PROPERTIES OF ACYLINDRICALLY HYPERBOLIC GROUPS AND THEIR SMALL CANCELLATION QUOTIENTS

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In memory of my grandfather,

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CHAPTER I

INTRODUCTION

I.1 Groups acting on hyperbolic spaces

One of the main goals of geometric group theory is to understand the relationship between the algebraic properties of a group and the geometric properties of the spaces on which the group acts. Of course, any group admits a trivial action on any space, so in order to derive any useful information, some restrictions need to be placed on the types of actions and spaces which can be considered. For finitely generated groups, it is quite natural to consider actions which are proper and cobounded. Here an action of a group G on a metric space (X,d) is called *proper* if for all $x \in X$ and all $N \in \mathbb{N}$,

$$|\{g \in G \mid d((x,gx) \leq N\}| < \infty$$

and an action is called *cobounded* if there exists a bounded subset $B \subseteq X$ such that

$$X = \bigcup_{g \in G} gB.$$

To any group G generated by a set S, one can associate to the pair (G,S) a metric space on which G acts coboundedly, called the *Cayley graph of G with respect to S* and denoted by $\Gamma(G,S)$. (For the definition of the Cayley graph see Chapter II.) If in addition the set S is finite, then the action of G on $\Gamma(G,S)$ is also proper. In fact, the reason that proper and cobounded actions are natural is the well-known Svarč–Milnor Lemma (see [16]):

Lemma I.1.1 (Svarč–Milnor Lemma)**.** *Suppose a group G acts properly and coboundedly on a metric space X. Then G is generated by a finite set S and X is quasi-isometric to* $\Gamma(G,S)$.

Recall that quasi-isometry is a coarse analogue of the notion of isometry between metric spaces; for details and motivation we refer to [33]. In particular, this lemma says that the space $\Gamma(G,S)$ is an invariant of the group itself and does not depend on the choice of finite generating set, at least up to quasi-isometry.

In his seminal paper [34], Gromov introduced the class of *hyperbolic groups* which revolutionized the study of geometric group theory. Gromov called a space *hyperbolic* if all geodesic triangles in the space were uniformly thin, and he called a group hyperbolic if it acts properly and coboundedly on a hyperbolic metric space (or equivalently, if the Cayley graph of G with respect to any finite generating set is hyperbolic). This definition was motivated by the fact that the "thin triangle" condition is shared both by simplicial trees and by manifolds of negative curvature. Thus, the notion of hyperbolicity connected both the "discrete" world of graphs and free groups and the "continuous" world of negatively curved manifolds and their fundamental groups.

In the same paper, Gromov also suggested the notion of *relatively hyperbolic groups.* Intuitively, relatively hyperbolic groups are similar to hyperbolic groups with the restriction that triangles are uniformly "thin" modulo some fixed collection of subgroups (called *peripheral subgroups*). For example if a complete Riemannian manifold with pinched negative sectional curvature is finite volume, then it is hyperbolic relative to the cusp subgroups. Also if a group admits a Bass-Serre decomposition with finite edge groups, then it is hyperbolic relative to the vertex groups; by a famous theorem of Stallings every infinitely ended finitely generated group carries such a structure.

In [12] this idea was elaborated on by Bowditch, who suggested a definition of relative hyperbolicity in terms of the dynamics of properly discontinuous isometric group actions on hyperbolic spaces. Alternatively, another definition was suggested by Farb in [28] who looked at the geometry of a certain graph associated to a group and a collection of subgroups, called the *coset graph*. Other equivalent definitions of relative hyperbolicity were given by Groves-Manning who looked at the hyperbolicity of the Cayley graph with certain "cusps" attached [32], and by Osin who gave an isoperimetric characterization of relative hyperbolicity [66].

Hyperbolic and relatively hyperbolic groups arise in a variety of contexts, and the presence of negative curvature has many important consequences for the structure of these groups. However, many groups which are not hyperbolic or relatively hyperbolic still admit natural and useful actions on hyperbolic metric spaces. For example, mapping class groups act on the curve complex, and $Out(F_n)$ acts on the free-splitting complex and the free-factor complex; all of these spaces have been shown to be hyperbolic [52] [10] [38], while in most cases these groups do not carry a hyperbolic or relatively hyperbolic structure [7]. These groups play an important role in low dimensional topology, and have many connections with the classical theory of the arithmetic group $Sl_n(\mathbb{Z})$ [48].

Motivated by these examples and others, different people have introduced and studied several other types of group actions on hyperbolic spaces, which typically involve some weak version of properness. We will mention just a few of these actions.

The action of a group G on a metric space (X,d) is called *acylindrical* if for all ε there exist $R>0$ and $N>0$ such that for all x, $y\in X$ with $d(x,y)\geq R$, the set

$$\{g\in G \mid d(x,gx)\leq\varepsilon, d(y,gy)\leq\varepsilon\}$$

contains at most N elements. This definition was given first by Sela [69] for the special case of groups acting on trees and in general by Bowditch [12]. One can think of this acylindricity condition as a version of properness for the induced action of G on $X\times X$ minus a "thick diagonal."

It is not hard to see that any proper, cobounded action is also acylindrical. Osin showed that the action of a relatively hyperbolic group on the relative Cayley graph is acylindrical [65], and Bowditch showed that the action of any "non-exceptional" mapping class group on the curve complex is acylindrical [13].

Notice, however, that any group action on a bounded space (for example, a point) trivially satisfies the aclyindricity condidtion. Since bounded spaces are always hyperbolic, there needs to be some condition to rule out trivial actions on bounded spaces. Hence we will usually only consider actions which are *non-elementary*. Recall that for a proper geodesic hyperbolic metric space X, the *Gromov boundary* ∂X is defined as the set of equivalence classes of infinite geodesic rays starting at a fixed base point, where two rays are equivalent if they are within bounded Hausdorff distance of each other. This definition can be naturally extended to any (not necessarily proper or geo7desic) hyperbolic metric space (see [34]). Then the action of a group G on a hyperbolic metric space is called *non-elementary* if for some (equivalently, any) $x \in X$, the Gromov boundary of the G-orbit of x contains at least three points (in which case, it contains infinitely many). For cobounded actions, this is equivalent to saying that ∂X contains at least three points. Note that a group action on a bounded space X cannot be non-elementary, since $\partial X = \emptyset$; for further motivation for only considering acylindrical actions which are non-elementary, see Theorem III.2.3.

A weaker condition (a priori) than acylindricity is the notion of an element satisfying *weak proper discontinuity* or WPD, introduced by Bestvina-Fujiwara in [11]. Recall that for a group G acting on a space X, $h \in G$ is called *loxodromic* if h admits an invariant, bi-infinite quasi-geodesic axis in X on which h acts as a non-trivial translation. Loosely speaking, a loxodromic element h satisfies WPD if the group action satisfies the aclyindricity condition when the points x and y belong to an axis of h. Thus if an action is acylindrical, then every loxodromic element satisfies WPD. For the (non-exceptional) mapping class groups acting on the curve complex, this means that every pseudo-anosov element satisfies WPD. Also, given a fully irreducible element g of $Out(F_n)$, Bestvina-Feighn [9] have constructed a hyperbolic complex on which $Out(F_n)$ acts such that g is a loxodromic WPD element.

In [24], Dahmani-Guirardel-Osin introduced the notion of a *hyperbolically embedded subgroup*. Loosely speaking, a subgroup H of a group G is said to be hyperbolically embedded if G acts on a hyperbolic metric space such that the action of H is proper, orbits of H are quasi-convex, and distinct orbits of H "quickly diverge." The idea here is that one replaces "global properness" with the "local properness" of the subgroup H. Dahmani-Guirardel-Osin showed that this condition led to a quite natural generalization of peripheral structure of subgroups of relatively hyperbolic groups, and they developed machinery for translating results about relatively hyperbolic groups to groups which contain non-degenerate (i.e. proper and infinite) hyperbolically embedded subgroups.

There is a clear progression when studying these types of actions on hyperbolic spaces; if a group acts acylindrically on a hyperbolic metric space, then every loxodromic element satisfies WPD. Also, if a group G acts on a hyperbolic metric space and G contains a loxodromic, WPD element g, then g is contained in a maximal virtually cyclic subgroup which is hyperbolically embedded in G [24]. In fact, the Bestvina-Fujiwara WPD condition was part of the motivation for the actions considered by Dahmani-Guirardel-Osin. However, a recent result of Osin [65] shows that if G contains a non-degenerate, hyperbolically embedded subgroup, then there exists a (possibly infinite) generating set $\mathscr{A} \subset G$ such that $\Gamma(G, \mathscr{A})$ is hyperbolic and the action of G on $\Gamma(G, \mathscr{A})$ is non-elementary and acylindrical. This gives the following theorem:

Theorem I.1.2. *[65] Let G be a group. The following are equivalent:*

1. *G admits a non-elementary, acylindrical action on a hyperbolic metric space.*
2. *G contains a loxodromic, WPD element with respect to a non-elementary action on a hyperbolic metric space.*
3. *G contains a non-degenerate hyperbolically embedded subgroup.*
4. *For some $\mathscr{A} \subset G$, $\Gamma(G, \mathscr{A})$ is hyperbolic and the action of G on $\Gamma(G, \mathscr{A})$ is non-elementary and acylindrical.*

That is, if you consider groups which act non-elementarily and acylindrically on hyperbolic metric spaces, groups which have non-elementary WPD actions on hyperbolic metric spaces, or groups which contain non-degenerate hyperbolically embedded subgroups, you will get *exactly the same class of groups*. Motivated by this theorem, we will introduce the following notation for this class.

Definition I.1.3. Let $\mathscr{AH}$ denote the class of groups which admit a non-elementary, acylindrical action on a hyperbolic metric space. We will call such groups *acylindrically hyperbolic*.

The term "acylindrically hyperbolic" is due to Osin [65]. In fact, there are other types of actions such as those studied by Hamenstädt [39] and Sistro [69] which lead to equivalent definitions of $\mathscr{AH}$. The fact that so many people have developed distinct definitions of the same class of groups is evidence that $\mathscr{AH}$ is worth studying. $\mathscr{AH}$ also contains many interesting examples of groups, such as:

(a) Non-elementary hyperbolic groups.

(b) Non-elementary groups hyperbolic relative to proper subgroups.

(c) All but finitely many mapping class groups of closed, orientable surfaces (possibly with punctures).

(d) $Out(F_n)$ for $n \geq 2$.

(e) Groups which act properly on proper $CAT(0)$ spaces and contain rank-1 elements, for example directly indecomposable non-cyclic right angled Artin groups.

(f) The Cremona group of birational transformations of the complex projective plane.

(g) All one-relator groups with at least three generators.

(h) Many fundamental groups of 3-manifolds. Specifically, if G is the fundamental group of a 3-manifold, Then G satisfies exactly one of the following:

   (1) $G \in \mathscr{AH}$.

   (2) G contains an infinite normal cyclic subgroup Z, and $G/Z \in \mathscr{AH}$.

   (3) G is virtually solvable.

(g) and (h) are shown in [57], and (e) is due to [69]. All other examples are from [24].

While being quite general and containing many interesting examples, $\mathscr{AH}$ is still restrictive enough to allow one to build an interesting theory. Indeed, modulo the equivalence of seemingly distinct definitions mentioned above, many results about acylindrically hyperbolic groups can be found in [11, 13, 24, 39, 65, 69]. The goal of this thesis will be to build some new results about acylindrically hyperbolic groups, and to generalize a version of small cancellation theory to this class of groups.

In the second chapter we will set our notation and give some background material. In the third chapter we will give the basic properties of acylindrically hyperbolic groups which we will need for the remaining chapters; in particular, we will present some of the different characterizations of $\mathscr{AH}$ and explore the relationship between them. The remaining three chapters will be dedicated to our main results, which we describe in detail in the next three sections.

Except for minor changes, Chapter IV and Section I.2 represent joint work with Osin which is published in [45]. Similarly, Chapter VI, Section I.4 and most of Section V.3 are joint with Osin and are published in [44]. Other results of Chapter V are more recent and will appear as [43].

I.2 Quasi-cocycles

Let V be a normed G-module. A map $q\colon G \to V$ is called a *quasi-cocycle* if there exists a constant $\varepsilon > 0$ such that for every $f, g \in G$ we have

$$\|q(fg) - q(f) - fq(g)\| \leq \varepsilon.$$

The vector space of all quasi-cocycles on G with values in V is denoted by $QZ^1(G,V)$.

The study of quasi-cocycles is partially motivated by the fact that the kernel of the comparison map $H_b^2(G,V) \to H^2(G,V)$ from the second bounded cohomology to the ordinary second cohomology with coefficients in V can be identified with the quotient $QZ^1(G,V)/(\ell^\infty(G,V) + Z^1(G,V))$, where $\ell^\infty(G,V)$ and $Z^1(G,V)$ are the subspaces of uniformly bounded maps and cocycles, respectively. In the last decade, techniques based on quasi-cocycles and bounded cohomology have led to new breakthroughs in the study of rigidity of group von Neumann algebras, measure equivalence and orbit equivalence of groups, and low dimensional topology (see [18, 21, 58, 68] and references therein).

If $V = \mathbb{R}$ with the trivial action of G, quasi-cocycles on G with values in V are called *quasimorphisms*. The classical examples are *counting quasimorphisms* of free groups introduced by Brooks [17]. Let F be a free group with a basis S and let w be a reduced word in $S \cup S^{-1}$. Given an element $g \in F$, denote by $c_w(g)$ the number of disjoint copies of w in the reduced representative of g. Then $h_w = c_w - c_{w^{-1}}$ defines a quasimorphism $F \to \mathbb{R}$ [17]. Observe that $h_w(g)$ extends the obvious cocycle (i.e., homomorphism) $H \to \mathbb{R}$ of the cyclic subgroup $H = \langle w \rangle \leq F$ that sends w^n to n for all $n \in \mathbb{Z}$.

This construction was extended to hyperbolic groups by Epstein-Fujiwara [27] and later to all groups which admit a WPD action on a hyperbolic metric space (or in our language, all acylindrically hyperbolic groups) by Bestvina-Fujiwara [11]. In fact, the motivation for the Bestvina-Fujiwara WPD condition was that it provided the "right" condition for a group acting on a hyperbolic metric space to allow one to build linearly independent counting quasimorphisms, and hence to guarantee that the kernel of the comparison map $H_b^2(G,\mathbb{R}) \to H^2(G,\mathbb{R})$ is infinite dimensional. As an application of this, they provide a new proof of a rigidity theorem of Farb-Kaimanovich-Masur which says that high rank lattices do not appear as subgroups of mapping class groups.

In Chapter IV, we will prove an extension theorem for quasi-cocycles defined on hyperbolically embedded subgroups. This can be considered as a generalization of the Bestvina-Fujiwara result, as their construction essentially extends certain quasimorphisms defined on (hyperbolically embedded) virtually cyclic subgroups. We state here a simplified version of our main result and refer to Theorem IV.2.2 for the full generality. A quasi-cocycle $q \in QZ^1(G,V)$ is called *anti-symmetric* if for every $g \in G$,

$$q(g^{-1}) = -g^{-1}q(g).$$

For a group G and a normed G-module V, let $QZ^1_{as}(G,V)$ denote the subspace of all anti-symmetric quasi-cocycles on G with coefficients in V.

Theorem I.2.1. *Let G be a group, H a hyperbolically embedded subgroup of G, V a normed G-module, U an H-submodule of V. Then there exists a linear map*

$$\iota\colon QZ^1_{as}(H,U) \to QZ^1_{as}(G,V)$$

such that for any $q \in QZ^1_{as}(H,U)$, we have $\iota(q)|_H \equiv q$.

It is well-known and easy to prove that every quasi-cocycle is anti-symmetric up to a bounded perturbation (see Lemma IV.0.6). In the notation of Theorem I.2.1, this gives the following.

Corollary I.2.2. *There exists a linear map $\kappa\colon QZ^1(H,U) \to QZ^1(G,V)$ such that for any $q \in QZ^1(H,U)$, $\kappa(q)|_H \in QZ^1(H,U)$ and*

$$\sup_{h \in H} \|\kappa(q)(h) - q(h)\| < \infty.$$

In Section IV.3, we obtain some other corollaries of our main result. Recall that the class $\mathscr{C}_{reg}$ of Monod-Shalom is the class of groups for which $H^2_b(G,\ell^2(G)) \neq 0$. This definition was proposed as cohomological characterization of the notion of "negative curvature" in group theory [61]. In [60] Monod and Shalom develop a rich rigidity theory with respect to measure equivalence and orbit equivalence of actions of groups in $\mathscr{C}_{reg}$. These results have a variety of applications to measurable group theory, ergodic theory and von Neumann algebras.

Another similar class of groups is the class $\mathscr{D}_{reg}$ introduced by Thom [71]. $G \in \mathscr{D}_{reg}$ if G is non-amenable and there exists some $q \in QZ^1(G,\ell^2(G))$ which is unbounded. Thom proved rigidity results about the subgroup structure of groups in $\mathscr{D}_{reg}$ and showed that this class is closely related to $\mathscr{C}_{reg}$. However neither inclusion is known to hold between these two classes.

Using Corollary I.2.2 and the fact that every group $G \in \mathscr{A}\mathscr{H}$ contains a virtually free (but not virtually cyclic) hyperbolically embedded subgroup [24], we recover the following result of Hamenstädt.

Corollary I.2.3. *For any $G \in \mathscr{A}\mathscr{H}$, the dimension of the kernel of the comparison map $H^2_b(G,\ell^p(G)) \to H^2(G,\ell^p(G))$ is infinite. In particular, $\mathscr{A}\mathscr{H} \subseteq \mathscr{C}_{reg} \cap \mathscr{D}_{reg}$.*

This corollary was first proved in [39]; however, it was not known that these results covered the same class of groups until the recent result of Osin [65] mentioned in the previous section. In addition, Bestvina, Bromberg, and Fujiwara [8] announced a proof showing that the dimension of the kernel of the comparison map $H^2_b(G,E) \to H^2(G,E)$ is infinite for any group which admits a WPD action on a hyperbolic metric space and any uniformly convex Banach G-module E. Thus the result of Bestvina, Bromberg, and Fujiwara is stronger than Corollary I.2.3.

As another application, we show that hyperbolically embedded subgroups are undistorted with respect to the stable commutator length, scl. For the definition of scl we refer to Section IV.3. Given a group G and a subgroup $H \leq G$ it is straightforward to see that $scl_G(h) \leq scl_H(h)$ for any $h \in [H,H]$, where scl_G and scl_H are the stable commutator lengths on $[G,G]$ and $[H,H]$, respectively.

On the other hand, recall that every torsion free group H can be embedded in a group G where every element is a commutator (see [49, Theorem 8.1] or [67] for a finitely generated version of such an embedding). In particular, scl_G vanishes on G, while scl_H can be unbounded on $[H,H]$. Thus, in general, there is no upper bound on scl_H in terms of scl_G. In what follows, we say that H is *undistorted in G with respect to the stable commutator length* if there exists a constant B such that for every $h \in [H,H]$, we have $scl_H(h) \leq Bscl_G(h)$.

Using Theorem I.2.1 and the Bavard duality, we obtain the following.

Corollary I.2.4. *Let G be a group, H a hyperbolically embedded subgroup of G. Then H is undistorted in G with respect to the stable commutator length.*

Even the following particular cases seem new. Recall that a subgroup $H \leq G$ is *almost malnormal* if $|H^g \cap H| < \infty$ for every $g \in G \setminus H$.

Corollary I.2.5. *Every almost malnormal quasiconvex subgroup of a hyperbolic group is undistorted with respect to the stable commutator length. In particular, so is every finitely generated malnormal subgroup of a free group.*

In Section IV.3 we show that the almost malnormality condition cannot be omitted even for free groups (see Remark IV.3.7).

I.3 Small cancellation theory

Classical small cancellation theory involves the study of groups which are given by presentations where the relations have a "small overlap" between each other. The basic ideas of small cancellation theory go back to the work of Dehn in the early 1900's and his formulation and solution to the word problem for surface groups. These ideas were further developed in the 60's and 70's by Greendlinger, Lyndon, Schupp, and others (see [49]). Small cancellation ideas also played an important role in Olshanskii's construction of various "exotic" groups, such as the Tarski monsters (see [63]).

In fact, the ideas of small cancellation theory are closely connected with the notion of hyperbolicity. Indeed, Gromov's definition of hyperbolic groups was motivated in part by the fact that hyperbolicity had been used *implicity* in the ideas of small cancellation theory going back to the work of Dehn. In fact, Dehn's solution to the word problem for surface groups involved embedding their Cayley graphs in the hyperbolic plane and applying the Gauss-Bonnet theorem to show that any closed loop must contain a large segment of a defining relator. One could then use this defining relator to find a shorter loop which represented the same group element. This process became known as *Dehn's Algorithm*, and it turns out that a group has a presentation where this algorithm can be used to solve the word problem if and only if that group is hyperbolic.

Gromov noticed that many small cancellation arguments could be simultaneously simplified and generalized by *explicitly* using hyperbolicity. As group presentations represent quotients of free groups, Gromov suggested that the theory could be adapted to study quotients of hyperbolic groups. This idea was formalized by several people, including Champetier [19], Delzant [25], Olshanskii [62], and others. Olshanskii's approach was generalized to relatively hyperbolic groups by Osin in [67].

In Chapter V we generalize Osin's version of small cancellation over relatively hyperbolic groups to the class of acylindrically hyperbolic groups. In particular, this verson of small cancellation covers (for sufficiently small λ) both the classical $C'(\lambda)$-condition for free groups, as well as the $C'(\lambda)$-condition for amalgamated products and HNN-extensions developed in [49]. We will leave the rather technical definitions of the small cancellation conditions for Chapter V; here we will state a version of our main small cancellation theorem and present some applications of this theorem. In order to state this theorem, we will need the notion of a suitable subgroup, which is a simplification of the notion of a G-subgroup given in [62].

Definition I.3.1. Given $G \in \mathscr{AH}$, a generating set $\mathscr{A}$ of G and a subgroup $S \leq G$, we will say that S is *suitable with respect to* $\Gamma(G,\mathscr{A})$ if the following holds:

1. $\Gamma(G,\mathscr{A})$ is hyperbolic and the action of G on $\Gamma(G,\mathscr{A})$ is acylindrical.
2. The action of S on $\Gamma(G,\mathscr{A})$ is non-elementary.
3. S does not normalize any finite subgroups of G.

We will further say that a subgroup is *suitable* if it is suitable with respect to some $\Gamma(G,\mathscr{A})$.

Given a generating set $\mathscr{A}$ of G, we denote by $B_{\mathscr{A}}(N)$ the subset of G which belongs to the ball of radius N centered at the identity in $\Gamma(G,\mathscr{A})$. The following is a simplification of our main technical theorem proved using small cancellation theory; for the full generality, see Theorem V.6.1:

Theorem I.3.2. *Suppose $G \in \mathscr{AH}$ and S is suitable with respect to $\Gamma(G,\mathscr{A})$. Then for any $\{t_1,\dots,t_m\} \subset G$ and $N \in \mathbb{N}$, there exists a group $\overline{G}$ and a surjective homomorphism $\gamma\colon G \to \overline{G}$ which satisfies*

(a) $\overline{G} \in \mathscr{AH}$.

(b) $\gamma|_{B_{\mathscr{A}}(N)}$ *is injective.*

(c) $\gamma(t_i) \in \gamma(S)$ *for* $i = 1,\dots,m$.

(d) $\gamma(S)$ *is a suitable subgroup of* $\overline{G}$.

(e) Every element of $\overline{G}$ of order n is the image of an element of G of order n.

Remark I.3.3. It should be noted that in [24] a geometric version of small cancellation is developed for $\mathscr{AH}$ based on Gromov's notion of a *rotating family of subgroups*. However, this theory focuses on the algebraic and dynamical properties of the generated normal subgroup, while our theory focuses on the resulting quotient group.

If G is finitely generated we can choose $t_1, \ldots, t_m$ to be a generating set and we get that the restricion of γ to S is surjective. A more general version of this observation can be found in Corollary V.6.5.

It is our hope that this theorem will become a useful tool for studying acylindrically hyperbolic groups. We will give a few applications of this theorem which illustrate the variety of problems to which it can be applied.

Our first application is to the study of Frattini subgroups of groups in $\mathcal{AH}$. The Frattini subgroup of a group G, denoted $Fratt(G)$, is defined as the intersection of all maximal subgroups of G, or as G itself if no such subgroups exist. It is not hard to show that the Frattini subgroup of G is exactly the set of *non-generators* of G, that is the set of $g \in G$ such that for any set X which generates G, $X \setminus \{g\}$ also generates G.

The study of the Frattini subgroup is related to the study of the *generation problem* and the *rank problem*. Given a group G and a subset $Y \subset G$, the generation problem is to determine whether Y generates G. The rank problem is to determine the smallest cardinality of a generating set of a given group G. Since $Fratt(G)$ consists of non-generators these problems can often be simplified by considering $G/Fratt(G)$. Hence these problems tend to be more approachable for classes of groups which have "large" Frattini subgroups. We will show, however, that this is not the case for acylindrically hyperbolic groups.

Theorem I.3.4. *Let $G \in \mathcal{AH}$. Then $Fratt(G) \leq K(G)$; in particular, the Frattini subgroup is finite.*

Here $K(G)$ denotes the maximal finite normal subgroup of G, also called the *finite radical*. Such a subgroup is shown to always exist in [24]. This theorem generalizes several previously known results. For example, it was known that the free product of any non-trivial groups has trivial Frattini subgroup [40], and that free products of free groups with cyclic amalgamation have finite Frattini subgroup [73]. Also, in [46] Kapovich proved that all subgroups of hyperbolic groups have finite Frattini subgroup. All of these groups are either virtually cyclic or belong to $\mathcal{AH}$.

Another useful Corollary of our small cancellation theorem is the following:

Corollary I.3.5. *Let $G_1, G_2 \in \mathcal{AH}$, with G_1 finitely generated, G_2 countable. Then there exists a non-virtually cyclic group Q and surjective homomorphisms $\alpha_i \colon G_i \to Q$ for $i = 1, 2$. If in addition G_2 is finitely generated, then we can choose $Q \in \mathcal{AH}$.*

This corollary provides a general way to construct interesting quotients. For example, since Property (T) is preserved under taking quotients and the existence of non-elementary hyperbolic groups with Property (T) is well-known, as an immediate consequense of Corollary I.3.5 we get:

Corollary I.3.6. *Every countable $G \in \mathcal{AH}$ has an infinite quotient with property (T).*

This generalizes a result of Gromov which says that all non-elementary hyperbolic groups have infinite property (T) quotients [34].

In addition, we can also apply Corollary I.3.5 to study the topology of marked group presentations. This topology provides a natural framework for studying groups which "approximate" a given class of groups. For example, Sela's limit groups, which were used in the solution of the Tarski problem, can be defined as the groups which are approximated by free groups with respect to this topology (see [20]). In [4], this topology is used to define a preorder on the space of finitely generated groups.

Let $\mathscr{G}_k$ denote the set of *marked k-generated groups*, that is $\mathscr{G}_k = \{(G, x_1, ..., x_k) \mid x_1, ..., x_k \in G, \langle x_1, ..., x_k\rangle = G\}$. Given a group G, let $[G]_k$ denote the (possibly empty) subset of $\mathscr{G}_k$ corresponding to the group G, and let $\overline{[G]}_k$ denote its closure with respect to the topology mentioned above. Let $\overline{[G]} = \bigcup_{k=1}^{\infty} \overline{[G]}_k$. Also, let $\mathscr{A}\mathscr{H}_0$ denote the class of aclyindrically hyperbolic groups G for which $K(G) = \{1\}$, and let $[\mathscr{A}\mathscr{H}_0] = \bigcup_{k=1}^{\infty}\{(G, x_1, ..., x_k) \in \mathscr{G}_k \mid G \in \mathscr{A}\mathscr{H}_0\}$.

Theorem I.3.7. *Let $\mathscr{C}$ be a countable subset of $[\mathscr{A}\mathscr{H}_0]$. Then there exists a finitely generated group D such that $\mathscr{C} \subset \overline{[D]}$.*

Roughly speaking, this theorem says that we can find a group D which approximates every group in $\mathscr{C}$. In the language of [4], the group D *preforms* G for every G which belongs to $\mathscr{C}$.

Finally, we show that our results can be used to build "exotic" quotient groups. Higman, B. H. Neumann and H. Neumann showed that any countable group G could be embedded in a countable group B in which any two elements are conjugate if and only if they have the same order [41]. Osin, using small cancellation over relatively hyperbolic groups, showed that the group B could be chosen to be finitely generated [67]; in fact, this allowed him to produce the first known examples of infinite, finitely generated groups with exactly two conjugacy classes. We show that any countable, torsion free $G \in \mathscr{A}\mathscr{H}$ has such a quotient group. Here we let $\pi(G) \subseteq \mathbb{N} \cup \{\infty\}$ be the set of orders of elements of G.

Theorem I.3.8. *Let $G \in \mathscr{A}\mathscr{H}$ be countable. Then G has an infinite, finitely generated quotient group C such that any two elements of C are conjugate if and only if they have the same order and $\pi(C) = \pi(G/K(G))$. In particular, if G is torsion free, then C has two conjugacy classes.*

A group is called *divisible* if for all $g \in G$ and $n \in \mathbb{N}$, there exists $x \in G$ such that $x^n = g$. A natural example of such a group is $\mathbb{Q}$; The first finitely generated examples were given by Guba [36].

More generally, a group G is called *verbally complete* if for any $k \geq 1$, any $g \in G$, and any freely reduced word $W(x_1, ..., x_k)$ there exists $g_1, ..., g_k \in G$ such that $W(g_1, ..., g_k) = g$ in the group G. The existence of finitely generated verbally complete groups was shown by Mikhajlovskii and Olshanskii [53], and Osin showed that every countable group could be embedded in a finitely generated verbally complete group [67].

Theorem I.3.9. *Let $G \in \mathscr{A}\mathscr{H}$ be countable. Then G has an infinite, finitely generated quotient group V such that V is verbally complete.*

I.4 Conjugacy growth

In the final chapter, we apply our small cancellation techniques to study conjugacy growth of finitely generated groups.

Let G be a group generated by a set X. Recall that the *word length* of an element $g \in G$ with respect to the generating set X, denoted by $|g|_X$, is the length of a shortest word in $X \cup X^{-1}$ representing g in the group G. If X is finite one can consider the *growth function* of G, $\gamma_G \colon \mathbb{N} \to \mathbb{N}$, defined by

$$\gamma_G(n) = |B_{G,X}(n)|,$$

where

$$B_{G,X}(n) = \{g \in G \mid |g|_X \leq n\}.$$

It was first introduced by Efremovic [26] and Svarč [69] in the 50's, rediscovered by Milnor [54] in the 60's, and served as the starting point and a source of motivating examples for contemporary geometric group theory. In Chapter VI we focus on a similar function $\xi_{G,X}\colon \mathbb{N} \to \mathbb{N}$ called the *conjugacy growth function* of G with respect to X. By definition $\xi_{G,X}(n)$ is the number of conjugacy classes in the ball $B_{G,X}(n)$.

It is straightforward to verify that $\gamma_{G,X}$ and $\xi_{G,X}$ are independent of the choice of a particular finite generating set X of G up to the following equivalence relation. Given $f, g\colon \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there exists $C \in \mathbb{N}$ such that $f(n) \leq g(Cn)$ for all $n \in \mathbb{N}$. Further f and g are *equivalent* (we write $f \sim g$) if $f \preceq g$ and $g \preceq f$. In what follows we always consider growth functions up to this equivalence relation and omit X from the notation.

The conjugacy growth function was introduced by Babenko [1] in order to study geodesic growth of Riemannian manifolds. Obviously free homotopy classes of loops in a manifold M are in 1-to-1 correspondence with conjugacy classes of $\pi_1(M)$. If M is a closed Riemannian manifold, the Svarč–Milnor Lemma then implies that $\xi_{\pi_1(M)}$ is equivalent to the function counting free homotopy classes of loops of given length in M. The later function serves as a lower bound for the geodesic growth function of M, which counts the number of geometrically distinct closed geodesics of given length on M. Moreover if M has negative sectional curvature, then all these functions are equivalent.

Geodesic growth of compact Riemannian manifolds has been studied extensively since the late 60's (see, e.g., [2, 3, 47, 50]). The most successful results were obtained in the case of negatively curved manifolds by Margulis [50, 51]. He proved that the number of primitive closed geodesics of length at most n on a closed manifold of negative sectional curvature is approximately equal to $e^{hn}/(hn)$, where h is the topological entropy of the geodesic flow on the unit tangent bundle of the manifold. Coornaert and Knieper [22, 23] proved a group theoretic analogue of this result and found an asymptotic estimate for the number of primitive conjugacy classes in a hyperbolic group similar to that from Margulis' papers. Our next theorem gives a similar result for all (finitely generated) acylindrically hyperbolic groups.

Recall that a conjugacy class of a group G is called *primitive* if some (or, equivalently, any) element g from the class is not a proper power, i.e., $h^n = g$ implies $n = \pm 1$. For a group G generated by a finite set X, let $\pi_G(n)$ denote the function counting primitive conjugacy classes in $B_{G,X}(n)$.

Theorem I.4.1. *Let $G \in \mathcal{AH}$ be finitely generated. Then $\xi_G \sim \pi_G \sim 2^n$.*

Theorem I.4.1 can be used to completely classify conjugacy growth functions of subgroups in mapping class groups (see Section VI.1).

Counting primitive conjugacy classes does not make much sense for general groups. For instance if the group G is torsion without involutions, then there are no primitive conjugacy classes in G at all. Moreover this can happen even for torsion free groups; for example, the finitely generated divisible groups constructed by Guba [36]. Thus in the context of abstract group theory it seems more natural to consider the conjugacy growth function ξ_G.

The algebraic study of the conjugacy growth function is strongly motivated by its similarity to the ordinary growth function. Recall that a function f is *exponential* if $f \sim 2^n$, *polynomial* if $f \sim n^d$ for some $d \in \mathbb{N}$, and *polynomially bounded* if $f \preceq n^d$ for some $d \in \mathbb{N}$. In [54], Milnor conjectured that γ_G is always either exponential or polynomial. Counterexamples to this conjecture were constructed by Grigorchuk in [30]. It turns out, however, that Milnor's dichotomy does hold for some important classes of groups including solvable and linear ones [55, 72, 74]. Gromov [35] proved that any group with polynomially bounded growth function contains a nilpotent subgroup of finite index. Combining this with a result of Bass [5] saying that every nilpotent group has a polynomial growth function, one can easily derive that if the growth function of a group is polynomially bounded, then it is in fact polynomial. Despite these advances, it is still far from being clear which functions can occur as growth functions of finitely generated groups. For instance it is unknown whether there exists a group G with non-polynomial growth function satisfying $\gamma_G \preceq 2^{\sqrt{n}}$. For a comprehensive survey we refer the interested reader to [31].

Recently some similar results were proved for the conjugacy growth function. Breuillard and Cornulier [15] showed that for a finitely generated solvable group G, the conjugacy growth function is either polynomially bounded or exponential and, furthermore, ξ_G is polynomially bounded if and only if G is virtually nilpotent. (For polycyclic groups this result was proved independently and simultaneously by the author in [42].) This dichotomy was also proved for finitely generated linear groups by Breuillard, Cornulier, Lubotzky, and Meiri [14]. Motivated by the Milnor conjecture, Guba and Sapir [37] suggested that 'natural' groups have either polynomially bounded or exponential conjugacy growth. They proved that many *HNN*-extensions and diagram groups, including the R. Thompson group F, have exponential conjugacy growth.

Note however that the ordinary and conjugacy growth functions can behave differently. For instance, the conjugacy growth of a nilpotent group is not necessarily polynomial. Indeed let H be the Heisenberg group

$$H = \mathrm{UT}_3(\mathbb{Z}) \cong \langle a, b, c \mid [a,b] = c, [a,c] = [b,c] = 1 \rangle.$$

Then it is fairly easy to compute that $\xi_H(n) \sim n^2 \log(n)$ (this example can be found in [1] and [37]). Note also that there exist finitely generated groups of exponential growth with finitely many conjugacy classes [63, Theorem 41.2] and even with 2 conjugacy classes [67]. Thus γ_G and ξ_G can be very far apart, actually on the opposite sides of the spectrum.

In Chapter VI we address the following realization problem: *Which functions can be realized (up to equivalence) as conjugacy growth functions of finitely generated groups*? Unlike in the case of ordinary growth, the realization problem for conjugacy growth admits a complete solution.

Theorem I.4.2. *Let G be a group generated by a finite set X, f the conjugacy growth function of G with respect to X. Then the following conditions hold.*

(a) f is non-decreasing.

(b) There exists $a \geq 1$ such that $f(n) \leq a^n$ for every $n \in \mathbb{N}$.

Conversely, suppose that a function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies the above conditions (a) and (b). Then there exists an infinite finitely generated group G such that $\xi_G \sim f$.

The first claim of the theorem is essentially trivial. Note, however, that even realizing simplest growth functions, e.g., $f(n) = \log n$, is nontrivial; moreover, we are not aware of any groups other than the ones constructed in this paper that have unbounded conjugacy growth functions satisfying $f(n) = o(n)$.

When speaking about asymptotic invariants of groups it is customary to ask whether these invariants are *geometric*, i.e. invariant under quasi-isometry. Many asymptotic invariants of groups are invariant under quasi-isometry up to suitable equivalence relations, e.g., the ordinary growth function, the Dehn function, and the asymptotic dimension growth function, just to name a few. However it turns out that the conjugacy growth function is not a geometric invariant in the strongest possible sense. More precisely, we construct the following example.

Theorem I.4.3. *There exists a finitely generated group G and a finite index subgroup $H \leq G$ such that H has 2 conjugacy classes while G is of exponential conjugacy growth.*

Since every (non-trivial) group has at least two conjugacy classes and at most exponential conjugacy growth, this theorem shows that the conjugacy growth of two quasi-isometric groups can be as far apart as possible. Note also that it is fairly easy to prove that for every finitely generated group G and a finite index subgroup $H \leq G$, one has $\xi_H \preceq \xi_G$.

The proofs of Theorems I.4.2 and I.4.3 will be accomplished by constructing groups which are direct limits of relatively hyperbolic groups. The main tool in constructing these sequences will be a more general version of Theorem I.3.2. While these results can be proved entirely using relatively hyperbolic groups, one still needs a stronger version of the small cancellation theorem than the one proved in [67] (see [44] for more details).

CHAPTER II

PRELIMINARIES

Notation We begin by standardizing the notation that we will use. Given a group G generated by a subset $S \subseteq G$, we denote by $\Gamma(G,S)$ the Cayley graph of G with respect to S. That is, $\Gamma(G,S)$ is the graph with vertex set G and an edge labeled by s between each pair of vertices of the form (g,gs), where $s \in S$. We let $|g|_S$ denote the *word length* of an element g with respect to S, that is $|g|_S$ is equal to the length of the shortest word in S which is equal to g in G. Similiarly, d_S will denote the *word metric* on G with respect to S, that is $d_S(h,g) = |h^{-1}g|_S$. Clearly $d_S(h,g)$ is the length of the shortest path in $\Gamma(G,S)$ from h to g. We denote the ball of radius n centered at the identity with respect to d_S by $B_{G,S}(n)$ (or simply $B_S(n)$ if omitting G does not lead to confusion); that is $B_{G,S}(n) = \{g \in G \mid |g|_S \leq n\}$. We will assume all generating sets are symmetric, that is $S = S \cup S^{-1}$. If p is a (combinatorial) path in $\Gamma(G,S)$, $\mathbf{Lab}(p)$ denotes its label, $\ell(p)$ denotes its length, p_- and p_+ denote its starting and ending vertex.

In general, we will allow metrics and length functions to take infinite value. For example, we will sometimes consider a word metric with respect to a subset Y which is not necesarily generating; in this case we set $d_Y(h,g) = \infty$ when $h^{-1}g \notin \langle Y \rangle$. Given two metric d_1 and d_2 on a set X, we say that d_1 is bi-Lipschitz equivalent to d_2 (and write $d_1 \sim_{Lip} d_2$) if for all $x,y \in X$, $d_1(x,y)$ is finite if and only if $d_2(x,y)$ is, and the ratios d_1/d_2 and d_2/d_1 are uniformly bounded on $X \times X$ minus the diagonal.

For a word W in an alphabet S, $\|W\|$ denotes its length. For two words U and V we write $U \equiv V$ to denote the letter-by-letter equality between them, and $U =_G V$ to mean that U and V both represent the same element of G. Clearly there is a one to one correspondence between words W in S and paths p in $\Gamma(G,S)$ such that $p_- = 1$ and $\mathbf{Lab}(p) \equiv W$.

The normal closure of a subset $K \subseteq G$ in a group G (i.e., the minimal normal subgroup of G containing K) is denoted by $\langle\langle K \rangle\rangle$. For group elements g and t, g^t denotes $t^{-1}gt$. We write $g \sim h$ if g is conjugate to h, that is there exists $t \in G$ such that $g^t = h$. We also say that g and h are *commensurable* if for some $n,k \in \mathbb{Z}$, $g^n \sim h^k$.

Van Kampen Diagrams. Recall that a *van Kampen diagram* Δ over a presentation

$$G = \langle \mathscr{A} \mid \mathscr{O} \rangle \tag{II.1}$$

is a finite, oriented, connected, simply–connected, planar 2–complex endowed with a labeling function $\mathbf{Lab}\colon E(\Delta) \to \mathscr{A}$, where $E(\Delta)$ denotes the set of oriented edges of Δ, such that $\mathbf{Lab}(e^{-1}) \equiv (\mathbf{Lab}(e))^{-1}$. Labels and lengths of paths are defined as in the case of Cayley graphs. Given a cell Π of Δ, we denote by $\partial\Pi$ the boundary of Π; similarly, $\partial\Delta$ denotes the boundary of Δ. The labels of $\partial\Pi$ and $\partial\Delta$ are defined up to a cyclic permutation. An additional requirement is that for any cell Π of Δ, the boundary label $\mathbf{Lab}(\partial\Pi)$ is equal to a cyclic permutation of a word $P^{\pm 1}$, where $P \in \mathscr{O}$. The van Kampen Lemma states that a word W over the alphabet $\mathscr{A}$ represents the identity in the group given by (II.1) if and only if there exists a diagram Δ over (II.1) such that $\mathbf{Lab}(\partial\Delta) \equiv W$ [49, Ch. 5, Theorem 1.1].

Remark II.0.4. For every van Kampen diagram Δ over (II.1) and any fixed vertex o of Δ, there is a (unique) combinatorial map γ: $Sk^{(1)}(\Delta) \to \Gamma(G, \mathscr{A})$ (where $Sk^{(1)}(\Delta)$ denotes the 1-skeleton of Δ) that preserves labels and orientation of edges and maps o to the vertex 1 of $\Gamma(G, \mathscr{A})$.

Hyperbolic spaces Here we will mention some well-known properties of hyperbolic spaces which will be used in the remaining chapters. A geodesic metric space X is called δ-*hyperbolic* if given any geodesic triangle T in X, each side of T is contain in the union of the closed δ-neighboorhoods of the other two sides. It is well-known that a space is hyperbolic if and only if it satisfies a linear isoperimetric inequality. This can be translated to the context of Cayley graphs of groups in the following way. A group presentation of G of the form (II.1) is called *bounded* if $\sup\{\|R\| \mid R \in \mathscr{O}\} < \infty$. Given a van Kampen diagram Δ over (II.1), let $Area(\Delta)$ denote the number of cells of Δ. Given a word W in $\mathscr{A}$ with $W =_G 1$, we let $Area(W) = \min_{\partial\Delta \equiv W}\{Area(\Delta)\}$, where the minimum is taken over all diagrams with boundary label W. The presentation (II.1) satisfies a linear isoperimetric inequality if there exists a contant L such that for all $W =_G 1$, $Area(W) \leq L\|W\|$. The following is well-known and can be easily derived from the results of Sec. 2, Ch. III.H in [16];

Theorem II.0.5. *Given a generating set $\mathscr{A}$ of a group G, the Cayley graph $\Gamma(G, \mathscr{A})$ is hyperbolic if and only if G has a bounded presentation of the form (II.1) which satisfies a linear isoperimetric inequality.*

Remark II.0.6. A group is called *hyperbolic* if for some finite $\mathscr{A}$, $\Gamma(G, \mathscr{A})$ is hyperbolic. In this case, a bounded presentation is necessarily finite. However, we will be interested in Cayley graphs $\Gamma(G, \mathscr{A})$ where $\mathscr{A}$ is infinite, which is why we work in this generality.

Given a subset T a geodesic metric space (X, d), we denote by $T^{+\sigma}$ the σ-neighborhood of T. T is called σ-*quasi-convex* if for any two elements $t_1, t_2 \in T$, any geodesic γ in X connecting t_1 and t_2 belongs to $T^{+\sigma}$. Let $\mathscr{Q} = \{Q_p\}_{p\in\Pi}$ be a collection of subsets of a metric space X. One says that $\mathscr{Q}$ is t-*dense* for $t \in \mathbb{R}_+$ if X coincides with the t-neighborhoods of $\bigcup_{p\in\Pi} Q_p$. Further $\mathscr{Q}$ is *quasi-dense* if it is t-dense for some $t \in \mathbb{R}_+$. Let us fix some positive constant c. A c-nerve of $\mathscr{Q}$ is a graph with the vertex set Π and with $p, q \in \Pi$ adjacent if and only if $d(Q_p, Q_q) \leq c$. Finally we recall that $\mathscr{Q}$ is *uniformly quasi-convex* if there exist σ such Q_p is σ-quasi-convex for any $p \in \Pi$. The lemma below is an immediate corollary of [12, Proposition 7.12].

Lemma II.0.7. *Let X be a hyperbolic space, and let $\mathscr{Q} = \{Q_p\}_{p\in\Pi}$ be a quasi-dense collection of uniformly quasi-convex subsets of X. Then there for any large enough c, the c-nerve of $\mathscr{Q}$ is hyperbolic.*

A path p in a metric space is called (λ, c)–*quasi–geodesic* for some $\lambda > 0$, $c \geq 0$, if

$$d(q_-, q_+) \geq \lambda l(q) - c$$

for any subpath q of p. p is called a k-local geodesic if any subpath of p of length at most k is geodesic.

Lemma II.0.8. *[16] Let r be a k-local geodesic in a δ-hyperbolic metric space for some $k > 8\delta$. Then r is a $(\frac{1}{3}, 2\delta)$-quasi-geodesic.*

The "thin triangle" condition can be easily translated to a "thin quadrangle" condition by simply drawing the diagonal and applying the thin triangle condition twice. The next lemma states this idea for quasi-geodesic quadrangles.

Lemma II.0.9. *[67, Corollary 3.3] For any $\delta \geq 0$, $\lambda > 0$, $c \geq 0$, there exists a constant $K = K(\delta, \lambda, c)$ with the following property. Let Q be a quadrangle in a δ–hyperbolic space whose sides are (λ, c)–quasi–geodesic. Then each side of Q belongs to the closed K–neighborhood of the union of the other three sides.*

The next lemma is a simplification of Lemma 10 from [64]. Here two paths p and q are called ε-close if either $d(p_-, q_-) \leq \varepsilon$ and $d(p_+, q_+) \leq \varepsilon$, or if $d(p_-, q_+) \leq \varepsilon$ and $d(p_+, q_-) \leq \varepsilon$.

Lemma II.0.10. *Suppose that the set of all sides of a geodesic n–gon $P = p_1 p_2 \ldots p_n$ in a δ–hyperbolic space is partitioned into two subsets A and B. Let ρ (respectively θ) denote the sum of lengths of sides from A (respectively B). Assume, in addition, that $\theta > \max\{\xi n, 10^3 \rho\}$ for some $\xi \geq 3\delta \cdot 10^4$. Then there exist two distinct sides $p_i, p_j \in B$ that contain 13δ-close segments of length greater than $10^{-3}\xi$.*

HNN-extensions and Amalgamated products. Given a group G containing two isomorphic subgroups A and B, the HNN-extension $G*_{A^t=B}$ is the group given by

$$G*_{A^t=B} = \langle G, t \mid t^{-1}at = \varphi(a),\ a \in A \rangle$$

where $\varphi \colon A \to B$ is an isomorphism. Recall that for a word W in the alphabet $\{G \setminus \{1\}, t\}$, a *pinch* (in the HNN-extension $G*_{A^t=B}$) is a subword of the form $t^{-1}at$ with $a \in A$ or tbt^{-1} with $b \in B$. a word $g_0 t^{\varepsilon_0} g_1 t^{\varepsilon_1} \ldots g_{n-1} t^{\varepsilon_{n-1}} g_n$, where each $g_i \in G$ and each $\varepsilon_i = \pm 1$, is called *reduced* if there are no pinches. Given such a word, we define its *t-length* as the number of occurrences of the letters t and t^{-1}. In $G*_{A^t=B}$, any pinch can be replaced by a single element of G. It follows that each element $w \in G*_{A^t=B}$ is equal to a reduced word. The converse to this statement is known as the Britton Lemma (see [49, Ch. 4, Sec.2]).

Lemma II.0.11 (Britton Lemma)**.** *Let W be a word in $\{G \setminus \{1\}, t\}$ with t-length at least 1 and no pinches. Then $W \neq 1$ in $G*_{A^t=B}$.*

An immediate consequence of this lemma is the well-known fact that G naturally embeds in the HNN-extension $G*_{A^t=B}$. We will use the following corollary of the Britton Lemma. Here a reduced word W is called *cyclicly reduced* if it is not conjugate to an element of shorter t-length, or equivalently, no cyclic shift contains a pinch.

Lemma II.0.12. *Let G be a group, and A, B isomorphic subgroups of G. Suppose that some $f \in G$ is not conjugate to any elements of $A \cup B$ in G. Then in the corresponding HNN-extension $G*_{A^t=B}$,*

1. *f is conjugate to another element $g \in G$ in $G*_{A^t=B}$ if and only if f and g are conjugate in G.*

2. *If f is primitive in G, then f is primitive in $G*_{A^t=B}$.*

Proof. Since f is not in A or B, if W is any reduced word then $W^{-1}fWg^{-1}$ contains no pinches. Thus, f is not conjugate to g. The second assertion will immediately follow if we can show that if $w \in G*_{A^t=B}$ such that $w^n \in G$, then either $w \in G$ or w^n is conjugate to an element of A or an element of B (here we identify G with its image in $G*_{A^t=B}$). To show this we induct on the t-length of the reduced form of w. If a reduced word representing w contains no t letters, then $w \in G$. Clearly $w^n \in G$ implies that w has even t-length, since the sum of the exponents of t letters must be 0. Suppose w has t-length 2. Then for some g_0, g_1, g_2, $\varepsilon \in \{0,1\}$, we have that $w = g_0t^{\varepsilon}g_1t^{-\varepsilon}g_2$. The Britton Lemma implies that $t^{-\varepsilon}g_2g_0t^{\varepsilon}$ must be a pinch or freely trivial, that is $t^{-\varepsilon}g_2g_0t^{\varepsilon} = h$ for some (possibly trivial) h in A or B. Without loss of generality, let $h \in A$, and note that this implies that $g_2g_0 \in B$. Then $w^n = g_0t^{\varepsilon}(g_1hg_1)^nt^{-\varepsilon}g_2$. Again, by the Britton Lemma $t^{\varepsilon}(g_1hg_1)^nt^{-\varepsilon}$ must be a pinch or trivial, and since the orientation of the t is reversed, we get that $t^{\varepsilon}(g_1hg_1)^nt^{-\varepsilon} = h'$ for some $h' \in B$. Finally, observe that $w^n = g_0h'g_2g_0g_0^{-1}$, thus $w^n \sim h'g_2g_0 \in B$.

Now suppose we have shown the above claim for all elements of $G*_{A^t=B}$ with shorter t-length then w. As before, $w = g_0t^{\varepsilon_0}...t^{-\varepsilon_0}g_n$, and $t^{-\varepsilon_0}g_ng_0t^{\varepsilon_0} = h$, for some $h \in A \cup B$. Let $u' = g_1t^{\varepsilon_1}...g_{n-1}h$. Now let u be a conjugate of u' which is cyclicly reduced. Since $u \sim u' = t^{-\varepsilon_0}g_0^{-1}wg_0t^{\varepsilon_0}$, we have that $u \sim w$ and so $u^n \sim w^n \in G$. Since u is cyclicly reduced, u^n is cyclicly reduced, hence $u^n \in G$. Since u has fewer t letters then w, by the inductive hypothesis, u^n (and thus w^n) is conjugate to an element of A or an element of B. □

Given groups A and B which contain isomorphic subgroups $K \leq A$ and $J \leq B$, the amalgamated product $A *_{K=J} B$ is the group given by

$$A *_{K=J} B = \langle A, B \mid a = \varphi(a),\ a \in K \rangle$$

where $\varphi\colon K \to J$ is an isomorphism. We will usually work with HNN-extensions, and when necessary translate these results to amalgamated products using the standard "retraction trick;" recall that a subgroup $H \leq G$ is called a *retract* if there exists a homomorphism $r\colon G \to H$ such that $r^2 = r$.

Theorem II.0.13. *[49] Let $P = A *_{K=J} B$, and let $G = (A * B)*_{K^t=J}$; that is, G is an HNN extension of the free product $A * B$. Then P is naturally isomorphic to the retract $\langle A^t, B \rangle \leq G$.*

CHAPTER III

ACYLINDRICALLY HYPERBOLIC GROUPS

III.1 Hyperbolically embedded subgroups

As we mentioned in the introduction, there are several equivalent characterizations of acylindrically hyperbolic groups. In this chapter we will present some of these characterizations and study the relationship between them. The first such characterization is the through the notion of hyperbolically embedded subgroups which was introduced in [24].

Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G. Set

$$\mathscr{H} = \bigsqcup_{\lambda\in\Lambda} (H_\lambda \setminus \{1\}). \tag{III.1}$$

Given a subset $X \subseteq G$ such that G is generated by X together with the union of all H_λ's, we denote by $\Gamma(G, X \sqcup \mathscr{H})$ the Cayley graph of G whose edges are labeled by letters from the alphabet $X \sqcup \mathscr{H}$. Note that some letters from $X \sqcup \mathscr{H}$ may represent the same element in G, in which case $\Gamma(G, X \sqcup \mathscr{H})$ has multiple edges corresponding to these letters.

We think of the Cayley graphs $\Gamma(H_\lambda, H_\lambda \setminus \{1\})$ as (complete) subgraphs of $\Gamma(G, X \sqcup \mathscr{H})$. For every $\lambda \in \Lambda$, we introduce a *relative metric* $\widehat{\mathrm{d}}_\lambda : H_\lambda \times H_\lambda \to [0, +\infty]$ as follows. Given $h, k \in H_\lambda$, let $\widehat{\mathrm{d}}_\lambda(h,k)$ be the length of a shortest path in $\Gamma(G, X \sqcup \mathscr{H})$ that connects h to k and has no edges in $\Gamma(H_\lambda, H_\lambda \setminus \{1\})$ (such paths are called *admissible*). If no such a path exists, we set $\widehat{\mathrm{d}}_\lambda(h,k) = \infty$. Clearly $\widehat{\mathrm{d}}_\lambda$ satisfies the triangle inequality. It is convenient to extend the metric $\widehat{\mathrm{d}}_\lambda$ the whole group G by assuming $\widehat{\mathrm{d}}_\lambda(f,g) \colon = \widehat{\mathrm{d}}_\lambda(f^{-1}g, 1)$ if $f^{-1}g \in H_\lambda$ and $\widehat{\mathrm{d}}_\lambda(f,g) = \infty$ otherwise. In case the collection consists of a single subgroup $H \leq G$, we denote the corresponding relative metric on H simply by $\widehat{d}$.

Definition III.1.1. Let G be a group, X a (not necessary finite) subset of G. We say that a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of G is *hyperbolically embedded in G with respect to X* (we write $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$) if the following conditions hold.

(a) The group G is generated by X together with the union of all H_λ's and the Cayley graph $\Gamma(G, X \sqcup \mathscr{H})$ is hyperbolic.

(b) For every $\lambda \in \Lambda$, $(H_\lambda, \widehat{\mathrm{d}}_\lambda)$ is a locally finite metric space; that is, any ball of finite radius in H_λ contains finitely many elements.

Further we say that $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in G and write $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$ if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$ for some $X \subseteq G$.

Examples III.1.2. (a) Let G be any group. Then $G \hookrightarrow_h G$. Indeed take $X = \emptyset$. Then the Cayley graph $\Gamma(G, X \sqcup H)$ has diameter 1 and $\widehat{d}(h_1, h_2) = \infty$ whenever $h_1 \neq h_2$. Further, if H is a finite subgroup of a group G, then $H \hookrightarrow_h G$. Indeed $H \hookrightarrow_h (G,X)$ for $X = G$. These cases are referred to as *degenerate*. In what follows we are only interested in non-degenerate examples.

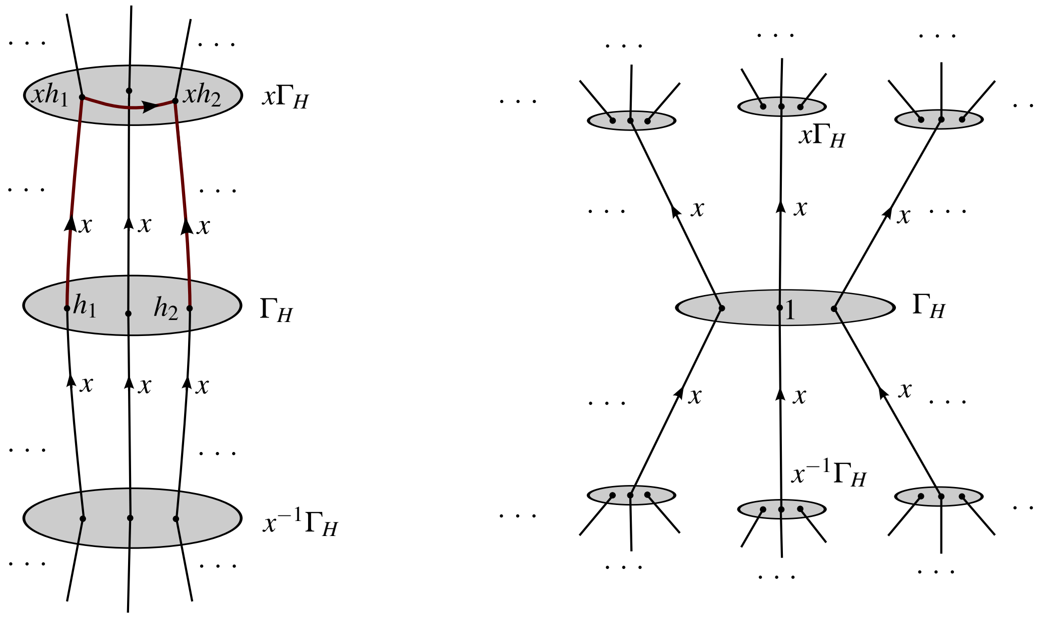

Figure III.1: Cayley graphs $\Gamma(G, X \sqcup H)$ for $G = H \times \mathbb{Z}$ and $G = H * \mathbb{Z}$.

(b) Let $G = H \times \mathbb{Z}$, $X = \{x\}$, where x is a generator of $\mathbb{Z}$. Then $\Gamma(G, X \sqcup H)$ is quasi-isometric to a line and hence it is hyperbolic. However $\widehat{d}(h_1, h_2) \leq 3$ for every $h_1, h_2 \in H$. Indeed let Γ_H denote the Cayley graph $\Gamma(H, H \setminus \{1\})$. In the shift $x\Gamma_H$ of Γ_H there is an edge (labeled by $h_1^{-1}h_2 \in H$) connecting h_1x to h_2x, so there is a path of length 3 connecting h_1 to h_2 and having no edges in Γ_H (see Fig. III.1). Thus if H is infinite, then $H \not\hookrightarrow_h (G, X)$. Moreover, a similar argument shows that $H \not\hookrightarrow_h G$.

(c) Let $G = H * \mathbb{Z}$, $X = \{x\}$, where x is a generator of $\mathbb{Z}$. In this case $\Gamma(G, X \sqcup H)$ is quasi-isometric to a tree (see Fig. III.1) and $\widehat{\mathrm{d}}_\lambda(h_1, h_2) = \infty$ unless $h_1 = h_2$. Thus $H \hookrightarrow_h (G, X)$.

The group G can also be regarded as a quotient group of the free product

$$F = (*_{\lambda \in \Lambda} H_\lambda) * F(X), \tag{III.2}$$

where $F(X)$ is the free group with the basis X. Let N denote the kernel of the natural homomorphism $F \to G$. If N is the normal closure of a subset $\mathscr{Q} \subseteq N$ in the group F, we say that G has *relative presentation*

$$\langle X, \mathscr{H} \mid \mathscr{Q} \rangle. \tag{III.3}$$

The relative presentation (III.3) is said to be *bounded* if $\sup\{\|R\| \mid R \in \mathscr{Q}\} < \infty$. Furthermore, it is called *strongly bounded* if in addition the set of letters from $\mathscr{H}$ which appear in relators $R \in \mathscr{Q}$ is finite.

Given a word W in the alphabet $X \sqcup \mathscr{H}$ such that W represents 1 in G, there exists an expression

$$W =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i \tag{III.4}$$

with the equality in the group F, where $R_i \in \mathscr{Q}$ and $f_i \in F$ for $i = 1, \ldots, k$. The smallest possible number k

in a representation of the form (III.4) is called the *relative area* of W and is denoted by $Area^{rel}(W)$.

Theorem III.1.3. *[24, Theorem 4.24] The subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ are* hyperbolically embedded in G with respect to X *if and only if there exists a strongly bounded relative presentation for G with respect to X and $\{H_\lambda\}_{\lambda\in\Lambda}$ and there is a constant $L > 0$ such that for any word W in $X \sqcup \mathscr{H}$ representing the identity in G, we have $Area^{rel}(W) \le L\|W\|$.*

Observe that the relative area of a word W representing 1 in G can be defined geometrically via van Kampen diagrams. Let G be a group given by the relative presentation (III.3) with respect to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. We denote by $\mathscr{S}$ the set of all words in the alphabet $\mathscr{H}$ representing the identity in the group F defined by (III.2). Then G has the ordinary (non–relative) presentation

$$G = \langle X \cup \mathscr{H} \mid \mathscr{S} \cup \mathscr{Q} \rangle. \tag{III.5}$$

A cell in van Kampen diagram Δ over (III.5) is called a *$\mathscr{Q}$–cell* if its boundary is labeled by a word from $\mathscr{Q}$. We denote by $N_{\mathscr{Q}}(\Delta)$ the number of $\mathscr{Q}$–cells of Δ. Obviously given a word W in $X \sqcup \mathscr{H}$ that represents 1 in G, we have

$$Area^{rel}(W) = \min_{\mathbf{Lab}(\partial\Delta)\equiv W} \{N_{\mathscr{Q}}(\Delta)\},$$

where the minimum is taken over all van Kampen diagrams with boundary label W. Thus, $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$ if G has a strongly bounded presentation with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ and all van Kampen diagrams over (III.5) satisfy a linear relative isoperimetric inequality.

In fact, hyperbolically embedded subgroups can be thought of as a natural generalization of relative hyperbolicity. Indeed, when this isoperimetric characterization of hyperbolically embedded subgroups is compared with Osin's isoperimetric characterization of relatively hyperbolic groups, the following theorem becomes obvious.

Theorem III.1.4. *[24] Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a collection of subgroups of a group G. Then G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ if and only if there exists a finite set $X \subset G$ such that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$.*

Based on this theorem, many basic properties of relatively hyperbolic groups can be translated to analogous results for groups with hyperbolically embedded subgroups. The following lemmas are examples of this process.

Lemma III.1.5. *[24] Suppose $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$. Then for all $g \in G$, the following hold:*

1. *If $g \notin H_\lambda$, then $|H_\lambda \cap H_\lambda^g| < \infty$.*
2. *If $\lambda \neq \mu$, then $|H_\lambda \cap H_\mu^g| < \infty$.*

Lemma III.1.6. *[24] Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups, and X_1, $X_2 \subseteq G$ relative generating sets of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ such that $|X_1 \triangle X_2| < \infty$. Then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X_1)$ if and only if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X_2)$.*

The following two lemmas are simplifications of [24, proposition 4.35] and [24, proposition 4.35] respectively.

Lemma III.1.7. *[24] Suppose $\{H_i\}_{i=1}^n \hookrightarrow_h G$, and $\{K_j\}_{j=1}^m \hookrightarrow_h H_1$. Then $\{H_i\}_{i=2}^n \cup \{K_j\}_{j=1}^m \hookrightarrow_h G$.*

Lemma III.1.8. *[24] If $H \hookrightarrow_h G$, then for any $t \in G$, $H^t \hookrightarrow_h G$.*

Lemma III.1.9. *[24] Let $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$. Then for each $\lambda \in \Lambda$, there exists a finite subset $Y_\lambda \subseteq H_\lambda$ such that $\widehat{\mathrm{d}}_\lambda$ is bi-Lipschitz equivalent to the word metric with respect to Y_λ.*

Recall that $K(G)$ denotes the maximal finite normal subgroup of a group G; [24] shows that such a subgroup exists for any group containing a non-degenerate hyperbolically embedded subgroup. Using Lemma III.1.5, it is not hard to show that for any $H \hookrightarrow_h G$, $K(G) \leq H$.

Lemma III.1.10. *[24] Suppose G contains a non-degenerate hyperbolically embedded subgroup. Then for all $n \geq 1$, G contains a free subgroup F_n of rank n such that $F_n \times K(G) \hookrightarrow_h G$.*

Components. Let $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$. Let q be a path in the Cayley graph $\Gamma(G, X \sqcup \mathscr{H})$. A (non-trivial) subpath p of q is called an H_λ-*subpath*, if the label of p is a word in the alphabet $H_\lambda \setminus \{1\}$. An H_λ-subpath p of q is an H_λ-*component* if p is not contained in a longer H_λ-subpath of q; if q is a loop, we require in addition that p is not contained in any longer H_λ-subpath of a cyclic shift of q. Further by a *component* of q we mean an H_λ-component of q for some $\lambda \in \Lambda$. If q is an H_λ-component of some path, then we define $\widehat{\ell}_\lambda(q) = \widehat{\mathrm{d}}_\lambda(q_-, q_+)$; similarly, for any $h \in H_\lambda$, $\widehat{\ell}_\lambda(h) = \widehat{\mathrm{d}}_\lambda(1,h)$).

Two H_λ-components p_1, p_2 of a path q in $\Gamma(G, X \sqcup \mathscr{H})$ are called *connected* if there exists a path c in $\Gamma(G, X \sqcup \mathscr{H})$ that connects some vertex of p_1 to some vertex of p_2, and $\mathbf{Lab}(c)$ is a word consisting only of letters from $H_\lambda \setminus \{1\}$. In algebraic terms this means that all vertices of p_1 and p_2 belong to the same left coset of H_λ. Note also that we can always assume that c has length at most 1 as every non-trivial element of H_λ is included in the set of generators. We say that a component p of a path q is *isolated* in q if p is not connected to any other components of q.

One important tool is the following, which is an easy consequence of [24] (see also [45, Lemma 2.4]). Recall that a (λ, c) quasi-geodesic n-gon is a closed path $p = p_1 p_2 ... p_n$ such that each p_i is a (λ, c) quasi-geodesic.

Lemma III.1.11. *There exists a constant $C = C(\lambda, c) > 0$ such that for any (λ, c) quasi-geodesic n-gon p in $\Gamma(G, X \sqcup \mathscr{H})$ and any isolated H_λ component a of p, we have $\widehat{\ell}_\lambda(a) \leq Cn$.*

III.2 Acylindrical and WPD actions

Recall the definition of an acylindrical action mentioned in Chapter I.

Definition III.2.1. Let G be a group acting by isometries on a metric space (X,d). We say that the action is *acylindrical* if for for all ε there exists $R > 0$ and $N > 0$ such that for all $x, y \in X$ with $d(x,y) \geq R$, the set

$$\{g \in G \mid d(x, gx) \leq \varepsilon, d(y, gy) \leq \varepsilon\}$$

contains at most N elements.

Given a group G acting on a hyperbolic metric space (X,d) and $g \in G$, the *translation length* of g is defined as $\tau(g) = \lim_{n\to\infty} \frac{1}{n} d(x, g^n x)$ for some (equivalently, any) $x \in X$. g is called *loxodromic* if $\tau(g) > 0$. Equivalently, an element is loxodromic if it admits an invariant, bi-infinte quasi-geodesic axis on which it acts as a non-trivial translation. An element g is called *elliptic* if some (equivalently, any) $\langle g\rangle$-orbit is bounded.

The next lemma characterizes acylindric isometries of hyperbolic metric spaces and is due to Bowditch.

Lemma III.2.2. *[13] Suppose G acts acylindrically on a hyperbolic metric space. Then every element of G is either elliptic or loxodromic.*

In fact, this result can be generalized to describe the actions of subgroups of a group acting acylindrically on a hyperbolic metric space. Recall that if g is loxodromic, then the orbit of g has exactly two limit points $\{g^{\pm\infty}\}$ on the boundary ∂X. Loxodromic elements g and h are called *independent* if the sets $\{g^{\pm\infty}\}$ and $\{h^{\pm\infty}\}$ are disjoint.

Theorem III.2.3. *[65] Suppose G acts acylindrically on a hyperbolic metric space. Then G satisfies exactly one of the following:*

1. *G has bounded orbits.*
2. *G is virtually cyclic and contains a loxodromic element.*
3. *G contains infinitely many pairwise independent loxodromic elements.*

Notice that the last condition holds if and only if the action of G is non-elementary.

In [11], Bestvina-Fujiwara defined a weak form of acylindricity, which they called *weak proper discontinuity condition*.

Definition III.2.4. [11] Let G be a group acting on a hyperbolic metric space X, and h a loxodromic element of G. We say h satisfies the *weak proper discontinuity condition* (or h is a *WPD element*) if for all $\kappa > 0$ and $x \in X$, there exists N such that

$$|\{g \in G \mid d(x, gx) < \kappa, d(h^N x, gh^N x) < \kappa\}| < \infty. \tag{III.6}$$

Note that if G acts acylinrically on a hyperbolic metric space, then every loxodromic element satisfies the WPD condition.

Using hyperbolicity, it suffices to verify the WPD condition with κ equal to some multiple of the hyperbolicity constant. The proof is the same as [24, Proposition 3.6].

Lemma III.2.5. *Let G be a group acting on a δ-hyperbolic metric space X, and h a loxodromic element of G. If for $\kappa = 100\delta$ and for all $x \in X$ there exists N such that (III.6) is satisfied, then h is a WPD element.*

Lemma III.2.6. *[24] Let G be a group acting on a hyperbolic metric space X, and let h be a loxodromic WPD element. Then h is containted in a unique, maximal elementary subgroup of G, called the* elementary closure *of h and denoted $E_G(h)$. Furthermore, for all $g \in G$, the following are equivalent:*

1. *$g \in E_G(h)$*
2. *There exists $n \in \mathbb{N}$ such that $g^{-1}h^n g = h^{\pm n}$*
3. *There exists k, $m \in \mathbb{Z} \setminus \{0\}$ such that $g^{-1}h^k g = h^m$*

Further, for some $r \in \mathbb{N}$,

$$E_G^+(h) = \{g \in G \mid \exists n \in \mathbb{N}, g^{-1}h^n g = h^n\} = C_G(h^r)$$

An element g of a group G is called *primitive* if g is not a proper power of an other element of G; that is, the equation $x^n = g$ has a solution in G if and only if $n = \pm 1$.

Corollary III.2.7. *Let f and g be primitive, loxodromic WPD elements in a torsion free group. Then f is commensurable with g if and only if $f^{\pm 1} \sim g$.*

Proof. If $f^k = (g^l)^x$, then $\langle f \rangle = E_G(f) = E_G(g^x) = \langle g^x \rangle$, thus $f^{\pm 1} = g^x$. □

The connection between WPD elements and hyperbolically embedded subgroups is given by the following results of [24].

Lemma III.2.8. *[24] Suppose G acts on a hyperbolic metric space X and h_1,...,h_n, is a collection of non-commensurable loxodromic WPD elements. Then $\{E_G(h_1), \ldots, E_G(h_n)\} \hookrightarrow_h G$.*

Also, when a subgroup $H \hookrightarrow_h (G,X)$, then there are usually many loxodromic, WPD elements with respect to the action of G on $\Gamma(G, X \sqcup H)$. The lemma shows explicitly how to find such elements.

Lemma III.2.9. *[24] Suppose $H \hookrightarrow_h (G,X)$ is non-degenerate and finitely generated. Then for all $g \in G$, there exist $h_1, \ldots, h_k$ such that $gh_1, \ldots, gh_k$ is a collection of non-commensurable, loxodromic WPD elements with respect to the action of G on $\Gamma(G, X \sqcup H)$. Moreover, if H contains an element of infinite order h, then each h_i can be chosen to be a power of h.*

The next theorem is a recent result of Osin which shows that hyperbolically embedded subgroups can be used to build aclyindrical actions.

Theorem III.2.10. *[65]*

Let G be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a finite collection of non-degenerate subgroups of G, X a subset of G such that $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G,X)$. Then there exists $Y \subseteq G$ such that $X \subseteq Y$ and the following conditions hold:

1. *$\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G,Y)$. In particular, $\Gamma(G, Y \sqcup \mathscr{H})$ is hyperbolic.*
2. *The action of G on $\Gamma(G, Y \sqcup \mathscr{H})$ is acylindrical.*

This action will also be non-elementary by Lemma III.2.9 and Theorem III.2.3. Since a non-elementary, aclyindric action on a hyperbolic metric space must contain loxodromic elements, combining Lemma III.2.8 and Theorem III.2.10 gives:

Corollary III.2.11. *A group G belongs to $\mathscr{A}\mathscr{H}$ if and only if G contains a non-degenerate hyperbolically embedded subgroup.*

As a consequence of this result and Theorem III.2.10, we can always choose the metric space from the definition of $\mathscr{A}\mathscr{H}$ to be a Cayley graph of G with respect to some (possibly infinite) generating set.

We will occasionally want to modify the Cayley graph on which G is acting without changing which element are loxodromic. The next two lemmas show how to do this.

Lemma III.2.12. *Suppose h is loxodromic with respect to the action of G on $\Gamma(G,\mathscr{A}_1)$ and $\mathscr{A} \subset \mathscr{A}_1$ generates G. Then h is a loxodromic with respect to the action of G on $\Gamma(G,\mathscr{A})$.*

Proof.

$$\lim_{n\to\infty}\frac{1}{n}d_{\mathscr{A}}(x,h^nx) \geq \lim_{n\to\infty}\frac{1}{n}d_{\mathscr{A}_1}(x,h^nx) > 0.$$

□

Lemma III.2.13. *Suppose G act acylindrically on $\Gamma(G,\mathscr{A})$, and B is a bounded subset of $\Gamma(G,\mathscr{A})$. Then the action of G on $\Gamma(G,\mathscr{A}\cup B)$ is acylindric and both actions have the same set of loxodromic elements.*

Proof. $\Gamma(G,\mathscr{A})$ and $\Gamma(G,\mathscr{A}\cup B)$ are quasi-isometric; all conditions are cleary preserved under quasi-isometries. □

III.3 A hyperbolic embeddability criterion

Next we will prove a criterion for a subgroup to be hyperbolically embedded with respect to a given generating set. This result is, in some sense, a weaker version of [24, Theorem 4.42]; However, in this theorem it is only shown that the subgroups are hyperbolically embedded with respect to some relative generating set, while we will need that the subgroups are hyperbolically embedded with respect to a specific generating set. It should be possible to repeat the proof of [24, Theorem 4.42] and keep track of the relative generating set produced there, but this would require quite a bit of technical detail and for our purposes a direct proof is easier.

An important part of our criterion is the notion of a collection of geometrically separated subgroups, which is inspired by the Bestvina-Fujiwara WPD condition.

Definition III.3.1. [24] Let G be a group acting on a metric space (X,d). A collection of subgroup $\{H_\lambda\}_{\lambda\in\Lambda} \leq G$ is called *geometrically separated* if for all $\varepsilon \geq 0$ and $x \in X$, there exists $R > 0$ such that the following holds. Suppose that for some $g \in G$ and some λ, $\mu \in \Lambda$,

$$diam(H_\mu(x) \cap (gH_\lambda(x))^{+\varepsilon}) \geq R$$

Then $\lambda = \mu$ and $g \in H_\lambda$.

Theorem III.3.2. *Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a finite collection of subgroup of G. Suppose that the following conditions hold.*

(a) G is generated by a (possibly infinite) set X such that $\Gamma(G,X)$ is hyperbolic.

(b) For every $\lambda \in \Lambda$, H_λ is quasi-convex in $\Gamma(G,X)$.

(c) $\{H_\lambda\}_{\lambda\in\Lambda}$ is geometrically separated.

Then the relative Cayley graph $\Gamma(G,X\sqcup\mathcal{H})$ is hyperbolic and there exists $C>0$ such that for every $\lambda\in\Lambda$, we have $\widehat{\mathrm{d}}_\lambda \sim_{Lip} \mathrm{d}_{\Omega_\lambda}$, where $\Omega_\lambda = \{h\in H_\lambda \mid |h|_X \le C\}$.

In particular, if every H_λ is locally finite with respect to d_X, then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$.

Proof. Let us first show that the graph $\Gamma(G,X\sqcup\mathcal{H})$ is hyperbolic. Let $\mathcal{Q}$ be the collection of all left cosets of subgroups H_λ, $\lambda\in\Lambda$. We think of $\mathcal{Q}$ as a collection of subsets of $\Gamma(G,X)$. Since Λ is finite and every H_λ is quasi-convex in (G,X), $\mathcal{Q}$ is uniformly quasi-convex. Clearly $\mathcal{Q}$ is quasi-dense. Hence by Lemma II.0.7 there exists $c\ge 1$ such that the c-nerve of $\mathcal{Q}$ is hyperbolic. Let Σ denote the nerve, and let $\widehat{\Gamma}$ be the conned-off graph of G with respect to X and $\{H_\lambda\}_{\lambda\in\Lambda}$. That is, $\widehat{\Gamma}$ is the graph obtained from $\Gamma(G,X)$ by adding one vertex v_{gH_λ} for each left coset of each subgroup H_λ and then adding an edge between v_{gH_λ} and each vertex of gH_λ.

Let d_Σ and $d_{\widehat{\Gamma}}$ denote the natural path metrics on Σ and $\widehat{\Gamma}$ respectively. It is easy to see that Σ and $\widehat{\Gamma}$ are quasi-isometric. Indeed let $\iota\colon V(\Sigma)\to V(\widehat{\Gamma})$ be the map which sends $gH_\lambda\in\mathcal{Q}$ to v_{gH_λ}. If $u,v\in V(\Sigma)$ are connected by an edge in Σ, then there exist elements g_1,g_2 of the cosets corresponding to u and v such that $d_X(g_1,g_2)\le c$ in $\Gamma(G,X)$. This implies that $d_{\widehat{\Gamma}}(\iota(u),\iota(v))\le c+2$. Hence $d_{\widehat{\Gamma}}(\iota(u),\iota(v))\le (c+2)d_\Sigma(u,v)$ for any $u,v\in V(\Sigma)$. On the other hand, it is straightforward to check that ι does not decrease the distance. Note that $\iota(V(\Sigma))$ is 1-dense in $\widehat{\Gamma}$. Thus ι extends to a quasi-isometry between Σ and $\widehat{\Gamma}$.

Further observe that $\widehat{\Gamma}$ is quasi-isometric to $\Gamma(G,X\sqcup\mathcal{H})$. Indeed the identity map on G induces an isometric embedding $V(\Gamma(G,X\sqcup\mathcal{H}))\to\widehat{\Gamma}$ whose image is 1-dense in $\widehat{\Gamma}$. Thus Σ is quasi-isometric to $\Gamma(G,X\sqcup\mathcal{H})$ and hence $\Gamma(G,X\sqcup\mathcal{H})$ is hyperbolic.

Now choose σ such that $\mathcal{Q}$ is σ-uniformly quasi-convex, fix $\lambda\in\Lambda$, and let p be an admissible path in $\Gamma(G,X\sqcup\mathcal{H})$ from h to h' such that $\ell(p)=\widehat{\mathrm{d}}_\lambda(h,h')$, where $h,\ h'\in H_\lambda$. Let e represent the H_λ-edge from h to h' in $\Gamma(G,X\sqcup\mathcal{H})$, and let c be the cycle pe^{-1}. Note that c has two types of edges; those labeled by elements of X and those labeled by elements of $\mathcal{H}$. Now for each edge of c labeled by an element of $\mathcal{H}$, we can replace this edge with a shortest path in X with the same endpoints. This produces a cycle c' which lives in $\Gamma(G,X)$. We consider $c'=q_1q_2...q_n$ as a geodesic n-gon where the sides conside of two types:

1. single edges of c' which represent X-edges of c.

2. geodesics in X which represent $\mathcal{H}$-edges of c.

Let q_n be the X geodesic which represents the edge e^{-1}. We will first show that $\ell(q_n)$ is bounded in terms of $\ell(p)$. Partion the sides of c' into A and B, where A consists of sides of the first type and B consists of sides of the second type. As in Lemma II.0.10, let Let ρ (respectively θ) denote the sum of lengths of sides from A (respectively B). Note that $n = \ell(c) = \ell(p)+1$, $\rho \le \ell(p)$, and $\theta \le \ell(q_n)$. Let R be the constant given by the definition of geometric separated subgroups for $\varepsilon = 13\delta + 2\sigma$. Choose $\xi = \max\{10^3(R+2\sigma), 10^3\ell(p), 3\delta \cdot 10^4(\ell(p)+1)\}$.

Suppose $\ell(q_n) \ge \xi$. Then we can apply Lemma II.0.10 to find two distinct B-sides, q_i and q_j of c' which contain 13δ-close segments of length at least $10^{-3}\xi \ge R+2\sigma$. This means that there exist vertices u_1, u_2 on q_i and v_1, v_2 on q_j, and paths s_1 and s_2 in $\Gamma(G,X)$ such that for $i = 1,2$, we have that $(s_i)_- = u_i$, $(s_i)_+ = v_i$, and $\ell(s_i) \le 13\delta$. We assume $i < j$, and let $g = \mathbf{Lab}(q_1...q_{i-1})$ and $g' = \mathbf{Lab}(q_1...q_{j-1})$ if $j < n$ and $g' = 1$ otherwise. Then $(q_i)_-, (q_i)_+ \in gH_\mu$ for some $\mu \in \Lambda$, and thus q_i belongs to the σ-neighborhood of gH_μ. Similarly, $(q_j)_-, (q_j)_+ \in g'H_\eta$ for some $\eta \in \Lambda$, and thus q_j belongs to the σ-neighborhood of $g'H_\eta$. Now for $i = 1,2$, choose vertices $u_i' \in gH_\mu$ such that $d_X(u_i,u_i') \le \sigma$ and $v_i' \in g'H_\eta$ such that $d_X(v_i,v_i') \le \sigma$. It follows that $d_X(u_i',v_i') \le 13\delta + 2\sigma = \varepsilon$. Also, $d_X(u_1',u_2') \ge (R+2\sigma) - 2\sigma = R$. Thus, by the definition of geometric separability, $\mu = \eta$ and $gH_\mu = g'H_\mu$.

Now, let h_i, h_j be the $\mathscr{H}$-edges of c corresponding to q_i, q_j. We have shown that these two edges belong to the same H_μ coset; hence, there exists and an edge f in $\Gamma(G, X \sqcup \mathscr{H})$ such that $f_- = (h_i)_-$ and $f_+ = (h_j)_+$. If $j < n$, we can replace the subpath of p from $(h_i)_-$ to $(h_j)_+$ by the single edge f, resulting in a shorter admissible path from h to h', which contradicts our assumption that $\ell(p) = \widehat{\mathrm{d}}_\lambda(h,h')$. If $j = n$, we get that $\mu = \lambda$ and $h_i \in gH_\lambda = g'H_\lambda = H_\lambda$, which violates the definition of an admissible path. Therefore, we conclude that $\ell(q_n) < \xi \le D\ell(p)$, where $D = \max\{10^3(R+2\sigma), 6\delta \cdot 10^4\}$.

Now denote the vertices of q_n by $h = v_0, v_1, ..., v_m = h'$. For each v_i, we can choose $h_i \in H_\lambda$ such that $d_X(v_i,h_i) \le \sigma$. It follows that $d_X(h_i,h_{i+1}) \le 2\sigma + 1$. Let $C = 2\sigma + 1$ and define Ω_λ accordingly. Note that

$$h^{-1}h' = (h^{-1}h_1)(h_1^{-1}h_2)...(h_{m-1}^{-1}h').$$

Since each $h_i^{-1}h_{i+1} \in \Omega_\lambda$, we have that $\mathrm{d}_{\Omega_\lambda}(h,h') \le m = \ell(q_n) \le D\ell(p) = D\widehat{\mathrm{d}}_\lambda(h,h')$. Finally, it is clear that $d_X(h,h') \le C\mathrm{d}_{\Omega_\lambda}(h,h')$. Since any path labeled only by X is admissible in $\Gamma(G, X \sqcup \mathscr{H})$, we get that $\widehat{\mathrm{d}}_\lambda(h,h') \le d_X(h,h') \le C\mathrm{d}_{\Omega_\lambda}(h,h')$, and thus $\widehat{\mathrm{d}}_\lambda \sim_{Lip} \mathrm{d}_{\Omega_\lambda}$.

□

Our main application of Theorem III.3.2 is due to the fact that all of the assumptions are satisfied by the elementary closures of a collection of pairwise non-commensurable loxodromic WPD elements; this is shown in the proof of [24, Theorem 6.8]. Thus, we have the following corollary.

Corollary III.3.3. *Suppose X is a generating set of G such that $\Gamma(G,X)$ is hyperbolic and $\{g_1,...,g_n\}$ is a collection of pairwise non-commensurable loxodromic WPD elements with respect to the action of G on $\Gamma(G,X)$. Then $\{E_G(g_1),...,E_G(g_n)\} \hookrightarrow_h (G,X)$.*

Remark III.3.4. We will only make use of the special case of Corollary III.3.3; however, the proof of Theorem III.3.2 is essentially the same as the proof of the special case, and we believe the more general statement may be of independent interest.

Finally, we will sometimes need to control which elements of G are elliptic.

Lemma III.3.5. *Let $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G,X)$, and let $a_1,...,a_m \in G$. Then there exists $Y \supseteq X$ such that*

1. $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,Y)$

2. *For each* $i = 1,..,m$, a_i *is elliptic with respect to the action of* G *on* $\Gamma(G, Y \sqcup \mathscr{H})$.

Proof. Clearly it suffices to prove the case when $m = 1$ and the general case follows by induction. By Theorem III.2.10 we can choose relative generating set $Y_0 \supset X$ such that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,Y_0)$ and G acts acylindrically on $\Gamma(G,Y_0 \cup \mathscr{H})$. If a is elliptic with respect to this action, we are done. Thus, by Lemma III.2.2 we can assume that a is loxodromic. Since the action is acylindric, all loxodromic elements satisfy WPD, so by Corollary III.3.3, $E_G(a) \hookrightarrow_h (G,Y_0 \cup \mathscr{H})$.

We claim that in fact, $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,Y_0 \sqcup E_G(a))$. Clearly the relative Cayley graph is hyperbolic, so we only need to verify that the relative metrics are locally finite. Fix $\lambda \in \Lambda$ and $n \in \mathbb{N}$. Let p be an admissible path of length at most n, and suppose p is the shortest admissible path between its endpoints. Let c be the cycle pe, were e is the edge in H_λ connecting the endpoints of p. If $x \in E_G(a)$ is the label of an edge of p, then x must be isolated in c; indeed e is not $E_G(a)$ component, and x cannot be connected to another component of p or we would have a shorter admissible path with the same endpoints. Thus by Lemma III.1.11, $\widehat{\ell}(x) \le C(n+1)$, where $C = C(1,0)$ is the constant from Lemma III.1.11. Since $E_G(a)$ is locally finite with respect to $\widehat{\ell}$, there is a finite set $\mathscr{F} \subset E_G(a)$ such that the label of any edge of any shortest admissible path of length at most n belongs to $Y_0 \sqcup \mathscr{F} \sqcup \mathscr{H}$. Thus, we may consider p as a path in $\Gamma(G,Y_0 \sqcup \mathscr{F} \sqcup \mathscr{H})$. Furthermore, by Lemma III.1.6, $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,Y_0 \sqcup \mathscr{F})$. Thus there are only finitely many elements of H which can be connected by a shortest admissible path of length at most n. Thus, H_λ is locally finite with respect to $\widehat{d_\lambda}$. It only remains to set $Y = Y_0 \sqcup E_G(a)$; clearly every $\langle a \rangle$ orbit has diameter 1 in $\Gamma(G,Y \sqcup \mathscr{H})$. □

CHAPTER IV

QUASI-COCYCLES

For a discrete group G, by a *normed G-module* we mean a normed vector space V (over some subfield of $\mathbb{C}$) endowed with a (left) action of the group G by isometries. Given a subgroup $H \leq G$, by an H-submodule of a G-module V we mean any H-invariant subspace of V with the induced action of H.

Let V be a normed G-module. A map $q\colon G \to V$ is called a *quasi-cocycle* if there exists a constant $\varepsilon > 0$ such that for every $f, g \in G$ we have

$$\|q(fg) - q(f) - fq(g)\| \leq \varepsilon.$$

The vector space of all quasi-cocycles on G with values in V is denoted by $QZ^1(G,V)$.

Recall that a quasi-cocycle $q \in QZ^1(G,V)$ is called *anti-symmetric* if

$$q(g^{-1}) = -g^{-1}q(g)$$

for every $g \in G$.

For a quasi-cocycle $q \in QZ^1(G,V)$ we define its *defect* $D(q)$ by

$$D(q) = \sup_{f,g \in G} \|q(fg) - q(f) - fq(g)\|. \tag{IV.1}$$

Note that

$$\|q(1)\| = \|q(1 \cdot 1) - q(1) - 1q(1)\| \leq D(q). \tag{IV.2}$$

We will use the following elementary fact.

Lemma IV.0.6. *Let G be a group, V a G-module. Then there exists a linear map*

$$\alpha\colon QZ^1(G,V) \to QZ^1_{as}(G,V)$$

such that for every $q \in QZ^1(G,V)$ we have

$$\sup_{g \in G} \|\alpha(q)(g) - q(g)\| < D(q).$$

Proof. Take $\alpha(q)(g) = \frac{1}{2}(q(g) - gq(g^{-1}))$. Verifying all properties is straightforward. Indeed for every $g \in G$, we have

$$\|\alpha(q)(g) - q(g)\| = \frac{1}{2}\| - q(g) - gq(g^{-1})\| \leq \frac{1}{2}\|q(1) - q(g) - gq(g^{-1})\| + \frac{1}{2}\|q(1)\| \leq D(q),$$

where the last inequality uses (IV.2). Further,

$$\alpha(q)(g^{-1}) = \frac{1}{2}(q(g^{-1}) - g^{-1}q(g)) = \frac{1}{2}g^{-1}(gq(g^{-1} - q(g)) = -g^{-1}\alpha(q)(g).$$

□

Bounded cohomology. Recall the definition of the bounded cohomology of a (discrete) group G with coefficients in an arbitrary normed G-module V. Let $C^n(G,V)$ be the vector space of n-cochains with coefficients in V, i.e., functions $G^n \to V$. The coboundary maps $d^n : C^n(G,V) \to C^{n+1}(G,V)$ are defined by the formula

$$\begin{aligned} d^n f(g_1,\dots,g_{n+1}) =& g_1 f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_{i-1},g_i g_{i+1},g_{i+2},\dots,g_{n+1}) \\ &+ (-1)^{n+1} f(g_1,\dots,g_n). \end{aligned}$$

Let $Z^n(G,V)$ and $B^n(G,V)$ denote the cocycles and coboundaries of this complex respectively; that is, $Z^n(G,V) = \operatorname{Ker} d^n$ and $B^n(G,V) = \operatorname{Im} d^{n-1}$ for $n > 0$ and $B^0(G,V) = 0$. Recall that the ordinary cohomology groups are defined by

$$H^n(G,V) := Z^n(G,V)/B^n(G,V).$$

Restricting to the subspaces $C^n_b(G,V)$ of $C^n(G,V)$ consisting of functions whose image is bounded with respect to the norm on V, we get the complex of bounded cochains. Similarly let $Z^n_b(G,V)$ and $B^n_b(G,V)$ denote its cocycles and coboundaries. Then the group

$$H^n_b(G,V) := Z^n_b(G,V)/B^n_b(G,V)$$

is called the *n-th bounded cohomology group of G with coefficients in V*.

Note that there is a natural map $c\colon H^n_b(G,V) \to H^n(G,V)$ which is induced by the inclusion map of the cochain complexes. This map is called the *comparison map*, and the kernel of c is denoted $EH^n_b(G,V)$. The following lemma is proved in [59] (see also [71]) in the case when V is a Banach space. The same proof works in the general case. We briefly sketch the argument for convenience of the reader.

Lemma IV.0.7. *Let G be a discrete countable group, V a normed G-module. Then there exists an exact sequence*

$$0 \to \ell^\infty(G,V) + Z^1(G,V) \to QZ^1(G,V) \xrightarrow{\delta} H^2_b(G,V) \xrightarrow{c} H^2(G,V),$$

where $\ell^\infty(G,V)$ is the vector space of all uniformly bounded functions $G \to V$.

Proof. We can identify $QZ^1(G,V)$ with the subspace of 1-cochains q for which $d^1 q$ is uniformly bounded, that is $d^1 q \in C^2_b(G,V)$. Since $d^2 \circ d^1 \equiv 0$, $d^1 q$ is in fact a bounded 2-cocycle. Let $\delta\colon QZ^1(G,V) \to H^2_b(G,V)$ denote the composition of d^1 and the natural quotient map $Z^2_b(G,V) \to H^2_b(G,V)$. Then δq represents a trivial element of $H^2_b(G,V)$ if and only if $d^1 q = d^1 p$ for some bounded cochain p, which means $p \in \ell^\infty(G,V)$

and $q-p \in Z^1(G,V)$. Further if q is a bounded 2-cocycle and $[q]_b \colon = q + B_b^2(G,V) \in H_b^2(G,V)$ is in the kernel of c, then $q = d^1 f$ for some 1-cochain f, which means $f \in QZ^1(G,V)$ and $\delta f = [q]_b$. □

IV.1 Separating cosets

Throughout this section, we denote by G a group with hyperbolically embedded collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$. Let X denote a subset of G such that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$. We also keep the notation $\mathscr{H}$ and $\Gamma(G, X \sqcup \mathscr{H})$ introduced in the previous chapter. By a coset of a subgroup we always mean a left coset.

We begin by introducing the notion of a separating coset for a pair of elements $f,g \in G$, which plays a crucial role in our construction.

Definition IV.1.1. We say that a path p in $\Gamma(G, X \sqcup \mathscr{H})$ penetrates a coset xH_λ for some $\lambda \in \Lambda$ if p decomposes as $p_1 a p_2$, where p_1, p_2 are possibly trivial, $(p_1)_+ \in xH_\lambda$, and a is an H_λ-component of p. If, in addition, $\widehat{\ell}_\lambda(a) > 3C$, where $C = C(1,0)$ is the constant from Lemma III.1.11, we say that *p essentially penetrates xH_λ*. Note that if p is geodesic, it penetrates every coset of H_λ at most once; in this case the vertices a_- and a_+ are called the *entrance and the exit points of p in xH_λ* and are denoted by $p_{in}(xH_\lambda)$ and $p_{out}(xH_\lambda)$, respectively.

Given two elements $f,g \in G$, we denote by $\mathscr{G}(f,g)$ the set of all geodesics in $\Gamma(G, X \sqcup \mathscr{H})$ going from f to g. Further we say that a coset xH_λ is *(f,g)-separating* if there exists a geodesic $p \in \mathscr{G}(f,g)$ that essentially penetrates xH_λ. For technical reasons we will also say xH_λ is (f,g)-separating whenever f and g are both elements of xH_λ and $f \neq g$; in this case we say xH_λ is *trivially (f,g)-separating*. The set of all (f,g)-separating cosets of H_λ is denoted by $S_\lambda(f,g)$.

The following lemma immediately follows from the definition and the facts that if $f,g,h \in G$ and $p \in \mathscr{G}(f,g)$, then $p^{-1} \in \mathscr{G}(g,f)$ and $hp \in \mathscr{G}(hf,hg)$.

Lemma IV.1.2. *For any $f,g,h \in G$ and any $\lambda \in \Lambda$, the following holds.*

(a) $S_\lambda(f,g) = S_\lambda(g,f)$.

(b) $S_\lambda(hf,hg) = \{hxH_\lambda \mid xH_\lambda \in S_\lambda(f,g)\}$.

The terminology in Definition IV.1.1 is justified by the first claim of following.

Lemma IV.1.3. *For any $\lambda \in \Lambda$, any $f,g \in G$ such that $f^{-1}g \notin H_\lambda$, and any (f,g)-separating coset xH_λ, the following hold.*

(a) Every path in $\Gamma(G, X \sqcup \mathscr{H})$ connecting f to g and composed of at most 2 geodesics penetrates xH_λ.

(b) For any $p,q \in \mathscr{G}(f,g)$, we have

$$\widehat{\mathrm{d}}_\lambda(p_{in}(xH_\lambda), q_{in}(xH_\lambda)) \leq 3C$$

and

$$\widehat{\mathrm{d}}_\lambda(p_{out}(xH_\lambda), q_{out}(xH_\lambda)) \leq 3C.$$

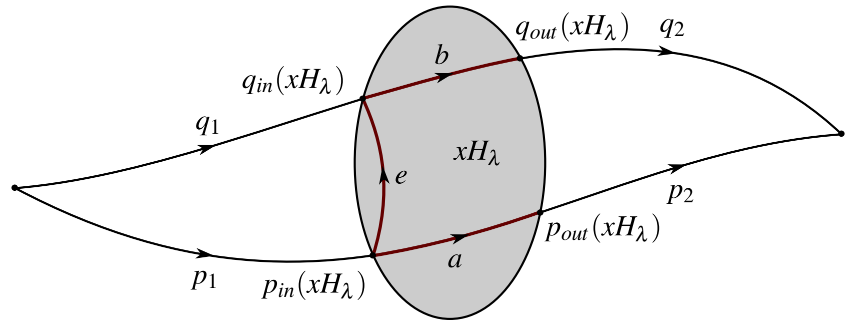

Figure IV.1:

Proof. Let $xH_\lambda \in S_\lambda(f,g)$ be (f,g)-separating coset. Since $f^{-1}g \notin H_\lambda$, xH_λ is non-trivially separating. Thus there exists a geodesic $p \in \mathscr{G}(f,g)$ that essentially penetrates xH_λ; let a denote the corresponding H_λ-component of p. Let r be any other path in $\Gamma(G, X \sqcup \mathscr{H})$ connecting f to g and composed of at most 2 geodesics. If a is isolated in the loop pr^{-1}, we obtain $\widehat{\ell}_\lambda(a) \leq 3C$ by Lemma III.1.11. This contradicts the assumption that p essentially penetrates xH_λ. Hence a is not isolated in pr^{-1}. Since p is geodesic, a cannot be connected to a component of p. Therefore a is connected to a component of r, i.e. r penetrates xH_λ.

Further let $p,q \in \mathscr{G}(a,b)$ and $xH_\lambda \in S_\lambda(f,g)$. By part (a) we have $p = p_1 a p_2$ and $q = q_1 b q_2$, where $(p_1)_+ \in xH_\lambda$, $(q_1)_+ \in xH_\lambda$ and a, b are H_λ-components of p and q, respectively (see Figure IV.1). (Of course, p_i or q_i, $i = 1,2$, can be trivial). Then a and b are connected. Let e be an edge or the trivial path connecting a_- to b_- and labeled by a letter from $H_\lambda \setminus \{1\}$. Applying Lemma III.1.11 to the geodesic triangle $p_1 e q_1^{-1}$, we obtain $\widehat{\mathrm{d}}_\lambda(e_-, e_+) \leq 3C$, which gives us the first inequality in (b). The proof of the second inequality is symmetric. □

Corollary IV.1.4. *For any $f,g \in G$ and any $\lambda \in \Lambda$, we have $|S_\lambda(f,g)| \leq \mathrm{d}_{X \sqcup \mathscr{H}}(f,g)$. In particular, $S_\lambda(f,g)$ is finite.*

In this section we will use the following elementary observation several times.

Lemma IV.1.5. *Let p be a geodesic in $\Gamma(G, X \sqcup \mathscr{H})$. Suppose that p penetrates a coset xH_λ. Let p_0 be the initial subpath of p ending at $p_{in}(xH_\lambda)$. Then $\ell(p_0) = \mathrm{d}_{X \sqcup \mathscr{H}}(p_-, xH_\lambda)$.*

Proof. Clearly $\mathrm{d}_{X \sqcup \mathscr{H}}(p_-, xH_\lambda) \leq \ell(p_0)$. Suppose that $\mathrm{d}_{X \sqcup \mathscr{H}}(p_-, xH_\lambda) < \ell(p_0)$. Since xH_λ has diameter 1 with respect to the metric $\mathrm{d}_{X \sqcup \mathscr{H}}$, we obtain

$$\mathrm{d}_{X \sqcup \mathscr{H}}(p_-, p_{out}(xH_\lambda)) \leq \mathrm{d}_{X \sqcup \mathscr{H}}(p_-, xH_\lambda) + 1 < \ell(p_0) + 1.$$

However we obviously have $\ell(p_0) + 1 = \mathrm{d}_{X \sqcup \mathscr{H}}(p_-, p_{out}(xH_\lambda))$. A contradiction. □

Definition IV.1.6. Given any $f,g \in G$, we define a relation $\preceq$ on the set $S_\lambda(f,g)$ as follows:

$$xH_\lambda \preceq yH_\lambda \quad \text{iff} \quad \mathrm{d}_{X \sqcup \mathscr{H}}(f, xH_\lambda) \leq \mathrm{d}_{X \sqcup \mathscr{H}}(f, yH_\lambda).$$

The next lemma is an immediate consequence of Lemma IV.1.3 and Lemma IV.1.5.

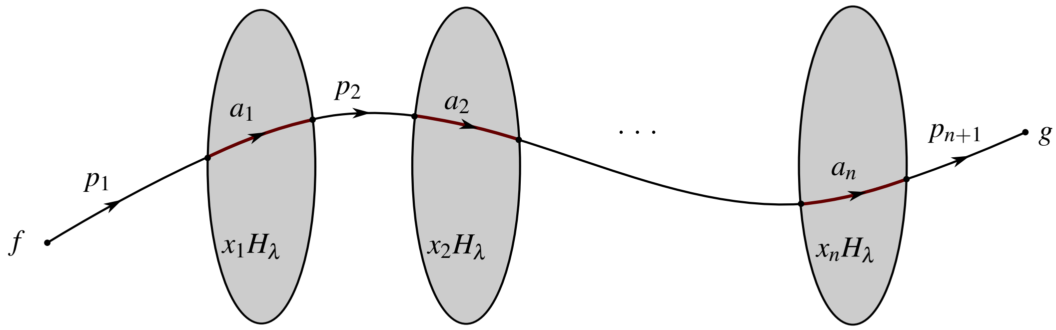

Figure IV.2:

Lemma IV.1.7. *For any $f,g \in G$ and any $\lambda \in \Lambda$, $\preceq$ is a linear order on $S_\lambda(f,g)$ and every geodesic $p \in \mathscr{G}(f,g)$ penetrates all (f,g)-separating cosets according to the order $\preceq$. That is, $S_\lambda(f,g) = \{x_1H_\lambda \preceq x_2H_\lambda \preceq \ldots \preceq x_nH_\lambda\}$ for some $n \in \mathbb{N}$ and p decomposes as*

$$p = p_1a_1 \cdots p_na_np_{n+1},$$

where a_i is an H_λ-component of p and $(p_i)_+ \in x_iH_\lambda$ for $i = 1,\ldots,n$ (see Fig. IV.2).

Given $f,g \in G$ and $xH_\lambda \in S_\lambda(f,g)$, we denote by $E(f,g;xH_\lambda)$ the set of ordered pairs of entrance-exit points of geodesics from $\mathscr{G}(f,g)$ in the coset xH_λ. That is,

$$E(f,g;xH_\lambda) = \{(p_{in}(xH_\lambda), p_{out}(xH_\lambda)) \mid p \in \mathscr{G}(f,g)\}.$$

Lemma IV.1.8. *For any $\lambda \in \Lambda$ and any $f,g,h,x \in G$, the following hold.*

(a) $E(g,f;xH_\lambda) = \{(v,u) \mid (u,v) \in E(f,g;xH_\lambda)\}$.

(b) $E(hf,hg;xH_\lambda) = \{(hu,hv) \mid (u,v) \in E(f,g;xH_\lambda)\}$.

(c) $|E(f,g;xH_\lambda)| < \infty$.

Proof. Parts (a) and (b) follow immediately from Lemma IV.1.2. To prove (c), note that if xH_λ trivially separates f and g, then $E(f,g;xH_\lambda) = \{(f,g)\}$. Further if xH_λ separates f and g non-trivially, fix any $(u,v) \in E(f,g;xH_\lambda)$. Then for any other $(u',v') \in E(f,g;xH_\lambda)$, we have $\widehat{\mathrm{d}}_\lambda(u,u') < 3C$ and $\widehat{\mathrm{d}}_\lambda(v,v') < 3C$ by part (b) of Lemma IV.1.3. Recall that $(H_\lambda, \widehat{\mathrm{d}}_\lambda)$ is a locally finite metric space by the definition of a hyperbolically embedded collection of subgroups. Hence $|E(f,g;xH_\lambda)| < \infty$. □

The main result of this section is the following.

Lemma IV.1.9. *For any $f,g,h \in G$ and any $\lambda \in \Lambda$, the set of all (f,g)-separating cosets of H_λ can be decomposed as*

$$S_\lambda(f,g) = S' \sqcup S'' \sqcup F,$$

where

(a) $S' \subseteq S_\lambda(f,h) \setminus S_\lambda(h,g)$ *and for every* $xH_\lambda \in S'$ *we have* $E(f,g;xH_\lambda) = E(f,h;xH_\lambda)$.

(b) $S'' \subseteq S_\lambda(h,g) \setminus S_\lambda(f,h)$ *and for every* $xH_\lambda \in S''$ *we have* $E(f,g;xH_\lambda) = E(h,g;xH_\lambda)$.

(c) $|F| \leq 2$.

Proof. First, if $|S_\lambda(f,g)| \leq 2$ the statement is trivial, so we can assume $|S_\lambda(f,g)| > 2$. Let

$$S_\lambda(f,g) = \{x_1H_\lambda \preceq x_2H_\lambda \preceq \ldots \preceq x_nH_\lambda\}.$$

We fix any geodesics $q \in \mathscr{G}(h,g)$ and $r \in \mathscr{G}(f,h)$. By the first claim of Lemma IV.1.3, every coset from $S_\lambda(f,g)$ is penetrated by at least one of q, r. Without loss of generality we may assume that at least one of the cosets from $S_\lambda(f,g)$ is penetrated by r. Let x_iH be the largest coset (with respect to the order $\preceq$) that is penetrated by r. Thus if $i < n$, then $x_{i+1}H$ is penetrated by q.

Let

$$S' = \{x_jH_\lambda \mid 1 \leq j < i\},$$

$$S'' = \{x_jH_\lambda \mid i+1 < j \leq n\},$$

and

$$F = S_\lambda(f,g) \setminus (S' \cup S'').$$

Obviously $|F| \leq 2$. It remains to prove (a) and (b). We will prove (a) only, the proof of (b) is symmetric.

Fix any $1 \leq j < i$. Let p be any geodesic from $\mathscr{G}(f,g)$. By Lemma IV.1.7, p decomposes as

$$p = p_1a_1p_2a_2p_3,$$

where a_1, a_2 are H_λ-components of p, $(p_1)_+ \in x_jH_\lambda$, and $(p_2)_+ \in x_iH_\lambda$. Similarly by the choice of i, r decomposes as

$$r = r_1br_2,$$

where b is an H_λ-component of r and $(r_2)_- \in x_iH_\lambda$ (see Fig. IV.3).

Since $(r_2)_-$ and $(p_2)_+$ belong to the same coset of H_λ, there exists a path e in $\Gamma(G, X \sqcup \mathscr{H})$ of length at most 1 such that $e_- = (p_2)_+$ and $e_+ = (r_2)_-$. By Lemma IV.1.5, we have $\ell(p_1a_1p_2) = \ell(r_1)$. Hence the path $t = p_1a_1p_2er_2$ has the same length as r, i.e., $t \in \mathscr{G}(f,h)$. Also,

$$p_{in}(x_jH_\lambda) = t_{in}(x_jH_\lambda) \tag{IV.3}$$

and

$$p_{out}(x_jH_\lambda) = t_{out}(x_jH_\lambda). \tag{IV.4}$$

So far all our arguments were valid for any $p \in \mathscr{G}(f,g)$. Since $x_jH_\lambda \in S_\lambda(f,g)$, there exists $p \in \mathscr{G}(f,g)$ that essentially penetrates x_jH_λ, i.e., $\widehat{\ell}_\lambda(a_1)) > 3C$ in the above notation. In this case t also essentially penetrates x_jH_λ. Thus $x_jH \in S_\lambda(f,h)$. Moreover since we have (IV.3) and (IV.4) for every $p \in \mathscr{G}(f,g)$, we obtain $E(f,g;x_jH_\lambda) = E(f,h;x_jH_\lambda)$.

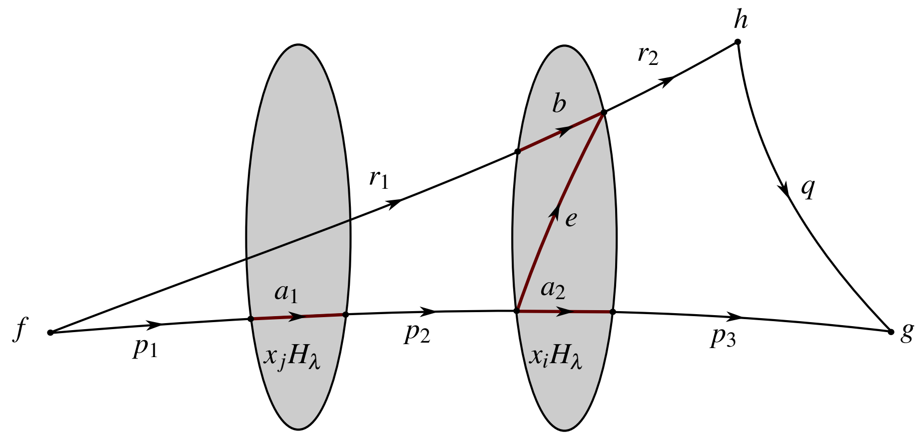

Figure IV.3:

To complete the proof of (a) it remains to show that $x_jH_\lambda \notin S_\lambda(h,g)$. Clearly $g \notin x_jH_\lambda$, or p would not be geodesic, so x_jH_λ does not trivially separate g and h. Thus, if $x_jH_\lambda \in S_\lambda(h,g)$ there must be a geodesic from h to g which essentially penetrates x_jH_λ. Hence by Lemma IV.1.3, every geodesic from h to g penetrates x_jH_λ, which means q penetrates x_jH_λ. Then using Lemma IV.1.5, the fact that every coset of H_λ has diameter 1 with respect to the metric $\mathrm{d}_{X\sqcup\mathscr{H}}$, and the triangle inequality, we obtain

$$\begin{aligned}\ell(q) &= \mathrm{d}_{X\sqcup\mathscr{H}}(h,x_jH_\lambda)+1+\mathrm{d}_{X\sqcup\mathscr{H}}(g,x_jH_\lambda)\\ &> \mathrm{d}_{X\sqcup\mathscr{H}}(h,x_iH_\lambda)+1+\mathrm{d}_{X\sqcup\mathscr{H}}(g,x_iH_\lambda)\\ &\geq \ell(r_2)+\mathrm{d}_{X\sqcup\mathscr{H}}((r_2)_-,(p_3)_-)+\ell(p_3)\\ &\geq \mathrm{d}_{X\sqcup\mathscr{H}}(h,g).\end{aligned}$$

Since one of the inequalities is strict, this contradicts the assumption that q is geodesic. □

IV.2 Extending quasi-cocycles

We keep all assumptions and notation from the previous section. For each $\lambda \in \Lambda$, let

$$\mathscr{F}_\lambda = \{h \in H_\lambda \mid h \in H_\mu \text{ for some } \mu \neq \lambda\}.$$

In particular, if $\{H_\lambda\}_{\lambda\in\Lambda}$ consists of a single subgroup H, the corresponding subset $\mathscr{F} = \emptyset$.

It follows from Lemma III.1.11 that every $h \in F_\lambda$ satisfies $\widehat{\mathrm{d}}_\lambda(1,h) \leq 2C$, where C is the constant from Lemma III.1.11. Indeed for every such h there is a loop e_1e_2 in $\Gamma(G, X\sqcup\mathscr{H})$, where e_1 is an edge labeled by $h \in H_\lambda\setminus\{1\}$ and e_2 is an edge labeled by the copy of h in $H_\mu\setminus\{1\}$ for some $\mu \in \Lambda$. Since the metric space $(H_\lambda, \widehat{\mathrm{d}}_\lambda)$ is locally finite by the definition of a hyperbolically embedded collection of subgroups, we obtain the following.

Lemma IV.2.1. *$|\mathscr{F}_\lambda| < \infty$ for all $\lambda \in \Lambda$.*

Also, for $q_\lambda \in QZ^1(H_\lambda, U_\lambda)$, let

$$K_\lambda = \max\{\|q_\lambda(g)\| : \widehat{\mathrm{d}}_\lambda(1,g) < 15C\}. \tag{IV.5}$$

Observe that the constant K_λ is well-defined by local finiteness of $(H_\lambda, \widehat{\mathrm{d}}_\lambda)$.

We can now state our main extension theorem in its full generality. Recall that for a quasi-cocycle q, $D(q)$ denotes its defect defined by (IV.1).

Theorem IV.2.2. *Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a hyperbolically embedded collection of subgroups of G, V a normed G-module. For each $\lambda \in \Lambda$, let U_λ be an H_λ-submodule of G. Then there exists a linear map*

$$\iota\colon \bigoplus_{\lambda\in\Lambda} QZ^1_{as}(H_\lambda, U_\lambda) \to QZ^1_{as}(G,V)$$

such that for any $q = (q_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} QZ^1_{as}(H_\lambda, U_\lambda)$ the following hold.

(a) For any $\lambda \in \Lambda$ and any $h \in H_\lambda \setminus \mathscr{F}_\lambda$, we have $\iota(q)(h) = q_\lambda(h)$. In particular, $\sup_{h\in H_\lambda} \|\iota(q)(h) - q_\lambda(h)\| < \infty$.

(b) $D(\iota(q)) \le \sum_\lambda (54K_\lambda + 66D(q_\lambda))$.

Notice that the sum in part (b) is finite because $q_\lambda \equiv 0$ for all but finitely many λ, and thus $K_\lambda = D(q_\lambda) = 0$ for all but finitely many λ. If G contains a singe hyperbolically embedded subgroup, Theorem IV.2.2 obviously reduces to Theorem I.2.1 mentioned in the introduction.

Throughout the rest of the section, we use the notation of Theorem IV.2.2. Although our proof can be entirely written in the language of quasi-cocycles, the following concept helps make some arguments more symmetric and easier to comprehend. In the definition below, we write $s(a) = t(a)$ for two partial maps $s,t\colon A \to B$ if the value $s(a)$ is defined if and only if $t(a)$ is, and these values are equal whenever defined.

Definition IV.2.3. A *partial bi-combing of G with coefficients in V* is a partial map $r\colon G\times G \to V$. We say that

(a) r is *G-equivariant* if $hr(f,g) = r(hf,hg)$ for any $f,g,h \in G$;

(b) r is *anti-symmetric* if $r(f,g) = -r(g,f)$ for any $f,g \in G$.

(c) r has *bounded area* if there exists a constant A such that for any $f,g,h \in G$ for which $r(f,g)$, $r(g,h)$, and $r(h,f)$ are defined, we have

$$\|r(f,g) + r(g,h) + r(h,f)\| \le A. \tag{IV.6}$$

The infimum of all A satisfying (IV.6) is called the *area* of r and is denoted by $A(r)$.

Let us fix $\lambda \in \Lambda$. Given $q_\lambda \in QZ^1_{as}(H_\lambda, U_\lambda)$, we define a partial map $r_\lambda \colon G \times G \to V$ by

$$r_\lambda(f,g) = f q_\lambda(f^{-1}g).$$

Thus $r_\lambda(f,g)$ is defined if and only if f and g belong to the same coset xH_λ.

Lemma IV.2.4. *The partial map $r_\lambda \colon G \times G \to V$ is an anti-symmetric equivariant partial bi-combing of G of area*

$$A(r_\lambda) \le D(q_\lambda). \tag{IV.7}$$

Proof. Equivariance of r_λ is obvious and anti-symmetry follows immediately from anti-symmetry of q_λ. By equivariance it suffices to verify the bounded area condition for the a triple $1, g, h \in G$. We have

$$\|r_\lambda(1,g) + r_\lambda(g,h) + r_\lambda(h,1)\| = \|q_\lambda(g) + g q_\lambda(g^{-1}h) - q_\lambda(h)\| \le D(q_\lambda).$$

□

Corollary IV.2.5. *For any $n \in \mathbb{N}$, any $x \in G$, and any $g_0, \dots, g_n \in xH_\lambda$, we have*

$$\left\| r_\lambda(g_0, g_n) - \sum_{i=1}^{n} r_\lambda(g_{i-1}, g_i) \right\| \le (n-1) D(q_\lambda).$$

Proof. The claim follows from anti-symmetry, the definition of area, and (IV.7) by induction. □

Our next goal is to construct a globally defined anti-symmetric bounded area G-equivariant bi-combing $\widetilde{r}_\lambda \colon G \times G \to V$ that extends r_λ. To this end, for each $f, g \in G$ and each coset xH_λ, we define the average

$$R_{av}(f,g;xH_\lambda) = \frac{1}{|E(f,g;xH_\lambda)|} \sum_{(u,v) \in E(f,g;xH_\lambda)} r_\lambda(u,v).$$

If $xH_\lambda \notin S_\lambda(f,g)$, we assume $R_{av}(f,g;xH_\lambda) = 0$. Note that $R_{av}(f,g;xH_\lambda)$ is well-defined since $E(f,g;xH_\lambda) < \infty$ by part (c) of Lemma IV.1.8.

Lemma IV.2.6. *For any $f, g, h, x \in G$, the following hold.*

(a) $R_{av}(f,g;xH_\lambda) = -R_{av}(g,f;xH_\lambda)$.

(b) $R_{av}(hf,hg;hxH_\lambda) = R_{av}(f,g;xH_\lambda)$.

(c) For any $(u,v) \in E(f,g;xH_\lambda)$, we have

$$\|r_\lambda(u,v) - R_{av}(f,g;xH_\lambda)\| \le 2D(q_\lambda) + 2K_\lambda. \tag{IV.8}$$

Proof. The first claim follows from parts (a) of Lemma IV.1.8 and anti-symmetry of r_λ. The second claim follows from parts (b) of Lemma IV.1.8 and the equivariance of r_λ.

To prove (c), note that for any $(u',v')\in E(f,g;xH_\lambda)$, we have

$$\max\{\widehat{\mathrm{d}}_\lambda(u,u'),\widehat{\mathrm{d}}_\lambda(v,v')\}\leq 3C$$

by Lemma IV.1.3. Thus, using the triangle inequality and applying Corollary IV.2.5 to elements $u,u',v',v\in xH_\lambda$, we obtain

$$\begin{aligned}\|r_\lambda(u,v)-r_\lambda(u',v')\|\leq & \quad \|r_\lambda(u,v)-r_\lambda(u,u')-r_\lambda(u',v')-r_\lambda(v',v)\| \\ + & \quad \|r_\lambda(u,u')\|+\|r_\lambda(v',v)\|\leq 2D(q_\lambda)+2K_\lambda.\end{aligned}$$

This obviously implies (IV.8). □

Let

$$\widetilde{r}_\lambda(f,g)=\sum_{xH_\lambda\in S_\lambda(f,g)} R_{av}(f,g;xH_\lambda).$$

Note that $\widetilde{r}_\lambda$ is well-defined as $S_\lambda(f,g)$ is finite for any $f,g\in G$ by Corollary IV.1.4.

Lemma IV.2.7. *The map $\widetilde{r}_\lambda\colon G\times G\to V$ is an anti-symmetric G-equivariant bi-combing of area*

$$A(\widetilde{r}_\lambda)\leq 66D(q_\lambda)+54K_\lambda. \tag{IV.9}$$

Proof. Equivariance and anti-symmetry of $\widetilde{r}_\lambda$ follow immediately from Lemma IV.1.2 and Lemma IV.2.6. In order to show that $\widetilde{r}_\lambda$ satisfies the bounded area condition, we need to estimate the norm of $\widetilde{r}_\lambda(f,g)+\widetilde{r}_\lambda(g,h)+\widetilde{r}_\lambda(h,f)$ uniformly on $f,g,h\in G$. Since $R_{av}(f,g;xH_\lambda)=0$ if $xH_\lambda\notin S_\lambda(f,g)$, we have

$$\widetilde{r}_\lambda(f,g)+\widetilde{r}_\lambda(g,h)+\widetilde{r}_\lambda(h,f)=\sum_{xH_\lambda\in G/H_\lambda}\rho(f,g,h;xH_\lambda),$$

where

$$\rho(f,g,h;xH_\lambda)\colon=R_{av}(f,g;xH_\lambda)+R_{av}(g,h;xH_\lambda)+R_{av}(h,f;xH_\lambda).$$

Of course, $\rho(f,g,h;xH_\lambda)$ is nontrivial only if $xH_\lambda\in S_\lambda(f,g)\cup S_\lambda(g,h)\cup S_\lambda(h,f)$.

Fix $f,g,h\in G$. We start by estimating $\rho(f,g,h;xH_\lambda)$ for cosets from $S_\lambda(f,g)$. Let $S_\lambda(f,g)=S'\sqcup S''\sqcup F$ be the decomposition provided by Lemma IV.1.9. Suppose first that $xH_\lambda\in S'$. Then $xH_\lambda\in S_\lambda(f,h)=S_\lambda(h,f)$ and $E(f,g;xH_\lambda)=E(f,h;xH_\lambda)$ by Lemma IV.1.9. Hence

$$R_{av}(f,g;xH_\lambda)=R_{av}(f,h;xH_\lambda)=-R_{av}(h,f;xH_\lambda). \tag{IV.10}$$

by Lemma IV.2.6 (a). On the other hand, Lemma IV.1.9 also states that $xH_\lambda\notin S_\lambda(h,g)=S_\lambda(g,h)$. Hence

$$R_{av}(g,h;xH_\lambda)=0. \tag{IV.11}$$

Summing up (IV.10) and (IV.11), we obtain $\rho(f,g,h;xH_\lambda)=0$. Similarly, $\rho(f,g,h;xH_\lambda)=0$ for any

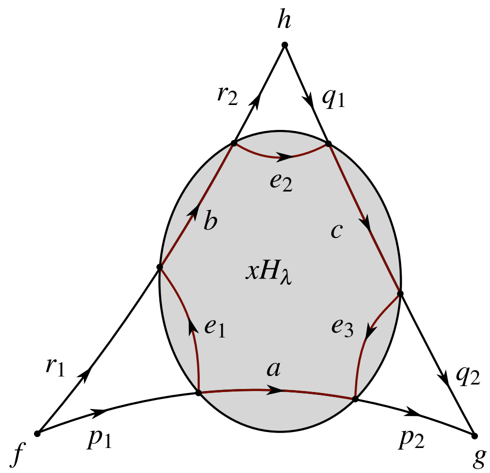

Figure IV.4:

$xH_\lambda \in S''$. Thus

$$\sum_{xH_\lambda \in S_\lambda(f,g)} \rho(f,g,h;xH_\lambda) = \sum_{xH_\lambda \in F} \rho(f,g,h;xH_\lambda). \qquad \text{(IV.12)}$$

Fix a coset $xH_\lambda \in F$ and any $p \in \mathscr{G}(f,g)$, $q \in \mathscr{G}(h,g)$, $r \in \mathscr{G}(f,h)$. There are three cases to consider.

Case 1: $xH_\lambda \in S_\lambda(g,h) \cap S_\lambda(h,f)$. In this case we have $p = p_1 a p_2$, $q = q_1 c q_2$, $r = r_1 b r_2$, where a, c, and b are H_λ-components of p, q, and r, respectively, corresponding to the coset xH_λ (i.e., $a_\pm, b_\pm, c_\pm \in xH_\lambda$). Let e_1, e_2, e_3 be paths of lengths at most 1 labeled by elements of H_λ and connecting a_- to b_-, b_+ to c_-, and c_+ to a_+ (see Fig. IV.4).

Since a geodesic in $\Gamma(G, X \sqcup \mathscr{H})$ can penetrate a coset of H_λ at most once, e_1 is either trivial or is an isolated component of a geodesic triangle (namely $p_1 e_1 r_1^{-1}$). The same holds true for e_1 and e_2. Hence by Lemma III.1.11, we obtain

$$\widehat{\mathrm{d}}_\lambda((e_i)_-, (e_i)_+) \le 3C, \quad i = 1,2,3. \qquad \text{(IV.13)}$$

In particular,

$$\|r_\lambda((e_i)_-, (e_i)_+)\| \le K_\lambda, \quad i = 1,2,3. \qquad \text{(IV.14)}$$

by the definition of K_λ (see (IV.5)). Using the triangle inequality, applying Lemma IV.2.5 to the vertices of the hexagon $e_1 b e_2 c e_3 a^{-1}$, and using (IV.14), we obtain

$$\|r_\lambda(a_-,a_+) + r_\lambda(b_+,b_-) + r_\lambda(c_+,c_-)\|$$

$$\le \left\| r_\lambda(a_-,a_+) - r_\lambda(b_-,b_+) - r_\lambda(c_-,c_+) - \sum_{i=1}^{3} r_\lambda((e_i)_-,(e_i)_+) \right\| + \left\| \sum_{i=1}^{3} r_\lambda((e_i)_-,(e_i)_+) \right\|$$

$$\le 5D(q_\lambda) + 3K_\lambda.$$

Now Lemma IV.2.6 (c) implies

$$\begin{aligned}\|\rho(f,g,h;xH_\lambda)\| &= \|R_{av}(f,g;xH_\lambda)+R_{av}(g,h;xH_\lambda)+R_{av}(h,f;xH_\lambda)\| \\ &\leq \|r_\lambda(a_-,a_+)+r_\lambda(c_+,c_-)+r_\lambda(b_+,b_-)\|+6(D(q_\lambda)+K_\lambda) \qquad \text{(IV.15)} \\ &\leq 11D(q_\lambda)+9K_\lambda.\end{aligned}$$

Case 2: $xH_\lambda \in S_\lambda(h,f)\setminus S_\lambda(g,h)$ or $xH_\lambda \in S_\lambda(g,h)\setminus S_\lambda(h,f)$. Since the proof in these cases is the same, we will only consider the case $xH_\lambda \in S_\lambda(h,f)\setminus S_\lambda(g,h)$. Let $p = p_1ap_2$, $r = r_1br_2$, and e_1 be as in Case 1 and let e be the path of length at most 1 in $\Gamma(G, X\sqcup\mathscr{H})$ connecting b_+ to a_+ and labeled by an element of H_λ. There are two possibilities to consider.

2a) First assume that e is isolated in the quadrilateral $ep_2q^{-1}r_2^{-1}$ (see Fig. IV.5). Then we have $\widehat{\mathrm{d}}_\lambda(e_-,e_+)\leq 4C$ by Lemma III.1.11 and hence

$$\|r_\lambda(e_-,e_+)\|\leq K_\lambda.$$

Note that (IV.14) remains valid for $i=1$. Applying Corollary IV.2.5 to the vertices of the quadrilateral e_1bea^{-1} as in Case 1 we obtain

$$\begin{aligned}\|r_\lambda(a_-,a_+)+r_\lambda(b_+,b_-)\|\leq\ & \|r_\lambda(a_-,a_+)-r_\lambda((e_1)_-,(e_1)_+)-r_\lambda(b_-,b_+)-r_\lambda(e_-,e_+)\| \\ &+\|r_\lambda((e_1)_-,(e_1)_+)\|+\|r_\lambda(e_-,e_+)\|\leq 3D(q_\lambda)+2K_\lambda.\end{aligned}$$

Since $xH_\lambda\notin S_\lambda(g,h)$, we have $R_{av}(g,h;xH_\lambda)=0$. Finally Lemma IV.2.6 (c) implies

$$\begin{aligned}\|\rho(f,g,h;xH_\lambda)\| &= \|R_{av}(f,g;xH_\lambda)+R_{av}(h,f;xH_\lambda)\| \\ &\leq \|r_\lambda(a_-,a_+)+r_\lambda(b_+,b_-)\|+4(D(q_\lambda)+K_\lambda) \qquad \text{(IV.16)} \\ &\leq 7D(q_\lambda)+6K_\lambda.\end{aligned}$$

2b) Suppose now that e is not isolated in the quadrilateral $ep_2q^{-1}r_2^{-1}$. Then e is connected to a component c of q. Let $q=q_1cq_2$ and let e_1 and e_2 be as in Case 1 (see Fig. IV.4). Then (IV.14) remains valid. In addition, we have $\widehat{\mathrm{d}}_\lambda(c_-,c_+)\leq 3C$ as $xH_\lambda\notin S_\lambda(g,h)$ and hence q can not essentially penetrate xH_λ. Hence $\|r_\lambda(c_-,c_+)\|\leq K_\lambda$. The reader can easily verify that arguing as in the Case 1 and then as in (IV.16), we can obtain

$$\|r_\lambda(a_-,a_+)+r_\lambda(b_+,b_-)\|\leq 5A(r_\lambda)+4K_\lambda$$

and consequently

$$\|\rho(f,g,h;xH_\lambda)\|\leq 9D(q_\lambda)+8K_\lambda. \qquad \text{(IV.17)}$$

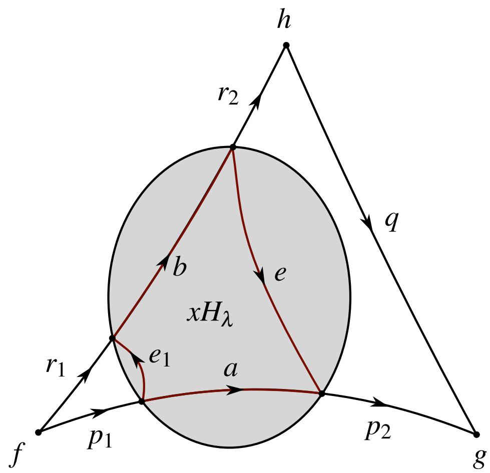

Figure IV.5:

Case 3: $xH_\lambda \notin S_\lambda(h,f)\cup S_\lambda(g,h)$. Let $p = p_1 a p_2$ be as in Cases 1 and 2. There are three possibilities to consider.

3a) a is an isolated component of $pq^{-1}r^{-1}$. In this case $\widehat{\mathrm{d}}_\lambda(a_-,a_+)\le 3C$.

3b) a is connected to a component of exactly one of q, r. For definiteness, assume that a is connected to a component b of r. Then, in the notation of Case 2 (see Fig. V.5), e is isolated in $ep_2q^{-1}r_2^{-1}$ and we have $\widehat{\mathrm{d}}_\lambda(e_-,e_+)\le 4C$ by Lemma III.1.11. As in Case 1, we have (IV.13) for $i=1$. Since $xH_\lambda \notin S_\lambda(h,f)$, r can not essentially penetrate xH_λ. Thus $\widehat{\mathrm{d}}_\lambda(b_-,b_+)\le 3C$. Applying the triangle inequality to the quadrilateral e_1bea^{-1}, we obtain

$$\widehat{\mathrm{d}}_\lambda(a_-,a_+)\le 10C.$$

3c) a is connected to a component b of r and a component c of q. Then in the notation of Case 1 and Fig. V.4, inequalities (IV.13) remain valid and we also have $\widehat{\mathrm{d}}_\lambda(b_-,b_+)\le 3C$ and $\widehat{\mathrm{d}}_\lambda(c_-,c_+)\le 3C$ as in Case 3b). Applying the triangle inequality to the hexagon $e_1be_2ce_3a^{-1}$, we obtain

$$\widehat{\mathrm{d}}_\lambda(a_-,a_+)\le 15C.$$

Thus, in all cases 3a) - 3c) we have $\|r_\lambda(a_-,a_+)\|\le K_\lambda$. Since $R_{av}(g,h;xH_\lambda)=R_{av}(h,f;xH_\lambda)=0$ in this case, using Lemma IV.2.6 (c) we obtain

$$\|\rho(f,g,h;xH_\lambda)\| = \|R_{av}(f,g;xH_\lambda)\| \le 2A(r_\lambda)+3K_\lambda. \tag{IV.18}$$

in Case 3.

Summarizing (IV.12), (IV.15), (IV.16), (IV.17), (IV.18), and taking into account that $|F|\le 2$, we obtain

$$\left\|\sum_{xH_\lambda\in S_\lambda(f,g)}\rho(f,g,h;xH_\lambda)\right\| = \left\|\sum_{xH_\lambda\in F}\rho(f,g,h;xH_\lambda)\right\| \le 22D(q_\lambda)+18K_\lambda.$$

Repeating the same arguments for $S_\lambda(h,f)$ and $S_\lambda(g,f)$ and summing up, we obtain

$$\|\widetilde{r}_\lambda(f,g)+\widetilde{r}_\lambda(g,h)+\widetilde{r}_\lambda(h,f)\| \le 66D(q_\lambda)+54K_\lambda.$$

$\square$

We are now ready to prove the main extension theorem.

Proof of Theorem IV.2.2. Let $q=(q_\lambda)_{\lambda\in\Lambda}\in \bigoplus_{\lambda\in\Lambda} QZ^1_{as}(H_\lambda,U_\lambda)$. For each $\lambda\in\Lambda$, let $\widetilde{r}_\lambda$ be the bi-combing constructed above and let $\widetilde{q}_\lambda(g)=\widetilde{r}_\lambda(1,g)$. Then $\widetilde{q}_\lambda\in QZ^1_{as}(G,V)$. Indeed we have

$$\begin{aligned}\|\widetilde{q}_\lambda(fg)-\widetilde{q}_\lambda(f)-f\widetilde{q}_\lambda(g)\| &= \|\widetilde{r}_\lambda(1,fg)-\widetilde{r}_\lambda(1,f)-f\widetilde{r}_\lambda(1,g)\| \\ &= \|\widetilde{r}_\lambda(1,fg)+\widetilde{r}_\lambda(f,1)+\widetilde{r}_\lambda(fg,f)\| \qquad \text{(IV.19)} \\ &\le A(\widetilde{r}_\lambda).\end{aligned}$$

Anti-symmetry of $\widetilde{q}_\lambda$ follows from that of $\widetilde{r}_\lambda$.

Further we define

$$\iota(q)=\sum_{\lambda\in\Lambda}\widetilde{q}_\lambda.$$

Since q is supported on only finitely many λ, $\iota(q)$ is equal to a finite linear combination of quasi-cocycles, so $\iota(q)\in QZ^1_{as}(G,V)$. It is easy to see that the maps $QZ^1_{as}(H_\lambda,U_\lambda)\to QZ^1_{as}(G,V)$ defined by $q_\lambda\mapsto\widetilde{q}_\lambda$ are linear. Hence so is ι.

If $h\in H_\lambda\setminus\mathscr{F}_\lambda$, then $S_\lambda(1,h)=\{H_\lambda\}$ and $S_\mu(1,h)=\emptyset$ for any $\mu\neq\lambda$. Obviously $E(1,h;H_\lambda)=\{(1,h)\}$. Thus $\widetilde{r}_\lambda(1,h)=r_\lambda(1,h)=q_\lambda(h)$ and $\widetilde{r}_\mu(1,h)=0$ whenever $\mu\neq\lambda$. Thus

$$\iota(q)(h)=\sum_{\mu\in\Lambda}\widetilde{q}_\mu(h)=\sum_{\mu\in\Lambda}\widetilde{r}_\mu(1,h)=q_\lambda(h).$$

This finishes the proof of (a). Part (b) follows from (IV.19) and (IV.9). $\square$

Remark IV.2.8. Our proof essentially uses the fact that the quasi-cocycles q_λ are anti-symmetric. In fact, our approach provably fails for non-anti-symmetric ones. This can be illustrated in the case when $G=F(x,y)$, the free group of rank 2, and $H=\langle x\rangle$. Indeed take $q\in QZ^1(H,\mathbb{R})$ defined by

$$q(x^n)=\begin{cases}1, & \text{if } n\ge 0,\\ 0, & \text{if } n<0.\end{cases}$$

Let $\widetilde{q}$ be the extension obtained as above using the subset $X=\{x,y\}$ of G. Take any $n\in\mathbb{N}$ such that $\widehat{d}(1,x^n)>3C$ (in fact, $C=0$ in this case, but we will not use this). Then it is straightforward to verify that $\widetilde{q}((yx^n)^k)=k$ while $\widetilde{q}((yx^n)^{-k})=\widetilde{q}((x^{-n}y^{-1})^k)=0$ for every $k\in\mathbb{N}$. This contradicts the quasi-cocycle identity as $k\to\infty$.

IV.3 Applications

Bounded cohomology. Our goal here is to prove Corollary I.2.3. We begin with an auxiliary result.

Proposition IV.3.1. *Let G be a group, H a hyperbolically embedded subgroup of G, V a G-module, and U an H-submodule of V. Suppose that there exists a continuous projection $\pi\colon V \to U$. Then there is a linear map $\varphi\colon QZ^1(H,U) \to EH^2_b(G,V)$ such that* $\operatorname{Ker}\varphi \subseteq \ell^\infty(H,U)+Z^1(H,U)$*. In particular,*

$$\dim H^2_b(G,V) \geq \dim EH^2_b(G,V) \geq \dim EH^2_b(H,U).$$

Proof. We define φ to be the composition $\delta\circ\kappa$, where κ is given by Corollary I.2.2 and δ is the natural map $QZ^1(G,V)\to EH^2_b(G,V)$ (see Lemma IV.0.7). Note that if $\varphi(q)=0$ for some $q\in QZ^1(H,U)$, then

$$\kappa(q) = h+b, \tag{IV.20}$$

where $b\in\ell^\infty(G,V)$ and $h\in Z^1(G,V)$. Since $\kappa(q)(x)\in U$ for all $x\in H$, composing both sides of this equality with π and restricting to H we obtain

$$\kappa(q)|_H = \pi\circ h|_H + \pi\circ b|_H.$$

Obviously $\pi\circ h|_H \in Z^1(H,U)$ and $\pi\circ b|_H\in\ell^\infty(H,U)$ since π is continuous. By Corollary I.2.2, $(q-\kappa(q)|_H)\in\ell^\infty(H,U)$, thus $q\in\ell^\infty(H,U)+Z^1(H,U)$. □

We are now ready to prove Corollary I.2.3.

Proof of Corollary I.2.3. It is easy to see that the assumptions of Lemma IV.3.1 hold in the case $V=\ell^p(G)$ and $U=\ell^p(H)$. It is well known that $\dim EH^2_b(H)=\infty$ for every virtually free group which is not virtually cyclic (see. e.g., [39]). To complete the proof it remains to note that every group $G\in\mathcal{AH}$ contains a virtually free but not virtually cyclic hyperbolically embedded subgroup by Lemma III.1.10. □

Stable commutator length. Let G be a group, and let $g\in[G,G]$. The the *commutator length* of g, denoted $cl_G(g)$, is defined as the minimal number of commutators whose product is equal to g in G. The *stable commutator length* is defined by

$$scl_G(g) = \lim_{n\to\infty}\frac{cl_G(g^n)}{n}.$$

It is customary to extend scl_G to all elements g for which have some positive power $g^n\in[G,G]$ by letting $scl_G(g)=\frac{scl(g^n)}{n}$. Basic facts and theorems about stable commutator length can be found in [18].

Following [18], we will denote space of *quasimorphisms* on G by $\widehat{Q}(G)$. Recall that this is the same as $QZ^1(G,\mathbb{R})$ where $\mathbb{R}$ is considered as a G-module with the trivial action. Note that in this setting Theorem I.2.1 says that any anti-symmetric quasimorphism on H can be extended to a quasimorphism on G.

A quasimorphism φ on G is called *homogeneous* if for all $g\in G$ and all $n\in\mathbb{Z}$, $\varphi(g^n)=n\varphi(g)$. In particular, all homogeneous quasimorphisms are anti-symmetric. We denote the subspace of homogeneous quasimorphisms by $Q(G)$. The connection between quasimorphisms and stable commutator length is provided by the Bavard Duality Theorem [6].

Theorem IV.3.2 (Bavard Duality Theorem)**.** *For any $g \in [G,G]$, there is an equality*

$$scl_G(g) = \sup_{\varphi \in Q(G)} \frac{\varphi(g)}{2D(\varphi)}. \tag{IV.21}$$

Where the supremum is taken over all homogeneous quasimorphisms of non-zero defect.

In fact, it is not hard to see that this supremum is always realized by some quasimorphism.

Given any quasimorphism φ, there is a standard way to obtain a homogeneous quasimorphism ψ, called the homogenization of φ. This is done by defining

$$\psi(g) = \lim_{n\to\infty} \frac{\varphi(g^n)}{n}.$$

Lemma IV.3.3 ([18, Corollary 2.59])**.** *Let $\varphi \in \widehat{Q}(G)$ with homogenization ψ. Then $D(\psi) \leq 2D(\varphi)$.*

Our plan for proving Corollary I.2.4 will be to take an element $h \in H$ and apply Bavard Duality to find a homogeneous quasimorphism which which realizes (IV.21) with respect to scl_H. Then we can use Theorem I.2.1 to extend this to a quasimorphism on all of G, then apply Bavard Duality again to find a lower bound on $scl_G(h)$. In order to do this we will need to understand the defect of the extended quasimorphism.

Let H be a group, and let $\xi : H \to H/[H,H] \otimes \mathbb{Q}$ be the natural map. A subset $Y \subseteq H$ will be called *nice* if Y can be decomposed as $Y = Y_1 \cup Y_2$ such that $\xi(Y_1)$ is linearly independent and $\xi|_{Y_2} \equiv 0$.

Lemma IV.3.4. *Every finitely generated subgroup of H has a nice finite generating set.*

Proof. Let H' be a finitely generated subgroup of H, and let X be a finite generating set of H'. Then $\xi(H')$ is a finitely generated subgroup of a torsion-free abelian group, and hence $\xi(H')$ is a finitely generated free abelian group. Let $\{v_1,...,v_n\}$ be a basis for $\xi(H')$ as a free abelian group and let $y_i \in H'$ be such that $\xi(y_i) = v_i$. Then for each $x \in X$, there exist integers $a_{x,1},...a_{x,n}$ such that $\xi(x) = \sum_{i=1}^{n} a_{x,i}v_i$. Let $\hat{x} = xy_1^{-a_{x,1}}...y_n^{-a_{x,n}}$. Now let $Y_1 = \{y_1,...y_n\}$, and let $Y_2 = \{\hat{x} \mid x \in X\}$. Then clearly $Y = Y_1 \cup Y_2$ is nice, and $\langle Y\rangle = \langle X\rangle = H'$.

□

Lemma IV.3.5 ([28, Theorem 16.1])**.** *Let B be a subgroup of an abelian group A, and let D be a divisible abelian group. Then every homomorphism from $B \to D$ can be extended to a homomorphism from $A \to D$.*

The reason we are interested in nice subsets is the following lemma.

Lemma IV.3.6. *For any group H, any nice finite subset $Y \subseteq H$, and any $\varphi \in Q(H)$, there exists $\varphi' \in Q(H)$ such that $\varphi'|_{[H,H]} \equiv \varphi|_{[H,H]}$, $D(\varphi') = D(\varphi)$, and for all $y \in Y$,*

$$|\varphi'(y)| \leq 2D(\varphi')scl_H(y).$$

Proof. Let $Y = Y_1 \cup Y_2$ be the decomposition given by the definition of a nice subset. If $y \in Y_2$, then there exists some n such that $y^n \in [H,H]$. Then for any $\varphi \in Q(H)$, Bavard Duality gives

$$|\varphi(y)| \leq 2D(\varphi) scl_H(y). \tag{IV.22}$$

Now, let $A = H/[H,H]$, and let B be the image of $\langle Y \rangle$ inside A. Let θ be the quotient map $\theta \colon H \to A$. Then by definition of nice subsets we can define a homomorphism $\alpha : B \to \mathbb{R}$ such that $\alpha(\theta(y)) = \varphi(y)$ for all $y \in Y_1$. Since $\mathbb{R}$ is divisible, Lemma IV.3.5 allows us to extend α to all of A. Composing α with θ gives a homomorphism $\beta : H \to \mathbb{R}$ which satisfies $\beta(y) = \varphi(y)$ for all $y \in Y_1$. Now we set $\varphi' = \varphi - \beta$. Since β vanishes on $[H,H]$, $\varphi'|_{[H,H]} \equiv \varphi|_{[H,H]}$. Since φ' is a shift of φ by a homomorphism, $D(\varphi') = D(\varphi)$. Furthermore, combining the fact that $\varphi'(y) = 0$ for all $y \in Y_1$ with (IV.22), we get that for all $y \in Y$,

$$|\varphi(y)| \leq 2D(\varphi') scl_H(y).$$

□

We are now ready to prove Corollary I.2.4.

Proof of Corollary I.2.4. Let $H \hookrightarrow_h (G,X)$, and by Lemma III.1.9 there exists Y' a finite subset of H such that the relative metric $\widehat{d}$ on H is bi-Lipschitz equivalent to the word metric with respect to Y'. By Lemma IV.3.4 the subgroup $\langle Y' \rangle$ has a nice finite generating set Y. Let d_Y be the word metric with respect to Y. Then d_Y is bi-Lipschitz equivalent to the relative metric $\widehat{d}$ on H, so there exists a constant L such that for all $f,g \in H$,

$$d_Y(f,g) \leq L\widehat{d}(f,g). \tag{IV.23}$$

Fix some $h \in [H,H]$, and let $\varphi \in Q(H)$ be the quasimorphism which realizes the Bavard Duality; that is, $scl_H(h) = \frac{\varphi(h)}{2D(\varphi)}$. Let φ' be the modified quasimorphism provided by Lemma IV.3.6.

Let $\iota : Q(H) \to \widehat{Q}(G)$ be map provided by Theorem IV.2.2. Then by part (b) of Theorem IV.2.2 we have

$$D(\iota(\varphi')) \leq 54K + 66D(\varphi')$$

where K is defined by $K = \max\{|\varphi'(k)| : \widehat{d}(1,k) < 15C\}$. However, by (IV.23) we get $K \leq \max\{|\varphi'(k)| : d_Y(1,k) < 15CL\}$. Inductively applying the definition of a quasimorphism along with Lemma IV.3.6, for any such k we get

$$|\varphi'(k)| \leq 15CL(D(\varphi') + 2D(\varphi') \max_{y \in Y}\{scl_H(y)\}).$$

That is, we have bound K as a constant multiple of $D(\varphi')$. Thus there exists a constant M (which is independet of φ') such that

$$D(\iota(\varphi')) \leq 54K + 66D(\varphi') \leq MD(\varphi'). \tag{IV.24}$$

Now, $\iota(\varphi')$ is a quasimorphism on G, and in order to apply Bavard Duality we homogenize $\iota(\varphi')$ to get a quasimorphism ψ, satisfying $D(\psi) \leq 2D(\iota(\varphi'))$. Then applying the definition of ψ, along with the homgeneity of φ' and the conditions of Theorem IV.2.2 gives

$$\psi(h) = \lim_{n\to\infty} \frac{\iota(\varphi')(h^n)}{n} = \varphi'(h) = \varphi(h).$$

Also, (IV.24) and Lemma IV.3.3 show that $D(\psi) \leq 2D(\iota(\varphi')) \leq 2MD(\varphi') = 2MD(\varphi)$. Applying Bavard Duality again gives

$$scl_G(h) \geq \frac{\psi(h)}{2D(\psi)} \geq \frac{\varphi(h)}{4MD(\varphi)} = \frac{1}{2M} scl_H(h).$$

□

Proof of Corollary I.2.5. If H is an almost malnormal quasi-convex subgroup of a hyperbolic group, then G is hyperbolic relative to H [12]. Hence H is hyperbolically embedded in G by [24, Proposition 2.4] and the claim follows from Corollary I.2.4. □

Remark IV.3.7. Note that the malnormality condition can not be dropped in Corollary I.2.5 even for free groups. For example, let $F = F(x,y,t)$ be the free group of rank 3 with basis $\{x,y,t\}$. In what follows we write a^b for $b^{-1}ab$ and $[a,b]$ for $a^{-1}b^{-1}ab$. Let $H = \langle x,y,x^t,y^t\rangle$ and let

$$h_k = [x,y]^{-k}[x^t,y^t]^k.$$

Since the subset $\{x,y,x^t,y^t\} \subseteq G$ is Nielsen reduced, the subgroup H is freely generated by x,y,x^t,y^t. Therefore $scl_H(h_k) = k + 1/2$ (see [18, Example 2.100]). On the other hand, we have

$$scl_G(h_k) = scl_G([x,y]^{-k}([x,y]^k)^t) = scl_G([[x,y]^k,t]) \leq 1.$$

Thus $scl_H(h_k)/scl_G(h_k) \to \infty$ as $k \to \infty$.

CHAPTER V

SMALL CANCELLATION THEORY

V.1 Small cancellation conditions

Given a set of words $\mathscr{R}$ in an alphabet $\mathscr{A}$, we say that $\mathscr{R}$ is *symmetrized* if for any $R \in \mathscr{R}$, $\mathscr{R}$ contains all cyclic shifts of $R^{\pm 1}$. Further, if G is a group generated by a set $\mathscr{A}$, we say that a word R is (λ, c)*–quasi–geodesic* in G if any path in the Cayley graph $\Gamma(G, \mathscr{A})$ labeled by R is (λ, c)–quasi–geodesic.

We begin by giving the small cancellation conditions introduced by Olshanskii in [62] and also used in [67], [56], [44].

Definition V.1.1. Let G be a group generated by a set $\mathscr{A}$, $\mathscr{R}$ a symmetrized set of words in $\mathscr{A}$. For $\varepsilon > 0$, a subword U of a word $R \in \mathscr{R}$ is called an ε*–piece* if there exists a word $R' \in \mathscr{R}$ such that:

(1) $R \equiv UV$, $R' \equiv U'V'$, for some V, U', V';

(2) $U' = YUZ$ in G for some words Y, Z in $\mathscr{A}$ such that $\max\{\|Y\|, \|Z\|\} \le \varepsilon$;

(3) $YRY^{-1} \ne R'$ in the group G.

Similarly, a subword U of $R \in \mathscr{R}$ is called an ε'*–piece* if:

(1′) $R \equiv UVU'V'$ for some V, U', V';

(2′) $U' = YU^{\pm 1}Z$ in the group G for some Y, Z satisfying $\max\{\|Y\|, \|Z\|\} \le \varepsilon$.

Definition V.1.2. We say that the set $\mathscr{R}$ satisfies the $C(\varepsilon, \mu, \lambda, c, \rho)$*–condition* for some $\varepsilon \ge 0$, $\mu > 0$, $\lambda > 0$, $c \ge 0$, $\rho > 0$, if

(1) $\|R\| \ge \rho$ for any $R \in \mathscr{R}$;

(2) any word $R \in \mathscr{R}$ is (λ, c)–quasi–geodesic;

(3) for any ε–piece of any word $R \in \mathscr{R}$, the inequality $\max\{\|U\|, \|U'\|\} < \mu \|R\|$ holds (using the notation of Definition V.1.1).

Further the set $\mathscr{R}$ satisfies the $C_1(\varepsilon, \mu, \lambda, c, \rho)$*–condition* if in addition the condition (3) holds for any ε'–piece of any word $R \in \mathscr{R}$.

Suppose that G is a group defined by

$$G = \langle \mathscr{A} \mid \mathscr{O} \rangle. \tag{V.1}$$

Given a set of words $\mathscr{R}$, we consider the quotient group of G represented by

$$\overline{G} = \langle \mathscr{A} \mid \mathscr{O} \cup \mathscr{R} \rangle. \tag{V.2}$$

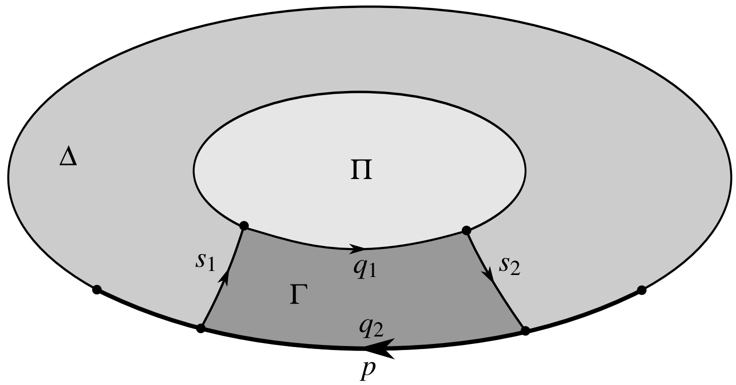

Figure V.1: Contiguity subdiagram.

A cell in a van Kampen diagram over (V.2) is called an *$\mathcal{R}$–cell* if its boundary label is a word from $\mathcal{R}$. Let Δ be a van Kampen diagram over (V.2), q a subpath of $\partial\Delta$, and Π an $\mathcal{R}$–cell of Δ. Suppose that there is a simple closed path

$$p = s_1 q_1 s_2 q_2 \tag{V.3}$$

in Δ, where q_1 is a subpath of $\partial\Pi$, q_2 is a subpath of q, and

$$\max\{l(s_1), l(s_2)\} \leq \varepsilon \tag{V.4}$$

for some constant $\varepsilon > 0$. By Γ we denote the subdiagram of Δ bounded by p. If Γ contains no $\mathcal{R}$–cells, we say that Γ is an *ε–contiguity subdiagram* (or simply a contiguity subdiagram if ε is fixed) of Π to the subpath q of $\partial\Delta$ and q_1 is the *contiguity arc* of Π to q. The ratio $\ell(q_1)/\ell(\partial\Pi)$ is called the *contiguity degree* of Π to q and is denoted by (Π, Γ, q). In case $q = \partial\Delta$, we talk about contiguity subdiagrams, etc., of Π to $\partial\Delta$. Since Γ contains no $\mathcal{R}$–cells, it can be considered a diagram over (V.1).

A van Kampen diagram Δ over (V.2) is said to be *reduced* if Δ has minimal number of $\mathcal{R}$–cells among all diagrams over (V.2) having the same boundary label. When dealing with a diagram Δ over (V.2), it is convenient to consider the following transformations. Let Σ be a subdiagram of Δ which contains no $\mathcal{R}$-cells, Σ' another diagram over (V.1) with $\mathbf{Lab}(\partial\Sigma) \equiv \mathbf{Lab}(\partial\Sigma')$. Then we can remove Σ and fill the obtained hole with Σ'. Note that this transformation does not affect $\mathbf{Lab}(\partial\Delta)$ and the number of $\mathcal{R}$-cells in Δ. If two diagrams over (V.2) can be obtained from each other by a sequence of such transformations, we call them *$\mathcal{O}$-equivalent*. [67, Lemma 4.4] provides an analogue to the well-known Greendlinger Lemma. We will make use of the more general version of this lemma appearing in the appendix of [67].

Lemma V.1.3 ([67], Lemma 9.7)**.** *Let G be a group with presentation (V.1). Suppose that the Cayley graph $\Gamma(G, \mathcal{A})$ of G is hyperbolic. Then for any $\lambda \in (0,1]$, $c \geq 0$, there exists $\varepsilon \geq 0$ such that for all $\mu \in (0, 1/16]$, there exists $\rho > 0$ with the following property. Let $\mathcal{R}$ be a symmetrized set of words in $\mathcal{A}$ satisfying the $C(\varepsilon, \mu, \lambda, c, \rho)$–condition, Δ a reduced van Kampen diagram over the presentation (V.2) such that $\partial\Delta = q_1 \cdots q_r$ for some $1 \leq r \leq 4$, where $q_1, \ldots, q_r$ are (λ, c)-quasi-geodesic. Assume that Δ has at least one $\mathcal{R}$–cell. Then up to passing to an $\mathcal{O}$-equivalent diagram, then there is an $\mathcal{R}$-cell Π of Δ and disjoint*

ε-contiguity subdiagrams Γ_j of Π to sections q_j, $j = 1, \dots, r$, of $\partial\Delta$ (some of them may be absent) such that

$$\sum_{j=1}^{r} (\Pi, \Gamma_j, q_j) > 1 - 13\mu.$$

This is actually a slight restatement of [67, Lemma 9.7], since we will need to choose ε independent of μ. However, this follows immediately from the choice of ε in the proof of this lemma (see [67, equation 36]). In fact, aside from the inductive proof of this lemma, [67] only makes use of the special case when $r = 1$; we will need the more general statement for the proof of Lemma V.3.7.

V.2 Small cancellation quotients

Is this section we prove various properties of small cancellation quotients. Analogous statements for relatively hyperbolic groups can be found in [67], and the proofs here are essentially the same. Throughout this section, we fix a group G and suppose $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$. By Theorem III.1.3, there exists a constant L such that G has a strongly bounded relative presentation $\langle X, \mathscr{H} \mid \mathscr{Q}\rangle$ which satisfies $Area^{rel}(W) \leq L\|W\|$ for any word W in $X \sqcup \mathscr{H}$ equal to the identity in G. Set $\mathscr{A} = X \sqcup \mathscr{H}$ and $\mathscr{O} = \mathscr{S} \cup \mathscr{Q}$, where $\mathscr{S}$ is defined as in (III.5). Hence G is given by the presentation (V.1). Let $\overline{G}$ denote the quotient of G given by the presentation (V.2).

Lemma V.2.1. *For any $\lambda \in (0,1]$, $c \geq 0$, $N > 0$, there exists $\mu > 0$, $\varepsilon > 0$, and $\rho > 0$ such that for any strongly bounded symmetrized set of words $\mathscr{R}$ satisfying the $C(\varepsilon, \mu, \lambda, c, \rho)$-condition, the following hold.*

1. *The restriction of the natural homomorphism $\gamma\colon G \to \overline{G}$ to $B_{\mathscr{A}}(N)$ is injective. In particular, $\gamma|_{\{H_\lambda\}_{\lambda\in\Lambda}}$ is injective.*
2. *$\{\gamma(H_\lambda)\}_{\lambda\in\Lambda} \hookrightarrow_h \overline{G}$.*

Proof. Clearly $\overline{G}$ is given by the strongly bounded relative presentation $\langle X, \mathscr{H} \mid \mathscr{Q} \cup \mathscr{R}\rangle$; let $Area_1^{rel}(W)$ (respectively, $Area_1^{rel}(\Delta)$) denote the relative area of a word W (respectively, diagram Δ) over this presentation. We will show that that any word W in $\mathscr{A}$ satisfies $Area_1^{rel}(W) \leq \alpha\|W\|$, where

$$\alpha = \frac{3L}{\lambda - 13\mu\lambda - 14\mu}.$$

We procede by induction on the length of W. Let p be a path in $\Gamma(G, \mathscr{A})$ labeled by W. First suppose that p is not a $(\frac{1}{2}, 0)$-quasi-geodesic. Then up to passing to a cyclic shift of W, we can assume that $p = p_0 p_1$, where $d_{\mathscr{A}}((p_0)_-, (p_0)_+) < \frac{\ell(p_0)}{2}$. Let $W = W_0 W_1$ be the corresponding decomposition of the word W. Let q be a geodesic in $\Gamma(G, \mathscr{A})$ from $(p_0)_-$ to $(p_0)_+$, and let U be the label of q. Then $W = (W_0 U^{-1})(U W_1)$. Now $W_0 U^{-1}$ represents 1 in G, and $\|W_0 U^{-1}\| \leq \ell(p_0) + \ell(q) \leq \frac{3\ell(p_0)}{2}$. Thus, $Area^{rel}(W_0 U^{-1}) \leq \frac{3L}{2}\ell(p_0)$. Now,

$\|UW_1\| \leq \ell(q)+\ell(p_1) \leq \frac{\ell(p_0)}{2}+\ell(p_1)$. Thus, applying the inductive hypothesis gives

$$Area_1^{rel}(W) \leq Area^{rel}(W_0U^{-1})+Area_1^{rel}(UW_1)$$

$$\leq \frac{3L}{2}\ell(p_0)+\alpha(\frac{\ell(p_0)}{2}+\ell(p_1))$$

$$\leq \alpha\|W\|$$

For $\alpha \geq 3L$.

Now suppose p is a $(\frac{1}{2},0)$-quasi-geodesic. Since the $C(\varepsilon,\mu,\lambda,c,\rho)$-condition becomes stronger as λ increases, without loss of generality we can assume $\lambda \leq \frac{1}{2}$, so p is a (λ,c)-quasi-geodesic. Thus, we can apply Lemma V.1.3 to find an $\mathscr{R}$-cell Π and an ε-contiguity subdiagram Γ from Π to $\partial\Delta$ such that $(\Pi,\Gamma,\partial\Delta)>1-13\mu$. Let $\partial\Gamma = s_1p_1s_2q_1$, where p_1 is a subpath of $\partial\Pi$, q_1 is a subpath of $\partial\Delta$, and $\ell(s_i)\leq\varepsilon$. Also let $\partial\Pi = p_0p_1$ and $\partial\Delta = q_0q_1$. Let Δ' be the subdiagram of Δ bounded by $q_0s_1^{-1}p_0s_2^{-1}$. Then

$$\ell(q_1)\geq d_{\mathscr{A}}((q_1)_-,(q_1)_+)\geq d_{\mathscr{A}}((p_1)_-,(p_1)_+)-2\varepsilon\geq\lambda\ell(p_1)-c-2\varepsilon\geq\lambda(1-13\mu)\ell(\partial\Pi)-c-2\varepsilon. \quad \text{(V.5)}$$

Also,

$$\ell(s_1^{-1}p_0s_2^{-1})\leq 13\mu\ell(\partial\Pi)+2\varepsilon. \quad \text{(V.6)}$$

Note that $\ell(s_1^{-1}p_0s_2^{-1})$ can be made smaller then $\ell(q_1)$ by taking sufficiently small μ and sufficiently large $\ell(\partial\Pi)$ (which can be done by taking large enough ρ). Thus, $\ell(\partial\Delta')\leq\ell(\partial\Delta)$. If n_1 is the number of cells in Δ' which contribute to the relative area of Δ, by the induction hypothesis, (V.5), and (V.6), we have

$$n_1 \leq \alpha\ell(\partial\Delta')\leq\alpha(\ell(\partial\Delta)-\ell(q_1)+\ell(s_1^{-1}p_0s_2^{-1}))$$

$$\leq \alpha(\ell(\partial\Delta)-(\lambda-13\mu\lambda-13\mu)\ell(\partial\Pi)+4\varepsilon+c).$$

Now, as q_1 is a $(\frac{1}{2},0)$ quasi-geodesic, we have

$$\ell(q_1)\leq 2d_{\mathscr{A}}((q_1)_-,(q_1)_+)\leq 2d_{\mathscr{A}}((p_1)_-,(p_1)_+)+4\varepsilon\leq 2(\ell(\partial\Pi))+4\varepsilon.$$

So,

$$\ell(\partial\Gamma)\leq\ell(p_1)+\ell(q_1)+2\varepsilon\leq 3(\ell(\partial\Pi))+6\varepsilon.$$

Thus if n_2 is the number of cells of Γ which contribute to the relative area of Δ, we have

$$n_2\leq L\ell(\partial\Gamma)\leq 3L\ell(\partial\Pi)+6\varepsilon\leq\alpha(\lambda-13\mu\lambda-14\mu)\ell(\partial\Pi)+6\varepsilon.$$

Thus,

$$Area_1^{rel}(\Delta) = n_1 + n_2 + 1 \leq \alpha(\ell(\partial\Delta) - (\lambda - 13\mu\lambda - 13\mu)\ell(\partial\Pi) + 4\varepsilon + c) + \alpha(\lambda - 13\mu\lambda - 14\mu)\ell(\partial\Pi) + 6\varepsilon + 1$$

$$\leq \alpha\ell(\partial\Delta) - \mu\ell(\partial\Pi) + 10\varepsilon + c + 1 \leq \alpha\ell(\partial\Delta).$$

when ρ is sufficiently large. Thus,

$$Area_1^{rel}(W) \leq Area_1^{rel}(\Delta) \leq \alpha\|W\|.$$

This completes the proof of the first condition. For the second condition, equation (V.5) along with the condition $\mu < \frac{1}{26}$ gives that

$$\|W\| \geq \frac{\lambda}{2}\rho - c - 2\varepsilon. \tag{V.7}$$

And thus is only remains to choose $\rho > \frac{2(N+c+2\varepsilon)}{\lambda}$. □

V.3 Torsion and conjugacy in the quotient

For this section, we keep the same assumptions about G, $\overline{G}$, and $\mathscr{A}$ as in the previous section; in addition, we assume that the action of G on $\Gamma(G,\mathscr{A})$ is acylindrical. This can be done with loss of generality by Theorem III.2.10. By a loxodromic element, we will mean an element which is loxodromic with respect to the action of G on $\Gamma(G,\mathscr{A})$. Also, let δ denote the hyperbolicity constant of $\Gamma(G,\mathscr{A})$. Recall that $\tau(g)$ denotes the translation length of the element g.

Theorem V.3.1. *[13] Suppose G acts acylindrically on a hyperbolic metric space. Then there exists $d > 0$ such that for all loxodromic elments g, $\tau(g) \geq d$.*

Given a word W in $\mathscr{A}$, we say that a word U is *W-periodic* if it is a subword of W^n for some $n \in \mathbb{Z}\setminus\{0\}$.

Corollary V.3.2. *Suppose that W is a word in $\mathscr{A}$ representing a loxodromic element $g \in G$ such that $|g|_{\mathscr{A}} = \|W\| \leq N$ for some $N > 0$. Then any path in $\Gamma(G,\mathscr{A})$ labeled by a W-periodic word is $(\frac{d}{N}, 2(N+d))$ quasi-geodesic.*

Proof. First observe that for any $n \in \mathbb{N}$,

$$|g^n|_{\mathscr{A}} \geq n\inf_i\left(\frac{1}{i}|g^i|_{\mathscr{A}}\right) \geq nd \geq \frac{d}{N}n|g|_{\mathscr{A}} \tag{V.8}$$

where d is the constant from Lemma V.3.1. Now suppose p is a path labeled by a W-periodic word. Let q be a maximal (maybe empty) subpath of p, labeled by W^n for some $n \in \mathbb{Z}$ (we identify W^0 with the empty word). Then, $\ell(p) \leq n|g|_{\mathscr{A}} + 2N$. Combining this with (V.8) and the triangle inequality, we get

$$d_{\mathscr{A}}(p_-, p_+) \geq d_{\mathscr{A}}(q_-, q_+) - 2N = |g^n|_{\mathscr{A}} - 2N \geq \frac{d}{N}n|g|_{\mathscr{A}} - 2N \geq \frac{d}{N}\ell(p) - 2N - 2d.$$

Since a subword of a W-periodic word is also W-periodic, we are done. □

Lemma V.3.3. *Suppose $g \in G$ is the shortest element in its conjugacy class and $|g|_{\mathscr{A}} > 8\delta$. Let W be a word in $\mathscr{A}$ representing g such that $\|W\| = |g|_{\mathscr{A}}$. Then W^n is a $(\frac{1}{3}, 2\delta)$ quasi-geodesic for all $n \in \mathbb{N}$. In particular, g is loxodromic.*

Proof. Since no cyclic shift of W can have shorter length then W, any path p labeled by W^n is a k-local geodesic where $k > 8\delta$, and hence W^n is a $(\frac{1}{3}, 2\delta)$ quasi-geodesic by Lemma II.0.8. □

Combining Corollary V.3.2 and Lemma V.3.3, we get uniform quasi-geodesic constants for any path labeled by W^n, where W represents a loxodromic element $g \in G$ and W is the shorest word representing an element in the conjugacy class of g. Specifically,

Corollary V.3.4. *There exists α and a such that the following holds: Let g be loxodromic and the shortest element in its conjugacy class, and let W be a word in $\mathscr{A}$ representing g such that $\|W\| = |g|_{\mathscr{A}}$. Then any path p labeled by W^n is a (α, a) quasi-geodesic for all $n \in \mathbb{N}$.*

Lemma V.3.5. *For any $\lambda \in (0,1]$, $c \geq 0$ there are $\mu > 0$, $\varepsilon \geq 0$, and $\rho > 0$ such that the following condition holds. Suppose that $\mathscr{R}$ is a symmetrized set of words in $\mathscr{A}$ satisfying the $C_1(\varepsilon, \mu, \lambda, c, \rho)$-condition. Let $\gamma \colon G \to \overline{G}$, where $\overline{G}$ is the quotient given by (V.2). Then every element of $\overline{G}$ of order n is the image of an element of G of order n.*

Proof. Note that it suffices to assume $\lambda < \alpha$ and $c > a$, as the $C_1(\varepsilon, \mu, \lambda, c, \rho)$-condition becomes stronger as λ increases and c decreases. Now we can choose μ, ε, and ρ satisfying the conditions of Lemma V.1.3 and Lemma V.2.1 with $N = 8\delta + 1$. Now suppose $\bar{g} \in \overline{G}$ has order n. Without loss of generality we assume that $\bar{g}$ is the shortest element of its conjugacy class. Let W be a shortest word in $\mathscr{A}$ representing $\bar{g}$ in $\overline{G}$, and let g be the preimage of $\bar{g}$ represented by W. Suppose towards a contradiction that $g^n \neq 1$.

Suppose first that g is elliptic. Then g^n is elliptic, and hence g^n is conjugate to an element h where $|h|_{\mathscr{A}} \leq 8\delta$ by Lemma V.3.3. But then $h \neq 1$ and $\gamma(h) = 1$, which contradicts the first condition of Lemma V.2.1.

Thus, we can assume that g is a loxodromic, and hence any path labeled by W^n is a (λ, c) quasi-geodesic by Lemma V.3.4. Now if Δ is a diagram over (V.2) with boundary label W^n, then Δ must contain $\mathscr{R}$-cells since $g^n \neq 1$. Now for sufficiently small μ and sufficiently large ε and ρ, Δ will violate the $C_1(\varepsilon, \mu, \lambda, c, \rho)$ condition; the proof of this is identical to the proof of [67, Lemma 6.3].

□

The next lemma provides a bound on contiguity degrees of $\mathscr{R}$-cells to paths with periodic labels and on a possible overlap between two contiguity subdiagrams to a geodesic if $\mathscr{R}$ satisfies a small cancellation condition.

Lemma V.3.6. *For any $\lambda \in (0,1]$, $c \geq 0$, $\varepsilon > 0$, and $N \in \mathbb{N}$, there exist constants $D = D(\varepsilon, \lambda, c, \delta, N)$ and $\varepsilon_1 \geq \varepsilon$ such that for all $\mu > 0$ and $\rho > 0$ and any set of words $\mathscr{R}$ satisfying the $C_1(\varepsilon_1, \mu, \lambda, c, \rho)$ condition, the following holds.*

(a) Let W be a word in $\mathscr{A}$ representing $g \in G$ such that $|g|_{\mathscr{A}} = \|W\| \leq N$. Then for any $R \in \mathscr{R}$ and any quadrangle $Q = s_1 q_1 s_2 q_2$ in $\Gamma(G, \mathscr{A})$, where $\ell(s_i) \leq \varepsilon$ for $i = 1, 2$, $\mathbf{Lab}(q_1)$ is a subword of R, and $\mathbf{Lab}(q_2)$ is W-periodic, we have $\ell(q_1) \leq D\mu\|R\| + D$.

(b) Let U and $V^{\pm 1}$ be disjoint subwords of some $R \in \mathscr{R}$, and let r be a geodesic path in $\Gamma(G, \mathscr{A})$. Suppose $q_1 s_1 r_1 t_1$ and $q_2 s_2 r_2 t_2$ are quadrangles in $\Gamma(G, \mathscr{A})$ such that $\mathbf{Lab}(q_1) \equiv U$, $\mathbf{Lab}(q_2) \equiv V$, r_1, r_2 are subpaths of $r^{\pm 1}$, and $\ell(s_i), \ell(t_i) \leq \varepsilon$ for $i = 1, 2$. Then the overlap between r_1 and r_2 is at most $\mu\|R\| + \varepsilon_1$.

Proof. Without loss of generality we can assume that s_1, s_2 are geodesic. Since the C_1 condition becomes stronger as λ increases and c decreases, it suffices to assume that $\lambda \leq \frac{d}{N}$ and $c \geq 2N + 2d$. Thus Q is a (λ, c)-quasi-geodesic quadrangle by Corollary V.3.2. Choose

$$\varepsilon_1 = 2(K + \varepsilon),$$

where $K = K(\lambda, c, \delta)$ is the constant provided by Lemma II.0.9.

Our proof of part (a) will closely follow the ideas from the proof of [67, Lemma 6.3]. Passing to a cyclic shift of $W^{\pm 1}$, we can assume q_2 is labeled by a prefix of W^n for some $n \in \mathbb{N}$. We will derive a contradiction under the assumption that q_1 is sufficiently long; the exact constant D can be easily extracted from the proof. First, note that the triangle inequality gives

$$\ell(q_1) \leq \frac{1}{\lambda}(d_{\mathscr{A}}((q_1)_-, (q_1)_+) + c) \leq \frac{1}{\lambda}(\ell(q_2) + 2\varepsilon + c). \tag{V.9}$$

Now, if $\ell(q_2) \leq \frac{4}{3}\|W\|$ we have

$$\ell(q_1) \leq \frac{1}{\lambda}\left(\frac{4}{3}N + 2\varepsilon\right) + c.$$

Thus, it suffices to assume $\ell(q_2) > \frac{4}{3}\|W\|$. Then we can decompose $\mathbf{Lab}(q_2)$ as $\mathbf{Lab}(q_2) \equiv UV_1UV_2$, where

$$\frac{\lambda^2 \ell(q_2)}{5} \leq \|U\| \leq \frac{\lambda^2 \ell(q_2)}{4} \tag{V.10}$$

and

$$\|V_1\| > \frac{\ell(q_2)}{3}. \tag{V.11}$$

Let $q_2 = u_1 v_1 u_2 v_2$ be the corresponding decomposition of the path q_2 (see Fig. V.2). Then by Lemma II.0.9, we can find an initial subpath r_1 of q_1^{-1} and a subpath r_2 of $q_1^{\pm 1}$ such that

$$d_{\mathscr{A}}((r_i)_\pm, (u_i)_\pm) \leq K + \varepsilon \tag{V.12}$$

for $i = 1, 2$. Now, we claim that for sufficiently long q_1, r_1 and r_2 will be disjoint. Indeed using (V.10) we obtain

$$\ell(r_1) \leq \frac{1}{\lambda}(d_{\mathscr{A}}((r_1)_-, (r_1)_+) + c) \leq \frac{1}{\lambda}(\ell(u_1) + 2\varepsilon + 2K + c) \leq \frac{\lambda \ell(q_2)}{4} + \frac{2\varepsilon + 2K + c}{\lambda}.$$

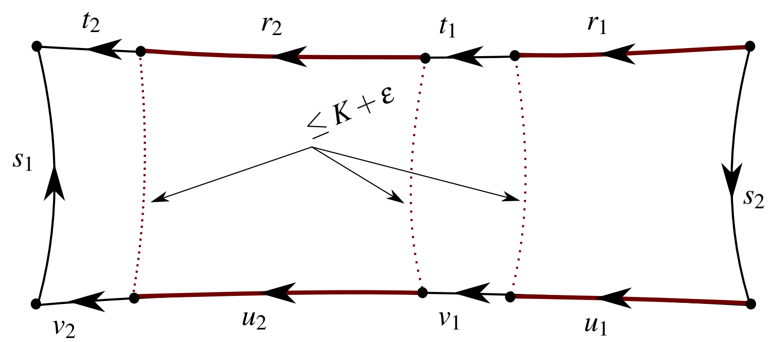

Figure V.2: Decompositions of q_1 and q_2 in the proof of Lemma V.3.6 (a).

However, if r_1 contains $(r_2)_-$, then by (V.11) we have

$$\ell(r_1) \geq d_{\mathscr{A}}((r_1)_-, (r_2)_-) \geq \lambda \ell(u_1 v_1) - c - 2\varepsilon - 2K \geq \frac{\lambda \ell(q_2)}{3} - c - 2\varepsilon - 2K.$$

These inequalities contradict each other for sufficiently large $\ell(q_2)$, which can be ensured if q_1 is long enough by (V.9).

Thus, we can decompose $q_1^{-1} = r_1 t_1 r_2^{\xi} t_2$, for some $\xi = \pm 1$ and t_1, t_2 where at least t_1 is non-empty. Let $\mathbf{Lab}(q_1)^{-1} \equiv R_1 T_1 R_2 T_2$ be the corresponding decomposition of the label of q_1^{-1}. Then by (V.12) we have $R_1 = Y_1 U Z_1$ and $R_2 = Y_2 U^{\pm 1} Z_2$ in G, where $\|Y_i\|, \|Z_i\| \leq K + \varepsilon$ for $i = 1, 2$. Thus, there exist Y, Z such that $\|Y\|, \|Z\| \leq 2(K+\varepsilon) = \varepsilon_1$ and $R_1 = Y R_2^{\pm 1} Z$ in G. Now, since $\mathscr{R}$ satisfies the $C_1(\varepsilon_1, \mu, \lambda, c, \rho)$–condition and R_1, R_2 are disjoint subwords of R, we have that $\|R_1\| \leq \mu \|R\|$. Finally, using (V.10) we obtain

$$\begin{aligned} \ell(q_2) &\leq \frac{5}{\lambda^2} \|U\| \leq \frac{5}{\lambda^3} (d_{\mathscr{A}}((u_1)_-, (u_1)_+) + c) \\ &\leq \frac{5}{\lambda^3} (\ell(r_1) + 2\varepsilon + 2K + c) \leq \frac{5}{\lambda^3} (\mu \|R\| + 2\varepsilon + 2K + c). \end{aligned}$$

Combining this with (V.9) produces a contradiction for sufficiently long q_1. This completes the proof of (a).

To prove (b) let r' denote the overlap of r_1 and r_2 with arbitrary orientation (see Fig. V.3). By Lemma II.0.9, we can choose points x_1, x_2 on q_1 such that $\mathrm{dist}((r')_-, x_1) \leq K + \varepsilon$ and $\mathrm{dist}((r')_+, x_2) \leq K + \varepsilon$. Similarly we choose y_1 and y_2 on q_2 satisfying the same conditions. Now, if $x_1 = x_2$ or $y_1 = y_2$, then

$$\ell(r') = d_{\mathscr{A}}((r')_-, (r')_+) \leq 2(K+\varepsilon) = \varepsilon_1.$$

Otherwise, we take p_1 to be the subpath of $q_1^{\pm 1}$ with endpoints x_1, x_2, and p_2 the subpath of $q_2^{\pm 1}$ with endpoints y_1, y_2. Then $d_{\mathscr{A}}((p_1)_\pm, (p_2)_\pm) \leq 2(K+\varepsilon)$. Thus, by the $C_1(\varepsilon_1, \mu, \lambda, c, \rho)$–condition, we have that $\ell(p_1) \leq \mu \|R\|$. Thus,

$$\ell(r') = d_{\mathscr{A}}((r')_-, (r')_+) \leq \mu \|R\| + 2(K+\varepsilon) = \mu \|R\| + \varepsilon_1.$$

□

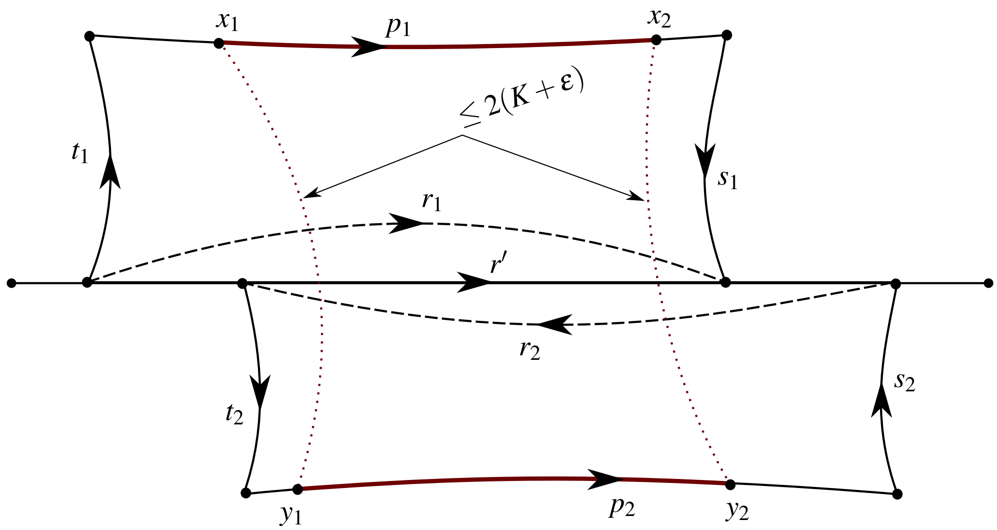

Figure V.3: The proof of Lemma V.3.6 (b).

Lemma V.3.7. *For any $\lambda \in (0,1]$, $c \geq 0$, and $N \in \mathbb{N}$, there exist $\varepsilon_1 > 0$, $\mu > 0$, and $\rho > 0$ such that the following holds. Suppose $\mathscr{R}$ satisfies the $C_1(\varepsilon_1, \mu, \lambda, c, \rho)$–condition. Let $\gamma\colon G \to \overline{G}$ be the natural epimorphism. Let g, h be loxodromic elements of G such that $|g|_{\mathscr{A}}, |h|_{\mathscr{A}} \leq N$, and $x \in G$ such that $\gamma(x^{-1}g^n x h^n) = 1$ in $\overline{G}$ but $x^{-1}g^n x h^n \neq 1$ in G. Then there exists y such that $|y|_{\mathscr{A}} < |x|_{\mathscr{A}}$, and $\gamma(y^{-1}g^n y h^n) = 1$ in $\overline{G}$. Furthermore, $x \in \langle g, y \rangle$ in $\overline{G}$.*

Proof. Again, as in the proof of Lemma V.3.6, it is sufficient to assume $\lambda \leq \frac{d}{N}$, $c \geq 2N + 2d$. Let ε be chosen according to Lemma V.1.3. Now choose D and ε_1 according to Lemma V.3.6. Recall that D is independent of μ and ρ. We will show that the conclusion of the lemma holds for sufficiently small μ and sufficiently large ρ.

Let W, V, and X be shortest words in $\mathscr{A}$ representing g, h, and x respectively. Let Δ be a reduced van Kampen diagram over (V.2) with $\partial\Delta = p_1 p_2 p_3 p_4$, with $\mathbf{Lab}(p_1)^{-1} \equiv \mathbf{Lab}(p_3) \equiv X$, $\mathbf{Lab}(p_2) \equiv W^n$, and $\mathbf{Lab}(p_4) \equiv V^n$. Then p_1 and p_3 are geodesic paths by our choice of X, and p_2 and p_4 are (λ, c) quasi-geodesics by Corollary V.3.2. Since $x^{-1}g^n x h^n \neq 1$ in G, Δ must contain an $\mathscr{R}$-cell. Since $\varepsilon_1 \geq \varepsilon$, $\mathscr{R}$ also satisfies $C_1(\varepsilon, \mu, \lambda, c, \rho)$, hence we can apply Lemma V.1.3 for $\mu \in (0, 1/16]$ and large enough ρ. That is, passing to an $\mathscr{O}$-equivalent diagram if necessary, we can find an $\mathscr{R}$-cell Π of Δ and disjoint ε-contiguity subdiagrams Γ_j of Π to p_j, $j = 1, \ldots, 4$, (some of which may be empty) such that

$$\sum_{j=1}^{4} (\Pi, \Gamma_j, p_j) > 1 - 13\mu. \tag{V.13}$$

Further by Lemma V.3.6,

$$(\Pi, \Gamma_2, p_2) + (\Pi, \Gamma_4, p_4) \leq 2\left(D\mu + \frac{D}{\ell(\partial\Pi)}\right). \tag{V.14}$$

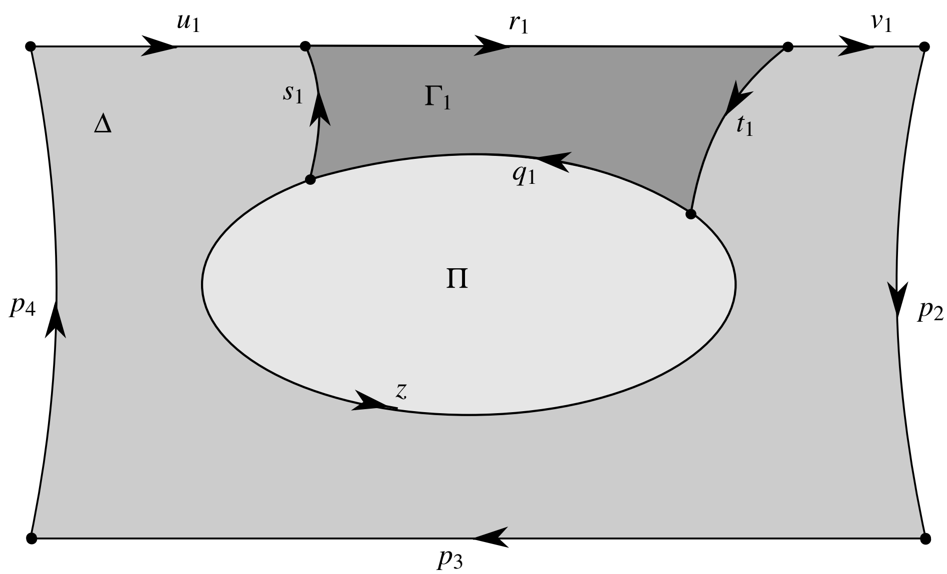

Figure V.4: Case 1 in the proof of Lemma V.3.7.

Thus, for at least one of $i = 1,3$, we have

$$(\Pi, \Gamma_i, p_i) > \frac{1}{2}\left(1 - (13+2D)\mu - \frac{2D}{\ell(\partial\Pi)}\right). \tag{V.15}$$

Without loss of generality, we assume this holds for $i = 1$. Let $\partial\Gamma_1 = s_1 r_1 t_1 q_1$, where $\ell(s_i) \leq \varepsilon$ for $i = 1,2$, q_1 is a subpath of $\partial\Pi$, r_1 is a subpath of p_1. There are two cases to consider.

Case 1. First suppose Γ_3 is empty. Then $(\Pi, \Gamma_1, q_1) > \left(1 - (13+2D)\mu - \frac{2D}{\ell(\partial\Pi)}\right)$, so $\ell(q_1) > (1-(13+2D)\mu)\ell(\partial\Pi) - 2D$. Let $\partial\Pi = q_1 z$ (see Fig. V.4). Then the path $s_1^{-1} z t_1^{-1}$ has the same start and end vertices as r_1. However,

$$\begin{aligned}\ell(s_1^{-1} z t_1^{-1}) &\leq \ell(\partial\Pi) - \ell(q_1) + 2\varepsilon < \ell(\partial\Pi) - (1-(13+2D)\mu)\ell(\partial\Pi) + 2D + 2\varepsilon \\ &= (13+2D)\mu\ell(\partial\Pi) + 2D + 2\varepsilon\end{aligned}$$

while

$$\begin{aligned}\ell(r_1) &\geq d_{\mathscr{A}}((q_1)_-, (q_1)_+) - 2\varepsilon \geq \lambda\ell(q_1) - c - 2\varepsilon \\ &> \lambda((1-(13+2D)\mu)\ell(\partial\Pi) - 2D) - c - 2\varepsilon.\end{aligned}$$

Thus, for sufficiently small μ and suffiently large $\ell(\partial\Pi)$ (which can be ensured by choosing large enough ρ), we will get that $\ell(r_1) > \ell(t_1 z s_1)$. However, this contradicts the fact that r_1 is a subpath of a geodesic p_1. Thus, we can assume that Γ_3 is non-empty.

Case 2. Let $\partial\Gamma_3 = s_2 r_2 t_2 q_2$, where $\ell(t_2), \ell(s_2) \leq \varepsilon$, q_2 is a subpath of $\partial\Pi$, and r_2 is a subpath of p_3. Also, let $\partial\Pi = q_1 z_1 q_2 z_2$. Now, we decompose $p_1 = u_1 r_1 v_1$ and $p_3 = u_2 r_2 v_2$. For definiteness we suppose that $\ell(v_2) \leq \ell(u_1)$ (in the case $\ell(v_2) \geq \ell(u_1)$ the proof is similar). Let Δ' be a copy of Δ. Given a path a in

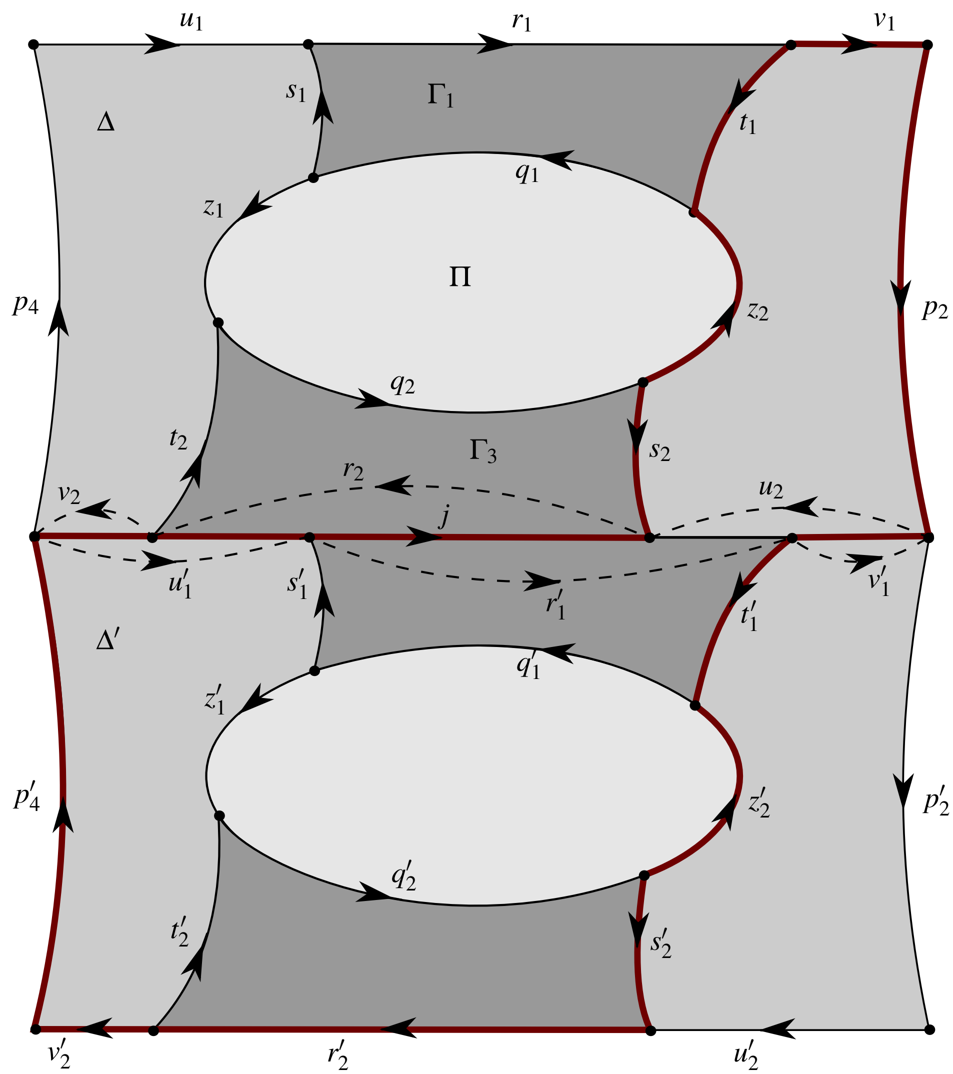

Figure V.5: Case 2 in the proof of Lemma V.3.7.

Δ, we denote its copy by a'. Let us glue Δ and Δ' by identifying p_3 to $(p_1')^{-1}$(see Fig. V.5). Let j denote the (possibly empty) intersection of r_1' and r_2^{-1}. Consider the path $p = v_2^{-1}r_2^{-1}s_2^{-1}z_2t_1^{-1}v_1$. We want to show that

$$\ell(p) < \|X\| = \ell(p_1). \tag{V.16}$$

Observe that $\ell(p) = \ell(p_1) - \ell(r_1) + \ell(j) + \ell(s_2^{-1}z_2t_1^{-1})$. Thus, we only need to show that $\ell(r_1) > \ell(j) + \ell(s_2^{-1}z_2t_1^{-1})$. By Lemma V.3.6, $\ell(j) \le \mu\ell(\partial\Pi) + \varepsilon_1$. Also,

$$\ell(z_2) \le \ell(\partial\Pi) - \ell(q_1) - \ell(q_3) \le (13+2D)\mu\ell(\partial\Pi) + 2D$$

by (V.13) and (V.14). Thus,

$$\begin{aligned}\ell(j) + \ell(s_2^{-1}z_2t_1^{-1}) &\le \mu\ell(\partial\Pi) + \varepsilon_1 + (13+2D)\mu\ell(\partial\Pi) + 2D + 2\varepsilon \\ &= (14+2D)\mu\ell(\partial\Pi) + \varepsilon_1 + 2D + 2\varepsilon.\end{aligned}$$

However, by (V.15)

$$\begin{gathered}\ell(r_1) \ge d_{\mathscr{A}}((q_1)_-, (q_1)_+) - 2\varepsilon \ge \lambda\ell(q_1) - c - 2\varepsilon > \\ \frac{\lambda}{2}(1-(13+2D)\mu)\ell(\partial\Pi) - \lambda D - c - 2\varepsilon.\end{gathered}$$

Thus, we will have $\ell(r_1) > \ell(j) + \ell(s_2^{-1}z_2t_1^{-1})$ as long as μ is sufficiently small and $\ell(\partial\Pi)$ is sufficiently large; the later condition can be guaranteed by choosing sufficiently large ρ. This completes the proof of (V.16).

Now let y be the element of $\overline{G}$ represented by $\mathbf{Lab}(p)$. By (V.16), we have $|y|_{\mathscr{A}} < |x|_{\mathscr{A}}$. Observe that $pp_2(p')^{-1}p_4'$ is a closed path (it is represented by the bold line on Fig. V.5), hence $yg^ny^{-1}h^n = \mathbf{Lab}(p)\mathbf{Lab}(p_2)\mathbf{Lab}((p')^{-1})\mathbf{Lab}(p_4') = 1$ in $\overline{G}$. Finally, observe that pp_2p_3 is also a closed path, so $yg^nx = 1$, therefore $x \in \langle g, y\rangle$. ☐

Corollary V.3.8. *For all $\lambda \in (0,1]$, $c \ge 0$ and $N \in \mathbb{N}$, there exist $\varepsilon_1 > 0$, $\mu > 0$, and $\rho > 0$ such that if $\mathscr{R}$ satisfies the $C_1(\varepsilon_1,\mu,\lambda,c,\rho)$ and γ is the natural epimorphism from G to $\overline{G}$, then the following conditions are satisfied:*

(a) If $g, h \in B_{G,\mathscr{A}}(N)$, then $\gamma(g) \sim \gamma(h)$ if and only if $g \sim h$.

(b) If $g \in B_{G,\mathscr{A}}(N)$ is loxodromic and $x \in G$, then there exists n such that $\gamma(x^{-1}g^nxg^{\pm n}) = 1$ if and only if $\gamma(x) \in \gamma(E_G(g))$.

Proof. First, choose ε_1, μ, ρ satisfying the conditions of Lemma V.3.7. Suppose g and h are non-conjugate elements of G which become conjugate in $\overline{G}$, and $g, h \in B_{G,\mathscr{A}}(N)$. For convenience we identify g and h with their images in $\overline{G}$. Suppose x is the shortest element in $\overline{G}$ satisfying $g^x = h$. But then by Lemma V.3.7 there exists a strictly shorter element y such that $g^y = h$, contradicting our choice of x. This proves (a).

Note that the "only if" part of (b) follows immediately from Lemma III.2.6. Now suppose $g \in B_{G,\mathscr{A}}(N)$ is a loxodromic element, and let x be a shortest element such that for some n, $\gamma(x^{-1}g^nxg^{\pm n}) = 1$ but $\gamma(x) \notin$

$\gamma(E_G(g))$. Thus, $x \notin E_G(g)$, so $x^{-1}g^nxg^{\pm n} \neq 1$ by Lemma III.2.6. so we can apply Lemma V.3.7 to find a strictly shorter element y satisfying $\gamma(y^{-1}g^nyg^{\pm n}) = 1$. By our choice of x, $\gamma(y) \in \gamma(E_G(g))$, so we can replace y with y' such that $y' \in E_G(g)$ and $\gamma(y) = \gamma(y')$. However, Lemma V.3.7 also gives that $\gamma(x) \in \langle \gamma(g), \gamma(y') \rangle \leq \gamma(E_G(g))$, a contradiction.

□

Remark V.3.9. If $g \in B_{G,\mathscr{A}}(N)$ is loxodromic and $\gamma(g)$ is a loxodromic, WPD element of $\overline{G}$, then part (b) of Corollary V.3.8 along with Lemma III.2.6 give that $\gamma(E_G(g)) = E_{\overline{G}}(\gamma(g))$.

V.4 Small cancellation words and suitable subgroups

Let $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G,X)$. We will consider the the set of words W in $X \sqcup \mathscr{H}$ which satisfy:

(W1) W contains no subwords of the form xy where x, $y \in X$.

(W2) If W contains $h \in H_\lambda$, then $\widehat{\ell}_\lambda(h) \geq 50C$, where $C = C(1,0)$ is the constant from Lemma III.1.11.

(W3) If W contains a subword h_1xh_2 (respectively, h_1h_2) where $x \in X$, $h_1 \in H_\lambda$ and $h_2 \in H_\mu$, then either $\lambda \neq \mu$ or the element of G represented by x does not belong to H_λ (respectively, $\lambda \neq \mu$).

Recall that paths p and q are called *oriented ε-close* if $d(p_-,q_-) \leq \varepsilon$ and $d(p_+,q_+) \leq \varepsilon$.

Lemma V.4.1. *[24, Lemma 4.27]*

1. *If p is a path in $\Gamma(G,X \sqcup \mathscr{H})$ labeled by a word which satisfies $(W1)-(W3)$, then p is a $\left(\frac{1}{4},1\right)$ quasi-geodesic.*

2. *For all $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists a constant $M = M(\varepsilon,k)$ such that if p and q are oriented ε-close paths in $\Gamma(G,X \sqcup \mathscr{H})$ whose labels satisfy $(W1)-(W3)$ and $\ell(p) \geq M$, then at least k consecutive components of p are connected to consecutive components of q.*

We will also consider words W which satisfy

(W4) All components of W belong to $H_\alpha \cup H_\beta$, where $H_\alpha \cap H_\beta = \{1\}$.

Lemma V.4.2. *Suppose p and q are oriented ε-close paths in $\Gamma(G,X \sqcup \mathscr{H})$ which are labeled by words which satisfy $(W1)-(W4)$. If $\ell(p) \geq M(\varepsilon,3)$. Then p and q have a common edge.*

Proof. By Lemma V.4.1, $p = p_1u_1u_2u_3p_2$ and $q = q_1v_1v_2v_3q_2$, where each u_i is a component of p connected to the component v_i of q (note that each component consists of a single edge). Without loss of generality we assume that u_1 and u_3 are H_α components and u_2 is an H_β component. Now if e is an edge from $(u_1)_+ = (u_2)_-$ to $(v_1)_+ = (v_2)_-$, then $\mathbf{Lab}(e) \in H_\alpha \cap H_\beta = \{1\}$. Thus, these vertices actually coincide, that is $(u_2)_- = (v_2)_-$. Similarly, $(u_2)_+ = (v_2)_+$, and since there is a unique edge labeled by an element of H_β between these vertices, we have that $u_2 = v_2$. □

Proposition V.4.3. *Suppose $W = xa_1..a_n$ satisfies $(W1)-(W4)$, where $x \in X \cup \{1\}$ and each $a_i \in H_\alpha \cup H_\beta$. Suppose, in addition, $a_1^{\pm 1}, ..., a_n^{\pm 1}$ are all distinct. Let $M = M(\varepsilon, 3)$ be the constant given by Lemma V.4.1. Then the set $\mathscr{R}$ of all cyclic shifts of $W^{\pm 1}$ satisfies the $C_1(\varepsilon, \frac{M}{n}, \frac{1}{4}, 1, n)$-condition.*

The proof is essentially the same as [67, Theorem 7.5].

Proof. Clearly $\mathscr{R}$ satisfies the first condition of Definition V.1.2. Lemma V.4.1 gives that $\mathscr{R}$ satisfies the second condition of Definition V.1.2. Now suppose U is an ε-piece of some $R \in \mathscr{R}$; without loss of generality we assume $\|U\| = \max\{\|U\|, \|U'\|\}$. Assume

$$\|U\| \geq \frac{M}{n}\|R\| \geq M \tag{V.17}$$

By the definition of a piece, there are oriented ε-close paths p and q in $\Gamma(G, X \sqcup \mathscr{H})$ such that $\mathbf{Lab}(p) \equiv U$, $\mathbf{Lab}(q) \equiv U'$. (V.17) gives that p and q satisfy the conditions of Lemma V.4.2, and thus p and q share a commone edge e. Thus, we can decompose $p = p_1 e p_2$ and $q = q_1 e q_2$; let $U_1 \mathbf{Lab}(e) U_2$ be the corresponding decomposition of U and $U_1' \mathbf{Lab}(e) U_2'$ the corresponding decomposition of U'. Let u be a path from q_- to p_- such that $\ell(u) \leq \varepsilon$, and let $Y = \mathbf{Lab}(u)$. Then

$$R \equiv U_1 \mathbf{Lab}(e) U_2 V$$

and

$$R' \equiv U_1' \mathbf{Lab}(e) U_2' V'.$$

Since $\mathbf{Lab}(e)$ only appears once in $W^{\pm}$, we have that R and R' are cyclic shifts of the same word and

$$U_2 V U_1 = U_2' V' U_1'.$$

Also $Y = U_1' U_1^{-1}$ since this labels the cycle $u p_1 q_1^{-1}$. Thus,

$$YRY^{-1} = U_1' U_1^{-1} U_1 \mathbf{Lab}(e) U_2 V U_1 (U_1')^{-1} = U_1' \mathbf{Lab}(e) U_2' V' = R'$$

which contradicts the definition of a ε-piece.

Similiarly, if U is an ε'-piece, then $R \equiv UVU'V'$, and arguing as above we get that U and U' share a common letter from $X \sqcup \mathscr{H}$. However each letter $a \in X \sqcup \mathscr{H}$ appears at most once in R, and if a appears then a^{-1} does not. □

Suitable subgroups Our goal now will be to give conditions under which we can find words which satisfying the conditions of Theorem V.4.3.

Fix $\mathscr{A} \subset G$ such that $\Gamma(G, \mathscr{A})$ is hyperbolic and G acts acylindrically on $\Gamma(G, \mathscr{A})$. For the rest of this section, unless otherwise stated a subgroup will be called non-elementary if it is non-elementary with respect to the action of on $\Gamma(G, \mathscr{A})$. Similarly, an element will be called loxodromic if it is loxodromic with respect to this action. In particular, all loxodromic elements will satisfy WPD.

Lemma V.4.4. *Suppose S is a non-elementary subgroup. Then for all $k \geq 1$, S contains pairwise non-commensurable loxodromic elements $f_1, ..., f_k$, such that $E(f_i) = E^+(f_i)$.*

Proof. We will basically follow the proof of [24, Lemma 6.16]. By Theorem III.2.3, since S is non-elementary, it contains a loxodromic element h, and an element g such that $g \notin E_G(h)$. By Lemma III.2.9, for sufficiently large n_1, n_2, n_2, $gh^{n_1}, gh^{n_2}, gh^{n_3}$ are pairwise non-commensurable loxodromic elements with respect to $\Gamma(G, \mathscr{A} \sqcup E_G(h))$, and by Lemma III.2.12 these elements are loxodromic with respect to $\Gamma(G, \mathscr{A})$. Thus, letting $H_i = E_G(gh^{n_i})$, we get that $\{H_1, H_2, H_3\} \hookrightarrow_h (G, \mathscr{A})$ by Corollary III.3.3. Now we can choose $a \in H_1 \cap S$, $b \in H_2 \cap S$ which satisfy $\widehat{\ell_1}(a) \geq 50C$ and $\widehat{\ell_2}(b) \geq 50C$, where C is the constant given by Lemma III.1.11. Then ab cannot belong to H_3 by Lemma III.1.11, so by Lemma III.2.9 we can find $c_1, ..., c_k \in H_3 \cap S$ such that $\widehat{\ell_3}(c_i) \geq 50C$ and the elements $f_i = abc_i$ are non-commensurable, loxodromic WPD elements with respect to the action of G on $\Gamma(G, \mathscr{A}_1)$, where $\mathscr{A}_1 = \mathscr{A} \sqcup H_1 \sqcup H_2 \sqcup H_3$. Next we will show that $E_G(f_i) = E_G^+(f_i)$. Suppose that $t \in E_G(f_i)$. Then for some $n \in \mathbb{N}$, $t^{-1} f_i^n t = f_i^{\pm n}$. Let $\varepsilon = |t|_{\mathscr{A}_1}$. Then there are oriented ε-close paths p and q labeled by $(abc_i)^n$ and $(abc_i)^{\pm n}$. Passing to a multiple of n, we can assume that $n \geq M$ where $M = M(\varepsilon, 2)$ is the constant provided by Lemma V.4.1. Then the labels of p and q satisfy $(W1)$-$(W3)$, so we can apply Lemma V.4.1 to get that p and q have two consecutive components. But then the label of q must be $(abc_i)^n$, because the sequences 123123... and 321321... have no common subsequences of length 2. Thus, $t^{-1} f_i^n t = f_i^n$, hence $t \in E_G^+(f_i)$. Finally, note that each f_i is loxodromic with respect to the action of G on $\Gamma(G, \mathscr{A})$ by Lemma III.2.12. $\square$

Now given a subgroup $S \leq G$, let $\mathscr{L}_S = \{h \in S \mid h \text{ is loxodromic and } E_G(h) = E_G^+(h)\}$. Now define $K_G(S)$ by

$$K_G(S) = \bigcap_{h \in \mathscr{L}_S} E_G(h).$$

The following lemma shows that $K_S(G)$ can be defined independently of $\Gamma(G, \mathscr{A})$.

Lemma V.4.5. *Let S be a non-elementary subgroup of G. Then $K_G(S)$ is the maximal finite subgroup of G normalized by S. In addition, for any infinite subgroup $H \leq S$ such that $H \hookrightarrow_h G$, $K_G(S) \leq H$.*

Proof. By Lemma V.4.4, $\mathscr{L}_S$ contains non-commensurable elements f_1 and f_2. Then by Lemma III.1.5 $K_G(S) \subset E_G(f_1) \cap E_G(f_2)$ is finite. $K_G(S)$ is normalized by S as the set $\mathscr{L}_S$ is invariant under conjugation by S and for each $g \in S$, $h \in \mathscr{L}_S$, $E_G(g^{-1}hg) = g^{-1}E_G(h)g$. Now suppose N is a finite subgroup of G such that for all $g \in S$, $g^{-1}Ng = N$. Then for each $h \in \mathscr{L}_S$, there exists n such that $N \leq C_G(h^n)$, and thus $N \leq E_G(h)$ for all $h \in \mathscr{L}_S$.

Suppose now that $H \leq S$ and $H \hookrightarrow_h G$. Then a finite-index subgroup of H centralizes $K_S(G)$, and hence $K_S(G) \leq H$ by Lemma III.1.5. $\square$

In our notation, the finite radical $K(G)$ is the same as $K_G(G)$. Now if S is a non-elementary subgroup of $G \in \mathscr{AH}$, then $S \in \mathscr{AH}$, so S has a finite radical $K(S)$. Clearly $K(G) \cap S \leq K(S) \leq K_G(S)$, but in general none of the reverse inclusions hold. Indeed suppose $S \in \mathscr{AH}$ with $K(S) \neq \{1\}$. Let $G = (S \times A) * H$, where A is finite and H is non-trivial. Then $K(G) = \{1\}$ and $K_G(S) = K(S) \times A$.

Lemma V.4.6. *Let S be a non-elementary subgroup of G. Then we can find non-commensurable, loxodromic elements $h_1,...,h_m$ such that $E_G(h_i) = \langle h_i \rangle \times K_G(S)$.*

Proof. First, since $K_G(S)$ is finite, we can find non-commensurable elements $f_1,...,f_k \in \mathscr{L}_S$ such that $K_G(S) = E_G(f_1) \cap ... \cap E_G(f_k)$, and we can further assume that $k \geq 3$. By Lemma III.3.3, $\{E_G(f_1),...,E_G(f_k)\} \hookrightarrow_h (G,\mathscr{A})$. Let $\mathscr{A}_1 = \mathscr{A} \sqcup E_G(f_1) \sqcup ... \sqcup E_G(f_k)$, and consider the action of G on $\Gamma(G,\mathscr{A}_1)$. For each $1 \leq i \leq k$ set $a_i = f_i^{n_i}$ where n_i is chosen such that

1. $E_G(f_i) = C_G(a_i)$
2. $h = a_1...a_k$ is a loxodromic WPD element with respect to $\Gamma(G,\mathscr{A}_1)$.
3. $\widehat{\ell_i}(a_i) \geq 50C$.

(Here $\widehat{\ell_i}$ denotes the relative length of elements of $E_G(f_i)$). The first condition can be ensured by Lemma III.2.6, the second by Lemma III.2.9 (note that $a_1...a_{k-1} \notin E_G(h_k)$ by lemma III.1.11), and the third by passing to a sufficiently high multiple of a power which satisfies the first two. We will show that, in fact, $E_G(h) = \langle h \rangle \times K_S(G)$. Let $t \in E_G(h)$, and let $\varepsilon = |t|_{\mathscr{A}_1}$. Then by Lemma III.2.6, there exists n such that

$$t^{-1}h^n t = h^{\pm n}. \tag{V.18}$$

Up to passing to a multiple of n, we can assume that

$$n \geq \frac{M}{k}$$

Where $M = M(\varepsilon,k)$ is the constant provided by Lemma V.4.1. Now (V.18) gives that there oriented ε-close paths p and q in $\Gamma(G,\mathscr{A}_1)$, such that p is labeled by $(a_1...a_k)^n$ and q is labeled by $(a_1...a_k)^{\pm n}$; notice that the label of these paths satisfy the conditions $(W1)-(W3)$. Furthermore, there is a path r connecting p_- to q_- such that $\mathbf{Lab}(r) = t$. Now we can apply Lemma V.4.1 to get k consecutive components of p connected to consecutive components of q. As in the proof of Lemma V.4.4, this gives that q is labeled by $(a_1...a_k)^n$ (not $(a_1...a_k)^{-n}$) since $k \geq 3$. Let $p = p_0u_1...u_kp_1$ and $q = q_0v_1...v_kq_1$, where each u_i is a component of p connected to the component v_i of q. Let e_0 be the edge which connects $(u_1)_-$ and $(v_1)_-$, and let e_i be the edge which connects $(u_i)_+$ to $(v_i)_+$. Let $c = \mathbf{Lab}(e_0)$.

Since $E_G(f_i) = C_G(a_i)$ for each $1 \leq i \leq k$, we get that c commutes with $\mathbf{Lab}(u_1) = \mathbf{Lab}(v_1)$. Thus, $c = \mathbf{Lab}(e_1)$, and repeating this arguement we get that $c = \mathbf{Lab}(e_i)$ for each $0 \leq i \leq k$. Thus, $c \in E_G(f_1) \cap ... \cap E_G(f_k) = K_G(S)$. Now observe that $\mathbf{Lab}(p_0) = (a_1...a_k)^l a_1....a_j$ and $\mathbf{Lab}(q_0) = (a_1...a_k)^m a_1....a_j$ for some $m,l \in \mathbb{N}$ and $0 \leq j \leq k$. Now $rq_0e_0^{-1}p_0^{-1}$ is a cycle in $\Gamma(G,\mathscr{A}_1)$ and c commutes with each y_i, so we get that

$$t = (a_1...a_k)^l a_1....a_j c a_j^{-1}...a_1^{-1}(a_1...a_k)^{-m} = h^{l-m}c$$

Thus, we have shown that $E_G(h) = \langle h \rangle K_S(G)$. Finally, note that all elements of $K_G(S)$ commute with each a_i and hence commute with h. Therefore, $E_G(h) = \langle h \rangle \times K_G(S)$. Now if we set $h_i = a_1...a_k^{l_i}$ for sufficiently large l_i, the elements $h_1...h_m$ will all be loxodromic, WPD elements with respect to $\Gamma(G,\mathscr{A}_1)$ by Lemma

III.2.9, and the same proof will show that each h_i will satisfy $E_G(h_i) = \langle h_i \rangle \times K_G(S)$. It only remains to note that these elements are all loxodromic with respect to $\Gamma(G,\mathscr{A})$ by Lemma III.2.12.

□

Definition V.4.7. A subgroup S of a group $G \in \mathscr{AH}$ is called *suitable with respect to* $\Gamma(G,\mathscr{A})$ if the following holds:

1. $\Gamma(G,\mathscr{A})$ is hyperbolic and the action of G on $\Gamma(G,\mathscr{A})$ is acylindrical.
2. The action of S on $\Gamma(G,\mathscr{A})$ is non-elementary.
3. $K_G(S) = \{1\}$.

We will further say that a subgroup is *suitable* if it is suitable with respect to some $\Gamma(G,\mathscr{A})$.

The next two results characterize suitable subgroups by the cyclic hyperbolically embedded subgroups they contain. The first is an immediate corollary of Lemma V.4.6 and Lemma III.3.3.

Corollary V.4.8. *Suppose S is suitable with respect to $\Gamma(G,\mathscr{A})$. Then for all $k \in \mathbb{N}$, S contains non-commensurable, loxodromic elements $h_1,\dots,h_k$ such that $E_G(h_i) = \langle h_i \rangle$ for $i = 1,\dots,k$. In particular, $\{\langle h_1 \rangle,\dots,\langle h_k \rangle\} \hookrightarrow_h (G,\mathscr{A})$.*

Proposition V.4.9. *If S contains an infinite order element h such that $\langle h \rangle$ is a proper subgroup of S and $\langle h \rangle \hookrightarrow_h (G,X)$, then S is suitable with respect to $\Gamma(G,\mathscr{A})$ for some $\mathscr{A} \supset X$.*

Proof. Note that Lemma III.1.5 gives that $\langle h \rangle$ does not have finite index in any subgroup of G, so S is not virtually cyclic. By Theorem III.2.10, there exists $X \subset Y \subset G$ such that $\langle h \rangle \hookrightarrow_h (G,Y)$ and the action of G on $\Gamma(G, Y \sqcup \langle h \rangle)$ is acylindrical; set $\mathscr{A} = Y \sqcup \langle h \rangle$. Now if $g \in S \setminus \langle h \rangle$, then there exists $n \in \mathbb{N}$ such that gh^n is loxodromic with respect to $\Gamma(G,\mathscr{A})$ by Lemma III.2.9. Since S is not virtually cyclic, the action of S on $\Gamma(G,\mathscr{A})$ is non-elementary by Theorem III.2.3. Finally, by Lemma V.4.5, $K_G(S)$ is a finite subgroup of $\langle h \rangle$, thus $K_G(S) = \{1\}$.

□

The next lemma follows from Proposition V.4.9 and Lemma III.1.7.

Lemma V.4.10. *Suppose $H \in \mathscr{AH}$, S is a suitable subgroup of H, and $H \hookrightarrow_h G$. Then S is a suitable subgroup of G.*

Notice a subgroup S will contain words which satisfy the conditions of Proposition V.4.3 if and only if S is suitable. Since G is a suitable subgroup of itself if and only if $K(G) = \{1\}$, the main obstruction to finding words which satisfying the assumptions of Theorem V.4.3 is the existence of finte normal subgroups. However, the following lemma (which is an easy exercise) shows that for most purposes this is a minor obstruction:

Lemma V.4.11. *Let $G \in \mathscr{AH}$. Then $G/K(G) \in \mathscr{AH}$ and $G/K(G)$ has trivial finite radical.*

V.5 Suitable subgroups of HNN-extensions and amalgamated products

Next we want to show that suitable subgroups can be controlled with respect to taking HNN-extensions and amalgamated products.

Lemma V.5.1. *Suppose S is suitable with respect to $\Gamma(G,\mathscr{A}_0)$, and A and B are cylic subgroups of G. Then there exists $\mathscr{A}_0 \subseteq \mathscr{A} \subset G$ such that $A \cup B \subset \mathscr{A}$ and S is suitable with respect to $\Gamma(G,\mathscr{A})$.*

Proof. By Corollay V.4.8, S contains an infinite order element y such that $\langle y\rangle \hookrightarrow_h (G,\mathscr{A}_0)$, and an element $g \in S\setminus\langle y\rangle$. By Lemma III.3.5, we can find a subset $\mathscr{A}_0 \subseteq Y_0 \subset G$ such that $\langle y\rangle \hookrightarrow_h (G,Y_0)$ and A and B are both elliptic with respect to the action of G on $\Gamma(G,Y_0 \sqcup \langle y\rangle)$. By Theorem III.2.10, we can find $Y \supset Y_0$ such that $\langle y\rangle \hookrightarrow_h (G,Y)$, and the action of G on $\Gamma(G,Y\sqcup\langle y\rangle)$ is acylindrical. Clearly A and B are still elliptic with respect to $\Gamma(G,Y\sqcup\langle y\rangle)$. By Lemma III.2.9, for some $n \in \mathbb{N}$, gy^n is loxodromic with respect to $\Gamma(G,Y\sqcup\langle y\rangle)$. Thus, the action of S on $\Gamma(G,Y\sqcup\langle y\rangle)$ is non-elementary by Theorem III.2.3. Letting $\mathscr{A} = Y\sqcup\langle y\rangle\cup A\cup B$, Lemma III.2.13 gives that the action of G on $\Gamma(G,\mathscr{A})$ is acylindric, and the action of S is still non-elementary, hence S is suitable with respect to $\Gamma(G,\mathscr{A})$. □

Proposition V.5.2. *Suppose S is a suitable subgroup of a group $G \in \mathscr{AH}$. Then for any isomorphic cyclic subgroups A and B of G, the corresponding HNN-extension $G*_{A^t=B}$ belongs to $\mathscr{AH}$ and contains S as a suitable subgroup.*

Proof. Let $\mathscr{A}$ be the set given by Lemma V.5.1 with respect to S, A and B. Since S is suitable with respect to $\Gamma(G,\mathscr{A})$, Corollary V.4.8 gives that S contains an element h which is loxodromic with respect to $\Gamma(G,\mathscr{A})$ and which satisfies $E_G(h) = \langle h\rangle$.

Let G_1 denote the HNN-extension $G*_{A^t=B}$; we identify G with its image in G_1. We will first show that $\Gamma(G_1,\mathscr{A}\cup\{t\})$ is a hyperbolic metric space. Since $\Gamma(G,\mathscr{A})$ is hyperbolic, by Theorem II.0.5 there exists a bounded presentation of G the form

$$\langle \mathscr{A} \mid \mathscr{S}\rangle \tag{V.19}$$

such that for any word W in $\mathscr{A}$ such that $W =_G 1$, the area of W over the presentation (V.19) is at most $L\|W\|$ for some constant L. Then G_1 has the presentation

$$\langle \mathscr{A}\cup\{t\} \mid \mathscr{S}\cup\{a^t = \varphi(a) \mid a \in A\}\rangle \tag{V.20}$$

where $\varphi\colon A \to B$ is an isomorphism. Note that (V.20) is still a bounded presentation, as we only added relations of length 4 (we use here that $A\cup B \subset \mathscr{A}$). We will show that (V.20) still satisfies a linear isoperimetric inequality, which is enough to show that $\Gamma(G_1,\mathscr{A}\cup\{t\})$ is a hyperbolic metric space by Theorem II.0.5.

Let W be a word in $\mathscr{A}\cup\{t\}$ such that $W =_{G_1} 1$. Let Δ be a minimal diagram over (V.20). Then is it well-known (and easy to prove) that every t-band of W starts and ends on $\partial\Delta$. Furthermore, since $A\cup B \subset \mathscr{A}$, minimality of Δ gives that each t-band consists of a single cell. Let $\Pi_1,\dots,\Pi_m$ denote the t-bands of Δ. Then $\Delta\setminus\bigcup\Pi_i$ consists of $m+1$ connected components $\Delta_1,\dots,\Delta_{m+1}$ such that each Δ_i is a diagram over (V.19).

Thus, for each i, $Area(\Delta_i) \leq L\ell(\partial\Delta_i)$. Clearly $m \leq \ell(\partial\Delta)$, and it is easy to see that

$$\sum_{i=1}^{m+1} \ell(\partial\Delta_i) = \ell(\partial\Delta).$$

It follows that

$$Area(\Delta) = \sum_{i=1}^{m+1} Area(\Delta_i) + m \leq \sum_{i=1}^{m+1} L\ell(\partial\Delta_i) + \ell(\partial\Delta) \leq (L+1)\ell(\partial\Delta).$$

Thus, $Area(W) \leq (L+1)\|W\|$, and hence $\Gamma(G_1, \mathscr{A} \cup \{t\})$ is a hyperbolic metric space.

Next, we will show that h is loxodromic with respect to the action of G_1 on $\Gamma(G_1, \cup\{t\})$. Observe that any shortest word W in $\mathscr{A} \cup \{t\}$ which represents an element of G contains no t letters. Indeed by Britton's Lemma if W represents an element of G and contains t letters, then it has a subword of the form $t^{-1}at$ for some $a \in A$ or a subword of the form tbt^{-1} for some $b \in B$. However, since $A \cup B \subset \mathscr{A}$, each of these subwords can be repalced with a single letter of $A \cup B$, contradicting the fact that W is a shortest word. Since h is loxodromic, if W is the shortest word in $\mathscr{A}$ representing h in G then any path p labeled by W^n is quasi-geodesic in $\Gamma(G, \mathscr{A})$. It follows that p is still quasi-geodesic in $\Gamma(G_1, \mathscr{A} \cup \{t\})$ thus h is loxodromic in G_1.

Finally, we will show that h satisfies the WPD condition. Let $\kappa = 100\delta$. By Lemma III.2.5, it suffices to verify (III.6) with respect to κ.

Choose M such that for all r_1 and r_2 satisfying if $|r_i|_{\mathscr{A}} \leq \kappa$ for $i = 1, 2$, then $r_1 A r_2 \cup r_1 B r_2 \subset B_{\mathscr{A}}(M)$. Now choose N such that $h^N \notin B_{\mathscr{A}}(M)$. Suppose $g \in G_1$ such that $d_{\mathscr{A} \sqcup \{t\}}(1, g) \leq \kappa$ and $d_{\mathscr{A} \sqcup \{t\}}(h^N, gh^N) \leq \kappa$. Consider the quadrilateral $s_1 p_1 (s_2)^{-1} (p_2)^{-1}$ in G_1 where $\ell(s_i) \leq \kappa$, $\mathbf{Lab}(s_1) = g$, and $\mathbf{Lab}(p_i) = h^N$. Without loss of generality, we assume each s_i and each p_i is a geodesic. As shown above, this means that no edges of p_i are labeled by $t^{\pm}$. Suppose that s_1 contains an edge labeled by $t^{\pm}$. Filling this quadrilateral with a van Kampen diagram Δ, for each edge of s_1 labeled by $t^{\pm}$, there exists a t-band connecting this edge to an edge of s_2. Let e be the last t edge of s_1, and let r_1 be the subpath of s_1 from e_+ to $(s_1)_+$. Similarly, let r_2 be the subpath of $(s_2)^{-1}$ from $(s_2)_+$ to f_-, where f is the edge of $(s_2)^{-1}$ connected to e by a t-band. Note that the label of the top edge of the t-band is an element of A or B; for concreteness we assume it is equal to an element $a \in A$. Now we have

$$h^N = r_1^{-1} a r_2$$

Moreover, this is equality in G which violates our choice of N. Therefore, s_1 must not contain any t-letters, and hence $g \in G$. Thus,

$$\{g \in G_1 \mid d_{\mathscr{A} \sqcup \{t\}}(1, g) < \kappa, d_{\mathscr{A} \sqcup \{t\}}(h^N, gh^N) < \kappa\} \subset \{g \in G \mid d_{\mathscr{A}}(1, g) < \kappa, d_{\mathscr{A}}(h^N, gh^N) < \kappa\}.$$

and this last set is finite because h satisfies WPD with respect to the action of G on $\Gamma(G, \mathscr{A})$. Thus, h is a loxodromic, WPD element with respect to the action of G_1 on $\Gamma(G_1, \mathscr{A} \cup \{t\})$, hence $E_{G_1}(h) \hookrightarrow_h G_1$ by Theorem III.2.8. Thus $G_1 \in \mathscr{A}\mathscr{H}$ by Theorem III.2.10.

Since h is loxodromic with respect to the action of G on $\Gamma(G, \mathscr{A})$, it is not conjugate with any elliptic element, in particular it is not conjugate with any element of A or B. It follows from Lemma III.2.6 and Lemma II.0.12 that $E_{G_1}(h) = E_G(h) = \langle h \rangle$. Therefore S is a suitable subgroup of G_1 by Proposition V.4.9. □

We now prove a similar result for amalgamated products using a standard retraction trick. The following lemma is a simplification of [24, Lemma 6.21]

Lemma V.5.3. *Suppose G is a group, R a subgroup which is a retract of G, and $H \leq R$ such that $H \hookrightarrow_h G$. Then $H \hookrightarrow_h R$.*

Proposition V.5.4. *Suppose $A \in \mathscr{A}\mathscr{H}$, S a suitable subgroup of A. Let $P = A *_{K=\varphi(K)} B$, where K is cyclic. Then $P \in \mathscr{A}\mathscr{H}$ and S is a suitable subgroup of P.*

Proof. Clearly, $A \hookrightarrow_h A * B$, so by Lemma V.4.10, if S is suitable in A, then S is suitable in $A * B$. By the previous lemma, S is suitable in the HNN extension $G = (A * B)*_{K^t=\varphi(K)}$. By Theorem II.0.13 P is isomorphic to $\langle A^t, B\rangle \leq G$ via an isomorphism which sends A to A^t and B to B. Futhermore, $\langle A^t, B\rangle$ is a retract of G. Thus if $h \in S \leq A$ satisfies $\langle h\rangle \hookrightarrow_h G$, then $\langle h^t\rangle \hookrightarrow_h G$ by Lemma III.1.8 and $\langle h^t\rangle \hookrightarrow_h \langle A^t, B\rangle$ by Lemma V.5.3. Thus S^t is a suitable subgroup of $\langle A^t, B\rangle$ by Proposition V.4.9, and passing to P through the isomorphism gives the desired result. □

V.6 Main theorem and applications

Our main technical small cancellation result is the following Theorem.

Theorem V.6.1. *Suppose $G \in \mathscr{A}\mathscr{H}$ and S is suitable with respect to $\Gamma(G, \mathscr{A})$. Then for any $\{t_1, ..., t_m\} \subset G$ and $N \in \mathbb{N}$, there exists a group $\overline{G}$ and a surjective homomorphism $\gamma\colon G \to \overline{G}$ which satisfies*

(a) $\overline{G} \in \mathscr{A}\mathscr{H}$

(b) $\gamma|_{B_{\mathscr{A}}(N)}$ *is injective*

(c) $\gamma(t_i) \in \gamma(S)$ *for* $i = 1, ..., m$.

(d) $\gamma(S)$ *is suitable with respect to* $\Gamma(\overline{G}, \mathscr{A}')$, *where* $\gamma(\mathscr{A}) \subset \mathscr{A}'$.

(e) Every element of $\overline{G}$ *of order* n *is the image of an element of* G *of order* n.

(f) For all $g, h \in B_{\mathscr{A}}(N)$, $g \sim h$ *if and only if* $\gamma(g) \sim \gamma(h)$

(g) If $g \in B_{G,\mathscr{A}}(N)$ *is loxodromic and* $x \in G$, *then there exists* n *such that* $\gamma(x^{-1}g^n x g^{\pm n}) = 1$ *if and only if* $\gamma(x) \in \gamma(E_G(g))$.

Remark V.6.2. It is easy to see from the proof of Theorem V.6.1 that

$$\overline{G} = G/\langle\langle t_1 w_1, \ldots, t_m w_m \rangle\rangle$$

for some elements $w_1, \ldots, w_m \in S$.

Proof. Clearly it suffices to prove the theorem with $m=1$, and the general statement follows by induction. Since S is suitable with respect to $\Gamma(G,\mathscr{A})$, by Corollary V.4.8 S contains infinite order elements h_1 and h_2 such that $\{\langle h_1\rangle,\langle h_2\rangle\}\hookrightarrow_h (G,\mathscr{A})$. Let $\mathscr{A}_1=\mathscr{A}\sqcup\langle h_1\rangle\sqcup\langle h_2\rangle$, and fix ε, μ, and ρ satisfying the conditions of Lemma V.2.1, Theorem V.3.5, and Corollary V.3.8 for $\lambda=\frac{1}{4}$ and $c=1$. Choose n such that $\frac{M}{2n}\leq\mu$ and $2n\geq\rho$, where $M=M(\varepsilon,3)$ is the constant given by Lemma V.4.1. Now if $m_1,...,m_n$ and $l_1,...,l_n$ to be sufficiently large, distinct positive integers, then the word

$$W=t^{-1}h_1^{m_1}h_2^{l_1}...h_1^{m_n}h_2^{l_n}$$

will satisfy all the assumptions of Proposition V.4.3 (here W is considered as a word in $\mathscr{A}_1$). Thus, the set $\mathscr{R}$ of all cyclic shifts of $W^{\pm1}$ satisfies the $C'(\varepsilon,\frac{M}{2n},\frac{1}{4},1,2n)$-condition by Proposition V.4.3. Let

$$\overline{G}=G/\langle\langle\mathscr{R}\rangle\rangle.$$

Lemma V.2.1 gives that γ is injective on $B_{\mathscr{A}_1}(N)$, and hence it is also injective on $B_{\mathscr{A}}(N)$. Lemma V.2.1 also gives that $\{\gamma(\langle h_1\rangle),\gamma(\langle h_2\rangle)\}\hookrightarrow_h\overline{G}$, thus $\overline{G}\in\mathscr{AH}$ by Corollary III.2.11. Theorem V.3.5 gives that every element of $\overline{G}$ of order n is the image of an element of G of order n. Corollary V.3.8 gives (f) and (g). Furthermore, since $t^{-1}h_1^{m_1}h_2^{l_2}...h_1^{m_n}h_2^{l_n}\in\mathscr{R}$, we have that $\gamma(t)=\gamma(h_1^{m_1}h_2^{l_2}...h_1^{m_n}h_2^{l_n})\in\gamma(S)$. Finally, Lemma V.2.1 gives that $\gamma(\langle h_1\rangle)=\langle\gamma(h_1)\rangle\hookrightarrow_h(\overline{G},\gamma(\mathscr{A}))$. Since $\gamma(h_2)\notin\langle\gamma(h_1)\rangle$, $\gamma(S)$ is suitable with respect to $\Gamma(\overline{G},\mathscr{A}')$ for some $\mathscr{A}'\supset\gamma(\mathscr{A})$ by Proposition V.4.9.

□

Remark V.6.3. Since the proof uses the same small cancellation conditions as [62] and [67], it follows from these papers that if G is non-elementary hyperbolic, then $\overline{G}$ is non-elementary hyperbolic, and if G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$, then $\overline{G}$ is hyperbolic relative to $\{\gamma(H_\lambda\}_{\lambda\in\Lambda}$.

Note that we can always choose N such that $B_{\mathscr{A}}(N)$ contains any given finite subset of G. We will next record some useful corollaries of this theorem.

Corollary V.6.4. *Suppose S is a subgroup of a finitely generated group G and for some infinite order elements h_1, $h_2\in S$ and some subgroup $H\leq G$, we have $\{\langle h_1\rangle,\langle h_2\rangle,H\}\hookrightarrow_h(G,X)$. Then for any $N\in\mathbb{N}$ and $\mathscr{A}=X\sqcup\langle h_1\rangle\sqcup\langle h_2\rangle\sqcup H$ there exists a group $Q\in\mathscr{AH}$ and a surjective homomorphism $\eta: G\to Q$ such*

1. *$\eta|_S$ is surjective.*
2. *$\eta|_{B_{\mathscr{A}}(N)}$ is injective*
3. *$\eta(H)\hookrightarrow_h Q$*
4. *$\eta(S)$ is a suitable subgroup of Q (equivalently, $K(Q)=\{1\}$).*
5. *Every element of Q of order n is the image of an element of G of order n.*

Proof. The proof is the same as Theorem V.6.1 with $t_1,...,t_m$ chosen as a generating set of G and W considered as a word in the alphabet $\mathscr{A}=X\sqcup\langle h_1\rangle\sqcup\langle h_2\rangle\sqcup H$; in this case Lemma V.2.1 also gives that $\eta(H)\hookrightarrow_h Q$. □

We will also make use of an infinitely generated version of this corollary; in this case we will take a sequence of groups, and we can not guarantee that the limit group will belong to $\mathscr{A}\mathscr{H}$, but only that it will not be finite or even virtually cyclic.

Corollary V.6.5. *Suppose $G \in \mathscr{A}\mathscr{H}$ is countable and S is suitable with respect to $\Gamma(G,\mathscr{A})$. Then for any $N \in \mathbb{N}$, there exists a non-virtually cyclic group Q and a surjective homomorphism $\eta: G \to Q$ such that*

1. *$\eta|_S$ is surjective.*
2. *$\eta|_{B_{\mathscr{A}}(N)}$ is injective*

Proof. By Lemma V.5.1, without loss of generality, we can assume that $\mathscr{A}$ contains infinite subgroups $\langle h\rangle$ and $\langle f\rangle$ such that $\langle h\rangle \cap \langle f\rangle = \{1\}$.

Let $G = \{1 = g_0, g_1, ...\}$. Let $G_0 = G$, and define a sequence of quotient groups

$$... \twoheadrightarrow G_i \twoheadrightarrow G_{i+1} \twoheadrightarrow ...$$

where the induced map $\eta_i: G \twoheadrightarrow G_i$ satisfies

1. $\eta_i(S)$ is suitable with respect to $\Gamma(\eta_i(G),\mathscr{A}_i)$, where $\eta_i(\mathscr{A}) \subset \mathscr{A}_i$.
2. $\eta_i(g_i) \in \eta_i(S)$.
3. $\eta_i|_{B_{\mathscr{A}}(N)}$ is injective.

Given G_i, we apply Theorem V.6.1 to G_i with $t = \eta_i(g_{i+1})$ and suitable subgroup $\eta_i(S)$. Theorem V.6.1 gives that the map $\gamma: G_i \to G_{i+1}$ will be injective on $B_{\mathscr{A}_i}(N)$ which contains $B_{\eta_i(\mathscr{A})}(N)$, and further for some $\mathscr{A}_{i+1} \supset \gamma(\mathscr{A}_i)$, $\gamma(\eta_i(S))$ is suitable with respect to $\Gamma(G,\mathscr{A}_{i+1})$. Hence the induced quotient map $\eta_{i+1} = \eta_i \circ \gamma$ will satisfy all of the above conditions. Let Q be the direct limit of this sequence (that is, $Q = G(0)/\bigcup \ker \eta_i$) and $\eta: G \to Q$ the induced epimorphism. Then for each $g_i \in G$, $\eta_i(g_i) \in \eta_i(S)$, thus $\eta(g_i) \in \eta(S)$. It follows that $\eta(S) = Q$. Finally, $\eta|_{B_{\mathscr{A}}(N)}$ is injective, since each η_i is injective on this set. Since η is injective on $\langle h\rangle \cup \langle f\rangle$, we get that Q is not virtually cyclic. □

Corollary V.6.6. *Let $G_1, G_2 \in \mathscr{A}\mathscr{H}$ with G_1 finitely generated, G_2 countable. Then there exists a non-virtually cyclic group Q and surjective homomorphims $\alpha_i: G_i \to Q$ for $i = 1,2$. In addition, if G_2 is finitely generated, then we can choose $Q \in \mathscr{A}\mathscr{H}$ with $K(Q) = \{1\}$, and if $K(G_i) = \{1\}$, then for any finite subset $\mathscr{F}_i \subset G_i$, we can choose α_i to be injective on $\mathscr{F}_i$.*

Proof. Since each G_i can be replaced with $G_i/K(G_i)$, it suffices to assume $K(G_i) = \{1\}$ for $i = 1,2$. Let $\mathscr{F}_i$ be any finite subset of G_i. Let $F = G_1 * G_2$, and let $\iota_i: G_i \to F$ be the natural inclusion. We will identify G_1 and G_2 with their images in F. By Corollary V.4.8, there exist $h_1, h_2 \in G_1$ such that $\{\langle h_1\rangle, \langle h_2\rangle\} \hookrightarrow_h G_1$. Since $\{G_1, G_2\} \hookrightarrow_h F$, Lemma III.1.7 gives that $\{\langle h_1\rangle, \langle h_2\rangle, G_2\} \hookrightarrow_h F$. Thus, $S = \langle h_1, h_2\rangle$ is suitable in F by Proposition V.4.9.

By Corollary V.6.4, there exists a group F' and a surjective homomorphism $\eta_1: F \to F'$, such that $\eta_1|_S$ is surjective, $\eta_1|_{G_2 \cup \mathscr{F}_1 \cup \mathscr{F}_2}$ is injective and $\eta_1(G_2) \hookrightarrow_h F'$. Since $S \subset G_1$, we also have that $\eta_1|_{G_1}$ is surjective.

From now on we identify G_2, $\mathscr{F}_1$, and $\mathscr{F}_2$ with their images in F'. Now since $K(G_2) = \{1\}$, G_2 will be a suitable subgroup of itself. Since $G_2 \hookrightarrow_h F'$, G_2 will be a suitable subgroup of F' by Lemma V.4.10. Now applying Corollary V.6.5 gives a non-virtually cyclic group Q and a surjective homomorphism $\eta_2 \colon F' \to Q$, such that $\eta_2|_{G_2}$ is surjective and $\eta_2|_{\mathscr{F}_1 \cup \mathscr{F}_2}$ is injective. Now since $\eta_1|_{G_1}$ is surjective and $\eta_2|_{G_2}$ is surjective, it follows that each of the compositions

$$G_i \overset{\iota_i}{\hookrightarrow} F \overset{\eta_1}{\twoheadrightarrow} F' \overset{\eta_2}{\twoheadrightarrow} Q$$

is surjective. Furthermore, each of these maps in injective on $\mathscr{F}_i$. Now if G_2 is finitely generated, we can apply Corollary V.6.4 to F' instead of Corollary V.6.5. Then we will get that $Q \in \mathscr{AH}$ and $\eta(G_2)$ is a suitable subgroup, thus $K(Q) \leq K_Q(\eta(G_2)) = \{1\}$. □

By fixing G_1 as a non-elementary, hyperbolic (hence finitely generated) group with Property (T), we get

Corollary V.6.7. *Every countable $G \in \mathscr{AH}$ has an infinite property (T) quotient.*

Frattini subgroups.

Definition V.6.8. The *Frattini subgroup $Fratt(G)$* of G is the intersection of all maximal subgroups of G, provided one such subgroup exists, otherwise $Fratt(G) = G$.

An element $g \in G$ is called a *non-generator* if for all $X \subset G$ such that $\langle X \rangle = G$, we have $\langle X \setminus \{g\} \rangle = G$. Conversely, if X is a generating set of G such that $\langle X \setminus \{g\} \rangle \neq G$, then we say that g is an *essential member* of the generating set X.

The following lemma is well-known.

Lemma V.6.9. *For any group G,*

$$Fratt(G) = \{g \in G \mid g \text{ is a non-generator of } G\}.$$

Lemma V.6.10. *Let $\varphi \colon G \to G'$ be a homomorphism. If $\varphi(g) \notin Fratt(\varphi(G))$, then $g \notin Fratt(G)$.*

Proof. Let X be a subset of G such that $\varphi(X)$ generates $\varphi(G)$ and $\varphi(g)$ is an essential member of this generating set. Then g is an essential member of the generating set $X \cup \ker(\varphi)$ of G. □

Theorem V.6.11. *Let $G \in \mathscr{AH}$. Then $Fratt(G) \leq K(G)$; in particular, the Frattini subgroup is finite.*

Proof. First, by Lemma V.4.11 and Lemma V.6.10 we can replace G with $G/K(G)$, so it suffices to assume $K(G) = \{1\}$.

Let $g \in G \setminus \{1\}$. Since $K(G) = \{1\}$, Corollary V.4.8 gives that G contains infinite order elements h_1 and h_2 such that $\langle h_1 \rangle \cap \langle h_2 \rangle = \{1\}$ and $\{\langle h_1 \rangle, \langle h_2 \rangle\} \hookrightarrow_h G$. In particular, this means that G contains some infinite order element h such that $\langle h \rangle \hookrightarrow_h G$, and such that $g \notin \langle h \rangle$. Let $S = \langle g, h \rangle$. By Proposotion V.4.9, S is a suitable subgroup of G. Now we can apply Corollary V.6.5 to find a non-virtually cyclic group Q and a homomorphism $\eta \colon G \to Q$ such that $\eta|_S$ is surjective, thus Q is generated by $X = \{\eta(g), \eta(h)\}$. Now

$\eta(g)$ is an essential member of the generating set X since Q is not cyclic, so $\eta(g) \notin Fratt(\eta(G))$. Therefore $g \notin Fratt(G)$ by Lemma V.6.10.

□

Topology of marked group presentations. Let $\mathcal{G}_k$ denote the set of *marked k-generated groups*, that is $\mathcal{G}_n = \{(G, x_1, \ldots, x_k) \mid x_1, \ldots, x_k \in G, \langle x_1, \ldots, x_k \rangle = G\}$. We will typically refer to elements of $\mathcal{G}_k$ simply as groups, although it should be understood that several elements of $\mathcal{G}_k$ will correspond to the same group. Now each element of $\mathcal{G}_k$ can be naturally associated to a normal subgroup N of the free group on k generators by the formula

$$G = F(x_1, \ldots, x_k)/N.$$

Given two normal subgroups N, M of the free group F_k, we can define a distance

$$d(N,M) = \begin{cases} \min\left\{\frac{1}{\|W\|} \mid W \in N \Delta M\right\} & \text{if } M \neq N \\ 0 & \text{if } M = N \end{cases}$$

This defines a metric (and hence a topology) on $\mathcal{G}_k$. It is not hard to see that this topology is equivalent to saying that a sequence $(G_n, X_n) \to (G, X)$ in $\mathcal{G}_k$ if and only if there are functions $f_n \colon \Gamma(G, X_n) \to \Gamma(G, X)$ which are label-preserving isometries between increasingly large neighboorhoods of the identity.

Given a group G, let $[G]_k$ denote the (possibly empty) subset of $\mathcal{G}_k$ corresponding to the group G, and let $\overline{[G]}_k$ denote its closure with respect to the topology mentioned above. Let $\overline{[G]} = \bigcup_{k=1}^{\infty} \overline{[G]}_k$. In the language of [4], a group $H \in \overline{[G]}$ if and only if G *preforms* H, that is for some generating set Y of H and some sequence of generating sets $X_1, \ldots$ of G,

$$\lim_{n\to\infty} (G, X_n) = (H, Y)$$

Where this limit is being taken in some fixed $\mathcal{G}_k$. Also, let $\mathcal{AH}_0$ denote the class of aclyindrically hyperbolic groups G for which $K(G) = \{1\}$, and let $[\mathcal{AH}_0] = \bigcup_{k=1}^{\infty} \{(G, x_1, \ldots, x_k) \in \mathcal{G}_k \mid G \in \mathcal{AH}_0\}$.

Theorem V.6.12. *Let $\mathcal{C}$ be a countable subset of $[\mathcal{AH}_0]$. Then there exists a finitely generated group D such that $\mathcal{C} \subset \overline{[D]}$.*

Proof. For each $(G, x_1, \ldots, x_k) \in \mathcal{C}$, let $X_G = \{x_1, \ldots, x_k\}$. Now enumerate all pairs (G_i, n_i) where G_i belongs to $\mathcal{C}$ and $n_i \in \mathbb{N}$. Let $Q_1 = G_1$, and suppose we have defined groups $Q_1, \ldots, Q_m$ and for each Q_k, we have surjective homomorphisms $\alpha_{(k,k)} \colon G_k \to Q_k$ and $\beta_{(k-1,k)} \colon Q_{k-1} \to Q_k$.

For $i \leq j$, let $\beta_{(i,j)}$ be the natural quotient map from Q_i to Q_j, and let $\alpha_{(i,j)} = \beta_{(i,j)} \circ \alpha_{(i,i)}$. Suppose that for each $1 \leq k \leq m$, Q_k satisfies

1. $Q_k \in \mathcal{AH}$ and $K(Q_k) = \{1\}$.

2. for each $1 \leq i \leq k$, $\alpha_{(i,k)}|_{B_{X_{G_i}}(n_i)}$ is injective.

Let $\mathcal{F} = \cup_{i=1}^{m} \alpha_{(i,m)}(B_{X_{G_i}}(n_i)) \subset Q_m$. Now, by Corollary V.6.6, there exists a group Q_{m+1} and surjective homomorphisms $\beta_{(m,m+1)} \colon Q_m \to Q_{m+1}$ and $\alpha_{(m+1,m+1)} \colon G_{m+1} \to Q_{m+1}$, such that $Q_{m+1} \in \mathcal{AH}$,

$K(Q_{m+1}) = \{1\}$, $\beta_{(m,m+1)}$ is injective on $\mathscr{F}$ and $\alpha_{(m+1,m+1)}$ is injective on $B_{X_{G_{m+1}}}(n_{m+1})$. Thus the the above conditions are satisfied for Q_{m+1}.

$$\begin{array}{ccccccccc} G_1 & & G_2 & & & & G_m & & \\ \downarrow \alpha_{(1,1)} & & \downarrow \alpha_{(2,2)} & & & & \downarrow \alpha_{(m,m)} & & \\ Q_1 & \overset{\beta_{(1,2)}}{\twoheadrightarrow} & Q_2 & \overset{\beta_{(2,3)}}{\twoheadrightarrow} & \dots & \overset{\beta_{(m-1,m)}}{\twoheadrightarrow} & Q_m & \twoheadrightarrow \quad \dots & \twoheadrightarrow D \end{array}$$

Now define D to be the direct limit of the sequence $Q_1, \dots$. That is, $D = Q_1 / \bigcup_{n=1}^{\infty} \ker \beta_{1,n}$. Let $\gamma_i \colon G_i \to D$ denote the composition of $\alpha_{(i,i)}$ and the natural quotient map from Q_i to D. Let $X_i = \gamma_i(X_{G_i})$. We will show that γ_i bijectively maps $B_{X_{G_i}}(n_i) \subset \Gamma(G_i, X_{G_i})$ to $B_{X_i}(n_i) \subset \Gamma(D, X_i)$. Clearly γ_i is surjective. now suppose $g, h \in B_{X_{G_i}}(n_i)$, $g \neq h$ and $\gamma_i(g) = \gamma_i(h)$. By construction, $\alpha_{(i,i)}(g) \neq \alpha_{(i,i)}(h)$. However $\alpha_{(i,i)}(gh^{-1}) \in \bigcup_{n=i}^{\infty} \ker \beta_{i,n}$, thus there must exists some $k \geq i$ such that $\beta_{(i,k)}(\alpha_{(i,i)}(g)) = \beta_{(i,k)} \alpha_{(i,i)}(h)$. But this means that $\alpha_{(i,k)}(g) = \alpha_{(i,k)}(h)$, which contradicts one of our injective assumptions. Thus, γ_i bijectively maps $B_{X_{G_i}}(n_i)$ to $B_{X_i}(n_i)$.

Now let $(G, X_G) \in \mathscr{C}$, and let (G_{i_j}, n_{i_j}) be the subsequence corrosponding to G.Now γ_{i_j} bijectively maps $B_{X_G}(n_{i_j}) \subset \Gamma(G, X_G)$ to $B_{X_{i_j}}(n_{i_j}) \subset \Gamma(D, X_{i_j})$.

Therefore,

$$\lim_{j \to \infty} (D, X_{i_j}) = (G, X_G).$$

□

Exotic quotients. Given a group G, let $\pi(G) \subset \mathbb{N} \cup \{\infty\}$ denote the set of orders of elements of G.

Theorem V.6.13. *Let $G \in \mathscr{AH}$ be countable. Then G has infinite quotient group C such that any pair of elements of C are conjugate if and only if they have the same order and $\pi(C) = \pi(G/K(G))$. In particular, if G is torsionfree, then C has two conjugacy classes.*

Proof. Since G can be replaced by $G/K(G)$, it suffices to assume $K(G) = \{1\}$.

Let $\mathscr{O} \subset G$ such that the orders of any two elements of $\mathscr{O}$ are different and $\pi(\mathscr{O}) = \pi(G)$. By Corollary V.4.8, G contains an infinite order element h such that $\langle h \rangle \hookrightarrow_h G$. Let $g \in G \setminus \langle h \rangle$, and let $S = \langle g, h \rangle$. Then S is suitable subgroup by Proposition V.4.9. Now enumerate G as $\{1 = g_0, g_1, \dots\}$. Let $G(0) = G$, and suppose we have constructed $G(n)$ and a surjective homomorphism $\alpha_n \colon G \to G(n)$ satisfying:

1. $G(n) \in \mathscr{AH}$.

2. $\alpha_n(S)$ is a suitable subgroup of $G(n)$.

3. $\pi(G(n)) = \pi(G)$.

4. For each $1 \leq i \leq n$, $\alpha_n(g_i)$ is conjugate to an element of $\alpha_n(\mathscr{O})$ and $\alpha_n(g_i) \in \alpha_n(S)$.

We construct $G(n+1)$ in two steps; first, let $f \in \mathscr{O}$ such that f has the same order as g_{n+1}. Now let $G(n+\frac{1}{2})$ be the HNN-extension with associated subgroups $\langle f \rangle$ and $\langle g_{n+1} \rangle$. By Lemma V.5.2, $\alpha_n(S)$ is a suitable subgroup of $G(n+\frac{1}{2})$. Applying Theorem V.6.1 to $G(n+\frac{1}{2})$ with $\alpha_n(S)$ as a suitable subgroup and $\{t, g_{n+1}\}$ as a finite set of elements and $N = 8\delta$ produces a group $G(n+1) \in \mathscr{A}\mathscr{H}$ and a surjective homomorphism $\gamma\colon G(n+\frac{1}{2}) \to G(n+1)$, such that $\gamma(t)$, $\gamma(g_{n+1}) \in \gamma(\alpha_n(S))$ and $\gamma(\alpha_n(S))$ is a suitable subgroup of $G(n+1)$. Also, if g has order k in $G(n+\frac{1}{2})$, then for each $1 \le i < k$, g^i is conjugate to an element of length at most 8δ by Lemma V.3.3. Hence the order of g in $G(n+1)$ is k, and combining this with Theorem V.6.1 gives $\pi(G(n+1)) = \pi(G(n+\frac{1}{2})) = \pi(G(n)) = \pi(G)$. Since $G(n+\frac{1}{2})$ is generated by $G(n)$ and t and $\gamma(t) \in \gamma(G(n))$, it follows that the restriction of γ to $G(n)$ is surjective. Let $\alpha_{n+1} = \gamma \circ \alpha_n$. Thus $G(n+1)$ will satifies the inductive assumptions. Let C be the direct limit of the sequence $G(1), \ldots$, and let $\alpha\colon G \to C$ be the natural quotient map. First note that for each $g_i \in G$, $\alpha_i(g_i) \in \alpha_i(S)$, thus $\alpha(g_i) \in \alpha(S)$. Therefore the restriction of α to S is surjective; in particular, C is two-generated. Since each $G(n)$ satisfies $\pi(G(n)) = \pi(G)$, we get that $\pi(C) = \pi(G)$. Finally suppose g_{i_1} and g_{i_2} have the same order in G. Then for some $f \in \mathscr{O}$, $\alpha_{i_1}(g_{i_1}) \sim \alpha_{i_1}(f)$ and $\alpha_{i_2}(g_{i_2}) \sim \alpha_{i_2}(f)$. Therefore $\alpha(g_{i_1}) \sim \alpha(f) \sim \alpha(g_{i_2})$.

□

Theorem V.6.14. *Let $G \in \mathscr{A}\mathscr{H}$ be countable. Then G has a finitely generated quotient V which is verbally complete.*

Proof. As before, it suffices to assume $K(G) = \{1\}$, and we can fix a two-generated suitable subgroup S of G. Enumerate all pairs $\{(g_1, v_1), \ldots\}$ where $g_i \in G$ and $v_i = v_i(x_1, \ldots)$ is a non-trivial freely reduced word in $F(x_1, \ldots)$. Let $G(0) = G$, and suppose we have constructed $G(n)$ and a surjective homomorphism $\alpha_n\colon G \to G(n)$ satisfying

1. $G(n) \in \mathscr{A}\mathscr{H}$.
2. $\alpha_n(S)$ is a suitable subgroup of $G(n)$.
3. The equation $g_i = v_i(x_1, \ldots)$ has a solution in $G(n)$
4. $\alpha_n(g_i) \in \alpha_n(S)$ for each $1 \le i \le n$.

Given $G(n)$, choose m such that v_{n+1} is a word in $x_1, \ldots, x_m$, and let $J = F(x_1, \ldots, x_m)$ if g_{n+1} has infinite order, and $J = \langle x_1, \ldots x_m \mid v_{n+1}^k = 1 \rangle$ if g_{n+1} has order k. In the case where g_{n+1} has order k, it is well-known that the order of v_{n+1} in J is k (see [49]). Thus the amalgamated product $G(n+\frac{1}{2}) = G(n) *_{g_{n+1}=v_{n+1}} J$ is well-defined in either case. By Lemma V.5.4, the $\alpha_n(S)$ is a suitable subgroup of $G(n+\frac{1}{2})$, so we can apply Theorem V.6.1 to get a group $G(n+1) \in \mathscr{A}\mathscr{H}$ and a surjective homomorphism $\gamma\colon G(n+\frac{1}{2}) \to G(n+1)$ such $\gamma(\alpha_n(S))$ is suitable, and $\{\gamma(x_1), \ldots, \gamma(x_m), \gamma(g_{n+1})\} \subset \gamma(\alpha_n(S))$. Since $G(n+\frac{1}{2})$ is generated by $\{G(n), x_1, \ldots, x_m\}$ and $\gamma(x_i) \in \gamma(G(n))$ for each $1 \le i \le m$, it follows that the restriction of γ to $G(n)$ is surjective. Thus there is a natural quotient map $\alpha_{n+1}\colon G \to G(n+1)$. It is easy to see that the inductive assumptions still hold in $G(n+1)$. Let V be the direct limit of the sequence $G(0), \ldots$, and let $\alpha\colon G \to V$ be the natural quotient map. For each $g \in G$, there exists n such that $\alpha_n(g) \in \alpha_n(S)$; thus, the restriction of α to S is surjective, so V is two-generated. It is straightforward to verify that V is verbally complete. □

CHAPTER VI

CONJUGACY GROWTH

VI.1 Conjugacy growth of acylindrically hyperbolic groups

Theorem VI.1.1. *Let $G \in \mathscr{AH}$ be a finitely generated. Then $\xi_G \sim \pi_G \sim 2^n$.*

Proof. Let $K = F_2 \times K(G) \leq G$ be the subgroup provided by Lemma III.1.10. Note first that if $g^m = f$ for some $g \in G$, $f \in K$, and $m \in \mathbb{Z} \setminus \{0\}$, then the intersection $K^g \cap K$ contains the subgroup $\langle f \rangle$. Hence if f has infinite order, $g \in K$ by Lemma III.1.5. Thus every element of K of infinite order that is primitive in K is also primitive in G. Furthermore, if two elements of K of infinite order are conjugate in G, then they are conjugate in K for the same reason. Thus we obtain $\pi_G \succeq \pi_K$ and the later function is obviously exponential (this also follows from the results of [22] as K is non-elementary hyperbolic). Since $\pi_G \preceq \xi_G \preceq 2^n$ for every finitely generated group G, we are done. □

Theorem VI.1.1 can be used to completely classify conjugacy growth functions of subgroups of certain groups, e.g., mapping class groups.

Corollary VI.1.2. *Let Σ be a (possibly punctured) closed orientable surface, G a subgroup of the mapping class group of Σ. Then either G is virtually abelian (in which case ξ_G is polynomial), or ξ_G is exponential.*

Proof. By Theorem 2.21 from [24], G is either virtually abelian, or has a finite index subgroup G_0 which surjects on a group with a non-degenerate hyperbolically embedded subgroup, that is an aclyindrically hyperbolic group. In the later case ξ_{G_0} is exponential by Theorem VI.1.1. It is straightforward to prove that one has $\xi_{G_0} \preceq \xi_G$ whenever $[G : G_0] < \infty$ (see, e.g., [42]). Hence the claim. □

VI.2 Constructing groups with specified conjugacy growth

The goal of this section is to prove Theorem I.4.2 For the rest of this chapter, we will work with relatively hyperbolic groups. Recall that G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ if and only if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$ for some finite set X. In this case, we call an element $g \in G$ *parabolic* if g is conjugate some an element of some H_λ. Also, any infinite order element which is not parabolic is loxodromic with respect to the action of G on $\Gamma(G, X \sqcup \mathscr{H})$; furthermore, this action is acylindrical [65].

We will also make use of following version of Proposition V.5.2 for torsion free relatively hyperbolic groups.

Lemma VI.2.1. *[44, Corollary 2.16] Let G be a torsion free group hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$, S a suitable subgroup of G, and g a loxodromic element of G. Then for any $h \in \mathscr{H}$, there is an isomorphism $\iota \colon E_G(g) \to \langle h \rangle$ and the corresponding HNN-extension $G*_{E_G(g)^t=\langle h\rangle}$ is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$. Furthermore, S is a suitable subgroup of $G*_{E_G(g)^t=\langle h\rangle}$.*

Groups with 2 conjugacy classes were first constructed in [67] as direct limits of relatively hyperbolic groups; our proof of Theorem V.6.13 is based on the same ideas. The proof of Theorem VI.2.3 is also based on these ideas; however its implementation is not automatic. Before proceeding to the proof of Theorem VI.2.3, we give a brief outline of the main difficulties which occur in adapting these ideas to our situation.

To prove Theorem VI.2.3, instead of trying to make all elements conjugate we want to control the number of conjugacy classes inside each ball with respect to a fixed finite generating set. So at the ith step of our construction we fix the desired number of conjugacy classes on the sphere of radius i (up to some constants), making all other elements of the sphere conjugate. The main problem, however, is that the conjugacy relations which we want to add may also produce "unwanted" conjugacy relations between elements we want to keep unconjugate. For instance conjugating two elements x and y, we also make x^n conjugate to y^n for all n. Induced conjugations of this particular type can be controlled by working with primitive conjugacy classes and making all elements in our group conjugate to all their nontrivial powers. However this does not solve the problem completely as "unwanted" conjugations can occur even between primitive elements. More precisely, the problem splits into two parts. When dealing with the sphere of radius i at step i, we have to make sure that

1) "Unwanted" conjugations do not occur inside the ball of radius $(i-1)$.

2) We keep enough non-conjugate primitive elements on spheres of radii $> i$ to continue the construction.

To overcome the first difficulty we "attach" a new parabolic subgroup with 2 conjugacy classes to a representative of each conjugacy class which we want to keep inside the ball of radius $(i-1)$. Then Lemma III.1.5 guarantees that such classes remain different at all steps of the inductive construction, and hence in the limit group.

The second part of the problem is more complicated and is typical for such inductive proofs. It is, in fact, the main obstacle in implementing the ideas from [67] in the proof of Theorem VI.2.3. To guarantee 2) we construct sets U_i of elements with ordinary word length i but relative length at most 4. Then parts (f) and (g) of Theorem V.6.1 come into play and allow us to control these elements during the small cancellation substep of each step; Lemma II.0.12 is used to control them during the HNN-extension substep.

Remark VI.2.2. Note that every torsion free group G with 2 conjugacy classes has exponential growth. Indeed every element $g \in G$ is conjugate to its square. If $g \neq 1$, this easily implies that the intersection of the cyclic subgroup $\langle g \rangle$ with a ball of radius n with respect to a fixed finite generating set of G has exponentially many elements.

Theorem VI.2.3. *Let G be a group generated by a finite set X, f the conjugacy growth function of G with respect to X. Then the following conditions hold.*

(a) f is non-decreasing.

(b) There exists $a \geq 1$ such that $f(n) \leq a^n$ for every $n \in \mathbb{N}$.

Conversely, suppose that a function $f \colon \mathbb{N} \to \mathbb{N}$ satisfies the above conditions (a) and (b). Then there exists an infinite finitely generated group G such that $\xi_G \sim f$.

Proof. The 'only if' part of the theorem is obvious. Let us prove the other one. Suppose $f\colon \mathbb{N}\to\mathbb{N}$ is a non-decreasing function such that $f \le a^n$ for some $a>1$. If $f\equiv 1$, the statement is obvious. Otherwise passing to an equivalent function if necessary, we can assume that $f(n)\ge 2$ for all n. Let $\bar{f}$ denote the function defined by $\bar{f}(n) = f(n) - f(n-1)$.

Let A be a finitely generated torsion free group with two conjugacy classes. Clearly, it suffices to assume that $f(n)\le \gamma_A(n)$, since γ_A is exponential by Remark VI.2.2. Set $G(1) = A*\langle h\rangle$, where h generates an infinite cyclic group. Let X' be a finite generating set for A. Then we take $X = X'\cup h^{-1}X'h\cup\{h\}$ as a finite generating set for $G(1)$. Let $B = h^{-1}Ah$, and fix $a_0\in A$ and $b_0\in B$ such that $|a_0|_X = |b_0|_X = 1$.

Further for each $i\ge 2$, we create a collection of subsets $U_i = \{ab : a\in A\setminus\{1\}, b\in B\setminus\{1\}, |ab|_X = i\}$. Note that all elements of $U = \bigcup_{i=1}^{\infty} U_i \cup \{a_0\}$ have word length at most 4 with respect to the generators $A\cup\{h\}$. Clearly $|U_n|\succeq \gamma_A(n)$, and since γ_A is exponential, there exists a constant L such that

$$|U_{Ln}| \ge (\gamma_{G(1)}(n-1))(\gamma_{G(1)}(n-1)+1) + \bar{f}(n). \tag{VI.1}$$

Suppose we have constructed a group $G(k)$, an epimorphism $\varphi_k\colon G(1)\to G(k)$, and a collection of subsets $\{a_0\} = W_1\subset \ldots \subset W_k\subset U$, such that the following conditions are satisfied.

(a) φ_k is injective on X, A, and U (so we identify these sets with their images in $G(k)$).

(b) $G(k)$ is hyperbolic relative to a collection $\mathscr{C}_k$ of proper subgroups with two conjugacy classes.

(c) $G(k)$ is a suitable subgroup of itself.

(d) $G(k)$ is torsion free.

(e) Every $g\in B_{G(k),X}(k-1)$ is parabolic in $G(k)$.

(f) Each element of W_k is parabolic, and there is exactly one element of W_k inside each parabolic conjugacy class. In particular, distinct elements of W_k are non-conjugate.

(g) For all $1\le n\le k$, $|W_n| = f(n)-1$, and for all $w\in W_n$, $|w|_X\le Ln$.

(h) if u, v are two different elements of $U^{\pm 1}$ and $u\sim v$ in $G(k)$, then $u\sim v\sim a_0$ in $G(k)$. Furthermore at most $\gamma_{G(1)}(k-1)$ elements of U are conjugate to a_0 in $G(k)$.

(i) For all $u\in U$ such that u is loxodromic in $G(k)$, u is also primitive in $G(k)$. In particular, $E_{G(k)}(u) = \langle u\rangle$.

Obviously (a)-(f) hold for $G(1)$ with φ_1 the identity map and $\mathscr{C}_1 = \{A\}$. Passing to an equivalent function, we can assume that $f(1) = 2$ without loss of generality. This gives (g) for $G(1)$. It is clear (e.g. from [49, Chapt. IV, Theorem 1.4]) that all elements of U are pairwise non-conjugate in $G(1)$. If $u\in U$ is loxodromic in $G(1)$, then $u\ne a_0$ and so $u = a_1h^{-1}a_2h$ for some $a_1, a_2\in A\setminus\{1\}$. The normal form theorem for free products [49, Chapt. IV, Theorem 1.2] implies that u is primitive in $G(1)$. Thus (h) and (i) also hold for $G(1)$.

Now we construct $G(k+1)$ in a sequence of four steps. The intermediate groups constructed in each step will be denoted as follows.

$$G(k) \overset{\iota_1}{\hookrightarrow} G'(k) \overset{\alpha_1}{\twoheadrightarrow} G''(k) \overset{\iota_2}{\hookrightarrow} G'''(k) \overset{\alpha_2}{\twoheadrightarrow} G(k+1).$$

Here ι_1 will be the natural embedding into an HNN-extension of the previous group, while ι_2 will be the natural embedding into an HNN-extension of a a free product, where the previous group is one of the factors. α_1 and α_2 will be epimorphisms which will correspond to taking a small cancellation quotient of the previous group. We will first show how to construct the group $G(k+1)$, and then verify that it satisfies all the inductive conditions.

Step 1. Let $g_1,...,g_n$ be the list of all elements in $G(k)$ such that $|g_i|_X = k$ and g_i is loxodromic in $G(k)$ for each $1 \leq i \leq n$. Note that $n \leq \bar{\gamma}_{G(1)}(k)$. Since $G(k)$ is torsion free, for each i there exists some h_i such that $E_{G(k)}(g_i) = \langle h_i \rangle$. Now we define $G'(k)$ as the multiple HNN-extension

$$G'(k) = \langle G(k), t_1,...,t_n \mid h_i^{t_i} = a_0 \rangle.$$

Let $\iota_1 \colon G(k) \hookrightarrow G'(k)$ be the natural embedding; for convenience we identify $G(k)$ with its image in $G'(k)$. Suppose $u,v \in U^{\pm 1}$ such that $u \sim v$ in $G'(k)$. If $u \sim v$ in $G(k)$, then by (h) $u \sim v \sim a_0$ in $G(k)$ and hence also in $G'(k)$. Otherwise, by Lemma II.0.12 either u or v is conjugate to some element h_i^m in $G(k)$. If this holds for u, we have $u \sim h_i^m \sim a_0^m \sim a_0$ in $G'(k)$, and similiarly if it holds for v. Thus, two elements of $U^{\pm 1}$ are either non-conjugate in $G'(k)$ or they are both conjugate to a_0.

Further if $u \sim a_0$ in $G'(k)$ but not in $G(k)$, then u must be conjugate to a power of some h_i in $G(k)$ by Lemma II.0.12. For each h_i, $u \sim h_i^m$ in $G(k)$ implies that $m = \pm 1$ since u is primitive in $G(k)$ by (i). By (h), for each $1 \leq i \leq n$, there is at most one element $u \in U$ conjugate to $h_i^{\pm 1}$ in $G(k)$. Thus, the number of elements of U conjugate to a_0 in $G'(k)$ is at most $\gamma_{G(1)}(k-1) + n \leq \gamma_{G(1)}(k)$. By Lemma II.0.12 and (i), all elements $u \in U$ which are loxodromic in $G'(k)$ are primitive in $G'(k)$. By Lemma VI.2.1, $G'(k)$ will be hyperbolic relative to $\mathscr{C}_k$ and $G(k)$ is suitable in $G'(k)$.

Step 2. Let $G''(k)$ be the quotient group of $G'(k)$ provided by applying Theorem V.6.1 to $G'(k)$ with $\{t_1,...,t_n\}$ as our finite set, $G(k)$ as our suitable subgroup, and $N = 4$. Let $\alpha_1 \colon G'(k) \twoheadrightarrow G''(k)$ be the corresponding epimorphism.

Since elements of $U \cup X$ all have relative length at most 4, α_1 will be injective on $U \cup X$, so we identify these sets with their images. The final two assertions of Theorem V.6.1 give that two elements of $U^{\pm 1}$ are conjugate in $G''(k)$ if and only if they were conjugate in $G'(k)$, and for each loxodromic $u \in U$, $\langle u \rangle = E_{G'(k)}(u) = E_{G''(k)}(u)$. Hence, all loxodromic elements of U are still primitive in $G''(k)$. Let us prove that $\alpha_1 \circ \iota_1$ is surjective. Since $G'(k)$ is generated by $G(k) \cup \{t_1,...,t_n\}$ and for each $1 \leq i \leq n$, $\alpha_1(t_i) \subset \alpha_1(G(k))$, we have that $G''(k)$ is generated by $\alpha_1(G(k))$. Thus, $\alpha_1 \circ \iota_1$ will be surjective, and $G''(k)$ will be finitely generated by (the image of) X. Theorem V.6.1 also gives that $\alpha_1(G(k)) = G''(k)$ will be a suitable subgroup of $G''(k)$.

Now let $U'_{L(k+1)}$ be the set of $u \in U$ such that $|u|_X = L(k+1)$ in $G''(k)$. Then $|U'_{L(k+1)}| \geq |U_{L(k+1)}|$; this follows from the fact that for each $u \in U_{L(k+1)}$, we have $|u|_X \leq L(k+1)$ in $G''(k)$, so we can choose $j \in \mathbb{N}$

such that $|ub_0^j|_X = L(k+1)$. An element $u \in U'_{L(k+1)}$ will be called *good* if for all elements v conjugate to u in $G''(k)$, we have $|v|_X \geq k+1$; otherwise it will be called *bad*. We want to show that $U'_{L(k+1)}$ contains at least $\bar{f}(k+1)$ good elements.

Indeed otherwise by (VI.1), $U'_{L(k+1)}$ must contain $(\gamma_{G(1)}(k))(\gamma_{G(1)}(k)+1)$ bad elements, each of which is conjugate to some element of X-length at most k. Since there are at most $\gamma_{G(1)}(k)$ such elements in $G''(k)$, there exists $V \subset U'_{L(k+1)}$ such that V contains $(\gamma_{G(1)}(k)+1)$ pairwise conjugate elements. Then all elements of V must be pairwise conjugate in $G'(k)$, and thus all elements of V are conjugate to a_0 in $G'(k)$. But this contradicts the fact that there are at most $\gamma_{G(1)}(k)$ elements of U conjugate to a_0 in $G'(k)$. Thus $U'_{L(k+1)}$ contains at least $\bar{f}(k+1)$ good elements.

Step 3. Let $W'_{k+1} = \{w_1, ..., w_s\}$ be a subset of the good elements of $U'_{L(k+1)}$ such that $s = |W'_{k+1}| = \bar{f}(k+1)$. Note that if u is a good element, then u is not conjugate to a_0, hence u is not conjugate to any other element of $U^{\pm 1}$. Thus all elements of W'_{k+1} are loxodromic and hence primitive; furthermore, they are pairwise non-commensurable by Corollay III.2.7. Then we define $W_{k+1} = W_k \cup W'_{k+1}$. Now, for each $1 \leq i \leq s$, let C_i be a torsion free group with two conjugacy classes, generated by $\{x_i, y_i\}$. Consider the group $G''(k) * (*_{i=1}^s C_i)$, which naturally contains an isometrically embedded copy of $G''(k)$. By Theorem III.1.4 and Lemma III.1.7, this group will be hyperbolic relative to $\mathscr{C}_{k+1}$, where $\mathscr{C}_{k+1} = \mathscr{C}_k \cup (\cup_{i=1}^s C_i)$. Also, clearly primitive elements of $G''(k)$ remain primitive in $G''(k) * (*_{i=1}^s C_i)$, and any two non-conjugate elements of $G''(k)$ remain non-conjugate. Since each w_i is primitive and loxodromic, we get that $E_{G''(k)*(*_{i=1}^s C_i)}(w_i) = \langle w_i \rangle$. Now we take a multiple HNN-extension and form the group

$$G'''(k) = \langle G''(k) * (*_{i=1}^s C_i), d_1, ..., d_s | w_i^{d_i} = x_i \rangle.$$

Let $\iota_2 \colon G''(k) \hookrightarrow G'''(k)$ denote the natural embedding, and again we identify $G''(k)$ with its image. Since the elements w_i are pairwise non-commensurable, by Lemma II.0.12 we can inductively apply Lemma VI.2.1 to get that $G'''(k)$ is hyperbolic relative to $\mathscr{C}_{k+1}$ and contains $G''(k)$ as a suitable subgroup.

Step 4. Finally, we obtain $G(k+1)$ as the quotient group of $G'''(k)$ by applying Theorem V.6.1 to the finite set $\{d_i, x_i, y_i : 1 \leq i \leq s\}$, suitable subgroup $G''(k)$, and $N = 4$. Let $\alpha_2 \colon G'''(k) \twoheadrightarrow G(k+1)$ be the corresponding epimorphism, and define $\varphi_{k+1} = \alpha_2 \circ \iota_2 \circ \alpha_1 \circ \iota_1 \circ \varphi_k$. Let us prove that φ_{k+1} is surjective. We have shown that $\alpha_1 \circ \iota_1$ is surjective. Similarly, since $G'''(k)$ is generated by $G''(k) \cup \{d_i, x_i, y_i : 1 \leq i \leq s\}$ and $\alpha_2(\{d_i, x_i, y_i : 1 \leq i \leq s\}) \subset \alpha_2(G''(k))$, we get that $G(k+1)$ is generated by $\alpha_2(G''(k))$, hence $\alpha_2 \circ \iota_2$ is surjective. Thus, φ_{k+1} is surjective being the composition of surjective maps.

Let us now prove that $G(k)$ satisfies all the inductive assumptions. First, Theorem V.6.1 gives that α_1 and α_2 are injective on all elements of relative length at most 4, which includes all elements in U, X and $\mathscr{C}_{k+1}$. Hence φ_{k+1} will be injective on these sets being the composition of these maps and the injective maps ι_1 and ι_2. $G'''(k)$ is hyperbolic relative to $\mathscr{C}_{k+1}$, and Theorem V.6.1 gives that $G(k+1)$ will be hyperbolic relative to $\mathscr{C}_{k+1}$ and $\alpha_2(G''(k)) = G(k+1)$ is a suitable subgroup of itself. Taking HNN-extensions and free products of torsion free groups gives torsion free groups, and combining this with Theorem V.6.1 gives that $G(k+1)$ will be torsion free. Clearly, every element of $B_{G(k),X}(k-1)$ is parabolic in $G'(k)$, thus they are also parabolic in $G(k+1)$. By construction, each $w \in W_{k+1}$ is parabolic in $G'''(k)$ and the conjugacy class of w corresponds to a unique parabolic subgroup. The definition of W_{k+1} gives that $|W_{k+1}| = f(k+1) - 1$, and for all $w \in W_{k+1}$, $|w|_X \leq L(k+1)$; clearly this also holds for all $1 \leq n \leq k$ as passing to quotient groups can only decrease word length. We have shown that in $G'(k)$, two elements of $U^{\pm 1}$ are conjugate if and only if they are both conjugate to a_0, and furthermore at most $\gamma_{G(1)}(k-1)$ elements of U are conjugate to a_0 in $G'(k)$. At all other steps non-conjugate elements of $U^{\pm 1}$ remain non-conjugate, so this also holds in $G(k+1)$. Finally, loxodromic elements of U are primitive in $G''(k) * (*_{i=1}^{s} C_i)$, and Lemma II.0.12 gives that they are primitive in $G'''(k)$. Hence for all loxodromic $u \in U$, $\langle u \rangle = E_{G'''(k)}(u) = E_{G(k+1)}(u)$ by Theorem V.6.1, so u is still primitive in $G(k+1)$. Thus, $G(k+1)$ satisfies all the inductive conditions.

Now, we take G to be the limit of this sequence of groups; that is, let $G = G(1)/N$, where $N = \bigcup_{i=1}^{\infty} \operatorname{Ker} \varphi_i$. We will show that every conjugacy class in G has a representative in $\bigcup_{k=1}^{\infty} W_k$. Suppose $g \in B_{G,X}(n)$, and let g_0 be a pre-image of g in $G(1)$ such that $|g_0|_X \leq n$. Then g_0 is parabolic in the group $G(n+1)$ by condition (e), so g_0 is conjugate to an element of W_{n+1} in $G(n+1)$ by (f). Hence g is conjugate to an element of W_{n+1} in G. Thus we have

$$\xi_G(n) \leq |W_{n+1}| + 1 = f(n+1) \leq f(2n).$$

On the other hand, all elements of W_n are pairwise non-conjugate, and for each $w \in W_n$, $|w|_X \leq Ln$. Hence $f(n) = |W_n| + 1 \leq \xi_G(Ln)$. Therefore $\xi_G \sim f$. □

VI.3 Conjugacy growth and subgroups of finite index

We now move to the proof of Theorem I.4.3. We start with an 'infinitely generated version' of Theorem I.4.3.

Lemma VI.3.1. *There exists a short exact sequence*

$$1 \to N \to C \to \mathbb{Z}_2 \to 1$$

such that the following hold.

- *(a) The group C is countable and torsion free.*
- *(b) The subgroup N has exactly 2 conjugacy classes.*
- *(c) There is a free subgroup $F \leq N$ of rank 2 and an element $a \in C$ such that for any two distinct elements $f_1, f_2 \in F$, af_1 and af_2 are not conjugate in C.*

Proof. We proceed by induction. Let $A_0 = \langle a, b, c \rangle$ be the free group of rank 3 and let $\varepsilon_0 \colon A_0 \to \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}_2$ be the natural epimorphism. Assume that A_n is already constructed together with an epimorphism

$$\varepsilon_n \colon A_n \to \mathbb{Z}_2.$$

Let K_n denote the kernel of ε_n. We enumerate all elements of $K_n = \{1, k_0, k_1, \ldots\}$ and let A_{n+1} be the multiple HNN-extension

$$\langle A_n, \{t_i\}_{i \in \mathbb{N}} \mid k_i^{t_i} = k_0 \rangle.$$

The map sending K_n and all stable letters to 1 (here 1 denotes the identity element of $\mathbb{Z}_2$) extends to a homomorphism $\varepsilon_{n+1}\colon A_{n+1} \to \mathbb{Z}_2$.

Let $C = \bigcup_{n=0}^{\infty} A_n$ and $N = \bigcup_{n=0}^{\infty} K_n$. Clearly N is a normal subgroup of index 2 in C. Since all nontrivial elements of K_n are conjugate in K_{n+1}, N has exactly 2 conjugacy classes. On the other hand, Lemma II.0.12 implies by induction that for any distinct $f_1, f_2 \in \langle b, c \rangle$, the elements af_1 and af_2 are not conjugate in A_n. Hence the same holds true in C. □

Theorem VI.3.2. *There exists a finitely generated group G and a finite index subgroup $H \leq G$ such that H has* 2 *conjugacy classes while G is of exponential conjugacy growth.*

Proof. Let

$$1 \to N \to C \stackrel{\varepsilon}{\to} \mathbb{Z}_2 \to 1$$

be the short exact sequence provided by Lemma VI.3.1. The desired group G is constructed as an inductive limit of relatively hyperbolic groups as follows. Let

$$G(0) = C * F(x,y),$$

where $F(x,y)$ is the free group of rank 2 generated by x and y. We enumerate all elements of

$$C = \{1 = c_0, c_1, c_2, \dots\}$$

and

$$G(0) = \{1 = g_0, g_1, g_2, \dots\}.$$

Without loss of generality we may assume that

$$\varepsilon(c_1) = 1. \qquad \text{(VI.2)}$$

Suppose that for some $i \geq 0$, the group $G(i)$ has already been constructed together with an epimorphism $\varphi_i\colon G(0) \to G(i)$ and an epimorphism $\alpha_i\colon G(i) \to \mathbb{Z}_2$. We use the same notation for elements $x, y, c_0, c_1, \dots, g_0, g_1, \dots$ and their images in $G(i)$. Assume that $G(i)$ satisfies the following conditions. It is straightforward to check these conditions for $G(0)$, the identity map $\varphi_0\colon G(0) \to G(0)$, and the epimorphism $\alpha_0\colon G(0) \to \mathbb{Z}_2$ which is induced by ε and the map sending x and y to 1.

(a) The restriction of φ_i to the subgroup C is injective. In what follows we identify C with its image in $G(i)$.

(b) $G(i)$ is hyperbolic relative to C.

(c) The elements x and y generate a suitable subgroup of $G(i)$.

(d) $G(i)$ is torsion free.

(e) In $G(i)$, the elements $c_0, \dots, c_i$ are contained in the subgroup generated by x and y.

(f) The diagram

$$\begin{array}{ccc} G(0) & \xrightarrow{\alpha_0} & \mathbb{Z}_2 \\ \varphi_i \downarrow & & \downarrow \mathrm{id} \\ G(i) & \xrightarrow{\alpha_i} & \mathbb{Z}_2 \end{array}$$

is commutative.

(g) In $G(i)$, for every $j = 1, \dots, i$, if $\alpha_i(g_j) = 1$ then the element g_j is conjugate to c_1 by an element of $\operatorname{Ker} \alpha_i$.

The group $G(i+1)$ is obtained from $G(i)$ in two steps.

Step 1. If g_{i+1} is a parabolic element of $G(i)$ or $\alpha_i(g_{i+1}) \neq 1$, we set $G'(i) = G(i)$. Otherwise, since $G(i)$ is torsion free, there is an isomorphism $\iota \colon E_{G(i)}(g_{i+1}) \to \langle c_1 \rangle$. Now we define $G'(i)$ to be the corresponding HNN–extension

$$G'(i) = \langle G(i), t \mid e^t = \iota(e), e \in E_{G(i)}(g_{i+1}) \rangle.$$

Then $G'(i)$ is hyperbolic relative to C and $\langle x, y \rangle$ is suitable in $G'(i)$ by Lemma VI.2.1. Note also that $G'(i)$ is torsion free being an HNN-extension of a torsion free group.

Step 2. We now apply Theorem V.6.1 to the group $G'(i)$, the subgroup $S = \langle x, y \rangle \leq G'(i)$, and the set of elements $\{t, c_{i+1}\}$ (or just $\{c_{i+1}\}$ if $G'(i) = G(i)$). Let $G(i+1) = \overline{G}$, where $\overline{G}$ is the quotient group provided by Theorem V.6.1. Since t becomes an element of $\langle x, y \rangle$ in $G(i+1)$, there is a naturally defined epimorphism $\varphi_{i+1} \colon G(0) \to G(i+1)$. Using Theorem V.6.1 and the inductive assumption it is straightforward to verify properties (a)–(e) for $G(i+1)$.

Observe that the group $G'(i)$ admits an epimorphism β_i to $\mathbb{Z}_2$ which sends the stable letter and $\operatorname{Ker}(\alpha_i)$ to 1. Indeed this follows immediately from the inductive assumption and our construction of $G'(i)$. By Remark V.6.2 and part (f) of the inductive assumption, the kernel of the natural epimorphism $G'(i) \to G(i+1)$ is contained in $\operatorname{Ker} \beta_i$. Hence β_i induces an epimorphism $\alpha_{i+1} \colon G(i+1) \to \mathbb{Z}_2$. Obviously (f) and (g) hold for $G(i+1)$.

Let $G = G(0)/M$, where $M = \bigcup_{i=1}^{\infty} \operatorname{Ker} \varphi_i$. By (d) G is torsion free. It is also easy to see that G is 2–generated. Indeed, $G(0)$ is generated by $x, y, c_1, c_2, \dots$ and hence condition (e) implies that G is generated by x and y.

Further notice that $M \leq \operatorname{Ker} \alpha_0$ by (f). Let $H = (\operatorname{Ker} \alpha_0)/M$. Then G/H is isomorphic to $G(0)/\operatorname{Ker} \alpha_0$, so $|G/H| = 2$. Let h be a nontrivial element of H. We take an arbitrary preimage $g \in G(0)$ of h. Observe that $\alpha_i(g) = 1$ for every i by (f). Hence (the image of) the element g becomes conjugate to c_1 by an element $\operatorname{Ker} \alpha_i$ at a certain step according to (g). Therefore, all non-trivial elements of H are conjugate in H.

Finally let F and a be the free subgroup and the element of C provided by Lemma VI.3.1, respectively. By part (c) of Lemma VI.3.1, parts (a), (b), (d) of the inductive assumption, and Lemma III.1.5, for any two distinct elements $f_1, f_2 \in F$, af_1 and af_2 are not conjugate in $G(i)$. Hence the same holds true in G. Since the natural map from C to G is injective by (a) and the (ordinary) growth function of F is exponential, the conjugacy growth function of G is exponential as well. □

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