THE ECONOMICS OF NETWORK FLOWS

By

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To my parents

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CHAPTER I

INTRODUCTION

We produce and consume goods and services that have gone through networks. Airlines, railroads, computer networks, and social networks are a few examples. However, those goods and services are not always beneficial to everyone. For example, firms may earn less profits because the products of their rival firms are brought to markets through supply chains. Revenue services may earn less tax revenue because taxable assets are transferred through financial networks and can be concealed in other countries. Websites and network service providers may suffer from malicious software sent through the Internet. Countries and their citizens may suffer from hazardous materials carried through transportation systems. In these contexts, competing products, concealed assets, malicious software, and hazardous materials are viewed as bads, as opposed to goods, because they are harmful to some economic agents. My research on the economics of network flows is motivated by (i) the possibility of bads being carried through networks, together with goods, (ii) the analysis of strategic behavior of agents in networks, and (iii) the experimental test of theoretical predictions about strategic behavior.

Chapter II develops a strategic model of network interdiction in a non-cooperative game of flow. An adversary, endowed with a bounded quantity of bads, chooses a flow specifying a plan for carrying bads through a network from a base to a target. Simultaneously, an agency chooses a blockage specifying a plan for blocking the transport of bads through arcs in the network. The bads carried to the target cause a target loss while the blocked arcs cause a network loss. The adversary earns and the agency loses from both target loss and network loss. The adversary incurs the expense of carrying bads. In this model I study Nash equilibria and find a power law relation between the probability and the extent of the target loss. My model contributes to the literature of game theory by introducing non-cooperative behavior into a Kalai-Zemel (cooperative) game of flow. My research also advances models and results on network interdiction.

Chapter III introduces a strategic network model where bads, such as infectious diseases and liquidity shocks, arise at a source node and may be transmitted to a sink node as a flow through a network. Two players wish to decrease network bads, which is determined by the extent of bads at the source node and by the maximum transmission of bads through the network. One player can act to mitigate the extent at the source node while the other player can act to reduce the maximum transmission through the network. Each player incurs the cost of action but benefits from the decrease in network bads. The bottlenecks of the network affect the maximum transmission of bads and the behavior of the players. I characterize efficient profiles and equilibria in terms of the bottlenecks. Interestingly, the player who acts in an equilibrium may not be the player who must act in the efficient profile. Overall, strategic inaction leads to inefficiency. Unless there is an efficient equilibrium where neither player acts, no efficient profile is an equilibrium and no equilibrium is efficient. I study cooperative solutions where the players jointly choose an efficient profile and make transfers to each other. Such cooperative solutions always exist.

Chapter IV builds a strategic model that exhibits the power law of conflict.[1] The power law of conflict is the name given to an empirical regularity that the frequency of conflict events, such as murders, insurgencies, and wars, scales as an inverse power of the severity of conflict events. In this model there are two adversarial players. Attacker can carry bads through a route to damage a target while Defender can block the route to stop

[1] This chapter is based on Chapter II and joint work with John Wooders. Besides allowing the study of comparative statices leading to a power law, this chapter aims to provide a theoretical foundation for empirical and experimental testing of the power law.

the transport of bads. In the unique equilibrium both Attacker and Defender choose mixed strategies. The comparative statics of the mixed strategy equilibrium yields a power law. The parameters of the power law can be estimated from data. The power law is a good fit to the Iraqi data in Global Terrorism Database.

CHAPTER II

STRATEGIC NETWORK INTERDICTION

Introduction

This chapter introduces a model with two players, an adversary and an agency, interacting strategically in a given network. The adversary is given a bounded quantity of bads at a base node and plans to carry bads to a target node. The adversary chooses a flow of bads that specifies a plan for carrying bads through the network from the base to the target. The agency is operating the network and wishes to stop the transport of bads to the target. The agency chooses a blockage of arcs that specifies a plan for stopping the transport of bads through the network. The bads carried to the target cause a target loss while the blocked arcs cause a network loss. The adversary earns and the agency loses from both target loss and network loss. The adversary incurs the expense of carrying bads.

In this model I analyze the equilibrium behavior of the players. If the bounded quantity of bads is small, there are pure strategy Nash equilibria. In these equilibria, the adversary carries bads up to the bounded quantity in a dispersed way through the network, but the agency does not block any arcs. If the bounded quantity of bads is either intermediate or large, there are mixed strategy Nash equilibria in which each player chooses only two pure strategies with positive probability. In these equilibria, the adversary carries no bads or carries a positive amount of bads to the target. Meanwhile, the agency blocks no arcs or blocks all the arcs necessary to make the target unreachable through the network. My analysis shows which arcs the agency blocks and how often she blocks them. My analysis also shows how the adversary carries bads through the network and how often he does.

In these Nash equilibria, the adversary successfully carries bads to the target if and only if the adversary carries a positive amount of bads to the target and the agency does not block any arcs. By computing the probability of this joint event, we can calculate the equilibrium probability of the target loss. If the bounded quantity of bads is either intermediate or large, there is a power law relation between the probability and the extent of the target loss. This theoretical finding is consistent with empirical evidence.[1]

This paper contributes to the game theory literature by introducing noncooperative behavior into a Kalai-Zemel network flow model. Kalai and Zemel (24) define a (transferable utility) cooperative game, called a flow game, where the worth of a coalition is defined as the value of a maximum flow in the network restricted to the members of the coalition.[2] Their main result is that a cooperative flow game is totally balanced and thus has a nonempty core (that is, there are distributions of the total payoff of the game that are stable against the formation of coalitions). The core of a flow game depends on the structure of a network and the ownership of arcs in the network. My framework differs in that players interact strategically. The agency owns and operates all arcs in a network while the adversary abuses the network.

This paper also contributes to the literature on network interdiction. Washburn and Wood (40) introduce a zero-sum game, where an evader chooses a path to move through a network and an interdictor chooses an arc at which to set up an inspection site. If the evader traverses a path that includes the inspected arc, the evader is detected with some exogenously given positive probability. Otherwise, the evader is not detected. Both players are allowed to choose mixed strategies. Given a mixed strategy profile, the interdiction

[1] In empirical research Bohorquez et al. (5) and Clauset et al. (11) show that the fatality distribution of terrorist events follows a power law.

[2] For other studies on cooperative flow games, see Kalai and Zemel (25), Granot and Granot (20), Potters et al. (34), and Reijnierse et al. (35).

probability is defined to be the average probability of the evader being detected. The evader aims to minimize the interdiction probability by choosing a path-selection mixed strategy, while the interdictor aims to maximize the interdiction probability by choosing an arc-inspection mixed strategy. By using linear programming and network flow techniques, Washburn and Wood (40) study the Nash equilibria of this game.[3]

My model differs from the existing models on network interdiction in four aspects:

(i) The definition of a network is different in that each arc has a capacity.

(ii) The adversary is endowed with a quantity of bads, which may, in equilibrium, be binding.

(iii) Both players have larger sets of strategies. The adversary chooses a flow rather than a path. If there are multiple paths in a network, the adversary can use them all at once. The agency chooses a blockage rather than an arc. That is, the agency can block multiple arcs at once.

(iv) My model is not a zero-sum game nor even a strictly competitive game.

Because of (i), we do not need to take the detection probability as given. In my model this probability is determined endogenously. By virtue of (ii), we can analyze how the adversary's resource constraint affects the adversary's and the agency's equilibrium behavior. By virtue of (iii), my model creates a more tractable environment and gives sharper results on equilibrium behavior. Because of (iv), we need to use a different solution technique to find equilibria. I exploit the idea that in any Nash equilibrium each player is indifferent between the pure strategies played with positive probability.

[3]Other than these papers, most of the literature on network interdiction deals with an interdictor's optimization problem subject to some budget constraints. See Cormican et al. (13), Israeli and Wood (22), and Wood (41).

Security in network games has attracted significant interest. For example, Ballester et al. (4) study the interaction between players whose payoffs depend on a network. They obtain a proportional relationship between how much effort a player exerts and how central the player's position is in the network. Baccara and Bar-Isaac (3) study the formation of networks between criminals and terrorists and find optimal policies for law enforcement agencies. Goyal and Vigier (19) study the design and protection of networks robust to attacks from outside on the networks' nodes.[4]

The remainder of this chapter is organized as follows. Section 2 develops a game-theoretic model of network interdiction. Section 3 studies the Nash equilibria of the model. Section 4 discusses my theoretical finding, together with empirical evidence, and also discusses future research topics.

Model

Two players, player 1 and player 2, strategically interact with each other in a given network. Players can be thought of as firms in the context of market competition, as a taxpayer and a revenue service in the context of tax evasion, as a malicious hacker and a network operator in the context of network security, or as a terrorist group and a security agency in the context of national security. Having these security applications in mind, player 1 is called an *adversary* and player 2 is called an *agency*.

[4] For a survey on other literature on networks, see Jackson (23).

Networks

A network consists of a set of nodes, N, a set of arcs, $A \subset N \times N$, and a (row) vector of arc capacities, $c := (c_{ij})_{(i,j)\in A}$. Each arc is an ordered pair of distinct nodes and has a positive capacity. For each $i, j \in N$ with $i \neq j$, if $(i, j) \in A$, node i is connected to node j through arc (i, j) with capacity $c_{ij} > 0$. Formally a *network* is defined as a collection (N, A, c).

Strategies

Player 1, the adversary, is given a *bound quantity* $q > 0$ of bads at a node. This node is called *base* s. Player 1 plans to carry bads to another node. This node is called *target* t. Player 1 chooses a flow of bads specifying a plan for carrying bads through network (N, A, c) from base s to target t.

For each $j \in N$, denote by $IA(j) := \{(i, j) : (i, j) \in A\}$ the set of the arcs coming into node j and by $OA(j) := \{(j, i) : (j, i) \in A\}$ the set of the arcs going out from node j.

Formally a *flow* of bads from base s to target t with bound quantity q in network (N, A, c) is a (column) vector $f := (f_{ij})'_{(i,j)\in A}$ satisfying the following constraints:

$$0 \leq f_{ij} \leq c_{ij} \qquad \text{for each } (i,j) \in A, \tag{II.1}$$

$$f_{is} = 0 \qquad \text{for each } (i,s) \in IA(s), \tag{II.2}$$

$$\sum_{(s,i)\in OA(s)} f_{si} \leq q \qquad \text{and} \tag{II.3}$$

$$\sum_{(i,j)\in IA(j)} f_{ij} - \sum_{(j,i)\in OA(j)} f_{ji} = 0 \qquad \text{for each } j \in N \setminus \{s,t\}. \tag{II.4}$$

Constraint (II.1) says that each arc flow is at least zero and at most the arc capacity. Constraint (II.2) says that each incoming flow to the base is zero. Constraint (II.3) says that the total outgoing flow from the base does not exceed the bound quantity. Constraint

(II.4) says that at each node, except for the base and the target, the total incoming flow equals to the total outgoing flow.

Denote by $F(s,t,q,N,A,c)$ the set of all flows of bads from base s to target t with bound quantity q in network (N,A,c). When there is no ambiguity, I use F instead of $F(s,t,q,N,A,c)$. Then the set of pure strategies for player 1 is denoted by F. By choosing a flow $f = (f_{ij})'_{(i,j)\in A} \in F$, player 1 carries f_{ij} amount of bads through arc (i,j).

The *value* of a flow is defined as the total incoming flow to the target less the total outgoing flow from the target. Thus, the value of a flow shows how many bads player 1 carries to the target. Let $v := (v_{ij})_{(i,j)\in A}$ be a (row) vector with $v_{it} = 1$ for each $(i,t) \in IA(t)$, $v_{ti} = -1$ for each $(t,i) \in OA(t)$, and $v_{ij} = 0$ for each $(i,j) \notin IA(t) \cup OA(t)$. Then the value of a flow $f \in F$ is calculated as

$$v \cdot f = \sum_{(i,t)\in IA(t)} f_{it} - \sum_{(t,i)\in OA(t)} f_{ti}. \tag{II.5}$$

Constraints (II.1) through (II.4) imply that the value of a flow is non-negative and constrained by the bound quantity. That is, for each $f \in F$, it holds that

$$0 \leq v \cdot f \leq q. \tag{II.6}$$

A flow $f^o \in F$ is the *zero flow* if f^o is the vector of zeros. A flow $f^\tau \in F$ is *trivial* if $v \cdot f^\tau = 0$. Notice that the zero flow f^o is trivial. A flow $f^* \in F$ is a *maximum flow* if for each $f \in F$, we have $v \cdot f^* \geq v \cdot f$. Notice that the value of a maximum flow is constrained by the bound quantity.

Player 2, the agency, wishes to stop the transport of bads to the target. Player 2 chooses a blockage of arcs specifying a plan for stopping the transport of bads through network (N,A,c) to target t. Formally a *blockage* of arcs in network (N,A,c) is a (column) vector $b := (b_{ij})'_{(i,j)\in A}$ with $b_{ij} \in \{0, 1\}$ for each $(i, j) \in A$.

Denote by $B(N, A, c)$ the set of all blockages of arcs in network (N, A, c). When there is no ambiguity, I use B instead of $B(N, A, c)$. Then the set of pure strategies for player 2 is denoted by B. By choosing a blockage $b = (b_{ij})'_{(i,j)\in A} \in B$, if $b_{ij} = 1$, player 2 blocks arc (i, j), and if $b_{ij} = 0$, player 2 does not block the arc. For each $b \in B$, denote by $A^b := \{(i, j) \in A : b = (b_{ij})'_{(i,j)\in A}$ and $b_{ij} = 1\}$ the set of all blocked arcs.

The *capacity* of a blockage is defined as the total capacity of the blocked arcs. Thus, the capacity of a blockage shows how much total arc capacity player 2 blocks in the network. The capacity of a blockage $b \in B$ is calculated as

$$c \cdot b = \sum_{(i,j)\in A} c_{ij} b_{ij}. \tag{II.7}$$

A *cut* $(C, \overline{C})$ in network (N, A, c) is a partition of the node set N with $s \in C$ and $t \in \overline{C}$. For each cut $(C, \overline{C})$, an arc $(i, j) \in A$ is a *cut arc* if $i \in C$ and $j \in \overline{C}$. That is, through a cut arc (i, j), node i in C is connected to node j in $\overline{C}$. For each cut $(C, \overline{C})$, denote by $A(C, \overline{C}) := \{(i, j) \in A : i \in C$ and $j \in \overline{C}\}$ the set of all cut arcs.

A blockage $b^o \in B$ is the *zero blockage* if b^o is the vector of zeros. A blockage $b \in B$ is a *cut blockage* if there is a cut $(C, \overline{C})$ such that $A(C, \overline{C}) = A^b$. A blockage $b^* \in B$ is a *minimum cut blockage* if for each cut blockage b, we have $c \cdot b^* \leq c \cdot b$.

The set of mixed strategies for player 1 is denoted by $\Delta(F)$ and the set of mixed strategies for player 2 is denoted by $\Delta(B)$.

Net Flows

We want to know how many bads player 1 *successfully* carries to the target when player 1 chooses a flow of bads and player 2 chooses a blockage of arcs. To answer this question I introduce the definition of net flows. For each flow of bads and each blockage of arcs, the net flow of bads to the target is obtained by (i) decomposing the flow of bads into cycle flows and path flows and (ii) removing all the cycle flows and all the path flows with blocked arcs.

An $s-t$ *path* in network (N, A, c) is a sequence of distinct nodes $i_1, \ldots, i_K$ such that $(i_k, i_{k+1}) \in A$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = s$ and $i_K = t$. In this case we say that the $s-t$ path *includes* arcs (i_1, i_2), ..., (i_{K-1}, i_K). A *cycle* in network (N, A, c) is a sequence of distinct nodes $i_1, \ldots, i_K$ such that $(i_k, i_{k+1}) \in A$ for each $k \in \{1, \ldots, K-1\}$ with $(i_K, i_1) \in A$. In this case we say that the cycle *includes* arcs (i_1, i_2), ..., (i_{K-1}, i_K), and (i_K, i_1). Denote by H the set of all $s-t$ paths and cycles in network (N, A, c).

The arc-path-cycle *incidence matrix* of (N, A, c) is $M := (m_{ah})_{a \in A, h \in H}$ with

$$m_{ah} = \begin{cases} 1 & \text{if } h \in H \text{ includes } a \in A; \\ 0 & \text{otherwise.} \end{cases}$$

A *cycle flow* is a flow of bads along a cycle. A *path flow* is a flow of bads along an $s-t$ path. By the *flow decomposition algorithm*, which will be presented in Appendix A, we can decompose a flow of bads into cycle flows and path flows.[5] Formally, for each $f \in F$, we can find a (column) vector $x := (x_h)'_{h \in H}$ such that $f = Mx$. That is, either along a cycle $h \in H$, or along an $s-t$ path $h \in H$, player 1 carries x_h amount of bads.

For vector x and each blockage b, let $x^b := (x_h^b)'_{h \in H}$ be a (column) vector with

$$x_h^b = \begin{cases} x_h & \text{if } h \text{ is an } s-t \text{ path including no blocked arcs;} \\ 0 & \text{otherwise.} \end{cases}$$

That is, only along an $s-t$ path $h \in H$ with no blocked arcs, player 1 successfully carries

[5] The flow decomposition algorithm is developed by Ford and Fulkerson (14). For a discussion see Ahuja et al. (1). In my model I use this algorithm to find net flows.

$x_h^b = x_h$ amount of bads to the target.

We are ready to define net flows. For each $f \in F$ and each $b \in B$, the *net flow* of bads to target t under flow f and blockage b is a (column) vector $f^b := (f_{ij}^b)'_{(i,j)\in A}$ such that $Mx^b = f^b$. Then the value of the net flow f^b is calculated as $v \cdot f^b$, which shows how many bads player 1 successfully carries to the target.

Notice that the net flow f^b under a flow f and a blockage b contains no cycle flows. Furthermore, the net flow f^{b^o} under a flow f and the zero blockage b^o contains all path flows but no cycle flows. A flow $f \in F$ is *acyclic* if $f = f^{b^o}$. Also notice that the net flow f^b under a flow f and a cut blockage b is the zero flow f^o. That is, if b is a cut blockage, for each $f \in F$, we have $f^b = f^o$.

Example 1

Suppose that a network is given as (N, A, c), where $N = \{s, i_1, i_2, t\}$ is the set of nodes, $A = \{(s, i_1), (s, i_2), (i_1, i_2), (i_2, t), (t, i_1)\}$ is the set of arcs, and $c = (c_{si_1}, c_{si_2}, c_{i_1i_2}, c_{i_2t}, c_{ti_1}) = (4, 1, 2, 5, 2)$ is the vector of arc capacities. A bound quantity is given as $q = 3$. Suppose that player 1 chooses a flow $f = (f_{si_1}, f_{si_2}, f_{i_1i_2}, f_{i_2t}, f_{ti_1})' = (1, 1, 2, 3, 1)'$. In Figure 1, each solid circle indicates a node, each arrow indicates an arc. In each pair of numbers, the first bold number indicates an arc flow, and the second light number indicates the arc capacity. In network (N, A, c) there are two $s - t$ paths s, i_1, i_2, t and s, i_2, t and one cycle i_1, i_2, t, i_1.

The arc-path-cycle incidence matrix of network (N, A, c) is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where the first column corresponds to path s, i_1, i_2, t, the second column corresponds to path s, i_2, t, and the third column corresponds to cycle i_1, i_2, t, i_1. By using the flow decomposition algorithm, we find a vector $x = (1, 1, 1)'$ such that $f = Mx$. Each entry of the vector x shows the amount of bads player 1 carries along path s, i_1, i_2, t, path s, i_2, t, and cycle i_1, i_2, t, i_1, respectively. Now suppose that player 2 chooses a blockage $b = (b_{si_1}, b_{si_2}, b_{i_1 i_2}, b_{i_2 t}, b_{ti_1})' = (0, 1, 0, 0, 0)'$. Then path s, i_1, i_2, t is the only $s-t$ path with no blocked arcs. Thus, $x^b = (1, 0, 0)'$. Therefore, the net flow of bads to target t under flow f and blockage b is $f^b = Mx^b = (1, 0, 1, 1, 0)'$ and the value of this net flow is $v \cdot f^b = 1$. $\square$

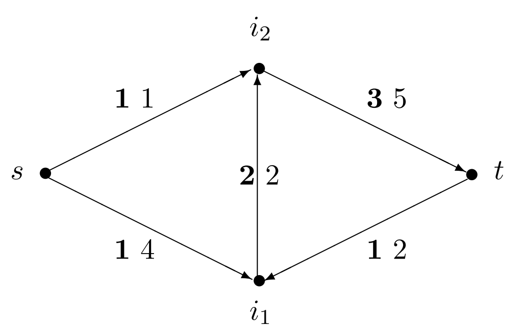

Figure 1. Network

Payoff Functions

The bads carried to the target cause a target loss. This target loss is determined by the value of the net flow of bads, $v \cdot f^b$, as well as by the *marginal target loss*, $\ell_t > 0$. For each $f \in F$ and each $b \in B$, the target loss amounts to $\ell_t(v \cdot f^b)$. Player 1 earns $\ell_t(v \cdot f^b)$ and player 2 loses the same amount from the target loss.

The blocked arcs cause a network loss. This network loss is determined by the capacity of the blockage of arcs, $c \cdot b$, as well as by the *marginal network loss*, $\ell_k > 0$. For each $b \in B$, the network loss amounts to $\ell_k(c \cdot b)$. Player 1 earns $\ell_k(c \cdot b)$ and player 2 loses the same amount from the network loss.

Player 1 incurs the expense of carrying bads. This expense is determined by the value of the flow of bads, $v \cdot f$, as well as by the *marginal expense of carrying bads*, $e > 0$. For each $f \in F$, the expense of carrying bads amounts to $e(v \cdot f)$.

Player 2 earns a constant worth of operating the network, w.

For each $(f, b) \in F \times B$, the *payoff function* of player 1 is defined as

$$u_1(f, b) = \ell_t(v \cdot f^b) + \ell_k(c \cdot b) - e(v \cdot f),$$

and the *payoff function* of player 2 is defined as

$$u_2(f, b) = w - \ell_t(v \cdot f^b) - \ell_k(c \cdot b).$$

For each $\sigma = (\sigma_1, \sigma_2) \in \Delta(F) \times \Delta(B)$, the expected payoff functions are $u_1(\sigma_1, \sigma_2) = E_\sigma[u_1(f, b)]$ and $u_2(\sigma_1, \sigma_2) = E_\sigma[u_2(f, b)]$. Since expected payoff functions are unique up to an affine transformation, without loss of generality, I assume that the marginal network loss equals to one, that is, $\ell_k = 1$.

Results

I analyze the equilibrium behavior of the players in the model. In a Nash equilibrium each player has no incentive to change his or her strategy. A strategy profile $(\sigma_1, \sigma_2) \in \Delta(F) \times \Delta(B)$ is a Nash equilibrium if for each $\sigma_1' \in \Delta(F)$ and each $\sigma_2' \in \Delta(B)$, $u_1(\sigma_1, \sigma_2) \geq u_1(\sigma_1', \sigma_2)$ and $u_2(\sigma_1, \sigma_2) \geq u_2(\sigma_1, \sigma_2')$.

Suppose that the marginal target loss is greater than the marginal expense of carrying bads.[6] The adversary has an incentive to carry bads through the network. Given this, I want to answer the following questions: Does the agency have any incentive to block arcs in the network? Which arcs does the agency block? And how often does the agency block the arcs?

To answer these questions I divide my analysis into three cases depending on the bound quantity. The bound quantity q is *small* if $q \leq (1/\ell_t)c \cdot b^*$, *intermediate* if $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$, and *large* if $c \cdot b^* < q$.

Denote by f^α an *acyclic maximum flow* with large bound quantity q in network (N, A, c). Because f^α is acyclic,

$$f^\alpha = (f^\alpha)^{b^o}. \tag{II.8}$$

Because q is large,

$$v \cdot f^\alpha = c \cdot b^*. \tag{II.9}$$

That is, the value of an acyclic maximum flow equals to the capacity of a minimum cut blockage. This equality is called the *max-flow min-cut theorem.*[7]

[6] If the marginal target loss is no greater than the marginal expense of carrying bads, the adversary has no incentive to carry bads through the network from the base to the target. Given this, the agency has no incentive to block arcs in the network. Thus, any trivial-flow zero-blockage strategy profile (f^τ, b^o) is a Nash equilibrium.

[7] Ford and Fulkerson (14) introduce the maximum flow problem in networks and show the max-flow

A flow $f^{\beta} \in F$ is a *binding flow* if $f^{\beta} = (q/(c \cdot b^{*}))f^{\alpha}$. Because f^{α} is acyclic, f^{β} is also acyclic. That is,

$$f^{\beta} = (f^{\beta})^{b^{o}}. \tag{II.10}$$

In addition the max-flow min-cut theorem (II.9) implies that

$$v \cdot f^{\beta} = q. \tag{II.11}$$

That is, the value of a binding flow equals to the bound quantity.

First, suppose that the bound quantity is small. We call (f^{β}, b^{o}) a *binding-flow zero-blockage strategy profile.* In any binding-flow zero-blockage strategy profile, if the bound quantity is small, each player has no incentive to change his or her strategy.

Proposition 1

If the bound quantity is small, that is, if $q \leq (1/\ell_t)c \cdot b^{*}$, *then any binding-flow zero-blockage strategy profile* (f^{β}, b^{o}) *is a Nash equilibrium.*

The proof of Proposition 1 is presented in Appendix B.

In any binding-flow zero-blockage Nash equilibrium, player 1 carries bads up to the bound quantity *in a dispersed way* through the network, but player 2 does not block any arcs in the network. The following is an example of this equilibrium.

Example 2

Consider network (N, A, c) in Example 1. Notice that $f^{\alpha} = (2, 1, 2, 3, 0)'$ is the only acyclic maximum flow and $b^{*} = (0, 1, 1, 0, 0)'$ is the only minimum cut blockage. Also notice that

min-cut theorem. For a detailed discussion see Ahuja et al. (1).

the capacity of the minimum cut blockage is $c \cdot b^* = 3$. Suppose that the marginal target loss is $\ell_t = 2$, the marginal expense of carrying bads is $e = 1$, and the bound quantity is $q = 1$. Then the binding flow is $f^\beta = (2/3, 1/3, 2/3, 1, 0)'$. Because the bound quantity is small, the binding-flow zero-blockage strategy profile (f^β, b^o) is a Nash equilibrium. However, if player 1 carries bads up to the bound quantity only through arcs (s, i_2) and (i_2, t), player 2 has the incentive to block arc (s, i_2). □

If the bound quantity is small and player 1 carries bads in a dispersed way through the network, then player 2 has no incentive to block arcs. However, if the bound quantity is *not* small, that is, if the bound quantity is either intermediate or large, then player 2 has an incentive to block arcs in the network. Now we want to know which arcs player 2 blocks and how often she blocks them. Consider the following example.

Example 3

Consider network (N, A, c) in Example 1. Suppose that player 2 chooses a cut blockage $b = (1, 1, 0, 0, 0)'$, that is, suppose that player 2 blocks all the arcs from the base. Then player 2 incurs a network loss of 5. However, if player 2 chooses the minimum cut blockage $b^* = (0, 1, 1, 0, 0)'$, she incurs a network loss of 3. Since both b and b^* are cut blockages, for each flow f, the net flow is the zero flow, that is, $f^b = f^o$ and $f^{b^*} = f^o$. Thus, the target loss is zero. Therefore, b is a dominated strategy for player 2. □

In general, if b is a cut blockage but not a minimum cut blockage and b^* is a minimum cut blockage, then b is dominated by b^* for player 2, that is, for each $f \in F$, $u_2(f, b) < u_2(f, b^*)$. Thus, it may be a dominated strategy for player 2 to block all the arcs from the base or to block all the arcs into the target. If player 2 blocks arcs, she blocks minimum cut arcs in the network.

Now imagine that player 2 blocks minimum cut arcs in the network with probability 1. Then player 1 has no incentive to carry bads through the network because he always fails to reach the target. If player 1 carries no bads to the target with probability 1, player 2 has no incentive to block the arcs. This is because she wants to avoid the network loss if there is no threat to the target. In turn, if player 2 blocks no arcs with probability 1, player 1 has an incentive to carry bads. If player 1 carries bads with probability 1, player 2 has an incentive to block arcs. In general, if the bound quantity is either intermediate or large, there is no pure strategy Nash equilibrium.

To study how often to block minimum cut arcs, I examine the mixed strategy Nash equilibria of the model. Now, suppose that the bound quantity is large.

A mixed strategy $\sigma_1^\lambda \in \Delta(F)$ is a *λ-scaled max-flow strategy*, or simply a *λ-flow strategy*, for player 1 if for some $\lambda \in [1/\ell_t, 1]$, $\sigma_1^\lambda(f^\tau) = 1 - 1/\lambda\ell_t$ and $\sigma_1^\lambda(\lambda f^\alpha) = 1/\lambda\ell_t$. By choosing a λ-flow strategy player 1 chooses a trivial flow f^τ with probability $1 - 1/\lambda\ell_t$ and a λ-scaled acyclic maximum flow λf^α with probability $1/\lambda\ell_t$. For example, if $\lambda = 1$, player 1 carries no bads from the base to the target with probability $1 - 1/\ell_t$ and carries the maximum possible amount of bads through the network with probability $1/\ell_t$. Here λ is a scale to adjust the probability and the amount of bads.

A mixed strategy $\sigma_2^* \in \Delta(B)$ is a *min-cut strategy* for player 2 if $\sigma_2^*(b^o) = e/\ell_t$ and $\sigma_2^*(b^*) = 1 - e/\ell_t$. By choosing a min-cut strategy player 2 chooses the zero blockage b^o with probability e/ℓ_t and a minimum cut blockage b^* with probability $1 - e/\ell_t$. That is, player 2 blocks no arcs with probability e/ℓ_t and blocks minimum cut arcs with probability $1 - e/\ell_t$.

We call $(\sigma_1^\lambda, \sigma_2^*)$ a *λ-flow min-cut strategy profile.*

Notice that player 1 chooses only two pure strategies f^τ and λf^α with positive probability. Given that player 2 chooses a min-cut strategy σ_2^*, by choosing a trivial flow f^τ, player 1 earns an expected payoff of

$$\begin{aligned} u_1(f^\tau, \sigma_2^*) &= \sigma_2^*(b^o)u_1(f^\tau, b^o) + \sigma_2^*(b^*)u_1(f^\tau, b^*) \\ &= (1 - e/\ell_t)(c \cdot b^*), \end{aligned}$$

because player 1 earns $u_1(f^\tau, b^o) = 0$ with probability $\sigma_2^*(b^o) = e/\ell_t$ and earns $u_1(f^\tau, b^*) = c \cdot b^*$ with probability $\sigma_2^*(b^*) = 1 - e/\ell_t$. Given a min-cut strategy σ_2^*, by choosing a λ-scaled acyclic maximum flow λf^α, player 1 earns an expected payoff of

$$\begin{aligned} u_1(\lambda f^\alpha, \sigma_2^*) &= \sigma_2^*(b^o)u_1(\lambda f^\alpha, b^o) + \sigma_2^*(b^*)u_1(\lambda f^\alpha, b^*) \\ &= (1 - e/\ell_t)(c \cdot b^*), \end{aligned} \tag{II.12}$$

because player 1 earns $u_1(\lambda f^\alpha, b^o) = (\ell_t - e)(v \cdot \lambda f^\alpha)$ with probability $\sigma_2^*(b^o) = e/\ell_t$ and earns $u_1(\lambda f^\alpha, b^*) = c \cdot b^* - e(v \cdot \lambda f^\alpha)$ with probability $\sigma_2^*(b^*) = 1 - e/\ell_t$. Thus, $u_1(f^\tau, \sigma_2^*) = u_1(\lambda f^\alpha, \sigma_2^*)$. By choosing a min-cut strategy σ_2^*, player 2 makes player 1 indifferent between the two pure strategies f^τ and λf^α.

Now notice that player 2 chooses only two pure strategies b^o and b^* with positive probability. Given that player 1 chooses a λ-flow strategy σ_1^λ, by choosing the zero blockage b^o, player 2 earns an expected payoff of

$$\begin{aligned} u_2(\sigma_1^\lambda, b^o) &= \sigma_1^\lambda(f^\tau)u_2(f^\tau, b^o) + \sigma_1^\lambda(\lambda f^\alpha)u_2(\lambda f^\alpha, b^o) \\ &= w - v \cdot f^\alpha, \end{aligned}$$

because player 2 earns $u_2(f^\tau, b^o) = w$ with probability $\sigma_1^\lambda(f^\tau) = 1 - 1/\lambda\ell_t$ and $u_2(\lambda f^\alpha, b^o) =$

$w - \lambda \ell_t(v \cdot f^\alpha)$ with probability $\sigma_1^\lambda(\lambda f^\alpha) = 1/\lambda\ell_t$. Given a λ-flow strategy σ_1^λ, by choosing a minimum cut blockage b^*, player 2 earns an expected payoff of

$$u_2(\sigma_1^\lambda, b^*) = w - c \cdot b^*, \tag{II.13}$$

because player 2 earns $w - c \cdot b^*$ whichever strategy player 1 chooses. Thus, the max-flow min-cut theorem (II.9) implies that $u_2(\sigma_1^\lambda, b^o) = u_2(\sigma_1^\lambda, b^*)$. By choosing a λ-flow strategy σ_1^λ, player 1 makes player 2 indifferent between the two pure strategies b^o and b^*.

In addition, we can show that for each player, these pure strategies are at least as good as any other pure strategies. Thus, in any λ-flow min-cut strategy profile, each player has no incentive to change his or her strategy.

Proposition 2

If the bound quantity is large, that is, if $c \cdot b^* < q$, then any λ-flow min-cut strategy profile $(\sigma_1^\lambda, \sigma_2^*)$ is a Nash equilibrium.

The proof of Proposition 2 is presented in Appendix B. The following is an example of λ-flow min-cut Nash equilibria.

Example 4

Consider network (N, A, c) in Example 1. Recall that $f^\alpha = (2, 1, 2, 3, 0)'$ is the acyclic maximum flow and $b^* = (0, 1, 1, 0, 0)'$ is the minimum cut blockage. Suppose that $\ell_t = 4$, $e = 1$, and $q = 5$. Because the bound quantity is large, any λ-flow min-cut strategy profile $(\sigma_1^\lambda, \sigma_2^*)$ is a Nash equilibrium. For instance, in a λ-flow min-cut Nash equilibrium with $\lambda = 1$, player 1 chooses the zero flow f^o with probability $\sigma_1^\lambda(f^o) = 3/4$ and the acyclic maximum flow f^α with probability $\sigma_1^\lambda(f^\alpha) = 1/4$, and player 2 chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/4$ and the minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 3/4$. In Figure 2, the bold numbers indicate the acyclic maximum flow, and the line segments indicate the minimum cut blockage. □

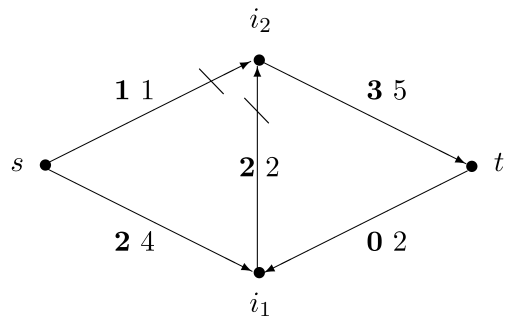

Figure 2. Maximum Flow and Blockage

In any λ-flow min-cut Nash equilibrium there is a power law relation between the probability and the extent of the target loss. In this equilibrium, player 1 successfully carries bads to the target if and only if player 1 chooses a λ-scaled acyclic maximum flow λf^α and player 2 chooses the zero blockage b^o. This joint event takes place with probability $(1/\lambda\ell_t)(e/\ell_t) = (1/\lambda)(e)\ell_t^{-2}$. Thus, with this probability, the bads carried to the target cause the target loss. Therefore, in any λ-flow min-cut Nash equilibrium, the *target loss probability* is $p_\lambda = (1/\lambda)(e)\ell_t^{-2}$.

In any λ-flow min-cut Nash equilibrium, if player 1 successfully carries bads to the target, the *target loss* amounts to $TL_\lambda = (\lambda\ell_t)(c \cdot b^*)$. Because $p_\lambda = (1/\lambda)(e)\ell_t^{-2}$ and $\ell_t = (1/\lambda)(1/(c \cdot b^*))TL_\lambda$,

$$p_\lambda = (\lambda)(e)(c \cdot b^*)^2(TL_\lambda)^{-2} \tag{II.14}$$

where $\lambda \in (1/\ell_t, 1]$. Furthermore, if $\lambda = (\ell_t)^{-\theta}$ for some $\theta \in [0, 1)$, equality (II.14) can be rewritten as

$$p_\lambda = (e)(c \cdot b^*)^{\frac{\theta-2}{\theta-1}} (TL_\lambda)^{-\frac{\theta-2}{\theta-1}} \tag{II.15}$$

because $p_\lambda = (e)(\ell_t)^{\theta-2}$ and $\ell_t = (c \cdot b^*)^{\frac{1}{\theta-1}} (TL_\lambda)^{-\frac{1}{\theta-1}}$. Thus, in any λ-flow min-cut Nash equilibrium with $\lambda = (\ell_t)^{-\theta}$ for some $\theta \in [0,1)$, the target loss probability p_λ is a *negative power function* of the target loss TL_λ. However, if $\lambda = 1/\ell_t$, then $p_\lambda = (e)\ell_t^{-1}$ and $TL_\lambda = c \cdot b^*$. Thus, if $\lambda = 1/\ell_t$, the equilibrium probability p_λ is independent of the target loss TL_λ.

Finally, suppose that the bound quantity is intermediate.

A mixed strategy $\sigma_1^\mu \in \Delta(F)$ is a *μ-scaled binding-flow strategy*, or simply a *μ-flow strategy*, for player 1 if for some $\mu \in [(1/\ell_t)(1/q)(c \cdot b^*), 1]$, $\sigma_1^\mu(f^\tau) = 1 - (1/\mu\ell_t)(1/q)(c \cdot b^*)$ and $\sigma_1^\mu(\mu f^\beta) = (1/\mu\ell_t)(1/q)(c \cdot b^*)$. By choosing a μ-flow strategy player 1 chooses a trivial flow f^τ with probability $1-(1/\mu\ell_t)(1/q)(c \cdot b^*)$ and a μ-scaled binding flow μf^β with probability $(1/\mu\ell_t)(1/q)(c \cdot b^*)$. For example, if $\mu = 1$, player 1 carries no bads from the base to the target with probability $1-(1/\ell_t)(1/q)(c \cdot b^*)$ and carries bads up to the bound quantity with probability $(1/\ell_t)(1/q)(c \cdot b^*)$. Here μ is a scale to adjust the probability and the amount of bads.

We call $(\sigma_1^\mu, \sigma_2^*)$ a *μ-flow min-cut strategy profile.*

Notice that player 1 chooses only two pure strategies f^τ and μf^β with positive probability. Given a min-cut strategy σ_2^*, by choosing a trivial flow f^τ, player 1 earns an expected payoff of $u_1(f^\tau, \sigma_2^*) = (1 - e/\ell_t)(c \cdot b^*)$. Given a min-cut strategy σ_2^*, by choosing a μ-scaled binding flow μf^{β}, player 1 earns an expected payoff of

$$\begin{aligned} u_1(\mu f^{\beta}, \sigma_2^*) &= \sigma_2^*(b^o) u_1(\mu f^{\beta}, b^o) + \sigma_2^*(b^*) u_1(\mu f^{\beta}, b^*) \\ &= (1 - e/\ell_t)(c \cdot b^*), \end{aligned} \tag{II.16}$$

because player 1 earns $u_1(\mu f^{\beta}, b^o) = (\ell_t - e)(v \cdot \mu f^{\beta})$ with probability $\sigma_2^*(b^o) = e/\ell_t$ and earns $u_1(\mu f^{\beta}, b^*) = c \cdot b^* - e(v \cdot \mu f^{\beta})$ with probability $\sigma_2^*(b^*) = 1 - e/\ell_t$. Thus, $u_1(f^{\tau}, \sigma_2^*) = u_1(\mu f^{\beta}, \sigma_2^*)$. By choosing a min-cut strategy σ_2^*, player 2 makes player 1 indifferent between the two pure strategies f^{τ} and μf^{β}.

Now notice that player 2 chooses only two pure strategies b^o and b^* with positive probability. Given a μ-flow strategy σ_1^{μ}, by choosing the zero blockage b^o, player 2 earns an expected payoff of

$$\begin{aligned} u_2(\sigma_1^{\mu}, b^o) &= \sigma_1^{\mu}(f^{\tau}) u_2(f^{\tau}, b^o) + \sigma_1^{\mu}(\mu f^{\beta}) u_2(\mu f^{\beta}, b^o) \\ &= w - c \cdot b^*, \end{aligned}$$

because player 2 earns $u_2(f^{\tau}, b^o) = w$ with probability $\sigma_1^{\mu}(f^{\tau}) = 1 - (1/\mu\ell_t)(1/q)(c \cdot b^*)$ and earns $u_2(\mu f^{\beta}, b^o) = w - (\mu\ell_t)q$ with probability $\sigma_1^{\mu}(\mu f^{\beta}) = (1/\mu\ell_t)(1/q)(c \cdot b^*)$. Given a μ-flow strategy σ_1^{μ}, by choosing a minimum cut blockage b^*, player 2 earns an expected payoff of

$$u_2(\sigma_1^{\mu}, b^*) = w - c \cdot b^*, \tag{II.17}$$

because player 2 earns $w - c \cdot b^*$ whichever strategy player 1 chooses. Thus, $u_2(\sigma_1^{\mu}, b^o) = u_2(\sigma_1^{\mu}, b^*)$. By choosing a μ-flow strategy σ_1^{μ}, player 1 makes player 2 indifferent between the two pure strategies b^o and b^*.

In addition, we can show that for each player, these pure strategies are at least as good as any other pure strategies. Thus, in any μ-flow min-cut strategy profile, each player has no incentive to change his or her strategy.

Proposition 3

If the bound quantity is intermediate, that is, if $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$*, then any* μ*-flow min-cut strategy profile* $(\sigma_1^\mu, \sigma_2^*)$ *is a Nash equilibrium.*

The proof of Proposition 3 is presented in Appendix B. The following is an example of μ-flow min-cut Nash equilibria.

Example 5

Consider network (N, A, c) in Example 1. Suppose that $\ell_t = 4$, $e = 1$, and $q = 3/2$. Notice that the binding flow is $f^\beta = (1, 1/2, 1, 3/2, 0)'$. Because the bound quantity is intermediate, any μ-flow min-cut strategy profile $(\sigma_1^\mu, \sigma_2^*)$ is a Nash equilibrium. For instance, in a μ-flow min-cut Nash equilibrium with $\mu = 1$, player 1 chooses the zero flow f^o with probability $\sigma_1^\mu(f^o) = 1/2$ and the binding flow f^β with probability $\sigma_1^\mu(f^\beta) = 1/2$, and player 2 chooses the zero blockage b^o with probability $\sigma_2^*(b^o) = 1/4$ and the minimum cut blockage b^* with probability $\sigma_2^*(b^*) = 3/4$. In Figure 3, the bold numbers indicate the binding flow, and the line segments indicate the minimum cut blockage. □

In any μ-flow min-cut Nash equilibrium the probability and the extent of the target loss show a power law relation. In this equilibrium, player 1 successfully carries bads to the target if and only if player 1 chooses a μ-scaled binding flow μf^β and player 2 chooses the zero blockage b^o. This joint event takes place with probability $(1/\mu\ell_t)(1/q)(c \cdot b^*)(e/\ell_t) =$

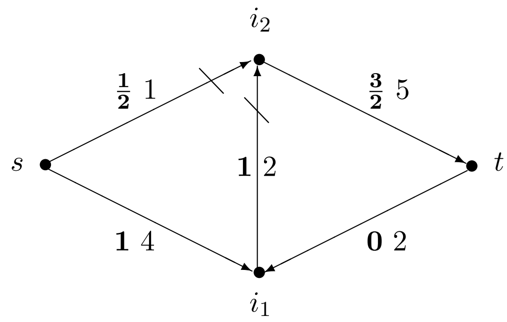

Figure 3. Binding Flow and Blockage

$(1/\mu)(1/q)(c \cdot b^*)(e)\ell_t^{-2}$. Thus, in any μ-flow min-cut Nash equilibrium, the *target loss probability* is $p_\mu = (1/\mu)(1/q)(c \cdot b^*)(e)\ell_t^{-2}$.

In any μ-flow min-cut Nash equilibrium, if player 1 successfully carries bads to the target, the *target loss* amounts to $TL_\mu = (\mu\ell_t)q$. Because $p_\mu = (1/\mu)(1/q)(c \cdot b^*)(e)\ell_t^{-2}$ and $\ell_t = (1/\mu)(1/q)TL_\mu$,

$$p_\mu = (\mu)(e)(q)(c \cdot b^*)(TL_\mu)^{-2} \tag{II.18}$$

where $\mu \in ((1/\ell_t)(1/q)(c \cdot b^*), 1]$. Furthermore, if $\mu = (q)^{-\theta}(c \cdot b^*)^{\theta}(\ell_t)^{-\theta}$ for some $\theta \in [0, 1)$, equality (II.18) can be rewritten as

$$p_\mu = (e)(q)(c \cdot b^*)^{-\frac{1}{\theta-1}}(TL_\mu)^{-\frac{\theta-2}{\theta-1}} \tag{II.19}$$

because $p_\mu = (e)(q)^{\theta-1}(c \cdot b^*)^{1-\theta}(\ell_t)^{\theta-2}$ and $\ell_t = (q)^{-1}(c \cdot b^*)^{\frac{\theta}{\theta-1}}(TL_\mu)^{-\frac{1}{\theta-1}}$. Thus, in any μ-flow min-cut Nash equilibrium with $\mu = (q)^{-\theta}(c \cdot b^*)^{\theta}(\ell_t)^{-\theta}$ for some $\theta \in [0, 1)$, the target loss probability p_μ is a *negative power function* of the target loss TL_μ. However, if $\mu = (1/\ell_t)(1/q)(c \cdot b^*)$, then $p_\mu = (e)\ell_t^{-1}$ and $TL_\mu = c \cdot b^*$. Thus, if $\mu = (1/\ell_t)(1/q)(c \cdot b^*)$, the equilibrium probability p_μ is independent of the target loss TL_μ.

Discussion

First I relate my results to empirical studies on terrorist events and then discuss related research in progress and further directions.

Fatality Distribution

Let z denote the number of fatalities in a terrorist event and let $p(z)$ denote the frequency of a terrorist event in which the number of fatalities is z. The fatality distribution of terrorist events follows a *power law* if for each $z \geq z_{\min}$,

$$p(z) \propto z^{-\gamma}$$

where $z_{\min}$ and γ are the parameters of the distribution. The estimates of the parameters are derived from data and denoted by $\hat{z}_{\min}$ and $\hat{\gamma}$.

Recent empirical studies show that the fatality distribution of terrorist events follows a power law. Clauset et al. (11) use the database of National Memorial Institute for the Prevention of Terrorism (MIPT) and conclude that the fatality distribution follows a power law. The estimate of the scaling parameter is $\hat{\gamma} = 2.38$. Bohorquez et al. (5) construct a data set on insurgent wars and conclude that for each insurgent war the fatality distribution follows a power law. The estimates of the scaling parameter are clustered around 2.5.

Recall that in any λ-flow min-cut Nash equilibrium with $\lambda = (\ell_t)^{-\theta}$ for some $\theta \in [0, 1)$, the target loss probability p_λ is a negative power function of the target loss TL_λ. Precisely, from equality (II.15),

$$p_\lambda = (e)(c \cdot b^*)^{\frac{\theta-2}{\theta-1}} (TL_\lambda)^{-\frac{\theta-2}{\theta-1}},$$

which can be rewritten as

$$p_\lambda(TL_\lambda) \propto (TL_\lambda)^{-\frac{\theta-2}{\theta-1}}.$$

To link this theoretical finding and empirical evidence I make two additional assumptions. Suppose that the target loss is measured by the number of fatalities and that the target loss probability is proportional to the frequency of a terrorist event.

Now suppose that the estimate of the scaling parameter, $\hat{\gamma} \geq 2$, is derived from data. By setting $\hat{\gamma} = \frac{\hat{\theta}-2}{\hat{\theta}-1}$ and solving for $\hat{\theta}$, we have $\hat{\theta} = \frac{\hat{\gamma}-2}{\hat{\gamma}-1}$. Notice that $\hat{\theta} \in [0,1)$. Therefore, in the λ-flow min-cut Nash equilibrium with $\lambda = (\ell_t)^{-\hat{\theta}}$, the fatality distribution is predicted to be

$$p_\lambda(TL_\lambda) \propto (TL_\lambda)^{-\hat{\gamma}}$$

and is consistent with data. Similarly, in the μ-flow min-cut Nash equilibrium with $\mu = (q)^{-\hat{\theta}}(c \cdot b^*)^{\hat{\theta}}(\ell_t)^{-\hat{\theta}}$, the predicted fatality distribution, $p_\mu(TL_\mu) \propto (TL_\mu)^{-\hat{\gamma}}$, is consistent with data.

Further Research

This paper presents a strategic model of network interdiction where two players have complete information and simultaneously choose their strategies. Building on this research we can study a model with incomplete information where players may not know each other's type. For example, a security agency may not know the strategies and payoffs of an adversary. This extension to incomplete information is, in my view, of clear importance. We can also study a model where players sequentially choose their strategies. For example, a security agency may observe an adversary's plots and choose her own strategy conditional on this observation or, alternatively, the agency may move first in setting up a security system. Both these approaches are subjects of my current and future planned research.

Appendix A

In this appendix I provide the flow decomposition algorithm.[8] A network is given as (N, A, c). For each $f \in F$, we can find a vector $x = (x_h)'_{h \in H}$ such that $f = Mx$. Initially we are given a flow f and the zero vector x. At each step, construct a sequence of distinct nodes, and obtain either an $s - t$ path or a cycle. Then we modify vector x and flow f. This algorithm terminates when the modified flow is the zero flow.

Algorithm 1 *Flow Decomposition*

Let $f = (f_{ij})'_{(i,j) \in A} \in F$ be given. Let $x = (x_h)'_{h \in H}$ be the vector of zeros.

At Step $k = 1, 2, \ldots$, if f is the zero flow, this algorithm terminates and yields vector x. If f is not the zero flow, there is an arc $(i, j) \in A$ with $f_{ij} > 0$.

(i) Start from base s. If there is $(i_1, i_2) \in A$ with $i_1 = s$ and $f_{i_1 i_2} > 0$, begin the construction of a sequence of distinct nodes with the two nodes i_1 and i_2. If there is $(i_2, i_3) \in A$ with $f_{i_2 i_3} > 0$, add node i_3 to the sequence. Repeat this until we add target t or a previously added node to the sequence. In the former case, an $s - t$ path is obtained and, in the latter case, a cycle is obtained. Denote the outcome by $h \in H$. Replace $x_h = 0$ with the minimum flow of the arcs included in h. Also, replace f_{ij} with $f_{ij} - x_h$ if h includes (i, j). Proceed to the next step.

[8] See Ahuja et al. (1) and Ford and Fulkerson (14) for reference.

(ii) If there is no $(i_1, i_2) \in A$ with $i_1 = s$ and $f_{i_1 i_2} > 0$, find another arc (i, j) with $f_{ij} > 0$. Start from node i. By applying the argument in (i), we obtain a cycle and modify vector x and flow f. Proceed to the next step. $\square$

Appendix B

Lemma 1

For each $(f, b) \in F \times B$, *it holds that* $v \cdot f^{b^o} - v \cdot f^b \leq c \cdot b$.

Proof. Let $f \in F$ be any flow. Because f^{b^o} is the net flow of bads to the target under flow f and the zero blockage b^o, for each $(i, j) \in A$, we have $f_{ij}^{b^o} \leq c_{ij}$. Thus, blocking arc (i, j) decreases the value of the net flow by at most c_{ij}. Therefore, for each $b = (b_{ij})'_{(i,j)\in A} \in B$, we have $v \cdot f^{b^o} - v \cdot f^b \leq \sum_{(i,j)\in A} c_{ij} b_{ij}$. $\square$

Lemma 2

If f^α *is an acyclic maximum flow with large bound quantity* q *in network* (N, A, c), *for each* $b \in B$, *it holds that* $v \cdot f^\alpha - v \cdot (f^\alpha)^b \leq c \cdot b$. *Furthermore, if* $q \leq (1/\ell_t) c \cdot b^*$ *and* f^β *is a binding flow, for each* $b \in B$, *it holds that* $\ell_t(v \cdot f^\beta) - \ell_t(v \cdot (f^\beta)^b) \leq c \cdot b$.

Proof. Lemma 1 implies that for each $b \in B$, $v \cdot (f^\alpha)^{b^o} - v \cdot (f^\alpha)^b \leq c \cdot b$. Because $f^\alpha = (f^\alpha)^{b^o}$ from equality (II.8), we have $v \cdot f^\alpha - v \cdot (f^\alpha)^b \leq c \cdot b$. Now multiplying both sides by $(\ell_t)(q/(c \cdot b^*))$, we have $(\ell_t)(q/(c \cdot b^*))(v \cdot f^\alpha - v \cdot (f^\alpha)^b) \leq (\ell_t)(q/(c \cdot b^*))(c \cdot b)$. Because f^β is a binding flow and $f^\beta = (q/(c \cdot b^*)) f^\alpha$, we have $(\ell_t)(q/(c \cdot b^*))(v \cdot f^\alpha - v \cdot (f^\alpha)^b) = \ell_t(v \cdot f^\beta) - \ell_t(v \cdot (f^\beta)^b)$. Because $q \leq (1/\ell_t) c \cdot b^*$, we have $(\ell_t)(q/(c \cdot b^*))(c \cdot b) \leq c \cdot b$. Thus, for each $b \in B$, we have $\ell_t(v \cdot f^\beta) - \ell_t(v \cdot (f^\beta)^b) \leq c \cdot b$. $\square$

Proof of Proposition 1. Suppose that $q \leq (1/\ell_t)c \cdot b^*$. We show that in any binding-flow zero-blockage strategy profile (f^β, b^o) each player has no incentive to change his or her strategy. Since $(f^\beta)^{b^o} = f^\beta$ from equality (II.10) and $v \cdot f^\beta = q$ from equality (II.11), we have $u_1(f^\beta, b^o) = (\ell_t - e)q$. Suppose that player 1 chooses any flow f. Since $v \cdot f^{b^o} \leq v \cdot f$ and $v \cdot f \leq q$,

$$\begin{aligned} u_1(f, b^o) &= \ell_t(v \cdot f^{b^o}) + c \cdot b^o - e(v \cdot f) \\ &\leq \ell_t(v \cdot f) - e(v \cdot f) \\ &\leq (\ell_t - e)q. \end{aligned}$$

Thus, player 1 has no incentive to change his strategy. Since $(f^\beta)^{b^o} = f^\beta$ from equality (II.10) and $v \cdot f^\beta = q$ from equality (II.11), we have $u_2(f^\beta, b^o) = w - (\ell_t)q$. Suppose that player 2 chooses any blockage b. Since $\ell_t(v \cdot f^\beta) - \ell_t(v \cdot (f^\beta)^b) \leq c \cdot b$ from Lemma 2 and $v \cdot f^\beta = q$ from equality (II.11),

$$\begin{aligned} u_2(f^\beta, b) &= w - \ell_t(v \cdot (f^\beta)^b) - c \cdot b \\ &\leq w - \ell_t(v \cdot f^\beta) \\ &= w - (\ell_t)q. \end{aligned}$$

Thus, player 2 has no incentive to change her strategy. Therefore, (f^β, b^o) is a Nash equilibrium. □

Proof of Proposition 2. Suppose that $c \cdot b^* < q$. In any λ-flow min-cut strategy profile $(\sigma_1^\lambda, \sigma_2^*)$ player 1 chooses only two pure strategies f^τ and λf^α with positive probability and player 2 chooses only two pure strategies b^o and b^* with positive probability. In addition each player is indifferent between the two pure strategies played with positive probability. Thus, to show that $(\sigma_1^\lambda, \sigma_2^*)$ is a Nash equilibrium, it suffices to show that (i) for each $f \in F$,

$u_1(\lambda f^{\alpha}, \sigma_2^*) \geq u_1(f, \sigma_2^*)$ and (ii) for each $b \in B$, $u_2(\sigma_1^{\lambda}, b^*) \geq u_2(\sigma_1^{\lambda}, b)$.

(i) We show that for each $f \in F$, $u_1(\lambda f^{\alpha}, \sigma_2^*) \geq u_1(f, \sigma_2^*)$. Let $f \in F$ be any flow. Calculate player 1's payoffs. Since $v \cdot f^{b^o} \leq v \cdot f$,

$$\begin{aligned} u_1(f, b^o) &= \ell_t(v \cdot f^{b^o}) + c \cdot b^o - e(v \cdot f) \\ &\leq (\ell_t - e)(v \cdot f). \end{aligned}$$

Since $v \cdot f^{b^*} = 0$,

$$\begin{aligned} u_1(f, b^*) &= \ell_t(v \cdot f^{b^*}) + c \cdot b^* - e(v \cdot f) \\ &= c \cdot b^* - e(v \cdot f). \end{aligned}$$

Since $\sigma_2^*(b^o) = e/\ell_t$ and $\sigma_2^*(b^*) = 1 - e/\ell_t$,

$$\begin{aligned} u_1(f, \sigma_2^*) &= \sigma_2^*(b^o) u_1(f, b^o) + \sigma_2^*(b^*) u_1(f, b^*) \\ &\leq (e/\ell_t)(\ell_t - e)(v \cdot f) + (1 - e/\ell_t)(c \cdot b^* - e(v \cdot f)) \\ &= (1 - e/\ell_t)(c \cdot b^*). \end{aligned}$$

From (II.12) we know that $u_1(\lambda f^{\alpha}, \sigma_2^*) = (1 - e/\ell_t)(c \cdot b^*)$. Thus, for each $f \in F$, $u_1(\lambda f^{\alpha}, \sigma_2^*) \geq u_1(f, \sigma_2^*)$.

(ii) We show that for each $b \in B$, $u_2(\sigma_1^{\lambda}, b^*) \geq u_2(\sigma_1^{\lambda}, b)$. Let $b \in B$ be any blockage. Calculate player 2's payoffs. Since $v \cdot (f^{\tau})^b = 0$,

$$\begin{aligned} u_2(f^{\tau}, b) &= w - \ell_t(v \cdot (f^{\tau})^b) - c \cdot b \\ &= w - c \cdot b. \end{aligned}$$

Since $v \cdot (\lambda f^\alpha)^b = \lambda(v \cdot (f^\alpha)^b)$,

$$\begin{aligned} u_2(\lambda f^\alpha, b) &= w - \ell_t(v \cdot (\lambda f^\alpha)^b) - c \cdot b \\ &= w - \lambda \ell_t(v \cdot (f^\alpha)^b) - c \cdot b. \end{aligned}$$

Since $\sigma_1^\lambda(f^\tau) = 1 - 1/\lambda\ell_t$, $\sigma_1^\lambda(\lambda f^\alpha) = 1/\lambda\ell_t$, and $v \cdot f^\alpha - v \cdot (f^\alpha)^b \leq c \cdot b$ from Lemma 2,

$$\begin{aligned} u_2(\sigma_1^\lambda, b) &= \sigma_1^\lambda(f^\tau)u_2(f^\tau, b) + \sigma_1^\lambda(\lambda f^\alpha)u_2(\lambda f^\alpha, b) \\ &= (1 - 1/\lambda\ell_t)(w - c \cdot b) + (1/\lambda\ell_t)(w - \lambda\ell_t(v \cdot (f^\alpha)^b) - c \cdot b) \\ &= w - c \cdot b - v \cdot (f^\alpha)^b \\ &\leq w - v \cdot f^\alpha. \end{aligned}$$

Then the max-flow min-cut theorem (II.9) implies that $u_2(\sigma_1^\lambda, b) \leq w - c \cdot b^*$. From (II.13) we know that $u_2(\sigma_1^\lambda, b^*) = w - c \cdot b^*$. Thus, for each $b \in B$, $u_2(\sigma_1^\lambda, b^*) \geq u_2(\sigma_1^\lambda, b)$.

Therefore, $(\sigma_1^\lambda, \sigma_2^*)$ is a Nash equilibrium. □

Proof of Proposition 3. Suppose that $(1/\ell_t)c \cdot b^* < q \leq c \cdot b^*$. In any μ-flow min-cut strategy profile $(\sigma_1^\mu, \sigma_2^*)$ player 1 chooses only two pure strategies f^τ and μf^β with positive probability and player 2 chooses only two pure strategies b^o and b^* with positive probability. In addition each player is indifferent between the two pure strategies played with positive probability. Thus, to show that $(\sigma_1^\mu, \sigma_2^*)$ is a Nash equilibrium, it suffices to show that (i) for each $f \in F$, $u_1(\mu f^\beta, \sigma_2^*) \geq u_1(f, \sigma_2^*)$ and (ii) for each $b \in B$, $u_2(\sigma_1^\mu, b^*) \geq u_2(\sigma_1^\mu, b)$.

(i) We show that for each $f \in F$, $u_1(\mu f^\beta, \sigma_2^*) \geq u_1(f, \sigma_2^*)$. Let $f \in F$ be any flow. Calculate player 1's payoffs. Since $v \cdot f^{b^o} \leq v \cdot f$,

$$\begin{aligned} u_1(f, b^o) &= \ell_t(v \cdot f^{b^o}) + c \cdot b^o - e(v \cdot f) \\ &\leq (\ell_t - e)(v \cdot f). \end{aligned}$$

Since $v \cdot f^{b^*} = 0$,

$$\begin{aligned} u_1(f, b^*) &= \ell_t(v \cdot f^{b^*}) + c \cdot b^* - e(v \cdot f) \\ &= c \cdot b^* - e(v \cdot f). \end{aligned}$$

Since $\sigma_2^*(b^o) = e/\ell_t$ and $\sigma_2^*(b^*) = 1 - e/\ell_t$,

$$\begin{aligned} u_1(f, \sigma_2^*) &= \sigma_2^*(b^o)u_1(f, b^o) + \sigma_2^*(b^*)u_1(f, b^*) \\ &\leq (e/\ell_t)(\ell_t - e)(v \cdot f) + (1 - e/\ell_t)(c \cdot b^* - e(v \cdot f)) \\ &= (1 - e/\ell_t)(c \cdot b^*). \end{aligned}$$

From (II.16) we know that $u_1(\mu f^\beta, \sigma_2^*) = (1 - e/\ell_t)(c \cdot b^*)$. Thus, for each $f \in F$, $u_1(\mu f^\beta, \sigma_2^*) \geq u_1(f, \sigma_2^*)$.

(ii) We show that for each $b \in B$, $u_2(\sigma_1^\mu, b^*) \geq u_2(\sigma_1^\mu, b)$. Let $b \in B$ be any blockage. Calculate player 2's payoffs. Since $v \cdot (f^\tau)^b = 0$,

$$\begin{aligned} u_2(f^\tau, b) &= w - \ell_t(v \cdot (f^\tau)^b) - c \cdot b \\ &= w - c \cdot b. \end{aligned}$$

Since $v \cdot (\mu f^\beta)^b = \mu(v \cdot (f^\beta)^b)$,

$$\begin{aligned} u_2(\mu f^\beta, b) &= w - \ell_t(v \cdot (\mu f^\beta)^b) - c \cdot b \\ &= w - \mu\ell_t(v \cdot (f^\beta)^b) - c \cdot b. \end{aligned}$$

Since $\sigma_1^\mu(f^\tau) = 1 - (1/\mu\ell_t)(1/q)(c \cdot b^*)$, $\sigma_1^\mu(\mu f^\beta) = (1/\mu\ell_t)(1/q)(c \cdot b^*)$, and $v \cdot (f^\beta)^b = (q/(c \cdot b^*))(v \cdot (f^\alpha)^b)$,

$$\begin{aligned} u_2(\sigma_1^\mu, b) &= \sigma_1^\mu(f^\tau)u_2(f^\tau, b) + \sigma_1^\mu(\mu f^\beta)u_2(\mu f^\beta, b) \\ &= w - c \cdot b - (1/q)(c \cdot b^*)(v \cdot (f^\beta)^b) \\ &= w - c \cdot b - (1/q)(c \cdot b^*)(q/(c \cdot b^*))(v \cdot (f^\alpha)^b) \\ &= w - c \cdot b - v \cdot (f^\alpha)^b \\ &\leq w - v \cdot f^\alpha. \end{aligned}$$

The last inequality comes from Lemma 2. Then the max-flow min-cut theorem (II.9) implies that $u_2(\sigma_1^\mu, b) \leq w - c \cdot b^*$. From (II.17) we know that $u_2(\sigma_1^\mu, b^*) = w - c \cdot b^*$. Thus, for each $b \in B$, $u_2(\sigma_1^\mu, b^*) \geq u_2(\sigma_1^\mu, b)$.

Therefore, $(\sigma_1^\mu, \sigma_2^*)$ is a Nash equilibrium. □

CHAPTER III

NETWORK BADS: HOW BOTTLENECKS MATTER

Introduction

When there is transmission of bads through a network, the structure of the network affects the behavior of economic players. Players may want to control the source of bads or the bottlenecks of the network. The structure of the network determines the bottlenecks, and in turn, the bottlenecks determine the transmission of bads through the network. Therefore, by controlling the bottlenecks, players can control the transmission of bads. However, players behave strategically and equilibrium outcomes may not be efficient.

For example, when water pollutants are carried through rivers and pipelines, public agencies can view the whole system of rivers and pipelines as a network and manage the bottlenecks of the network. They can also manage the source of pollution. If public agencies manage the source and the network separately, they may not take action that leads to efficient outcomes.[1]

Cooperative approaches through transfers between players can help achieve efficient outcomes. Sharing human resources and monetary funds can be examples of transfers. During the outbreak of Severe Acute Respiratory Syndrome (SARS) in 2003, by sharing human resources, public health agencies worked together to prevent the disease. Also, in the ongoing debt crisis, European governments are seeking cooperative agreements includ-

[1] When infectious diseases are transmitted through social networks between individuals, public agencies may want to control social networks to reduce the transmission of diseases. For studies on disease control, see Glass et al. (17) and Viboud et al. (38). When liquidity shocks are contagious through financial networks between banks, liquid banks may want to control financial networks to reduce the contagion of shocks. For studies on financial contagion, see Allen and Gale (2) and Leitner (26).

ing the creation and expansion of a bailout fund, the European Financial Stability Facility (EFSF), to resolve the crisis.

This paper examines a model where bads arise at a source node and may be transmitted to a sink node through a network. Two players, player 1 and player 2, wish to decrease the extent of bads transmitted to the sink node. Player 1 can act to mitigate the Extent of Bads at the Source (EBS) while player 2 can act to reduce the Maximum Transmission of Bads (MTB) through the network. The minimum of the mitigated EBS and the reduced MTB is assumed to determine the extent of bads transmitted to the sink node, which is called the extent of network bads. Each player incurs the cost of action but benefits from the decrease in the extent of network bads. Each player's action to decrease the extent of network bads can be viewed as a public good.

I characterize efficient strategy profiles and equilibria in terms of the bottlenecks of the network and the extent of bads at the source, as well as costs and benefits. In an efficient profile the joint payoff is maximized. In an equilibrium each player maximizes his own payoff given the other player's strategy.

In every efficient profile and in every equilibrium, either (a) player 1 acts to mitigate the EBS or (b) player 2 acts to reduce the MTB through the network or (c) neither player acts. Because the minimum of the mitigated EBS and the reduced MTB determines the extent of network bads, it is neither an efficient profile nor an equilibrium for both players to act. Thus, one player may free-ride on the other player's action. In case (a), player 2 free-rides on player 1's action. In case (b), player 1 free-rides on player 2's action.

If either player 1 or player 2 acts, in an efficient profile, the marginal cost of action equals to the marginal joint loss from network bads, whereas in an equilibrium, the marginal cost equals to the marginal private loss. Thus, in case (a) and in case (b), the equilibrium level of action is less than the efficient level.

For further comparison, my analysis is divided into three cases depending on the bottleneck capacity. In a network a bottleneck is an arc used to its capacity when the maximum transmission occurs. The total capacity of bottlenecks, or the bottleneck capacity, determines the maximum transmission of bads through the network. Whether the bottleneck capacity is small, intermediate, or large depends on the extent of bads at the source and the efficient level of action.

If the bottleneck capacity is small, the maximum transmission of bads (MTB) is less than the extent of bads at the source (EBS) mitigated to the efficient level. Thus, even if player 1 mitigates the EBS to the efficient level, the extent of network bads does not decrease. Similarly, when he mitigates to the equilibrium level, the extent does not decrease. Therefore, in every efficient profile and in every equilibrium, player 1 does not act at all. Either (b) player 2 acts or (c) neither player acts.

If the bottleneck capacity is large, the EBS is less than or equal to the MTB reduced to the efficient level. Thus, even if player 2 reduces the MTB to the efficient level, the extent of network bads does not decrease. Therefore, in every efficient profile, player 2 does not act at all. Either (a) player 1 acts or (c) neither player acts. However, in equilibrium, player 1 always acts. That is, there is no equilibrium where player 2 acts or neither player acts.

If the bottleneck capacity is intermediate, the MTB is greater than or equal to the EBS mitigated to the efficient level, and the EBS is greater than the MTB reduced to the efficient level. Thus, both players can decrease the extent of network bads to the efficient level. In every efficient profile, either (a) player 1 acts or (b) player 2 acts. Costs and benefits determine who takes action in an efficient profile. However, in equilibrium, both players may take no action or wrong action while they try to free-ride on each other's action. Specifically, there may be an inefficient equilibrium where neither player acts. More interestingly, the player who acts in an equilibrium may not be the player who must act in the efficient profile. That is, the efficient player may not act in equilibrium. Overall, strategic inaction leads to inefficiency.

Unless there is an efficient equilibrium where neither player acts, no efficient profile is an equilibrium and no equilibrium is efficient. Thus, players 1 and 2 may not have incentives to choose an efficient profile. To provide incentives for efficiency, I introduce cooperative solutions, where players 1 and 2 jointly choose an efficient profile and make transfers to each other. Each player is better off by using a cooperative solution than by maximizing his payoff alone. I show that such cooperative solutions always exist. Therefore, both players can always achieve efficiency by using cooperative solutions. If the bottleneck capacity is intermediate, in every cooperative solution, the efficient player acts to decrease the extent of network bads and gets a positive net transfer from the other.

The contribution of this chapter is threefold.

First, this chapter contributes to the literature on public goods in networks by considering public bads transmitted through networks. Public goods, and also public bads, are transmitted through various networks. Knowledge and information can be examples of public goods.[2] Infectious diseases and liquidity shocks can be examples of public bads. In the literature there are two different approaches: The network interaction approach assumes that networks are given and fixed; the network formation approach assumes that networks are chosen by players. Bramoullé and Kranton (6) is an example of the former; Galeotti

[2]Conley and Udry (12) show how farmers learn agricultural practices from other farmers.

and Goyal (16) is an example of the latter.[3] A third approach is explored in this chapter: A network is given but not fixed. Some player may have limited discretion to control the network. For example, suppose that a disease breaks out in one area connected through a transportation network. This disease can be transmitted through the network to another area. An agency may want to reduce the transmission of the disease by controlling the network. The agency's efforts to reduce the transmission can be viewed as public goods in the network.

Second, this chapter contributes to the literature on adversarial networks by introducing non-adversarial players. While these players are affected by network bads, they have different tools to deal with the bads. Thus, it is important to coordinate the players' behavior to achieve better outcomes. In the literature, however, it is assumed that players are adversarial. Washburn and Wood (40) study a zero-sum game between an evader and an interdictor in a given network. The evader chooses a path to move through the network while the interdictor chooses an arc in the network to stop the evader. Baccara and Bar-Isaac (3) study a game between a law enforcement agency and a criminal organization. The agency allocates its resources to detect criminals while the criminals form an organization to carry out illegal activities. In this chapter, players are non-adversarial. Players coordinate with each other to decrease network bads. Cooperative solutions are studied to help the coordination.

Third, this chapter contributes to the literature on flow games by introducing strategic behavior. Kalai and Zemel (24) introduce a coalitional game, called a flow game, where the worth of a coalition is defined as the value of a maximum flow through the network restricted to the members of the coalition. Every flow game is totally balanced

[3] Cho (9) also studies a network model for public goods. For a review of the literature, see Goyal (18) and Jackson (23).

and thus has a non-empty core. A minimum cut is used to find a core allocation.[4] In this chapter, players make strategic choices to decrease the extent of network bads, which is constrained by the value of a maximum flow through a network. Equilibrium strategies, as well as efficient strategies, are constructed on the concept of a minimum cut.

The rest of this chapter is organized as follows. Section 2 develops a strategic network model. Section 3 analyzes efficient profiles and equilibria and introduces cooperative solutions. Section 4 concludes.

Model

Let $N := \{1, 2, \ldots, n\}$ be the set of nodes with $n \geq 2$. Node 1 is called the source and node n is called the sink. Let $A \subseteq \{(i, j) \in N \times N : i \neq j\}$ be the set of arcs, where each arc is an ordered pair of distinct nodes. Each arc (i, j) is directed from node i to node j. The Extent of Bads at the Source (EBS) is described by a number $x > 0$. These bads are transmitted through an arc from one node to another node. For each $(i, j) \in A$, the maximum transmission of bads through arc (i, j) from node i to node j is described by a number $y_{ij} > 0$, which is called the capacity of arc (i, j).[5] Let $y := (y_{ij})_{(i,j)\in A}$ be the vector of arc capacities. The collection (N, A, y) is called a *network*.

Two players, player 1 and player 2, wish to decrease the extent of bads transmitted to the sink. Both players choose their strategies simultaneously and independently.

Player 1 can act to mitigate the extent of bads at the source. Player 1 chooses a *mitigation strategy* m with $0 \leq m \leq x$. The set of mitigation strategies for player 1 is

[4]For studies on flow games, see Granot and Granot (20), Kalai and Zemel (25), Potters et al. (34), and Reijnierse et al. (35). Also, see Granot and Maschler (21) and Van den Nouweland et al. (37) for related studies on spanning network games.

[5]This capacity can be viewed as a plausible upper bound for the transmission of bads through arc (i, j). For a discussion of plausible upper bounds in risk analysis, see Paté-Cornell (33).

denoted by M. By choosing $m \in M$, player 1 mitigates the extent of bads at the source to $x - m$.

Player 2 can act to reduce the maximum transmission of bads through network (N, A, y). Player 2 chooses a *reduction strategy* $r := (r_{ij})_{(i,j)\in A}$ with $0 \leq r_{ij} \leq y_{ij}$ for each $(i, j) \in A$. The set of reduction strategies for player 2 is denoted by R. By choosing $r = (r_{ij})_{(i,j)\in A} \in R$, player 2 reduces the maximum transmission of bads through arc (i, j) to $y_{ij} - r_{ij}$.

For each $(m, r) \in M \times R$, the extent of bads transmitted to the sink is called the extent of *network bads* and denoted by $e(m, r)$. How to define $e(m, r)$ will be explained later.

Both players lose $l(e(m, r))$ from $e(m, r)$, where $l(\cdot)$ is a *loss function* with $l(0) = 0$. In other words, both players benefit from the decrease in the extent of network bads. However, each player incurs the cost of action. Player 1 pays $c_1(m)$ for $m \in M$, where $c_1(\cdot)$ is a *mitigation cost function* with $c_1(0) = 0$. Player 2 pays $c_2\left(\sum_{(i,j)\in A} r_{ij}\right)$ for $r = (r_{ij})_{(i,j)\in A} \in R$, where $\sum_{(i,j)\in A} r_{ij}$ is the capacity of reduction strategy r, and $c_2(\cdot)$ is a *reduction cost function* with $c_2(0) = 0$. Assume that $l(\cdot)$, $c_1(\cdot)$, and $c_2(\cdot)$ are twice continuously differentiable, strictly increasing, and strictly convex. That is, $l'(\cdot) > 0$, $l''(\cdot) > 0$, $c_1'(\cdot) > 0$, $c_1''(\cdot) > 0$, $c_2'(\cdot) > 0$, and $c_2''(\cdot) > 0$. Players earn a constant worth w.

For each $(m, r) \in M \times R$, the payoff of player 1 is

$$u_1(m, r) = w - c_1(m) - l(e(m, r))$$

and the payoff of player 2 is

$$u_2(m, r) = w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(e(m, r)).$$

Here the transmission of bads creates a public bad. Both mitigation strategy and reduction strategy are public goods to decrease the public bad.

A strategy $\tilde{m} \in M$ is a *best response* of player 1 to $r \in R$ if for each $m \in M$, $u_1(\tilde{m}, r) \geq u_1(m, r)$. For each $r \in R$, let $BR_1(r) \subseteq M$ be the set of player 1's best responses to r. A strategy $\tilde{r} \in R$ is a *best response* of player 2 to $m \in M$ if for each $r \in R$, $u_2(m, \tilde{r}) \geq u_2(m, r)$. For each $m \in M$, let $BR_2(m) \subseteq R$ be the set of player 2's best responses to m. A strategy profile $(\tilde{m}, \tilde{r}) \in M \times R$ is an *equilibrium* if $\tilde{m}$ is a best response to $\tilde{r}$ and $\tilde{r}$ is a best response to $\tilde{m}$.

For each $(m, r) \in M \times R$, the joint payoff is $U(m, r) = 2w - c_1(m) - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(e(m, r))$. A strategy profile $(\tilde{m}, \tilde{r}) \in M \times R$ is *efficient* if for each $(m, r) \in M \times R$, $U(\tilde{m}, \tilde{r}) \geq U(m, r)$. That is, an efficient (strategy) profile maximizes the joint payoff.

Now I define $e(m, r)$ the extent of network bads. The transmission of bads from the source to the sink is modeled as a flow through a network. Formally, for each $r \in R$, a flow through network $(N, A, y - r)$ is a vector $f := (f_{ij})_{(i,j)\in A}$ satisfying the following constraints:

$$0 \leq f_{ij} \leq y_{ij} - r_{ij} \qquad \text{for each } (i, j) \in A; \tag{III.1}$$

$$f_{i1} = 0 \qquad \text{for each } (i, 1) \in A; \tag{III.2}$$

$$f_{ni} = 0 \qquad \text{for each } (n, i) \in A; \tag{III.3}$$

$$\sum_{j:(j,i)\in A} f_{ji} = \sum_{j:(i,j)\in A} f_{ij} \qquad \text{for each } i \in N \setminus \{1, n\}. \tag{III.4}$$

Constraint (III.1) says that each arc flow f_{ij} is non-negative and constrained by the reduced arc capacity $y_{ij} - r_{ij}$. Constraint (III.2) says that there is no arc flow to the source. Constraint (III.3) says that there is no arc flow from the sink. Constraint (III.4) says that at each node i, except for the source and the sink, the total arc flow to node i equals to the total arc flow from node i. That is, there is no change in the total arc flow at node $i \neq 1, n$. For each $r \in R$, let $F(r)$ be the set of all flows through network $(N, A, y - r)$.

For each $r \in R$, the value of a maximum flow through network $(N, A, y - r)$ is defined as

$$f(r) = \max \sum_{(i,n) \in A} f_{in} \quad \text{subject to } f \in F(r). \tag{III.5}$$

That is, $f(r)$ is the maximum total arc flow to the sink through network $(N, A, y - r)$. The value of a maximum flow through network $(N, A, y - r)$ is also called the Maximum Transmission of Bads (MTB).

For each $(m, r) \in M \times R$, the extent of network bads is defined as

$$e(m, r) = \min\{x - m, f(r)\}.$$

That is, the extent of network bads is the minimum of $x - m$, the mitigated extent of bads at the source, and $f(r)$, the reduced maximum transmission of bads. For each $(m, r) \in M \times R$, if $x - m \leq f(r)$, the extent of network bads is $e(m, r) = x - m$. If $f(r) \leq x - m$, however, the extent of network bads is $e(m, r) = f(r)$.

The following example shows how to find the extent of network bads.

Example 6

Suppose that a network is given as (N, A, y), where $N = \{1, 2, 3, 4\}$ is the set of nodes, $A = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$ is the set of arcs, and $y = (y_{12}, y_{13}, y_{23}, y_{24}, y_{34}) = (50, 10, 10, 10, 50)$ is the vector of arc capacities. The extent of bads at the source is given as $x = 36$. Suppose that player 1 chooses a mitigation strategy $m = 16$ and player 2 chooses a reduction strategy $r = (r_{12}, r_{13}, r_{23}, r_{24}, r_{34}) = (10, 10, 0, 0, 0)$. Thus, $x - m = 20$ and $y - r =$

$(40, 0, 10, 10, 50)$. Note that there is a unique maximum flow $f = (f_{12}, f_{13}, f_{23}, f_{24}, f_{34}) = (20, 0, 10, 10, 10)$ through network $(N, A, y-r)$. The value of the maximum flow is $f(r) = 20$. Thus, the extent of network bads is $e(m, r) = 20$. In Figure 4 each solid circle indicates a node and each arrow indicates an arc. In each pair of numbers the first bold number shows the maximum flow and the second light number shows the reduced arc capacity. □

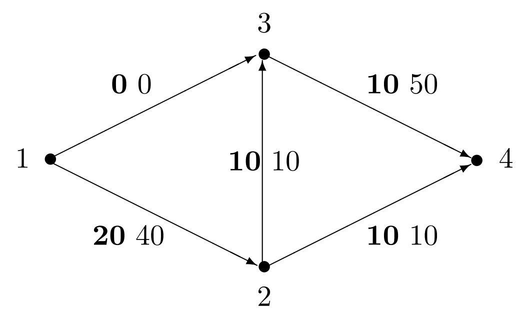

Figure 4. Maximum Flow

The vector of zeros $\underline{0}$ is called the *zero reduction strategy*. The value of a maximum flow through network (N, A, y) is denoted by $f(\underline{0})$.

A cut $C := (C_1, C_n)$ is an ordered partition of the node set N with $1 \in C_1$ and $n \in C_n$. For each cut $C = (C_1, C_n)$, an arc $(i, j) \in A$ is a cut arc of C if $i \in C_1$ and $j \in C_n$. For each cut $C = (C_1, C_n)$, let $A(C) := \{(i, j) \in A : i \in C_1 \text{ and } j \in C_n\}$ be the set of all cut arcs of C.

For each cut C, the value of a maximum flow through network (N, A, y) is less than or equal to the total capacity of cut arcs of C. Formally, for each cut C, it holds that

$$f(\underline{0}) \leq \sum_{(i,j) \in A(C)} y_{ij}. \tag{III.6}$$

For each $r \in R$, a cut C^r is a *minimum cut* in $(N, A, y - r)$ if for each cut C,

$$\sum_{(i,j)\in A(C^r)} y_{ij} - r_{ij} \leq \sum_{(i,j)\in A(C)} y_{ij} - r_{ij}. \tag{III.7}$$

The left hand side of inequality (III.7) is called the capacity of a minimum cut in $(N, A, y-r)$.

A minimum cut in a network determines the bottlenecks of the network. For each $r \in R$, if C^r is a minimum cut in network $(N, A, y - r)$, then $A(C^r)$ is the set of all cut arcs of C^r. Each cut arc $(i, j) \in A(C^r)$ is called a *bottleneck* of network $(N, A, y - r)$. The capacity of a minimum cut is also called the *bottleneck capacity.*

The *max-flow min-cut theorem* says that for each $r \in R$, the value of a maximum flow through network $(N, A, y-r)$ equals to the capacity of a minimum cut in the network.[6] Formally, for each $r \in R$, it holds that

$$f(r) = \sum_{(i,j)\in A(C^r)} y_{ij} - r_{ij}. \tag{III.8}$$

In other words, the maximum transmission of bads through network $(N, A, y-r)$ equals to the bottleneck capacity of the network.

From inequality (III.6) and the max-flow min-cut theorem (III.8), for each $r \in R$, we can show that

$$f(\underline{0}) - f(r) \leq \sum_{(i,j)\in A} r_{ij}. \tag{III.9}$$

A reduction strategy $r \in R$ is a *bottleneck reduction strategy* if there is a minimum cut C in network (N, A, y) such that for each $(i, j) \in A(C)$, $r_{ij} > 0$, and for each $(i, j) \notin A(C)$, $r_{ij} = 0$. Note that each cut arc $(i, j) \in A(C)$ is a bottleneck of network (N, A, y) and player 2 reduces the maximum transmission of bads through the bottlenecks.

[6]Ford and Fulkerson (14) introduce the maximum flow problem (III.5) and show the max-flow min-cut theorem. For a detailed discussion, see Ahuja et al. (1).

From the max-flow min-cut theorem (III.8), if $r \in R$ is a bottleneck reduction strategy, we can show that

$$\sum_{(i,j) \in A} r_{ij} = f(\underline{0}) - f(r). \qquad \text{(III.10)}$$

The following is an example of a bottleneck reduction strategy.

Example 7

Consider network (N, A, y) in Example 6. There is a unique minimum cut $C = (\{1, 2\}, \{3, 4\})$ in network (N, A, y). The set of all cut arcs of C is $A(C) = \{(1, 3), (2, 3), (2, 4)\}$. Suppose that player 2 chooses a bottleneck reduction strategy $r = (0, 10, 5, 5, 0)$. The value of a maximum flow through network $(N, A, y - r)$ is $f(r) = 10$. In Figure 5, the bold numbers indicate the maximum flow, and the light numbers indicate the reduced arc capacities. □

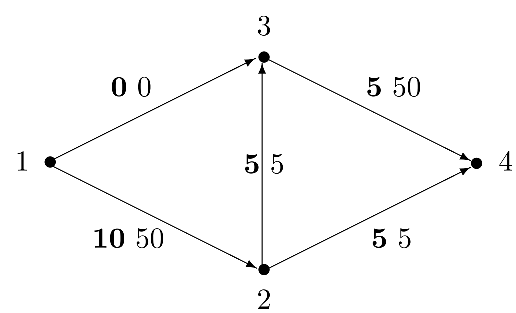

Figure 5. Bottleneck Reduction Strategy

Analysis

I want to analyze strategic behavior in my model. Specifically, how does network (N, A, y) affect strategic behavior? How much mitigation and reduction do players choose? Do players have incentives for efficiency? To answer these questions, I characterize efficient (strategy) profiles and equilibria.

Recall that x is the extent of bads at the source and $f(\underline{0})$ is the maximum transmission of bads through network (N, A, y). I make two assumptions on the loss and cost functions.

(A1) The marginal private loss at x is greater than the marginal mitigation cost at 0, i.e., $l'(x) > c_1'(0)$. The marginal joint loss at 0 is less than the marginal mitigation cost at x, i.e., $2l'(0) < c_1'(x)$.

(A2) The marginal private loss at $f(\underline{0})$ is greater than the marginal reduction cost at 0, i.e., $l'(f(\underline{0})) > c_2'(0)$. The marginal joint loss at 0 is less than the marginal reduction cost at $f(\underline{0})$, i.e., $2l'(0) < c_2'(f(\underline{0}))$.

Efficient Profiles

There are two necessary conditions for efficient profiles. In every efficient profile there is at most one player who acts to decrease the extent of network bads.

Remark 1

For each $(m,r) \in M \times R$, if both players act, that is, if $m \neq 0$ and $r \neq \underline{0}$, then (m,r) is not efficient. To see this, let $(m,r) \in M \times R$ be such that $m \neq 0$ and $r \neq \underline{0}$. Consider two cases. First, if $x - m \leq f(r)$, then $e(m,r) = x - m$ and $U(m,r) = 2w - c_1(m) - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(x-m)$. Because $c_2(\cdot)$ is strictly increasing and $r \neq \underline{0}$, we have $U(m,r) < U(m,\underline{0})$. Second, if $f(r) \leq x - m$, then $e(m,r) = f(r)$ and $U(m,r) = 2w - c_1(m) - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(f(r))$. Because $c_1(\cdot)$ is strictly increasing and $m \neq 0$, we have $U(m,r) < U(0,r)$. In both cases, (m,r) is not efficient. Henceforth, we only consider $(m,r) \in M \times R$ with $m = 0$ or $r = \underline{0}$ or both. $\square$

From inequality (III.9), for each $r \in R$, it holds that $\sum_{(i,j)\in A} r_{ij} \geq f(\underline{0}) - f(r)$. If (m,r) is an efficient profile such that $f(r) \leq x - m$, then $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. In words, there is no wasted reduction capacity to decrease the extent of network bads.

Remark 2

For each $(m,r) \in M \times R$ with $f(r) \leq x - m$, if $\sum_{(i,j)\in A} r_{ij} > f(\underline{0}) - f(r)$, then (m,r) is not efficient. To see this, let $(m,r) \in M \times R$ be such that $\sum_{(i,j)\in A} r_{ij} > f(\underline{0}) - f(r)$. Also, let $(m,\tilde{r}) \in M \times R$ be such that $\sum_{(i,j)\in A} \tilde{r}_{ij} = f(\underline{0}) - f(\tilde{r})$ and $f(\tilde{r}) = f(r)$. Because $f(r) \leq x - m$, we have $U(m,r) = 2w - c_1(m) - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(f(r))$. Because $f(\tilde{r}) = f(r)$ and $f(r) \leq x - m$, we have $U(m,\tilde{r}) = 2w - c_1(m) - c_2\left(\sum_{(i,j)\in A} \tilde{r}_{ij}\right) - 2l(f(\tilde{r}))$. Because $\sum_{(i,j)\in A} \tilde{r}_{ij} < \sum_{(i,j)\in A} r_{ij}$ and $f(\tilde{r}) = f(r)$, we have $U(m,r) < U(m,\tilde{r})$. Thus, (m,r) is not efficient. Henceforth, if $f(r) \leq x - m$, we only consider $(m,r) \in M \times R$ with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. $\square$

Define $\bar{m}$ and $\bar{q}$ for efficient profiles.

Let $\bar{m}$ be a level of mitigation such that $2l'(x-\bar{m}) = c_1'(\bar{m})$. That is, the marginal joint loss at $x - \bar{m}$ equals to the marginal mitigation cost at $\bar{m}$. Let $\bar{U}_1(m) = 2w - c_1(m) - 2l(x-m)$. The function $\bar{U}_1(m)$ is maximized at $m = \bar{m}$. The existence of a unique value for $\bar{m}$ is guaranteed by the assumption (A1). Note that $0 < \bar{m} < x$.

Let $\bar{q}$ be a level of reduction such that $2l'(f(\underline{0}) - \bar{q}) = c_2'(\bar{q})$. That is, the marginal joint loss at $f(\underline{0}) - \bar{q}$ equals to the marginal reduction cost at $\bar{q}$. Let $\bar{U}_2(q) = 2w - c_2(q) - 2l(f(\underline{0}) - q)$, where $0 \leq q \leq f(\underline{0})$. The function $\bar{U}_2(q)$ is maximized at $q = \bar{q}$. The existence of a unique value for $\bar{q}$ is guaranteed by the assumption (A2). Note that $0 < \bar{q} < f(\underline{0})$.

Let $\bar{R} := \{\bar{r} \in R : \sum_{(i,j)\in A} \bar{r}_{ij} = \bar{q}$ and $\bar{q} = f(\underline{0}) - f(\bar{r})\}$ be a set of reduction strategies with capacity $\bar{q}$. From equality (III.10), if $r \in R$ is a bottleneck reduction strategy with capacity $\sum_{(i,j)\in A} r_{ij} = \bar{q}$, then $r \in \bar{R}$.

My analysis is divided into three cases depending on the bottleneck capacity of network (N, A, y), which equals to the maximum transmission of bads through the network. The bottleneck capacity of network (N, A, y) is *small* if $f(\underline{0}) < x - \bar{m}$, *intermediate* if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, and *large* if $x \leq f(\bar{r})$.

If the bottleneck capacity is small, that is, if $f(\underline{0}) < x - \bar{m}$, the maximum transmission of bads (MTB) through network (N, A, y) is less than the extent of bads at the source (EBS) mitigated to the efficient level. Thus, even if player 1 acts to mitigate the EBS to the efficient level $x - \bar{m}$, the extent of network bads does not decrease. Therefore, in every efficient profile, player 1 does not act at all. Whether player 2 acts depends on the loss and cost functions. The following proposition summarizes this result. All proofs are presented in Appendix A.

Proposition 4

If the bottleneck capacity is small, that is, if $f(\underline{0}) < x - \bar{m}$*, every efficient profile is* $(0, \underline{0})$ *or* $(0, \bar{r})$*. Furthermore, if* $2l(f(\underline{0})) < c_2 \left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r}))$*, then* $(0, \underline{0})$ *is the only efficient profile. However, if* $2l(f(\underline{0})) > c_2 \left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r}))$*, then* $(0, \bar{r})$ *is the only efficient profile.*

If the bottleneck capacity is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, the MTB through network (N, A, y) is greater than or equal to the EBS mitigated to the efficient level, and the EBS is greater than the MTB through network $(N, A, y - \bar{r})$. Thus, both players can decrease the extent of network bads to the efficient level. Therefore, in every efficient profile, either player 1 or player 2 acts to decrease the extent of network bads. There is no efficient profile in which neither player acts. Whether player 1 acts or player 2 acts depends on the loss and cost functions.

Proposition 5

If the bottleneck capacity is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ *and* $f(\bar{r}) < x$*, every efficient profile is* $(0, \bar{r})$ *or* $(\bar{m}, \underline{0})$*. Furthermore, if* $c_2 \left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) < c_1(\bar{m}) + 2l(x - \bar{m})$*, then* $(0, \bar{r})$ *is the only efficient profile. However, if* $c_2 \left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) > c_1(\bar{m}) + 2l(x - \bar{m})$*, then* $(\bar{m}, \underline{0})$ *is the only efficient profile.*

The followings are examples of efficient profiles.

Example 8

Consider network (N, A, y) in Example 6. The MTB through network (N, A, y) is $f(\underline{0}) = 30$. Suppose that $x = 36$. Assume that $l(e(m, r)) = (e(m, r))^2$, $c_1(m) = m^2$, and $c_2\left(\sum r_{ij}\right) = \left(\sum r_{ij}\right)^2$. Because $4(x - \bar{m}) = 2\bar{m}$ and $x = 36$, we have $\bar{m} = 24$. Because $4(f(\underline{0}) - \bar{q}) = 2\bar{q}$ and $f(\underline{0}) = 30$, we have $\bar{q} = 20$. Suppose that player 2 chooses $\bar{r} = (0, 10, 5, 5, 0)$. The MTB through network $(N, A, y - \bar{r})$ is $f(\bar{r}) = 10$. Because $x - \bar{m} = 12 \leq 30 = f(\underline{0})$ and $f(\bar{r}) = 10 < 36 = x$, the bottleneck capacity is intermediate. Because $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) = 600 < 864 = c_1(\bar{m}) + 2l(x - \bar{m})$, from Proposition 5, $(0, \bar{r})$ is efficient. □

In the example above, $(0, \bar{r})$ is efficient and player 2 reduces the MTB to the efficient level $f(\bar{r})$. However, in the example below, $(\bar{m}, \underline{0})$ is efficient and player 1 mitigates the EBS to the efficient level $x - \bar{m}$.

Example 9

Consider Example 8 but now assume that $c_2\left(\sum r_{ij}\right) = 4\left(\sum r_{ij}\right)^2$. Because $4(f(\underline{0}) - \bar{q}) = 8\bar{q}$ and $f(\underline{0}) = 30$, we have $\bar{q} = 10$. Suppose that player 2 chooses $\bar{r} = (0, 4, 3, 3, 0)$. The MTB through $(N, A, y - \bar{r})$ is $f(\bar{r}) = 20$. Because $x - \bar{m} = 12 \leq 30 = f(\underline{0})$ and $f(\bar{r}) = 20 < 36 = x$, the bottleneck capacity is intermediate. Because $c_2\left(\sum \bar{r}_{ij}\right) + 2l(f(\bar{r})) = 1200 > 864 = c_1(\bar{m}) + 2l(x - \bar{m})$, from Proposition 5, $(\bar{m}, \underline{0})$ is efficient. □

If the bottleneck capacity is large, that is, if $x \leq f(\bar{r})$, the EBS is less than or equal to the MTB through network $(N, A, y - \bar{r})$. Thus, even if player 2 acts to reduce the MTB to the efficient level $f(\bar{r})$, the extent of network bads does not decrease. Therefore, in every efficient profile, player 2 does not act at all. Whether player 1 acts depends on the loss and cost functions.

Proposition 6

If the bottleneck capacity is large, that is, if $x \leq f(\bar{r})$*, every efficient profile is* $(0, \underline{0})$ *or* $(\bar{m}, \underline{0})$*. Furthermore, if* $2l(x) < c_1(\bar{m}) + 2l(x - \bar{m})$*, then* $(0, \underline{0})$ *is the only efficient profile. However, if* $2l(x) > c_1(\bar{m}) + 2l(x - \bar{m})$*, then* $(\bar{m}, \underline{0})$ *is the only efficient profile.*

To sum up, in every efficient profile, either (a) player 1 acts to mitigate the extent of bads at the source or (b) player 2 acts to reduce the maximum transmission of bads through the network or (c) neither player acts. In case (a), player 2 free-rides on player 1's mitigation effort. The efficient level of mitigation is $\bar{m}$, where the marginal joint loss equals to the marginal mitigation cost. In case (b), player 1 free-rides on player 2's reduction effort. The efficient level of reduction is $\sum_{(i,j) \in A} \bar{r}_{ij} = \bar{q}$, where the marginal joint loss equals to the marginal reduction cost.

As in Proposition 4, if $(0, \underline{0})$ is the only efficient profile, the extent of network bads is $e(0, \underline{0}) = f(\underline{0})$. As in Propositions 4 and 5, if $(0, \bar{r})$ is the only efficient profile, the extent is $e(0, \bar{r}) = f(\bar{r})$. As in Propositions 5 and 6, if $(\bar{m}, \underline{0})$ is the only efficient profile, the extent is $e(\bar{m}, \underline{0}) = x - \bar{m}$. As in Proposition 6, if $(0, \underline{0})$ is the only efficient profile, the extent is $e(0, \underline{0}) = x$. Later I will compare these with the extent of network bads in equilibrium.

Equilibria

Define $\hat{m}$ for player 1's best response. Let $\hat{m}$ be a level of mitigation such that $l'(x - \hat{m}) = c_1'(\hat{m})$. That is, the marginal private loss at $x - \hat{m}$ equals to the marginal mitigation cost at $\hat{m}$. Let $\hat{u}_1(m) = w - c_1(m) - l(x - m)$. The function $\hat{u}_1(m)$ is maximized at $m = \hat{m}$. The existence of a unique value for $\hat{m}$ is guaranteed by the assumption (A1). Note that $0 < \hat{m} < x$ and $\hat{m} < \bar{m}$.

Lemma 3

The set of player 1's best responses to $r \in R$ is

$$BR_1(r) = \begin{cases} \{0\} & \text{if } f(r) < x - \hat{m}; \\ \{0\} & \text{if } x - \hat{m} \le f(r) < x \text{ and } l(f(r)) < c_1(\hat{m}) + l(x - \hat{m}); \\ \{0, \hat{m}\} & \text{if } x - \hat{m} \le f(r) < x \text{ and } l(f(r)) = c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x - \hat{m} \le f(r) < x \text{ and } l(f(r)) > c_1(\hat{m}) + l(x - \hat{m}); \\ \{\hat{m}\} & \text{if } x \le f(r). \end{cases}$$

Given $r \in R$, the maximum transmission of bads through network $(N, A, y - r)$ is $f(r)$, and player 1's best response depends on $f(r)$. Player 1's best response to $r \in R$ is $m = 0$ or $m = \hat{m}$ or both.

There is a necessary condition for player 2's best responses, which is analogous to Remark 2. If $r \in R$ is a best response to $m \in M$ such that $f(r) \le x - m$, then $\sum_{(i,j) \in A} r_{ij} = f(\underline{0}) - f(r)$. Thus, there is no wasted reduction capacity to decrease the extent of network bads.

Remark 3

For each $(m, r) \in M \times R$ with $f(r) \le x - m$, if $\sum_{(i,j) \in A} r_{ij} > f(\underline{0}) - f(r)$, then r is not a best response to m. To see this, let $r \in R$ be such that $\sum_{(i,j) \in A} r_{ij} > f(\underline{0}) - f(r)$. Also, let $\tilde{r} \in R$ be such that $\sum_{(i,j) \in A} \tilde{r}_{ij} = f(\underline{0}) - f(\tilde{r})$ and $f(\tilde{r}) = f(r)$. Because $f(r) \le x - m$, we have $u_2(m, r) = w - c_2\left(\sum_{(i,j) \in A} r_{ij}\right) - l(f(r))$. Because $f(\tilde{r}) = f(r)$ and $f(r) \le x - m$, we have $u_2(m, \tilde{r}) = w - c_2\left(\sum_{(i,j) \in A} \tilde{r}_{ij}\right) - l(f(\tilde{r}))$. Because $\sum_{(i,j) \in A} \tilde{r}_{ij} < \sum_{(i,j) \in A} r_{ij}$ and $f(\tilde{r}) = f(r)$, we have $u_2(m, r) < u_2(m, \tilde{r})$. Thus, r is not a best response to m. Henceforth, if $f(r) \le x - m$, we only consider $r \in R$ with $\sum_{(i,j) \in A} r_{ij} = f(\underline{0}) - f(r)$. $\square$

Define $\hat{q}$ for player 2's best responses. Let $\hat{q}$ be a level of reduction such that $l'(f(\underline{0}) - \hat{q}) = c_2'(\hat{q})$. That is, the marginal private loss at $f(\underline{0}) - \hat{q}$ equals to the marginal reduction cost at $\hat{q}$. Let $\hat{u}_2(q) = w - c_2(q) - l(f(\underline{0}) - q)$, where $0 \leq q \leq f(\underline{0})$. The function $\hat{u}_2(q)$ is maximized at $q = \hat{q}$. The existence of a unique value for $\hat{q}$ is guaranteed by the assumption (A2). Note that $0 < \hat{q} < f(\underline{0})$ and $\hat{q} < \bar{q}$.

Let $\hat{R} := \{\hat{r} \in R : \sum_{(i,j)\in A} \hat{r}_{ij} = \hat{q} \text{ and } \hat{q} = f(\underline{0}) - f(\hat{r})\}$ be a set of reduction strategies with capacity $\hat{q}$. From equality (III.10), if $r \in R$ is a bottleneck reduction strategy with capacity $\sum_{(i,j)\in A} r_{ij} = \hat{q}$, then $r \in \hat{R}$.

Lemma 4

The set of player 2's best responses to $m \in M$ is

$$BR_2(m) = \begin{cases} \{\underline{0}\} & \text{if } x - m \leq f(\hat{r}); \\ \{\underline{0}\} & \text{if } f(\hat{r}) < x - m \text{ and } l(x-m) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})); \\ \{\underline{0}\} \cup \hat{R} & \text{if } f(\hat{r}) < x - m \text{ and } l(x-m) = c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})); \\ \hat{R} & \text{if } f(\hat{r}) < x - m \text{ and } l(x-m) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})). \end{cases}$$

Given $m \in M$, the mitigated extent of bads is $x - m$, and player 2's best response depends on $x - m$. Player 2's best response to $m \in M$ is $r = \underline{0}$ or $r = \hat{r}$ or both.

From Lemma 3, player 1's best response is $m = 0$ or $m = \hat{m}$. From Lemma 4, player 2's best response is $r = \underline{0}$ or $r = \hat{r}$. Consider strategy profiles $(0, \underline{0})$, $(0, \hat{r})$, $(\hat{m}, \underline{0})$, and $(\hat{m}, \hat{r})$. At a glance we can see that $(\hat{m}, \hat{r})$ is not an equilibrium. If $(\hat{m}, \hat{r})$ is an equilibrium, $\hat{m}$ is a best response to $\hat{r}$. From Lemma 3, if $\hat{m}$ is a best response to $\hat{r}$, it is necessary that $x - \hat{m} \leq f(\hat{r})$. From Lemma 4, if $x - \hat{m} \leq f(\hat{r})$, then $\hat{r}$ is not a best response to $\hat{m}$. This is a contradiction.

Now there remain three strategy profiles $(0, \underline{0})$, $(0, \hat{r})$, and $(\hat{m}, \underline{0})$ for equilibria. In these strategy profiles there is at most one player who acts to decrease the extent of network bads. In strategy profile $(0, \underline{0})$, neither player acts. In strategy profile $(0, \hat{r})$, player 2 acts to reduce the maximum transmission of bads through network (N, A, y) by $\sum_{(i,j)\in A} \hat{r}_{ij} = \hat{q}$. In strategy profile $(\hat{m}, \underline{0})$, player 1 acts to mitigate the extent of bads at the source by $\hat{m}$.

We want to know when these strategy profiles are equilibria. From Lemma 3, player 1's best response to $r \in R$ depends on $f(r)$. From Lemma 4, player 2's best response is $r = \underline{0}$ or $r = \hat{r}$. There are six cases for equilibria. Table 1 shows these cases. In Table 1 there are three contradictions. Because $f(\hat{r}) < f(\underline{0})$, if $f(\underline{0}) < x - \hat{m}$, we have $f(\hat{r}) < x - \hat{m}$, which contradicts $x - \hat{m} \leq f(\hat{r})$ and $x \leq f(\hat{r})$. Similarly, if $f(\underline{0}) < x$, we have $f(\hat{r}) < x$, which contradicts $x \leq f(\hat{r})$.

Table 1. Six Cases for Equilibria

$r = \underline{0}$	$r = \hat{r}$	Results
	$f(\hat{r}) < x - \hat{m}$	Proposition 7
$f(\underline{0}) < x - \hat{m}$	$x - \hat{m} \leq f(\hat{r}) < x$	Contradiction
	$x \leq f(\hat{r})$	Contradiction
	$f(\hat{r}) < x - \hat{m}$	Proposition 8
$x - \hat{m} \leq f(\underline{0}) < x$	$x - \hat{m} \leq f(\hat{r}) < x$	Proposition 9
	$x \leq f(\hat{r})$	Contradiction
	$f(\hat{r}) < x - \hat{m}$	Proposition 10
$x \leq f(\underline{0})$	$x - \hat{m} \leq f(\hat{r}) < x$	Proposition 11
	$x \leq f(\hat{r})$	Proposition 12

Proposition 7 *Suppose that* $f(\underline{0}) < x - \hat{m}$.

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \underline{0})$ is the only equilibrium.

(ii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

Because $f(\underline{0}) < x - \hat{m}$, from Lemma 3, we know that $\hat{m}$ is not a best response to $\underline{0}$. Thus, $(\hat{m}, \underline{0})$ is not an equilibrium. In Proposition 7, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$.

Proposition 8 *Suppose that* $x - \hat{m} \leq f(\underline{0}) < x$ *and* $f(\hat{r}) < x - \hat{m}$.

(i) If $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \underline{0})$ is the only equilibrium.

(ii) If $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

(iii) If $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(iv) If $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})) > l(x - \hat{m})$, then $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(v) If $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$ and $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

If $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, from Lemma 3, we know that $\hat{m}$ is not a best response to $\underline{0}$. Thus, $(\hat{m}, \underline{0})$ is not an equilibrium. Therefore, in (i) and (ii) of Proposition 8, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$.

However, if $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, from Lemma 3, we know that 0 is not a best response to $\underline{0}$. Thus, $(0, \underline{0})$ is not an equilibrium. Therefore, in (iii) through (v) of Proposition 8, every equilibrium is $(0, \hat{r})$ or $(\hat{m}, \underline{0})$.

The followings are examples of equilibria.

Example 10

Recall Example 8. Since $2(x-\hat{m}) = 2\hat{m}$ and $x = 36$, we have $\hat{m} = 18$. Since $2(f(\underline{0})-\hat{q}) = 2\hat{q}$ and $f(\underline{0}) = 30$, we have $\hat{q} = 15$. If player 2 chooses $\hat{r} = (0,5,5,5,0)$, the MTB through $(N, A, y-\hat{r})$ is $f(\hat{r}) = 15$. From (iv) of Proposition 8, both $(0,\hat{r})$ and $(\hat{m},\underline{0})$ are equilibria. □

In the example above, there are two equilibria $(0,\hat{r})$ and $(\hat{m},\underline{0})$. In Example 8, however, $(0,\bar{r})$ is efficient. Player 2 does not act in the equilibrium $(\hat{m},\underline{0})$ but he must act in the efficient profile $(0,\bar{r})$. If player 2 does not act by choosing $\underline{0}$, knowing that player 1's best response to $\underline{0}$ is $\hat{m}$, player 2 can free-ride on player 1's action in the equilibrium $(\hat{m},\underline{0})$.

Proposition 9 *Suppose that* $x - \hat{m} \leq f(\hat{r})$ *and* $f(\underline{0}) < x$.

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x-\hat{m})$, then $(0,\underline{0})$ is the only equilibrium.

(ii) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x-\hat{m})$, then $(\hat{m},\underline{0})$ is the only equilibrium.

(iii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x-\hat{m})$, then $(0,\hat{r})$ is the only equilibrium.

(iv) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x-\hat{m}) > l(f(\hat{r}))$, then $(0,\hat{r})$ and $(\hat{m},\underline{0})$ are the only pure strategy equilibria.

(v) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x-\hat{m})$, then $(\hat{m},\underline{0})$ is the only equilibrium.

If $f(\underline{0}) < x$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, from Lemma 4, $\hat{r}$ is not a best response to 0. Thus, $(0, \hat{r})$ is not an equilibrium. In (i) and (ii) of Proposition 9, every equilibrium is $(0, \underline{0})$ or $(\hat{m}, \underline{0})$.

If $f(\underline{0}) < x$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, from Lemma 4, $\underline{0}$ is not a best response to 0. Thus, $(0, \underline{0})$ is not an equilibrium. In (iii) through (v) of Proposition 9, every equilibrium is $(0, \hat{r})$ or $(\hat{m}, \underline{0})$.

The followings are examples of equilibria.

Example 11

Recall Example 9. Recall Example 9. Since $2(x - \hat{m}) = 2\hat{m}$ and $x = 36$, we have $\hat{m} = 18$. Since $2(f(\underline{0}) - \hat{q}) = 8\hat{q}$ and $f(\underline{0}) = 30$, we have $\hat{q} = 6$. If player 2 chooses $\hat{r} = (0, 2, 2, 2, 0)$, the MTB through $(N, A, y - \hat{r})$ is $f(\hat{r}) = 24$. From (iv) of Proposition 9, both $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are equilibria. $\square$

In the example above, there are two equilibria $(0, \hat{r})$ and $(\hat{m}, \underline{0})$. In Example 9, however, $(\bar{m}, \underline{0})$ is efficient. Player 1 does not act in the equilibrium $(0, \hat{r})$ but he must act in the efficient profile $(\bar{m}, \underline{0})$. If player 1 does not act by choosing 0, knowing that player 2's best response to 0 is $\hat{r}$, player 1 can free-ride on player 2's action in the equilibrium $(0, \hat{r})$.

Proposition 10 *Suppose that $x \leq f(\underline{0})$ and $f(\hat{r}) < x - \hat{m}$.*

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r})) > l(x - \hat{m})$, then $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(iii) If $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(0, \hat{r})$ is the only equilibrium.

Proposition 11 *Suppose that* $x \leq f(\underline{0})$ *and* $x - \hat{m} \leq f(\hat{r}) < x$.

(i) If $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, then $(\hat{m}, \underline{0})$ is the only equilibrium.

(iii) If $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$, then $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

If $x \leq f(\underline{0})$, from Lemma 3, we know that 0 is not a best response to $\underline{0}$. Thus, $(0, \underline{0})$ is not an equilibrium. Therefore, in Propositions 10 and 11, every equilibrium is $(0, \hat{r})$ or $(\hat{m}, \underline{0})$.

Proposition 12 *If* $x \leq f(\hat{r})$, *then* $(\hat{m}, \underline{0})$ *is the only equilibrium.*

If $x \leq f(\hat{r})$, from Lemmas 3 and 4, we know that $\hat{m}$ is the only best response to $\underline{0}$ and $\hat{r}$, and $\underline{0}$ is the only best response to $\hat{m}$. Thus, in Proposition 12, $(\hat{m}, \underline{0})$ is the only equilibrium.

To sum up, in every equilibrium, either (a) player 1 acts to mitigate the extent of bads at the source or (b) player 2 acts to reduce the maximum transmission of bads through the network or (c) neither player acts. In case (a), player 2 free-rides on player 1's mitigation effort. The equilibrium level of mitigation is $\hat{m}$, where the marginal private loss equals to the marginal mitigation cost. In case (b), player 1 free-rides on player 2's reduction effort. The equilibrium level of reduction is $\sum_{(i,j)\in A} \hat{r}_{ij} = \hat{q}$, where the marginal private loss equals to the marginal reduction cost.

Now I compare equilibria with efficient profiles. Table 2 summarizes the results.

If either (a) player 1 acts or (b) player 2 acts, in an efficient profile, the marginal cost of action equals to the marginal joint loss. In an equilibrium, the marginal cost of action equals to the marginal private loss. Thus, in case (a), the equilibrium level of mitigation is less than the efficient level, that is, $\hat{m} < \bar{m}$. Also, in case (b), the equilibrium level of reduction is less than the efficient level, that is, $\sum \hat{r}_{ij} < \sum \bar{r}_{ij}$.

From Proposition 4, if the bottleneck capacity of network (N, A, y) is small, that is, if $f(\underline{0}) < x - \bar{m}$, every efficient profile is $(0, \underline{0})$ or $(0, \bar{r})$. From Proposition 7, if $f(\underline{0}) < x - \hat{m}$, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$. Because $f(\underline{0}) < x - \bar{m}$ implies $f(\underline{0}) < x - \hat{m}$, if the bottleneck capacity of network (N, A, y) is small, every equilibrium is $(0, \underline{0})$ or $(0, \hat{r})$. Thus, in every efficient profile and in every equilibrium, either (b) player 2 acts or (c) neither player acts. There may be an efficient equilibrium $(0, \underline{0})$ in which neither player acts.

From Proposition 5, if the bottleneck capacity of network (N, A, y) is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, every efficient profile is $(0, \bar{r})$ or $(\bar{m}, \underline{0})$. Thus, in every efficient profile, either (a) player 1 acts or (b) player 2 acts. However, no efficient profile is an equilibrium and no equilibrium is efficient. There may be an equilibrium $(0, \underline{0})$ in which neither player acts. Further, as shown in Examples 10 and 11, there may be an equilibrium where the player in action is not the player who must act in the efficient profile.

From Proposition 6, if the bottleneck capacity of network (N, A, y) is large, that is, if $x \leq f(\bar{r})$, every efficient profile is $(0, \underline{0})$ or $(\bar{m}, \underline{0})$. Thus, in every efficient profile, either (a) player 1 acts or (c) neither player acts. From Proposition 12, if $x \leq f(\hat{r})$, then $(\hat{m}, \underline{0})$ is the only equilibrium. Because $x \leq f(\bar{r})$ implies $x \leq f(\hat{r})$, if the bottleneck capacity of network (N, A, y) is large, $(\hat{m}, \underline{0})$ is the only equilibrium. Thus, in equilibrium, player 1 always acts. However, the equilibrium level of action is less than the efficient level.

Table 2. Efficient Profiles (EF) versus Equilibria (EQ)

Bottleneck capacity	Profiles	(a) Player 1 acts	(b) Player 2 acts	(c) Neither acts
Small	EF	No EF	$(0, \bar{r})$	$(0, \underline{0})$
	EQ	No EQ	$(0, \hat{r})$	$(0, \underline{0})$
Intermediate	EF	$(\bar{m}, \underline{0})$	$(0, \bar{r})$	No EF
	EQ	$(\hat{m}, \underline{0})$	$(0, \hat{r})$	$(0, \underline{0})$
Large	EF	$(\bar{m}, \underline{0})$	No EF	$(0, \underline{0})$
	EQ	$(\hat{m}, \underline{0})$	No EQ	No EQ

Unless $(0, \underline{0})$ is an efficient equilibrium, no efficient profile is an equilibrium and no equilibrium is efficient. The exception is observed only if the bottleneck capacity is small. Therefore, except for this case, we can conclude that players 1 and 2 do not have incentives to choose efficient profiles.

In equilibrium $(0, \underline{0})$, the extent of network bads is $e(0, \underline{0}) = f(\underline{0})$. Thus, the maximum transmission of bads through network (N, A, y) is allowed.

In equilibrium $(0, \hat{r})$, the extent of network bads is $e(0, \hat{r}) = f(\hat{r})$. In efficient profile $(0, \bar{r})$, the extent is $e(0, \bar{r}) = f(\bar{r})$. Because $f(\bar{r}) < f(\hat{r})$, the extent of network bads is greater in equilibrium $(0, \hat{r})$.

In equilibrium $(\hat{m}, \underline{0})$, the extent of network bads is $e(\hat{m}, \underline{0}) = x - \hat{m}$. In efficient profile $(\bar{m}, \underline{0})$, the extent is $e(\bar{m}, \underline{0}) = x - \bar{m}$. Because $\hat{m} < \bar{m}$, the extent of network bads is greater in equilibrium $(\hat{m}, \underline{0})$.

Cooperative Solutions

In my strategic model players may not have incentives to choose efficient profiles. Now transfers are allowed between players. We want to know if it is possible to make transfers that provide incentives to choose efficient profiles. I introduce a coalitional game and study cooperative solutions for the game.

A *coalition* $S \subseteq \{1,2\}$ is a non-empty set of players. A *coalitional game* v is a function that associates with each coalition S a number $v(S)$, which is called the value of coalition S. In my model a coalitional game v is defined as follows:

$$\begin{aligned} v(\{1\}) &= \max_{m \in M} u_1(m, \underline{0}); \\ v(\{2\}) &= \max_{r \in R} u_2(0, r); \\ v(\{1,2\}) &= \max_{(m,r) \in M \times R} U(m, r). \end{aligned}$$

That is, each player maximizes his payoff without any reduction or mitigation from the other if he belongs to a singleton coalition. Players 1 and 2 maximize the joint payoff if they belong to the same coalition.

Player 1 maximizes $u_1(m, \underline{0})$ by choosing a best response to $\underline{0}$. From Lemma 3, player 1's best response is 0 or $\hat{m}$. If 0 is a best response to $\underline{0}$, then $v(\{1\}) = u_1(0, \underline{0})$. If $\hat{m}$ is a best response to $\underline{0}$, then $v(\{1\}) = u_1(\hat{m}, \underline{0})$.

Player 2 maximizes $u_2(0, r)$ by choosing a best response to 0. From Lemma 4, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. If $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$.

Players 1 and 2 maximize the joint payoff $U(m, r)$ by choosing an efficient profile.

The coalitional game v is *superadditive* if $v(\{1\}) + v(\{2\}) \leq v(\{1,2\})$. An *allocation* (a_1, a_2) is a vector of payoffs with transfers, where a_i is the allocation of player $i = 1, 2$. An allocation (a_1, a_2) is in the set of *imputations* of the coalitional game v if $a_1 \geq v(\{1\})$, $a_2 \geq v(\{2\})$, and $a_1 + a_2 = v(\{1,2\})$. If the coalitional game v is superadditive, the set of imputations of v is non-empty.

Proposition 13

The coalitional game v is superadditive.

If (m, r) is an efficient profile and (a_1, a_2) is an allocation in the set of imputations of v, players 1 and 2 jointly choose (m, r) and make transfers in a way that player $i = 1, 2$ gets a net transfer of $a_i - u_i(m, r)$. Each player is better off by using this cooperative solution than by maximizing his payoff alone. Therefore, it is possible to make transfers that provide incentives for efficiency.

Remark 4

If the bottleneck capacity is intermediate, in every cooperative solution, the efficient player acts to decrease the extent of network bads and gets a positive net transfer from the other. If player 1 is the efficient player, he gets at least $v(\{1\}) - u_1(\bar{m}, \underline{0})$ from player 2. However, if player 2 is the efficient player, he gets at least $v(\{2\}) - u_2(0, \bar{r})$ from player 1. $\square$

For every coalitional game with two players, superadditivity is equivalent to convexity. Thus, the coalitional game v is convex. Because every convex game is totally balanced and every totally balanced game is a flow game, the coalitional game v is a flow game. Kalai and Zemel (24) show how to construct a flow game.

Conclusion

This chapter introduces a strategic network model where bads are transmitted from a source to a sink as a flow through a network. Two players wish to decrease network bads. One player can act to mitigate the extent of bads at the source. The other player can act to reduce the maximum transmission of bads through the network. Each player incurs the cost of action but benefits from the decrease in network bads.

I characterize efficient profiles and equilibria in this model. The equilibrium level of action is less than the efficient level. Interestingly, the player who acts in an equilibrium may not be the player who must act in the efficient profile. Further, there may an equilibrium where neither player acts. However, this equilibrium may not be efficient. Therefore, strategic inaction may lead to inefficiency.

Unless there is an efficient equilibrium where neither player acts, no efficient profile is an equilibrium and no equilibrium is efficient. Thus, the players may not have incentives to choose efficient profiles. I also study cooperative solutions where the players jointly choose an efficient profile and make transfers to each other. The players can achieve efficiency by using cooperative solutions.

An extension of this model is to consider multiple network players. Networks may be controlled by various players. Coordination between players is a key to reduce the transmission of bads through networks. Another extension is to study the transmission of network goods. Online data streaming can be an example. Content providers, such as Netflix and YouTube, and network service providers, such as AT&T and Verizon, have different interests that may lead to inefficient use of networks. In these extensions cooperative solutions can help achieve social efficiency.

Appendix A

Proof of Proposition 4. Suppose that $f(\underline{0}) < x - \bar{m}$. From Remark 1, we only consider $(m, r) \in M \times R$ with $m = 0$ or $r = \underline{0}$. First, let $(0, r)$ be any profile with $m = 0$. Because $f(\underline{0}) < x - \bar{m}$, we have $f(r) < x$. Thus, the extent is $e(0, r) = f(r)$, and the joint payoff is $U(0, r) = 2w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(f(r))$. From Remark 2, we only consider $(0, r) \in M \times R$ with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. Now the joint payoff is $U(0, r) = 2w - c_2(f(\underline{0}) - f(r)) - 2l(f(r))$. Because $f(\underline{0}) < x - \bar{m}$, we have $f(\bar{r}) < x$. Also, $f(r) < x$. Thus, the joint payoff is maximized when $f(r) = f(\bar{r})$. The maximum joint payoff is $U(0, \bar{r}) = 2w - c_2(\sum \bar{r}_{ij}) - 2l(f(\bar{r}))$.

Second, let $(m, \underline{0})$ be any profile with $r = \underline{0}$. If $x - m \leq f(\underline{0})$, the extent is $e(m, \underline{0}) = x - m$, and the joint payoff is $U(m, \underline{0}) = 2w - c_1(m) - 2l(x - m)$. Because $x - m \leq f(\underline{0})$ and $f(\underline{0}) < x - \bar{m}$, the joint payoff is maximized at $m = x - f(\underline{0})$. The maximum joint payoff is $U(x - f(\underline{0}), \underline{0}) = 2w - c_1(x - f(\underline{0})) - 2l(f(\underline{0}))$. If $f(\underline{0}) \leq x - m$, the extent is $e(m, \underline{0}) = f(\underline{0})$, and the joint payoff is $U(m, \underline{0}) = 2w - c_1(m) - 2l(f(\underline{0}))$, which is maximized at $m = 0$. The maximum joint payoff is $U(0, \underline{0}) = 2w - 2l(f(\underline{0}))$. Because $f(\underline{0}) < x - \bar{m}$, we have $x - f(\underline{0}) > 0$. Because $c_1(\cdot)$ is strictly increasing, we have $U(x - f(\underline{0}), \underline{0}) < U(0, \underline{0})$.

Thus, if the bottleneck capacity is small, that is, if $f(\underline{0}) < x - \bar{m}$, every efficient profile is $(0, \underline{0})$ or $(0, \bar{r})$. Further, if $2l(f(\underline{0})) < c_2(\sum \bar{r}_{ij}) + 2l(f(\bar{r}))$, then $U(0, \underline{0}) > U(0, \bar{r})$, and $(0, \underline{0})$ is the only efficient profile. However, if $2l(f(\underline{0})) > c_2(\sum \bar{r}_{ij}) + 2l(f(\bar{r}))$, then $U(0, \underline{0}) < U(0, \bar{r})$, and $(0, \bar{r})$ is the only efficient profile. $\square$

Proof of Proposition 5. Suppose that $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$. From Remark 1, we only consider $(m, r) \in M \times R$ with $m = 0$ or $r = \underline{0}$. First, let $(0, r)$ be any profile with $m = 0$. If $x \leq f(r)$, the extent is $e(0, r) = x$, and the joint payoff is $U(0, r) = 2w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(x)$, which is maximized at $r = \underline{0}$. The maximum joint payoff is $U(0, \underline{0}) = 2w - 2l(x)$. If $f(r) \leq x$, the extent is $e(0, r) = f(r)$, and the joint payoff is $U(0, r) = 2w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(f(r))$. From Remark 2, we only consider $(0, r)$ with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. Now the joint payoff is $U(0, r) = 2w - c_2(f(\underline{0}) - f(r)) - 2l(f(r))$. Because $f(r) \leq x$ and $f(\bar{r}) < x$, the joint payoff is maximized when $f(r) = f(\bar{r})$. The maximum joint payoff is $U(0, \bar{r}) = 2w - c_2(\sum \bar{r}_{ij}) - 2l(f(\bar{r}))$.

Second, let $(m, \underline{0})$ be any profile with $r = \underline{0}$. If $x - m \leq f(\underline{0})$, the extent is $e(m, \underline{0}) = x - m$, and the joint payoff is $U(m, \underline{0}) = 2w - c_1(m) - 2l(x - m)$. Because $x - m \leq f(\underline{0})$ and $x - \bar{m} \leq f(\underline{0})$, the joint payoff is maximized at $m = \bar{m}$. The maximum joint payoff is $U(\bar{m}, \underline{0}) = 2w - c_1(\bar{m}) - 2l(x - \bar{m})$. If $f(\underline{0}) \leq x - m$, the extent is $e(m, \underline{0}) = f(\underline{0})$, and the joint payoff is $U(m, \underline{0}) = 2w - c_1(m) - 2l(f(\underline{0}))$, which is maximized at $m = 0$. The maximum joint payoff is $U(0, \underline{0}) = 2w - 2l(f(\underline{0}))$.

Now compare the joint payoffs for profiles $(0, \underline{0})$, $(0, \bar{r})$, and $(\bar{m}, \underline{0})$. If $x \leq f(\underline{0})$, we have $U(0, \underline{0}) = 2w - 2l(x)$ and $U(\bar{m}, \underline{0}) = 2w - c_1(\bar{m}) - 2l(x - \bar{m})$. Thus, $U(0, \underline{0}) < U(\bar{m}, \underline{0})$. If $f(\underline{0}) \leq x$, we have $U(0, \underline{0}) = 2w - 2l(f(\underline{0}))$ and $U(0, \bar{r}) = 2w - c_2(\sum \bar{r}_{ij}) - 2l(f(\bar{r}))$. Thus, $U(0, \underline{0}) < U(0, \bar{r})$. In both cases, $(0, \underline{0})$ is not efficient.

Therefore, if the bottleneck capacity is intermediate, that is, if $x - \bar{m} \leq f(\underline{0})$ and $f(\bar{r}) < x$, every efficient profile is $(0, \bar{r})$ or $(\bar{m}, \underline{0})$. Further, if $c_2(\sum \bar{r}_{ij}) + 2l(f(\bar{r})) < c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0, \bar{r}) > U(\bar{m}, \underline{0})$, and $(0, \bar{r})$ is the only efficient profile. However, if $c_2(\sum \bar{r}_{ij}) + 2l(f(\bar{r})) > c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0, \bar{r}) < U(\bar{m}, \underline{0})$, and $(\bar{m}, \underline{0})$ is the only efficient profile. $\square$

Proof of Proposition 6. Suppose that $x \le f(\bar{r})$. From Remark 1, we only consider $(m, r) \in M \times R$ with $m = 0$ or $r = \underline{0}$. First, let $(0, r)$ be any profile with $m = 0$. If $x \le f(r)$, the extent is $e(0, r) = x$, and the joint payoff is $U(0, r) = 2w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(x)$, which is maximized at $r = \underline{0}$. The maximum joint payoff is $U(0, \underline{0}) = 2w - 2l(x)$. If $f(r) \le x$, the extent is $e(0, r) = f(r)$, and the joint payoff is $U(0, r) = 2w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - 2l(f(r))$. From Remark 2, we only consider $(0, r) \in M \times R$ with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. Now the joint payoff is $U(0, r) = 2w - c_2\left(f(\underline{0}) - f(r)\right) - 2l(f(r))$. Because $f(r) \le x$ and $x \le f(\bar{r})$, the joint payoff is maximized when $f(r) = x$. The maximum joint payoff is $U(0, r) = 2w - c_2\left(f(\underline{0}) - x\right) - 2l(x)$. Because $x \le f(\bar{r})$, we have $f(\underline{0}) - x > 0$. Because $c_2(\cdot)$ is strictly increasing, we have $U(0, r) < U(0, \underline{0})$.

Second, let $(m, \underline{0})$ be any profile with $r = \underline{0}$. Because $x \le f(\bar{r})$, we have $x - m < f(\underline{0})$. Thus, the extent is $e(m, \underline{0}) = x - m$, and the joint payoff is $U(m, \underline{0}) = 2w - c_1(m) - 2l(x - m)$. Because $x \le f(\bar{r})$, we have $x - \bar{m} < f(\underline{0})$. Also, $x - m < f(\underline{0})$. Thus, the joint payoff is maximized at $m = \bar{m}$. The maximum joint payoff is $U(\bar{m}, \underline{0}) = 2w - c_1(\bar{m}) - 2l(x - \bar{m})$.

Thus, if the bottleneck capacity is large, that is, if $x \le f(\bar{r})$, every efficient profile is $(0, \underline{0})$ or $(\bar{m}, \underline{0})$. Further, if $2l(x) < c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0, \underline{0}) > U(\bar{m}, \underline{0})$, and $(0, \underline{0})$ is the only efficient profile. However, if $2l(x) > c_1(\bar{m}) + 2l(x - \bar{m})$, then $U(0, \underline{0}) < U(\bar{m}, \underline{0})$, and $(\bar{m}, \underline{0})$ is the only efficient profile. □

Proof of Lemma 3. Let $r \in R$ be any reduction strategy. We divide into three cases.

Case 1. Suppose that $f(r) < x - \hat{m}$. For each $m \in M$, if $x - m \le f(r)$, the extent of network bads is $e(m, r) = x - m$, and the payoff of player 1 is $u_1(m, r) = w - c_1(m) - l(x - m)$. Because $x - m \le f(r)$ and $f(r) < x - \hat{m}$, the payoff is maximized at $m = x - f(r)$, and the maximum payoff is $u_1(x - f(r), r) = w - c_1(x - f(r)) - l(f(r))$.

For each $m \in M$, if $f(r) \leq x - m$, the extent is $e(m, r) = f(r)$, and the payoff is $u_1(m, r) = w - c_1(m) - l(f(r))$. Because $c_1(\cdot)$ is strictly increasing, the payoff is maximized at $m = 0$, and the maximum payoff is $u_1(0, r) = w - l(f(r))$.

Because $x - f(r) > \hat{m}$ and $\hat{m} > 0$, we have $x - f(r) > 0$. Because $c_1(\cdot)$ is strictly increasing, we have $u_1(0, r) > u_1(x - f(r), r)$. Thus, $m = 0$ is the only best response.

Case 2. Suppose that $x - \hat{m} \leq f(r) < x$. For each $m \in M$, if $x - m \leq f(r)$, the extent of network bads is $e(m, r) = x - m$, and the payoff of player 1 is $u_1(m, r) = w - c_1(m) - l(x - m)$. Because $x - m \leq f(r)$ and $x - \hat{m} \leq f(r)$, the payoff is maximized at $m = \hat{m}$, and the maximum payoff is $u_1(\hat{m}, r) = w - c_1(\hat{m}) - l(x - \hat{m})$.

For each $m \in M$, if $f(r) \leq x - m$, the extent is $e(m, r) = f(r)$, and the payoff is $u_1(m, r) = w - c_1(m) - l(f(r))$. Because $c_1(\cdot)$ is strictly increasing, the payoff is maximized at $m = 0$, and the maximum payoff is $u_1(0, r) = w - l(f(r))$.

If $l(f(r)) < c_1(\hat{m}) + l(x - \hat{m})$, then $u_1(0, r) > u_1(\hat{m}, r)$. Thus, $m = 0$ is the only best response. If $l(f(r)) = c_1(\hat{m}) + l(x - \hat{m})$, then $u_1(0, r) = u_1(\hat{m}, r)$. Thus, both $m = 0$ and $m = \hat{m}$ are the best responses. If $l(f(r)) > c_1(\hat{m}) + l(x - \hat{m})$, then $u_1(\hat{m}, r) > u_1(0, r)$. Thus, $m = \hat{m}$ is the only best response.

Case 3. Suppose that $x \leq f(r)$. Because $x \leq f(r)$, for each $m \in M$, we have $x - m \leq f(r)$. Thus, the extent of network bads is $e(m, r) = x - m$, and the payoff of player 1 is $u_1(m, r) = w - c_1(m) - l(x - m)$. The payoff is maximized at $m = \hat{m}$. Thus, $m = \hat{m}$ is the only best response. □

Proof of Lemma 4. Let $m \in M$ be any mitigation strategy. We divide into two cases.

Case 1. Suppose that $x - m \leq f(\hat{r})$. For each $r \in R$, if $x - m \leq f(r)$, the extent of network bads is $e(m, r) = x - m$, and the payoff of player 2 is $u_2(m, r) = w -$

$c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(x-m)$. Because $c_2(\cdot)$ is strictly increasing, the payoff is maximized at $r = \underline{0}$, and the maximum payoff is $u_2(m, \underline{0}) = w - l(x-m)$.

For each $r \in R$, if $f(r) \leq x - m$, the extent is $e(m, r) = f(r)$, and the payoff is $u_2(m, r) = w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(f(r))$. From Remark 3, we only consider $r \in R$ with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. Now the payoff is $u_2(m, r) = w - c_2(f(\underline{0}) - f(r)) - l(f(r))$. Because $f(r) \leq x - m$ and $x - m \leq f(\hat{r})$, the payoff is maximized when $f(r) = x - m$. The maximum payoff is $u_2(m, r) = w - c_2(f(\underline{0}) - x + m) - l(x - m)$.

Because $x - m \leq f(\hat{r})$ and $f(\hat{r}) < f(\underline{0})$, we have $x - m < f(\underline{0})$. That is, $f(\underline{0}) - x + m > 0$. Because $c_2(\cdot)$ is strictly increasing, we have $u_2(m, \underline{0}) > u_2(m, r)$. Thus, $\underline{0}$ is the only best response.

Case 2. Suppose that $f(\hat{r}) < x - m$. For each $r \in R$, if $x - m \leq f(r)$, the extent of network bads is $e(m, r) = x - m$, and the payoff of player 2 is $u_2(m, r) = w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(x-m)$. Because $c_2(\cdot)$ is strictly increasing, the payoff is maximized at $r = \underline{0}$, and the maximum payoff is $u_2(m, \underline{0}) = w - l(x-m)$.

For each $r \in R$, if $f(r) \leq x - m$, the extent is $e(m, r) = f(r)$, and the payoff is $u_2(m, r) = w - c_2\left(\sum_{(i,j)\in A} r_{ij}\right) - l(f(r))$. From Remark 3, we only consider $r \in R$ with $\sum_{(i,j)\in A} r_{ij} = f(\underline{0}) - f(r)$. Now the payoff is $u_2(m, r) = w - c_2(f(\underline{0}) - f(r)) - l(f(r))$. Because $f(r) \leq x - m$ and $f(\hat{r}) < x - m$, the payoff is maximized when $f(r) = f(\hat{r})$. The maximum payoff is $u_2(m, \hat{r}) = w - c_2\left(\sum \hat{r}_{ij}\right) - l(f(\hat{r}))$.

If $l(x-m) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $u_2(m, \underline{0}) > u_2(m, \hat{r})$. Thus, $\underline{0}$ is the only best response. If $l(x-m) = c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $u_2(m, \underline{0}) = u_2(m, \hat{r})$. Thus, $\underline{0}$ and $\hat{r}$ are the best responses. If $l(x-m) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $u_2(m, \hat{r}) > u_2(m, \underline{0})$. Thus, $\hat{r}$ is the only best response. $\square$

Proof of Proposition 7. Because $f(\underline{0}) < x - \hat{m}$, we have $BR_1(\underline{0}) = \{0\}$. Because $f(\hat{r}) < f(\underline{0})$ and $f(\underline{0}) < x - \hat{m}$, we have $f(\hat{r}) < x - \hat{m}$ and $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. As in (i), if $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $BR_2(0) = \{\underline{0}\}$, and $(0, \underline{0})$ is the only equilibrium. As in (ii), if $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, then $BR_2(0) = \hat{R}$, and $(0, \hat{r})$ is the only equilibrium. □

Proof of Proposition 8. (i) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\underline{0}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, we have $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. Because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \{\underline{0}\}$. Thus, $(0, \underline{0})$ is the only equilibrium.

(ii) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\underline{0}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, we have $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. Also, because $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \hat{R}$. Thus, $(0, \hat{r})$ is the only equilibrium.

(iii) Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. Also, because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \{\underline{0}\}$. Because $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $l(x - \hat{m}) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$. Also, because $f(\hat{r}) < x - \hat{m}$, we have $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(iv) Because $f(\hat{r}) < x - \hat{m}$, we have $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. Also, because $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \hat{R}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x - \hat{m}) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(v) Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. Because $l(x - \hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) +$

$l(f(\hat{r}))$, we have $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$ and $BR_2(0) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x-\hat{m}) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(\hat{m}) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. The only equilibrium is $(0, \hat{r})$. $\square$

Proof of Proposition 9. Because $x - \hat{m} \leq f(\hat{r})$ and $f(\underline{0}) < x$, we have $x - \hat{m} \leq f(\underline{0}) < x$ and $x - \hat{m} \leq f(\hat{r}) < x$.

(i) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, $BR_1(\underline{0}) = \{0\}$. Because $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, we have $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$. Also, because $x - \hat{m} \leq f(\hat{r}) < x$, $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \{\underline{0}\}$. Thus, $(0, \underline{0})$ is the only equilibrium.

(ii) Because $f(\hat{r}) < x$ and $l(x) < c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\hat{r})$, we have $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(iii) Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\underline{0}) = \{0\}$. Because $l(f(\underline{0})) < c_1(\hat{m}) + l(x - \hat{m})$, we have $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$. Also, because $x - \hat{m} \leq f(\hat{r}) < x$, we have $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \hat{R}$. Thus, $(0, \hat{r})$ is the only equilibrium.

(iv) Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) > c_2\left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \hat{R}$. Because $x - \hat{m} \leq f(\underline{0}) < x$ and $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, we have $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(v) Because $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, we have $l(f(\underline{0})) > c_1(\hat{m}) + l(x - \hat{m})$. Also, because $x - \hat{m} \leq f(\underline{0}) < x$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\hat{r}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$.

The only equilibrium is $(\hat{m}, \underline{0})$. $\square$

Proof of Proposition 10. (i) Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. Also, because $l(x) < c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \{\underline{0}\}$. Because $l(x) < c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $l(x - \hat{m}) < c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$. Also, because $f(\hat{r}) < x - \hat{m}$, we have $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x \leq f(\underline{0})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) Because $f(\hat{r}) < x - \hat{m}$, we have $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x - \hat{m}$, we have $f(\hat{r}) < x$. Also, because $l(x) > c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \hat{R}$. Because $x \leq f(\underline{0})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x - \hat{m}) < c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(0, \hat{r})$ and $(\hat{m}, \underline{0})$ are the only pure strategy equilibria.

(iii) Because $l(x - \hat{m}) > c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $l(x) > c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$. Also, because $f(\hat{r}) < x$, we have $BR_2(0) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$ and $l(x - \hat{m}) > c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(\hat{m}) = \hat{R}$. Because $f(\hat{r}) < x - \hat{m}$, $BR_1(\hat{r}) = \{0\}$. Thus, $(0, \hat{r})$ is the only equilibrium. $\square$

Proof of Proposition 11. (i) Because $f(\hat{r}) < x$ and $l(x) < c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \{\underline{0}\}$. Because $x - \hat{m} \leq f(\hat{r})$, we have $BR_2(\hat{m}) = \{\underline{0}\}$. Because $x \leq f(\underline{0})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(ii) Because $x \leq f(\underline{0})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) > c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\hat{r}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium.

(iii) Because $x - \hat{m} \leq f(\hat{r}) < x$ and $l(f(\hat{r})) < c_1(\hat{m}) + l(x - \hat{m})$, we have $BR_1(\hat{r}) = \{0\}$. Because $f(\hat{r}) < x$ and $l(x) > c_2 \left(\sum \hat{r}_{ij}\right) + l(f(\hat{r}))$, we have $BR_2(0) = \hat{R}$. Because $x \leq f(\underline{0})$, we have $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x - \hat{m} \leq f(\hat{r})$, $BR_2(\hat{m}) = \{\underline{0}\}$. The only pure strategy equilibria are $(0, \hat{r})$ and $(\hat{m}, \underline{0})$. $\square$

Proof of Proposition 12. Because $x \le f(\hat{r})$, we have $x < f(\underline{0})$ and $BR_1(\underline{0}) = \{\hat{m}\}$. Because $x \le f(\hat{r})$, we have $BR_1(\hat{r}) = \{\hat{m}\}$. Because $x \le f(\hat{r})$, we have $x - \hat{m} < f(\hat{r})$ and $BR_2(\hat{m}) = \{\underline{0}\}$. Thus, $(\hat{m}, \underline{0})$ is the only equilibrium. □

Proof of Proposition 13. We show that $v(\{1\}) + v(\{2\}) \le v(\{1, 2\})$. First, suppose that $x - \hat{m} \le f(\underline{0}) < x$.

From Lemma 3, player 1's best response is 0 or $\hat{m}$. If 0 is a best response to $\underline{0}$, then $v(\{1\}) = u_1(0, \underline{0})$. Because $f(\underline{0}) < x$, we have $u_1(0, \underline{0}) = w - l(f(\underline{0}))$. If $\hat{m}$ is a best response to $\underline{0}$, then $v(\{1\}) = u_1(\hat{m}, \underline{0})$. Because $x - \hat{m} \le f(\underline{0})$, we have $u_1(\hat{m}, \underline{0}) = w - c_1(\hat{m}) - l(x - \hat{m})$. Thus, $v(\{1\}) = w - l(f(\underline{0}))$ or $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$.

From Lemma 4, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. Because $f(\underline{0}) < x$, we have $u_2(0, \underline{0}) = w - l(f(\underline{0}))$. If $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$. Because $f(\hat{r}) < f(\underline{0})$ and $f(\underline{0}) < x$, we have $u_2(0, \hat{r}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. Thus, $v(\{2\}) = w - l(f(\underline{0}))$ or $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. We divide into four cases.

Case 1. If $v(\{1\}) = w - l(f(\underline{0}))$ and $v(\{2\}) = w - l(f(\underline{0}))$, we have

$$v(\{1\}) + v(\{2\}) = 2w - 2l(f(\underline{0})) = U(0, \underline{0}) \le v(\{1, 2\}).$$

Case 2. If $v(\{1\}) = w - l(f(\underline{0}))$ and $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$, because $f(\hat{r}) < f(\underline{0})$ implies $v(\{1\}) < w - l(f(\hat{r}))$, we have

$$v(\{1\}) + v(\{2\}) < 2w - c_2(\sum \hat{r}_{ij}) - 2l(f(\hat{r})) = U(0, \hat{r}) \le v(\{1, 2\}).$$

Case 3. If $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$ and $v(\{2\}) = w - l(f(\underline{0}))$, because

$x - \hat{m} \leq f(\underline{0})$ implies $v(\{2\}) \leq w - l(x - \hat{m})$, we have

$$v(\{1\}) + v(\{2\}) \leq 2w - c_1(\hat{m}) - 2l(x - \hat{m}) = U(\hat{m}, \underline{0}) \leq v(\{1, 2\}).$$

Case 4. If $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$ and $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$, we consider two subcases. If $x - \hat{m} \leq f(\hat{r})$, then $v(\{2\}) \leq w - l(x - \hat{m})$. Thus,

$$v(\{1\}) + v(\{2\}) \leq 2w - c_1(\hat{m}) - 2l(x - \hat{m}) = U(\hat{m}, \underline{0}) \leq v(\{1, 2\}).$$

However, if $f(\hat{r}) \leq x - \hat{m}$, then $v(\{1\}) \leq w - l(f(\hat{r}))$. Thus,

$$v(\{1\}) + v(\{2\}) \leq 2w - c_2(\sum \hat{r}_{ij}) - 2l(f(\hat{r})) = U(0, \hat{r}) \leq v(\{1, 2\}).$$

Second, suppose that $f(\underline{0}) < x - \hat{m}$. From Lemma 3, 0 is the only best response to $\underline{0}$, and $v(\{1\}) = u_1(0, \underline{0})$. Because $f(\underline{0}) < x - \hat{m}$ and $x - \hat{m} < x$, $u_1(0, \underline{0}) = w - l(f(\underline{0}))$ and $v(\{1\}) = w - l(f(\underline{0}))$.

From Lemma 4, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. Because $f(\underline{0}) < x$, we have $u_2(0, \underline{0}) = w - l(f(\underline{0}))$. Thus, $v(\{2\}) = w - l(f(\underline{0}))$. As in Case 1, we have $v(\{1\}) + v(\{2\}) \leq v(\{1, 2\})$.

However, if $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$. Also, if $\hat{r}$ is a best response to 0, we have $f(\hat{r}) < x$ and $u_2(0, \hat{r}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. Thus, $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. As in Case 2, we have $v(\{1\}) + v(\{2\}) \leq v(\{1, 2\})$.

Third, suppose that $x \leq f(\underline{0})$. From Lemma 3, $\hat{m}$ is the only best response to $\underline{0}$, and $v(\{1\}) = u_1(\hat{m}, \underline{0})$. Because $x - \hat{m} < x$ and $x \leq f(\underline{0})$, $u_1(\hat{m}, \underline{0}) = w - c_1(\hat{m}) - l(x - \hat{m})$ and $v(\{1\}) = w - c_1(\hat{m}) - l(x - \hat{m})$.

From Lemma 4, player 2's best response is $\underline{0}$ or $\hat{r}$. If $\underline{0}$ is a best response to 0, then $v(\{2\}) = u_2(0, \underline{0})$. Because $x \leq f(\underline{0})$, we have $u_2(0, \underline{0}) = w - l(x)$. Thus, $v(\{2\}) = w - l(x)$. Because $x - \hat{m} < x$, we have $v(\{2\}) < w - l(x - \hat{m})$. Thus, as in Case 3, we have

$v(\{1\}) + v(\{2\}) < 2w - c_1(\hat{m}) - 2l(x - \hat{m}) \leq v(\{1, 2\})$.

However, if $\hat{r}$ is a best response to 0, then $v(\{2\}) = u_2(0, \hat{r})$. Also, if $\hat{r}$ is a best response to 0, we have $f(\hat{r}) < x$ and $u_2(0, \hat{r}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. Thus, $v(\{2\}) = w - c_2(\sum \hat{r}_{ij}) - l(f(\hat{r}))$. As in Case 4, we have $v(\{1\}) + v(\{2\}) \leq v(\{1, 2\})$.

Therefore, v is superadditive. $\square$

CHAPTER IV

THE POWER LAW OF CONFLICT

Introduction

Power law distributions have been observed in various contexts. Distributions of wealth, city populations, academic citations, and stock market returns are a few examples. Clauset et al. (10) and Gabaix (15) present empirical examples exhibiting power law distributions.

In particular, the *power law of conflict* is an empirical regularity that the frequency of a conflict event scales as an inverse power of the severity of the conflict event. Precisely, the frequency $f(x)$ of a conflict event with severity $x \geq x_{\min}$ scales as $f(x) \propto x^{-\alpha}$, where $x_{\min}$ is the minimum level above which the power law holds and α is the scaling parameter of the power law. The power law of conflict is first observed by Richardson (36) in a broad context that includes murders and wars.

There are no theoretical explanations for the power law of conflict. It is a complex task to understand what causes conflict, and in turn, the power law of conflict. There may be cultural, economic, historical, and political circumstances. However, the power law of conflict has been universally observed across time and region. Cederman (8) observes the power law in interstate wars between 1820 and 1997. Bohorquez et al. (5) and Clauset et al. (11) also observe the power law in terrorist events and insurgent wars in recent decades. Thus, there may be common circumstances that lead to the power law of conflict across time and region.

The power law of conflict may be caused by strategic behavior of players in an adversarial relationship. With cultural, economic, historical, and political circumstances, an adversarial relationship may evolve between players. Strategic behavior of these players leads to the power law of conflict. See Figure 6. To test the hypothesis that strategic bahavior causes the power law, we need to develop a simple model amenable to experimental analysis. The purpose of this chapter is to provide a theoretical foundation for experimental and empirical testing.

We build a strategic model with a unique mixed strategy equilibrium. There are two adversarial players, *Attacker* and *Defender*. Given a quantity of bads, Attacker chooses how many bads to carry through a route to a target. Simultaneously and independently, Defender chooses whether to block the route to stop the transport of bads to the target. When Attacker successfully carries bads to the target, that is, when Attacker carries bads to the target and Defender does not block the route, the target is damaged. Defender suffers from target loss, which is determined by the amount of bads and the *marginal target loss*. Attacker gains from target loss, which is scaled up by the *target loss power*. Attacker incurs the cost of carrying bads while Defender incurs the cost of blocking the route.

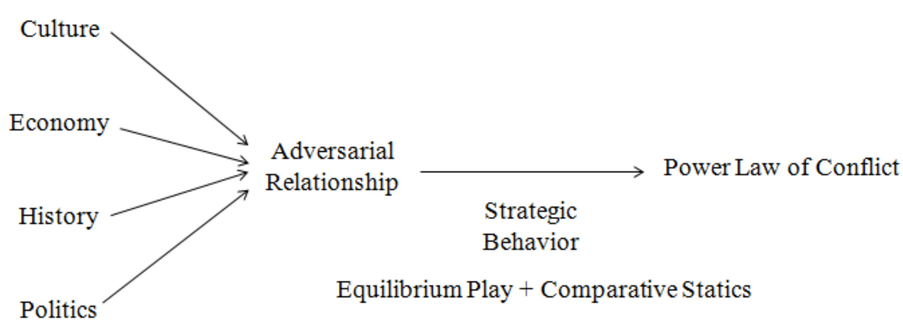

Figure 6. Common Circumstances

In equilibrium both players choose mixed strategies. Attacker carries no bads or the maximum amount of bads while Defender blocks the route or not. Target loss is caused when Attacker carries the maximum amount of bads and Defender does not block the route. The probability of target loss can be calculated as a function of target loss. If the marginal target loss is a random variable distributed uniformly on an interval, the probability distribution of target loss follows a power law. The scaling parameter of the power law is determined by the target loss power.

The driving force of the power law in our model is strategic behavior of players in an adversarial relationship. Both players create unpredictability in equilibrium because they are adversarial against each other. For example, if Defender's action is perfectly predictable, Attacker will take advantage of Defender. Thus, both players make their actions unpredictable by playing mixed strategies. Furthermore, changes in mixed strategies, due to changes in the marginal target loss, generate a power law distribution. That is, in our model, equilibrium play and its comparative statics yield the power law of conflict.

The parameters of the power law are estimated from data by using the procedure of Clauset et al. (10). Our data are based on Global Terrorism Database (28). The power law is a good fit to the data.

The rest of this chapter is organized as follows. Section 2 presents a strategic model. Section 3 analyzes equilibrium behavior in this model. Section 4 studies a power law. Section 5 discusses future research.

Model

There is a route that connects *Source* to *Target*. Two players, *Attacker* and *Defender*, choose their strategies simultaneously and independently. See Figure 7 for an illustration.

Attacker is endowed with a *quantity* $q \geq 1$ of bads at Source. Attacker chooses a strategy a with $0 \leq a \leq q$. The set of strategies for Attacker is denoted by A. By choosing $a \in A$, Attacker carries an amount a of bads from Source to Target.

Defender wishes to stop the transport of bads to Target. Defender chooses a strategy $b \in \{0, 1\}$. The set of strategies for Defender is denoted by B. If $b = 1$, Defender blocks the route, and if $b = 0$, Defender does not block the route.

Players are allowed to choose mixed strategies. The set of mixed strategies for Attacker is denoted by $\Delta(A)$ and the set of mixed strategies for Defender is denoted by $\Delta(B)$.

If Defender does not block the route, that is, if $b = 0$, Attacker successfully carries an amount a of bads to Target. The bads carried to Target cause a target loss. Defender loses la from the target loss, where $l > 1$ denotes a *marginal target loss.* Attacker earns $(la)^k$, where $k > 1$ denotes a *target loss power.*

If Defender blocks the route, that is, if $b = 1$, Attacker fails to carry bads to Target. No target loss is caused. However, Defender incurs a *cost* $c \geq 1$ of blocking the route.

If Attacker carries an amount a of bads to Target, he incurs an expense ea, where $e > 0$ denotes a *marginal expense.* Assume that $l > e$.

Attacker earns a constant worth w_1 and Defender earns w_2.

For each $(a, b) \in A \times B$, Attacker's payoff is

$$u_1(a, b) = \begin{cases} w_1 + (la)^k - ea & \text{if } b = 0; \\ w_1 - ea & \text{if } b = 1, \end{cases}$$

and Defender's payoff is

$$u_2(a, b) = \begin{cases} w_2 - la & \text{if } b = 0; \\ w_2 - c & \text{if } b = 1. \end{cases}$$

For each $\sigma = (\sigma_1, \sigma_2) \in \Delta(A) \times \Delta(B)$, Attacker earns an expected payoff of $u_1(\sigma_1, \sigma_2) = E_\sigma[u_1(a, b)]$ and Defender earns an expected payoff of $u_2(\sigma_1, \sigma_2) = E_\sigma[u_2(a, b)]$.

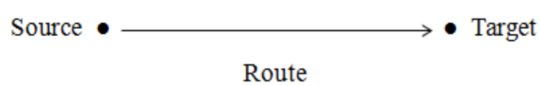

Figure 7. Route from Source to Target

Equilibrium Analysis

To analyze mixed strategy equilibrium, we assume that $c < lq$. In words, the blocking cost is less than the maximum target loss. However, if $c > lq$, strategy profile $(q, 0)$ is the only equilibrium. Thus, in equilibrium, Attacker carries the maximum amount of bads and Defender does not block the route.

Proposition 14

Any amount a with $0 < a < q$ is a dominated strategy for Attacker.

Proof. Let a be any strategy with $0 < a < q$. Let σ_1 be a mixed strategy such that $\sigma_1(0) + \sigma_1(q) = 1$. We show that a is dominated by σ_1. If Defender chooses $b = 0$, then $u_1(\sigma_1, 0) = w_1 + \sigma_1(q)((lq)^k - eq)$ and $u_1(a, 0) = w_1 + (la)^k - ea$. Thus, if $\sigma_1(q) > \frac{(la)^k - ea}{(lq)^k - eq}$, we have $u_1(\sigma_1, 0) > u_1(a, 0)$. If Defender chooses $b = 1$, then $u_1(\sigma_1, 1) = w_1 - \sigma_1(q)(eq)$ and $u_1(a, 1) = w_1 - ea$. Thus, if $\sigma_1(q) < \frac{a}{q}$, we have $u_1(\sigma_1, 1) > u_1(a, 1)$. Because $\frac{(la)^k - ea}{(lq)^k - eq} < \frac{a}{q}$ and $\frac{a}{q} < 1$, there is $\sigma_1(q)$ with $\frac{(la)^k - ea}{(lq)^k - eq} < \sigma_1(q) < \frac{a}{q}$. Thus, a is dominated by σ_1. $\square$

After removing dominated strategies, our model can be represented in a matrix game. See Table 3.

Table 3. Matrix Game

$1 \setminus 2$	0	1
0	w_1, w_2	$w_1, w_2 - c$
q	$w_1 + (lq)^k - eq, w_2 - lq$	$w_1 - eq, w_2 - c$

For Attacker, $a = 0$ is the best response to $b = 1$ and $a = q$ is the best response to $b = 0$. The set of rationalizable strategies for Attacker is $\{0, q\}$. For Defender, $b = 0$ is the best response to $a = 0$ and $b = 1$ is the best response to $a = q$. The set of rationalizable strategies for Defender is $\{0, 1\}$. Furthermore, there is no pure strategy equilibrium.

Now we study mixed strategy equilibrium. Let σ_1^* be a mixed strategy such that $\sigma_1^*(0) = 1 - c/(lq)$ and $\sigma_1^*(q) = c/(lq)$. Let σ_2^* be a mixed strategy such that $\sigma_2^*(0) = (eq)/(lq)^k$ and $\sigma_2^*(1) = 1 - (eq)/(lq)^k$.

Proposition 15

Strategy profile (σ_1^*, σ_2^*) is the only equilibrium.

Proof. It suffices to show that each player makes the other player indifferent between the pure strategies played with positive probability. By choosing σ_1^*, Attacker makes Defender indifferent between $b = 0$ and $b = 1$. Given σ_1^*, Defender's payoffs are

$$\begin{aligned} u_2(\sigma_1^*, 0) &= \sigma_1^*(0)u_2(0,0) + \sigma_1^*(q)u_2(q,0) \\ &= (1 - c/(lq))(w_2) + (c/(lq))(w_2 - lq) \\ &= w_2 - c \end{aligned}$$

and

$$\begin{aligned} u_2(\sigma_1^*, 1) &= \sigma_1^*(0)u_2(0,1) + \sigma_1^*(q)u_2(q,1) \\ &= w_2 - c. \end{aligned}$$

Thus, $u_2(\sigma_1^*, 0) = u_2(\sigma_1^*, 1)$. By choosing σ_2^*, Defender makes Attacker indifferent between $a = 0$ and $a = q$. Given σ_2^*, Attacker's payoffs are

$$\begin{aligned} u_1(0, \sigma_2^*) &= \sigma_2^*(0)u_1(0,0) + \sigma_2^*(1)u_1(0,1) \\ &= w_1 \end{aligned}$$

and

$$\begin{aligned} u_1(q, \sigma_2^*) &= \sigma_2^*(0)u_1(q,0) + \sigma_2^*(1)u_1(q,1) \\ &= ((eq)/(lq)^k)(w_1 + (lq)^k - eq) + (1 - (eq)/(lq)^k)(w_1 - eq) \\ &= w_1 \end{aligned}$$

Thus, $u_1(0, \sigma_2^*) = u_1(q, \sigma_2^*)$. Therefore, (σ_1^*, σ_2^*) is the only equilibrium. $\square$

In equilibrium, Attacker carries no bads with probability $1-c/(lq)$ and carries the maximum amount of bads with probability $c/(lq)$. Defender does not block the route with probability $(eq)/(lq)^k$ and blocks the route with probability $1-(eq)/(lq)^k$.

In equilibrium, Attacker earns a payoff of w_1 and Defender earns a payoff of w_2-c. Thus, equilibrium (σ_1^*, σ_2^*) is Pareto dominated by strategy profile $(0,0)$, where Attacker earns w_1 and Defender earns w_2. However, strategy profile $(0,0)$ is not an equilibrium, because Attacker is better off by choosing $a=q$.

In equilibrium, target loss is caused when Attacker carries the maximum amount of bads and Defender does not block the route. Let $L=lq$ and let $C=ceq$. Target loss $X=L$ occurs with probability CL^{-k-1} and no target loss $X=0$ occurs with probability $1-CL^{-k-1}$. Thus, given L, equilibrium target loss $X|L$ follows a binomial distribution. Precisely,

$$P(X=0|L)=1-CL^{-k-1} \text{ and } P(X=L|L)=CL^{-k-1}. \tag{IV.1}$$

Power Law

The power law of conflict states that the frequency of a conflict event scales as an inverse power of the severity of the event. Precisely, if the severity X of a conflict event is measured by a number $x \geq x_{\min}$, that is, if $X=x$, the probability of the event conditional on $X>0$ scales as

$$p(X=x|X>0) \propto x^{-\alpha},$$

where $x_{\min}$ is the minimum level above which the power law holds and α is the scaling parameter of the power law.

We want to show that the probability distribution of target loss X conditional on $X > 0$ follows a power law.[1]

Suppose that all parameters are given and fixed except the marginal target loss l, which is a random variable distributed uniformly between $l_{\min}$ and $l_{\max}$. Suppose that $c/q < l_{\min}$ and $l_{\min} < l_{\max}$.

When l is realized, it is observed by both players. In equilibrium, target loss $X = L = lq$ occurs with probability CL^{-k-1} and no target loss $X = 0$ occurs with probability $1 - CL^{-k-1}$.

Because l is uniformly distributed between $l_{\min}$ and $l_{\max}$, L is uniformly distributed between $L_{\min} = ql_{\min}$ and $L_{\max} = ql_{\max}$. Precisely, the probability distribution of L is $f_L(L) = \frac{1}{L_{\max} - L_{\min}}$.

We present the Cumulative Distribution Function (CDF) of target loss X conditional on $X > 0$.

Proposition 16

For each x with $L_{\min} \leq x \leq L_{\max}$, the CDF of target loss X conditional on $X > 0$ is

$$P(X \leq x | X > 0) = \frac{L_{\min}^{-k} - x^{-k}}{L_{\min}^{-k} - L_{\max}^{-k}}, \tag{IV.2}$$

where $L_{\min}$, $L_{\max}$, and k are the parameters of the distribution function.

[1] In empirical data, such as Global Terrorism Database (28), the severity of a conflict event is measured by the number of fatalities in the event. Conflict events with no fatalities are often ignored by media and may not be included in empirical data that are based on media reports. Thus, we study a power law distribution conditional on $X > 0$, that is, on a positive number of fatalities.

Proof. Note that for each $x > 0$,

$$P(X \leq x | X > 0) = \frac{P(0 < X \leq x)}{P(X > 0)} = \frac{P(X \leq x) - P(X = 0)}{1 - P(X = 0)}.$$

Calculate $P(X \leq x)$ and $P(X = 0)$. Note that

$$P(X \leq x | L) = \begin{cases} 1 & \text{if } x \geq L; \\ 1 - CL^{-k-1} & \text{if } x < L. \end{cases}$$

Or equivalently,

$$P(X \leq x | L) = 1 - (CL^{-k-1}) \cdot 1(x < L),$$

where $1(\cdot)$ is an indicator function.

Because $\int_{L_{\min}}^{L_{\max}} f_L(L) dL = 1$, for each x with $L_{\min} \leq x \leq L_{\max}$,

$$\begin{aligned} P(X \leq x) =& \int_{L_{\min}}^{L_{\max}} P(X \leq x | L) f_L(L) dL \\ =& \int_{L_{\min}}^{L_{\max}} (1 - (CL^{-k-1}) \cdot 1(x < L)) f_L(L) dL \\ =& 1 - \int_{L_{\min}}^{x} (CL^{-k-1}) \cdot 1(x < L) f_L(L) dL - \int_{x}^{L_{\max}} (CL^{-k-1}) \cdot 1(x < L) f_L(L) dL. \end{aligned}$$

Because $1(x < L) = 0$ for $L_{\min} \leq L \leq x$ and $1(x < L) = 1$ for $x \leq L \leq L_{\max}$, for x with $L_{\min} \leq x \leq L_{\max}$,

$$\begin{aligned} P(X \leq x) =& 1 - \int_{x}^{L_{\max}} (CL^{-k-1}) f_L(L) dL \\ =& 1 - \int_{x}^{L_{\max}} (CL^{-k-1}) \left(\frac{1}{L_{\max} - L_{\min}} \right) dL \\ =& 1 - \left(\frac{C}{L_{\max} - L_{\min}} \right) \left(\frac{x^{-k} - L_{\max}^{-k}}{k} \right). \end{aligned}$$

Also,

$$
\begin{aligned}
P(X=0) &= \int_{L_{\min}}^{L_{\max}} P(X=0|L) f_L(L) dL \\
&= \int_{L_{\min}}^{L_{\max}} \left(1 - CL^{-k-1}\right) \left(\frac{1}{L_{\max} - L_{\min}}\right) dL \\
&= 1 - \left(\frac{C}{L_{\max} - L_{\min}}\right) \left(\frac{L_{\min}^{-k} - L_{\max}^{-k}}{k}\right).
\end{aligned}
$$

Thus,

$$
P(X \le x) - P(X=0) = \left(\frac{C}{L_{\max} - L_{\min}}\right) \left(\frac{L_{\min}^{-k} - x^{-k}}{k}\right)
$$

and

$$
1 - P(X=0) = \left(\frac{C}{L_{\max} - L_{\min}}\right) \left(\frac{L_{\min}^{-k} - L_{\max}^{-k}}{k}\right).
$$

Therefore, for each x with $L_{\min} \le x \le L_{\max}$, the CDF of target loss X conditional on $X > 0$ is

$$
P(X \le x | X > 0) = \frac{L_{\min}^{-k} - x^{-k}}{L_{\min}^{-k} - L_{\max}^{-k}},
$$

where $L_{\min}$, $L_{\max}$, and k are parameters. $\square$

By differentiating the CDF (IV.2), we can show that the Probability Distribution Function (PDF) of target loss X conditional on $X > 0$ follows a power law. Precisely, for each x with $L_{\min} \le x \le L_{\max}$,

$$
p(X = x | X > 0) = \frac{dP(X \le x | X > 0)}{dx} = \left(\frac{k}{L_{\min}^{-k} - L_{\max}^{-k}}\right) x^{-k-1}, \tag{IV.3}
$$

where $L_{\min}$ is the minimum level above which the power law holds and $k+1$ is the scaling parameter of the power law. Interestingly, the scaling parameter is determined by the target loss power k.

The Complementary Cumulative Distribution Function (CCDF) of X conditional on $X > 0$ is

$$P(X > x|X > 0) = 1 - P(X \leq x|X > 0) = \frac{x^{-k} - L_{\text{max}}^{-k}}{L_{\text{min}}^{-k} - L_{\text{max}}^{-k}}. \tag{IV.4}$$

We want to estimate parameters L_{min}, L_{max}, and k from data. We use Global Terrorism Database (28) and focus on terrorist events occurred in a single country. Suppose that target loss is measured by the number of fatalities in a terrorist event.

In Iraq there were 2233 terrorist events from 1976 to 2007. The number of fatalities ranged from 1 to 202. Figure 8 shows the empirical CCDF of the Iraqi data.

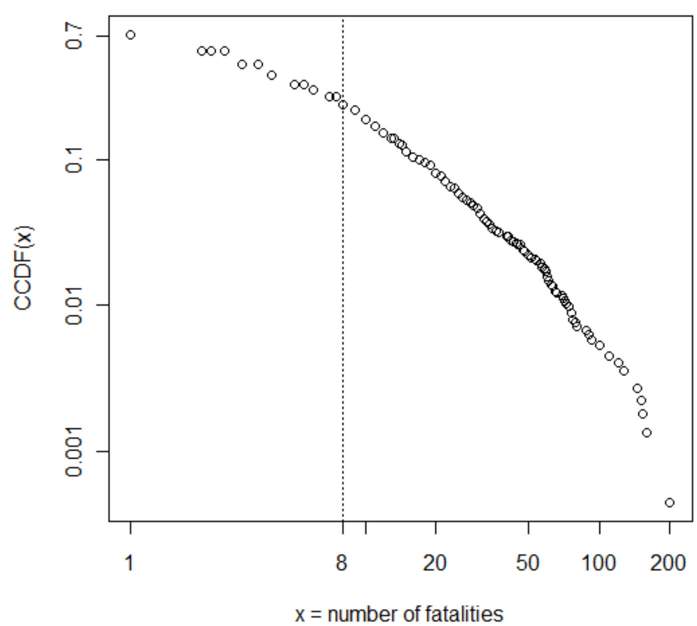

Figure 8. Empirical Distribution

We assume that $\hat{L}_{\max}$ equals to the maximum number of fatalities in the data and estimate $L_{\min}$ and k by using the two-step procedure of Clauset et al. (10). In the first step we take $L_{\min}$ as given and estimate k by the method of maximum likelihood. Note that estimate k depends on the value of $L_{\min}$. To make this clear, we denote the estimate by $k(L_{\min})$. In the second step we choose the value of $\hat{L}_{\min}$ that makes the probability distributions of the data and the model as close as possible above the minimum level $\hat{L}_{\min}$. The Kolmogorov-Smirnov or KS statistic is used to define the distance between two probability distributions. Finally, the estimate for k is $\hat{k} = k(\hat{L}_{\min})$. This two-step procedure will be presented more precisely in Appendix A.

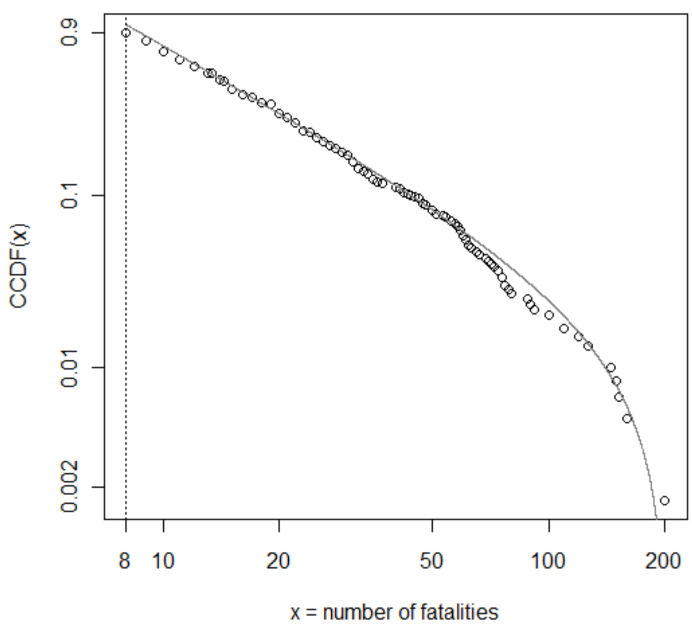

Figure 9. Power Law Distribution

From the Iraqi data we estimate parameters as $\hat{L}_{\min} = 8$, $\hat{L}_{\max} = 202$, and $\hat{k} = 1.27$. Figure 9 shows the empirical CCDF above the minimum level $\hat{L}_{\min} = 8$ in black circles as well as the model CCDF (IV.4) with estimated parameters in a gray curve. The KS statistic is 0.01 between these two distributions. Our model is a good fit to the data.[2]

Future Research

This chapter examines how strategic behavior leads to the power law of conflict. Equilibrium play and its comparative statics yield a power law. The scaling parameter of the power law is determined by the target loss power. The parameters of the power law can be estimated from data. We cannot, however, empirically test whether equilibrium play and its comparative statics yield a power law because we cannot observe whether players use equilibrium mixed strategies. In empirical data on conflict events, we only observe the realized (and reported) actions of players.[3] The model developed in this chapter, however, can be used to test the power law of conflict in the laboratory. In experiments we can test if subjects use equilibrium mixed strategies and if subjects adjust strategies as the experimenter adjusts payoffs.[4]

We can also test if laboratory data exhibit a power law distribution. The power law of conflict has been studied only in empirical data. It will be interesting to study the

[2]During the period of the data there may be "regime shifts" that affect the nature of conflict in Iraq. If we restrict the data to the period between 2003 and 2007, there were 2149 events with the maximum number of fatalities being 202. Parameters are estimated as $\hat{L}_{\min} = 10$, $\hat{L}_{\max} = 202$, and $\hat{k} = 1.40$ with the KS statistic 0.005. Thus, it may give a better fit to restrict the data to a specific time period.

[3]In other contexts, especially in sports, it is possible to track down players' actions to infer their strategies. Walker and Wooders (39) obtain field data from matches between professional tennis players and test if the players follow equilibrium mixed strategies. Palacios-Huerta (31) studies penalty kicks in professional soccer games.

[4]Mixed strategy play has been studied in experiments. For example, see Brown and Rosenthal (7), Levitt et al. (27), Ochs (29), O'Neill (30), Palacios-Huerta and Volij (32), and Wooders (42).

power law in laboratory data. If we observe the power law in experiments, we may be able to support our theory that strategic behavior in conflict events causes the power law.

Another interesting line of research would be to make the payoff functions of Chapter II strictly convex so that the equilibrium would be unique and then investigate the comparative statics properties in the general case.

Appendix A

We estimate parameters $L_{\min}$ and k by using the two-step procedure of Clauset et al. (10).

In the first step we take $L_{\min}$ as given. From data $\{x_i\}_{i=1}^n$ with $x_i \geq L_{\min}$ for $i = 1, \ldots, n$, we estimate k by the method of maximum likelihood. From (IV.3), the PDF of target loss X conditional on $X > 0$ is

$$p(X = x|X > 0) = \left(\frac{k}{L_{\min}^{-k} - L_{\max}^{-k}}\right) x^{-k-1}.$$

Thus, the probability that the data $\{x_i\}_{i=1}^n$ is drawn from the PDF is

$$p(\{x_i\}|k) = \prod_{i=1}^{n} \left(\frac{k}{L_{\min}^{-k} - L_{\max}^{-k}}\right) x_i^{-k-1},$$

which is also called the likelihood of the data. Given $L_{\min}$, the Maximum Likelihood Estimator (MLE) for k is

$$k(L_{\min}) = \arg\max_k p(\{x_i\}|k). \tag{IV.5}$$

However, it is impossible to solve for $k(L_{\min})$ analytically. We can use a numerical method to find $k(L_{\min})$.

In the second step we choose the value of $\hat{L}_{\min}$ that makes the probability distributions of the data and the model as close as possible above $\hat{L}_{\min}$. Given $L_{\min}$ and $k = k(L_{\min})$, let $F(x)$ denote the CDF (IV.2) of our model. Given $L_{\min}$, let $G(x)$ denote the empirical CDF of the data $\{x_i\}_{i=1}^{n}$. The Kolmogorov-Smirnov or KS statistic is defined as the maximum difference in absolute value between $F(x)$ and $G(x)$. That is,

$$D = \max_{x} |F(x) - G(x)|. \tag{IV.6}$$

Because both $F(x)$ and $G(x)$ depend on $L_{\min}$, the KS statistic D also depends on $L_{\min}$. To make this clear, we denote D by $D(L_{\min})$. Our estimate $\hat{L}_{\min}$ is the value of $L_{\min}$ that minimizes $D(L_{\min})$. That is,

$$\hat{L}_{\min} = \arg\min_{L_{\min}} D(L_{\min}).$$

Finally, the estimate for k is $\hat{k} = k(\hat{L}_{\min})$.

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