55] Joseph Warren Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton: Princeton University Press, 1990), 165–66.

extended the proof to apply to real numbers.[56] (The diagonal proof is typically described as a proof by contradiction. According to this description, Cantor assumed (contrary to what was to be proved) that the totality of all possible series E_μ could be enumerated in a countable table, an assumption that was then proven false by the diagonalization.)

To apply Cantor's diagonal proof to the real numbers, it can be seen that every real number can be interpreted as an infinite binary sequence of zeroes and ones and then interpreted as a point in any real number interval, such as [0, 1). On this basis, the set of real numbers can be substituted into the place of the series of E_μ in the preceding discussion. Just as a diagonal series was defined there, proving the set to be uncountable, so too a diagonal can be defined across the set of real numbers. The diagonalization procedure can also be applied to base 10 or other systems. Cantor designated the countable infinity of natural and rational numbers as $\aleph_0$, and then demonstrated that the uncountable infinity of real numbers has cardinality $2^{\aleph_0}$ (discussed below).

To adapt Cantor's diagonalization to the casino question, all that is needed is to substitute the strings of roulette outcomes into the place of real numbers in the preceding discussion (whether in binary or decimal form). Once this is done, a diagonal can be drawn across the roulette outcomes, just as Cantor did with the series of E_μ and the real numbers. The countable array of E_μ is referred to as the diagonal set, whereas the complete uncountable series is referred to as the anti-diagonal set. Cantor's diagonal set, diagonalization, and anti-diagonal set provide a complete modeling of roulette outcome strings, their diagonalization and the anti-diagonal set of potential outcome strings. Thus a demonstration is given as to the disparity between the

[56] Ibid., 165–68.

countable infinity of the diagonal set of roulette outcomes on the casino floor and the anti-diagonal set of the uncountable infinity of their total probability space.

Note that the expressions “roulette outcomes” or “outcome strings” refer to the actual outcomes that result “on the casino floor.” The expression “potential outcome strings” refers to the set of all outcome possibilities as expressed mathematically within the probability measure space that measures the probabilities of the actual outcomes. The diagonal argument proves that there is not a 1-1 mapping between the diagonal set of outcome strings on the casino floor and the set of their potential outcomes strings within the anti-diagonal set of the measure space that pertains to the casino.

Because of the finitist-infinitist controversy involving different conceptions of rational and irrational numbers, it will be helpful to differentiate these number systems in the most straightforward noncontroversial terms possible. Accordingly, rational numbers can be defined as having finite decimal expansions, except for those that repeat, and irrationals as having infinite decimal expansions. The $\aleph_0$ casino has been developed thus far as isomorphic to all real numbers, taking rational and irrational numbers together. In order to allow the casino discussion to represent natural, rational and irrational numbers as effectively as possible, the $\aleph_0$ casino can be modified to contain not only an infinite number of random devices each of which produces an infinite outcome string, but also to contain an infinite number of random devices each of which produces finite outcome strings of arbitrary length, including null activity. Diagonals can be drawn across finite outcome strings just as with infinite outcome strings. (The effect in either case is to point to a greater totality than previously defined.) With this supplementation, the probability space of the anti-diagonal set that pertains to the $\aleph_0$ casino is isomorphic to the real numbers. This probability space can also allow for partitions that are isomorphic to rational and irrational numbers, whether of the [0, 1) interval or of the entire continuum (depending on the metric of the probability space).

Though the casino question was initially framed in terms of roulette wheels, it can be extended to any sort of random event. For instance, the casino question can be restated in terms of tossing coins, though this would be a less familiar way to depict an idealized casino. This extension is not limited to random events involving replacement (such that the same event can occur repeatedly) but also to events involving non-replacement, such as games of cards. Thus by further extension, the casino discussion provides a comprehensive model by which random phenomena of all sorts can be understood in terms of countably infinite casino outcome strings, giving rise in turn to a diagonal procedure. Thus any sort of random phenomena can be treated by means of transfinite probability. The infinite construction of the casino question implies that there is no ultimate difference as to whether random events within the casino have chances of 1 out of 38, 1 out of 6, 1 out of 52, etc. Accordingly, casino outcome strings can be normalized to binary or decimal expression and modeled within the probability space of the real interval [0, 1). The $\aleph_0$ casino designates this first transfinite casino, which is limited to $\aleph_0$ random devices each of which produces $\aleph_0$ outcomes.

Once casino outcome strings are normalized within the probability space of the [0, 1) real interval, the question can be addressed as to whether the cardinality of potential outcome strings produced by the casino diagonalization is the same as of the cardinality of the real numbers of the [0, 1) interval, namely, $2^{\aleph_0}$. The isomorphism between the anti-diagonal probability space of the $\aleph_0$ casino and the real numbers raises the question as to how exact is the isomorphism between the two. This question is particularly pressing in light of the continuum problem, in which it has not been determined whether the cardinality of points on the real continuum is the first cardinal number to follow the cardinality of natural numbers or if some other cardinal number might intervene. That is, does $2^{\aleph_0}$ (the number of points on the continuum) equal $\aleph_1$ (the first cardinal number after $\aleph_0$) or some other transfinite number? Because the isomorphism here involves extremely distinct species of mathematical entities (numbers vs. probabilities), it might be thought that the mere isomorphism of a diagonal on numbers and on probabilities is insufficient to guarantee that the resultant cardinalities are the same in each case. The logic of indeterminate probabilities might somehow be related to the indeterminacy of the continuum problem., implying that the diagonal on probabilistic outcome strings might not lead to any particular stable cardinality, or if so, that this cardinality might not be the same as what results from an isomorphic diagonal on numbers.

In comparing the isomorphism of the diagonals of transfinite numbers and of transfinite probability, it is immediately clear that both cases are closely connected to the power set function. This commonality owes to the significance this function plays within combinatorial analysis. The power set function of a set *A* is expressed as $P(A)$, which equals 2^n, where *n* equals the number of elements in *A*. The base number 2 in this expression represents bivalence (true or false, yes or no, 1 or 0), such that the exponent *n* signifies the number of times the bivalent number 2 is multiplied on itself, thereby yielding 2^n. With respect to the set {a, b, c}, the power set function specifies that there are 2^3 combinatorial selections from this set. The set of these selection possibilities is as follows: {∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}. Thus the power set function is of crucial significance to combinatorial analysis. When Cantor applied this combinatorial sense to set theory, he spoke of the power set of a given set *A* as the set of all subsets of *A*. Additionally, he showed through what is known as Cantor's theorem (first presented in 1891) that the power set function can be extended to infinite sets (assuming infinite sets are well ordered). In applying the power set function to the set *N* of natural numbers (where *N* has cardinality $\aleph_0$), the resultant $2^{\aleph_0}$ is an arithmetic expression referring to the cardinality of all combinatorial permutations of the natural numbers, namely, the set of all subsets of the natural numbers.[57] In this way, Cantor's diagonal procedure and power set function work coordinately with one another. Negatively, the diagonal procedure demonstrates that the set of all subsets of the natural numbers cannot be put into a 1-1 mapping with the natural numbers. Positively, the power set function shows that the cardinality implied by this diagonal is $2^{\aleph_0}$. It is obvious that the combinatorial logic that applies to the natural numbers and to the power set of the natural numbers also applies to the casino discussion. The casino question considers the set of all possible outcomes from an infinite number of random devices in such a way as to imply that these possible outcomes do not have the same logical structure as countable infinity but as the power set of countable infinity. Thus not only is the probability space of the $\aleph_0$ casino isomorphic to Cantor's discussion of real numbers with respect to diagonalization, but also with respect to the combinatorial logic of Cantor's power set function as applied to infinite sets. Just as Cantor showed that the set of real numbers has cardinality $2^{\aleph_0}$ (wherein real numbers correspond to all the subsets of natural numbers), so too a casino of $\aleph_0$ randomized devices producing finite and $\aleph_0$ outcomes implies through combinatorial analysis a probability space of $2^{\aleph_0}$ potential outcome strings. These $2^{\aleph_0}$ potential strings correspond to the set of all subsets of the natural numbers as pertaining to outcomes of randomized devices within the $\aleph_0$ casino.

The isomorphism between real numbers and the probability space of the $\aleph_0$ casino is not limited to the diagonal argument and power set function, but extends to an identity relation

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[57] Mary Tiles, *The Philosophy of Set Theory: An Introduction to Cantor's Paradise* (Oxford: Basil Blackwell, 1989) 98; Rudy Rucker, *Infinity and the Mind: The Science and Philosophy of the Infinite* (Princeton: Princeton University Press, 1995), 239.

between real numbers and potential outcome strings of the $\aleph_0$ casino. Not only do real numbers and potential outcome strings have the same cardinality of $2^{\aleph_0}$, but there exists a 1-1 mapping between real numbers of the [0, 1) interval and the anti-diagonal set of potential strings pertaining to the $\aleph_0$ casino. This bijection results from the fact that both real numbers and potential strings are defined in terms of permutations of the natural numbers – more specifically, in terms of the set of all $2^{\aleph_0}$ subsets of natural numbers. This condition implies that any combinatorial string of natural numbers of either finite or $\aleph_0$ length satisfies the definition of being interpreted either as a real number or as a potential outcomes string from the $\aleph_0$ casino. The 1-1 mapping between real numbers and potential strings can thus be defined on the identity relation of combinatorial strings of natural numbers in which the mapping goes either from real numbers to potential outcome strings or from potential outcome strings to real numbers. The anti-diagonal set of potential strings pertaining to the diagonal set of the $\aleph_0$ casino produce an image of the real numbers of the [0, 1) interval, just as the real numbers of the [0, 1) interval produce an image of the anti-diagonal set of potential strings pertaining to the $\aleph_0$ casino.

With the foregoing discussion in view, it is abundantly clear that the probability space of the $\aleph_0$ casino does not consist of $\aleph_0$ potential outcome strings but of $2^{\aleph_0}$ potential outcome strings. Because Cantor's transfinite arithmetic is well suited to retain infinitesimal values, there is no need to adopt the measure theoretic interpretation of classical probability with respect to vanishing magnitudes. With an acceptance of infinitesimals, Cantor's transfinite theory answers the casino question quite differently from classical probability. According to transfinite terms, if $\aleph_0$ roulette wheels are spun $\aleph_0$ times, the chance that at least one such roulette wheel will correspond exactly to a specified target string of $\aleph_0$ outcomes is equal to $\aleph_0$ out of $2^{\aleph_0}$. If only one roulette wheel is spun $\aleph_0$ times, the chance that it will perfectly match the specified target string is 1 out of $2^{\aleph_0}$. Classical probability treats both of these probabilities as infinitesimals that vanish to zero, even though these values differ from one another by an infinite magnitude. Thus, regarding the findings of classical probability as imprecise, the present discussion elicits a theory of transfinite probability.

There are two obvious questions which present themselves at this point. One is to specify a measure space for the anti-diagonal set of the $\aleph_0$ casino, and the other is to consider whether transfinite probabilities exist with cardinalities beyond the $\aleph_0$ casino and its measure space. This presents a problem of the cart and the horse. A measure space as just considered can indeed be specified by means of the nonstandard analysis of Abraham Robinson (see below), though these resources would not remain sufficient for probabilities of succeeding transfinite casinos. The discussion here will thus proceed on an explorative basis, considering the general architectonic features of transfinite probability in informal terms. This consideration will encounter an escalating series of needs, each of which seeks fulfillment in a subsequent discussion. For instance, the measure space for a general theory of transfinite probability seeks expression in the theory of hypercomplex-transfinite numbers as presented in the following chapter. The present discussion thus does not attempt to finalize technical considerations as it goes along, but rather to conduct a study into the feasibility of the general plan. The same is to be said for the discussions below in chapters 5 and 6. The study, therefore, is an informal work in progress.

Consider now whether transfinite probability is restricted to initial values of $\aleph_0$ and $2^{\aleph_0}$, or whether it admits of higher transfinite values. (For ease of expression, the continuum hypothesis will be adopted here, wherein $2^{\aleph_0} = \aleph_1$. The generalized continuum hypothesis will also be adopted for the same reason.) It is conceivable that the $\aleph_0$ casino might be followed by an $\aleph_1$

casino, and then again by an $\aleph_2$ casino, such that the theory of transfinite probability becomes housed within an infinite series of transfinite casinos. The $\aleph_0$ casino is the first casino, defined as containing $\aleph_0$ randomized devices, each producing $\aleph_0$ outcomes. The $\aleph_0$ casino also contains $\aleph_0$ randomized devices that produce outcomes of arbitrary finite length, including null activity. It was shown that this $\aleph_0$ casino admits of a diagonalization, indicating that the probability space of the $\aleph_0$ casino consists of an anti-diagonal set of $2^{\aleph_0}$ potential outcome strings, now referred to as $\aleph_1$ potential outcome strings. But according to the rules of casino management, probabilities of $\aleph_1$ cannot be wagered in the $\aleph_0$ casino, inasmuch that such a hyper-cardinality as $\aleph_1$ is inexpressible within the $\aleph_0$ casino. Thus the casino question itself cannot be wagered in the $\aleph_0$ casino but must be referred to the $\aleph_1$ casino, if such a casino can be adequately defined. The question thus arises as to whether a general procedure can be found for constructing an infinite series of transfinite casinos beyond the $\aleph_0$ casino. Indeed, an $\aleph_1$ casino and its subsequent anti-diagonal set can be produced if both the number of potential outcome strings and the length of these strings are increased to $\aleph_1$.[58] Just as with the $\aleph_0$ casino, the $\aleph_1$ casino can be represented by two axes, each of which has a unit length of $\aleph_1$. Because the real number interval [0, 1) contains $2^{\aleph_0}$ points (which by virtue of the continuum hypothesis equal $\aleph_1$), then such a representation of the $\aleph_1$ casino can be given by two perpendicular axes of real line segments, one axis representing the $\aleph_1$ casino potential outcome strings, and the other representing the $\aleph_1$ potential outcomes on each string (i.e. the length of each string). The $\aleph_1$ casino as just defined is sufficient to express the anti-diagonal set of the $\aleph_0$ casino, but not merely because it contains $\aleph_1$ strings but also in that it contains the full $\aleph_1$ length of each $\aleph_1$ string. From this perspective, the

[58] An anti-diagonal set would not result if either the number of strings or length of strings remains $\aleph_0$. I am indebted to my friend Steve Owen for clarification on this point.

$\aleph_0$ casino is a twice-truncated form of the $\aleph_1$ casino. The $\aleph_1$ casino can thus be viewed as the pre-existing potentiality from which the $\aleph_0$ casino is actuated as a proper subset.[59]

What then of progressing from the $\aleph_1$ casino to the $\aleph_2$ casino? Here a problem arises. According to standard analysis of the continuum, the cardinality of points in any line or line segment is $2^{\aleph_0}$ and no more. As a consequence, the ease of drawing up the $\aleph_0$ and $\aleph_1$ casinos comes to an abrupt end with the $\aleph_2$ casino. It might be thought that construction of succeeding transfinite casinos can be accomplished only logically or abstractly but not geometrically, owing to the essential geometric nature of axes and the diagonalization procedure. The present discussion is thus compelled to contemplate abandoning its dependence on geometry. Yet there is a caveat here. Surreal number theory as developed by John Conway places the entire system of surreal-transfinite cardinal numbers along a single continuum,[60] thereby allowing the extraction of any desired cardinality from within the surreal number continuum. Surreal numbers thus provide a basis for constructing the $\aleph_2$ casino and beyond. However, surreal number theory has never received appreciable acceptance toward interpreting scientific systems, especially quantum mechanics. At least on a preliminary basis, then, transfinite probability can proceed in reliance upon surreal number theory until finding support elsewhere. Because the theory of transfinite probability is presented here in conjunction with a theory of hypercomplex-transfinite (HT) numbers, it is claimed that transfinite probability can ultimately be modeled in terms of HT

[59] The present discussion requires extension of the standard definition of *string.* “A string (or word) is a finite sequence of symbols juxtaposed.” John Hopcroft and Jeffrey Ullman, *Introduction to Automata Theory, Languages and Computation* (Reading, MA: Addison-Wesley, 1979), 1. The present discussion first amends the definition of “string” to apply to strings of infinite length, whether countable or uncountable. Strings may thus have length $\aleph_0$, $\aleph_1$, etc.

[60] Donald E. Knuth, *Surreal Numbers: How Two X-Students Turned On to Pure Mathematics and Found Total Happiness* (Reading, Mass.: Addison-Wesley, 1974), 99–105.

numbers rather than of surreal numbers. The complete presentation of transfinite probability thus cannot be accomplished without the subsequent presentation of HT numbers.

With respect then to the $\aleph_2$ casino, surreal number theory provides for a perpendicular representation of two $\aleph_2$ axes that provide the basis of the diagonal set of the $\aleph_2$ casino. Surreal number theory allows further for drawing a diagonalization across these $\aleph_2$ axes, thus providing the anti-diagonal basis for the $\aleph_3$ casino. Note that in each casino, the diagonalization does not belong to the diagonal set of the originating casino but to the anti-diagonal set of the succeeding casino. In this way, the diagonalization of one casino becomes the baseline on which the succeeding casino is constructed. In keeping with this progression, the cardinality of points in the diagonalization exceeds the limit of the diagonal casino but is the same as the cardinality of points in the axes of the anti-diagonal casino.

An iterative procedure is now available to construct an infinite series of transfinite casinos. Generally, the $\aleph_\alpha$ casino can be defined as a transfinite casino admitting of two $\aleph_\alpha$ axes, one of which represents the number of randomized devices in the casino, the other representing the number of potential outcomes of each device. For any $\aleph_\alpha$ casino, a diagonalization can be produced such that an anti-diagonal set of potential outcome strings is indicated. This anti-diagonal set contains the cardinality of probabilities pertaining to the $\aleph_\alpha$ casino, though this cardinality cannot itself be expressed within the $\aleph_\alpha$ casino. Thus the anti-diagonal set of the $\aleph_\alpha$ casino seeks expression in and becomes the basis of the $\aleph_{\alpha+1}$ casino. For every $\aleph_\alpha$ casino, the $\aleph_{\alpha+1}$ casino is the anti-diagonal casino (where $2^{\aleph_\alpha} = \aleph_{\alpha+1}$). So too the $\aleph_{\beta-1}$ casino is a proper subset of the $\aleph_\beta$ casino. Thus a procedure has been produced by which an infinite series of transfinite casinos can be generated. If GCH is assumed, $2^{\aleph_\alpha} = \aleph_\beta$, $2^{\aleph_\beta} = \aleph_\gamma$, etc.[61]

It is important to note that the diagonalization procedures described here have a different force from that as employed by Cantor. Cantor's diagonalization demonstrates simply that there is no possible mapping between the natural and real numbers; i.e., that any such supposed mapping is a pseudo-mapping. But the diagonalization procedures described here with reference to transfinite casinos allow for mappings on a ramified basis, specifying where particular mappings can and cannot be accomplished. This extended ramification is possible inasmuch that transfinite probability provides for a geometric representation of line segments with points of any surreal-transfinite cardinality. The diagonalization procedure can thus be defined in relation to the power set function such that the mapping disallowed through diagonalization is made possible through the ramified power set function. Cantor demonstrated that the total combinatorial possibilities of $\aleph_0$ natural numbers have a cardinality greater than $\aleph_0$, as shown by his diagonalization on the natural numbers, and yet this combinatorial set finds expression through the power set function within the real numbers, in which it is seen that the set of all subsets of the natural numbers have cardinality $2^{\aleph_0}$. In the same way, the total combinatorial probability space of the $\aleph_0$ casino has a cardinality greater than the natural numbers, as shown by the diagonalization drawn on the $\aleph_0$ casino, and yet also has a cardinality of $2^{\aleph_0}$, as specified by the power set function. Generally then, the total probability space pertaining to the $\aleph_\alpha$ casino cannot be expressed within the $\aleph_\alpha$ casino, as demonstrated by the diagonalization drawn on the $\aleph_\alpha$ casino, but can be expressed by the power set function as applied to the $\aleph_\alpha$ casino and leading to the $\aleph_{\alpha+1}$ casino. Cantor's diagonalization effected a modus tollens; it showed that a 1-1 mapping does not exist between the natural and real numbers. But the same procedure in the present discussion

[61] For all values greater than $\aleph_1$, the construction of transfinite casinos requires the axiom of choice (AC).

leads to a modus ponens, inasmuch that a constructive relation can be defined between the diagonalization procedure of geometry and the power set function of algebra. The diagonalization drawn in the $\aleph_{\alpha}$ casino delimits the totality of that casino (modus tollens) but then gives rise to an anti-diagonal set of greater totality in the $\aleph_{\alpha+1}$ casino (modus ponens). But more specifically, the diagonalization drawn on the $\aleph_{\alpha}$ casino has $\aleph_{\alpha+1}$ points, making it an appropriate baseline for the $\aleph_{\alpha+1}$ casino. The geometric diagonalization function and the algebraic power set function are precisely coordinated to work in tandem with each other through a ramified zigzag[62] theory in order to transition from the $\aleph_{\alpha}$ casino to the $\aleph_{\alpha+1}$ casino. The two functions form an iterative pair in that the $\aleph_{\alpha+1}$ casino is the complement of the $\aleph_{\alpha}$ casino.

Transfinite Probability vs. Classical Probability

In order to attempt to deal comprehensively with questions of infinity, classical probability has been defined in more specific terms than set theory and cardinality. As formulated by A. N. Kolmogorov in 1931, classical probability is often presented as if composed of two approaches, one of discrete phenomena and the other of continuous phenomena, though these two approaches are ultimately integrated under the single rubric of measure theory. Measure theory is defined on measure spaces of the real semi-interval [0, 1), and thus measure theory can appeal to the canonical ordering of numbers whereas set theory alone cannot. A discrete problem is one such as the probability of randomly selecting an even number from the set of all natural numbers. Classical probability treats discrete problems such as this in terms of statistical induction. Statistical induction shows that as the size of sample sets of random selections from the natural numbers becomes arbitrarily large, i.e., approaches infinity, the percentage of even numbers to

[62] The zigzag theory will be discussed in greater detail in chapter 5.

natural numbers approaches 0.5. (This question can be treated in terms of measure theory once the sample space is translated to measure spaces on the real line continuum.) A continuous problem asks such questions as: if randomly selecting a real number from the real semi-interval [0, 1), what is the probability that the selected number will be taken from the semi-interval [0, 0.25)? Because the set {[0, .25), [.25, 1)} is both countable and measurable, and because the measure of the interval [0, 0.25) is one fourth the measure of [0, 1), it follows that classical probability describes the probability of a random selection from the real interval [0, 1) coming from the interval [0, 0.25) is 0.25.

Theoretical problems begin to become evident within classical probability in the thesis of vanishing magnitudes, a thesis exemplified in both discrete and continuous contexts. As an example of the discrete, consider the probability of randomly selecting a particular natural number from the set of all natural numbers. Since there are infinitely many naturals, the probability of selecting any particular number would appear to be extremely small. But yet there should seemingly be some non-zero probability of selecting any particular natural number, inasmuch that no natural number seems an impossible result of a random selection. But such a value harkens to infinitesimals, and infinitesimals are not incorporated within the Wierstrass-Dedekind conception of the real number continuum as adopted by measure theory and classical probability. Unlike natural numbers, rational numbers are dense, i.e. between any two rational numbers there exists another rational number. Yet although the rational numbers are dense, they are not sufficiently dense to form a continuum, as was first discovered to the alarm of the Pythagorean community. Unlike natural and rational numbers, irrational numbers aggregate into a measure space and are thus collectively measurable. Because natural and rational numbers are not measurable (either individually or more particularly collectively), they fail the fundamental criterion of measure theory and are thus interpreted as vanishing magnitudes.

As for a continuous example of vanishing magnitudes, if considering a random selection from the real interval [0, 1), classical probability maintains not only that the probability of selecting any particular rational number is zero, but that the probability of any of the infinitely many rational numbers being selected is also zero, i.e. that the randomly selected number cannot be rational but must be irrational, even though there are an infinite number of rational numbers within the interval. This example vividly illustrates the striking nature of the thesis of vanishing magnitudes in that the entire infinity of rational numbers within the [0, 1] interval is made to vanish from before the inspection of measure theory and classical probability. In the same way, classical probability interprets the probability of randomly selecting any particular irrational number from the [0,1) interval as a vanishing magnitude with probability zero.[63]

Because the real line continuum contains the natural, rational and irrational numbers, it bridges the gulf between two types of infinity, the countable infinity of natural and rational numbers, and the uncountable infinity of irrational numbers. Inasmuch that the countable infinity of natural and rational numbers is dwarfed by the uncountable infinity of the irrationals, it is appropriate at least in some sense to regard the irrationals as the most prominent portion of the continuum. But, as broadly conceived, this would not seem to justify the conclusion that the other constitutive elements of the continuum are to be counted as nothing.

The existence of competing models of the continuum whereby a point can be interpreted as either a nothing or a something has been a prominent feature of mathematical history. Zeno of Elea began this chapter of thought by arguing that conceptions of the point are fundamentally paradoxical. One view holds that a point is equivalent to a process of infinite division, as if by a

[63] Patrick Billingsley, *Probability and Measure,* 2nd ed. (New York: John Wiley & Sons, 1986), 16–32.

transcendent action or supertask, a view that lends itself to a Platonist understanding. Another view is that a point is only an abstract entity, the asymptotic conclusion of a process that is ongoing but never completed. This view of the limit is found in Aristotle. This *something-nothing* controversy has been especially acute in considering whether the continuum is built up of points as if built up of somethings or if the continuum is built up of points as if built up of nothings. Archimedes was an interesting figure in this debate. He agreed with Aristotle that points are only limits but found himself going against this view when he intuitively apprehending infinitesimals in solving problems of parabolas.

The infinitesimal calculus of Isaac Newton and Gottfried Leibniz, introduced during the second half of the seventeenth century, allowed for unprecedented achievements in physics, but was also met with deep suspicion from such critics as George Berkeley in that it presented infinitesimals in a contradictory fashion as both infinitely small and equal to zero. Despite their initial unrefined character, Leibniz believed that infinitesimals could be consistently integrated into a more general framework, even as imaginary numbers (originally viewed as contradictory) had been incorporated into a rigorous theory of complex numbers. His inability to develop such a systematic treatment for infinitesimals prompted him to characterize the problem of points and real line continuum as a “labyrinth of the mind.” While Leibniz continued to believe that the foundations of calculus should rest on infinitesimals, Newton vacillated between defining calculus in terms of infinitesimals or in terms of limits.

In the nineteenth century, calculus came to be so rigorously and satisfactorily defined in terms of limits by Augustin Cauchy and Karl Wierstrass that there seemed little reason to countenance infinitesimals, although Cauchy himself continued to accept infinitesimals, not as numbers but as variables. With the ascendance of the Wierstrass-Dedekind continuum, infinitesimals were once again effectively banished from mathematics, except that it still remained unclear whether a theory of limits was sufficient for all questions of points and the continuum. But during the late nineteenth century, Cantor developed his highly celebrated theory of transfinite numbers that depicts an infinite hierarchy of increasing orders of infinity, a theory whose definitions and assumptions are directly dependent on the concept of actual infinity. Inasmuch that Cantor described actual infinity in terms of the very large, it was possible that someone else might define actual infinity in terms of the very small. Abraham Robinson accomplished this task in 1966, analyzing the continuum in terms of infinitesimals. In doing so, Robinson vindicated Leibniz's aspirations for a general theory that rendered infinitesimals intelligible. Though controversies endure as to whether actual infinity should be understood in realist terms (as believed by Cantor) or in nominalist terms (as believed by Robinson), it has become recognized that there are now two consistent and viable interpretations of the continuum, the continuum of limits as per Wierstrass and Dedekind known as standard analysis, and the continuum of infinitesimals as per Robinson known as nonstandard analysis.

Acceptance of Robinson's nonstandard analysis has been strangely mixed. It has been enthusiastically endorsed by many mathematicians and yet marginalized by the status quo. Reference to Robinson's theory as "nonstandard" does not suggest an ancillary standing to standard analysis, but points rather to Robinson's theoretical indebtedness to Thoralf Skolem's proof of 1934, which describes a nonstandard interpretation of arithmetic. In fact, mathematicians familiar with nonstandard analysis generally regard it as superior to standard analysis. Speaking after Robinson at the Institute for Advanced Study in 1973, Kurt Gödel described Robinson's theory as "the analysis of the future." Gödel remarked as follows:

> I would like to point out a fact that was not explicitly mentioned by Professor Robinson, but seems quite important to me; namely that non-standard analysis

> frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. This is true, e.g., also for the proof of the existence of invariant subspaces for compact operators, disregarding the improvement of the result; and it is true in an even higher degree in other cases. This state of affairs should prevent a rather common misinterpretation of non-standard analysis, namely the idea that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.
>
> One reason is the just mentioned simplification of proofs, since simplification facilitates discovery. Another, even more convincing reason, is the following: Arithmetic starts with the integers and proceeds by successively enlarging the number system by rational and negative numbers, irrational numbers, etc. But the next quite natural step after the reals, namely the introduction of infinitesimals, has simply been omitted. I think in coming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus.[64]

Contrary, then, to the beliefs of the founders of classical probability (individuals such as A. N. Kolmogorov, Émile Borel and Henri Lebesgue, who published their principal works during the first third of the twentieth century), there can be no presumption as to the truthfulness of standard analysis, and thus the possibility arises from nonstandard analysis that points of the continuum are better described as somethings rather than nothings, a possibility that imperils the veracity of classical probability. The pivotal issue is that where standard analysis defines points as vanishing to nothings, nonstandard analysis defines points as somethings. The focus of the present discussion thus becomes a comparison of the nonexistence claims of standard analysis and classical probability versus the existence claims of nonstandard analysis and transfinite probability.

To understand better how the historical development of number systems is inverted by classical probability, consider how various species of numbers or points on the continuum have been generally understood apart from claims of classical probability. It is the natural numbers,

[64] Quoted in Abraham Robinson, *Non-standard Analysis* (Amsterdam: North-Holland, 1966), xvi. Robinson provides much of the historical overview of the continuum as presented in the discussion above.

together with their finite complications into rational numbers, that have been viewed as the most rudimentary and accessible portion of the continuum. The irrational numbers, because of their infinite decimal expansions, have been regarded as more remote and, in some sense, non-constructive. Insofar then as mathematics has been taken to be a foundationalist endeavor, the natural and rational numbers rank among the most fundamental of objects, objects formulated within primitive finite intuition, but which come to be surrounded on the continuum by the infinitely-based irrational numbers. Thus due to the infinite construction required for each irrational number, it is irrational numbers that strain the limits of mathematical existence, and not the naturals or rationals. Unlike the vast majority of mathematicians who have no qualms in accepting actual infinity, finitists insist that all mathematics must be defined in terms of finite construction, and thus they are driven to understand the decimal expansion of irrational numbers as only potentially infinite, and the cardinality of all infinite sets as infinite in only an indeterminate sense. Thus the finitist seeks to supplant the standing of actual infinity as traditionally conceived within the decimal expansions of irrational numbers in favor of what must be somehow construed as finite and humanly constructive.[65] As the leading nineteenth-century finitist Leopold Kronecker famously declared, "God made the integers; all else is the work of man." Though finitists and infinitists disagree as to the legitimacy of actual infinity, they agree that, insofar as numbers are constructed from out of the null set, construction of the continuum begins straightforwardly with the naturals, continues in the same way with the rationals, but then concludes somewhat mysteriously with the irrationals. That is, despite other disagreements, finitists, infinitists and the bulk of mathematical history generally agree that natural and rational numbers are more primitive than irrational numbers, are in some sense

[65] See for example the finitist binary tree as an interpretation of the continuum. Tiles, *Philosophy of Set Theory*, 66.

logically prior to them, and that it is irrational numbers that are not only the most elusive ingredients of the continuum, but are chiefly responsible for its controversial nature. As set within this context, the disregard of classical probability toward natural and rational numbers via the description of vanishing magnitudes comprises what must be considered a radical departure from previous thought.

In mathematical contexts, then, other than classical probability and measure theory, natural and rational numbers constitute such a fundamental and accessible portion of the continuum that it would be unthinkable to describe them as at once having an infinite cardinality and also as vanishing to nonexistence. Yet because classical probability is defined in terms of a particular emphasis on measure theory, it produces this sort of strange equivocation. Whereas the argument above stressed that classical probability employs a particular interpretation of measure theory, even though current evidence suggests a rival interpretation is superior, the current argument asserts that, even if nonstandard analysis is removed from consideration, measure theory as based on standard analysis could never be regarded as a sufficient basis for a satisfactory theory of probability. The concern here is the advisability of defining probability wholly in terms of a single criterion, and whether that criterion is sufficient for appropriate expectations for a science of probability. That is, should the fact that natural and rational numbers vanish from the continuum in a particular measure theoretic sense be taken as evidence to suggest that they vanish from the continuum in a more general sense so as to have a zero probability of random selection from within the continuum? The best evidence of mathematical history and current understanding suggest not.

Because of its complicated structure, the continuum has historically demanded a number of criteria by which its heterogeneous elements are described. Viewed constructively, the continuum begins with the naturals, proceeds through the rationals, and concludes with the irrationals and then infinitesimals. Viewed by cardinality, the continuum is dominated by the uncountable infinity of irrationals and infinitesimals, but also populated by the countable infinity of naturals and rationals. Viewed by the measure theory of standard analysis, natural and rational numbers are totally insubstantial, while only irrational numbers are sufficient to aggregate into measurable units. The chief fault of classical probability is that it privileges and gives exclusive reliance to a partial criterion as if it were sufficient to take full inventory of the continuum, while in fact there are other criteria that offer their own distinct contributions toward development of the continuum. Even if the standard analytic interpretation of measure theory were not rivaled by nonstandard analysis, it would fail to give a comprehensive description of the continuum. In maintaining that any one of the infinity of rational numbers within the [0, 1) interval has no greater probability of selection than numbers that do not even exist within the interval, classical probability manifests what might be considered not only an imprecision but a significant incoherence. If numbers exist within a domain, then it follows that it is possible for them to exist within that domain, since existence implies an unimpeded condition of possibility. Yet classical probability impedes such possibilities by describing rational numbers within the [0, 1) interval with the same zero probability (or impossibility) as numbers that do not exist within the interval, the latter numbers being truly impossible to be selected as if from within the interval.

Classical probability reflects the belief that, if the physical universe is populated only by countably many objects, then the probability space of its interrelated possibilities must also be countable. Then out of recognition that the abstract entities of real numbers are uncountable, classical probability attempts to portray a comprehensive stance toward infinity, first by straddling uncountable and countable infinities, then by deflating uncountable infinity by emptying it of its positive significance, and finally by rendering all calculations within finitist terms. Classical probability speaks the language of uncountable infinity, but then sloughs off implications of uncountable infinity as can be accepted only by finitist philosophy. It is a historical oddity that finitists have been so marginalized by mainstream thought with respect to number theory but yet achieved such extreme prominence with respect to probability. If the finitist program of classical probability is not already discredited for the sake of vanishing magnitudes, then it should clearly be so by reason of the casino discussion. Transfinite probability implies that classical probability fails to respond satisfactorily to relevant questions of infinity. Though the chief analog to the casino discussion is Cantor's diagonalization procedure, it can also be seen that the casino discussion stands in the same relation to classical probability as irrational numbers to the Pythagorean doctrine of discrete magnitudes.

One additional point should be made by way of differentiating transfinite probability from classical probability. Classical probability is predicated on the assumption of unitary total probability. This means that the measure spaces of the probability space add up to a total probability of 1. This condition is certainly true in many empirical cases, but the assumption fails in transfinite probability. It is incorrect to say within transfinite probability that as the sample sets become arbitrarily large, the total probability space converges to 1 (where 1 is the totality of all probabilities). The diagonalization procedure, as applied within the transfinite casinos, demonstrates that there is a hard distinction to be made between empirical activities and the totality of all probabilities. Probabilistic activities are observable within the diagonal set but the total probability pertaining to that set is expressible only within the anti-diagonal set. Transfinite probability argues that there is a fundamental disconnect between the logic of finite sets and of infinite sets.[66]

In addition to conforming to the basic parameters of a non-classical probability theory, transfinite probability is intended as falling within the bourgeoning class of quantum probabilities. One advocate of quantum probability characterizes the relationship of classical probability and quantum probability as follows:

> Recent results suggest that quantum mechanical phenomena may be interpreted as a failure of standard probability theory and may be described by a Bayesian complex probability theory…. So why is there any reason to doubt probability theory? Here I think that there is a historical effect: probability theory may actually be failing all the time, it's just that the situations where a failure occurs are called "quantum mechanical phenomena" and thus appear in physics conferences instead of in probability theory conferences. This suggests that perhaps there is something wrong with probability theory after all, and that this may be where quantum mechanical effects come from.[67]

Karl Popper also argues for a reformulation of probability and quantum mechanics. He writes, "The solution of the problem of interpreting probability theory is fundamental for the interpretation of quantum theory; for quantum theory is a probabilistic theory."[68] Stanley Gudder

[66] There is an echo of Kant here. Kant claimed that certain questions (the ideas of reason) can be formulated but not answered within the bounds of theoretical reason but can only be addressed by practical reason. Transfinite probability offers the same sort of partitioning of question and answer in terms of the diagonal and anti-diagonal sets, although the partitioning of question and answer is much tighter in transfinite probability than in Kant.

[67] Saul Youssef, "Quantum Mechanics as an Exotic Probability Theory," in *Maximum Entropy and Bayesian Methods: Santa Fe, New Mexico, U.S.A.*, ed. Kenneth M. Hanson and Richard N. Silver (Dordrecht, The Netherlands: Kluwer Academic Publishers, 1996), 237.

[68] Popper claims that the idea of a statistical interpretation in quantum mechanics is correct but lacks clarity. He writes, "The consequence of this lack of clarity, the usual interpretation of probability in physics oscillates between two extremes: an *objective* purely statistical interpretation and a *subjective* interpretation in terms of our incomplete knowledge, or of the available information.… In the orthodox Copenhagen interpretation of quantum theory we find the same oscillation between an objective and a subjective interpretation: *the famous intrusion of the observer into physics.*" Karl R. Popper, "The Propensity Interpretation of the Calculus of Probability, and the Quantum Theory," in *Logic, Probability, and Epistemology: The Power of Semantics,* ed. Sahotra Sarkar (New York: Garland, 1996), 135.

agrees that the commonplace view of quantum uncertainty and statistical induction “does more harm than good,” and that what is needed is a ramping up of mathematical rigor.[69]

Transfinite Probability and Geometry

Since the time of Descartes, algebra and geometry have been understood as different aspects of a single unified system of analytic geometry. Algebraic equations can be expressed geometrically, and so also geometrical graphs can be expressed algebraically. In the same vein, the algebraic expressions of classical probability are represented geometrically. This is the process by which standard analysis of the continuum provides a measure space for classical probability. From such a perspective, the potential failure of expressing transfinite probability in geometric terms (with scientific rigor) poses a significant shortcoming. But transfinite probability is not the first occasion of confronting such an impasse. Cantor himself experienced this same failure in regard to his transfinite theory. When Cantor devised his theory of transfinite numbers, he sundered the senses of cardinality and ordinality so fundamentally as to produce two distinct sets of infinite numbers: the transfinite cardinal numbers ($\aleph_0$, $\aleph_1$, $\aleph_2$, etc.) and the transfinite ordinal numbers (ω_0, ω_1, ω_2, etc.). The frustration experienced by Cantor was that after prying apart the tightly-connected senses of cardinality and ordinality, he was unable to reintegrate the two in a significant way. In taking the power set of $\aleph_0$, he was faced with $2^{\aleph_0}$, but exactly what transfinite value is that? As 2^3 reduces to 8, Cantor wanted to show that $2^{\aleph_0}$ reduces to $\aleph_1$, though he had no means to do so. As far as Cantor knew, $2^{\aleph_0}$ might just as well equal $\aleph_2$ or $\aleph_3$, or some other transfinite value. The ordering of the cardinal numbers in this sense eluded him. This problem was particularly acute in that not only did it reflect a failing of

[69] Stanley R. Gudder, *Quantum Probability* (Boston: Academic Press, 1988), ix.

transfinite arithmetic, but it entailed the inability to resolve such a fundamental question as to the number of points on the real continuum. The Cartesian unification of algebra and geometry was thus falling apart.

As a result of Cantor's inability to solve the continuum problem, he advanced the hypothesis that $2^{\aleph_0}$ (the number of points on the real continuum) equals $\aleph_1$, that is, that the cardinality of points on the real continuum is the next cardinal number to follow $\aleph_0$. This claim became known as the continuum hypothesis (CH)[70] It has been a flash point of controversy ever since. Kurt Gödel proved in 1938 that, given ZF set theory, it is impossible to disprove CH. Conversely, Paul Cohen proved in 1963, again in reference to ZF set theory, that CH can never be proven true (it is safe to reject CH without being contradicted by ZF). As a consequence of these opposing results, CH has come to be described as undecidable; that is, the truth or falsity of CH is independent of the axioms of ZF set theory. So too, the generalized continuum hypothesis (GCH) is undecidable. GCH states the general formula that $2^{\aleph_\alpha} = \aleph_\beta$, that is, that $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, etc. But even further, the axiom of choice (AC) has also been shown to be independent of ZF. This is particularly troublesome for Cantor, since AC lies at the heart of his theory.[71] In addition to these issues of undecidability, transfinite theory introduces two paradoxes of totality, the Burali-Forti paradox relating to the ordinal theory and Cantor's paradox relating to the cardinal theory.[72] In all then, Cantor's transfinite theory is widely accepted within mainstream mathematics, and yet is made dubious because of theoretical problems extending from its founding assumptions to its superstructure.

[70] Dauben, *Georg Cantor,* 280–81, 289, 297, 334–35.

[71] Tiles, *Philosophy of Set Theory,* 175–91.

[72] Dauben, *Georg Cantor,* 248, 259–62 (Burali-Forti paradox); 241–47, 262 (Cantor paradox); Tiles, *Philosophy of Set Theory,* 114-17.

It may seem initially, then, that the pursuit of transfinite probability is destined to suffer certain inevitable difficulties inherited from Cantor's original theory. The inability to provide a geometric model for transfinite probability would entail the inability of providing a model for transfinite measure space. But the situation is quite different, due to the fact that the theory of transfinite probability as advanced here is presented in tandem with a theory of hypercomplex-transfinite (HT) numbers. Consequently, the proposal here is that the theory of transfinite probability can achieve geometric representation by means of HT number theory. HT numbers will be discussed in the following chapter. After that, the ensuing discussion will address issues of transfinite probability and HT numbers jointly.

CHAPTER 5

HYPERCOMPLEX-TRANSFINITE NUMBERS

In considering the progression of number systems from natural numbers to integers, rationals, irrationals and then to complex numbers, William Rowan Hamilton wondered whether the complex numbers represented the culmination of the development of number systems. If the one-dimensional real number line could be extended into a two-dimensional complex plane, he thought, why could not the complex plane be extended into further dimensions as well? Driven by the confidence this ought to be so, he explored the possibility of three-dimensional numbers, though without success. But then in 1843, he discovered he could produce an algebra for a four-dimensional number system. He named these numbers quaternions. This discovery was made possible when Hamilton realized he must abrogate the commutative law of multiplication, namely, $a \times b = b \times a$.[73] The development of quaternions made Hamilton the discoverer of hypercomplex numbers. The next hypercomplex system to be discovered was the octonions, a discovery made independently by John T. Graves and Arthur Cayley. The eight-dimensional octonions require the abrogation not only of the commutative law of multiplication but also the associative law of addition, namely, $a + (b + c) = (a + b) + c$. It was then discovered that an infinite series of hypercomplex systems can be defined that follow the pattern of dimension

[73] The loss of multiplicative commutation holds for matrix multiplication as well. Werner Heisenberg described quantum mechanics in terms of matrices, while Paul Dirac descried quantum mechanics by means of quaternions. According to both of these mathematical interpretations, quantum mechanics fails to exhibit multiplicative commutation.

doubling, and thus hypercomplex numbers can be found in 4, 8, 16, 32 dimensions, and on to infinity. It was also discovered that, subsequent to the octonions, no hypercomplex system offers a normed division. Within systems of normed division, the quotient of two non-zero numbers cannot equal zero.[74]

As stand-alone systems, both hypercomplex and transfinite numbers exhibit not only structural gaps but brute givens that seem to defy penetrating analysis. But as proposed below, when set alongside one another and interconnected in a particular way, these systems dovetail so as to fill out what is missing in the other. These systems can be thought of as two tables of information, each of which is torn and incomplete, but when put together can be read across as one intelligible whole. The project at hand is thus to sketch out a unified hypercomplex-transfinite (HT) number system and to consider various implications that follow.

It is clear that hypercomplex and transfinite numbers can be discussed separately. The question here is how they might be discussed jointly. Because both systems originate with elementary concepts of number, arising from considerations of natural, rational and irrational numbers, such a basis of comparison is possible. Figure 1 offers such a comparison.

[74] For a brief overview of hypercomplex numbers, see Keith Devlin, *Mathematics: The New Golden Age* (New York: Columbia University Press, 1999), 76–77. For greater detail, see I. L. Kantor and A. S. Solodovnikov, *Hypercomplex Numbers: An Elementary Introduction to Algebras,* trans. A. Shenitzer (New York: Springer-Verlag, 1989).

Figure 1: THE HYPERCOMPLEX-TRANSFINITE ASSUMPTION

Number Class	First Principle of Generation	Dimension	Number System	Cardinality	Second Principle of Generation	Third Principle (Limitation or Interruption)
1	0	$2^0 = 1$	Natural Rational	$\aleph_0$	ω_0	Begin (next) row with ω_0
		$2^0 = 1$	Real	$2^{\aleph_0} = ?$		
		$2^1 = 2$	Complex			
		$2^2 = 4$	Quaternion			
		$2^3 = 8$	Octonion			
		$2^4 = 16$	Sedenion			
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Notice several features about this table. First, much of the table is incomplete. The data shown here is limited to what is commonly recognized as pertaining to both systems, prior to any theorizing done here. Second, columns three and four are those that apply directly to hypercomplex numbers (including the real and complex numbers as their precursors). The two extreme columns on either side (first and second, sixth and seventh) are taken directly from Cantor's work on transfinite numbers. These column headings (Number Class, First Principle of Generation, Second Principle of Generation, and Third Principle – Principle of Limitation) are Cantor's own terms.[75] The fifth column (Cardinality) is neutral, applying equally well to hypercomplex and transfinite numbers. Such is the basic layout of the table.

Importantly, the commitment to provide information within Figure 1 as corresponding to both hypercomplex and transfinite numbers prevents the inclusion of information which might be singularly applied to one number system or the other. There is no question, for instance, that columns one and two, six and seven, are to be completed according to the ordinal sequence of numbers, as per transfinite theory. However, this information is omitted from the table because it cannot be put down as corresponding to columns three and four as regarding the standard analysis of complex and hypercomplex numbers. Likewise, column five might be thought, by relation to column four, to be filled out by the value $2^{\aleph_0}$ (which accords with the standard analysis of complex and hypercomplex numbers), except that this information would conflict with what is omitted in the table from the Cantorian columns. Notice also that the inability to offer a final disposition to the second row of the Cardinality column by relating it to any other column in the table gives expression to the continuum problem.

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[75] Dauben, *Georg Cantor*, 97–99; Tiles, *Philosophy of Set Theory*, 104–06.

Figure 2: HYPERCOMPLEX-TRANSFINITE NUMBERS

Number Class	First Principle of Generation	Dimension	Number System	Cardinality	Second Principle of Generation	Third Principle (Limitation or Interruption)
1	0	$2^0 = 1$	Natural Rational	$\aleph_0$	ω_0	Begin next row with ω_0
2	ω_0	$2^0 = 1$	Real	$2^{\aleph_0} = \aleph_1$	ω_1	Begin next row with ω_1
3	ω_1	$2^1 = 2$	Complex	$2^{\aleph_1} = \aleph_2$	ω_2	Begin next row with ω_2
4	ω_2	$2^2 = 4$	Quaternion	$2^{\aleph_2} = \aleph_3$	ω_3	Begin next row with ω_3
5	ω_3	$2^3 = 8$	Octonion	$2^{\aleph_3} = \aleph_4$	ω_4	Begin next row with ω_4
6	ω_4	$2^4 = 16$	Sedenion	$2^{\aleph_4} = \aleph_5$	ω_5	Begin next row with ω_5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The question arises as to what pattern or sense, if any, might be contained within Figure 1. Does the fragmentary data provided here allow the missing information to be supplied in such a way as to extend beyond existing theory? That is, does the data provide a basis for combining hypercomplex and transfinite numbers synthetically in a way that is not possible analytically from within either number system itself? Figure 2 suggests that such unification is possible.

Several points can be made concerning the details of Figure 2. First, the HT table should be viewed as an electronic spreadsheet wherein each cell is calculated by functions relating to other values in the table. Not only should these functions be evaluated by specific results but also in terms of the relationships they imply between different columns of the table, and as to whether these relations reflect a significant rationale for HT theory. Various columns in the table relate to other columns that come before and after. As column three takes input from column two and calculates its operation in finite terms, so column five takes input from column two and calculates its operation in transfinite terms as specified by column six. More specifically, the third column expresses the function $2^n = d$ (where d signifies dimensionality) and where n is read from the discriminant in the second column (the first principle of generation). Similarly, the fifth column expresses the same 2^n function, though within a transfinite context, suggesting that the common 2^n function be expressed more appropriately as the 2^x function. With reference then to the fifth column, this column expresses the transfinite function $2^x = c$ (where c signifies cardinality) where x is read from the discriminant in the second column (the first principle of generation), whereupon the cardinality $2^{\aleph_x}$ is calculated by reading x in connection with the discriminant in the sixth column (the second principle of generation).

Second, attention should be directed most critically to the center three columns of the table. Here it is that the fundamental proposal for the HT numbers finds beginning. The initial assumption of HT theory is that every number system of column four be defined such that the dual conceptions of dimensionality and cardinality are both defined through a unified covariant 2^x function, wherein the parameters of dimensionality and cardinality receive the same input but calculate their output on different scales so as to produce different results. Dimensionality is calculated through the 2^x function in finite terms, and cardinality is calculated through the 2^x function in transfinite terms. As for dimensionality, the 2^x function reads the value x from the first principle of generation (column two) and produces a finite result that doubles at each level. The finite results of column four are 1, 2, 4, 8, 16, etc. Likewise, the cardinality column reads the value x from the first principle of generation (column two), but because this value becomes subscripted to a transfinite number $2^{\aleph_x}$, the cardinality value is calculated not in finite terms but by taking the subscripted value from the second principle of generation (column six). The transfinite results of the 2^x function as relating to cardinality (column five) are thus $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, etc. Because the first and second principles of generation are coupled to both dimensionality and cardinality through the 2^x function, they serve to establish the bounds for each number system. The first principle of generation establishes the floor and the second principle the ceiling for each domain. (The first principle is one of beginning, the second of delimitation, and the third of recursion or ramification.) For example, HT complex numbers are defined by this table as two-dimensional numbers whose elements have cardinality $\aleph_2$, quaternions as four-dimensional numbers whose elements have cardinality $\aleph_3$, etc. The assertion here is that 2^x is a general, archetypal function which operates to produce both the dimensionality and cardinality of each number system in such a way as to forge a unity between hypercomplex and transfinite numbers. To restate in brief, the proposal here is that the 2^x function is the generative force which produces and unifies number systems with respect to dimensionality and cardinality. This function receives the same input for each of these parameters (taken from the first principle of generation) but in applying the same function to different parameters, the 2^x function yields diverse results, depending on which side of infinity a calculation is made; depending on whether x is finite so as to be calculated by elementary algebra under the first principle of generation and thus doubled, or infinite so as to be calculated (as defined by the HT table) with reference to the second principle of generation, and thus sequenced.

Third, it can be judged that the HT table is sufficiently textured and nuanced so as to avoid the appearance of superficiality. One way to observe this is to notice that the natural, rational and real numbers are all designated as one-dimensional number systems, but the real numbers are designated not in Cantor's first number class but the second. Likewise, the distinction between the numbers of the first two columns is suggestive of subtleties within the table. Additional complications such as these will become apparent as the discussion proceeds. Evidence such as this serves to rebut potential criticism that the table is simplistic.

Fourth, in reviewing the question of hypercomplex dimension doubling in the light of the HT table, the dimensional gaps that previously seemed inexplicable now appear explained by the accompanying sequential increases of transfinite cardinality. Thus the HT numbers seem to fill in missing data of the hypercomplex numbers. Likewise, insofar as the transfinite numbers have never been given geometric interpretation, they have lacked a geometric model by which they can either be visualized or connected to the rest of the mathematical superstructure. Their connection to the hypercomplex numbers overcomes this shortcoming. Thus the HT numbers offer a means by which the hypercomplex and transfinite numbers can be made more intelligible, adding support to the claim that they constitute a single, unified system.

Fifth, the HT table does not assume the continuum hypothesis, even though the statement $2^{\aleph_0} = \aleph_1$ is made in the second row; similarly for GCH. These statements appear in the HT table not as assumptions but because they result from within the logical relations within the HT table (Figure 2) as described below in relation to the HT power set function. These relations are predicated on the HT assumption that the 2^x function forges a unification of hypercomplex and transfinite numbers with respect to dimensionality and cardinality and other related terms in the HT table. The HT assumption is not axiomatic, in the rich sense that it represents a clear and distinct idea, but is quasi-empirical, and thus looks to its results for confirmation. The possibility of this quasi-empirical assumption presents itself in terms of the dimensionality of the natural, rational, real, complex and hypercomplex number systems and in terms of the cardinality of the natural, rational and real number systems. These are the details that are included within Figure 1, and then it is also within Figure 1 that the HT assumption is made. The data filled in within Figure 2 (beyond what is in Figure 1) is what results from having made the HT assumption in Figure 1. Therefore, it can be seen that HT theory implies both CH and GCH, but does not assume either. Because GCH implies AC, HT implies both GCH and AC. Throughout the remainder of this study, the truthfulness of GCH and AC will be affirmed, though not by assuming either, but because they are entailed by HT theory.

Sixth, as seen from the point of view of standard analysis of the complex and hypercomplex numbers, HT theory might seem to advance an enriched claim that runs afoul of existing theory. The HT table claims that there are $\aleph_1$ points on the real number line, $\aleph_2$ in the HT complex plane, $\aleph_3$ HT quaternions, etc. Standard analysis holds that real, complex and hypercomplex systems are all limited to $2^{\aleph_0}$ ($\aleph_1$) numbers. What is to be made of this disparity? Mathematics is full of varying systems whose theorems are incompatible with one another, such as Euclidean vs. non-Euclidean geometries. There is therefore no difficulty in recognizing one complex and hypercomplex system for standard analysis and another for HT theory. Standard analysis has demonstrated that a 1-1 mapping exists between points of the real continuum and the complex plane, thereby implying that real and complex numbers have the same cardinality. HT theory offers no challenge to mappings such as this, although it questions whether such a mapping is solely determinative as to whether additional criteria might be considered.

Seventh, as to a more substantive comparison between standard analysis and HT theory as to the discrepancy just discussed, what explanation can be given for the extra points supplied by HT theory? How are these extra points to be described? The only (visible) access to the HT plane is that provided by the *x* and *y* axes. Since each axis has points of cardinality $\aleph_1$, then inasmuch as the HT plane is understood as a cross-product of these axes, there appears to be only $\aleph_1$ points in the HT plane. In fact, insofar as points are held to be accessible to polynomial equations, HT theory agrees with standard analysis that there are only $\aleph_1$ accessible points within the HT plane. But since a fundamental claim of HT theory is that there can be distinguished two senses by which points are said to be in the plane, one by which points are made to populate the plane and another by which some or all of these points become accessible to direct inspection, it can be thought that more points are put into the plane than what can be treated in terms of a cross-product of the *x* and *y* axes. That is, because HT theory defines numbers by a dual aspect of both cardinality and dimensionality, the interpretation is possible that the covariant HT function of cardinality and dimensionality puts the materiality of $\aleph_2$ points into the plane, while the dimensionality of the system allows only $\aleph_1$ of them to become accessible in polynomial terms. With respect to polynomial equations, *de re* knowledge is possible for $\aleph_1$ points in the HT complex plane, though *de dicto* knowledge of $\aleph_2$ points in the HT complex plane is possible by means of the HT table. HT theory thus advances a theory of inaccessible numbers.[76] Note that the question here is not treated strictly in analytic terms. It may be thought that if the plane is simply the cross-product of two $\aleph_1$ axes, then the plane is obviously a closed quadratic field with $\aleph_1$ points. But HT theory considers the question both synthetically and analytically, and thus conceives of the HT complex plane as more than simply a closed quadratic field.

Eighth, HT theory is highly compatible with surreal number theory. Surreal theory is peculiar in that it places the entire series of infinite cardinal numbers on a single continuum. Surreal theory is also distinctive in that it defines numbers geometrically in terms of left and right. Thus by treating infinite numbers within fundamental terms of dimensionality and cardinality, HT and surreal theory speak the same language. The chief difference between the two is that HT theory defines each infinite cardinal number system within its own dimensional space, whereas surreal theory makes them all cohabit on a single horizontal continuum. For these reasons, a synthesis of HT and surreal theory into a unified hypercomplex-transfinite-surreal (HTS) number theory can be easily realized, chiefly by relocating surreal cardinalities greater than $\aleph_1$ from the surreal continuum so as to be distributed iteratively throughout HT dimensions.[77]

Ninth, it can be seen that novelties introduced within HT theory do more to maintain patterns of symmetry with elementary number systems than to break

[76] Interestingly, reference to inaccessible numbers also appears in Cantor's transfinite theory. Cantor's theory implies the existence of inaccessible, hyper-inaccessible and hyper-hyper-inaccessible transfinite numbers. Tiles, *Philosophy of Set Theory*, 130, 180-81, 195.

[77] As described by Knuth, surreal numbers are created on particular days of creation, in analogy with Genesis 1. (See above at chapter 4 note 7.) According to the present proposal for synthesizing HT and surreal theory, numbers resulting from the $\aleph_0$ day and $\aleph_1$ day of creation would remain on the HTS continuum. Subsequently, each successive infinite day of creation would produce numbers for a higher dimension, placing numbers of the $\aleph_2$ day in the HTS complex plane, the $\aleph_3$ day in HTS quaternions, etc.

pattern with them. Consider the progression of elementary and HT number systems with respect to their decimal expansions as presented below.

Numbers	*Decimal Expansion*
Naturals	0
Rationals	>0 and $\leq\aleph_0$ ($\aleph_0$ if repeating)
Irrationals	$\aleph_0$
HT Complex	$\aleph_1$
HT Quaternions	$\aleph_2$
$\vdots$	$\vdots$

This table shows that HT numbers complete a pattern begun by elementary number systems.[78] But what does it mean for HT numbers to have decimal expansions of length $\aleph_1$ and longer? Are not HT complex numbers two-dimensional, HT quaternions four-dimensional, etc.? This situation seems to reflect a certain freedom of expression wherein HT numbers can be conceived of either in linear or multidimensional terms. This freedom of expression is particularly convenient in that a synthesis of HT and surreal numbers was just proposed. Whereas HT numbers are characteristically expressed in multi-dimensional terms, surreal numbers are expressed in linear terms. It is therefore natural to think that a unified HTS system would have the freedom to express numbers by either method. This freedom of expression resembles the fact that rational numbers can be written either in linear decimal form or as fractions. So too, natural numbers can be written whole or as fractions. Thus the freedom of expression wherein HT/HTS numbers can be written in linear or in multidimensional terms is not unprecedented. Rather, such a linear or sequential ordering is exactly what would be expected if a well-ordering of HT numbers is thought to exist, as is true under AC.

Tenth, the expanded sense of HT numbers as conceived in linear terms offers a constructive explanation of why there are more points in the HT complex plane than in the standard complex

[78] The values here are not to be confused with the cardinality of each number system. The values are different in each case. The decimal expansion is smaller than the cardinality for each number system.

plane, and so too for succeeding HT systems, as compared to their hypercomplex counterparts. Unlike the standard complex plane, the HT complex plane is not a closed quadratic field. HT theory holds that surds exist within the HT plane relative to the cross product of real and imaginary numbers, just as irrational surds exist within the real number system relative to rational numbers. Moreover, just as the real number continuum contains transcendental numbers which cannot be captured within algebraic terms, so the HT plane contains transcendental numbers which cannot be expressed in polynomial terms. If HT theory be thought deviant or contradictory for reason of having $\aleph_2$ points in the HT complex plane, it can be noted that surreal theory is even "worse" in that surreal theory constructs all infinite cardinal numbers on a single dimension.

Eleventh, the surds in HT numbers would seem to require that the cardinality of Dedekind cuts[79] be iterated so as to allow $\aleph_1$ cuts in the real line, $\aleph_2$ cuts in the HT complex plane, $\aleph_3$ cuts in HT quaternions, etc. However, the literal sense of *cut* loses its meaning when extended thus beyond the real number line, the literal constructive sense of cuts giving way to mathematical impenetrability. Borrowing then from surreal number theory (as to "days of creation"), it can be said that $\aleph_1$ HT/HTS cuts are made on the $\aleph_1$ day of creation, $\aleph_2$ cuts on the $\aleph_2$ day of creation, etc. But insofar as the HT plane consists of both a real and imaginary axis, HT theory in particular can define a logical construction as occurring in imaginary time that breaks away from the "arrow of time" as canonically modeled by the real number line. HTS cuts can thus be understood as constructed in imaginary space-time. HT theory and surreal theory seem then to sharpen one another with respect to the idea of mathematical constructions occurring within imaginary time.

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[79] Tiles, *Philosophy of Set Theory*, 84–86.

But an alternative understanding of cuts is also possible, one in which HT theory borrows less from surreal theory than just mentioned. The $\aleph_1$ cuts of the real line can be construed as superintended by the $\aleph_2$ numbers in the HT plane, just as the $\aleph_2$ cuts in the HT plane are superintended by the $\aleph_3$ numbers in HT quaternions, etc. This superintendence suggests that the higher dimensions are in some sense antecedent to the lower dimensions. Recall that in transfinite probability, the $\aleph_0$ casino was said to be subtracted out of the preexisting $\aleph_1$ casino, the $\aleph_1$ casino subtracted out of the preexisting $\aleph_2$ casino, etc. This suggestion runs in the opposite direction of foundationalist number theory, though such a reversal is in keeping with the sense of imaginary space-time. Just as imaginary numbers transcend the law of signs as maintained by the real numbers, so too HT imaginary numbers transcend the strict relation of antecedent and consequent as suggested by the real number line, at least insofar as the real number line is used to model the arrow of time. Indeed, these two considerations are highly analogous (the minus and plus of the law of signs represents the past and future of the arrow of time). Because HT number theory can interpret the law of signs and the arrow of time jointly, the force of this interpretation is such that the prioricity of one HT number system is not fully determinate with respect to another. One mathematical construction can be regarded as progressing bottom-up while a different construction progresses top-down, the two approaches disagreeing as to what is antecedent and what is consequent. HT number systems thus resemble Einstein's theory of relativity in that different observers in different HT frames of reference cannot agree on what comes before or after. Because of this disruption in temporality (and because temporality is fundamental to the classical concept of causality), HT theory also resembles quantum mechanics in that both undermine the classical relationship of cause and effect. These circumstances bring the resources of HT theory to bear on questions of space-time and causality.

Twelfth, and by way of further emphasis on what was just said, HT theory seeks to incorporate a conception of imaginary time as an important and even fundamental aspect of the theory. Peano claimed that every natural number has a successor, thereby creating the impression that mathematical constructions are to be conceived, in some sense, as occurring in real time. The possibility of imaginary time is of crucial significance toward constructing fractal geometry and other infinite and transfinite supertasks. Beyond mathematics, a proper conceptualization of imaginary time would also be of great significance to quantum physics.

The Imaginary Unit, Zigzag Theory and HT Functions

As typical of elementary number systems, the real numbers give rise to certain formulations that cannot be solved from within the real numbers. For example, the polynomial equation $(x + 2)^2 = -9$ has no real solution. This inadequacy owes to the law of signs, wherein the square of a real number cannot equal a negative. The complex number system was devised in order to overcome this limitation. However, the conceptual transition from the real to the complex numbers has been associated with a significant amount of controversy. Central to this controversy is the logical import of the imaginary number i, which in some sense contravenes the law of signs as understood from within the real number system. As represented by the complex plane, complex numbers constitute a closed quadratic field. Not only can complex numbers solve all quadratic equations but all polynomial equations in a single variable with real or complex coefficients.

Complex numbers take the form $a + bi$ where a is real and bi is imaginary. If the imaginary part of a complex number is zero, the complex number can be expressed as a real number. If the real part of a complex umber is zero, the complex number can be expressed as an imaginary number. Thus both real and imaginary numbers can be interpreted as proper subsets of the complex number system. Crucially, the imaginary number i is defined as satisfying the expression $i^2 = -1$. This is the algebraic bridge that unites real and imaginary numbers. The complex plane thus defines the x and y axes in terms of a specific algebraic relation, which differentiates the complex plane from the Cartesian plane. Based on this algebra, the polynomial example stated above can be solved by two different complex numbers: $-2 + 3i$ and $-2 + -3i$ (commonly expressed as $-2 -3i$.

According to standard definition, a complex number z takes the form $a + bi$ where a and b are real numbers, where a is the real part and b is the imaginary part, and where i is the imaginary unit. The imaginary unit is defined as satisfying the expression $i^2 = -1$. The values b and i receive separate algebraic treatment, as is apparent in cases such as $2i$, $3i$, etc., or in cases of exponentiation. For example, the expression bi^2 equals $(bi)^2$, which equals $(b^2)(i^2)$. Thus for any b, $bi^2 = (b^2)(-1)$. This is the algebraic reasoning that brings imaginary numbers from the realm of the "absurd" (as some critics have claimed) into a fully integrated complex number system. To note an additional feature of imaginary numbers, squaring an imaginary number always yields a negative real number. Regardless of whether the imaginary part b is positive or negative, b^2 is positive, and thus $(b^2)(-1)$ is negative.

According to standard interpretation, complex numbers are treated geometrically as two dimensional numbers, combining the x-axis of real numbers with the y-axis of imaginary numbers. A complex number $a + bi$ can thus be represented as the point (a, b) in the complex plane. Further, the complex plane allows for the imaginary number i to be treated in terms of a polar coordinate system. On this basis, the function of squaring a complex number, such as moving from i to i^2, is interpreted as the rotation of the point i through the plane by 90º counterclockwise (this rotation regarded as an extension of vector analysis). This model provides direct geometric justification for the algebraic claim that $i^2 = -1$. Apart from such a model as this, the imaginary unit i has been disparaged as an algebraic figment, as if contrived out of thin air. But the logical structure of the polar coordinate system dispels such an objection.

The historical struggle of coming to terms with the number i has resulted not from any intrinsic difficulty of algebra or of the polar coordinate system but from the accidental historical circumstance in which the x-axis was initially described as *real.* If the real number system is *real*, as if in a deep metaphysical sense, then it is all too easy for the opposing y-axis to be thought of as *imaginary,* an apt description for that which opposes reality. As the thinking went, the real numbers maintain that a number multiplied by itself cannot produce a negative number, and so any number system that contradicts this claim must be *imaginary*. But if the x and y axes had been described merely as *horizontal* and *vertical* respectively, much controversy might have been avoided. With an appropriate understanding of the imaginary axis, numbers of the complex plane can be understood as no more imaginary or less real than other mathematical entities, particularly those of the real numbers. This is especially true in that the real number system poses certain expressions that it cannot satisfy, though all such expressions can be satisfied by the complex number system. For this reason, complex numbers seem to be more real or at least more intelligible than real numbers. (Note that Plato’s criterion of reality was in *Republic* Book 6 was intelligibility.

The property wherein the real number system is able to formulate certain expressions that it cannot satisfy is not peculiar to real numbers. Much rather, this property is typical of other elementary number systems. Delineating this property through elementary number systems gives rise to what can loosely be described as Russell’s zigzag theory. To begin with, consider the natural numbers. For any natural numbers m and n, $m + n$ yields a natural number, and so the natural numbers are closed under addition. But the same is not true for the inverse operation of subtraction. In cases where $m > n$, $n – m$ cannot be satisfied from within the natural numbers, and thus the natural numbers are open under subtraction. By virtue of this inadequacy, natural numbers express the need for negative numbers. Accordingly, the integers fulfill what is lacking from within the natural numbers by providing both positive and negative numbers. Integers are closed under addition and subtraction, and also closed under multiplication, but are open under the inverse operation of division. Integers thus require the rational numbers to supply the notion of fractional numbers which is missing from within them. Rational numbers are closed under addition, subtraction, multiplication and division, and also closed under exponentiation, but are open under the inverse operation of roots. For instance, the square root of any prime number is an irrational number. The rational numbers are thus inadequate unto themselves, thus giving rise to the real numbers, which contain both rational and irrational numbers. Like the rationals, the real numbers are closed under all familiar algebraic operations, except for roots. For instance, real numbers cannot provide square roots for negative real numbers. Although real numbers may give the appearance that square roots of negative real numbers are impossible, polynomial equations and complex numbers show that such roots are both needed and achievable. Thus, just as with other elementary number systems, real numbers give rise to expressions that they cannot satisfy, though this inadequacy is fulfilled by a succeeding number system. In all these cases, the succeeding number system contains what is both familiar and alien to the preceding number system. Each given system is closed under a particular positive function but open under its inverse. The succeeding system is closed under that inverse operation, and thus constitutes the complement of the original system. The characteristic feature of Russell's zigzag theory is that a given set is open under a diagonalization argument, for which a complementary set is closed under that function, only to have the zigzag repeat itself. On this interpretation, elementary number systems can be described in terms of diagonal and anti-diagonal sets.

The question arises whether complex numbers continue the same dynamic of zigzag as exhibited in elementary number systems. They do not, but not because they satisfy all expressions within the system. Complex numbers can solve all polynomial equations in a single variable with real or complex coefficients but not those with two or more variables. But unlike elementary number systems, complex numbers do not introduce a novel positive function under which they are closed, only to be made open by the inverse of the function. Thus the zigzag pattern exhibited within elementary number systems comes to an end within the complex number system. For this reason, complex numbers are often regarded as the culmination of algebraic progression, even though they fail to achieve algebraic closure. This is a strange circumstance. The complex number system received a great deal of criticism during its historical development because of controversy surrounding the number *i*, yet it seems they might be critiqued more appropriately for breaking the zigzag progression and also for failing to provide closure for all polynomial expressions. HT theory claims that these two failures are related, and thus seeks to remedy both by proposing a common solution.

In light of the preceding discussion, a fundamental objective of HT theory is to reconceive the standard complex number system in such a way that the zigzag progression exhibited within elementary numbers continues throughout all HT number systems. In order to accomplish this objective, two proposals will be advanced below. The first is to reconceive the imaginary unit *i*. The second is to define an algebraic function with corresponding inverse under which the HT complex numbers are first closed, then open, and then to have the zigzag pattern continue throughout all HT number systems.

Toward reconceiving the imaginary unit *i*, although i and $-i$ are both square roots of -1, the standard definition of the imaginary unit gives i priority over $-i$. According to the standard complex plane, there is only one nodal point that connects the imaginary axis to the real axis through exponentiation, and this nodal point is i. The point $-i$ is certainly algebraically available to the real axis, though only as derived from i, ($i \times -1 = -i$). This prioritization reflects the desire for a unitary fundamentality with respect to the imaginary unit. But because i is only one of two square roots of -1, the desire for unitary absolutism in this context might not be desirable within HT number theory. Accordingly, HT number theory thus redefines the imaginary unit, not as satisfying the expression $i^2 = -1$ but as satisfying the expression $i = \pm\sqrt{-1}$. The result is that the HT complex plane has two equally fundamental imaginary units, i and $-i$, both of which connect the imaginary axis to the real axis. Neither of these imaginary units is derived from the other, but they spring together from a common source.[80] As seen here, HT number theory tightens the algebraic relation between the real and imaginary axes as compared to the standard complex number system, and in so doing, provides greater flexibility and access in moving from one axis to the other.

[80] The larger issue here is the battle between linear and nonlinear logic. The acceptance of two equal and opposite imaginary units might be thought to make the interface between the real and imaginary axes unstable, as if to make algebraic foundations spin out of control. But the polar coordinate system already involves the notion of rotation through the plane, and so, in the first instance, the purpose of defining two opposing imaginary units is to facilitate this process of polar rotation. Beyond this, two imaginary units enhance the algebraic underpinnings of fractal geometry, as well as providing direct theoretical support for quantum mechanics.

Toward defining functions by which HT numbers continue the zigzag feature of elementary number systems, such functions can be defined as extensions of the power set function and of the diagonalization function. Thus the HT power set function is defined below as a constructive (positive) function, and its inverse is defined as the HT diagonal function.

The HT power set function is the same as Cantor's power set function, except that the HT power set function includes the concept of dimensionality, which is not present within the standard power set function. As shown in the HT table, HT theory defines both dimensionality and cardinality in terms of the same 2^x function, though in relation to different Principles of Generation, thus producing different algebraic results. The HT power set function thus operates through the 2^x function with respect both to dimensionality and cardinality, thereby synthetically conjoining these two aspects into a single covariant function. This dual synthesis is the fundamental construction giving rise to the HT table. Whereas transfinite theory and the power set function are somewhat non-constructive, in that they lack geometric representation, HT theory and the HT power set function are highly constructive in that their algebraic operations can be modeled geometrically.

Because the HT power set function is highly constructive, in that it defines a hierarchical system of dimensionality by which to support transfinite cardinal and ordinal numbers, it allows for a much more constructive understanding of the diagonalization procedure than provided for by Cantor. Cantor's use of the diagonal was a proof by contradiction, showing (in the first instance) that a 1-1 mapping does not exist between the natural numbers and the power set of the natural numbers. For lack of geometric representation, Cantor could not develop this procedure constructively, though he conceived of the diagonalization as iterated within abstract terms (lacking any geometric representation). HT theory, however, is able to provide a comprehensive application of the HT diagonal function as modeled in geometric terms for all HT number systems. The HT power set function and HT diagonal function can thus be paired as complementary functions within a structure of HT ramification and zigzag. As the positive function, the HT power set function constructs each HT number system in conjunction with Cantor's first principle of generation (as per the HT table) such that for any ω_a, the dimensionality of the number system equals 2^a and its cardinality equals $2^{\aleph_\alpha}$ such that $2^{\aleph_\alpha} = \aleph_{a+1,}$ in conjunction with the second principle of generation. Through mathematical induction on the HT power set function (in connection with the second principle of generation), it becomes clear that the HT table implies GCH, such that $2^{\aleph_\alpha} = \aleph_\beta$, that is, that $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_1$, etc. As the inverse this of function, the HT diagonal function runs through the entire HT number system $\aleph_a$, bringing it to exhaustion, and thus pointing to the anti-diagonal $\aleph_{a+1}$ (or $\aleph_\beta$) system that cannot be contained within the diagonal system $\aleph_a$. In this way, the HT diagonal function brings about the ramification of the ensuing HT power set function, this ramification and zigzag continuing to infinity. In each case, $\aleph_\beta$ is the complement of $\aleph_a$.

The HT diagonal function raises the question as to how the diagonalization procedure is to be specified in advance for all HT number systems. Is each diagonal to be drawn throughout the entirety of each dimensional space or simply through a critical portion of it? The answer is found in the identity function for standard complex and hypercomplex numbers. This identity is defined in terms of the absolute values of each number system.[81] This implies, taking the HT complex numbers as an example, that the HT diagonal for the HT complex plane is drawn only through the first quadrant of the plane, this quadrant containing the absolute values of the *x* and *y* axes. The diagonal for the HT complex plane is thus drawn from the number 1 on the *x*-axis to the

[81] Kantor and Solodovnikov, *Hypercomplex Numbers,* v–vi.

number i on the y-axis. In moving to higher dimensions, each diagonal in lower dimensions remains valid in higher dimensions. For instance, in moving from HT complex numbers to HT quaternions, the transition is made from numbers of the form $a + bi$ to numbers of the form $a + bi + cj + dk$, where a, b, c and d are real numbers and i, j, and k are imaginary. In HT quaternions, the diagonal axis already given for HT complex numbers between a and bi remains valid, meaning that new axes must be drawn to connect between bi and cj, and then between cj and dk. In similar fashion, all the diagonal axes in HT quaternions remain valid for HT octonions. The procedure thus begun can be continued in moving from HT quaternions to HT octonions, etc. In advancing from a number system of x dimensions to the succeeding system of $2x$ dimensions, defining diagonal axes in x dimensions can always be continued in higher dimensions by connecting to succeeding imaginary numbers in sequential order. For any HT number system with x dimensions, the number of diagonal axes for that number system is $x - 1$. These diagonalization axes intersect the positive imaginary numbers for every HT number system, meaning that these axes bisect the absolute values of each HT number system. Thus a schema of diagonalization procedures has been defined for all HT number systems.

However, given that the discussion above allowed for a certain freedom of expression of HT numbers in either linear or multi-dimensional terms, it is also valid to express HT/HTS numbers and their diagonalization procedures as simple line segments, as provided for by surreal number theory. In this case, the length (or decimal expansion) of numbers within each diagonal set relates to the length of the diagonalization (and of other numbers in the succeeding anti-diagonal set) as follows:

Number System	*Diagonal*	*Anti-Diagonal*
Reals	$\aleph_0$	$\aleph_1$
HT Complex	$\aleph_1$	$\aleph_2$
HT Quaternions	$\aleph_2$	$\aleph_3$
HT Octonions	$\aleph_3$	$\aleph_4$
$\vdots$	$\vdots$	$\vdots$

This table shows that the diagonal for each HT number system is not *autological* with respect to the diagonal set for which it is drawn but is *heterological* to it. That is, the diagonalization produces a number that cannot be expressed in the diagonal set. The diagonal set is thus open under the HT diagonal function, though this logical openness becomes closed through the ramified zigzag of the HT power set function in the anti-diagonal set. The subsequent HT diagonal of this anti-diagonal set repeats the process, and so the iteration continues for infinitely many HT number systems.

HT Arithmetic, Measure Theory and Chaos

Significantly, transfinite arithmetic involves certain vagaries that can be resolved in terms of HT number theory. Consider that in transfinite arithmetic, subtracting the set of even integers from the set of all integers is expressed in the equation $\aleph_0 - \aleph_0 = \aleph_0$. But what if the set of all integers is subtracted from the set of all integers? Transfinite arithmetic is not equipped to differentiate the semantic content of which infinity is subtracted from which infinity, and so is unable to answer appropriately for different circumstances. But operations on HT numbers can be defined with reference to particular semantic content. For example, HT number theory allows for operations of the HT power set function φ such that:

φ of the natural numbers $N = \mathbb{R}$,

φ of the real numbers $\mathbb{R}$ = the HT complex numbers,

φ of the HT complex numbers equals the HT quaternions, etc.

These basic HT operations are specified within the HT table. But if these operations are valid, then HT theory clearly has the power to differentiate whether $\aleph_0 - \aleph_0$ is to equal $\aleph_0$ or 0, as based on the semantic content. This algebraic refinement can be expressed in terms of an HT measure theory, and then applied to transfinite probability.

Beyond this consideration of HT algebra and measure theory, since so much has been said regarding freedom of expression, it is possible to define the iteration of HT number systems in terms of dimension doubling. On this basis,

$\mathbb{R}^2 = \text{the HT complex numbers,}$

$\text{HT complex numbers}^2 = \text{the HT quaternions, etc.}$

But there is a possible confusion here. $\mathbb{R}^2$ is also a valid operation within standard analysis, though it leads to the closed quadratic field of the standard complex plane. This need not be a worry; each operation is valid within its respective domain. The point here is simply to state that, in HT theory, squaring a number system can be interpreted as squaring the number of dimensions so as to produce the subsequent number system, for which the cardinality of that system is specified by the HT table.

Unlike the standard complex and hypercomplex number systems, HT number theory provides an adequate basis for modeling transfinite probability. The discussion above in chapter 4 (concerning transfinite probability) made clear that any possible outcome string within the probability space of the $\aleph_1$ casino can be mapped to and modeled by a real number, and so too any real number can be mapped to and modeled by a potential outcome string of the $\aleph_1$ casino. This 1-1 mapping and co-imaging is valid not only between the real numbers and the $\aleph_1$ casino, but between every HT number system and its corresponding transfinite casino. That is, a 1-1 mapping and co-imaging also exists between the HT complex numbers and the $\aleph_2$ casino, the HT quaternions and the $\aleph_3$ casino, etc. Thus the hierarchy of all HT number systems can produce an image of all transfinite probabilities, and so too the entire hierarchy of transfinite probabilities can produce an image of all HT number systems. This is a startling outcome. Taken together, these isomorphic systems provide a basis for a theory of HT modality, discussed below in chapter 6.

Finally, the 2^x function that is fundamental to HT theory is also fundamental to chaos theory and fractal geometry. Even more, the logical features of the Mandelbrot fractal are strikingly similar to those of the HT table. These logical features are the same in that, in both cases, the 2^x function combines with Cantor's three principles in advancing through ordinal and cardinal sequences. This shared logical structure can be seen by comparing the HT table of Figure 2 with the Mandelbrot fractal of Appendix 1.

CHAPTER 6

HT THEORY AND THE SET THEORETIC PARADOXES

The modern paradoxes of set theory bear a strong resemblance to paradoxes of similar construction which have been known from antiquity. In order to set these modern paradoxes in a broad context so as to analyze them better, a few representative paradoxes will be described here.

Epimenides wrote, “All Cretans are liars.” But because Epimenides was a Cretan, his statement makes himself to be a liar. Eubulides was interested in this and other paradoxes. He offered a variation of the liar paradox known as the pseudomenon. He wrote, “A man says he is lying [*pseudomenon*]. Is what he says true or false?” These two variations of the Liar share the same circular logic. If the statements are true, then they are false; if they are false, then they are true.

The liar paradox is malleable, and has been expressed in a number of variations. The following version presents the paradox in a streamlined and tight construction:

The statement within this box is false.

Another formulation goes in the opposite direction, expanding the paradox into a couplet:

Socrates: “What Plato is about to say is true.”

Plato: “What Socrates just said is false.”

Bertrand Russell formulated the Russell paradox in 1901. What is now known as the *Russell set* is defined as the set of all sets, and only those sets, that are not members of themselves. The question arises whether the Russell set is a member of itself. In order to qualify as a member of itself, the Russell set must not be a member of itself. So if it is not a member of itself, it is; if it is, then it is not.

In the wake of the Russell paradox, a number of similar paradoxes were formulated which share the same logical structure. Two of these paradoxes are particularly similar to each other. The barber paradox describes a barber who shaves all those men and only those men of a particular village who do not shave themselves. But does the barber shave himself? If he does not, then he does; but if he does, then he does not. Similarly, the librarian paradox describes a librarian who observes that reference books within the reference room are of two types: those that refer to themselves and those that do not. In order that every reference book in the reference room should have some reference given to it, the librarian compiles a new reference book that refers to all those reference books and only those reference books that do not refer to themselves. But does the new reference book refer to itself? Again, if it does not, then it does; but if it does, then it does not. The intended reference book must be either incomplete (by reason of omitting itself) or contradictory (by reason of including itself).

Kurt Grelling and Leonard Nelson published a paradox in 1908, referred to as the Grelling or Grelling-Nelson paradox. This paradox defines an adjective as "autological" if and only if it describes itself (is true of itself). An adjective is "heterological" if it does not describe itself (is not true of self). Adjectives such as "English," "polysyllabic" and "pronounceable" are autological in that they apply to themselves. Adjectives such as "French," "monosyllabic" and "unpronounceable" are heterological in that they do not describe themselves. The question arises whether the term "heterological" is autological or heterological. If "heterological" is autological (true of itself), then "heterological" is not heterological, and thus not true of itself. But if “heterological” is heterological (not true of itself), then it is autological, and thus true of itself. “Heterological” is heterological only if it is not; but if it is not, then it is. (Syntactically, a paradoxical effect does not arise in analyzing “autological,” which may be interpreted as either autological or heterological without contradiction.) The Grelling-Nelson paradox can be transformed into the Russell paradox. This requires substituting “autological” for Russell’s “member of itself” and “heterological” for Russell’s “not a member of itself.” The Russell set can then be expressed as the set of all sets that are heterological. The contradiction is triggered by considering whether the modified Russell set is autological or heterological.

The Barry paradox was first published by Russell, who attributed it to G. G. Barry. The paradox takes different forms, depending on the exact language used in expressing it. One characteristic expression is, “the smallest positive integer not definable in fewer than twelve words.”[82] This expression contemplates the set of positive integers whose elements require twelve words or more for their definition. Assuming this set of numbers is well ordered, then there is a least such positive integer, and this is the number specified by the Barry formula. But yet if the Barry formula succeeds in defining such a least undefinable number, it has thereby defined that number with fewer than twelve words, since the Barry formula consists of eleven words. Thus the Barry number implies a contradiction. Unlike the other paradoxes described here, the Barry paradox can be diagnosed rather easily in that the paradox equivocates as to what is meant by “definable.” Yet even so, the paradox exhibits remarkable staying power. George Boolos formalized the Barry paradox in in 1989 to produce a simplified version of Gödel’s incompleteness theorem.

[82] Another formulation of this paradox states, “the least number not definable with less than eighteen syllables.”

Beyond the paradoxes that resemble the Liar, there are two paradoxes of transfinite number theory: Cantor's paradox and the Burali-Forti paradox. Together with the Russell paradox, these three make up the fundamental paradoxes of set theory. Cantor's paradox refers to the universal cardinal set, and the Burali-Forti paradox refers to the universal ordinal set. Transfinite theory depends for its development on infinite recursion of the power set function. According to this function, for any infinite set A, the power set of A yields a subsequent ordinal set of greater cardinality. But in considering the universal sets, for which the universal ordinal set is thought to have no successor and the universal cardinal set is thought to have no cardinality greater than itself, there is no well-defined principle by which continued application of the power set function is to be suspended. As a consequence, the power set function of naive set theory is relentlessly applied to the universal cardinal and ordinal sets, thereby implying cardinal and ordinal values that contradict what were otherwise presented as universal. Cantor's power set function thus cannot be consistently defined in relation to universal sets.

The assortment of paradoxes described here might not represent a single general problem. The Russell paradox certainly presents a logical problem that concerns mathematical foundations, though the same is not obviously true for the pseudomenon in a box. Some paradoxes might simply be appreciated as contrivances that resist logical clarification, though which also present no threat to the philosophy of logic or set theory. But the range of diversity of these paradoxes is certainly interesting. What is the range of the particular paradoxes that is of specific interest to mathematical foundations, and what is the cluster of issues within that range which are responsible for causing the difficulty? These questions have been debated for more than one hundred years.

Avoiding the Paradoxes

There have been a number of proposed solutions to Russell's paradox specifically and to the Liar class of paradoxes generally. Frank Ramsey asserted in 1928 that the paradoxes result exclusively from their semantic content but do not demonstrate that logic fails within purely syntactic terms. This view is easy to believe with respect to the Barry paradox, but more difficult to reconcile with Russell's paradox, which seems to be much more syntactic. But blaming the paradoxes on their semantic content became more difficult after Gödel adapted the liar paradox in his famous proofs of 1930 and 1931. Like the librarian paradox, Gödel demonstrated that, upon pain of contradiction, the theorems of first-order logic and number theory must remain incomplete. Further, Gödel's proof demonstrates that the syntactic and semantic sides of first-order logic are much more interconnected than what had been thought, such that it is problematic to conceive of first-order logic in purely syntactic terms.[83] On the same point, if the Barry paradox is to be picked out as especially blameworthy for its semantic content, then why could a revised version of it produce a simplified version of Gödel's proof? At any rate, the syntactic-semantic distinction is highly controversial because of the paradoxes and so has failed to yield any consensus opinion.

For a number of years, Russell's responses to the paradoxes underwent continual change. One of his first responses to the paradoxes was his theory of definite descriptions. Consider the sentence, "The present king of France is bald." The person referred to here is a figment, since France is no longer a monarchy. What then of the sentence, "The present king of France is not

[83] Wilfrid Sellars believed that semantic implications penetrated into what might be thought to be purely syntactic. For this reason, he described the Russell and similarly derived paradoxes as semantic. He writes, "Notice that, although these derivatives parallel the Russell moves, they do so in a way which brings out the essential involvement of the semantical concept of truth…. Russell's paradox is at bottom, a semantical paradox to be handled by semantic distinctions." *Philosophical Perspectives: Metaphysics and Epistemology* (Atascadero, CA: Ridgeview, 1967), 110.

bald." This too refers to a figment. But yet the law of the excluded middle implies that one of these sentences must be true. Russell's solution was to analyze these statements by means of the existential quantifier, thus yielding the formulation: "There is an *x* and *x* has a certain property." It is then clear that both sentences are false in that they fail to satisfy the terms of the existential quantifier. And so Russell began the process of analyzing propositions and sentences in terms of logical structure, a process which led to his theory of types. A key feature of type theory is that it does not allow sets to be members of themselves, a strategy which served to avoid the self-referential paradoxes. Russell revised his early type theory to allow for finite ramification, and then others produced theories of transfinite ramification. Certain versions of ramified type theory were inconsistent in that they admitted of Cantorian diagonal arguments. Like Russell's theory of definite descriptions, these approaches seek the primary logical subject that takes on predicates but which itself is not predicated of another subject. This general project extends from Aristotle, through Kant, Leibniz, Frege, Russell, Ryle, Carnap and perhaps the whole of analytic philosophy.

Another theory pursued by Russell was his zigzag theory. Russell wrote very little on this, and so his presentation of it was sketchy. Zigzag theory rejected types, and so returned to Russell's earlier view of the universal scope of logic, except that certain restrictions are to be placed on the axiom of comprehension (when the axiom leads to contradictions). But the essential nature of the zigzag theory is summed up as follows: "The idea here is that the propositional functions which do not determine classes are all derived by a diagonal construction so that they have a zigzag form."[84]

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[84] Alasdair Urkuhart, "Russell's Zigzag Path to the Ramified Theory of Types," *Russell: The Journal of the Bertrand Russell Archives* 8 (Summer-Winter 1988): 85–87.

The three paradoxes of set theory do not all necessarily seem to stand on the same footing. The Cantor and Burali-Forti paradoxes apply to the high mathematics of Cantor's transfinite theory, whereas the Russell paradox strikes directly at the foundations of logic and set theory. There is good reason to think, then, that these paradoxes might be unrelated and can be treated separately. The critical issue with the Russell paradox might be self-reference, which is not readily apparent within the other two. But from another perspective, all three paradoxes involve the concept of totality, each one referring to "all sets" in absolute terms. On this basis, it can be thought that the concept of totality or universality presents a fundamental problem which exerts itself throughout first-order logic, set theory and transfinite number theory. This latter possibility suggests that the three paradoxes of set theory are deeply interconnected and in need of a common solution.

HT Theory and Cantor's Theorem

It is interesting that the logic of Russell's paradox is embedded within the structure of Cantor's theorem, though the positive force of the theorem is not diminished by it. How is it that the logic of the Russell paradox is so negative within set theory, and yet constructive in this particular case? Similarly, how is it that the Barry paradox can be adapted in developing a simplified version of Gödel's incompleteness theorem? With these positive utilizations of the paradoxes in view, is it possible that a constructive sense of the paradoxes might be discovered within these contexts and then adapted to HT theory?

Cantor's theorem demonstrates that for any set L, finite or infinite, the power set of L has cardinality greater than the cardinality of L. The critical move of this theorem is to prove that a 1-1 mapping cannot exist between the set L and its power set $P(L)$. Cantor assumed, contrary to what was to be proved, that such a mapping does exist; that is, that every element *x* in *L* can be paired in a 1-1 mapping with its correlate subset *x** of *P(L)* and vice versa. The condition therefore arises that particular elements *x* in *L* may or may not be members of their correlates *x** in *P(L).* Therefore, there exists a set within *P(L)* consisting of all the elements *x* in *L* that satisfy the condition of not being members of their correlate *x** in *P(L).* Let this set be identified as *r**. The set *r** is a subset of *L* and thus a member of *P(L).* Therefore, in the putative mapping between *L* and *P(L), r** is the correlate of *r* in *L*. Now consider whether *r* is a member of its correlate *r**. With this question, the logic of the Russell paradox is triggered. If *r* is a member of *r**, then it violates the defining characteristic of *r**, since *r** consists only of elements of *L* which are not members of their correlates in *P(L).* But if *r* is not an element of *r**, then it thereby qualifies for membership in *r**, since *r** consists of all elements of *L* which are not members of their correlates in *P(L)*. Cantor's theorem thus proves that a 1-1 mapping cannot exist between *L* and *P(L),* inasmuch that a mapping between *r* and *r** is impossible. Because it is clear that the cardinality of *P(L)* is greater than or equal to that of *L*, in that for every *x* in *L*, {x} is a subset of *P(L),* it becomes clear from the impossibility of mapping between *L* and *P(L)* that the cardinality of *P(L)* is strictly greater than *L,* and thus *P(L)* can be identified as set *M.*[85]

What can be learned from the logic of the Russell paradox embedded within Cantor's theorem, which may be applicable to HT theory? In answering this question, consider the degree of failure within the theorem of mapping set *L* to *P(L)*. The theorem proves that a 1-1 mapping between the two sets is not possible. Yet it can be asked exactly how much of a mapping is and is not possible within this context. Cantor's theorem begins by showing that every *x* in *L* can be mapped 1-1 to {x} in *P(L),* so this much of a mapping is possible. Consider that if set *L* is the set of natural numbers, then a 1-1 mapping of $\aleph_0$ numbers between *L* and *P(L)* can be accomplished.

[85] Tiles, *Philosophy of Set Theory,* 63–64; Rucker, *Infinity and the Mind,* 256.

But now notice that the functionality of the logic of the Russell paradox within Cantor's theorem is synonymous with Cantor's diagonal argument. In fact, it is commonly accepted that this paradoxical logic in Cantor's theorem is a logical expression equivalent to Cantor's diagonal. On this basis, it is valid to extend the terms "diagonal set" and "anti-diagonal set" from the diagonal discussion so as to apply within Cantor's theorem. It then becomes clear that, with respect to Cantor's theorem, there exists a 1-1 mapping of elements x in the diagonal set L with their correlate subsets {x} in the anti-diagonal set $P(L)$, but that no such mapping is possible between the diagonal set L and the entire anti-diagonal set $P(L)$. It is clear that a constructive relation exists between the paradoxical logic of Cantor's theorem and the diagonal procedure that can be extended throughout infinitely many HT number systems. It follows that the diagonal interpretation of the Russell logic within Cantor's theorem can be subsumed into the HT diagonal function, and that the positive force of the power set function within Cantor's theorem can be subsumed by the HT power set function. In mapping values from the diagonal set L to the anti-diagonal set $P(L)$, where L has cardinality $\aleph_\alpha$ and $P(L)$ has cardinality $\aleph_\beta$, it is possible to map $\aleph_\alpha$ numbers between L and $P(L)$ by means of the HT power set function.

Beyond the isomorphism of the Russell paradox within Cantor's theorem to the diagonal, is there yet a deeper sense in which this occurrence of the Russell paradox has implications for HT theory? The discussion of the HT diagonal function and the HT power set function (in chapter 5 above) defined a relation between these two functions. Is it possible that this same relation is discoverable within an adaptation of the Russell paradox of Cantor's theorem? Recall that the Russell paradox can be stated within the autological-heterological terms of the Grelling-Nelson paradox. Therefore, just as language of "diagonal set" and "anti-diagonal set" can be extended to the Russell paradox of Cantor's theorem from the diagonal procedure, so too the terms "autological" and "heterological" can be extended to the Russell paradox of Cantor's theorem with the same effect. Note that in defining a 1-1 mapping of elements *x* in the diagonal set *L* to their correlate sets {x} in the anti-diagonal set *P(L),* this mapping can be defined as an autological mapping; that is, each element *x* in *L* is mapped to its autological correlative set *{x}* in *P(L)* for which *x* is the singleton in {x}. All other sets in the anti-diagonal set *P(L)* which are not correlate sets of a singleton element *x* in *L* are defined as heterological to the diagonal set *L.* Accordingly, elements of set *L* can be described as both autological and as the diagonal set, and so too elements of *P(L)* can be described as both heterological and as the anti-diagonal set. Because of the naturalness with which the Russell paradox of Cantor's theorem negotiates the terms "diagonal," "anti-diagonal," "autological," and "heterological," it is clear that the logical structure of the Russell paradox as adapted within HT number theory gives expression to both the HT diagonal function and to the HT power set function.

Even as the ramification of HT number systems can be defined in terms of diagonal and anti-diagonal sets, so too this ramification can be defined in terms of autological and heterological sets. In taking these two nomenclatures together, the anti-diagonal set of one system becomes the diagonal set of the following system. Within the new diagonal set, the previous distinction of autological and heterological is done away with, such that numbers formerly heterological to one another within the preceding anti-diagonal set become autological with one another in the new diagonal set. For example, the HT complex numbers are the anti-diagonal set of the real numbers, such that there are $\aleph_2$ numbers in the HT complex plane that are heterological to the $\aleph_1$ numbers of the real continuum. But once the HT complex numbers are presented as the diagonal set, the heterological distinction between real numbers and HT complex numbers is dissolved, such that both of these number systems are autological with one another with respect to the new anti-diagonal set of HT quaternions. HT quaternions are now the heterological set to HT complex numbers, including the real numbers contained within them. This is so because, in each case that a diagonal set is to be mapped into the succeeding anti-diagonal set, all the nested number systems within the present diagonal set are mapped in autological fashion to their correlates within the anti-diagonal set, regardless as to what dimensional number systems are involved. This is to say that each number system is mapped to itself within the subsequent number system. The real numbers are mapped to themselves within HT complex numbers, the HT complex numbers are mapped to themselves within HT quaternions, etc. For any HT number system considered as a diagonal set of x dimensions, there is a succeeding anti-diagonal number system of $2x$ dimensions in which x dimensions are autological to the diagonal set and x dimensions are heterological to it. The x dimensions of the anti-diagonal set that are autological to the diagonal set are autological because these x dimensions of the anti-diagonal set are the same x dimensions as of the diagonal set. Autological and heterological relations change from one number system to the next, such that the heterological relation toward the anti-diagonal set is overcome in the succeeding diagonal set. But although the autological relation increases in cardinality with each number system, the succeeding anti-diagonal set presents an even greater cardinality of heterological numbers. The heterological relation consists of x heterological dimensions set against x autological dimensions, but yet of $2^{\aleph_\alpha}$ heterological numbers set against $\aleph_\alpha$ numbers.

The development of HT theory has presented a number of cases in which freedom of expression has allowed different conceptualizations for mathematical operations or entities. One such example is that of HT algebra, in which squaring an HT number system can be regarded as tantamount to performing the HT power set function on the same system (discussed in chapter

5). This algebraic freedom of expression is relevant in the present discussion. It is clear that the dimension doubling and cardinal sequencing of the number systems as just described with respect to autological and heterological sets is indicative of number systems as described in the HT table and treated in terms of the HT power set function. Yet it is also true that the dimension doubling of these number systems can be described in shorthand as squaring the number of dimensions from one number system to the next, allowing the cardinality to be specified in terms of HT theory.

In reviewing the relation of the logic of the Russell paradox within Cantor's theorem to the reconstruction of its logical structure as advanced within HT theory, the critical difference is that Cantor's theorem considers a possible mapping, but then allows the diverse elements of the mapping to butt heads in a paradox of modus tollens. HT theory contemplates a similar mapping, but avoids the contradiction by redirecting this mapping through a ramified zigzag, thus allowing for a modus ponens. The head-butting of the Russell paradox within Cantor's theorem results from conflating into a single dimension what HT theory separates out into ramified dimensions. It is clear that HT theory does not avoid the Russell paradox of Cantor's theorem, but constructively engages and refashions it. Such a refashioning is possible for HT number theory, but not for standard complex and hypercomplex numbers, which offer no increase in transfinite cardinalities.

As stated earlier in the present study, a major objective of HT number theory is to take up the zigzag pattern as begun in elementary number systems and continue it throughout HT dimensions. The discussion in chapter 5 stated that the zigzag pattern can be implemented within HT theory in terms of (1) the HT diagonal function, which brings the diagonal set to a point of exhaustion, and (2) the HT power set function, which produces a ramification to the anti-diagonal set, which in turn becomes the new diagonal set, suitable for further diagonalization and iteration. Because the HT diagonal function can be defined in the HT complex plane as drawn from the real number 1 to the imaginary number *i*, and then because succeeding HT diagonals can be defined across imaginary numbers throughout HT dimensions (as described in chapter 5), it is apparent that the imaginary numbers assume a prominent position within HT theory. These imaginary numbers can be interpreted as nodal points connecting the inner space of HT dimensions, demarcating the geometric space in which the HT diagonal function and HT power set function operate in tandem with one another. Because the logic of the Russell paradox within Cantor's theorem can be treated constructively within HT theory, it is possible for the autological and heterological language of the paradox to be understood as mapped out by the nodal points of HT dimensions. This is a remarkable claim, in that it offers a possible basis by which to treat the paradoxes of set theoretic foundations. If a ramified zigzag pattern exists within HT dimensions as implemented within the logic of the Russell paradox within Cantor's theorem, then perhaps this ramified zigzag can be extended to the Russell paradox itself.

What then of the possibility of applying the technique developed here to the Russell paradox within set theoretic foundations? In order to consider this, the Russell paradox must be considered in terms of universality, and thus discussed in conjunction with the Cantor and Burali-Forti paradoxes. The question of universality raises the question as to the extent HT numbers can be defined. Do the geometric conditions provided by HT dimensionality impose limits on the size of HT numbers beyond what is present in transfinite theory? How large are HT numbers and what is their maximum limit? In order to consider these questions, HT numbers must be considered in maximal terms.

HT Numbers, Totality and Modality

The discussion of HT numbers thus far has been limited to considerations of HT number systems that have 2^x dimensions, wherein x has been spoken of only as finite. Hence the question arises as to whether HT number systems admit of a transfinite number of dimensions. It may seem that HT numbers have achieved their full range of systems when HT dimensionality is specified as $\aleph_0$, and so this critical point might represent the maximal limit of HT numbers insofar as they are able to find dimensional support. Indeed, it may be thought that this is the juncture at which HT theory incurs the set theoretic paradoxes. But this would be a disappointing alternative. HT theory is predicated on infinitist principles, so there is no cause to shrink back from the prospects of a transfinite number of dimensions. It seems, then, that HT number systems can be conceived of most appropriately as having a finite, infinite and transfinite number of dimensions.

But then another question arises of special significance. If HT number systems admit of finite, $\aleph_0$, $\aleph_1$, $\aleph_2$ dimensions, etc., then *where* are dimensions of $\aleph_1$ and greater located with reference to the $\aleph_0$ dimensions? Do the $\aleph_1$ dimensions fit into the interior space between the sequence of $\aleph_0$ dimensions (as real numbers fill in the continuum between natural numbers) or do they constitute a space over and beyond the $\aleph_0$ dimensions? At issue here is the relation between HT and GCH. In the discussion thus far, HT numbers have lent support to GCH in claiming that $2^{\aleph_0} = \aleph_1$, making $\aleph_1$ the *next* cardinal number to follow $\aleph_0$. But if HT numbers now advance the claim that there are uncountably infinite dimensions between $\aleph_0$ and $\aleph_1$, then $\aleph_1$ would no longer be the first transfinite cardinal to follow $\aleph_0$, thereby negating GCH, or at least making it more complicated than normally thought.[86] The continuum problem asks how many points are on the real continuum. But the question here is whether transfinite numbers themselves constitute a continuum. This latter possibility might seem an appealing alternative. Given that the development of HT theory has never been intended as a defense of GCH, it might be that HT numbers argue ultimately for the undecidability of GCH, suggesting that the difference between the truthfulness and falsity of GCH is aspectual, depending on whether transfinite numbers are considered in analogy to the natural numbers or in analogy with the reals.

So then how evenhanded should HT be with respect to the undecidability of GCH? HT numbers can clearly offer support to Gödel's affirmation of GCH, but what does HT theory owe Cohen regarding the negation of GCH? HT theory might be conceived as a house with two fronts, one conforming to the specifications of Gödel, the other of Cohen. Yet it is significant to note that (with limited focus to the HT table and thus the HT affirmation of GCH), HT entails GCH, which in turn entails AC. Like GCH, AC is also undecidable as based on ZF. It seems that HT is not undecided on AC but completely affirming of it. And it may be that Cohen's rejection of CH was based on indeterminacy in ZF that owed to the undecidability of AC. Mary Tiles writes as follows:

> In ZF the power set of an infinite set is impredicatively defined and its membership is thus under-determined relative to the remaining axioms. The inclusion of the power set axiom thus leaves an indeterminacy in what is, from the point of view of ZF, to count as a set. It is this under-determination which makes it possible to produce models of ZF in which CH holds and models in which it fails, and also renders the force of AC indeterminate.[87]

[86] It may be that the truth of GCH consists strictly in the claim that $2^{\aleph_0} = \aleph_1$, and that $\aleph_1$ is the next number to follow $\aleph_0$ only with reference to the series of alephs, but that there is an additional geometric sense in which $\aleph_1$ is not the first to follow $\aleph_0$. This would be analogous to the relation of natural numbers to rational and real numbers, wherein 1 is the first natural number to follow 0 but not the first number in absolute terms to do so.

[87] Tiles, *Philosophy of Set Theory*, 216.

It seems then that HT numbers can be interpreted as an inner model of ZF that entails both GCH and AC, thus discounting the indeterminacy of ZF that allows for Cohen's negation of CH. The infinity of HT dimensions need not be interpreted as occupying space between $\aleph_0$ and $\aleph_1$ (thereby lending support to the negation of CH), but can be interpreted as occupying space beyond the countable sequence of transfinite dimensions. However, both this question and the undecidability of GCH with respect to ZF and HT cannot be resolved here, and thus must be left as open questions.

Defining HT numbers in terms of an infinite and even transfinite number of dimensions requires greater abstraction than number systems with finitely many dimensions, though defining these systems can be done according to the pattern of preceding number systems. Just as a finite number of dimensions can be defined in sequential order wherein the real axis is first, the imaginary axis is second, the axes of HT quaternions are third and fourth, etc., so too a transfinite number of dimensions can be defined as ω_0, ω_1, ω_2, etc. It seems possible, then, for HT numbers to be defined with a transfinite number of dimensions without limit, and thus there are no constraints imposed on the development of HT number systems with respect to geometric representation. Accordingly, the development of HT number systems is as open-ended and limitless as is that of transfinite numbers. The only constraint on the progression of transfinite and HT numbers is the consideration of totality as relating to the set theoretic paradoxes.

What then of the set theoretic paradoxes? There is clearly an inclination toward conceptualization of the universal set, though the paradoxical implications of this conceptualization are well known. But because HT number theory has successfully treated the logic of the Russell paradox within Cantor's theorem, it is possible that the same or similar techniques might be employed here.

Let the universal set of HT number theory be designated as Ω. Now consider that the present study has proposed theories of both HT numbers and transfinite probability. Because HT number theory and transfinite probability have been defined in isomorphic terms, there exists a 1-1 mapping between their corresponding domains. The discussion above in chapter 4 made clear that any possible outcome string within the probability space of the $\aleph_1$ casino can be mapped to and modeled by a real number, and so too any real number can be mapped to and modeled by a potential outcome string of the $\aleph_1$ casino. As further developed in chapter 5, this 1-1 mapping and co-imaging is valid not only between the real numbers and the $\aleph_1$ casino but between every HT number system and its corresponding transfinite casino. That is, a 1-1 mapping and co-imaging exists between:

Real numbers and the $\aleph_1$ casino,

HT complex numbers and the $\aleph_2$ casino,

HT quaternions and the $\aleph_3$ casino, etc.

Based on this isomorphic relation, it is clear that the universal set of HT number theory has a 1-1 mapping and co-imaging relation with a corresponding universal set of transfinite probability. Therefore, let Ω_+ now designate the universal set of HT numbers and let Ω_- designate the universal set of transfinite probability. Two universal sets now exist as polar opposites within a structure of HT modality.

The existence of two universal sets within HT modality raises the possibility of duplicating the paradoxes of set theory. Accordingly, let the Cantor, Burali-Forti and Russell paradoxes be duplicated so as to apply *mutatis mutandis* within transfinite probability. HT modality now consists of six theoretic paradoxes. The objective of the present discussion is to resolve these paradoxes so as to produce a unified theory of HT modality.

Next, let the HT diagonal function be applied to both Ω_+ and Ω_-. Clearly, this operation renders the two universal sets extremely unstable. But even further, this move seems especially hazardous, in that the HT diagonal function has been defined in tandem with the HT power set function. But notice what follows. An iterated application of the HT power set function is impossible with respect to an algebraic operation between Ω_+ and Ω_-. This function implies an increase in both dimensionality and cardinality, yet Ω_+ and Ω_- are isomorphic with each other, having the same dimensionality and cardinality. Consequently, these two universal sets cannot relate algebraically to one another as input or output through the HT power set function. The HT power set function has been zeroed out. As delimited within HT modality, justification has now been given for suspending the HT power set function. Because no such suspension for the power set function can be given within set theory, the present circumstance seems to provide a much needed refinement on the set theoretic paradoxes. The HT diagonal function has rendered Ω_+ and Ω_- unstable, yet this instability has not fallen into contradiction with the HT power set function.

How then can Ω_+ and Ω_- relate to one another algebraically? A start on this question has been made in having performed HT diagonal functions on both. But what are the implications of this function in the present context? How are these diagonals to be conceived? A transfinite number of instances of HT diagonals are available for comparison. It has been a universal property of HT diagonals to transcend the diagonal sets to which they pertain. That is, an HT diagonal is never constructed within its diagonal set but within its anti-diagonal set. Put another way, every HT diagonal is heterological to its diagonal set but autological to the anti-diagonal set. Therefore, if this rule is to remain consistent in the present case, the diagonals within Ω_+ and Ω_- are not produced from within themselves but from a heterological source. Number and probability are of course heterological to one another as modal opposites. What then is the significance of the diagonals on Ω_+ and Ω_-? Based on the logic of the HT diagonal function, the two universal sets have each marked its heterological opposite with its own autological sign. Number theory, conceived as its own diagonal set, has been marked by its anti-diagonal set of probability, and so too probability, conceived as its own diagonal set, has been marked by its anti-diagonal set of number theory. The two heterological species of mathematical objects have squared off with one another in advance of a modal synthesis. But where is the algebra by which to effect such a synthesis?

The present situation is peculiar in that two HT diagonals have been defined, but yet no increase in dimensionality or cardinality is applicable. Also, the HT diagonals present a horizontal relation rather than vertical. So what algebraic operation is suitable for two bilateral universal sets, Ω_+ and Ω_-? Consider the possibility of addition, such that $\Omega_+ + \Omega_- = 2\ \Omega$, or perhaps even that $\Omega_+ + \Omega_- = 0$. Addition and multiplication are applicable for elements of the same species of mathematical objects, and so the present objects certainly do not qualify for these operations. Exponentiation is the most likely candidate, inasmuch that exponentiation and roots are the most powerful algebraic means of advancing from one number system to another, indeed, the only operations capable of doing so for systems greater than the real numbers. But the HT power set function has already been excluded. However, the preceding discussion in chapter 5 described a near equivalence between the HT power set function and that of squaring. These two functions have been described as shorthand equivalents, and thus interchangeable in advancing from one HT system to another. Perhaps now is a critical juncture at which the near equivalence between them is teased apart. Consider then that, within the context of HT modality, the universal sets Ω_+ and Ω_- yield the expression Ω^2. What does this mean? This question may be treated in analogy with the continuum problem, wherein it is asked what $2^{\aleph_0}$ reduces to. In the same way, the present question considers what Ω^2 reduces to. Again, the possibility of a contradiction presents itself. But is a contradiction here inevitable?

One of the many controversial issues touched on in the present study is the syntactic-semantic distinction. What then are the syntactic and semantic implications of Ω^2, and is there an interpretation in which they can be taken together? Syntactically, the present objective is to treat both the diagonal and anti-diagonal distinction and the autological-heterological distinction in such a way as to continue the ramified zigzag pattern as explicated within the Russell logic of Cantor's theorem. If done correctly, this logical progression should be able to engage and resolve the set theoretic paradoxes of HT modality. Because an HT diagonal can be defined for both Ω_+ and Ω_-, the interpretation follows that the opposing universal sets are squaring off as if to negotiate their autological and heterological distinctions. The logic of this encounter is nothing unprecedented for either number or probability. Each system has negotiated autological and heterological conditions throughout all of their dimensional progressions. The key difference here is that, rather than negotiating these conditions in a vertical progression, each system is now negotiating autological and heterological conditions within a horizontal bilateral embrace. Semantically, number can be described as the metric of what is. Probability can be described as the metric between what is not, but what might come to be. One is a science of being, the other of nonbeing. The two universal sets are alien to each other, and yet each one can produce an exact image of the other. In terms of both syntax and semantics, the polar opposites are recognizing one another as dialectical opposites. This is the syntactic and semantic significance of the expression Ω^2 within the present context.

The essential force of the Russell logic of Cantor's theorem as described in HT theory is to map the totality of autological and heterological terms between the diagonal and anti-diagonal sets. Within the present context, there are two diagonal sets, two diagonals and two anti-diagonal sets. The presence of an HT diagonal on Ω_+ and on Ω_- indicates that these two sets are both mapping all their sets on to the other. Each universal set is acting simultaneously on the other, making each universal set both subject and object. Therefore, as signified by Ω^2, all the sets of Ω_+ are mapped to their autological syntactical correlate sets within Ω_-, and so too all the sets in Ω_- are mapped to their autological syntactical correlate sets in Ω_+. That is, the syntactic force of Ω^2 maps every set of Ω_+ to its autological place within its heterological universal set, and every set of Ω_- to its autological place within its heterological universal set. This mapping perfectly maps every set of Ω_+ and Ω_- without remainder. Every set has been mapped according to a perfect dovetailing of autological and heterological conditions. This bilateral mapping abolishes the syntactic distinction between number and probability.

On the semantic side, the nature of number is determinate being. The nature of probability is indeterminate nonbeing. Within their heterological encounter, number seeks to enumerate the totality of all probabilities, thereby giving its own mark of determinacy and being to probability. In the same encounter, probability places its mark of indeterminacy and nonbeing on number, thereby eroding the determinate character of number. In this bilateral exchange and blurring of properties, the semantic distinction between number and probability is abolished. Thus as to syntax and semantics, the modal distinctions of number and probability are lost. Put more constructively, a transcendental synthesis of being and nonbeing in mathematical terms is hereby made possible.

On the basis of the syntactic and semantic conditions described here, what is the only numeric outcome of Ω^2 that is logically and algebraically consistent? The claim of HT modality as advanced here is that, where Ω^2 involves Ω_+ and Ω_-, $\Omega^2 = 0$. This claim is in keeping with the account of the via negativa offered above in chapter 1 in which being and nonbeing are described as contraries. For Neoplatonists of the via negativa, the fusion of being and nonbeing is the basis of becoming. In HT modality, the origin of mathematical becoming is the null set. HT modality thus offers the analogy that numbers erupt out of the null set as if emerging from a mathematical big bang of being and nonbeing. The fusion of being and nonbeing is regarded as analogous to the fusion of matter and antimatter. HT modality thus yields the algebraic result that the square root of 0 equals Ω.

The present discussion engages the Russell paradox in a clear manner. The ramified zigzag treats the Russell paradox in the same way as in Cantor's theorem, placing emphasis on (1) diagonal and anti-diagonal sets, and (2) the semantics of autological and heterological. Two details were changed here in the treatment of the Russell paradox from that of the Russell logic of Cantor's theorem. First, the function of squaring was used here as a near equivalent to the HT power set function (this near equivalence was discussed in chapter 5). Second, the ramified zigzag made a U-turn in returning to its origin at the null set rather than continuing to make a vertical ascent. This extreme zigzag is necessitated by Cantor's paradox, but also in keeping with the proposed synthesis of being and nonbeing. The HT ramified zigzag described here thus resolves the Cantor and Russell paradoxes. The HT ramified zigzag pattern progresses from totality through modality to the null set. (Describing the HT treatment of the Burali-Forti paradox is a bit more complicated, and so will be discussed a bit later.)

HT theory produces a grand circuit of ramified zigzag, which begins with the null set, continues through the natural numbers, integers, rationals, irrationals, HT complex numbers, all HT number systems, HT totality, HT modality, and then returns to the null set, the point at which it began. Each one of these zigzags has been construed at some point as violating that which was regarded as absolute. If the dialectical synthesis of being and nonbeing proposed here seems disquieting, consider that this zigzag simply fills out the pattern as exhibited elsewhere. The two crucial innovations of the present study have been: (1) to define the zigzag from real to imaginary numbers as involving an iterative increase in cardinality, and (2) to define a synthesis of number and probability within HT modality so as to engage and finesse the set theoretic paradoxes. The zigzag insight involved in navigating these two junctures is to recognize that the number *i* is to the law of signs what the Russell paradox is to modality. These logical progressions are the same in that the same ramified zigzag moves through both. Rejecting the zigzag in moving from totality to the null set because of an absolutist conception of modality is like rejecting the zigzag in moving from real to imaginary numbers because of a commitment to the law of signs. The HT ramified zigzag transcends the distinction of being and nonbeing just as imaginary numbers transcend the distinction of plus and minus.

The most crucial point at which criticism of the present discussion might be directed is the algebraic claim that $\Omega^2 = 0$. This claim is satisfactory in terms of its semantic interpretation, and also because of the critical role it plays within the grand ramified zigzag pattern. But insofar as the algebra is considered in analytic terms, it seems rather ad hoc and unjustified. The algebraic operation considered here is singular, whereas algebra tends to deal with operations that are valid in infinitely many cases, at least insofar as exponentiation is concerned. This, then, seems to be the weakest link of the discussion. To make up for this shortcoming, attention will be directed to Gaussian integers and the class number problem.

In 1796, Gauss proved the quadratic reciprocity law, using what are now referred to as Gaussian integers. These numbers take the form of complex numbers, except that they are limited to integers on the real and imaginary axes. Gauss showed that these integers obey their own version of the fundamental theorem of arithmetic, reducing to their peculiar system of prime numbers or "irreducibles." Gauss proved this in the "unique factorization theorem." The Gaussian integers proved to be of significant interest, and then it turned out that they can be modified so as to be expressed as half integers. Subsequently, it became clear that the imaginary unit can be redefined for these numbers, such that each substituted value of *i* gives rise to its own number system. It is then meaningful to ask for which of these substituted values a reasonable number system results. In Gauss' lifetime, it was known that there were nine values for which a unique factorization theorem exists, including 1, 2, 3, 7, 11, 19, 43, 67 and 163. It has been proved that a tenth such number does not exist. It is then possible to consider the margin by which other numbers substituted for *i* fall short of unique factorization. Depending on this margin of failure, the substituted values of *i* are put into different classes. Class number 1 is for numbers that produce unique factorization, class number 2 for numbers that barely fail, class number 3 for numbers that fail a bit more, etc. Since the time of Gauss, mathematicians have been interested in deep and complicated properties relating to the class number system. One interesting consideration is to determine the largest number for each number class. In 1983, Zagier and Gross produced a large equation that determined the greatest number for each number class, thereby solving Gauss' class number problem. As Keith Devlin writes, "What is remarkable is the fact that a single curve somehow controls the behavior of an infinite family of number systems."[88]

It is interesting that both the class number problem and HT theory give special emphasis to the number *i*. The HT diagonal function is defined in terms of imaginary numbers, and consequently the HT power set function and Cantor's theorem were interpreted in the same way. The class number problem suggests that the underlying value of *i* can be reset or rejigged for infinitely many number systems. It is of great interest for the present study, then, that the grand HT ramified zigzag can be interpreted as providing a mechanism for resetting the value of *i* for the class number problem. In putting these two considerations together, the synthetic claim can be made that each iteration of the grand HT ramified zigzag is responsible for resetting the value of *i* in the subsequent HT number system. Consider now that a synthesis has already been proposed between HT and surreal numbers. Surreal number theory defines the null set a consisting of two sets, null set left and null set right. HTS theory seeks to define the HTS null set in terms of a vertical dimension, top set 0 and bottom set –*i*. HTS number theory diverges from the standard definition of the null set by adding a negative imaginary number. HT theory redefined the imaginary unit in chapter 5 so as to be expressed as $i = \pm\sqrt{-1}$. The claim was made then that this definition provides greater access and flexibility in moving between the real and imaginary axes. The present discussion provides a critical example of such flexibility. This provides the very basis by which the value of *i* is reset within the ensuing number system. The plan thus emerges for a unified hypercomplex-transfinite-surreal-Gaussian (HTSG) number system.

[88] Devlin, *Mathematics: The New Golden Age*, 75–82.

To refine the algebraic claim made above, $\Omega^2 = 0$ within HT modality theory, where Ω and 0 are in the same HT number system. But from the perspective of HTSG, this is a crude understanding, both in terms of algebra and in that HT modality does not offer an adequate understanding of the Burali-Forti paradox. Thus for HTSG, $\Omega^2 = 0$, where Ω and 0 are *not* in the same HT number system but rather, 0 is in the succeeding HTSG number system. In saying that the square root of 0 equals Ω, the force of Ω^2 is in the HTSG number system prior to the null set under consideration. This refinement of HTSG relative to HT provides a basis for solving the Burali-Forti paradox.

Like Cantor's paradox, the Burali-Forti paradox is triggered if a *next* ordinal number is produced. But within HTSG theory, there is not a *next* ordinal number with respect to the universal set of a particular HTSG number system, for the ramified zigzag returns to the null set of the succeeding HTSG number system. Accordingly, the concept of totality must be delimited to a particular HTSG number system. Thus for any HTSG_α number system, the ordinal following the universal ordinal for that system is not a greater ordinal of HTSG_α but the first ordinal (the null set) of HTSG_β. The logic here is the same as what Russell reluctantly accepted in turning to a ramified theory of types, having first rejected the axiom of comprehension. It is also similar to Cantor's response to the same paradox. Cantor concluded that the ordinal numbers are well ordered but do not form a set. In adapting Cantor's claim here to NBG set theory, the transfinite ordinals form a class but not a set. The ramified HT zigzag theory thus emerges from HT number theory to HT modality, and then through all HTSG systems of number and modality.

Before Cantor developed his transfinite theory, infinity was regarded as an undifferentiated totality. Cantor showed that is not the case, and that infinity can be described on a scale of increasing orders which stretch infinitely beyond simple infinity. The present study makes the same claim with respect to totality. There is not a single totality, as grasped by the first mention of a universal set, but a vast spectrum of HTSG totalities which dwarf our own "universal" set theory by a margin infinitely larger than that by which all the galaxies of the universe dwarf our own Milky Way. It is this totality of totalities, if the expression may be allowed, which shows the need for rejecting an absolutist understanding of the axiom of comprehension.

Gödel's incompleteness theorem made a sharp distinction between syntax and semantics that is overcome in HT theory. This is because the HT ramified zigzag can preserve the semantic content that is lost from a diagonal number system by retrieving it from the anti-diagonal system. Accordingly, HT theory ramifies the Gödel statements just as with the Russell logic of Cantor's theorem and the universal statements of set theory. Within any HTSG number system, all statements of universality are to be qualified as to which HT number system or HTSG universal sets are under consideration. Gödelian truths that cannot be expressed within a particular HT diagonal number system can be expressed within the succeeding anti-diagonal system.

In considering again the algebraic validity of the expression $\Omega^2 = 0$, the claim was made above that this algebraic claim was a weak link in the theory, due to this operation having so little analytic support. But a great deal more analytic support has now been given. It has become clear that HTSG makes this claim for every HTSG modal system. The discussion in chapter 5 states that HT algebra can be defined in measure theoretic terms, based on the semantic inspection of the particular numbers involved in the algebraic operation. To recall an example, the meaning of $\aleph_0 - \aleph_0$ cannot be treated in purely syntactic terms, but must distinguish, for example, between (1) subtracting the set of even integers from the set of integers, and (2) subtracting the set of integers from the set of integers. This is a fundamental difference between HT theory and Cantor's transfinite theory. The additional semantic power attributed to HT theory here is due to the close connection the theory maintains between algebra and geometry. Such direct semantic interpretation is not available for transfinite probability within itself, but is made available to it through a synthesis with HT/HTSG number theory. Transfinite probability can thus be interpreted in terms of HTSG measure theory, providing a basis for HTSG probability and HTSG modality. Because of the close association shown here between algebra and geometry, and the ability of HTSG theory to inspect semantic values, the algebraic operation $\Omega^2 = 0$ is valid for all HTSG modal operations. The syntactic and semantic context of this operation is always such that the autological and heterological aspects of Ω_+ and Ω_- coalesce in such a way as to recognize each other as their dialectical opposites.[89]

What now of the status of the grand HT ramified zigzag? Where does it go from here? This question will be discussed in chapter 7.

[89] The validity of this algebraic operation is conceived in terms of a delimited universe of discourse or range of significance, and so is not construed as having universal applicability. The range of significance is determined semantically.

CHAPTER 7

IMPLICATIONS OF HT MODALITY

HT modality has significant implications for a variety of considerations, including contemporary physics, traditional metaphysics, epistemology, ethics, information theory and beyond. Some of these implications are discussed below.

HT Modality, Quantum Physics and Quantum Computing

Paul Dirac developed a mathematical quantum system in which square roots on the *x*-axis were interpreted through relativity theory as implying the existence of negative particles. These anti-matter particles were discovered shortly afterward by Carl Anderson, earning Dirac and Anderson the Nobel Prize in Physics. The same sort of reconceptualization through square roots is possible for the *y*-axis. The number *i* is a fundamental component of quantum equations. The proposal was made in chapter 5 above that the HT imaginary unit be defined as $\pm\sqrt{-1}$, making both *i* and –*i* satisfy this expression in equally fundamental fashion. When applied to quantum equations, one result of this renormalization is to allow for imaginary time. Accordingly, time can be interpreted as positive and negative on the *x*-axis, and imaginary on the *y*-axis. The most immediate application for this claim is to resolve problems of "spooky action at a distance" as concerning the EPR paradox. Within this quantum context, a single instant of imaginary time can be calibrated as equal to one Planck unit of time.[90] Once this understanding of imaginary time is provided for within quantum mechanics, a similar conceptualization can allow for an HT number generator that operates in imaginary time. A number generator operating in imaginary time contrasts sharply with Peano arithmetic, wherein each number has a "successor," as if algebraic constructions are limited to linear time. HT number theory thus obtains license to construct algebraic operations in imaginary time (most particularly those involving supertasks).

The quantum renormalization stated here counters Hawking's no boundary condition in which the quantum singularity of the big bang is set in imaginary time, such that it is ungrammatical to ask what happened prior to time, as if asking what is north of the North Pole.[91] The renormalization claim advanced here critically treats i and imaginary time in a way that Hawking does not. But what else must be supplied from HT theory toward addressing the quantum context? The HT diagonal function can be extended to quantum theory so as to be logically and algebraically equivalent to the function of collapsing the quantum wave packet. Collapsing the quantum wave is of course what leads to the uncertainty principle within the Copenhagen interpretation. The Copenhagen collapsed wave results in voiding possibilities within the sine wave on the standard complex plane. However, collapsing the same sine wave on the HT complex plane (as adapting the quantum equations appropriately in HT theory) would void only $\aleph_1$ possibilities, while leaving $\aleph_2$ possibilities within the anti-diagonal HT wave function as found in higher dimensions. Quantum results from HT theory would thereby vary

[90] Imaginary time within the EPR context should be conceived of as operating in the time frame of a Planck unit.

[91] Stephen Hawking, *A Brief History of Time,* 136-39.

dramatically from Copenhagen positivism. (Of course, this claim is merely hypothetical at this point, pending further development of HT theory.)

Hawking proposed in his Information paradox that information is lost from the universe through the singularity at the center of a black hole. Leonard Susskind and others counter that the information is smeared along the event horizon, so as never to be lost inside the core. Hawking has changed his position to claim information is not lost in a singularity because it is preserved in parallel universes. Both Hawking and Susskind agree that information cannot be processed through a singularity. HT theory proposes a more powerful number theory such that what has hitherto been interpreted as singularities might now be interpreted as polarities (again, pending further development of the theory). From the point of view of HT theory, information is not lost from within a black hole but is lost from within the standard complex plane. This information should be retained within the HT complex plane or higher HT number systems.

But there is a surprising implication here. The HT wave function seemingly allows for hidden variables as hypothesized by Einstein. This is odd in that HT theory advocates the same sort of determinate-indeterminate uncertainty as commonly interpreted within quantum mechanics. Is HT theory now arguing against its own modal outlook? Not ultimately, because any determinacy provided by HT number theory would collapse within the dialectical synthesis of HT modality.

Inasmuch that the discussion here is concerned with processing quantum information, it can be noted the universal Turing machine can be adapted within the HT quantum context. The universal Turing machine can be analyzed in terms of an infinite array of universal Turing machines and then subjected to a diagonal argument, thus producing a theory of HT computing. Then, because Turing's halting problem was defined in analogy with Gödel's incompleteness theorem, the same HT ramified zigzag that applies to Cantor's theorem, set theoretic paradoxes and Gödelian truths can also be used to solve the halting problem. Note that an HTSG quantum computer operates in imaginary time, as described above. This computing model should be sufficient to prevent information from escaping from a black hole, and thus would be of a power to solve Hawking's information paradox. The HT ramified zigzag pattern can thus be taken as extending through mathematics, computer science, and quantum physics (including black holes and the big bang).

HT Modality and Metaphysics

Laplace was a strict determinist, and yet he was one of the pioneering developers of probability theory. This was because Laplace sought to understand determinism systematically, and so interpreted contingency as the attenuated limits of determinism. Leibniz' modal outlook was quite different. He understood both necessity and contingency as having distinct modal avenues to God. For Leibniz, the ontological argument is an a priori proof for the necessary existence of God. Conversely, the cosmological argument is an a posteriori proof for cosmic contingencies and possibilities receiving ultimate grounding in God's necessary existence. For Laplace, contingency is a mere privation. For Leibniz, contingency possesses a distinct metaphysical quality from determinism and its privation. And because Leibniz believed that efficient and final causes also terminate with God, he conceived of all modal realities within a theological context. For Leibniz, God is the God of necessary being, causal being, appetitive becoming and contingent becoming.[92] But even so, Leibniz' view of contingency in itself is not that different from Laplace's; Leibniz simply thought that contingency in itself was inadequate

[92] G. W. Leibniz, *The Monadology*, §29–52; 77–90.

unto itself apart from a metaphysical grounding. Others, such as Hume, give a more robust understanding of contingency as perhaps ungrounded in ontology.

With Laplace, Leibniz and Hume serving as exemplars of traditional modal thought, what sort of construction is envisioned by HT modality? HT theory clearly agrees with Hume against Leibniz and Laplace that contingency offers a distinct modal nature than what is a mere privation of necessity and determinism. In terms of cosmological thermodynamics, Laplace and Leibniz would have regarded infinite entropy as the heat death of the universe. But Hume would have regarded this as an open empirical question. The discovery of Bose-Einstein condensation shows, in fact, that the continued "privation" of order and determinism leads to an "opposite" state which, as Plato wrote in *Sophist*, has its own (positive) nature. On this view, entropy, lack of order and contingency are more than mere privations, thus contradicting the views of Laplace and Leibniz.

HT modality rejects the claim that necessity or contingency can be conceptualized appropriately in isolation from one another. HT modality asserts rather than modal categories are to conceived as within a quantum blur of imaginary time. The discussion in chapter 6 claimed that number and probability operate on one another simultaneously such that each is subject and object with respect to the other. The same claim is advanced here with respect to the quantum context of cause and effect. If quantum mechanics is subject to imaginary time, as claimed above with respect to the EPR paradox, then imaginary time would have major consequences for classical linear conceptions of cause and effect.

Discussions of quantum mechanics, imaginary time and quantum computing raise the question of the millennium problem concerning the equality of P and NP. There are two crucial considerations of this problem. First is the qualitative or metaphysical consideration, and second is the quantitative or computational consideration. If it should turn out that $P \neq NP$, then the qualitative or metaphysical implications would not be very interesting, since they would conform to mainstream expectation. But if it should turn out that $P = NP$ (as asserted here), then the metaphysical implications would be stunning. When computer science entertains the possibility of supplementing a deterministic computing machine with a black box, as if computational input should be provided by an oracle, the black box is taken to be as nondeterministic as if operating by pure chance, even though the black box or oracle can be stipulated as guessing correctly 100% of the time. The metaphysics of this deterministic computer on the one hand as juxtaposed with a random black box on the other is the same metaphysical juxtaposition contemplated within HT modality. Put another way, Neoplatonists of the via negativa have contemplated the metaphysical juxtaposition described here for more than two thousand years. Even more, they have understood the metaphysical question in such a way as to argue for the equality of P and NP, except that the quantitative and computational aspect of the question has been unknown to them. But if it should turn out that the metaphysical vision of the via negativa is correct, then what would be the computational implications? Engaging in metaphysical speculation with the via negativa would perhaps not interest most computer scientists. However, if quantum computing provides real world circumstances that agree with the metaphysics of the via negativa, then the present discussion would be much more than a metaphysical abstraction. The discussion above concerning an HTSG quantum computer operating in imaginary time is the sort of quantum reality by which such a result would be possible. Such a quantum computer would be able to solve NP problems efficiently in polynomial time, just as easily as imaginary time can communicate signals involving spooky action at a distance. However, it is not clear that this information could always be accessed from within our Newtonian frame of reference. The de re results might remain inaccessible within a black hole, for instance.

The consideration just given to P and NP was expressed in both qualitative and quantitative terms. Significantly, these are the same terms of the present broader discussion. Chapter 6 considered HT modality in quantitative (algebraic) terms and now chapter 7 is considering the theory in qualitative (philosophical) terms. The purpose for this comparison has been to determine whether the two accounts are congruent. The qualitative descriptions of HT modality are entirely consistent with the algebraic operations of the theory. Additionally, the quantitative and qualitative conditions of the present theory overlap perfectly with real world considerations of quantum physics and the question of P vs. NP. This is a remarkable convergence.

What then is the claim here as per necessity, determinism and contingency? In chapter 6 above, a modal synthesis was proposed between the universal set Ω_+ of HT number theory and the universal set Ω_- of transfinite probability. The synthesis proposed there was a quantitative synthesis expressed algebraically. The objective here is to describe the same synthesis qualitatively through concepts. The claim here is that necessity and contingency are each the modal ground of the other. This is the metaphysical basis for the equality of P and NP. This metaphysical synthesis pertains to quantum physics and to cause and effect just as the modal synthesis in chapter 6 pertains to mathematics.

The discussion here harkens back to the discussion in chapter 1 dealing with the classical conception of *being* and the alternative conception as advanced by the via negativa. The discussion there was cast in terms of Plato and the Neoplatonist tradition. The discussion here can be expressed in terms of one of the most prominent members of that tradition, namely, Hegel. Hegel spoke vividly of the synthesis of Being and Nonbeing in his *Science of Logic,* but also developed the dialectical logic for this synthesis in his *Phenomenology of Spirit.*

Consider now that, within a Hegelian context, the synthesis of being and nonbeing as discussed quantitatively in chapter 6 can be given a deeper and more natural analysis because of the specific dialectical features of Hegel's syntax and concepts. In the quantitative discussion, Ω_+ and Ω_- were said to be dialectical opposites in which each had marked its heterological opposite with its own autological sign by means of the HT diagonal function. In Hegel, opposing subjects engaging one another as heterological predicates is the ideal syntactical construction. Hegel describes being and nonbeing as opposing subject and object of the other. Within this synthesis, Being and Nonbeing are autological as subject but heterological as object. For classical thought, this construction reflects a deviant logic. For traditional thought, this sort of dialectical construction amounts to syntactical paradox. From Aristotle to Descartes to Kant to Frege-Russell, the received tradition seeks to retain the strict subject-object relation in which the subject (as ontologically primary) receives predicates (as ontologically secondary). The chief exception here is for a subject to receive itself as predicate, as in Aristotle's "nous nousing nous." This is an example of the magisterial autonomous subject, Aristotle's version of a Platonic form, "itself by itself with itself." But in either case, the Parmenidean tradition construes a primary substance as the subject that receives predicates. [93]

More deeply, the Parmenidean grammar of subject-object conveys the stable semantic content of a concept. But in Hegel, the syntax does not produce a static *concept*, but a synthesis of opposing concepts into a dynamic *notion.* Neither of the concepts "understands itself" until united with its opposite. That is, the concepts are unstable when autonomous and autological, but

[93] Leibniz writes, "It is indeed true that when several predicates are attributed to a single subject and this subject is attributed to no other, it is called an individual substance" *Discourse on Metaphysics,* §8; in *Philosophical Essays,* trans. Roger Ariew and Daniel Garber. (Indianapolis: Hackett Publishing, 1989), 40–41.

are rendered intelligible in a heterological synthesis. The semantic contents or identities within the opposing concepts are blended so as to lose their individual distinctiveness within the notion.

Hegel describes the master-slave relationship as a failed dialectic on the part of the master but a successful one on the part of the slave. The master fails to recognize the slave as other, that is, as someone in the same image as himself. In his estimation, the slave is nothing but a means to the master's ends. But the slave is led to a greater self-consciousness because of his oppression, and thus gains insight that is hidden from the master. To generalize on this example in a manner differently from Hegel, the Parmenidean tradition routinely fails to recognize dialectical opposites. From the Parmenidean point of view, such positive force should not be exerted from the negative. Yet this is precisely the claim made by Hegel. Hegel describes the negative as having the same sort of constructive power as described above in chapters 5 and 6 with respect to the HT ramified zigzag theory. The HT zigzag as proposed herein is offered as an instantiation of Hegel's commitment to the negative. Because of this commitment on Hegel's part, he criticized Kant for failing to recognize the constructive power expressed within Kant's antinomies. Anselm's ontological argument describes God as perfectly composed of the positive attribute of necessary being, though without any acknowledgement of the negative corollary of nonbeing. Descartes made the same presumption in the Fifth Meditation. All of these claims are Eleatic errors as viewed from the via negativa.

The power of the negative presents a profound metaphysical problem. Hegel conceived of it as having power to construct, but yet it also deconstructs. The via negativa and apophatic traditions are quite often spoken of in synonymous terms. However, there is a sense in which they may be teased apart. Those who call themselves apophatic might tend to be committed to the mandate of always saying no, as if the sole metaphysical task is always to deconstruct. But the via negativa can be construed very differently. Rather than being committed to a negative method, one can follow Hegel and HT ramified zigzag in producing positive constructions.

Is it possible to reconcile the opposing views of classical thought and the via negativa? In fact, the critical logical structure described above in which Ω_{+} and Ω_{-} acted on one another simultaneously as opposing subject and object can be construed by means of the Löwenheim-Skolem theorem. Whereas Ω_{+} may be construed as subject in the standard interpretation acting on Ω_{-}, a nonstandard interpretation is also possible in which Ω_{-} is the subject that acts on Ω_{+}. To refine this dialectical relation further, these mutual actions can be construed as occurring in imaginary time, whether in a physical system (in terms of a Plank unit) or in a logical-mathematical context (described above in chapter 1 in relation to Plato's *Parmenides).* This convergence demonstrates that the present discussion of HTSG modality and quantum computing has yet not succeeded in working beyond the via negativa parameters upon which this study has been based. Further, the Löwenheim-Skolem theorem is seen here to be fundamental in bridging between standard logic and HTSG quantum weirdness.

Within the via negativa, the status of Plato's form of the Good and the Summum bonum are quite problematic. In *Republic,* Plato is explicit that the form of the Good was at the apex of the Analogy of the Divided Line. Within that analogy, Plato regarded all other forms as permitting some degree of contrariety and heteronomy. But this is not true of the form of the Good in *Republic,* and it seems that the form of Beauty in *Symposium* and *Phaedo* was to be understood in the same Parmenidean fashion. Plato never identified the Good with Being, as was done by Augustine and other medieval philosophers and theologians. Plato seems to have conceived of the supreme task of philosophy to construct a dialectical system in which at least the forms of Good and Beauty are kept above the tumult of contrariety. But it also seems that Plato was a rational mystic, and his mysticism and critical awareness were such as to suggest that he may have thought this an impossible task. In medieval philosophy, the apophatic tradition reached such a high pitch that any predication of God whatsoever was subjected to negation.

The plight of the via negativa is sufficiently worrisome as to make it advisable to consider all manner of critical issues. Nietzsche wrote:

> In fact, nothing up to now has been more naively persuasive than the error of being, as it was formulated by the Eleatics, for instance: after all, it has on its side every word, every sentence we speak! Even the opponents of the Eleatics fell prey to the seduction of their concept of being – among others, Democritus did so in inventing his *atom.* "Reason" in language: oh, what a tricky old woman she is! I'm afraid we're not rid of God because we still believe in grammar.[94]

But strangely, Plato expressed this worry before Nietzsche did. In his Seventh Letter, Plato expressed the worry that grammar and language are inadequate expressions for metaphysical truth.[95] In light of a rare agreement between Plato and Nietzsche, and the apophatic predilection to negate everything, what is to be made of this predicament? The mystic and the skeptic make strange bedfellows. Hegel's power of the negative is a long way from Adorno's *Negative Dialectics.* Is Hegel's deviant use of language (as measured from Parmenides) moving in the right direction? Does grammar remain valid when pushed to its dialectical limits, or should it be pushed beyond its limits so as to be negated?

HT Modality and Epistemology

The ability of HT number theory to interpret the logic of the Russell paradox within Cantor's theorem, and then the similar ability of HT modality to interpret the set theoretic

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[94] Friedrich Nietzsche, *Twilight of the Idols.*

[95] Kenneth Sayre, "Plato's Seventh Letter" in *Platonic Writings, Platonic Readings*, Charles L. Griswold, Jr., ed. (New York: Routledge, 1988), 93-109.

paradoxes in meaningful fashion suggests that the theory is still on its feet in difficult territory. Perhaps attention can be given then to important considerations of the characteristics and further development of HT theory.

HT theory is remarkably similar to Kant's philosophy in that both are synthetic a priori. But there are major differences. Kant intends his theory to be a priori in absolute terms. HT theory is a priori only in relative terms. More specifically, HT theory is a quasi-empirical synthetic a priori theory. Kant's a priori deductions seek to progress apodictically from positive to positive. HT theory progresses quasi-empirically from negative to negative through HT ramification and zigzag. Kant seeks absolute autonomy. HT theory seeks a dialectical blend of autonomy and heteronomy.

The axiom of comprehension is a particularly worrisome source of intellectual error. This axiom has prompted many individuals to believe they can grasp an absolute totality by a simple extension of a predicate across an unqualified universe of thought. Russell struggled to maintain this axiom in the face of the foundational crisis, believing that the battle for reason was tantamount to the battle for this very axiom. This axiom is fundamental to the quest for universality, which in turn is fundamental to logical certainty. What is more, the Parmenidean application of this axiom precludes the possibility of the dialectical method of such vital interest to the present study. The axiom of comprehension puts analysis and autonomy in the places where HT theory puts synthesis and heteronomy.[96] In fact, the objective of replacing this axiom with respect to the axiom of unitary total probability was the motivation that prompted the development of transfinite probability.

[96] HT theory rejects neither analysis nor autonomy, but combines them dialectically with synthesis and heteronomy.

The axiom of comprehension has been relentlessly applied to both number theory and probability theory. Consider a particularly striking case. Based on the axiom of comprehension as applied not only universally but trans-universally, many contemporary physicists and cosmologists believe in a multiverse of parallel universes. Such extrapolation of probabilistic principles is made without the least hint of logical contrariety. But conversely, another group of physicists are struggling with the negative implications of quantum probability and are publishing articles in which they are attempting to come to terms with the concept of negative probability. These physicists have faced up to the threat of negation and are not falling uncritically into the axiom of comprehension.[97] The present study would seem to have relevance for their interests. This study suggests that the attempt for a negative probability is perhaps a defective dialectic, but what should be sought instead is a theory of imaginary probability as expressed on the *y*-axis. Imaginary probability and imaginary space-time would, if true, preclude the validity of the multiverse as deduced in association with the axiom of comprehension.

The preceding discussion should make clear why HT theory is characterized by philosophical restraint, and therefore reflects a philosophy of humility. This claim might seem to clash with the tremendous hierarchical structure of the theory, except that the theory was produced through the activity of the negative which continually negates itself through HT ramified zigzag. This theory has rejected the axiom of comprehension at every turn, and has grown large only through the power of negation. But the present theory also reflects a philosophy of humility by insisting on the distinction between de dicto and de re. As Knuth writes of surreal numbers, "Only God can finish the calculations, but we can finish the proofs."[98]

[97] Saul Youssef of Florida State University maintains a list of articles on this topic on his website.
[98] Knuth, *Surreal Numbers,* 104.

HT Modality and Ethics

What conditions must be presupposed as a sufficient basis for the self? The self that is completely determined by others is entirely passive, so as to lack autonomy and agency. The equality of P and NP brackets the world of necessity and determinism such that the autonomy of the self is not diminished by external forces of nature. This provides a basis for personal agency though not for extreme libertarianism. A self that is relentlessly committed to its own autonomy and liberty cares too much for self and too little for others. Absolute autonomy in either a Platonic form or in interpersonal relations allows one to recognize oneself in others, thus leading to alienation and a reduced self.

The development of HT numbers in chapter 5 could not have proceeded apart from the autological-heterological distinction, as is also true of the synthesis of HT modality in chapter 6. Yet this logical syntax reduces to the relation of self and other. The logical development of the theory depicts an infinite amount of heterology and difference that is overcome by and subsumed within autology and sameness, while preserving individual distinctiveness. The progression reflects the upward movement of the *psyche* in Plato's *Symposium.* As Plato writes, the soul "is led by another into the mystery of Love." Love is predicated on alterity and heteronomy, both of which can be described as heterological. Thus in addition to reflecting a logical structure of humility, HT theory expresses a philosophy of alterity, mutual dependence and love.

APPENDIX 1

Mandelbrot Set Chaos

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The Mandelbrot set is the set of points in the complex c-plane that do not go to infinity when iterating $z_{n+1} = z_n^2 + c$ starting with $z = 0$. One can avoid the use of complex numbers by using $z = x + iy$ and $c = a + ib$, and computing the orbits in the ab-plane for the 2-D mapping

$$x_{n+1} = x_n^2 - y_n^2 + a$$
$$y_{n+1} = 2x_n y_n + b$$

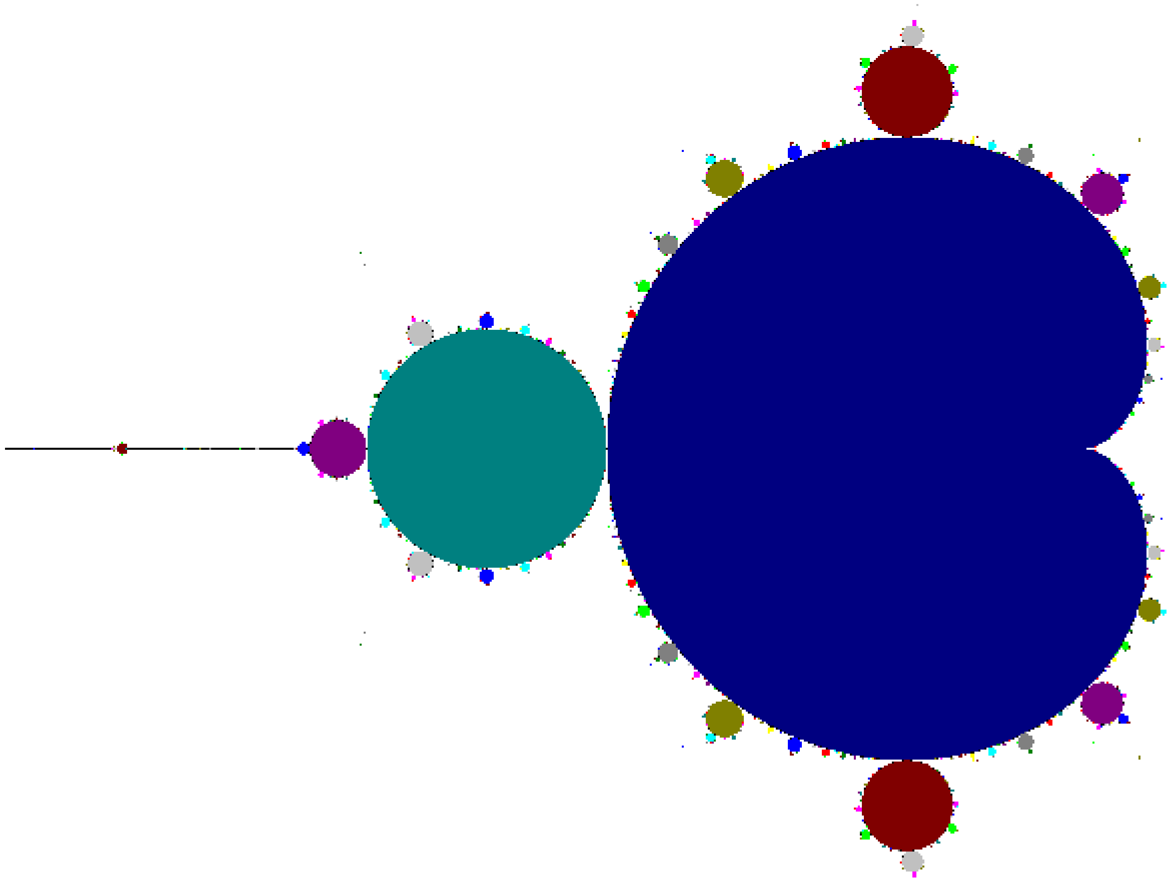
with initial conditions $x = y = 0$ (or equivalently $x = a$ and $y = b$). It can be proved that the orbits are unbounded if $|z| > 2$ (i.e., $x^2 + y^2 > 4$). The boundary of the Mandelbrot set is a very complicated fractal with a Hausdorff dimension of 2. Bounded orbits may attract to a fixed point, a periodic cycle, or they may be chaotic. More details and references for the Mandelbrot set are included in the sci.fractals FAQ.

There are also various ways to express the Mandelbrot set in terms of a single time-delayed scalar variable. One such representation is

$$y_{n+1} = 2y_n[(y_n - b)^2/4y_{n-1}^2 - y_{n-1}^2 + a] + b$$

with initial conditions $y_0 = b$, $y_{-1} = 0$.

An interesting question is to ask for what values of (a, b) are the orbits chaotic. This question was addressed numerically using a BASIC program MANCHAOS.BAS which has been compiled with PowerBASIC. The program does period-checking for periods up to 16,000 and considers the orbit to be chaotic if it is bounded but not periodic up to the limit checked. The following figure shows the output of that program.



In this figure, white represents unbounded orbits, black represents chaotic orbits, dark blue represents fixed points, and other periodic cycles are plotted modulo the remaining 13 colors. The chaotic regions appear to be restricted to the boundary of the set and to a portion of the real (x) axis toward the left. Along the x-axis, the dynamics are governed by the 1-D map, $x_{n+1} = x_n^2 + a$, which is known to have chaotic solutions over most of the range $-2 < a < -1.4011$. The number of apparently chaotic orbits off the real axis shrinks as period-checking is extended to higher periods, seemingly tending to zero.

If chaotic orbits are limited to the boundary of the Mandelbrot set, as appears to be the case, then they occur with a probability less than or equal to the probability that a point falls on the boundary of the set. Apparently it has not been proved whether the boundary of the Mandelbrot set has non-zero area. Similarly, it is not known whether a point in any finite region of the *ab*-plane has a non-zero probability of exhibiting chaos. It appears that the probability is extremely small and very likely zero.

The few black points off the x-axis may just be examples of transient chaos; they may eventually go to infinity or settle into a periodic cycle, although some of these cases have been followed for over 10^{10} iterations. Some of these aperiodic points can be shown to have a Lyapunov exponent of exactly zero (see note by Jay Hill below). For these points, the orbits are aperiodic and fractal, and the separation of orbits with two nearby initial conditions fluctuates, but it doesn't grow, exponentially or otherwise. These points seemingly fail to satisfy the sensitive dependence on initial conditions that is usually the defining characteristic of chaos. This is in contrast to most orbits in the range $-2 < a < -1.4011$, $b = 0$, which are truly chaotic. Thus the Mandelbrot set, with all its complexity, apparently admits a negligibly small number of truly chaotic orbits.

Several interesting responses to the above observations were posted on the newsgroup sci.fractals:

On 26 June 1997, Bob Beland (bg570@FreeNet.Carleton.CA) wrote:

"Points in the Mandelbrot plane come in several types:

- Outside set, nowhere near it: Orbit diverges to infinity
- Inside set, not on boundary: Orbit converges to a finite attracting point or cycle
- Edge of set, tip or branch of filament: Orbit converges on finite repelling cycle
- Edge of set, other filament points, with irrational external angle: Chaotic orbit
- Edge of set, cusp of a cardioid or bud attachment point (One or more rational external angles): parabolic, orbit goes to weakly attracting cycle or point.
- Edge of set component, not on a cusp, perhaps at the limit of an alternating series of buds, with irrational external angle: Orbit chaotic. (Julia set has Siegel disks.)"

On 26 June 1997, Jay R. Hill (Jay.R.Hill@cpmx.saic.com) wrote:

"There are special values of *c* with convergent behavior at 'rational' points on the boundary of a component. And there are 'irrational' points on the boundary that are chaotic. But even these are at the edge of chaos, with a Lyapunov exponent equal to zero. On 16 Feb 1994 I posted a list of these chaotic points. Four of them are $c = 0.33 + 0.06i$, $0.37 + 0.16i$, $-0.23 + 0.64i$, $-0.47 + 0.54i$. As for these being only transiently chaotic, I have followed them for more than 10^7 iterations. They are still going, forever, in their chaotic path. Their paths are fractal, by the way, and fun to plot."

On 3 July 1997, Jay R. Hill (Jay.R.Hill@cpmx.saic.com) added:

"The value of *c* [$0.33 + 0.06i$) is exactly on the edge of the cardioid with theta related to the 3-4-5 triangle. For the Mandelbrot Set the Lyapunov exponent is When *lambda* = 0, this implies If the cycle ever becomes periodic, the product will go to zero violating [2]. There is a path normal the component for which inside the component *lambda* is negative and outside positive. On the edge it must be zero and cannot ever be a cycle.

$$c = 0.5 * \exp(i*theta) - 0.25 * \exp(2*i*theta).$$

$$lambda = \lim (1/N) \text{ sum}\{ \log2 \mid 2*z[n] \mid \} \qquad [1]$$

$$\lim \text{ prod } (2*z[n]) = 1. \qquad [2]$$

The iteration orbit follows a fractal path always finding a new spot between earlier ones. If that means asymptotic to a cycle, the cycle has infinite length. I wouldn't call that a cycle. The *lambda*=0 orbits are very different from the periodic orbits located at the touch points between components. When *lambda*=0, we get a fractal path. The others have star patterns which develop as the iterations converge to the limit cycle. The number of star 'spikes', the period of the orbit, is the same as the period of the 'outer' attached component."

On August 5, 1997, Paul Derbyshire (ao950@FreeNet.Carleton.CA) wrote:

"The book The Beauty Of Fractals calls these hypothetical objects "queer components". It is strongly suspected the ordinary $z^n + c$ Mandelbrots do not have these. Other Mandelbrots for more complex (non polynomial) formulas likely can have them. They would be associated, I guess, with Siegel disk and Herman ring attractors. (Say, what is a Herman ring? They are mentioned briefly anywhere Siegel disks are, but Siegel disks are described in detail and Herman rings are not. It is mentioned that Herman rings do not occur in $z^2 + c$ though.)"

On August 6, 1997, Peter T. Wang (peterw@cco.caltech.edu) wrote:

"Yes, the boundary of the m-set contains points which correspond to j-sets of Siegel disk type; in a sense, the orbit of the iteration is quasiperiodic. Instead of being attracted to some n-cycle, an iterated point in the basin of attraction orbits an "invariant circle" about some fixed point in the plane (we're talking about the function space, not the parameter space of the m-set). Anyway, what this basically amounts to is that the "period," if you say it has one, is effectively infinite. It is bounded behavior but no periodicity occurs. However, I wouldn't say the behavior is chaotic."

http://sprott.physics.wisc.edu/chaos/manchaos.htm

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