

MONEY AND FINANCE IN DYNAMIC MODELS WITH  
SEARCH FOR INFORMATIONAL FRICTIONS

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To my beloved wife, Jianmei

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# CHAPTER I

## INTRODUCTION

Like other fields of science, most of fundamental economic theories characterize an ideal frictionless economy abstracted from the complexity of the real world. Although they help us understand the quintessence of the economics, one must not simply ignore the important consequences of various frictions in the classical economic models. Search and informational frictions are among the ones that attracts a number of economists.

The alleviation of search frictions is among the crucial roles for money as a medium of exchange. In Chapter II, we investigate how the introduction of money would improve the technology choice problem in a search-theoretic model. Chapter III illustrates a new asset pricing approach in an economy with heterogeneous prior beliefs, where the stock prices may deviate from the expected fundamental values, and speculators can make profits from stock price manipulations provided the presence of informational frictions. An empirical wavelet analysis of stock market comovements in Chapter IV reveals some features of the informational spillovers from NASDAQ and S&P500 to some Eastern Asian markets.

### **1.1 Search Frictions and Technology Choice in a Monetary Economy**

It is well known that the existence of search frictions in trade gives birth to fiat money. The path-breaking work by Kiyotaki and Wright (1989, 1991, and 1993) vividly illustrates this process within a random-matching framework. By alleviating the "double coincidence of wants" problem in a barter economy, the introduction of money as a medium of exchange greatly facilitates the trade and hence foster faster economic development.

Although there is a growing literature on the role of money in search equilibrium, few of them emphasize another associated but distinct problem: the effect of search frictions on technology choice and the role of money in this aspect. The separation of the beneficiaries and cost bearers of the goods, a direct impact from specialization, would distort the producers' technology choice in the presence of severe search frictions, especially in a decentralized trading environment. When the expected waiting period between trades is too long, the time cost for rejecting low-quality goods and waiting for a better one will be unaffordable. When the acceptability is independent on the quality of the goods, producers are likely to choose the low-cost technology, even when it is socially suboptimal.

Attribute to the introduction of money into the economy, search frictions in trading are considerably mitigated, which reduces the time costs for being selective about the quality of goods. As demonstrated in Chapter II, if the search frictions are low enough to grant the premium goods higher acceptability than the low-quality ones, producers are more likely to choose the high technology. As a consequence, it is easier to restore first-best technology choices in a monetary economy than in a pure barter economy. The result is robust in both the simplified case with instantaneous production and the more realistic non-instantaneous production scenario.

## **1.2 Asset Pricing with Informational Frictions**

Even in a more developed economy, the presence of informational frictions may still have an important impact on individual behavior and equilibrium outcomes. In Chapter III, we introduce a heterogeneous agents framework to study how the market price of an asset may deviate from their fundamental values as long as one or two decades, and how the informational frictions may lead to profitable speculative manipulations via signal distortions.

Note that the general equilibrium framework proposed in Chapter III is among the first few efforts in exploring approaches toward the computation of markets prices in an economy with heterogeneous prior beliefs, where the asset pricing formulas based on representative agent

framework only suggest asset valuations to each investor. Owing to the assumed heterogeneity, the expected market prices are likely to differ from the expected asset values, and the discrepancies must be priced when the agents are active in stock transactions and/or portfolio adjustments. Hence, the market price contains a non-fundament component, which explains the deviation from the corresponding expected fundamental values.

On the basis of our proposed heterogeneous agents framework, speculators may bid up the stock prices in a market with informational frictions, pretending that they receive better signals than what they really have. Subsequent investors would be misinformed and raise the stock prices accordingly. Signal distortions emerging from some market traits, such as the price fluctuation limits, can reduce the cost of speculative biddings and thus make this kind of price manipulation profitable. As a result, we can justify some speculations within our proposed framework.

### **1.3 Wavelet Analysis of Stock Market Comovements**

Due to informational frictions, the previous performances in one stock market could be regarded as important signals to investors in another stock market. This kind of information spillover is crucial for the understanding of financial contagion. In Chapter IV, we investigate the pattern of stock market comovements across time scales empirically with the help of wavelet analysis.

While it is difficult to control all the linear and/or non-linear impacts from fundamental factors, wavelet analysis provides an alternative approach. After employing wavelet filters to decompose the time series of stock indices over time scales, we can disentangle high frequency components from the low frequency ones. Based on the belief that long-run effects from fundamental factors are mostly captured by low frequency components, we can have a better idea about how the non-fundamental factors would influence stock market comovements in the short run.

The study on stock markets in US, Japan, Hong Kong, Taiwan, and Mainland China provides interesting empirical patterns of comovements among them across the time scales. First, we find the level of comovements varies across the time scales, while short-run wavelet correlation coefficients are significantly positive among open and/or semi-open markets. This suggests that there exist some short-run comovements mostly generated by non-fundamental factors.

Secondly, the markets sharing similar fundamental factors comove more substantially in the long run, implying that fundamentals (in a broad sense) are still the central piece of stock prices.

Thirdly, market openness matters. Open and semi-open markets in Japan, Hong Kong and Taiwan are quite sensitive to the US markets at the monthly to quarterly level, while the essentially closed Chinese mainland markets are literally uncorrelated with most of the other markets. Moreover, the level of short-run comovements seems increasing in the degree of market openness.

Finally, our results are robust in the sense that the comovement patterns computed based on measures of non-linear interdependence, such as Kendall's tau and Spearman's rho, are qualitatively similar to those suggested by the linear dependence measure (i.e. correlation coefficient).

# CHAPTER II

## MONEY, TECHNOLOGY CHOICE AND PATTERN OF EXCHANGE IN SEARCH EQUILIBRIUM<sup>1</sup>

“When the division of labour has been once thoroughly established, it is but a very small part of a man's wants which the produce of his own labour can supply. He supplies the far greater part of them by exchanging that surplus part of the produce of his own labour, which is over and above his own consumption, for such parts of the produce of other men's labour as he has occasion for.” (Adam Smith, *The Wealth of Nations*, Book I, Chapter IV, paragraph 1)

### 2.1 Introduction

In this paper, we develop a search-theoretic framework to study how money in a decentralized trading environment may affect *technology choice* and *decentralized exchange* patterns in the presence of trade frictions. Since the seminal work of Kiyotaki and Wright (1989,1991,1993), there has been a growing literature on money in search equilibrium, emphasizing that the use of a medium of exchange minimizes the time/resource costs associated with searching for exchange opportunities, hence alleviating the "double coincidence of wants" problem with barter.<sup>2</sup> While the study of the role of money in facilitating trade has generated considerable insights towards understanding the origin and use of money, its roles in promoting higher production technology remain largely unexplored.

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<sup>1</sup> This Chapter is based on the joint work with Ping Wang and Haibin Wu.

<sup>2</sup> The literature of barter versus money with a formal model of exchange is pioneered by Jones (1976). In the prototypical search model of money following Kiyotaki and Wright, exchange is characterized by one-for-one swaps of goods and money, implying fixed prices, under which the optimal inflation issue can be studied using the arguments by Li (1995). Extensions of the Kiyotaki-Wright model with divisible goods but indivisible money to include pricing include Trejos and Wright (1995) and Shi (1995). More recent attempts to characterize pricing behavior and the distribution of cash permit divisible goods and money. For a brief survey, the reader is referred to Rupert, Schindler and Wright (2001, footnote 1) and papers cited therein.

Different from the canonical Walrasian monetary growth models, the search-theoretic framework allows us to provide a deep structure to help understand more clearly how money affects technology choice in decentralized trade.<sup>3</sup> Under this setup, we can inquire (i) *whether the presence of trade frictions grants the high technology disadvantageous* and (ii) *whether money encourages adoption of the high technology*. In particular, our paper models explicitly the production *process* of quality-differentiated goods in a way that enables low-quality goods to be produced and traded in equilibrium *even under perfect observability of goods quality*. Money, by improving decentralized trades, can mitigate the high technology's cost-disadvantage and hence encourage the implementation of a more advanced technology. Our paper therefore provides a plausible channel through which money can generate a real effect via technology choice.

More specifically, we consider a continuous-time search and random-matching model with three groups of agents: producers, goods traders and money traders. For tractability, both goods and money are indivisible, and each non-producing agent has only one unit of space to store either goods or money. There are two clusters of goods: high quality and low quality, with each cluster consisting of a continuum of varieties. While high quality goods yield greater consumption values, they incur a production time delay and a greater production resource cost. At any point in time, each producer must choose between the two technologies and can only produce one unit of the good of a particular type. Upon a successful production, a producer becomes a goods trader with a commodity of a particular quality. The quality of goods is public information to all traders. Each buyer consumes only a subset of varieties, exclusive of those self-produced, and forms a *best response* to accepting goods of different quality within the desired subset.

The way through which money influences technology choice can be illustrated intuitively. Since the deepening of specialization entails some time for a consumer to buy the output from a producer, we have to consider the inventory costs incurred. If the use of money can save consumers' transactions time to search for desired commodities, the time costs of inventories can be reduced. This makes the high technology's disadvantage

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<sup>3</sup>Our paper is thus in sharp contrast with the ad hoc setup of money-in-the-production-function. Also note that in the canonical Walrasian frameworks, it is difficult to illustrate how search frictions would affect the acceptability of goods with different qualities. In contrast, the search-theoretic framework performs well in this regard.

in manufacturing and time delay costs less significant, thus creating an intensive margin in favor of the high technology. Since only producers take into account the underlying inventory costs, this transactions time effect becomes more important when production takes longer time. In conducting the welfare analysis, we employ the autarky technology choice as the benchmark. Note that in the absence of search frictions, producers' *first-best* technology choice is the same as their autarky counterparts, which may thus be referred to as *autarky efficiency*.<sup>4</sup> It is in this regard that the good produced by the high technology is never the worse good in terms of consumer's valuation. The presence of search frictions distorts the producer's technology choice and by comparing this with the first-best outcome in a frictionless economy, we can then investigate whether the introduction of money can restore the efficient technology choice outcome.

A principal contribution of our paper to the existing literature on money and product quality is that low quality goods may be produced and traded in equilibrium without relying on incomplete information. In particular, we establish the possible coexistence of two locally stable *pure* and one locally unstable *mixed strategy* equilibria with active trade, depending on the society's initial endowment of money. By examining equilibrium and welfare outcomes, we obtain the following properties. First, when production is instantaneous, the mixed strategy equilibrium, if it coexists with the pure strategy high-technology equilibrium, is Pareto-dominated, and features a positive relationship between the fraction of high technology producers and the society's endowment of money. Moreover, autarky efficiency is both sufficient and necessary for the high technology equilibrium to Pareto dominate the low technology one. Second, when production takes time, the high technology equilibrium has higher level of social welfare than the low one if, in addition to autarky efficiency, the high technology's delay cost is not too large and the social endowment of money is sufficiently high. The introduction of money affects producers' technology choice, by mitigating the high technology's disadvantage in production time delay. Finally, by deriving the optimal quantity of money under each equilibrium, we identify an important source of social inefficiency caused by search frictions that can lead to an *under-investment* in the advanced technology in decentralized equilibrium.

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<sup>4</sup>The reader should be warned that the term "autarky efficiency" is referred to technology choice rather than the pattern of exchange.

## Literature Review

In the money search literature, there are papers considering two types of traded goods, including Williamson and Wright (1994), Kim (1996), Berentsen and Rocheteau (2004), and Trejos (1997, 1999). However, in these models, the low quality good is always undesirable under perfect observability, as it is costless to produce and generates no consumption value, compared to a high quality good yielding a strictly positive net utility gain. In order for both goods to be traded, private information about the goods quality is therefore assumed. In contrast, our paper shows the possibility of producing low quality goods in equilibrium *even under perfect observability of goods quality* by modelling more explicitly the process of production of the two quality-differentiated goods, thus complementing previous studies by proposing an alternative and plausible underlying driving force that permits low-quality goods to be produced and traded in equilibrium.

There is a small but growing literature studying money and technology choice in search equilibrium. The first branch of this literature considers endogenous choice of the *horizontal* scopes of production specialization in the absence of quality differentiation. In Shi (1997), agents can produce desired goods at a higher cost than those for trade. Money enhances decentralized trade and thus creates a gain from specialization. A similar effect is considered by Reed (1998) where there is a trade-off between devoting time to trade and to maintaining production skills. Recently, Camera, Reed and Waller (2003) allow agents to choose whether to be a “jack of all trade” or a “master of one” in which money again advances individuals’ specialization in a decentralized trading environment. In Laing, Li and Wang (2003), a multiple-matching framework is developed where trade frictions manifest themselves in limited consumption variety and via a positive feedback between shopping and work effort decisions, money creation may have a positive effect on productive activity.

The second branch, to which our paper belongs, analyzes the endogenous choice between *vertically* differentiated production technologies under a given scope of specialization. The only previous study to our knowledge is by Kim and Yao (2001) who introduce money into a search model with divisible and heterogeneous goods. In their paper, production is instantaneous, the focus is exclusively on the mixed strategy equilibrium, whereas the proportions of high and low technology producers are exogenously given. In



contrast, our paper considers the more general case of *non-instantaneous production* to allow for different production time under different technologies and examines *both* mixed and pure strategy equilibria, by maintaining the simplifying assumption of indivisible goods. Moreover, we allow money traders to determine whether they would accept either type or both types of goods and, as a consequence, the proportion of producers using high/low technology is *endogenous*. Furthermore, we study the welfare implications under various equilibria and with different initial social endowment of money.

## 2.2 The Basic Model

The basic structure extends that of Kiyotaki and Wright (1993). Time is continuous. There are a continuum of infinitely-lived agents whose population is normalized to one. Based on their activities, agents are divided into three different categories at any point in time: producers, goods traders and money traders. Both goods and money are indivisible. Each non-producing agent has only one unit of space that may be used to store either a unit of commodity or a unit of money.

There are two groups of goods: high quality (type- $H$ ) and low quality (type- $L$ ). Each group consists of a continuum of varieties whose characteristic location can be indexed on a unit circumference. At any point in time, each producer can produce only one unit of the good of a particular type. Upon producing a commodity, a producer becomes a goods trader instantaneously. Thus, producers can be classified as type- $H$  or type- $L$ , as are goods traders. The type of agents (and hence the quality of goods) is assumed to be public information to all traders.

Money is storable but cannot be consumed or produced. At the beginning of time, there are  $M \in (0, 1)$  units of money in the economy, so we have a measure of  $M$  money traders due to the unit-storage-space assumption. Thus, letting  $N_0$ ,  $N_H$ ,  $N_L$ , and  $N_m$ , respectively, be the measures of producers, type- $H$  goods traders, type- $L$  goods traders, and money traders,<sup>5</sup> population identity implies:

$$N_m + N_H + N_L + N_0 = 1 \tag{2.1}$$

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<sup>5</sup>Due to the assumption of unit storage space and the indivisibility of money,  $N_m = M$ .

The proportion of type- $H$  goods traders to all goods traders, denoted  $h$ , and the fraction of money traders to all traders, denoted  $\mu$ , can thus be expressed as:

$$h = \frac{N_H}{N_H + N_L} \quad (2.2)$$

$$\mu = \frac{N_m}{N_m + N_H + N_L} = \frac{M}{1 - N_0}. \quad (2.3)$$

Traders match with each other according to a Poisson process characterized by the arrival rate parameter,  $\beta$ . Because that the probability for a particular pair of traders to rematch is zero in our continuum economy and that no authority exists to enforce the repayment of credits or IOU's, sellers must accept money in the absence of double coincidence of wants. Throughout the main text of the paper, we follow Trejos (1997) and Kim and Yao (2001) to focus exclusively on pure monetary equilibrium, that is, barter exchanges are not allowed in the basic framework. A discussion of the pure barter economy is relegated to Appendix.

### 2.2.1 Production Technology

There are two types of technologies. The high technology can produce a unit of the high quality good at a (utility) cost of  $\varepsilon$ , while the low technology incurs a lower manufacturing cost of  $\delta\varepsilon$  (with  $0 < \delta < 1$ ) to produce one unit of the low quality good.

The two technologies also differ in the arrival rates of the respective outputs. Specifically, the product arrival of the low technology follows a Poisson process with arrival rate of  $\alpha$ , while that of the high technology has an arrival rate of  $\eta\alpha$  (with  $0 < \eta < 1$ ).

### 2.2.2 Preferences

Following the convention of the money-search literature, we assume that no agent would consume the good he or she produces. Moreover, each agent gains positive utility only by consuming a subset of the varieties of each type (called a consumable set), whose measure is denoted by  $x$ . Thus,  $x$  can be regarded as a taste specialization index.

	Type- $H$	Type- $L$
Utility from consumption	$U$	$\theta U$
Production cost	$\varepsilon$	$\delta\varepsilon$
Output arrival rate	$\eta\alpha$	$\alpha$

$(0 < \theta, \delta, \eta < 1)$

Table 2.1: Key Parameters of Technologies and Preferences

Despite their taste heterogeneity, all agents have identical utility functional forms. While the consumption of the first unit of a high quality good within the consumable set yields a utility  $U > 0$ , any additional unit at a given point in time would not generate any extra value. Similarly, the consumption of the first unit of a low quality good within the consumable set gives a utility of  $\theta U$  (with  $0 < \theta < 1$ ). The key technology and preference parameters are summarized in Table 2.1.

To ensure non-trivial technological choice, we impose:

**Assumption 1:**  $U > \theta U > \varepsilon > \delta\varepsilon$ .

That is, both types of products deliver positive net values to the economy. The assumption of  $\theta U > \varepsilon$  guarantees the existence of mixed strategy equilibrium, as we will show later.

### 2.2.3 Value Functions

Denote the probability with which a money trader will accept type- $i$  goods as  $\pi_i$  ( $i = H, L$ ), and  $\Pi_i$  as the average probability of acceptability in the economy (which is taken as parametrically given by all individual traders). Denote the discount rate by  $r$ . Further denote  $V_i$  as the asset value of a type- $i$  agent, where  $i = 0, H, L, m$  represents producers, type- $H$  goods traders, type- $L$  goods traders, and money traders, respectively.

We are now well equipped to set up the Bellman equations, displayed for simplicity by assuming steady states (as in the conventional money and search literature):

$$rV_0 = \max\{\alpha(V_L - V_0 - \delta\varepsilon), \eta\alpha(V_H - V_0 - \varepsilon)\} \quad (2.4)$$

$$rV_i = \beta\mu x \Pi_i (V_m - V_i), \quad i = H, L \quad (2.5)$$

$$rV_m = \beta(1-\mu)x \left[ h \max_{\pi_H} \{ \pi_H(U + V_0 - V_m) \} + (1-h) \max_{\pi_L} \{ \pi_L(\theta U + V_0 - V_m) \} \right]. \quad (2.6)$$

Equation (2.4) states that the flow value of a producer is the maximum incremental value, over the two technologies, from the producer state to the goods trader state net of the corresponding production cost, upon a successful arrival of the product (measured by  $\alpha$  and  $\eta\alpha$ , respectively).

Recall that with a flow probability  $\beta$ , a type- $i$  goods trader can meet another trader who will be a money trader with probability  $\mu$ . The chance for this money trader to like the goods trader's product is  $x$ , which will be accepted with probability  $\Pi_i$ . Thus, as indicated by (2.5), the flow value of a type- $i$  goods trader is the incremental value from exchanging the product for money, which is the differential,  $V_m - V_i$ , multiplied by the flow probability,  $\beta\mu x\Pi_i$ .

Similarly, the flow probability for a money trader to meet a type- $H$  goods trader whose commodity is within the consumable set is  $\beta(1-\mu)xh$  and that to meet a type- $L$  goods trader is  $\beta(1-\mu)x(1-h)$ . The flow value of meeting a type- $i$  goods trader is the flow utility ( $U$  and  $\theta U$ , for  $i = H, L$ , respectively) plus the incremental value from the money trader state to the producer state ( $V_0 - V_m$ ). A money trader may stay put (by not accepting the good, i.e.,  $\pi_i = 0$ ) or accept the trade with probability  $\pi_i > 0$  (which is the best response by the money trader, possibly less than one). Thus, this flow value must be multiplied by the corresponding acceptance probability, as displayed in (2.6).

It is convenient to define by  $\Delta_i$  ( $i = H, L$ ) the producer's effective discount factors over the expected span of the production process and by  $\rho_i$  ( $i = H, L$ ) the goods trader's effective discount factors for the expected waiting period for sales.

$$\Delta_H \equiv \frac{\eta\alpha}{\eta\alpha + r}; \quad \Delta_L \equiv \frac{\alpha}{\alpha + r} \quad (2.7)$$

$$\rho_H \equiv \frac{\beta\mu x\Pi_H}{\beta\mu x\Pi_H + r}; \quad \rho_L \equiv \frac{\beta\mu x\Pi_L}{\beta\mu x\Pi_L + r}. \quad (2.8)$$

Given the Poisson process,  $\frac{1}{\eta\alpha}$  is the average waiting time for production and  $\frac{r}{\eta\alpha}$  is the discount rate over the expected span of the production process, thus yielding the producer's effective discount factors,  $\Delta_H$ . Similar explanations apply to  $\Delta_L$ ,  $\rho_H$  and  $\rho_L$ .

Accordingly, we can rewrite the value functions (2.4) and (2.5) in a cleaner manner,

$$rV_0 = \max\{\Delta_L(V_L - \delta\varepsilon), \Delta_H(V_H - \varepsilon)\} \quad (2.9)$$

$$V_i = \rho_i V_m, \quad i = H, L. \quad (2.10)$$

## 2.3 Equilibria with Instantaneous Production

We begin by considering a special case with instantaneous production ( $\alpha \rightarrow \infty$ ), which enables a complete analytic analysis of the *steady-state monetary equilibrium*. With instantaneous production, we have  $N_0 = 0$ , and, from (2.3),  $\mu = M$ . Moreover, (2.7) implies  $\Delta_H = \Delta_L = 1$  and hence (2.9) can be rewritten as:

$$V_0 = \max\{(V_L - \delta\varepsilon), (V_H - \varepsilon)\}. \quad (2.11)$$

### 2.3.1 Money Trader's Best Response

To solve the equilibrium under instantaneous production, first consider the money traders. A money trader's best responses  $\pi_H$  and  $\pi_L$  are determined according to the following:

$$\pi_H \begin{cases} = 0, & \text{if } U + V_0 - V_m < 0 \\ \in (0, 1), & \text{if } U + V_0 - V_m = 0 \\ = 1, & \text{if } U + V_0 - V_m > 0 \end{cases} \quad (2.12)$$

$$\pi_L \begin{cases} = 0, & \text{if } \theta U + V_0 - V_m < 0 \\ \in (0, 1), & \text{if } \theta U + V_0 - V_m = 0 \\ = 1, & \text{if } \theta U + V_0 - V_m > 0 \end{cases} \quad (2.13)$$

Thus, in the case of  $U + V_0 - V_m = 0$  or  $\theta U + V_0 - V_m = 0$ , the corresponding best response ( $\pi_H$  or  $\pi_L$ ) constitutes a mixed strategy.

In equilibrium, the individual's best response agrees with the average behavior in the economy, that is,

$$\pi_i = \Pi_i, \quad (2.14)$$

for  $i = H, L$ .

### 2.3.2 Existence

We focus on the case of nondegenerate equilibrium in which all agents in the economy participate in trade actively. Thus, a producer must have positive payoff,

$$\max\{(V_L - \delta\varepsilon), (V_H - \varepsilon)\} > 0 \quad (2.15)$$

Moreover, a money trader must buy at least one type of the commodities. This is valid under the following active equilibrium condition:

$$\max\{U + V_0 - V_m, \theta U + V_0 - V_m\} > 0 \quad (2.16)$$

The strict inequality is required in order to ensure the validity of condition (2.15).

Since  $\theta < 1$ , this condition requires  $U + V_0 - V_m > 0$ , and thus  $\pi_H = 1$ , which means the money trader will fully accept the type- $H$  goods. Based on the three different best responses towards the acceptability of the type- $L$  goods, we can have three equilibria: (A)  $\pi_L^A = 0$ ; (B)  $\pi_L^B \in (0, 1)$ ; and (C)  $\pi_L^C = 1$ . We use superscript  $A$ ,  $B$ , and  $C$  to denote each equilibrium whenever it is necessary. Also, we can define the effective discount factor for the purchasing period (when always accepting a good) as:

$$\rho_m \equiv \frac{\beta(1 - \mu)x}{\beta(1 - \mu)x + r}. \quad (2.17)$$

It is not difficult to solve the asset values ( $V_0, V_H, V_L, V_m$ ) from the linear equation system (2.6), (2.10) and (2.11), and the solutions are summarized in Table 2.2. We can interpret the solutions intuitively with the effective discount factors defined in (2.8) and (2.17). In equilibrium A, for example, the producer bears the manufacturing cost instantaneously but should wait for both the selling and purchasing periods, so his utility in one production cycle is  $\rho_H \rho_m U - \varepsilon$ . Since the effective discount factor for one production cycle is  $\rho_H \rho_m$ , the summation of infinite geometric series yields the solution in the first

	Equilibrium A	Equilibrium B	Equilibrium C
$V_0$	$\frac{\rho_H^A \rho_m^A U - \varepsilon}{1 - \rho_H^A \rho_m^A}$	$\frac{\rho_H^B \theta U - \varepsilon}{1 - \rho_H^B}$ , or $\frac{\rho_L^B \theta U - \delta \varepsilon}{1 - \rho_L^B}$	$\frac{\rho_L^C \rho_m^C \theta U - \delta \varepsilon}{1 - \rho_L^C \rho_m^C}$
$V_H$	$\frac{\rho_H^A \rho_m^A (U - \varepsilon)}{1 - \rho_H^A \rho_m^A}$	$\frac{\rho_H^B (\theta U - \varepsilon)}{1 - \rho_H^B}$	$V_L^C$
$V_L$	0	$\frac{\rho_L^B (\theta U - \delta \varepsilon)}{1 - \rho_L^B}$	$\frac{\rho_L^C \rho_m^C (\theta U - \delta \varepsilon)}{1 - \rho_H^C \rho_m^C}$
$V_m$	$\frac{\rho_m^A (U - \varepsilon)}{1 - \rho_H^A \rho_m^A}$	$\frac{\theta U - \varepsilon}{1 - \rho_H^B}$ , or $\frac{\theta U - \delta \varepsilon}{1 - \rho_L^B}$	$\frac{\rho_m^C (\theta U - \delta \varepsilon)}{1 - \rho_L^C \rho_m^C}$
$\Pi_L$	0	$\pi_L^B$	1
$h$	1	$h^B$	0
$M$	$S^a$	$S^B$	$S^C$

Table 2.2: Solutions for the Instantaneous production Case

cell in Table 2.2, where other cells can be derived in an analogous fashion.<sup>6</sup>

Utilizing these asset values, we next turn to deriving the best responses of the agents and checking the corresponding conditions on the parameters. Define  $Q \equiv \frac{(\beta x + r)r\varepsilon}{\beta^2 x^2 (U - \varepsilon)}$  and impose:

**Assumption 2:**  $Q \max \left\{ \frac{\delta U - \delta \varepsilon}{\theta U - \delta \varepsilon}, 1 \right\} < \frac{1}{4}$ .

**Assumption 3:**  $\frac{1}{\theta U - \varepsilon} + \frac{\theta}{1 - \theta} < \frac{\beta x}{r}$ .

We first examine the two pure strategy equilibria (A and C). In equilibrium A, no producer would choose the low technology since it yields negative flow value to producers ( $h = 1$ ). We can show from (2.8) and (2.10) that  $V_L = 0$ . From (2.13), we know that  $\pi_L = 0$ , if  $\theta U + V_0 - V_m < 0$ , or,  $\theta U < \rho_M^A U - (1 - \rho_M^A)V_0$ , as  $V_M = \rho_M^A(U + V_0)$  when the high technology is chosen. We now define:

$$M_1 \equiv \max \left\{ 1 - \frac{(\beta x + r)(\theta U - \varepsilon)}{\beta x (U - \varepsilon)}, 0 \right\} \quad (2.18)$$

and  $M_2 \leq 0.5$  such that

$$M_2(1 - M_2) = \frac{(\beta x + r)r\varepsilon}{\beta^2 x^2 (U - \varepsilon)}, \quad (2.19)$$

<sup>6</sup>Note that  $\rho_H^A$ ,  $\rho_H^B$ , and  $\rho_L^C$  have the same functional form with respect to  $\mu$  and exogenous parameters, so do  $\rho_m^A$  and  $\rho_m^C$ . However, the argument  $\mu$  may differ.

which has real roots under Assumption 2. We can then establish:

**Lemma 1:** (Equilibrium A) *Equilibrium A exists if  $S^A \equiv (0, M_1) \cap (M_2, 1 - M_2) = (M_2, \min(1 - M_2, M_1)) \neq \emptyset$  and  $M \in S^A$ , where  $M \in (0, M_1)$  iff  $\theta U < \rho_M^A U - (1 - \rho_M^A)V_0^A$ .*

*Proof.* All proofs are in Appendix.

Note that by accepting the type- $L$  goods, a money trader can enjoy the utility of  $\theta U$ . By rejecting the trade and waiting for type- $H$  goods in the next trade, the money trader will have the discounted utility,  $\rho_M^A U$ , at the cost of delayed production,  $(1 - \rho_M^A)V_0$ . Hence when  $M \in (0, M_1)$ , or equivalently,  $\theta U < \rho_M^A U - (1 - \rho_M^A)V_0$ , it is a money trader's best response to reject a trade with a type- $L$  producer. Intuitively, in an economy swamped by too much money, money traders would buy any type of goods as soon as possible since they cannot afford the long waiting period for the second chance. This is particularly important when the difference in the quality is not large enough to make the waiting worthwhile. This effect is due primarily to the presence of search frictions in which too much money can crowd out the advanced technology. Thus, it may be referred to as the *nonselectivity effect*.

The requirement that  $M \in (M_2, 1 - M_2)$  is to ensure positive producer payoffs. If the amount of initial money endowment is either too small or too large, then the probability of successful trades is too low. This *transactions cost effect* caused by search frictions implies that no producers would find profitable to use the high technology given the high manufacturing cost. Thus, if the economy has insufficient amount of money initially, the introduction of money can serve to mitigate the transactions cost effect and hence encourage producers to adopt the advanced technology. It may be noted that the transactions cost effect is also present in Shi (1997), where an insufficient endowment of money may discourage agents from trading in a decentralized market and instead make them stay in autarky. By mitigating search frictions, money enlarges the extent of the market, encouraging horizontal production specialization in Shi (1997) while enhancing vertical production quality in ours.

The solution of equilibrium C is quite similar to that of equilibrium A. Observe that when  $\pi_L = \Pi_L = 1$ , equation (2.5) results in  $V_H^C = V_L^C$ , as well as  $\rho_H^C = \rho_L^C$ . The



producer would definitely choose the low technology to minimize his cost, which means  $h = 0$ . After solving for the values, we find that since  $U > \theta U > V_m^C - V_0^C$ , for any  $M \in (0, 1)$ , equilibrium C exists as long as  $V_0^C > 0$ . Define  $M_3 \leq 1/2$  such that

$$M_3(1 - M_3) = \frac{(U - \varepsilon)\delta Q}{\theta U - \delta\varepsilon}, \quad (2.20)$$

which has real roots under Assumption 2. Then we have:

**Lemma 2:** (Equilibrium C) *Equilibrium C exists if  $M \in S^C \equiv (M_3, 1 - M_3)$ . Moreover,  $S^C \supseteq S^A$  if  $0 < \delta \leq \theta < 1$ .*

Equilibrium B is a bit more complicated. The money trader's mixed strategy implies  $\theta U + V_0^B - V_m^B = 0$ . Based on the fact that the producers are indifferent between the two technologies, we can solve the money trader's acceptability of low quality goods,

$$\pi_L^B = \Pi_L^B \equiv 1 - \frac{(1 - \delta)\varepsilon}{\rho_H(\theta U - \delta\varepsilon)}, \quad (2.21)$$

and the equilibrium proportion of type- $H$  goods in the market,

$$h^B \equiv \frac{(\beta\mu x + r)(\theta U - \varepsilon)}{\beta(1 - \mu)x(1 - \theta)U}. \quad (2.22)$$

It is easily seen that  $\pi_L^B$  is increasing in  $\mu$  and thus  $M$ . Moreover,  $h^B$  is increasing in  $\mu$  and thus  $M$ , which implies as the amount of money increases in the economy, there are more people holding type- $H$  goods. Define,

$$M_4 \equiv \frac{r\varepsilon}{\beta x(\theta U - \varepsilon)}, \quad (2.23)$$

we obtain:

**Lemma 3:** (Equilibrium B) *Equilibrium B exists if  $S^B \equiv (M_4, M_1) \neq \emptyset$  and  $M \in S^B$ . Moreover,  $S^B \subseteq S^A$ .*

Under Assumptions 2 and 3,  $S^j \neq \emptyset$  ( $j = A, B, C$ ) and with the aid of Lemmas 1-3 and Proposition A1 in Appendix, we can establish:

**Proposition 1:** (Existence, Stability and Characterization) *Under Assumptions 1-3, a steady-state monetary equilibrium exists, which possesses the following properties, depending on the society's initial endowment of money  $M$ :*

- (i)  $\pi_L = 0$  with  $M \in S^A$  (equilibrium A);
- (ii)  $\pi_L \in (0, 1)$  with  $M \in S^B$  (equilibrium B);
- (iii)  $\pi_L = 1$  with  $M \in S^C$  (equilibrium C).

*Moreover, multiple equilibria may arise. Among the three equilibria, equilibrium A and C are locally stable, while equilibrium B is locally unstable. Furthermore, by mitigating the transactions cost effect, the introduction of money encourages investment in the advanced technology and the emergence of equilibrium A. By contrast, only type-L technology will be chosen in a pure barter economy with instantaneous production.*

As to the existence, Assumptions 2 and 3 ensure the nonemptiness of  $S^C$  and  $S^B$ , respectively, whereas both assumptions together guarantee that  $S^A$  is nonempty. From Lemma 3, when  $M \in S^B$ , the mixed strategy equilibrium B always co-exists with the pure strategy equilibrium A (as  $S^B \subseteq S^A$ ). Moreover, when  $0 < \delta \leq \theta < 1$ , both pure strategy equilibria co-exist if  $M \in S^A$  (as  $S^A \subseteq S^C$ ) while all three types of equilibria co-exist if  $M \in S^B$ .

The possibility for low quality goods to be produced and traded contributes to the existing literature on money and product quality in which only high quality goods arise in equilibrium unless information is asymmetric between buyers and sellers. The co-existence of these various types of equilibria with active trade is also new to the literature. Notably, for some given sets of social endowment of money, the equilibrium outcome is indeterminate where the underlying equilibrium selection mechanism is due entirely to *self-fulfilling prophecies*. For example, should a producer expect that traders are less selective in good quality (animal spirits), he or she would be more inclined toward employing the low technology. As a result, there will be more traders with low quality goods and the probability for money holders to locate a high quality good becomes lower. This implies that money holders would tend to be more generous toward accepting a low quality good, so the expectations are self-fulfilled.

The two pure strategy equilibria are both locally stable, since small disturbances in the acceptability of the type- $L$  goods cannot affect the producer's choice. However, equilibrium B is locally unstable. To see this we can simply disturb  $\Pi_L$ . If the agents believe  $\Pi_L$  to be a bit larger (smaller),  $V_L$  would be higher (lower). Thus the producer will prefer the low (high) technology, thereby leading to equilibrium  $C$  ( $A$ ). This is consistent with the finding in Wright (1999) where mixed strategy equilibria are always unstable in an evolutionary sense in the search-theoretic model of money with indivisible goods and indivisible money.

Equilibrium  $B$  in our model can be compared with the mixed strategy equilibrium in Kim and Yao (2001): When both types of products co-exist, the share of type- $H$  goods ( $h$ ) and the level of social welfare are increasing in the quantity of money supply ( $M$ ).

### 2.3.3 Welfare Implications

Note that in the absence of search frictions, the producers would make the same technology choice (*first-best*) as their autarky counterparts, while the existence of search frictions can distort the producer's choices. Hence we employ the first best choice as the benchmark of our welfare analysis, and investigate whether the introduction of money can reinstall the *autarky efficiency*.

Due to the assumption of instantaneous production, only the goods and money traders are considered in the commonly used equally weighted steady-state social welfare function. Observe that  $M \in [0, M_1]$  is equivalent to  $V_m^A > V_m^B$ , which implies  $V_H^A > V_H^B > V_L^B$ , and  $V_0^A > V_0^B$ , *pointwise* with respect to  $M$ . Since  $S^B \subseteq S^A$ , for any value of  $M \in S^B$ , there is always an equilibrium with  $\pi_L = 0$  (equilibrium  $A$ ) that Pareto dominates the mixed strategy equilibrium. Since this equilibrium is locally unstable and Pareto-dominated in its existence region (see the following subsection), we put more effort on comparing the two pure strategy equilibria,  $A$  and  $C$ .

By examining these two equilibria, we find that both goods traders and money traders prefer (pointwise with respect to  $M$ ) the technology with autarky efficiency, i.e., that with the higher net-of-cost utility. The Pareto ranking in this case is straightforward because the producers are of measure zero. In general, it may be useful to compare the

	Equilibrium $A$	Equilibrium $C$
Monetary	$\beta x M(1 - M)(U - \varepsilon)$	$\beta x M(1 - M)(\theta U - \delta \varepsilon)$
Autarky	$(U - \varepsilon)$	$(\theta U - \delta \varepsilon)$

Table 2.3: Flow Values of Welfare in the Instantaneous Production Case

steady-state social welfare instead of Pareto rankings:

$$Z = N_0 V_0 + N_L V_L + N_H V_H + N_m V_m. \quad (2.24)$$

We assume that a social planner can set the initial amount of money  $M$  to maximize  $Z$ . Hence we compare the maximal welfare in equilibria  $A$  and  $C$ .

The flow values of social welfare for both equilibria are shown in Table 2.3.<sup>7</sup> The *optimal quantity of money* can be easily solved as  $\min\{1/2, M_1\}$  for equilibrium  $A$  and  $1/2$  for equilibrium  $C$ .<sup>8</sup> If  $M_1 > 1/2$  (which holds when  $\theta$  is sufficiently small), the welfare comparison between Equilibrium  $A$  and  $C$  is again equivalent to autarky efficiency. Otherwise, the social planner would choose the high technology only when it provides sufficiently more net utility than the low technology, that is,

$$\frac{U - \varepsilon}{\theta U - \delta \varepsilon} \geq \frac{1/4}{M_1(1 - M_1)} > 1.$$

From (2.18),  $M_1$  is decreasing in  $\theta$  and independent of  $\delta$ . Therefore, when the quality difference is sufficiently small, the social planner could still support the production of type- $L$  goods, even when the type- $H$  goods provides more net utility. On the contrary, the production cost differential (captured by  $\delta$ ) does not play any role, which is a result of the take-it-or-leave-it offer to buyers whose only concern is the quality of the goods. Under instantaneous production, the social planner can do no better than the autarky efficiency outcome, with a frictional exchange process being introduced. This conclusion would no longer be true if goods production also takes time (see Section 2.4 below).

<sup>7</sup>We use flow values here just because the stock values of social welfare for autarky economy are infinite, and hence incomparable, due to instantaneous production.

<sup>8</sup>Rigorously speaking, since the admissible set is not closed, the optimal quantity of money does not exist if  $M_1 < 1/2$ . For illustrative purposes, however, we will refer to the welfare maximizer over the closure of the admissible set as the optimal quantity of money.

**Proposition 2:** (Welfare and Optimal Quantity of Money) *Equilibrium B is always Pareto dominated by equilibrium A either pointwise with respect to  $M$  or in the sense of equally weighted social welfare maximization. The comparison between equilibrium A and C possesses the following properties:*

- (i) *under pointwise Pareto criterion, it is equivalent to the case of autarky efficiency;*
- (ii) *under social welfare maximization,*
  - a. *it is equivalent to autarky efficiency if  $M_1 > 1/2$ ,*
  - b. *the social planner is less likely to adopt the high technology than autarky efficiency if  $M_1 \leq 1/2$ ;*
- (iii) *the socially optimal quantity of money is  $\min\{1/2, M_1\}$  for equilibrium A and  $1/2$  for equilibrium C.*

Recall the nonselectivity effect that too much money may discourage the adoption of the high technology. By accounting for this, the social planner must set the optimal quantity of money for equilibrium A at a lower level than for equilibrium C. Also due to the presence of the nonselectivity effect, the optimal quantity of money in our paper may be lower than that obtained by Kiyotaki and Wright (1993) under an exogenously given production technology (which is  $1/2$ ).

## 2.4 Non-instantaneous Production

When production is not instantaneous, *i.e.*, when  $\alpha$  is finite, there are a nontrivial steady-state mass of producers. Thus,  $\mu > M$  and this creates great algebraic complexity. Nonetheless, this exercise allows us to gain additional insights into how the introduction of money could improve technological development.

### 2.4.1 Steady-State Monetary Equilibrium

Based on the active equilibrium condition (2.16) we once more obtain:  $\pi_H = 1$ , which means money trader will fully accept the type- $H$  goods in equilibrium. Based on the three different best responses to accepting type- $L$  goods, we again have three equilibria: ( $AA$ ):  $\pi_L^{AA} = 0$ ; ( $BB$ ):  $\pi_L^{BB} \in (0, 1)$ ; and ( $CC$ ):  $\pi_L^{CC} = 1$ , where the labelings  $AA$ ,  $BB$ , and  $CC$  correspond to  $A$ ,  $B$ , and  $C$ , in the instantaneous production case.

To solve the population distribution in the steady state, we equate the outflows and inflows from and to the population of goods and money traders to yield:

$$\Lambda\eta\alpha N_0 = \beta\mu x\Pi_H N_H \quad (2.25)$$

$$(1 - \Lambda)\alpha N_0 = \beta\mu x\Pi_L N_L \quad (2.26)$$

$$\beta\mu x(\Pi_L N_L + \Pi_H N_H) = \beta(1 - \mu)x[h\Pi_H + (1 - h)\Pi_L]N_m \quad (2.27)$$

where  $\Lambda$  is the proportion of producers employing the high technology. From equation (2.25) and (2.26), and using  $\pi_H = \Pi_H = 1$ , we can derive:

$$\Lambda = \frac{h}{h + \eta(1 - h)\Pi_L} \quad (2.28)$$

Observe that  $\Lambda$  is strictly increasing in  $h$ , satisfying:  $\lim_{h \rightarrow 0} \frac{\Lambda}{h} = \frac{1}{\eta}$ , and  $\lim_{h \rightarrow 1} \frac{\Lambda}{h} = 1$ .

Now  $\mu$  is no longer equal to  $M$ . Since the expressions in terms of  $M$  are complex, our strategy is to establish the existence of various types of equilibria based on the values of  $\mu$ . Once this is done, we can derive the corresponding values of  $M$  by utilizing the following monotone increasing relationship between  $M$  and  $\mu$ , which can be obtained by combining equations (2.25) and (2.27) to yield,  $\Lambda\eta\alpha(1 - \frac{M}{\mu}) = \beta\mu xh(\frac{M}{\mu} - M)$ , or,

$$M = \frac{\mu\eta\alpha(\Lambda/h)}{\beta x\mu(1 - \mu) + \eta\alpha(\Lambda/h)} \quad (2.29)$$

The expression could be simplified with the aid of the limiting properties under equilibrium  $AA$  or  $CC$ . As a result, the population distribution will be determined by only three endogenous variables,  $h$ ,  $\mu$ , and  $\Pi_L$ , since from (2.1), (3.8), (2.3), (2.28) and (2.29),

all population masses can be expressed in terms of  $h$ ,  $\mu$  and  $M$ .<sup>9</sup>

As before, we can solve the system using the discount rates  $\Delta_H$  and  $\Delta_L$  (see Table 2.4), where the equilibrium acceptability of type- $L$  goods in equilibrium  $BB$  is:<sup>10</sup>

$$\pi_L^{BB}(\mu) = \frac{1}{\beta\mu x} \frac{\beta\mu x \eta(\alpha + r)\theta U - \{(\beta\mu x + r)\eta\alpha + r[(\beta\mu x + r)(\eta - \delta) - \delta\eta\alpha]\}\varepsilon}{[(\beta\mu x + r) + \eta(\alpha - \beta\mu x)]\theta U + [(\beta\mu x + r)(\eta - \delta) - \delta\eta\alpha]\varepsilon}, \quad (2.30)$$

and the related proportion of type- $H$  goods in the market becomes:

$$h^{BB}(\mu) = \frac{r(\theta U - \Delta_H \varepsilon)}{(1 - \rho_H \Delta_H)\beta(1 - \mu)x(1 - \theta)U}. \quad (2.31)$$

Although  $h^{BB}$  is increasing in  $\mu$ , the relationship between  $\pi_L^{BB}$  and  $\mu$  is no longer monotone.<sup>11</sup>

The values in equilibria  $AA$  and  $CC$  listed in Table 2.4 can be explained intuitively. Note that the effective discount factors indicate the time costs over the respective waiting periods (production, selling, and buying). Take  $V_0^{AA}$  as an example. As the producers must wait for all the three waiting periods, the utility should be discounted by all the three factors,  $\Delta_H$ ,  $\rho_H$ , and  $\rho_m$ . Meanwhile, the production cost is generated at the end of the production period, so only  $\Delta_H$  is attached to it. This provides the producer's value in one cycle,  $\Delta_H \rho_H^{AA} \rho_m^{AA} U - \Delta_H \varepsilon$ . The value is then obtained by simply dividing the one-cycle value by one minus the discount factor for a cycle,  $\Delta_H \rho_H^{AA} \rho_m^{AA}$ .

Repeating the same steps as in the previous section, one can derive parameter regions for  $\mu$  (instead of  $M$ ) to support each type of equilibrium. As shown in the Appendix, we have:  $S^{AA} = (0, \mu_1) \cap (M_2, 1 - M_2)$ , where  $\mu_1$  solves:

$$(1 - \theta)U = \frac{(\beta\mu x + r)r(U - \Delta_H \varepsilon)}{\beta^2 x^2 \mu(1 - \mu)(1 - \Delta_H) + r\beta x + r^2}; \quad (2.32)$$

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<sup>9</sup>Actually we express all the values in terms of  $\mu$ . In contrast to the instantaneous production case,  $\mu$  is now endogenously determined. It seems that it is no longer suitable to use  $\mu$  to characterize the existence conditions. However, equation (2.29) enables us to relate  $\mu$  with the exogenous variable  $M$  in each equilibrium. All the conditions in terms of  $\mu$  can be transformed into expressions in  $M$  accordingly.

<sup>10</sup>Like Table 2, we have  $\rho_H^{AA}$ ,  $\rho_H^{BB}$ , and  $\rho_L^{CC}$  with the same functional form.  $\rho_m^{AA}$  and  $\rho_m^{CC}$  also have same functional form. The argument  $\mu$  may differ.

<sup>11</sup>The reader can easily check that the solution of  $\pi_L^{BB}$  reduces to  $\pi_L^B$  with  $\alpha \rightarrow \infty$  and  $\eta \rightarrow 1$ .

	Equilibrium AA	Equilibrium BB	Equilibrium CC
$V_0$	$\Delta_H \frac{\rho_H^{AA} \rho_m^{AA} U - \varepsilon}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H}$	$\Delta_H \frac{\rho_H^{BB} \theta U - \varepsilon}{1 - \rho_H^{BB} \Delta_H}$ , or $\Delta_L \frac{\rho_L^{BB} \theta U - \delta \varepsilon}{1 - \rho_L^{BB} \Delta_L}$	$\Delta_L \frac{\rho_L^{CC} \rho_m^{CC} \theta U - \delta \varepsilon}{1 - \rho_L^{CC} \rho_m^{CC} \Delta_L}$
$V_H$	$\frac{\rho_H^{AA} \rho_m^{AA} (U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H}$	$\frac{\rho_H^{BB} (\theta U - \Delta_H \varepsilon)}{1 - \rho_H^{BB} \Delta_H}$	$V_L^{CC}$
$V_L$	0	$\frac{\rho_L^{BB} (\theta U - \Delta_L \delta \varepsilon)}{1 - \rho_L^{BB} \Delta_L}$	$\frac{\rho_L^{CC} \rho_m^{CC} (\theta U - \Delta_L \delta \varepsilon)}{1 - \rho_L^{CC} \rho_m^{CC}}$
$V_m$	$\frac{\rho_m^{AA} (U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H}$	$\frac{\theta U - \Delta_H \varepsilon}{1 - \rho_H^{BB} \Delta_H}$ , or $\frac{\theta U - \Delta_L \delta \varepsilon}{1 - \rho_L^{BB} \Delta_L}$	$\frac{\rho_m^{CC} (\theta U - \Delta_L \delta \varepsilon)}{1 - \rho_L^{CC} \rho_m^{CC} \Delta_L}$
$\Pi_L$	0	$\pi_L^{BB}$	1
$h$	1	$h^{BB}$	0
$\mu$	$S^{AA}$	$S^{BB}$	$S^{CC}$

Table 2.4: Solutions for the Non-instantaneous Production Case

$S^{BB} = (M_4, \mu_1)$ , and,  $S^{CC} = S^C$ . Analogous to Lemma 1,  $\mu \in (0, \mu_1)$  iff  $\theta U < \rho_M^{AA} U - (1 - \rho_M^{AA}) V_0^{AA}$ . Based on the proofs in Appendix, we can establish the existence regions of these equilibria:

**Proposition 3:** (Existence) *Under Assumptions 1-3, a steady-state monetary equilibrium exists. Depending on the society's initial endowment of money  $M$ , it possesses the following properties:*

- (i)  $\pi_L = 0$  with  $\mu \in S^{AA}$  (equilibrium AA);
- (ii)  $\pi_L \in (0, 1)$  with  $\mu \in S^{BB}$  (equilibrium BB);
- (iii)  $\pi_L = 1$  with  $\mu \in S^{CC}$  (equilibrium CC);

where multiple equilibria may arise.

## 2.4.2 Welfare Implications

Due to its complexity, we will not conduct welfare analysis based on the endowment of money ( $M$ ). Rather, we restrict our attention to the case where the fraction of money traders ( $\mu$ ) is identical in different types of equilibria. With this modification, we still have equilibrium AA Pareto dominates equilibrium BB. However the welfare comparison between equilibria AA and CC is a bit more sophisticated now. Let us derive the social



	Equilibrium <i>AA</i>	Equilibrium <i>CC</i>
Monetary	$\frac{\eta\alpha b(U - \varepsilon)}{r(b + \eta\alpha)}$	$\frac{\alpha b(\theta U - \delta\varepsilon)}{r(b + \alpha)}$
Autarky	$\frac{(\eta\alpha + r)U - \eta\alpha\varepsilon}{r}$	$\frac{(\alpha + r)\theta U - \alpha\delta\varepsilon}{r}$
	$b \equiv \beta x\mu(1 - \mu)$	

Table 2.5: Present Values of Welfare in the Non-Instantaneous Production Case

welfare for the respective equilibria as follows:

$$Z^{AA} = \frac{\eta\alpha b}{b + \eta\alpha} \left( \frac{U - \varepsilon}{r} \right) \quad (2.33)$$

$$Z^{CC} = \frac{\alpha b}{b + \alpha} \left( \frac{\theta U - \delta\varepsilon}{r} \right). \quad (2.34)$$

where  $b \equiv \beta x\mu(1 - \mu)$ .

Obviously the optimal fraction of money traders satisfies  $\mu = 0.5$  in each case, provided that  $\mu_1 \geq 0.5$ , from which the optimal quantity of money can be derived using equation (2.29). For pointwise comparison of social welfare between different types of equilibria with respect to  $\mu$ , we still have the net utility terms,  $U - \varepsilon$  versus  $\theta U - \delta\varepsilon$  as in the instantaneous production case. However, the slow production process makes the high technology less attractive than the low technology as the multiplier on the right-hand side of (2.33) is less than that of (2.34) if  $\eta < 1$ . When the net utility gain from undertaking the high technology is positive and sufficiently large to overcome the disadvantage from a non-instantaneous production process, the welfare under equilibrium *AA* is greater than that under equilibrium *CC*.

Meanwhile, the autarkic values in the respective equilibria are

$$W^{AA} = \frac{U - \Delta_H \varepsilon}{1 - \Delta_H} = \frac{(\eta\alpha + r)U - \eta\alpha\varepsilon}{r} \quad (2.35)$$

$$W^{CC} = \frac{\theta U - \Delta_L \delta\varepsilon}{1 - \Delta_L} = \frac{(\alpha + r)\theta U - \alpha\delta\varepsilon}{r}. \quad (2.36)$$

Again, the comparison between the two values depends crucially on the net utility gain versus the loss in a non-instantaneous production process. Formally, we define  $q \equiv \frac{\theta U - \delta\varepsilon}{U - \varepsilon}$

and calculate two critical values for  $\eta$ ,

$$\eta_Z(\theta, \delta) = q - \frac{q\alpha(1-q)}{\alpha + b - \alpha q} \quad \text{and} \quad \eta_W(\theta, \delta) = q - \frac{r(1-\theta)U}{\alpha(U-\varepsilon)},$$

such that  $Z^{AA} > Z^{CC}$  iff  $\eta > \eta_Z$ , and that  $W^{AA} > W^{CC}$  iff  $\eta > \eta_W$ . Note that when  $q \geq 1$ , type- $L$  technology is always chosen in both monetary and autarky economies, since it provides same net utility but requires less production time.<sup>12</sup> In the case where  $q < 1$ , we can show that  $\eta_Z > \eta_W$ . As a result, the introduction of money can completely restore the first-best technology choice only when  $\mu_1 > 1/2$  and  $\eta > \eta_Z$ .

Nonetheless, the introduction of money does improve the efficiency of technology choices if  $\frac{\beta x^2}{\beta x^2 + r} < \theta < \frac{\beta x(1-\mu)}{\beta x(1-\mu) + r}$ ,  $\mu < \mu_1$  and  $\eta > \eta_Z$  (see Proposition A2 in Appendix and the proof therein). Under  $\frac{\beta x^2}{\beta x^2 + r} < \theta$ , only can the type- $L$  technology be chosen in a pure barter economy, as the utility gain from consuming the high-tech good is not sufficient to overcome the time delay cost. While the equalities,  $\theta < \frac{\beta x(1-\mu)}{\beta x(1-\mu) + r}$  and  $\mu < \mu_1$ , ensure that the type- $H$  technology can be adopted in a monetary economy, the condition  $\eta > \eta_Z$  guarantees that adopting the high type technology results in higher welfare than adopting the low one. In this case, search frictions grant too much disadvantage for producers to adopt the high technology under barter; the introduction of money can fully mitigate such disadvantage to encourage the employment of the advanced technology in equilibrium. Summarizing,

**Proposition 4:** (Money and Technology Choice) *When  $\eta > \eta_Z$ , technology choice in a monetary economy is autarky efficient. By further assuming  $\frac{\beta x^2}{\beta x^2 + r} < \theta < \frac{\beta x(1-\mu)}{\beta x(1-\mu) + r}$  and  $\mu < \mu_1$ , the introduction of money improves the efficiency of technology choice by encouraging the adoption of the high technology that cannot be chosen in the pure barter economy.*

As long as the type- $H$  goods provide more utility and the nonselectivity effect is sufficiently small ( $\mu_1 \geq 0.5$ ), autarky efficiency is a *sufficient* (but not necessary) condition for equilibrium  $AA$  to dominate  $CC$  in social welfare sense. As a consequence, the monetary economy can improve technological development.

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<sup>12</sup>It is also true in the pure barter economy.

**Proposition 5:** (Welfare under Non-instantaneous Production) *While equilibrium AA always Pareto dominates equilibrium BB, it leads to higher welfare than equilibrium CC if  $\eta > \eta_Z$ , and  $\mu_1 \geq 0.5$ . The optimal quantity of money is  $\min\{[2 + \frac{\beta x}{2\eta\alpha}]^{-1}, M(\mu_1)\}$  for equilibrium AA with  $M = \frac{\mu\eta\alpha}{\beta x\mu(1-\mu)+\eta\alpha}$  and  $[2 + \frac{\beta x}{2\alpha}]^{-1}$  for equilibrium CC.*

Notice that the results of social welfare comparison are essentially driven by the values of goods and money traders. Provided that the two technologies provide the same values to producers in autarky, the sellers and buyers in the monetary exchange economy would prefer the high one (pointwise with respect to  $\mu$ ), since

$$\frac{1 - \Delta_H}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} > \frac{1 - \Delta_L}{1 - \rho_L^{CC} \rho_m^{CC} \Delta_L}.$$

However, in terms of Pareto criteria, we must also examine the welfare of producers, whose relative gain from employing the high technology can be written as:

$$\begin{aligned} V_0^{AA} - V_0^{CC} &= \left( \frac{1 - \Delta_H}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} W^{AA} - \frac{1 - \Delta_L}{1 - \rho_L^{CC} \rho_m^{CC} \Delta_L} W^{CC} \right) - (1 - \theta)U \\ &= \left[ \frac{1 - \rho_L^{CC} \rho_m^{CC} \Delta_L}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} \frac{U - \Delta_H \varepsilon}{\theta U - \Delta_L \delta \varepsilon} - 1 \right] \frac{\theta U - \Delta_L \delta \varepsilon}{1 - \rho_L^{CC} \rho_m^{CC} \Delta_L} - (1 - \theta)U. \end{aligned}$$

The term in the square bracket is similar to the value comparison for goods and money traders, but the last term may upset such a comparison if  $\theta$  is sufficiently lower than one. This last term can be viewed as the difference in inventory costs per unit of goods, which is driven by the time-consuming trading period in the monetary economy with search frictions. Thus, even when the high technology provides a higher autarkic value, the producers may still prefer the low technology when frictional exchanges are taken into account.

Another interesting finding is that the producer's gains from employing the high technology relevant for decentralized equilibrium ( $V_0^{AA} - V_0^{CC}$ ) need not be maximized at the welfare-optimizing quantity of money. In particular, we can identify a *time-saving effect* from  $\frac{1}{1 - \rho_L^{CC} \rho_m^{CC} \Delta_L}$ , which is increasing in  $\mu(1 - \mu)$ . In fact, it is the only effect in the case of instantaneous production, since  $\Delta_H = \Delta_L = 1$ . When production takes time, there also exists a *mitigation effect*, which is decreasing in  $\mu(1 - \mu)$  as long as it

takes more time to produce the type- $H$  goods ( $\Delta_H < \Delta_L$ ).<sup>13</sup> Intuitively, a longer waiting period to trade would mitigate the disadvantage of the high technology in production time to a greater extent.

When the expected trading period approaches to its minimum ( $\mu$  tends to 0.5), the mitigation effect may be strong enough to dominate the time-saving effects under some parameter values. Figure 2.2 illustrates a numerical example, in which the sign of producers' gain depends on the quantity of money and the mitigation effect dominates the time saving effect near the optimal quantity of money. The resultant social inefficiency from the presence of a strong mitigation effect leads to a negative gain from employing the high technology and hence an under-investment in that technology. Summarizing,

**Proposition 6:** (Under-investment in the High Technology under Non-instantaneous Production) *With a sufficiently short expected trading period and an endowment of money near its optimal level, producers tend to under-invest in the high technology in decentralized equilibrium.*

## 2.5 Conclusion

An interesting message our model has delivered is that the use of money affects producers' technology choices in favor of the high technology in the instantaneous production model and that such an effect is reinforced if production takes time. Moreover, we identify a social inefficiency caused by search frictions leading to under-investment in the advanced technology in decentralized equilibrium. Furthermore, in the case of mixed strategy equilibrium, the share of high-technology output is increasing in the quantity of money.

The implication of our model could go beyond the technology choice issue. Should we regard the high technology as a production plan of high volume, and the low technology as one with low volume, it becomes a binary output quantity model, where the utilities, manufacturing costs and production times are all increasing in the scale of production. This may shed light on the possibility of multiple equilibria in the multiple consumption units or divisible goods setup. For instance, in a simple case with constant return and cost

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<sup>13</sup>This effect is via the term,  $\frac{1-\rho_H^{CC}\rho_m^{CC}\Delta_L}{1-\rho_H^{AA}\rho_m^{AA}\Delta_H} = \frac{\Delta_L}{\Delta_H} + \frac{1}{1-\rho_H^{AA}\rho_m^{AA}\Delta_H}(1 - \frac{\Delta_L}{\Delta_H})$ .

to scale, the highest possible volume of output is best in the sense of social welfare. The optimal volume of output will be determined by the relevant set of parameters (similar to  $S^A$ ), which depends on the quantity of money in the economy.

Due to the decentralized exchange mechanism, we have multiple equilibria while the one with coexistence of both technologies is unstable and Pareto-dominated. One may wonder whether this finding is robust under an alternative, directed-search framework (with a high- and a low-quality submarket). Our preliminary results suggest that we may have a unique stable equilibrium with coexistence of both technologies under proper conditions. Moreover, when production is instantaneous, an increase in the quantity of money tends to encourage high-quality goods production, but does not affect the thickness of each market.

As one of the central features of the model, perfect observability is assumed throughout. To another extreme, if buyers cannot detect the quality of the commodities trade at all, then  $V_H$  always equals  $V_L$  and producers will always choose the cost-saving technology without investing in the high technology. In the case of partial observability, we expect similar results as in Trejos (1997). In particular, if the high technology has adequate relative efficiency over the low, then the buyers would prefer type- $H$  goods whenever they are able to identify its quality. It is therefore straightforward to conclude that the presence of private information will not eliminate the positive role of money in production efficiency as long as partial observability is preserved.

Figure 2.1: Steady-State Flow Chart

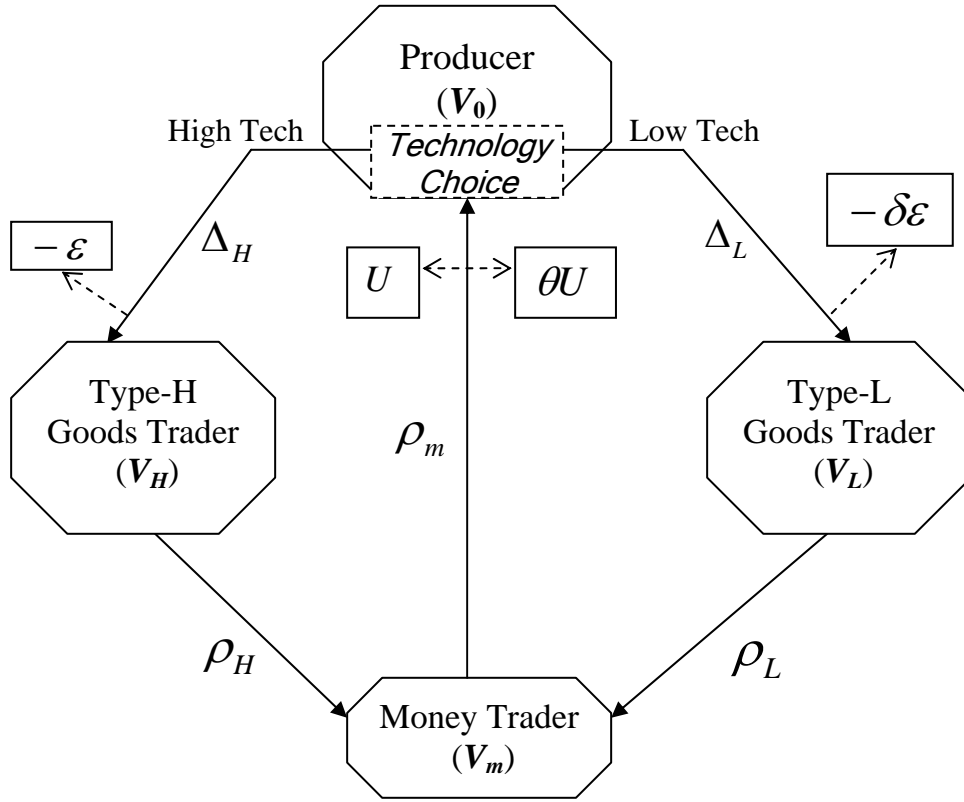
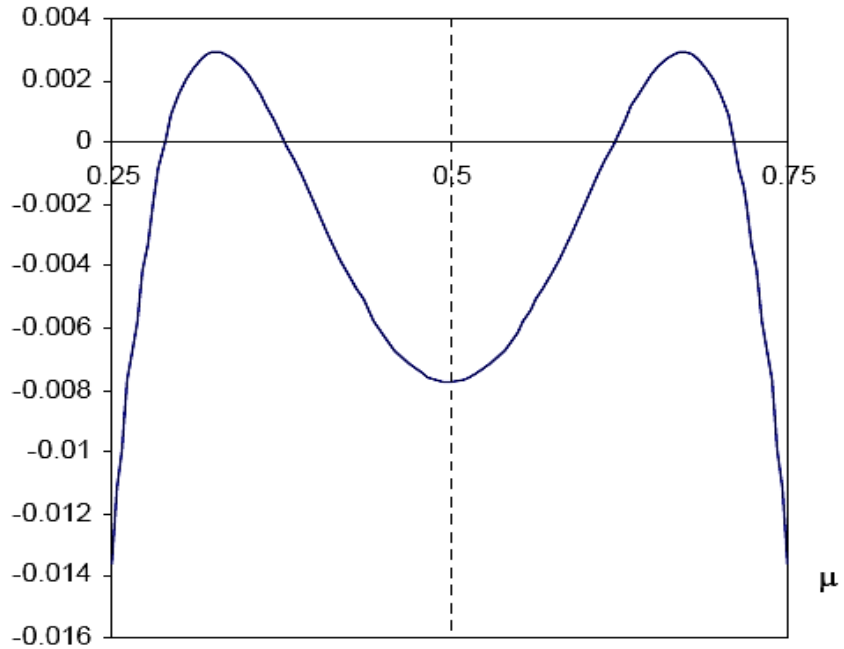


Figure 2.2: Producers' Net Gains from Investing in High Technology



## Appendix

### A. Technology Choice in a Pure Barter Economy

In this appendix, we investigate the technology choice issue in the scenario of a pure barter economy. On the basis of the notation we employ in Section II, we can set up the related values functions:

$$rV_0 = \max\{\alpha(V_L - V_0 - \delta\varepsilon), \eta\alpha(V_H - V_0 - \varepsilon)\}, \quad (\text{A1})$$

$$rV_H = \beta x^2 [h\Pi_{HH} \max_{\pi_{HH}}\{\pi_{HH}(U + V_0 - V_H)\} + (1-h)\Pi_{LH} \max_{\pi_{HL}}\{\pi_{HL}(\theta U + V_0 - V_H)\}], \quad (\text{A2})$$

$$rV_L = \beta x^2 [h\Pi_{HL} \max_{\pi_{LH}}\{\pi_{LH}(U + V_0 - V_L)\} + (1-h)\Pi_{LL} \max_{\pi_{LL}}\{\pi_{LL}(\theta U + V_0 - V_L)\}], \quad (\text{A3})$$

where  $\pi_{i,j}$  indicates the probability for  $i$ -type goods trader to accept  $j$ -type commodities. The equilibrium population equations are

$$\Lambda\eta\alpha N_0 = \beta x^2 [h\Pi_{HH}\pi_{HH}^* + (1-h)\Pi_{LH}\pi_{HL}^*]N_H, \quad (\text{A4})$$

$$(1-\Lambda)\alpha N_0 = \beta x^2 [h\Pi_{HL}\pi_{LH}^* + (1-h)\Pi_{LL}\pi_{LL}^*]N_L. \quad (\text{A5})$$

The active equilibrium condition similar to condition (2.16) yields

$$\Pi_{HH} = \pi_{HH}^* = \Pi_{LH} = \pi_{LH}^* = 1. \quad (\text{A6})$$

As a result, we can rewrite equation (A2) and (A3) as

$$rV_H = \beta x^2 [h(U + V_0 - V_H) + (1-h)\pi_{HL}^*(\theta U + V_0 - V_H)], \quad (\text{A7})$$

$$rV_L = \beta x^2 [h\Pi_{HL}(U + V_0 - V_L) + (1-h)\Pi_{LL}\pi_{LL}^*(\theta U + V_0 - V_L)], \quad (\text{A8})$$

and solve  $\Lambda$  as a function of  $h$

$$\Lambda = \frac{h[h + (1-h)\pi_{HL}^*]}{h[h + (1-h)\pi_{HL}^*] + (1-h)\eta[h\Pi_{HL} + (1-h)\Pi_{LL}\pi_{LL}^*]}. \quad (\text{A9})$$

	Equilibrium A <sup>b</sup>	Equilibrium B <sup>b</sup>	Equilibrium C <sup>b</sup>
$V_0$	$\max\{\Delta_H(V_H - \varepsilon), \Delta_L(V_L - \delta\varepsilon)\}$	$\Delta_H \frac{\theta U - \varepsilon}{1 - \Delta_H}$	$\Delta_L \frac{\rho_b \theta U - \delta\varepsilon}{1 - \rho_b \Delta_L}$
$V_H$	$\frac{h\beta x^2(U - \Delta_H \varepsilon)}{h\beta x^2(1 - \Delta_H) + r}$	$\frac{\theta U - \Delta_H \varepsilon}{1 - \Delta_H}$	$\frac{\rho_b(\theta U - \Delta_L \delta\varepsilon)}{1 - \rho_b \Delta_L}$
$V_L$	$\frac{(1 - h)\beta x^2(\theta U - \Delta_L \delta\varepsilon)}{(1 - h)\beta x^2(1 - \Delta_L) + r}$	$\frac{\alpha + r}{r} \eta(\theta U - \varepsilon) + \delta\varepsilon$	$\frac{\rho_b(\theta U - \Delta_L \delta\varepsilon)}{1 - \rho_b \Delta_L}$
$\pi_{HL}^*$	0	$\pi_b$	1
$\pi_{LL}^*$	1	1	1
$h$	$h_s$ or 1	$h_b$	0

Table 2.6: Solutions for Pure Barter Economy with Non-Instantaneous Production

### Instantaneous Production

In the instantaneous production case,  $V_0 = \max\{(V_L - \delta\varepsilon), (V_H - \varepsilon)\}$ . Observe that  $\theta U + V_0 - V_H \geq \theta U - \varepsilon > 0$  under Assumption 1. Therefore  $\Pi_{HL} = \pi_{HL}^* = 1$ . Similarly  $\theta U + V_0 - V_L \geq \theta U - \delta\varepsilon > 0$ , and  $\Pi_{LL} = \pi_{LL}^* = 1$ . From (A7) and (A8), we can find that  $V_L = V_H$ , which means only the low technology would be chosen, since  $V_L - \delta\varepsilon > V_H - \varepsilon$ . The results can be summarized as follows:

**Proposition A1** (Pure Barter with Instantaneous Production) *Under instantaneous production with pure barter exchange and Assumption 1, only the type-L technology will be chosen even when it provides less net utilities than the type-H technology.*

Note that, as described in Proposition 2, the social planner will choose the technologies providing more utilities in the monetary economy under Assumption 1-3 and the condition  $M_1 > 1/2$ , it is obvious that the introduction of money does improve the efficiency in technology choice.

### Non-Instantaneous Production

When we have non-instantaneous production, it is a bit more complicated. If  $V_H \leq V_L$ , the producers will choose only the low technology, which requires less production cost and shorter production time. From equation (A7) and (A8) as well as  $h = 0$ , we can find that

$$V_H = \frac{\beta x^2 \pi_{HL}^* (\theta U + V_0)}{r + \beta x^2 \pi_{HL}^*} \quad \text{and} \quad V_L = \frac{\beta x^2 \Pi_{LL} \pi_{LL}^* (\theta U + V_0)}{r + \beta x^2 \Pi_{LL} \pi_{LL}^*}$$

Hence  $\pi_{HL}^* \leq \Pi_{LL} \pi_{LL}^*$ . Meanwhile, we must have  $\theta U + V_0 - V_H \geq \theta U + V_0 - V_L$ , which



implies that  $\pi_{HL}^* \geq \pi_{LL}^* \geq \Pi_{LL}\pi_{LL}^*$ . Since  $\pi_{HL}^* = \pi_{LL}^* = 0$  leads to  $V_L = 0$  and  $V_0 < 0$ , the only possible case is  $\pi_{HL}^* = \pi_{LL}^* = 1$ , where  $V_H = V_L$ , and the producers only choose the low technology ( $h = 0$ ).

If  $V_H > V_L$ , we have  $\theta U + V_0 - V_H < \theta U + V_0 - V_L$ , and thus  $\pi_{HL}^* \leq \pi_{LL}^*$ . Note that we cannot have both mixed strategies at the same time. Therefore, we have only four cases to discuss: (1)  $\pi_{HL}^* = \pi_{LL}^* = 1$ ; (2)  $0 < \pi_{HL}^* < \pi_{LL}^* = 1$ ; (3)  $0 = \pi_{HL}^* \leq \pi_{LL}^* < 1$ ; and (4)  $0 = \pi_{HL}^* < \pi_{LL}^* = 1$ .

**Case 1:**  $\pi_{HL}^* = \pi_{LL}^* = 1$ . It implies  $V_H = V_L$ , and the producers only choose the low technology ( $h = 0$ ). The solutions are provided as equilibrium  $C^b$  in Table 2.6 with

$$\rho_b = \frac{\beta x^2}{\beta x^2 + r}. \quad (\text{A10})$$

In this case, we need  $\theta U + V_0 - V_L > 0$ , which always holds under Assumption 1. Meanwhile,  $V_0 > 0$  requires

$$\frac{\delta \varepsilon}{\theta U} < \frac{\beta x^2}{\beta x^2 + r}.$$

**Case 2:**  $0 < \pi_{HL}^* < \pi_{LL}^* = 1$ . The immediate implication is

$$\theta U + V_0 - V_H = 0. \quad (\text{A11})$$

Based on equation (A11), we can rewrite the value functions as

$$V_H = \frac{\beta x^2 h (1 - \theta) U}{r}, \quad (\text{A12})$$

$$V_0 = \frac{\beta x^2 h (1 - \theta) U}{r} - \theta U, \quad (\text{A13})$$

$$V_L = \frac{\beta x^2 [h \Pi_{HL} + (1 - h)] + r \Pi_{HL}}{\beta x^2 [h \Pi_{HL} + (1 - h)] + r} V_H. \quad (\text{A14})$$

Observe from (A13) that  $V_0 > 0$  implies  $h > 0$  and consequently  $rV_0 = \eta\alpha(V_H - V_0 - \varepsilon) = \eta\alpha(\theta U - \varepsilon)$  in the case of positive production time. We can combine it with equation

(A13) to obtain the proportion of type- $H$  goods

$$h_b = \frac{(\eta\alpha + r)\theta U - \eta\alpha\varepsilon}{\beta x^2(1 - \theta)U}. \quad (\text{A15})$$

If  $h = h_b < 1$ , we can substitute (A15) into the expressions of  $V_H$  and  $V_0$

$$V_H = \frac{(\eta\alpha + r)\theta U - \eta\alpha\varepsilon}{r} = \frac{\theta U - \Delta_H\varepsilon}{1 - \Delta_H} \quad (\text{A16})$$

and

$$V_0 = \frac{\eta\alpha(\theta U - \varepsilon)}{r} = \Delta_H \frac{\theta U - \varepsilon}{1 - \Delta_H}. \quad (\text{A17})$$

In order to make the producers indifferent between the two technologies, we need

$$V_L - V_0 - \delta\varepsilon = \eta(V_H - V_0 - \varepsilon). \quad (\text{A18})$$

With the help of equations (A11), (A14) (A16), and (A17) we can convert equation (A18) into

$$\frac{1 - \Pi_{HL}}{\beta x^2[h_b\Pi_{HL} + (1 - h_b)] + r} = \frac{(1 - \eta)\theta U - (\delta - \eta)\varepsilon}{(\eta\alpha + r)\theta U - \eta\alpha\varepsilon}.$$

and compute the solution for the cross-type acceptability, denoted as  $\pi_b$ . Note that  $\pi_b < 1$  as long as  $\theta U > \delta\varepsilon$ . Actually this equilibrium is unstable if we disturb the acceptability  $\Pi_{HL}$  slightly away from its equilibrium level. Note that  $\Pi_{HL} \geq 0$  only when

$$\frac{(1 - \theta)U}{(\beta x^2 + r)(1 - \theta)U - (\eta\alpha + r)\theta U + \eta\alpha\varepsilon} \geq \frac{(1 - \eta)\theta U - (\delta - \eta)\varepsilon}{(\eta\alpha + r)\theta U - \eta\alpha\varepsilon} \quad (\text{A19})$$

$$\frac{(1 - \theta)U}{(1 - \eta)\theta U - (\delta - \eta)\varepsilon} \geq \frac{(\beta x^2 + r)(1 - \theta)U}{(\eta\alpha + r)\theta U - \eta\alpha\varepsilon} - 1$$

The other subcase with  $h_b = 1$  requires some particular cost-utility ratio to satisfy equation (A15). Moreover, we need  $\Pi_{HL} < \pi_b$  to discourage the producers from choosing the low technology. As a consequence, this equilibrium does not hold generically.

**Case 3:**  $0 = \pi_{HL}^* \leq \pi_{LL}^* < 1$ . Now we have  $V_L = 0$  and  $V_H = \frac{\beta x^2(U - \Delta_H\varepsilon)}{\beta x^2(1 - \Delta_H) + r}$ . Hence the producers will only choose the high technology. Note that we need  $\theta U + V_0 < V_H$ , which

requires  $\theta U < \rho_b U - (1 - \rho_b)V_0$ , or

$$\Delta_H < 1 - \frac{r(\theta U - \varepsilon)}{\beta x^2(1 - \theta)U - r\varepsilon}, \text{ or } \eta\alpha < \frac{\beta x^2(1 - \theta)U - r\theta U}{\theta U - \varepsilon}. \quad (\text{A20})$$

Similar to the discussion following Lemma 1, one must guarantee that it is not too costly to wait for the next trade, instead of producing right now. Observe that, given Assumption 1 and  $\theta < \rho_b$ , we have  $\rho_b U > \varepsilon$ , which implies  $V_0 > 0$ .

**Case 4:**  $0 = \pi_{HL}^* < \pi_{LL}^* = 1$ . It demands  $V_L < \theta U + V_0 < V_H$ . While the cross-type acceptability is zero, we have separating equilibrium with

$$V_H = \frac{h\beta x^2(U - \Delta_H \varepsilon)}{h\beta x^2(1 - \Delta_H) + r},$$

$$V_L = \frac{(1 - h)\beta x^2(\theta U - \Delta_L \delta \varepsilon)}{(1 - h)\beta x^2(1 - \Delta_L) + r},$$

and

$$V_0 = \max\{\Delta_H(V_H - \varepsilon), \Delta_L(V_L - \delta \varepsilon)\}$$

The condition  $V_L < \theta U + V_0 < V_H$  requires  $\theta U + \Delta_H(V_H - \varepsilon) < V_H$ , or

$$h > h_0 \equiv \frac{r(\theta U - \Delta_H \varepsilon)}{\beta x^2(1 - \Delta_H)(1 - \theta)U}.$$

Note that  $h_0 < 1$  iff the conditions given in Case 3 are satisfied.

Moreover, we need  $V_0 > 0$ , which implies

$$h > h_1 \equiv \frac{\varepsilon r}{\beta x^2(U - \varepsilon)}$$

when the type- $H$  technology is chosen, or

$$h < h_2 \equiv 1 - \frac{\delta \varepsilon r}{\beta x^2(\theta U - \delta \varepsilon)}$$

when the producers employ the low technology. So one of the necessary condition for the

coexistence of both technologies is

$$h_2 \geq \max\{h_0, h_1\}. \quad (\text{A21})$$

Note that  $h_2 \geq h_1$  implies

$$\frac{\delta\varepsilon}{\theta U} \leq \frac{\beta x^2 U - (\beta x^2 + r)\varepsilon}{(\beta x^2 + r)U - (\beta x^2 + 2r)\varepsilon},$$

where the right-hand side is less than  $\frac{\beta x^2}{\beta x^2 + r}$ . As a result, being a subset of the existence region for high-technology only equilibrium, the existence region for coexistence in Case 4 is also a subset of the existence region in Case 1.

Since an increase in  $h$  leads to bigger  $V_H$  and smaller  $V_L$ , the function  $f(h) = \Delta_H(V_H - \varepsilon) - \Delta_L(V_L - \delta\varepsilon)$  is strictly increasing in  $h$ . Moreover, it is easy to find that, under the necessary condition (A21),  $f(h_2) > 0$  and  $f(h_1) < 0$ . Consequently, there exists a unique  $h_s \in (h_1, h_2)$ , such that  $f(h_s) = 0$ . This solution is only valid when

$$h_s \geq h_0. \quad (\text{A22})$$

As a result, there are two possible equilibria conditional on the parameters. The separating equilibrium exists only when conditions (A20), (A21) and (A22) are all satisfied, while the producers would choose high technology when we have both (A20) and  $\varepsilon r \leq \beta x^2(U - \varepsilon)$ , or  $\frac{\varepsilon}{U} < \frac{\beta x^2}{\beta x^2 + r}$ . Obviously, the existence region of the high-technology only equilibrium is a subset of that of the low-technology only equilibrium when  $0 < \delta \leq \theta < 1$ , which resembles Lemma 2.

**Proposition A2** (Pure Barter with Non-instantaneous Production) *Under instantaneous production with pure barter exchange and Assumption 1, there exist multiple equilibria where the admissible sets of equilibria vary with different primitives of the economy. Equilibrium  $C^b$  (low technology only) exists as long as  $\frac{\delta\varepsilon}{\theta U} < \frac{\beta x^2}{\beta x^2 + r}$ . The mixed-strategy equilibrium and the separating equilibrium are unstable. Parameters satisfying  $\theta < \frac{\beta x^2}{\beta x^2 + r}$  and inequality (A20) feature the adoption of the high technology.*

In case 1, the number of producers and goods holder satisfies  $\alpha N_0 = \beta x^2 N_L$ . Hence

$N_0 = \frac{\beta x^2}{\alpha + \beta x^2}$ , and  $N_L = \frac{\alpha}{\alpha + \beta x^2}$ . As a result, social welfare becomes

$$W_b^C = \alpha \frac{\beta x^2 (\theta U - \delta \varepsilon)}{r(\alpha + \beta x^2)}.$$

Similarly, social welfare in the case of high-technology only equilibrium is

$$W_b^A = \eta \alpha \frac{\beta x^2 (U - \varepsilon)}{r(\eta \alpha + \beta x^2)}.$$

When both equilibria coexist, we must have

$$\eta > \eta_A \equiv \frac{q \beta x^2}{\alpha + \beta x^2 - \alpha q},$$

where  $q = \frac{\theta U - \delta \varepsilon}{U - \varepsilon}$ , as previously defined.

## B. Proofs

In this appendix, we provide detailed mathematical derivations of some fundamental relationships and propositions presented in the main text.

*Proof of Lemma 1:*

In equilibrium  $A$ , we need  $\pi_L = 0$ , and hence  $\theta U + V_0 - V_m < 0$ . Using the solutions provided in Table 2, we can obtain

$$\theta U + \frac{\rho_H^A \rho_m^A U - \varepsilon}{1 - \rho_H^A \rho_m^A} - \frac{\rho_m^A (U - \varepsilon)}{1 - \rho_H^A \rho_m^A} < 0$$

or

$$\theta U - \varepsilon + \frac{\rho_H^A \rho_m^A (U - \varepsilon)}{1 - \rho_H^A \rho_m^A} - \frac{\rho_m^A (U - \varepsilon)}{1 - \rho_H^A \rho_m^A} < 0.$$

Therefore,

$$\frac{\theta U - \varepsilon}{U - \varepsilon} < \frac{\rho_m^A (1 - \rho_H^A)}{1 - \rho_H^A \rho_m^A}$$

Employing the definition of (2.8) and (2.17), we can multiply  $(\beta \mu x + r)[\beta(1 - \mu)x + r]$  to both the numerator and the denominator. Now we have

$$\frac{\theta U - \varepsilon}{U - \varepsilon} < \frac{\beta(1 - \mu)x}{\beta x + r}$$

or

$$M < M_1 \equiv 1 - \frac{(\beta x + r)(\theta U - \varepsilon)}{\beta x(U - \varepsilon)} \quad (\text{B1})$$

where we use the equilibrium result that  $\mu = M$ .

In addition, we also need the producer's value to be positive, *i.e.*

$$\frac{\rho_H^A \rho_m^A U - \varepsilon}{1 - \rho_H^A \rho_m^A} > 0.$$

Hence

$$\frac{\varepsilon}{U} > \rho_H^A \rho_m^A = \frac{\beta^2 x^2 \mu(1 - \mu)}{\beta^2 x^2 \mu(1 - \mu) + (\beta x + r)r}$$

or

$$\mu(1 - \mu) > Q \equiv \frac{(\beta x + r)r\varepsilon}{\beta^2 x^2 (U - \varepsilon)}. \quad (\text{B2})$$

Observe that the quadratic equation given by the equality in (B2) has two real roots within the interval  $(0, 1)$ , if Assumption 2 holds. To differentiate the two roots, we define the smaller root to be  $M_2$ . As a result, condition (B2) can be written as  $M_2 < M < 1 - M_2$  in equilibrium.

In conclusion, the existence region for equilibrium  $A$  is given by  $M < M_1$  and  $M_2 < M < 1 - M_2$ .

*Proof of Lemma 2:*

The derivation of the existence region is analogous to that of condition (B2). We only have to replace  $U$  and  $\varepsilon$  with  $\theta U$  and  $\delta\varepsilon$  respectively. In addition, if  $0 < \delta \leq \theta < 1$  and Assumption 1 holds,

$$\frac{(\beta x + r)r\varepsilon}{\beta^2 x^2 (U - \varepsilon)} = \frac{(\beta x + r)r\delta\varepsilon}{\beta^2 x^2 (\delta U - \delta\varepsilon)} \geq \frac{(\beta x + r)r\delta\varepsilon}{\beta^2 x^2 (\theta U - \delta\varepsilon)}.$$

As a result,  $S^A \subseteq S^C$ .

*Derivation of  $h^B$  and  $\pi^B$ :*

Since  $\theta U + V_0^B - V_m^B = 0$ , we can rewrite the money holder's value (2.6) as

$$rV_m = \beta(1 - \mu)xh(1 - \theta)U.$$

Based on the solution listed in Table 2, we have

$$h^B = \frac{r}{\beta(1-\mu)x(1-\theta)U} \frac{\theta U - \varepsilon}{1 - \rho_H^B} = \frac{(\beta\mu x + r)(\theta U - \varepsilon)}{\beta(1-\mu)x(1-\theta)U}$$

While the producers are indifference between the two technologies, the two solutions of  $V_0^B$  listed in Table 2 should be the same, *i.e.*

$$\frac{\rho_H^B \theta U - \varepsilon}{1 - \rho_H^B} = \frac{\rho_L^B \theta U - \delta \varepsilon}{1 - \rho_L^B} = \frac{\rho_L^B (\theta U - \delta \varepsilon)}{1 - \rho_L^B} - \delta \varepsilon.$$

Note that

$$\frac{\rho_L^B}{1 - \rho_L^B} = \frac{\beta\mu x \Pi_L}{r} = \frac{\rho_H^B}{1 - \rho_H^B} \Pi_L.$$

Therefore

$$\begin{aligned} \frac{\rho_H^B \theta U - \varepsilon}{1 - \rho_H^B} &= \frac{\rho_H^B (\theta U - \delta \varepsilon)}{1 - \rho_H^B} \Pi_L - \delta \varepsilon \\ \pi^B = \Pi_L &= \frac{\rho_H^B \theta U - \varepsilon + (1 - \rho_H^B) \delta \varepsilon}{\rho_H^B (\theta U - \delta \varepsilon)} = 1 - \frac{(1 - \delta) \varepsilon}{\rho_H^B (U - \delta \varepsilon)} \end{aligned}$$

*Proof of Lemma 3:*

The conditions for existence come from the requirement of  $V_0 > 0$ , and  $h^B, \pi^B \in (0, 1)$ , where  $h^B$  and  $\pi^B$  are given by equation (2.22) and (2.21), respectively. Assumption 1 implies that  $h^B > 0$ , while the condition  $h^B < 1$  is equivalent to  $\mu = M < M_1$ . The latter comes from the fact that

$$\begin{aligned} (\beta x M_1 + r)(\theta U - \varepsilon) &= \left[ \beta x - \frac{(\beta x + r)(\theta U - \varepsilon)}{U - \varepsilon} + r \right] (\theta U - \varepsilon) \\ &= (\beta x + r) \frac{(1 - \theta)U}{U - \varepsilon} (\theta U - \varepsilon) \\ &= \beta x (1 - M_1) (1 - \theta) U \end{aligned}$$

and that  $h^B$  is increasing in  $\mu$ .

Meanwhile,  $V_0 > 0$  iff

$$\rho_H^B = \frac{\beta\mu x}{\beta\mu x + r} > \frac{\varepsilon}{\theta U}$$

or

$$\mu > M_4 \equiv \frac{r\varepsilon}{\beta x(\theta U - \varepsilon)}. \quad (\text{B3})$$

Observe that condition (B3), along with Assumption 1, implies that

$$\pi^B > 1 - \frac{(1 - \delta)\theta U}{\theta U - \delta\varepsilon} = \frac{\delta(\theta U - \varepsilon)}{\theta U - \delta\varepsilon} > 0,$$

while Assumption 1 also implies that  $\pi^B < 1$ .

Now consider the relationship between  $S^B$  and  $S^A$ . We know that  $S^B$  is non-empty, iff  $M_4 < M_1$ . Observe that, with  $Q \equiv \frac{(\beta x + r)r\varepsilon}{\beta^2 x^2 (U - \varepsilon)}$ , we have

$$M_4(1 - M_1) = \frac{r\varepsilon}{\beta x(\theta U - \varepsilon)} \frac{(\beta x + r)(\theta U - \varepsilon)}{\beta x(U - \varepsilon)} = Q. \quad (\text{B4})$$

Hence  $M_1(1 - M_1) > M_4(1 - M_1) = Q$ , and  $M_4(1 - M_4) > M_4(1 - M_1) = Q$ . By Lemma 1,  $M_1 \in S^A$ , and  $M_4 \in S^A$ . Consequently,  $S^B = (M_4, M_1) \subseteq S^A$ .

*Proof of Proposition 1:*

Since the stability is proved in the body text, only remaining work is to show that all the existence regions are non-empty under Assumption 1-3. Given Assumption 2, we know that  $\frac{1}{2} \in (M_2, 1 - M_2)$ , and  $\frac{1}{2} \in S^C$ . Now we need to establish  $M_2 < M_1$ . One sufficient condition is that  $Q < M_1(1 - M_1)$ , which boils down to

$$(U - \varepsilon)r\varepsilon < (\theta U - \varepsilon)[\beta x(1 - \theta)U - r(\theta U - \varepsilon)],$$

or

$$\frac{1}{\theta U - \varepsilon} + \frac{\theta}{1 - \theta} < \frac{\beta x}{r}.$$

Note that  $Q < M_1(1 - M_1)$  and equation (B4) imply  $M_4 < M_1$ . As a result, Assumption 1-3 guarantee that  $S^B \neq \emptyset$ .

*Derivation of the social welfare in the instantaneous production case:*



In equilibrium  $A$ , the social welfare

$$\begin{aligned}
Z^A &= MV_m^A + (1 - M)V_H^A = M \frac{\rho_m^A(U - \varepsilon)}{1 - \rho_H^A \rho_m^A} + (1 - M) \frac{\rho_H^A \rho_m^A(U - \varepsilon)}{1 - \rho_H^A \rho_m^A} \\
&= \frac{U - \varepsilon}{1 - \rho_H^A \rho_m^A} \rho_m^A [\rho_H^A + (1 - \rho_H^A)M] = \frac{U - \varepsilon}{(\beta x + r)r} \beta(1 - \mu)x(\beta \mu x + rM) \\
&= \frac{\beta x M(1 - M)(U - \varepsilon)}{r},
\end{aligned}$$

where the last equality employs the equilibrium result that  $\mu = M$ . Analogously, we can derive

$$Z^B = \frac{\beta x M(1 - M)(\theta U - \delta \varepsilon)}{r}.$$

*Proof of Proposition 2:*

For each  $M \in S^B$ ,  $M < M_1$  and  $\rho_H^A = \rho_H^B$ . We have

$$\frac{V_m^A}{V_m^B} = \frac{\rho_m^A(1 - \rho_H^B)}{1 - \rho_H^A \rho_m^A} \frac{U - \varepsilon}{\theta U - \varepsilon} = \frac{\beta(1 - \mu)xr}{(\beta x + r)r} \frac{U - \varepsilon}{\theta U - \varepsilon} > 1$$

and hence  $V_H^A = \rho_H^A V_m^A > \rho_H^B V_m^B = V_H^B$ . While the producers are indifferent between the two technologies,  $V_H^B - \varepsilon = V_L^B - \delta \varepsilon$ . Consequently  $V_H^B > V_L^B$ . So the goods trader's value in equilibrium  $A$  is always higher than that in equilibrium  $B$ . To the producers, we also have  $V_0^A = V_H^A - \varepsilon > V_H^B - \varepsilon = V_0^B$ . With the knowledge that  $S^B \subseteq S^A$ , we can conclude that equilibrium  $A$  Pareto dominates equilibrium  $B$  either for same  $M$  or at the optimal quantity of money. The other parts are straightforward.

*Derivation of  $h^{BB}$  and  $\pi^{BB}$ :*

Since  $\theta U + V_0^B - V_m^B = 0$ , we can rewrite the money holder's value (2.6) as

$$rV_m = \beta(1 - \mu)xh(1 - \theta)U.$$

Based on the solution listed in Table 4, we have

$$h^{BB} = \frac{r}{\beta(1 - \mu)x(1 - \theta)U} \frac{\theta U - \Delta_H \varepsilon}{1 - \rho_H^{BB} \Delta_H}$$

While the producers are indifference between the two technologies, two solutions for

$V_0^{BB}$  listed in Table 4 should be the same. Since  $\theta U + V_0^{BB} - V_m^{BB} = 0$ , we can also equate two solutions for money holder's value

$$\frac{\theta U - \Delta_H \varepsilon}{1 - \rho_H^{BB} \Delta_H} = \frac{\theta U - \Delta_L \delta \varepsilon}{1 - \rho_L^{BB} \Delta_L}$$

Therefore

$$\rho_L^{BB} \Delta_L = \rho_H^{BB} \Delta_H + \frac{\Delta_H \varepsilon - \Delta_L \delta \varepsilon}{\theta U - \Delta_H \varepsilon} (1 - \rho_H^{BB} \Delta_H)$$

$$\frac{\rho_H^{BB}}{1 - \rho_H^{BB}} \Pi_L = \frac{\rho_L^{BB} \Delta_L}{\Delta_L - \rho_L^{BB} \Delta_L} = \frac{(\theta U - \Delta_H \varepsilon) \rho_H^{BB} \Delta_H - (\Delta_H \varepsilon - \Delta_L \delta \varepsilon) (1 - \rho_H^{BB} \Delta_H)}{(\theta U - \Delta_H \varepsilon) (\Delta_L - \rho_H^{BB} \Delta_H) + (\Delta_H \varepsilon - \Delta_L \delta \varepsilon) (1 - \rho_H^{BB} \Delta_H)}$$

$$\begin{aligned} \pi^{BB} &= \Pi_L = \frac{r}{\beta \mu x} \frac{(\theta U - \Delta_H \varepsilon) \rho_H^{BB} \Delta_H - (1 - \rho_H^{BB} \Delta_H) (\Delta_H - \Delta_L \delta) \varepsilon}{(\theta U - \Delta_H \varepsilon) (\Delta_L - \rho_H^{BB} \Delta_H) + (1 - \rho_H^{BB} \Delta_H) (\Delta_H - \Delta_L \delta) \varepsilon} \\ &= \frac{r}{\beta \mu x} \frac{\rho_H^{BB} \Delta_H \theta U - (\Delta_H - \Delta_L \delta + \rho_H^{BB} \Delta_H \Delta_L \delta) \varepsilon}{(\Delta_L - \rho_H^{BB} \Delta_H) \theta U + (\Delta_H - \Delta_L \delta + \rho_H^{BB} \Delta_H \Delta_L \delta - \Delta_H \Delta_L) \varepsilon} \end{aligned}$$

After substituting the expressions of the effective discount factors, we can obtain the result given in the main text. Note that when  $\Delta_H = \Delta_L = 1$ ,

$$\begin{aligned} \pi^{BB} &= \frac{r}{\beta \mu x} \frac{\rho_H^{BB} \theta U - (1 - \delta + \rho_H^{BB} \delta) \varepsilon}{(1 - \rho_H^{BB}) \theta U - (1 - \rho_H^{BB}) \delta \varepsilon} \\ &= \frac{r}{\beta \mu x} \frac{(\theta U - \delta \varepsilon) \rho_H^{BB} - (1 - \delta) \varepsilon}{(\theta U - \delta \varepsilon) (1 - \rho_H^{BB})} \\ &= \frac{(\theta U - \varepsilon) \rho_H^{BB} - (1 - \delta) \varepsilon}{(\theta U - \delta \varepsilon) \rho_H^{BB}} = \pi^B \end{aligned}$$

*Proof of Proposition 3:*

By comparing the solution for producer's values ( $V_0$ ) in Table 2 and 4, we can find that the condition for  $V_0 > 0$  would not change in the non-instantaneous production case. However, in Equilibrium AA, the condition  $\theta U + V_0 - V_m < 0$  leads to

$$\theta U + \Delta_H \frac{\rho_H^{AA} \rho_m^{AA} U - \varepsilon}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} - \frac{\rho_m^{AA} (U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} < 0$$

or

$$\frac{(1 - \rho_m^{AA})(U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} < (1 - \theta) U$$

Note that the left-hand side is strictly increasing in  $\mu$ , since

$$\begin{aligned}
\frac{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H}{1 - \rho_m^{AA}} &= 1 + \frac{\rho_m^{AA} - \rho_H^{AA} \rho_m^{AA} \Delta_H}{1 - \rho_m^{AA}} = 1 + \frac{\rho_m^{AA}(1 - \rho_H^{AA} \Delta_H)}{1 - \rho_m^{AA}} \\
&= 1 + \frac{\rho_m^{AA}(1 - \Delta_H)}{1 - \rho_m^{AA}} + \frac{\rho_m^{AA}(1 - \rho_H^{AA}) \Delta_H}{1 - \rho_m^{AA}} \\
&= 1 + \frac{\beta(1 - \mu)x}{r}(1 - \Delta_H) + \frac{\beta(1 - \mu)x}{\beta\mu x + r} \Delta_H.
\end{aligned}$$

Denote  $\mu_1 = \mu_1(\Delta_H)$  as the solution for

$$(1 - \theta)U = \frac{(1 - \rho_m^{AA})(U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} = \frac{(\beta\mu x + r)r(U - \Delta_H \varepsilon)}{\beta^2 x^2 \mu(1 - \mu)(1 - \Delta_H) + r\beta x + r^2}. \quad (\text{B5})$$

Hence we need  $\mu < \mu_1$  to guarantee  $\theta U + V_0 - V_m < 0$ . By Assumption 1,  $\theta U > \varepsilon$ . Hence  $\mu_1 < 1$ . When  $\Delta_H = 1$ ,

$$1 - \frac{\theta U - \varepsilon}{U - \varepsilon} = \frac{(1 - \theta)U}{U - \varepsilon} = \frac{\beta\mu x + r}{\beta x + r} = 1 - \frac{\beta(1 - \mu)x}{\beta x + r}.$$

Hence  $\mu_1(1) = M_1$ . Moreover,

$$\begin{aligned}
&\frac{(1 - \rho_m^{AA})(U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} - \frac{(1 - \rho_m^{AA})\varepsilon}{\rho_H^{AA} \rho_m^{AA}} \\
&= (1 - \rho_m^{AA}) \frac{\rho_H^{AA} \rho_m^{AA}(U - \Delta_H \varepsilon) - (1 - \rho_H^{AA} \rho_m^{AA} \Delta_H)\varepsilon}{(1 - \rho_H^{AA} \rho_m^{AA} \Delta_H)\rho_H^{AA} \rho_m^{AA}} \\
&= (1 - \rho_m^{AA}) \frac{\rho_m^{AA} \rho_H^{AA} U - \varepsilon}{(1 - \rho_H^{AA} \rho_m^{AA} \Delta_H)\rho_H^{AA} \rho_m^{AA}} \geq 0
\end{aligned}$$

as long as  $V_0^{AA} > 0$ . It means the right-hand side of (B5) is just a constant plus a term that is increasing in  $\Delta_H$ . Recall that this term is also strictly increasing in  $\mu$ . Therefore the implicit function  $\mu_1(\Delta_H)$  given by (B5) is decreasing in  $\Delta_H$ , and  $\mu_1(\Delta_H) \geq \mu_1(1) = M_1$  in non-instantaneous production case, where  $\Delta_H < 1$ .

As a consequence, Assumptions 1-3 are sufficient for all the existence regions to be nonempty in the case of non-instantaneous production.

*Derivation of the social welfare in the non-instantaneous production case:*

Consider equilibrium AA with  $h = 1$  first. From equation (2.25)-(2.29), along with

the population identity  $N_m + N_H + N_L + N_0 = 1$  and  $N_m = M$  in equilibrium, we can solve

$$N_0 = \frac{\mu - M}{\mu} \text{ and } N_H = \frac{M(1 - \mu)}{\mu}.$$

Based on the equation (2.9), (2.10) and the solutions listed in Table 4, we have

$$\begin{aligned} Z^{AA} &= \frac{\mu - M}{\mu} V_0^{AA} + \frac{M(1 - \mu)}{\mu} V_H^{AA} + M V_m^{AA} \\ &= \frac{\mu - M}{\mu} \Delta_H (\rho_H^{AA} V_m^{AA} - \varepsilon) + \frac{M(1 - \mu)}{\mu} \rho_H^{AA} V_m^{AA} + M V_m^{AA} \\ &= V_m^{AA} \left[ \frac{\mu - M}{\mu} \Delta_H \rho_H^{AA} + \frac{M(1 - \mu)}{\mu} \frac{\beta \mu x}{\beta \mu x + r} + M \right] - \frac{\mu - M}{\mu} \Delta_H \varepsilon \\ &= \frac{\rho_m^{AA} (U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} \left[ \frac{\mu - M}{\mu} \Delta_H \rho_H^{AA} + M \frac{\beta x + r}{\beta \mu x + r} \right] - \frac{\mu - M}{\mu} \Delta_H \varepsilon \\ &= \frac{\mu - M}{\mu} \frac{\rho_m^{AA} (U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} \left[ \Delta_H \rho_H^{AA} + \frac{M \mu}{\mu - M} \frac{\beta x + r}{\beta \mu x + r} \right] - \frac{\mu - M}{\mu} \Delta_H \varepsilon \end{aligned}$$

Recall that, when  $h = 1$ ,  $M = \frac{\mu \eta \alpha}{\beta x \mu (1 - \mu) + \eta \alpha}$ , and hence,

$$\frac{\mu - M}{\mu} = \frac{\beta x \mu (1 - \mu)}{\beta x \mu (1 - \mu) + \eta \alpha} \quad \text{and} \quad \frac{M \mu}{\mu - M} = \frac{\eta \alpha}{\beta x (1 - \mu)}$$

As a consequence,

$$\begin{aligned} \frac{\mu}{\mu - M} Z^{AA} &= \frac{\rho_m^{AA} (U - \Delta_H \varepsilon)}{1 - \rho_H^{AA} \rho_m^{AA} \Delta_H} \left[ \Delta_H \rho_H^{AA} + \frac{\eta \alpha}{\beta x (1 - \mu)} \frac{\beta x + r}{\beta \mu x + r} \right] - \Delta_H \varepsilon \\ &= \frac{(U - \Delta_H \varepsilon) \eta \alpha [\beta \mu x \beta x (1 - \mu) + (\eta \alpha + r)(\beta x + r)]}{\beta^2 x^2 \mu (1 - \mu) r + (\eta \alpha + r)(r \beta x + r^2)} - \Delta_H \varepsilon \\ &= \frac{(U - \Delta_H \varepsilon) \eta \alpha}{r} - \Delta_H \varepsilon \\ &= \frac{\eta \alpha U - \Delta_H \varepsilon (\eta \alpha + r)}{r} \\ &= \frac{\eta \alpha (U - \varepsilon)}{r} \end{aligned}$$

and

$$Z^{AA} = \frac{\mu - M}{\mu} \frac{\eta \alpha (U - \varepsilon)}{r} = \frac{\beta x \mu (1 - \mu)}{\beta x \mu (1 - \mu) + \eta \alpha} \frac{\eta \alpha (U - \varepsilon)}{r}.$$

We can compute  $Z^{BB}$  analogously.

*Derivation of the social welfare in barter economy:*

Recall that in the low-technology only equilibrium, the number of producers and goods holder satisfies  $\alpha N_0 = \beta x^2 N_L$ , which implies  $N_0 = \frac{\beta x^2}{\alpha + \beta x^2}$ , and  $N_L = \frac{\alpha}{\alpha + \beta x^2}$ . Hence the social welfare becomes

$$\begin{aligned}
W_b^C &= \frac{\beta x^2}{\alpha + \beta x^2} \Delta_L \frac{\rho_b \theta U - \delta \varepsilon}{1 - \rho_b \Delta_L} + \frac{\alpha}{\alpha + \beta x^2} \frac{\rho_b (\theta U - \Delta_L \delta \varepsilon)}{1 - \rho_b \Delta_L} \\
&= \frac{\beta x^2 \Delta_L (\rho_b \theta U - \delta \varepsilon) + \alpha \rho_b (\theta U - \Delta_L \delta \varepsilon)}{(\alpha + \beta x^2)(1 - \rho_b \Delta_L)} \\
&= \frac{(\beta x^2 \Delta_L + \alpha) \rho_b \theta U - (\beta x^2 + \alpha \rho_b) \Delta_L \delta \varepsilon}{(\alpha + \beta x^2)(1 - \rho_b \Delta_L)} \\
&= \alpha \frac{(\beta x^2 + \alpha + r) \rho_b \theta U - (\beta x^2 + \alpha \rho_b) \delta \varepsilon}{(\alpha + \beta x^2)(\alpha + r - \alpha \rho_b)} \\
&= \alpha \frac{(\beta x^2 + \alpha \rho_b)(\theta U - \delta \varepsilon)}{(\alpha + \beta x^2)(\alpha + r - \alpha \rho_b)} = \alpha \frac{\beta x^2 (\theta U - \delta \varepsilon)}{r(\alpha + \beta x^2)}
\end{aligned}$$

Analogously, the social welfare in the high-technology only equilibrium

$$W_b^A = \eta \alpha \frac{\beta x^2 (U - \varepsilon)}{r(\eta \alpha + \beta x^2)}.$$

*Proof of  $\eta_Z > \eta_b > \eta_W$  given  $\beta x \mu (1 - \mu) > \beta x^2$ , and  $q < 1$ :*

It is easy to show that  $\eta_Z > \eta_b$  iff  $\beta x \mu (1 - \mu) > \beta x^2$ .

Comparing  $\eta_b$  and  $\eta_W$ ,

$$\begin{aligned}
\eta_b - \eta_W &= \frac{\beta x^2 q}{\beta x^2 + \alpha - \alpha q} - q + \frac{r(1 - \theta)U}{\alpha(U - \varepsilon)} \\
&= -\frac{\alpha(1 - q)q}{\beta x^2 + \alpha - \alpha q} + \frac{r(1 - \theta)U}{\alpha(U - \varepsilon)}
\end{aligned}$$

Given  $q < 1$ , we know  $\eta_b > \eta_W$  iff

$$\begin{aligned}
k(q) &= -\alpha(1 - q)q + x(b + \alpha - \alpha q) \\
&= \alpha q^2 - \alpha q(1 + x) + x(b + \alpha) > 0,
\end{aligned}$$

where we employ the short-hand notation  $x \equiv \frac{r(1 - \theta)U}{\alpha(U - \varepsilon)} > 0$ . Observe that  $k(1) > 0$ ,  $k(0) > 0$ . To have at least a real root between 0 and 1, we need  $0 < \frac{1+x}{2} < 1$ , and a

positive discriminant. However, when  $0 < x < 1$ , the discriminant

$$\begin{aligned} D &= \alpha^2(1+x)^2 - 4\alpha(b+\alpha) \\ &= \alpha[\alpha(1+x)^2 - 4(b+\alpha)] < 0. \end{aligned}$$

As a consequence,  $\eta_b > \eta_W$ , for all  $0 < q < 1$ .

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# CHAPTER III

## NON-FUNDAMENTAL ASSET PRICING UNDER HETEROGENEOUS PRIOR BELIEFS

“The actual price at which any commodity is commonly sold is called its market price. It may either be above, or below, or exactly the same with its natural price.” (Adam Smith, *The Wealth of Nations*, Book I, Chapter VII, paragraph 7)

### 3.1 Introduction

Since the seminal paper of Lucas (1978), present value approach has been widely used by financial analysts to calculate the fundamental values of stocks. Recently, observing the discrepancies between stock prices and their underlying present values, economists provide several revised asset pricing approaches, such as the presence of bubbles, stochastic discounting, or unconventional behavior suggested by psychological evidence.<sup>1</sup> While most of them try to compute the *values* of assets that are bought and held *forever*, the representative agent framework they employed guarantees that the market prices are the same as the valuations.<sup>2</sup>

In an economy with heterogeneous agents, the above mentioned approaches fail to determine market prices directly since they are essentially asset *valuation* methods for each investor. Stemming from the ideas initiated by Harrison and Kreps (1978) and Morris (1996),

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<sup>1</sup> Campbell (2000), Barberis and Thaler (2003) list some of the empirical puzzles and provide nice surveys about the explanations on the basis of stochastic discount factors, arbitrage constraint, nonstandard behavior suggested by psychological evidence. The surveys and discussions about bubbles can be found in Blanchard and Fischer (1989, Chapter 5) and Brunnermeier (2001 Section 2.3).

<sup>2</sup> We use the term "bubble" in a narrow sense, since the existence of bubbles in most bubble literatures, such as Blanchard and Fischer (1985, Chapter 5), Diba and Grossman (1988), Santos and Woodford (1997), requires an explosive Ponzi process and an infinite horizon. Since it has no direct (or explicit) link with market trading opportunities so far, we may consider it as a part of the asset values. Note that it is difficult to justify negative bubbles in these models. In contrast, the deviation from fundamental values in our model can be positive and negative, does not require any explosive process and can exist even in a finite horizon.

this paper is among the first few that propose a dynamic general equilibrium framework to compute the *market prices* under heterogeneous prior beliefs and even endogenize profitable speculative manipulations. We show that investors submit a trading price according to a *non-fundamental asset pricing formula*, instead of the expected fundamental values. Intuitively, since the stocks are sold at the expected market prices instead of investors' expected asset value, there should be an additional term accounting for the reselling opportunity or portfolio adjustment requirement in a market with active traders.<sup>3</sup>

One may ask: “why investors have to actively adjust their portfolios?” We know that rational expectations are usually conditional on available information. When new signals arrive, investors updated their beliefs about some important stock pricing inputs such as the firm’s productivity, the sector’s prospect, Greenspan’s opinion about the economy, etc. The optimal portfolio has to change accordingly. While portfolio adjustment is impossible in a representative agent model, since stock prices will jump to the level with no trading opportunity due to identical valuations of the asset for both buyers and sellers, it is much easier to justify transactions among heterogeneous agents.

Moreover, due to the law of iterated expectations (LIE), the expected capital gain is zero under the representative agent framework.<sup>4</sup> In contrast, in an economy with heterogeneous agents, where marginal agents determine the asset prices, the characteristic of marginal agents is likely to change over time, making LIE inapplicable. As demonstrated in this paper, during the learning period (when the true state has yet been revealed), the anticipation of forthcoming signals now results in a generically non-trivial expected capital gain from active portfolio adjustments. Based on the discussion in Harrison and Kreps

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link with market trading opportunities so far, we may consider it as a part of the asset values. Note that it is difficult to justify negative bubbles in these models. In contrast, the deviation from fundamental values in our model can be positive and *negative*, does *not* require any explosive process and can exist even in a *finite* horizon.

<sup>3</sup>For example, mutual fund managers have to review and adjust their portfolio more than once in every fiscal year, in an effort to catch up or outperform the competitors to attract potential investors. Long-term investors, such as Warren Buffet, can buy the stock and then leave the market. However, stock prices recorded in the stock exchange only reflect the prices of the assets under adjustment.

<sup>4</sup>The expected capital gain can be non-zero in the presence of some additional explosive process (bubbles). However, the growth rate of expected bubble term must be the exactly same as the gross return for risk-free assets, which is hard to be justified in reality. Hence we don't take any explosive processes into account.

(1978), we regard this component as non-fundamental since the trading opportunities are crucial to its presence.

As one of the main results of this paper, investors would *not* buy the stocks at their expected present values (EPVs). Instead, the new asset pricing formula proposed in this paper contains the above-mentioned non-fundamental component. Moreover, the stock prices may persistently deviate from their dividend-based fundamental values for years or even decades, as illustrated in several empirical findings.<sup>5</sup>

Another key feature of the proposed framework in this paper is the endogenization of profitable speculative manipulations. The Bayesian learning structure embedded in our model, along with the assumption of asymmetric information and inference of private signals from previous stock performance, explains the presence of positive short-run serial correlation in the stock market returns,<sup>6</sup> as well as the existence of the “feedback loops” described in Shiller (2000).<sup>7</sup> As one step ahead, we demonstrate that speculators can make use of the “feedback loops” to profit from manipulating stock prices at least in a stock market with price fluctuation limits. Intuitively, speculators can bid up the stock prices to hit the upper bounds, pretending that they receive better signals than what they really have. Subsequent investors would be misinformed and raise the stock prices accordingly. The boundedness of stock prices can lead to severe signal distortions, and thus make this kind of price manipulation profitable.

There are interesting policy implications related to the above finding. Being a measure employed by more than 16 stock markets, including Tokyo Stock Exchange, to *prevent* violent price fluctuations due to speculative trading, daily fluctuation limits turns out to encourage speculations in the long run. Our result provides a new theoretical support for empirical findings, such as Kim and Rhee (1997), about the ineffectiveness of price limits.

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<sup>5</sup>Campbell, Lo and Mackinlay (1997, Figure 7.2) estimate the expected dividend components in CRSP based on a vector autoregressive (VAR) model and demonstrate a sustained negative deviation in the period of 1910-1927 and two positive ones in 1958-1975 and 1985-1994. Similar observation can be obtained in Shiller (2000, Figure 9.1), which shows that the latest sustained positive deviation from the ex post dividend present value is still in the ascendant at least in 2000.

<sup>6</sup>Campbell, Lo and Mackinlay (1997, Chapter 2) find the autocorrelations for the first lag of stock returns to be positive and statistically significant in CRSP data, while Cutler, Poterba and Summers (1991) provide similar evidence in 12 other countries.

<sup>7</sup>The feedback loop means that a rise in stock prices is more likely to be followed by another rise.

Other interesting findings include the non-monotone relationship between the magnitude of the portfolio adjustment component and the quality of signaling noises, and the consequences of signaling distortions due to boundedness of stock prices or higher dimensions of signaling noises. Apart from those, there can be more possible extensions in future studies to make current framework closer to reality.

In terms of methodology, this paper provides the *first* effort to introduce overlapping generations framework into the study on asset pricing with heterogeneous agents, and features the intriguing depiction of price changes during the learning process.<sup>8</sup> Specifically, our proposed model setup can explain not only why the asset prices may deviate from their fundamental values during the learning period, but also how the prices would converge to their fundamental values when the beliefs approaches the truth. As a result, we no longer rely on exogenous booms and busts of bubbles to explain the relationship between asset prices and their fundamental values.

Despite of several possible dimensions of heterogeneity in investors' traits, this paper focuses on heterogeneous prior beliefs just to illustrate how the heterogeneity among investors can affect the asset prices. As a matter of fact, recent empirical evidence provided by Anderson, Ghycels and Juergens (2005) suggests that heterogeneous beliefs among financial analyst matter in asset pricing. While the financial analysts are likely to have similar information set in a competitive market, we regard heterogeneous prior beliefs as a good proxy for heterogeneity among investors. Nonetheless, we believe that our framework is friendly to other heterogeneities, and delegate them to future studies.

### Literature Review

Harrison and Kreps (1978) and Morris (1996) point out analogous rationale of our paper by suggesting that the opportunity to resell the stock to more optimistic investors would lead to a *positive* deviation from the fundamental values, based on the assumption that the group of most optimistic agents have sufficient financial resources to buy all the available stocks. This paper demonstrates that both positive and negative deviations are possible under a more general wealth distribution where each agent has limited financial

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<sup>8</sup>Note that the overlapping generations framework is more closer to reality than it seems. Regarding the living periods of agents as the life cycle of a fixed portfolio, we can mimic real-life economy by revising the assumptions on wealth distribution and information structure.

resources.

As compared with other studies on asset pricing under heterogeneous beliefs, this paper differs from the works by Varian (1985), Abel (1989), Detemple and Murthy (1994) in terms of aggregation method, which we regard as the crux of the matter in heterogeneous agent models. Following the basic microeconomic approach, we compute the total amount demanded/supplied for each price and then clear the market. This method is similar to Miller (1977), where the beliefs of marginal agents determine the stock prices. As a result, our model behaves differently from those employing average beliefs or average prices of all agents.

As a matter of fact, the market microstructure literatures provide another framework to study the impact from heterogeneous agents and lead to similar results as ours.<sup>9</sup> For instance, Du (2003) demonstrates that stock prices may over- or under-react when investors have heterogeneous beliefs. Allen and Gale (1992) illustrate how the speculators make profit by manipulating stock prices, while Aggarwal and Wu (2003) extend the model and study the empirical evidence from the US cases identified by SEC. However, the demand and supply schemes for each type of traders are exogenously given in the market microstructure framework. In contrast, our general equilibrium settings make it possible to derive the demand scheme based on investors' optimization behavior.<sup>10</sup>

Our model setup also benefits greatly from the latest development about the impact of learning on asset pricing. Bulkley and Tonks (1989) suggest that dividend announcements have an additional indirect effect to stock pricing via adjusting the estimation of dividend growth. Timmermann (1996) illustrates this idea by simulating an estimation-based asset pricing model. Pastor and Veronesi (2004) argue that the uncertainty about dividend growth rate, due to its convexity in dividend growth in the asset pricing formula, would lead to a higher expected present value of a firm than that based on expected dividend growth. Although these papers also study the information content in the announcement of earnings or dividends, they still rest on the conventional representative agent framework, thus fail to investigate speculative manipulations due to the ignorance

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<sup>9</sup>A comprehensive review of market microstructure models can be found in O'Hara (1995).

<sup>10</sup>Although supply is inelastic in the current two-period overlapping generations framework, we can easily make it elastic in the extended multi-period settings.

of expected capital gains.

Bayesian learning structure and signal distortions introduced in this paper are inspired by the social learning literatures discussed in Brunnermeier (2001) and Chamley (2003b), most of which assume *exogenous* payoff schemes.<sup>11</sup> In contrast, this paper provides a general equilibrium framework to determine price process endogenously.

## 3.2 The Model

Consider a prototypical two-period overlapping generations model with two assets. One is a risk-free asset with infinite supply and a constant gross return of  $R$  with  $R > 1$ . The investors can also buy common stocks issued by the *one and only* listed company.<sup>12</sup> Shortselling is forbidden on both markets.

At time 0, the firm is listed in the stock market by an initial public offer (IPO) to generation 0 after the circulation of its prospectus about the its pre-IPO performance. The IPO price per share is  $P_0 = 1$ , while the number of outstanding shares,  $S$ , depends on the volume of applications. For simplicity, the shares are assumed to be perfectly divisible. After the issuance, the number of shares is fixed but the price,  $P_t$ , can change over time.

Each generation, indexed by the date of entrance to the stock market, constitutes a continuum of agents with a Lebesgue measure of one. Each agent, assumed to be risk neutral, is initially endowed with one unit of capital. For tractability, borrowing is not allowed in the economy.<sup>13</sup> As we will see later, it is already difficult to find analytic solutions in this simple economy. Nonetheless, there are full of interesting results even

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<sup>11</sup>Chamley (2003b) reviews several frameworks where investors can infer private signals from other agents' behaviors. For instance, in the information cascades modeled by Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992), individual learning would be outvoiced by the observation of the history. Avery and Zemsky (1998) show that unbounded price adjustment could lead efficient learning and thus prevent herd behavior, while the introducing another dimension of signaling noises may deter the learning process. Chamley (2003a) investigates speculative manipulation in foreign exchange markets, while Abreu and Brunnermeier (2003) study the endogenous timing to break a bubble.

<sup>12</sup>We can regard the firm as a portfolio of many stocks. It is not difficult to allow for more than one listed firms, but the loss in tractability outweighs the marginal gain.

<sup>13</sup>It is not difficult to extend the model to allow for borrowing. Note that the collateral requirements would set some boundary for the amount to borrow. In this sense, we expect that the results would not change qualitatively.

within such a stylized setting.

At period  $t > 0$  (from time  $t$  to  $t + 1$ ), the young agents (generation  $t$ ) enter the stock market and bid for the common stocks sold by the old agents (generation  $t - 1$ ). If the bidding price is lower than the market price, an agent would invest all the money in the risk-free asset.<sup>14</sup> At period  $t + 1$ , they sell all the assets and enjoy their retirement on the beach. The stock exchange only generates one price each period after collecting all the orders. For tractability purpose, consumption is assumed to be valued only in the second period of their life,<sup>15</sup> which simplifies our model by imposing forced savings on the young investors and inelastic supply of stocks.

### 3.2.1 States and Signals

The information structure follows the social learning literatures concerning binary states and Gaussian signals.<sup>16</sup> Before the IPO, the nature chooses, once for all, the quality of the listed firm,  $i \in \{H, L\}$ , which is unknown to any agents. The firm's realized profit per share  $s_t$  announced in period  $t$  is randomly drawn from a Gaussian distribution with mean  $\theta_i$ , variance  $\sigma^2$  and a probability density function (pdf) of  $\phi(\cdot; \theta_i, \sigma^2)$ .<sup>17,18</sup> To ensure the results to be nontrivial, we assume that the expected dividends satisfy

$$\theta_L < R - 1 < \theta_H. \tag{3.1}$$

The dividend payout ratio is set as 100 percent, i.e. all the profits generated in that period are paid to the current shareholders as dividend before the market opens. We assume the earnings and dividends at period  $t$  are known only to current shareholders

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<sup>14</sup>Since we have a continuum of agents, those who are indifferent between investing in the two assets are of measure zero. Hence there is no need to take mixed strategies into account.

<sup>15</sup>Actually this would be the optimal decision for agents with a linear utility function and a subjective discounting rate lower than the risk-free rate.

<sup>16</sup>Following the social learning literatures, the model focus on signal extraction, since the signals are exogenous.

<sup>17</sup>In fact,  $s_t$  is the corporate earnings in period  $t - 1$ . Current index indicates that it is essentially a signal in period  $t$ , which simplifies the notations.

<sup>18</sup>The noises can follow any other distributions. Gaussian distribution is chosen due to its simplicity in calculation and easier comparison with other social learning models. Admittedly, this assumption violates limited liability for investments in stock market, but we may consider the negative dividends as seasoned equity offer. Nonetheless, the results of this paper would still hold for other distributions, such as truncated Gaussian distribution.

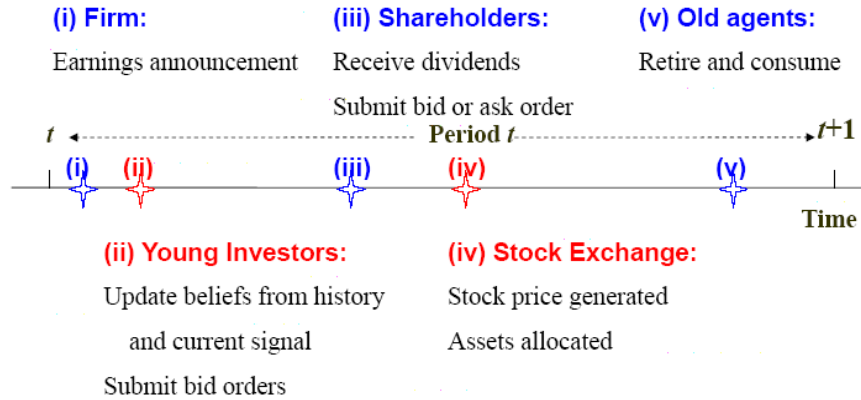


Figure 3-1: Sequence of Events

(generation  $t - 1$ ) and potential stock buyers (generation  $t$ ). While the true quality of the firm is unknown, the realized profit per share also serves as a signal.

### 3.2.2 The Sequence of Events

As we described above, agents in generation  $t$  are born at time  $t$  with endowment of one unit of capital as well as heterogeneous prior beliefs. At period 0, the firm issue the stocks to young agents in generation 0. The sequence of events within period  $t > 0$ , can be summarized as follows:

- (i) The firm announces its profit per share  $s_t$  to the young generation (generation  $t$ ), and pay the same amount as dividends to the shareholders (generation  $t - 1$ );
- (ii) After doing their research on the history of stock price performance and current signal of the company's performance  $s_t$ , the young generation update their beliefs about the company's quality, and then bid for the stocks;
- (iii) After receiving the dividends, the shareholders of the old generation submit market price orders of their holding stocks;
- (iv) The market generates a price  $P_t$  for the stock according to the bidding mechanism described below, where the losers, whose bidding price is lower than the market price, invest in the risk-free assets;
- (v) The old generation (generation  $t - 1$ ) retire and consume.



### 3.2.3 Evolution of Beliefs

Agents are endowed with prior beliefs,  $\mu_{-1}$ , about the probability of the good state when they are born.<sup>19</sup> The prior beliefs are heterogeneous for agents within the same generation, but its distribution, characterized by a cumulative density function (cdf) of  $F_{-1}(\cdot)$ , is same for each generation, and is assumed to be common knowledge. Later, we refer to this distribution as *birth distribution* of prior beliefs.

Similar to all the models of herding behavior, we assume that the agents can observe the history of prices and the number of shares,  $h_t = \{S, P_0, P_1, \dots, P_{t-1}\}$ , but the firm's past earnings are *not* recorded.<sup>20</sup> Hence, the young agents have to make an inference from  $h_t$  to obtain the estimated signals  $\Omega_t = \{\hat{s}_0, \hat{s}_1, \dots, \hat{s}_{t-1}\}$ . We set  $\Omega_0 = \emptyset$  for notational purposes. In contrast, the firm announces its latest performance,  $s_t$ , to the young investors of generation  $t$ . Knowing  $h_t$  and  $s_t$ , the young agents update their beliefs to a distribution with a cdf of  $F(\cdot; \Omega_t, s_t)$ .<sup>21</sup>

On the basis of Bayesian inference in Chamley (2003b, section 2.1), a  $t$ -generation agent with a prior belief of  $\mu_{-1}$  would update their beliefs to  $\mu_t$  such that

$$\frac{\hat{\mu}_\tau}{1 - \hat{\mu}_\tau} = m(\hat{s}_\tau) \frac{\hat{\mu}_{\tau-1}}{1 - \hat{\mu}_{\tau-1}}, \quad \tau = 0, 1, \dots, t-1 \quad (3.2)$$

$$\frac{\mu_t}{1 - \mu_t} = m(s_t) \frac{\hat{\mu}_{t-1}}{1 - \hat{\mu}_{t-1}}, \quad (3.3)$$

where

$$m(s_t) = \frac{\phi(s_t; \theta_H, \sigma^2)}{\phi(s_t; \theta_L, \sigma^2)}.$$

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<sup>19</sup>The endowment of prior beliefs is exogenous in this model. We can regard it as different interpretation about the future of the firm and the industry from the information contained in the prospectus due to heterogeneous knowledge endowment within each generation.

<sup>20</sup>Although this assumption is somewhat unrealistic as that the history of a listing firm's earnings is observable, it greatly simplifies the model and provides an easier comparison with the conventional EPV approach. In fact, without signals distortion, we do have perfect learning, so the observability of earnings does not affect the results about sustained deviations from fundamentals. In the cases with signal distortions, the demonstration of profitability in speculative manipulation can easily apply for other type of signals, such as earnings manipulation by the listed firm. We could have relaxed this assumption but it would complicate the paper by introducing another economy.

<sup>21</sup>In the case of perfect learning, we have  $\Omega_t = \Omega_{t-1} \cup \{s_t\}$ . However, it does not hold when we have signal distortions and the inferred signals no longer equal to the true ones.

Since the signals are normally distributed, we can rewrite the updating multiplier as

$$m_t = m(s_t) = \exp \left[ \frac{(\theta_H - \theta_L)(2s_t - \theta_H - \theta_L)}{2\sigma^2} \right]. \quad (3.4)$$

Observe that the current beliefs would also be heterogeneous within each generation due to the heterogeneity in their prior beliefs. Moreover, although each generation has the same distribution of prior beliefs, the current beliefs could differ a lot among different generations due to different history they can observe.

In order to derive the cdf of the posterior beliefs, we define the cumulative updating multiplier

$$\hat{M}_t = m(s_t) \prod_{\tau=0}^{t-1} m(\hat{s}_\tau).$$

Hence, the agents with a posterior belief of  $\mu_t$  have a prior belief

$$\mu_{-1} = \frac{\mu_t}{\hat{M}_t + (1 - \hat{M}_t)\mu_t},$$

and the cdf of posterior beliefs satisfies

$$F(\mu_t; \Omega_t, s_t) = F_{-1} \left( \frac{\mu_t}{\hat{M}_t + (1 - \hat{M}_t)\mu_t} \right), \quad (3.5)$$

as long as we can infer  $\hat{s}_\tau$  from  $h_{\tau+1}$ , for  $\tau = 1, 2, \dots, t-1$ . In the following subsection, we describe one of the bidding mechanisms enabling us to do so.

### 3.2.4 Bidding Mechanism in the Stock Market

The stock market collects the orders from buyers and sellers in each period, and then generates only one price for each trading date.<sup>22</sup> After comparing the expected payoffs from risk-free asset and the stocks, young investors would like to hand in *limit price order* in accordance with their beliefs. If all agents are truthful, agents with higher beliefs will bid for a higher price. On the other hand, the sellers must submit a *market price order*

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<sup>22</sup>This simplification makes the model more tractable. Actually this assumption, as well as the bidding mechanism we describe here, is in line with the applications in some of over-the-counter (OTC) markets.

because they have to sell all the stocks to enjoy their fruits in retirement.<sup>23</sup>

The market collects all the orders and determines the market price  $P_t$  such that the number of shares from bidding orders above  $P_t$  equals  $S$ . With  $\mu_t^*$  indicating the *cutoff belief* for the marginal agents, the number of winning buyers is  $1 - F(\mu_t^*; \Omega_t, s_t)$ . Recall that each agent has one unit of capital and the total market value of the company is  $SP_t$  at time  $t$ . Hence we obtain the market price as a function of cutoff beliefs. It also depends on all the inferred and observed signals.

$$P_t = \frac{1 - F(\mu_t^*; \Omega_t, s_t)}{S}, \quad (3.6)$$

where

$$S = SP_0 = 1 - F(\mu_0^*; \Omega_0, s_0). \quad (3.7)$$

The winning buyers have to purchase the stock at a price of  $P_t$ , while the other young investors would invest in the risk-free asset. Theoretically, we can solve  $\mu_t^*$  as a function of  $P_t$  from (3.6)

$$\mu_t^* = \varphi(P_t; \Omega_t, s_t) = F^{-1}(1 - SP_t; \Omega_t, s_t) \quad (3.8)$$

### 3.3 Bayesian Learning Equilibrium Pricing Process

#### 3.3.1 General Results

For the marginal agent at time  $t$ , the expected gross rate of return on equity  $y_t$  equates the yield from risk-free assets,

$$y_t = \frac{E_t[s_{t+1} + P_{t+1} | \mu_t^*]}{P_t} = \frac{\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L + E_t[P_{t+1} | \mu_t^*, \Omega_{t+1}]}{P_t} = R. \quad (3.9)$$

Note that the number of shares per agent,  $S/[1 - F(\mu_t^*; \Omega_t, s_t)]$ , cancels out from the numerator and denominator. The expectations are taken on both the states of nature  $\theta$  and the subsequent signals  $s_t$ . In the light of equation (3.6) and the approach to generate

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<sup>23</sup>Since this market is essentially a limit-order market, there is no bid-ask spread due to the absence of market makers.

$F(\cdot; \Omega_t, s_t)$ , we can multiply both sides by  $SP_t$ , and rewrite equation (3.9) as

$$R[1 - F(\mu_t^*; \Omega_t, s_t)] = S[\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L] + 1 - E[F(\mu_{t+1}^*; \Omega_{t+1}, s_{t+1}) | \mu_t^*, \Omega_{t+1}] \quad (3.10)$$

The left-hand side is the opportunity cost of the capitals invested in the stock market in period  $t$ , while the right-hand side is the expected return from the dividends and the expected market value of the stocks in period  $t + 1$ .

Equation (3.8) and (3.10) characterize the *Bayesian learning equilibrium price* (BLEP).<sup>24</sup> Meanwhile, based on the present value approach, we can define

$$V_i = \frac{\theta_i}{R - 1}, \quad i = H, L,$$

and rewrite equation (3.9) as

$$P_t = [\mu_t^* V_H + (1 - \mu_t^*) V_L] + \frac{E_t[P_{t+1} | \mu_t^*, \Omega_{t+1}] - P_t}{R - 1}. \quad (3.11)$$

Comparing BLEP with the expected present value

$$V_t^E = \mu_t^* V_H + (1 - \mu_t^*) V_L, \quad (3.12)$$

we find that only when  $P_t = E_t[P_{t+1} | \mu_t^*, \Omega_t]$ , i.e. when the expected capital gain is zero, the price is exactly the expectation of the probable present values for the marginal stock investors. However we can show below that the expected capital gain is actually non-zero almost all the time.<sup>25</sup>

Note that, in a representative agent model,  $E_t[\mu_{t+1} | \mu_t, \Omega_{t+1}] = \mu_t$  due to the law of iterated expectations (LIE). While the EPV formula is linear in  $\mu_t$ , we have  $E_t[P_{t+1} | \mu_t^*, \Omega_{t+1}] = P_t$ , which means the expected capital gain is zero. However, in an economy with heterogeneous beliefs, the prior beliefs of the marginal agents (who determine the stock prices) would change over time, as illustrated in Lemma 1.

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<sup>24</sup>Hereafter, we call these two equations as BLEP equations.

<sup>25</sup>The term ‘‘almost all the time’’ means that it is of measure zero for the expected capital gain to be zero. The following terms with ‘‘almost ...’’ have similar meanings.

**Lemma 1: (*ex post* effects)** *If the signal is in favor of the good (bad) state in period  $t + 1$ , i.e.  $s_{t+1}$  is larger (smaller) than the mid point between  $\theta_H$  and  $\theta_L$ , then:*

- (i) *there is a positive (negative) capital gain and more (less) stock investors;*
- (ii) *the prior beliefs of marginal agents vary over time in almost all the cases;*
- (iii) *cutoff beliefs are higher (lower) than before.*

*Mathematically, results (i) and (ii) means that an encouraging (a discouraging) signal implies*

$$\frac{m_{t+1}\mu_t^*}{1 + (m_{t+1} - 1)\mu_t^*} > (<)\mu_{t+1}^* > (<)\mu_t^*, \quad (3.13)$$

*where  $m_{t+1} = m(s_{t+1})$  is the updating multiplier defined in equation (3.4).*

*Proof:* All the proofs are relegated to Appendix B.

Since the LIE only applies to the same agent or agents with the same prior beliefs, the argument in a representative agent model is no longer valid. Observe that, if EPV were the solution of the BLEP equations, we could substitute equation (3.5) and (3.6) into the asset pricing formula to obtain

$$1 - F_{-1} \left( \frac{\mu_t^*}{\hat{M}_t + (1 - \hat{M}_t)\mu_t^*} \right) - \mu_t^*SV_H - (1 - \mu_t^*)SV_L = 0.$$

and derive a solution  $\mu_t^* = \mu^*(\hat{M}_t; S)$  as a result.<sup>26</sup> Although the general form of  $F_{-1}$  makes rigorous proof difficult, it is much easier to show that the necessary condition for EPV to be the solution of BLEP equations,  $E_t[\mu_{t+1}^* | \mu_t^*, \Omega_{t+1}] = E_t[\mu^*(m_{t+1}\hat{M}_t) | \hat{M}_t] = \mu_t^*$  does not hold if  $F_{-1}$  takes some specific functional forms, including the one we assume in the following subsection.

**Proposition 1: (Deviation from EPV)** *In an economy with heterogeneous prior beliefs, stock prices set by the marginal agents differ from its EPV (conditional on the cutoff beliefs) by a nontrivial expected capital gain component almost all the time until the announcement of true quality.*

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<sup>26</sup>Note that the left-hand side is decreasing in  $\mu_t^*$ , and increasing in  $\hat{M}_t$ ,  $\mu^*(\hat{M}_t; S)$  is an increasing function in  $\hat{M}_t$ .

Note that the additional expected capital gain term is a non-fundamental component since its existence relies heavily on reselling the stock.<sup>27</sup> Since speculators care much more about expected capital gains rather than dividends, the presence of expected capital gain component in the asset pricing formula makes our social learning framework a good platform to analyze speculators' behaviors.

### 3.3.2 Uniform Likelihood Ratio Distribution of Prior Beliefs

Assume that the likelihood ratio of prior belief,  $\mu_{-1}/(1 - \mu_{-1})$ , follows a uniform distribution on  $[0, b_{-1}]$ .<sup>28</sup> As a consequence, the likelihood of the belief of generation  $t$ ,  $\mu_t/(1 - \mu_t)$ , is also uniformly distributed on the support of  $[0, b_t]$ . We call this family of distribution as *uniform likelihood ratio distribution* (ULR distribution). It enables us to employ only one parameter to characterize  $F(\cdot; \Omega_t, s_t)$

$$F(\mu_t; \Omega_t, s_t) = \frac{\mu_t}{b_t(1 - \mu_t)}. \quad (3.14)$$

The corresponding probability density function (pdf) is given by

$$f(\mu_t; \Omega_t, s_t) = \frac{1}{b_t(1 - \mu_t)^2}, \quad (3.15)$$

and the upper bound of belief, or *maximal belief*, is

$$\mu_t^{\max} = \frac{b_t}{1 + b_t} \quad (3.16)$$

With the help of equation (3.2) and (3.4), we know that  $b_t$  can be written as a function

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<sup>27</sup>We don't call it bubble (in a narrow sense), since there is no explosive associated with this type of deviation from fundamental.

<sup>28</sup>We chose this family of distribution just to simplify the subsequent computation since it is closed under belief updating. Theoretically, other families of single-parameter distributions can also be employed. However, we believe that the qualitative results would not change.

of estimated signals  $\{\hat{s}_\tau\}_{\tau=0}^{t-1}$  and the latest signal  $s_t$ :

$$\begin{aligned} b_t &= b_{-1} m(s_t) \prod_{\tau=0}^{t-1} m(\hat{s}_\tau) \\ &= b_{-1} \exp \left[ \frac{\theta_H - \theta_L}{2\sigma^2} \left[ 2s_t + 2 \sum_{\tau=0}^{t-1} \hat{s}_\tau - (t+1)\theta_H - (t+1)\theta_L \right] \right] \end{aligned} \quad (3.17)$$

Note that we have one-to-one mapping between two sequences  $\{\Omega_t \cup \{s_t\}\}$  and  $\{b_t\}$ , and one can easily infer the previous signals from the knowledge about the sequence  $\{b_t\}$ .

Under the ULR distribution, we can rewrite equation (3.6) as

$$SP_t = 1 - \frac{\mu_t^*}{b_t(1 - \mu_t^*)},$$

or

$$\mu_t^* = \frac{b_t(1 - SP_t)}{1 + b_t(1 - SP_t)}, \quad (3.18)$$

with

$$\lim_{b_t \rightarrow 0} \mu_t^* = 0; \quad \lim_{b_t \rightarrow \infty} \mu_t^* = 1.$$

Employing equation (3.18) and the no-arbitrage condition

$$RP_t = [\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L] + \mu_t^* E[P_{t+1} | b_t, \theta_H] + (1 - \mu_t^*) E[P_{t+1} | b_t, \theta_L].$$

we can obtain the BLEP process,  $P_t = P(b_t)$ , from

$$\begin{aligned} & RP(b_t) \{1 + b_t[1 - SP(b_t)]\} - b_t[1 - SP(b_t)]\theta_H - \theta_L \\ &= \int_{-\infty}^{+\infty} P(b_{t+1,L}) \phi(s_{t+1}; \theta_L, \sigma^2) ds_{t+1} + b_t[1 - SP(b_t)] \int_{-\infty}^{+\infty} P(b_{t+1,H}) \phi(s_{t+1}; \theta_H, \sigma^2) ds_{t+1}. \end{aligned} \quad (3.19)$$

where the number of issued shares,  $S$ , is given by

$$\begin{aligned} & R[1 + b_0(1 - S)] - b_0(1 - S)\theta_H - \theta_L \\ &= \int_{-\infty}^{+\infty} P(b_{1,L}) \phi(s_1; \theta_L, \sigma^2) ds_1 + b_0(1 - S) \int_{-\infty}^{+\infty} P(b_{1,H}) \phi(s_1; \theta_H, \sigma^2) ds_1. \end{aligned}$$

Observe that the BLEP process characterized in equation (3.19) satisfies that

$$\lim_{b_t \rightarrow 0} P(b_t) = V_L; \lim_{b_t \rightarrow \infty} P(b_t) = V_H.$$

when  $SV_H < 1$ .<sup>29</sup> It means that, in the absence of uncertainty, equation (3.19) is identical with the present value approach.

### 3.3.3 Numerical Solution

The analytic solution for equation (3.19) is difficult to obtain. However, the numerical solution can be computed based on the brute-force (or iteration) method using EPV prices as the initial values on the right-hand side of equation 3.19. The explanation is quite intuitive. Suppose that the investors expect that firm's true quality would be revealed after  $k$  periods, when the stock price should equal to its EPV. If the sequence of price functions converges as  $k$  approaches to infinity, the limit function would be the solution for the case that the true quality is never revealed. When  $k$  is finite, the price function changes as  $k$  changes.

To compute the EPV pricing in a market with heterogeneous prior beliefs, we set initially the stock price at the expected present value for the marginal investors. The no-arbitrage condition is no longer effective for calculating this initial function since we have demonstrated that the EPV approach fails to be arbitrage-free.

While the IPO price  $P_0 = 1$ , we can calculate  $\mu_0^*$  from (3.12) and the EPV assumption  $P_0 = V_0^E$ ,

$$\mu_0^* = \frac{1 - V_L}{V_H - V_L}. \quad (3.20)$$

As a result, we can use equations (3.7), and (3.14) to infer  $b_0$  from the volume of the issued stock

$$b_0 = \frac{1 - V_L}{(V_H - 1)(1 - S)} \quad (3.21)$$

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<sup>29</sup>This is likely to be the case in reality, where the market value of all stocks is always smaller than capitals available.



For the subsequent periods, we can equate the prices in (3.6) and (3.12) to obtain

$$\frac{1 - F(\mu_t^*; \Omega_t, s_t)}{S} = \mu_t^* V_H + (1 - \mu_t^*) V_L \quad (3.22)$$

and then derive the relationship between  $b_t$  and  $\mu_t^*$  with the help of equation (3.14). Substituting  $\mu_t^*$  as a function of  $b_t$  into equation (3.6) yields

$$SP_t = \frac{1}{2} \left[ 1 + SV_H + \frac{1}{b_t} - \sqrt{\left(1 - \frac{1}{b_t} - SV_H\right)^2 + \frac{4}{b_t}(1 - SV_L)} \right] \quad (3.23)$$

$$\mu_t^* = 1 - \frac{2}{(b_t + 1 - SV_H b_t) + \sqrt{(b_t + 1 - SV_H b_t)^2 + 4b_t S(V_H - V_L)}} \quad (3.24)$$

for  $t = 1, 2, \dots$ . We show in Appendix A that both  $P_t$  and  $\mu_t^*$  are increasing in  $b_t$ .

With the above initial pricing function, we calculate the numerical results with parameter values of  $R = 1.01$ ,  $V_H = 1.8$ ,  $V_L = 0.6$ ,  $b_0 = 1$ ,  $S = 0.5$ ,  $\sigma = (\theta_H - \theta_L)/q$  and  $q = 1$ . Note that the return of the risk-free assets is close to quarterly or semiannual risk-free return in the States, we can interpret each period as 3 or 6 months. This is also in line with the quarterly or semiannual reporting requirements for listing firms. In this example, the signals are quite rough since the probability of misleading signals is just 0.4085 if we employ the middle point of  $\theta_H$  and  $\theta_L$  as the critical point. The iteration method provides a reasonable convergence rate for the pricing function. For instance, after 30 iterations, the supnorm distance between adjacent pricing functions is only  $1.3001 \times 10^{-5}$ , while stock prices range from 0.6 to 1.8.

Figures 3.2 and 3.3 illustrate the relationship between EPV and BLEP. We can find that the EPV approach would underprice the stock as much as 5.15% of the BLEP when the firm is widely believed to be good and overprices it as much as 2.14% when the most investors are pessimistic. The ratios are quite substantial compared with the risk-free rate.

Based on this result, we can have some idea about the dynamics of stock prices. Note that  $b_t$  will change over time since it is based on the realization of the random signal  $s_t$ . When the firm is of a good (bad) quality, its stock prices would be more likely to stay higher (lower) than its EPV until the truth is revealed. In this case, we can observe a

sustained deviation from the stock's EPV, as illustrated in the example in figure 3.4. This result agrees with the phenomena of sustained high prices during the bubble period and seemingly endless low prices in a bearish market. Intuitively, if the marginal investors believe that the quality of the firm is good (bad), the signal in the subsequent period is more likely to be encouraging (discouraging). Hence he would expect a capital gain (loss), which explains the sustained deviation.

Observe that the deviation from EPV is biggest when there is a widespread rumor in the market rather than the announcement of the truth. The reason lies in the fact that when buying at a price quite close to, for instance,  $V_H$ , the surprisingly adverse news could lead to a larger expected loss even when the probability is small, since the price drops would be quite drastic, which is similar to the black sheep effect in other learning literature, such as Chamley (2003b, page 73). Proposition 2 summarizes the findings so far.

**Proposition 2: (Sustained Deviation from EPV)** *There exist sustained deviations from EPV in finite periods. Widespread rumors in the market give rise to larger deviations.*

### 3.3.4 Signaling Noises

A noteworthy remark is that the quality of signal would greatly change results quantitatively, although not qualitatively. Recall that the standard deviation of noise is set as  $\sigma = (\theta_H - \theta_L)/q$ , which means the two true states are  $q$  standard deviations apart. Hence we have a signal with better quality if  $q$  is large. When the signals are rather accurate with  $q = 4$ , investors are less confused by the signals, and thus stock prices are not far from the corresponding EPVs. Actually, when  $q$  approaches to infinity, the agents know the true type each period, and there is no deviation from the EPVs. When the signals are less informative with  $q = 1/4$ , the investors cannot have a good anticipation about the subsequent signals, the expected capital gain component is also small. Extremely, when  $q$  tends to zero, the variance of the signal  $\sigma^2$  approaches to infinity, and the signal is of no use any more. In this case  $E_t[P_{t+1}|\mu_t^*, \Omega_{t+1}] = P_t$ , and the BLEP converges to the EPV. figure 3.5 demonstrate how the price deviation changes with the quality of the

signals.

In addition, the quality of signaling noises would also affect the convergence rate toward the truth. figure 3.6 demonstrates the square root of mean squared errors (RMSE) of the maximal beliefs based on a simulation with 1000 replications. It takes only 5 period for RMSE to fall below  $10^{-4}$  when  $q = 4$ , 15 periods when  $q = 2$ , and 58 periods when  $q = 1$ . In the cases of  $q = 1/2$  and  $1/4$ , the RMSE is still as high as 0.0770 and 0.2656 respectively after 100 periods. If we take a period as 3 or 6 months, the convergence rate for  $q < 1$  is fairly slow (about 20 years).

### 3.3.5 Speculative Manipulation

Equation (3.23) essentially provides a pricing function  $P_t = P(b_t)$ . We have shown that this function is strictly increasing. This provides us an opportunity to study the possibility of speculative manipulation. The  $t$ -generation agents with a belief just below the cutoff may consider submitting a higher bidding price in an attempt to pretend that the signal is better than it actually is. Through Bayesian learning, the next generation would like to offer a higher price. In short, these investors can sell the stocks at a higher price in the subsequent period, but they have to bid up their purchasing price to disguise the quality of the firm. This is similar to the *feedback loop*, an important effect in creating speculative bubbles suggested by Shiller (2000). Previous explanations of this effect rely either on adaptive expectation or some psychological factors, such as overconfidence, which need to relax individual rationality. However, it can be easily justified in this social learning framework.

Suppose the agents manage to create an illusion that the highest belief is  $b_t + \delta$ . The expected yield becomes

$$y_t^{SP} = \frac{\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L}{P(b_t + \delta)} + \frac{E_t\{P[(b_t + \delta)m(s_{t+1})]|\mu_t^*, \Omega_t\}}{P(b_t + \delta)}. \quad (3.25)$$

The first term captures the *dilution effect*, where the expected rate of return from dividends is diluted by the higher purchasing price. The second term represents the *capital gain effect*. The dilution effect is obviously negative, but the capital gain effect is generally ambiguous depending on the property of the price function. We have feedback loops,

because an increase in current prices would make the subsequent generation adjust their beliefs upward and thus raise the prices in the next period. However, whether it is profitable to manipulate the stock prices in this manner depends crucially on the gains from a higher stock price in the next period net of the costs from bidding up current price.

figure 3.7 shows that, the dilution effect is unsurprisingly negative, but rather small in magnitude; moreover, under the current settings, the capital gain effect turns out to be negative as well, which means the costs of speculative manipulations are bigger than its gains. Thus, while we do identify feedback loops, it is not strong enough to grant the speculative manipulations profitable. The results are summarized in Proposition 3.

**Proposition 3: (Speculative Manipulation)** *In the benchmark setting, feedback loops cannot lead to profitable speculative manipulations.*

With negative profits, one cannot justify the existence of speculative manipulations. An possible explanation is that investors can have signals as good in quality as the preceding ones, hence they are less likely to be fooled. Another reason is that the changes in the beliefs by bidding up the prices is small relative to the costs. If we change the assumption about the signaling noises, speculative manipulations could be profitable. Section 4 provides an example in this direction.

An immediate implication from Proposition 3 is that stock prices would perfectly reveal previous signals since no one would manipulate the stock prices. Due to the one-to-one mapping from maximal beliefs to stock prices, we can infer the history of beliefs from the records of stock prices, and hence figure out all the previous signals. Actually, with continuous and unbounded actions and only one dimension of signals, previous signals would be perfectly inferred, as pointed out by Avery and Zemsky (1998). Nonetheless, we still find sustained stock price deviations from EPV, which is more remarkable given perfect learning.

In fact, the deviation from EPV stems from the treatment of corporate earnings as signals, instead of truth. The conceived data generation process changes with the beliefs, while other works, such as the experiments on estimation-based asset pricing model by Timmermann (1996), rested on the ex ante belief that the conceived data generation process is stationary, although it changes with time ex post.

There are at least two ways to make learning imperfect and lead to signal distortions: one is to limit the fluctuation in the stock prices, and another is to increase the dimension of signals. We will show that, in at least one situation, speculative manipulations are profitable, which leads to endogenous noises within the system.

## 3.4 Signal Distortions

### 3.4.1 Stock Markets with Fluctuation Limits

In several emerging markets, fluctuation limits are introduced to avoid drastic changes in stock prices.<sup>30</sup> In the baseline model, stock prices completely reveal the signals. In contrast, we only have partial revelation when stock price hits the limit. The boundedness in prices turns out to be important to make the speculative manipulations profitable.

For simplicity and illustrative purposes, only *limit-ups* are considered here.<sup>31</sup> With limit-ups, stock exchange generates the prices according to the formula

$$P_t = \min\left\{\frac{1 - F(\mu_t^*; \Omega_t, s_t)}{S}, \lambda P_{t-1}\right\}, \quad (3.26)$$

where  $\lambda$  is one plus the fluctuation limits for price rises. When the stock price hits the upper limit  $\lambda P_{t-1}$ , each investor can only invest a proportion  $\alpha_t$  of his capital endowment, where

$$\alpha_t = \frac{S \lambda P_{t-1}}{1 - F(\mu_t^*; \Omega_t, s_t)}.$$

The remaining  $1 - \alpha_t$  would be invested in the risk-free asset. Assume that  $\alpha_t$  is not

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<sup>30</sup>The fluctuation limit is also called as daily price limit or daily trading limit, which stipulates that the stock prices can only fluctuate daily within a band computed based on the previous closing prices. It is more widely employed in futures markets as well as in many stock markets, including those in Japan, Korea, Mainland China, Taiwan, Malaysia, Thailand, Mexico, Austria, Belgium, France, Greece, Italy, Netherlands, Spain, Switzerland, Turkey. Most of them are listed in Roll (1989). As one of the largest stock markets in the world, Tokyo Stock Exchange imposes daily price limits in an effort to prevent “excessively violent price fluctuations due to an imbalance in the buy/sell equilibrium or due to speculative trading.”

<sup>31</sup>Limit-ups are referred to the limits for price rises, and limit-downs means the limit for price drops. In a stock market with limit-downs, at least some stock holders fail to sell all of their stocks, and have to sell the remaining in the subsequent trading day. Hence it requires a multi-period (at least three-period) overlapping generation framework, which makes the model more complicated without much contribution.

recorded, which is in line with the practice of the emerging stock markets mentioned above. When the stock price is exactly at the limit-up level, we can compute the threshold value of the cutoff beliefs as

$$\mu_t^{\text{up}} = F^{-1}(1 - S\lambda P_{t-1}; \Omega_t, s_t).$$

For simplicity, we again assume that the likelihood ratio is uniformly distributed. Then we have

$$SP_t = \min\left\{1 - \frac{\mu_t^*}{b_t(1 - \mu_t^*)}, S\lambda P_{t-1}\right\},$$

where the price function in this case depends not only on  $b_t$  but also on  $P_{t-1}$ . We know that better current signal leads to a higher stock price. Hence, as long as  $\lambda P_{t-1} < V_H$ , there exist a threshold value of current signal  $s_t^{\text{up}}$  such that the price hits the limit-up level if and only if  $s_t \geq s_t^{\text{up}}$ . As a result, when  $s_t < s_t^{\text{up}}$ ,  $P_t$  and  $\mu_t^*$  satisfies

$$\mu_t^* = \frac{b_t(1 - SP_t)}{1 + b_t(1 - SP_t)}$$

and

$$\frac{\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L + E_t[P_{t+1} | \mu_t^*, b_t]}{P_t} = R;$$

when  $s_t \geq s_t^{\text{up}}$ , we have

$$P_t = \lambda P_{t-1}$$

and

$$\frac{\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L + E_t[P_{t+1} | \mu_t^*, \tilde{b}_t]}{\lambda P_{t-1}} = R.$$

Observe that we use  $\tilde{b}_t$  instead of  $b_t$  in the second case, since the limit-up price results in signal distortion. When stock price hits the upper bound, subsequent generation fails to observe the true signal. All they can infer is the probabilities for stock prices to hit the upper limit when firm is of good or bad quality. Therefore the updating multiplier turns out to be

$$\tilde{m}(s_t^{\text{up}}) = \frac{1 - \Phi(s_t^{\text{up}}; \theta_H, \sigma^2)}{1 - \Phi(s_t^{\text{up}}; \theta_L, \sigma^2)},$$

where  $\Phi$  is the Gaussian cdf.

From the model, we can find that stock prices hitting fluctuation limits would distort the revelation of private signals. More specifically, since  $m(s_t^{\text{up}}) < \tilde{m}(s_t^{\text{up}})$  for Gaussian distribution,<sup>32</sup> the perceived signal has an upward bias when the true signal is close to  $s_t^{\text{up}}$ . In the case when the true signal is lower but sufficiently close to  $s_t^{\text{up}}$ , some investors can pretend that they have received  $s_t^{\text{up}}$  instead of the true signal in an effort to make use of the upward bias, which makes speculative manipulations more profitable than the baseline model.

figure 3.8 illustrates the region of profitable speculative manipulations when current signal is slightly below the level to make the price hit the fluctuation limit. Since the difference between the would-be price without speculative manipulations and the upper limit is small, the manipulation cost is low. However, the gains from the manipulations would be quite substantial when the previous price is low as a result of the large magnitude of the upward bias. Numerically, we do find the region, as shown in figure 3.8, for the speculative manipulations to be profitable. We summarize the result in Proposition 4.

**Proposition 4: (Active Speculative Manipulation)** *With fluctuation limits, there exist cases in which the speculative manipulations are profitable.*

Note that these emerging markets initially introduce the fluctuation limits for the purpose of reducing the possibility of speculations. While achieving this purpose within a trading day, it could may turn out to encourage speculative manipulations over a longer period of time.

### 3.4.2 Financial Frenzies

We next introduce animal spirits in such a way that a proportion  $(1 - N_t)$  of young agents at period  $t$ , would always invest in the stock markets regardless of the signals received (the remaining agents of proportion  $N_t$  behave the same as before). The proportion  $N_t$  has the value of  $n$  with probability  $\pi$ , and equals 1 otherwise. The distribution of frenzied agents is assumed independent of the prior beliefs. While the realization of  $N_t$  is private information, its distribution is public knowledge. Note that we now have two-dimensional

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<sup>32</sup>It is based on the fact that the hazard rate,  $\frac{\phi(x)}{1-\Phi(x)}$ , is increasing for Gaussian distribution.

signaling noises, but have only a single dimensional action reflected by stock price. The additional noises lead to signal distortion.<sup>33</sup>

For simplicity, we employ the uniform likelihood ratio distribution again. With financial frenzies, the measure of normal agents becomes

$$F(\mu_t; \Omega_t, s_t, N_t) = \frac{N_t \mu_t}{b_t(1 - \mu_t)}.$$

Stock exchange now generates the stock price given by

$$SP_t = 1 - \frac{N_t \mu_t^*}{b_t(1 - \mu_t^*)}, \quad (3.27)$$

and the no-arbitrage condition becomes

$$\frac{\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L + E_t[P_{t+1} | \mu_t^*, \tilde{b}_t]}{P_t} = R. \quad (3.28)$$

Note that the subsequent agents can only figure out that the value of the updating multiplier is  $m(s_t)/N_t$ . Since they do not know the value of  $N_t$ , they must consider both cases, and infer two possible signals  $s_t^n$  and  $s_t^1$ , where

$$m(s_t^n; N_t) = \frac{nm(s_t)}{N_t}, \quad m(s_t^1; N_t) = \frac{m(s_t)}{N_t}$$

As a consequence, the perceived updating multiplier changes to

$$\tilde{m}(s_t^n, s_t^1) = \frac{\pi \phi(s_t^n; \theta_H, \sigma^2) + (1 - \pi) \phi(s_t^1; \theta_H, \sigma^2)}{\pi \phi(s_t^n; \theta_L, \sigma^2) + (1 - \pi) \phi(s_t^1; \theta_L, \sigma^2)}.$$

Note that the difference between perceived updating multiplier,  $\tilde{m}(s_t^n, s_t^1)$ , and the one based on true signals,  $m(s_t)$ , leads to signal distortions in the learning process. The reason lies in the fact that the mapping from two-dimensional signaling noises to the one-dimensional stock prices fails to be one-to-one. figure 3.9 shows that the signal distortions

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<sup>33</sup>We can also introduce financial distress analogously. For example, we can assume that a fraction of agents would always stay away from the stock market whatever signal he receives. However, it is a bit more complicated since we have to adjust the total shares available to rational agents, while in the financial distress model,  $n_t$  only appears in the cumulative distribution function.



caused by financial frenzies slow down the convergence. Similar to the baseline model, we can obtain the BLEP process from equation (3.27) and (3.28) by substituting out  $\mu_t^*$ . However, in this case, the signal distortions are not large enough to make the speculative manipulations profitable. Proposition 5 summarizes the results.

**Proposition 5: (Slower Convergence)** *The introduction of financial frenzies results in signal distortions, which slow down the rate of convergence toward the truth.*

In reality, we could possibly have ten or more dimensions of signaling noises, which would make the convergence rate even lower, and lengthen the period of sustained deviations from EPV. This may help explain why the stocks would deviate from their fundamental values as long as two decades.

### 3.5 Conclusion

In this paper we establish an overlapping generations model with Bayesian learning about the listed firm's quality, and find that the arbitrage-free prices can have sustained deviation from the related EPVs. In a bullish market, the stocks are priced higher than that implied by the present value of subsequent dividend flows. During the recession, we can find sustained existence of underpriced stocks.

Since we find that speculative manipulations can be profitable at least in the case with fluctuation limits, the subsequent agents would take this endogenous noises into account. In the our model, it would not change much in the region of signals leading to upper limits, however, it would be interesting and important in other cases.

Moreover, there could be other types of signal distortions making speculative manipulations profitable. Possible candidates include changing the binary animal spirit model into a continuous one. However, it takes much longer to compute double integrals numerically.

We can also change the informational structure in the baseline model. For example, the introduction of partial access to the signals would slow down the convergence rate of cutoff beliefs toward the truth. A three-period overlapping generations settings would greatly enrich the analysis of stock market behaviors. In contrast to the passive selling

from the old investors in current model, we would have more elastic supply since the mid-aged investors can choose whether to hold or sell. The idea is close to the occupational choice model proposed by Banerjee and Newman (1993). We expect that the increase in the dimension of strategy space provides more room for speculative manipulations. Some of the preliminary works are shown in Appendix B.

The consideration of more than one firms or stock markets is another interesting extension for this paper. If there is some correlation among the realized signals of the firms, investors would update their beliefs of one firm based on the stock price performance of other firms. As a consequence, there will be substantial comovements among different stocks or stock markets, which is in line with the observation of the stock market performance and help us understand the mechanism of financial contagions.

Providing the first asset pricing formula including a non-trivial expected capital gain component, our framework is more suitable for future studies on the impacts from taxes on capital gain. It seems that this kind of taxes would discourage positive deviations from fundamentals in a booming market, but enhance the negative deviations in a bearish market. However the impact from a progressive capital gain taxes demands more thorough studies.

More generally, we can employ the social learning framework to revisit the adaptive expectation literature. In fact, adaptive expectation can be regarded as one of the short-memory learning processes. By doing so, we can find more applications for social learnings.

Figure 3.2: Stock Prices and Cutoff Beliefs

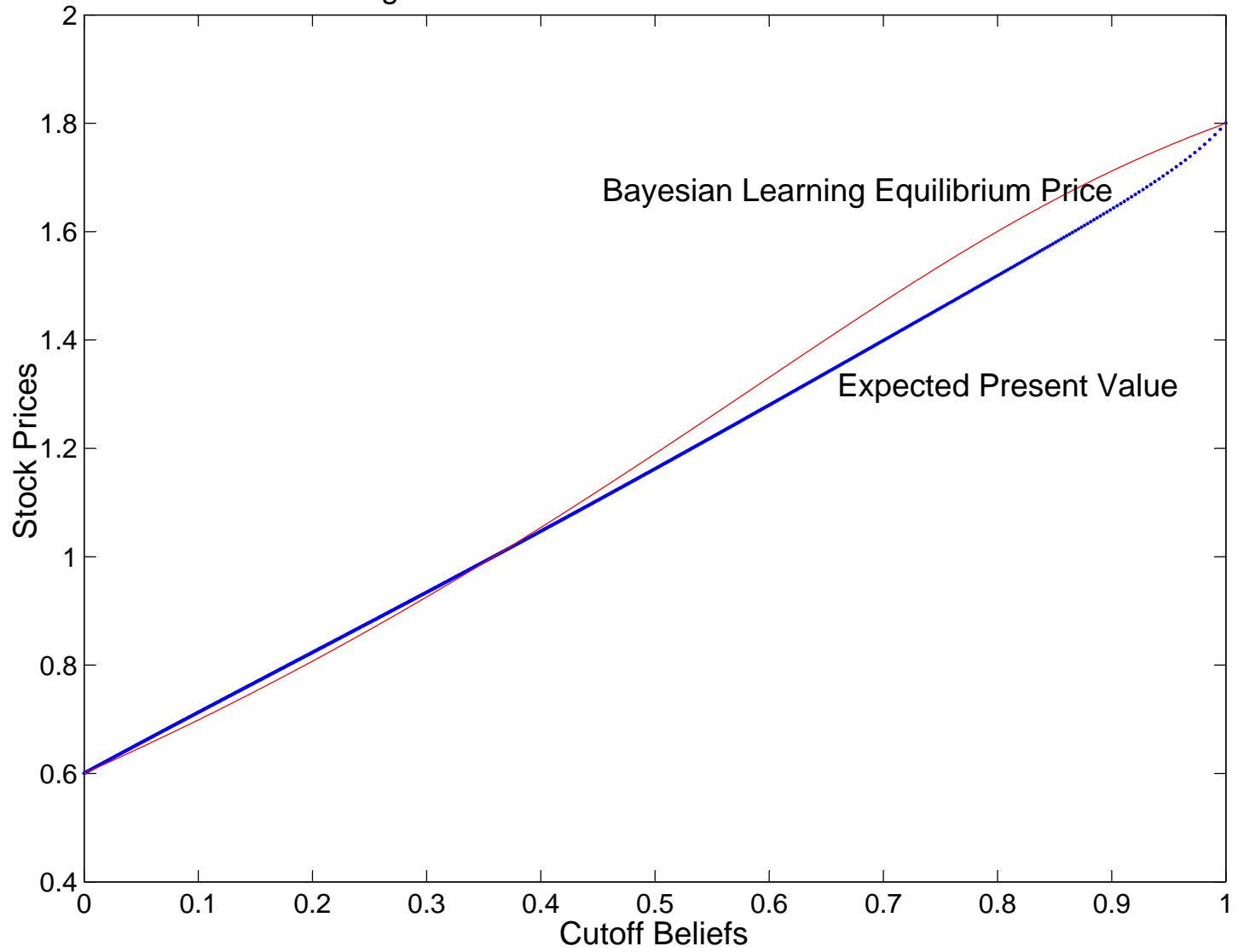


Figure 3.3: Ratio between Bayesian Learning Prices and EPV

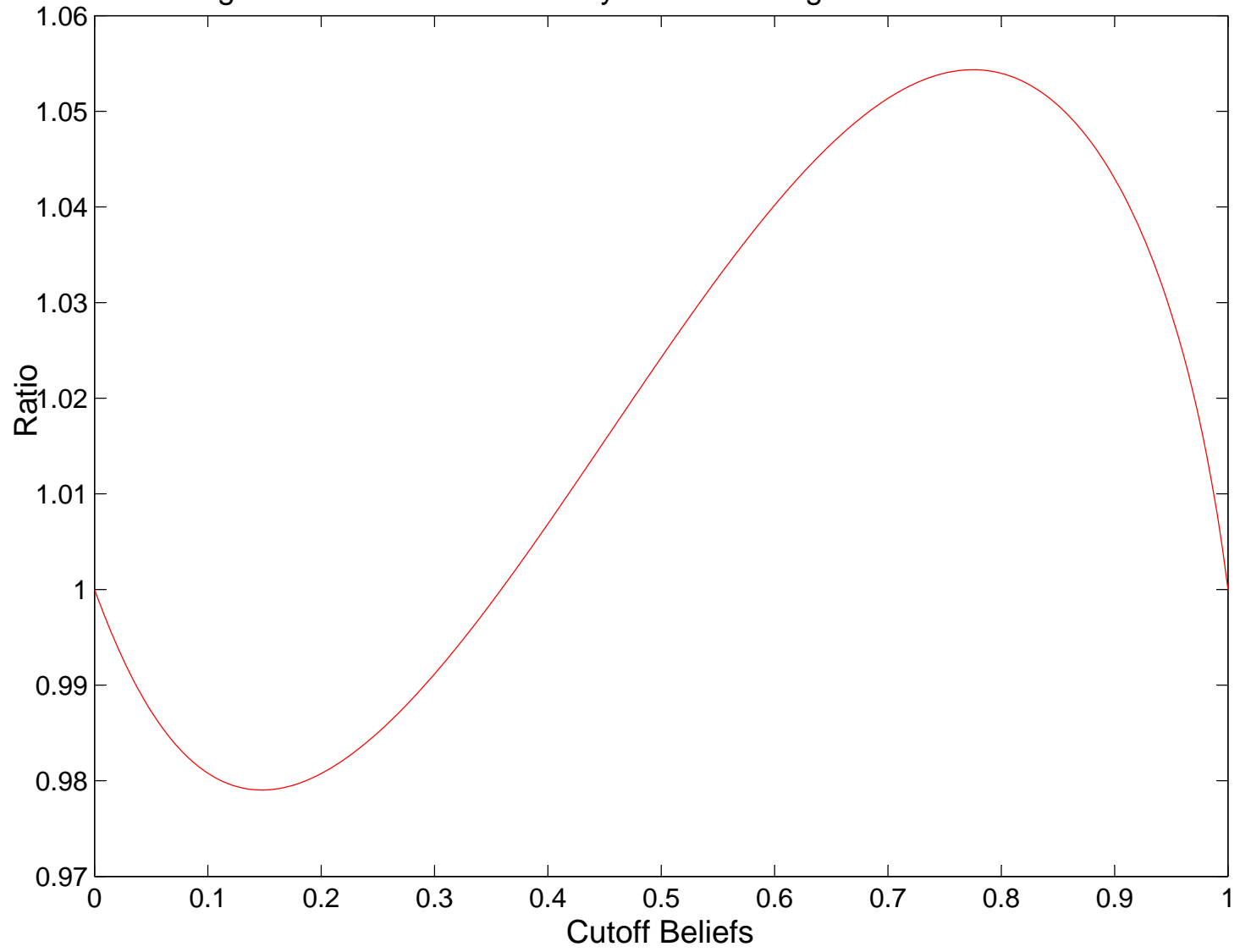


Figure 3.4: An Example of Price Series

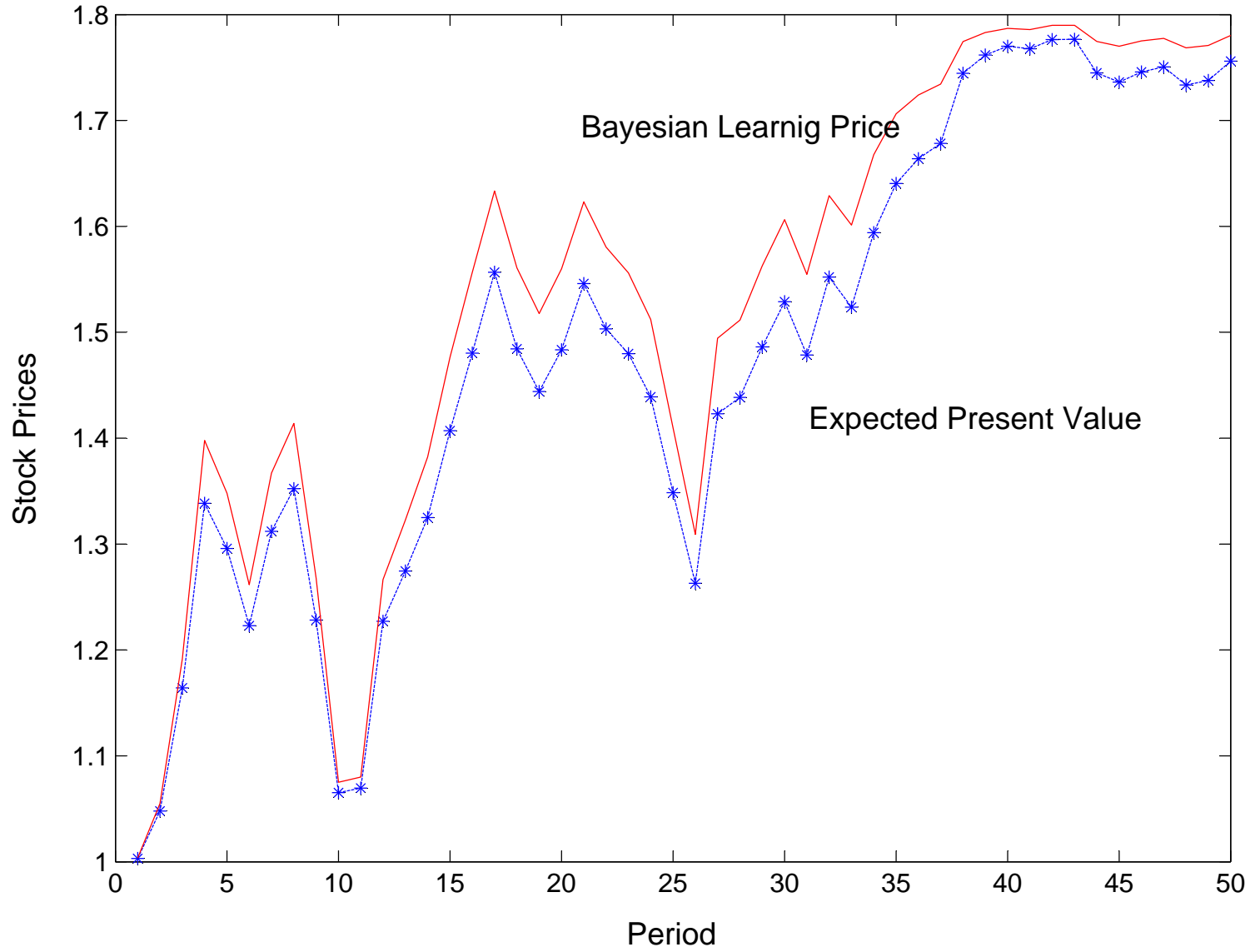


Figure 3.5: Price Ratio for Different Qualities of Signaling Noises

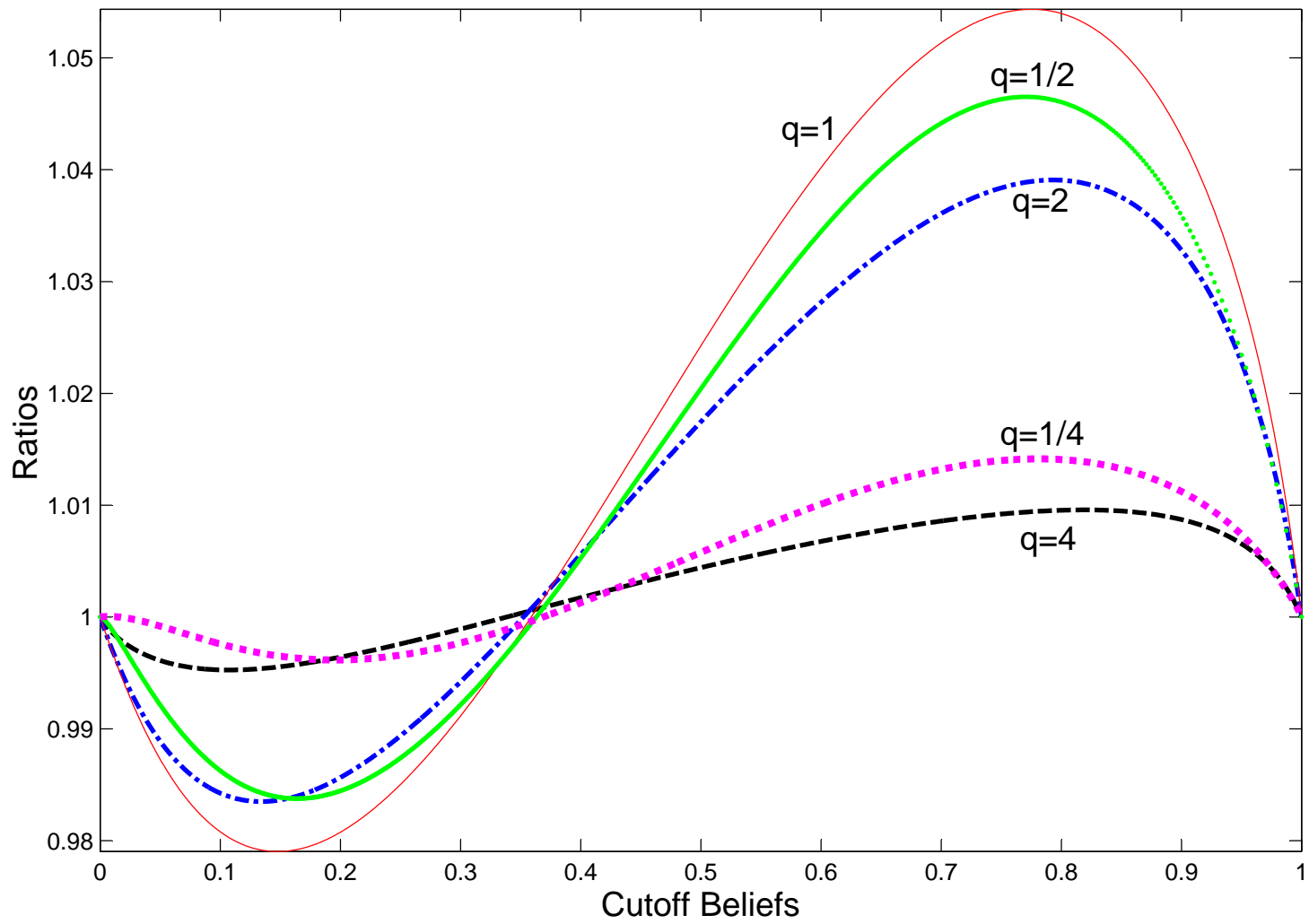


Figure 3.6: Convergence for Different Qualities of Noises

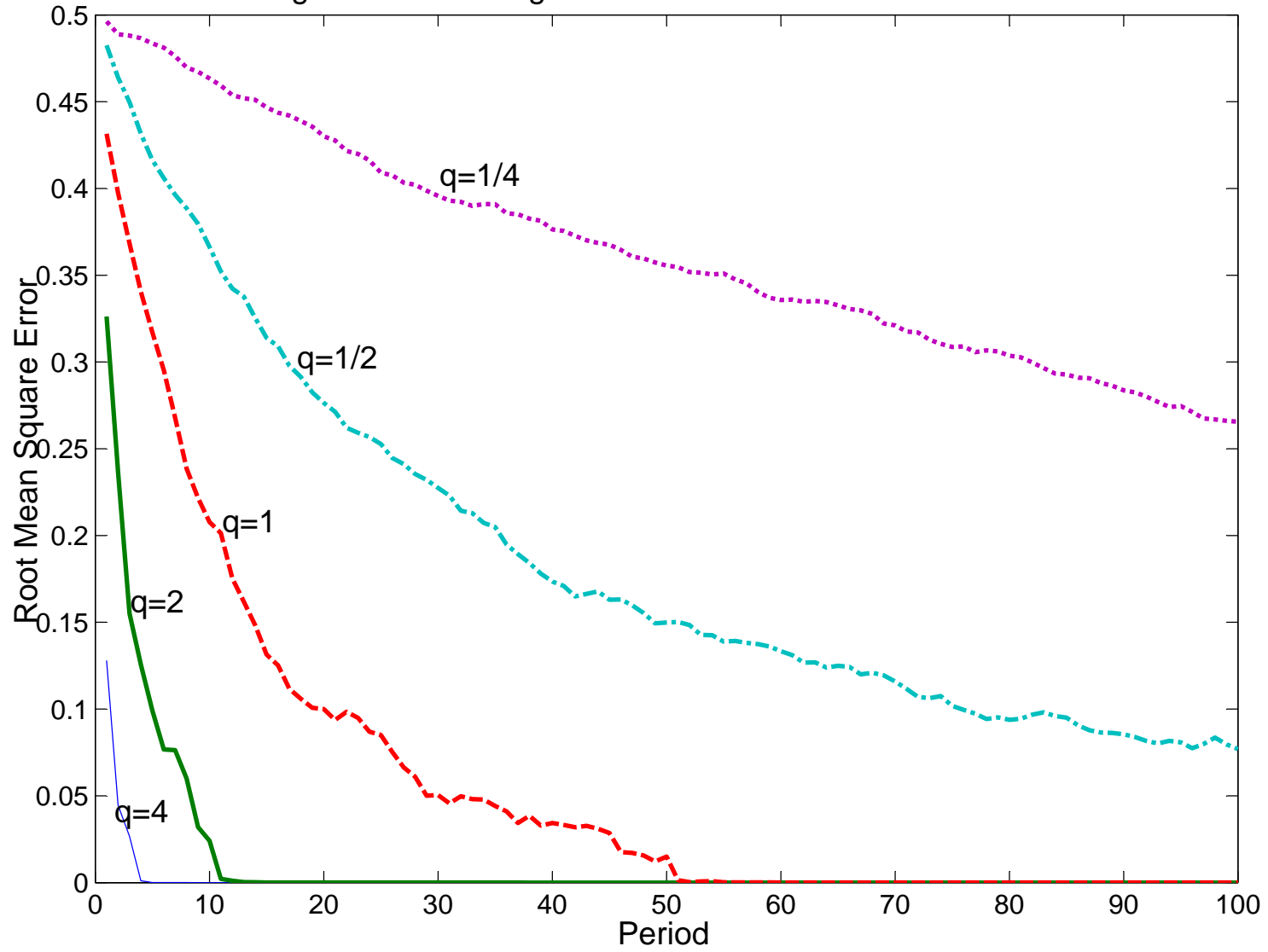
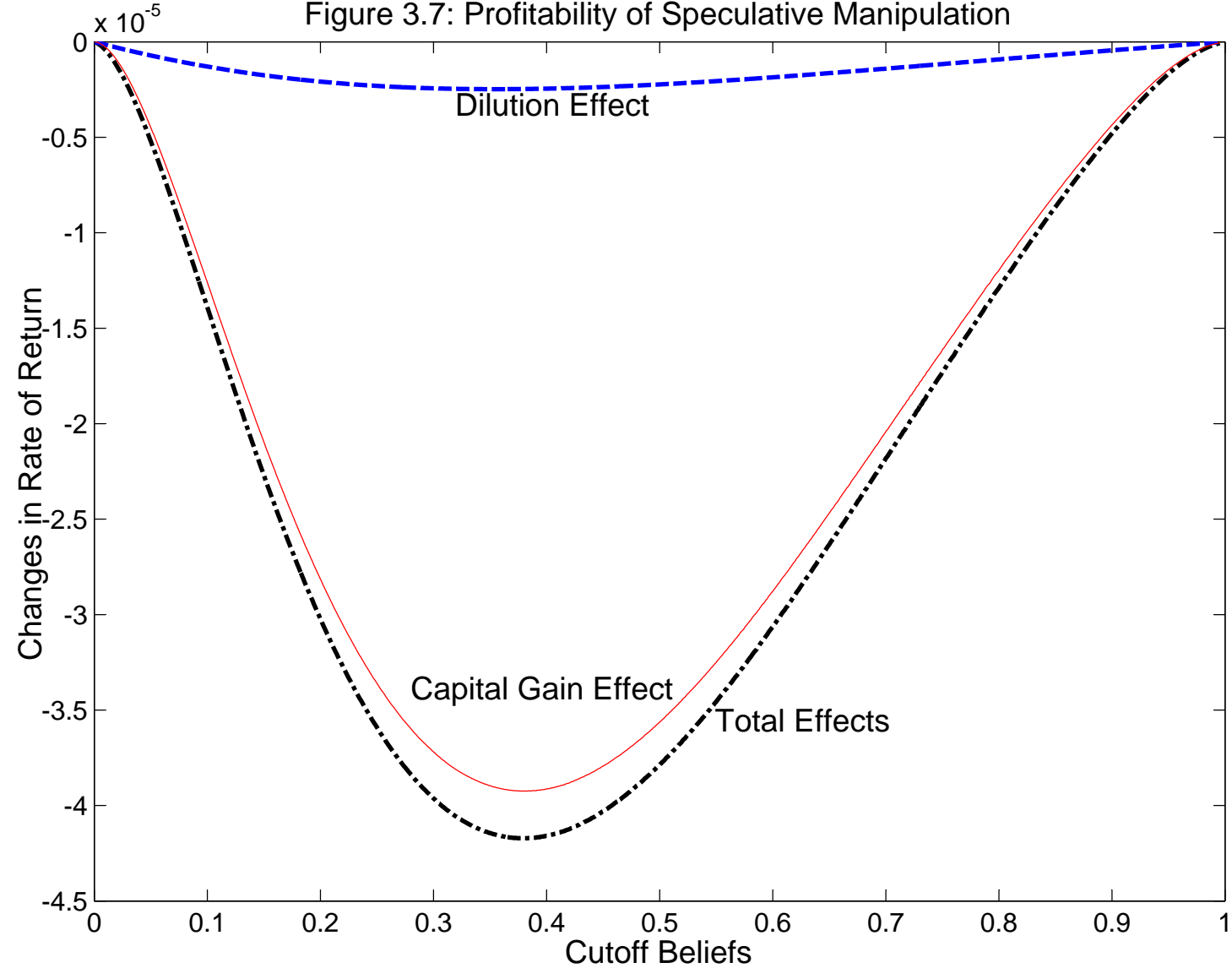


Figure 3.7: Profitability of Speculative Manipulation





**Figure 3.8: Regions of Profitable Speculative Manipulation with Price Fluctuation Limit**

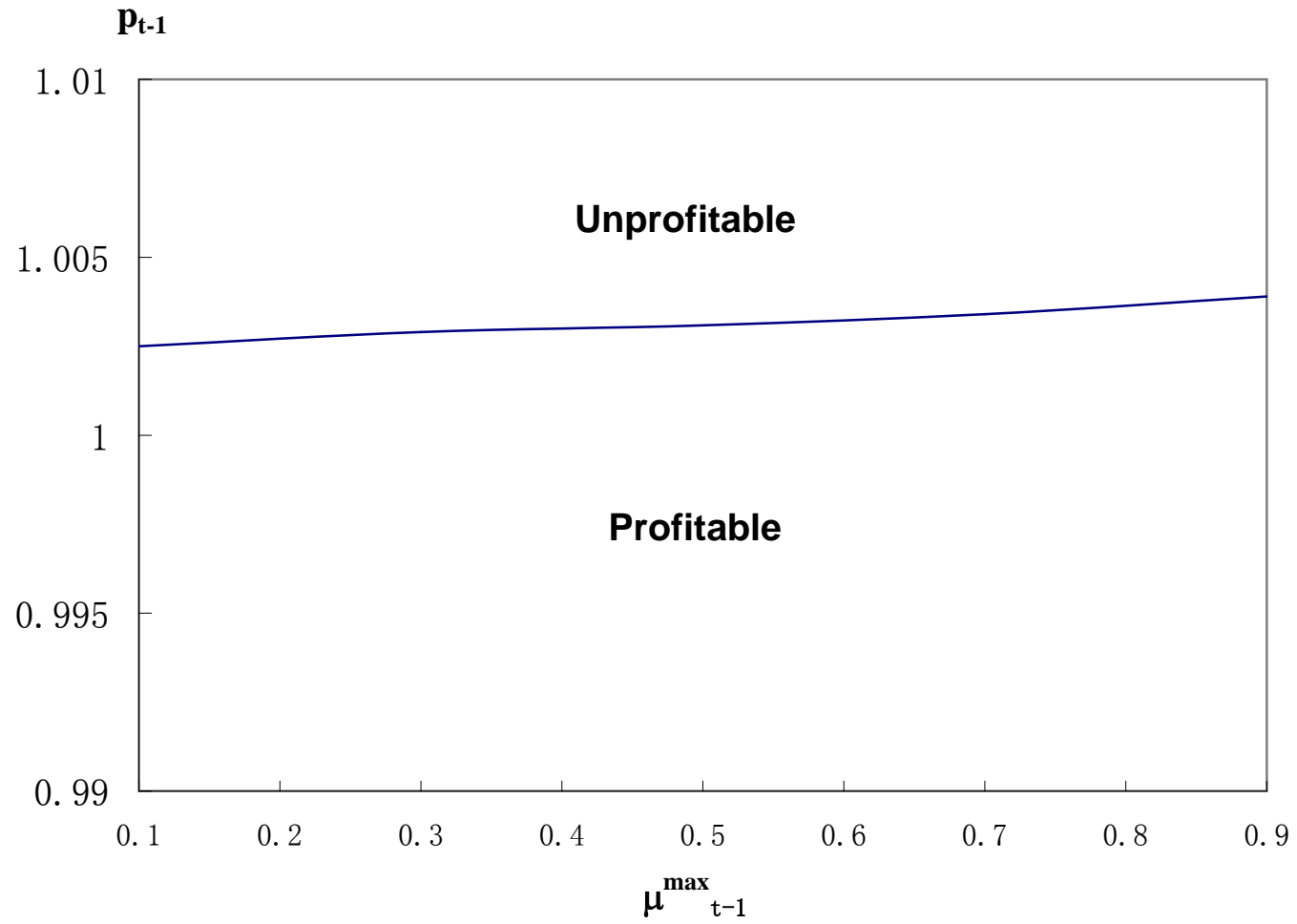
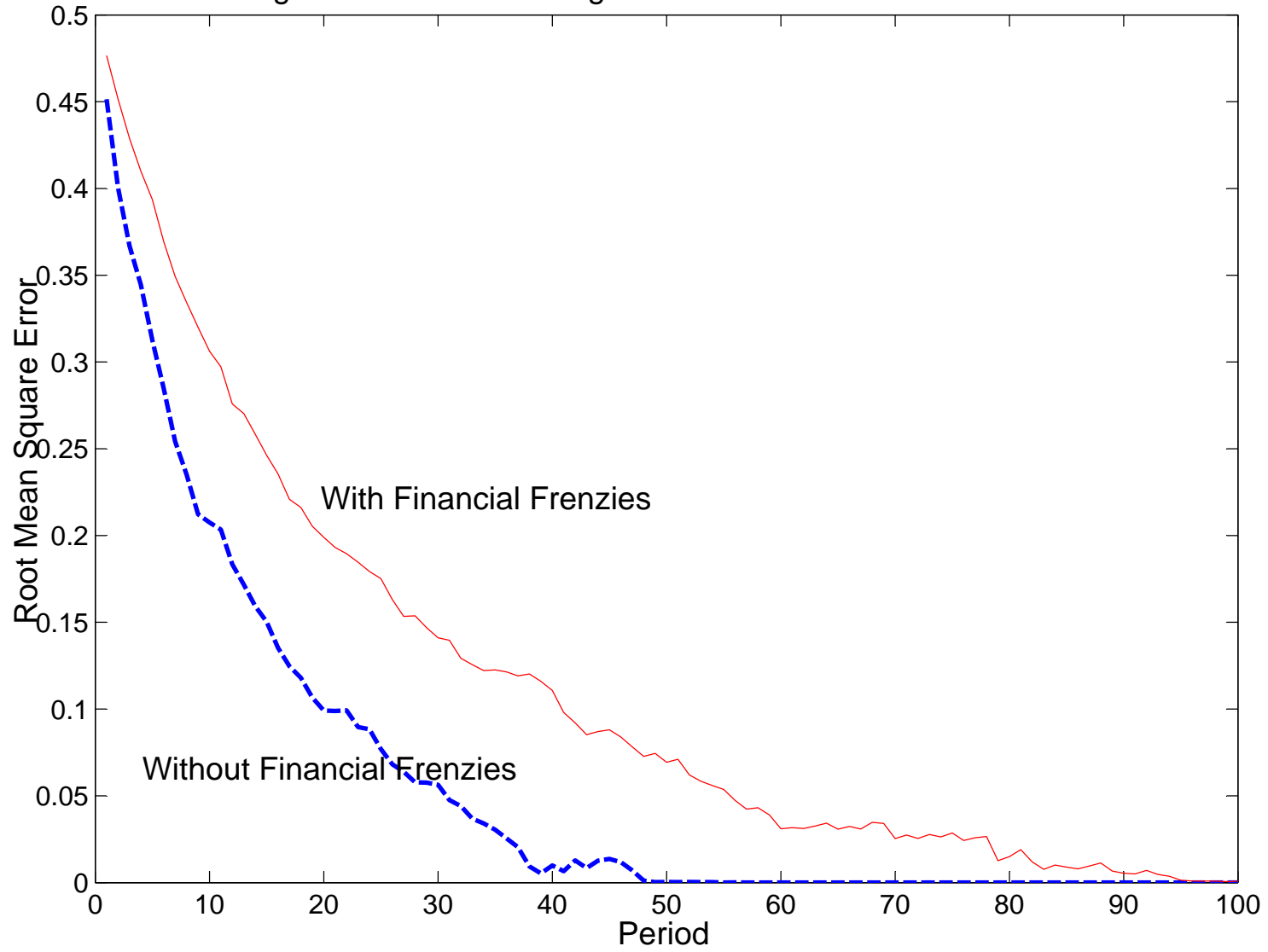


Figure 3.9: Slow Convergence Rate with Financial Frenzies



## Appendix

### A. Possible Extensions of the Model

In the above model, the complete access to the signals discourages the possibility of speculative manipulations. Hence, one of the interesting extensions would be the imposition of partial access to the signals. We expect the convergence rate to be much slower, and the speculative manipulations to be more likely.

Another concern of the main model is the forced selling for the old generation. If we extend the model to a three-period overlapping generations framework, it would be more close to reality.

#### A1. Partial Access to the Signals

The major difference from the baseline model is that the young agents has a probability  $\pi_t$  to observe the signals, while the uninformed agents would have a belief  $\mu_t^N = \hat{\mu}_{t-1}$ . As a result, the number of winning buyers is  $\pi_t[1 - F(\mu_t^{S*}; \Omega_{t-1}, s_t)] + (1 - \pi_t)[1 - F(\mu_t^{N*}; \Omega_{t-1})]$ , where  $\mu_t^{N*}$  and  $\mu_t^{S*}$  stand for the beliefs of marginal agents in each group respectively. Recall that each agent has one unit of capital and the total market value of the company is  $SP_t$  at time t. Hence we obtain the market price as a function of cutoff belief. It also depends on the  $F_t^N$  and  $F_t^S$ .

$$P_t(\mu_t^*; \sigma_{t+1}) = \frac{\pi_t[1 - F(\mu_t^{S*}; \Omega_{t-1}, s_t)] + (1 - \pi_t)[1 - F(\mu_t^{N*}; \Omega_{t-1})]}{S}, \quad (\text{A1})$$

where

$$S = SP_0 = \pi_0[1 - F(\mu_0^{S*}; s_0)] + (1 - \pi_0)[1 - F(\mu_0^{N*}; \Omega_0)] \quad (\text{A2})$$

Now we have no-arbitrage condition for each type of agents

$$y_t = \frac{\mu_t^{S*}\theta_H + (1 - \mu_t^{S*})\theta_L + E_t[P_{t+1}|\mu_t^{S*}, \Omega_{t-1}, s_t]}{P_t} = R, \quad (\text{A3})$$

and

$$y_t = \frac{\mu_t^{N*}\theta_H + (1 - \mu_t^{N*})\theta_L + E_t[P_{t+1}|\mu_t^{N*}, \Omega_{t-1}]}{P_t} = R. \quad (\text{A4})$$

If we assume the likelihood ratio to be uniformly distributed, then

$$SP_t = SP(b_t; b_{t-1}) = \pi_t \left[ 1 - \frac{\mu_t^{S*}}{b_t(1 - \mu_t^{S*})} \right] + (1 - \pi_t) \left[ 1 - \frac{\mu_t^{N*}}{b_{t-1}(1 - \mu_t^{N*})} \right] \quad (\text{A5})$$

We can obtain the relationship between the two cutoff beliefs

$$\begin{aligned} & (\mu_t^{S*} - \mu_t^{N*})(\theta_H - \theta_L) \\ = & \mu_t^{N*} E_t[P(b_{t+1}; b_t)|b_{t-1}, \theta_H] + (1 - \mu_t^{N*}) E_t[P(b_{t+1}; b_t)|b_{t-1}, \theta_L] \\ & - \mu_t^{S*} E_t[P(b_{t+1}; b_t)|b_t, \theta_H] - (1 - \mu_t^{S*}) E_t[P(b_{t+1}; b_t)|b_t, \theta_L] \end{aligned} \quad (\text{A6})$$

Theoretically, we can solve the  $\mu_t^{S*}$  and  $\mu_t^{N*}$  from (A5) and (3.29) and then put them back to either (A3) or (A4) to solve the BLEP process. However it is even more difficult to solve this model.

## A2. Three-Period Overlapping Generations Framework

Assume that each agent will live for three period, young, adult and old. While they are born with an endowment of one unit of capital and only consume at the end of the third period, they invest when young, modify their investment portfolio when adult, and sell all assets when old.

At period 0, the listed firm issues new shares to the young and adult investors, hence

$$\begin{aligned} S &= SP_0 = [1 - F(\mu_{0,0}^*; \Omega_0)] + R[1 - F(\mu_{-1,0}^*; \Omega_0)] \\ &= (1 + R) \left[ 1 - \frac{\mu_{0,0}^*}{b_0(1 - \mu_{0,0}^*)} \right] \end{aligned}$$

where the two cutoff beliefs,  $\mu_{0,0}^{S*}$  and  $\mu_{-1,0}^{S*}$ , are equal, since the expected price in the next period are the same. We also impose the assumption of uniform distribution for the likelihood ratio. Here the first subscript stands for the birth date of the agent, while the second one indicates the current period.

A good way to obtain the stock market clearing condition is to imagine that the adults sell all the shares and then make their portfolio decision again. Now the market value of the stocks is given by the young and adult agents' stock investment. To the adult agents,

if the number of investors increases, i.e.  $1 - \frac{\mu_{t-1,t}^*}{b_t(1 - \mu_{t-1,t}^*)} > 1 - \frac{\mu_{t-1,t-1}^*}{b_{t-1}(1 - \mu_{t-1,t-1}^*)}$ , the previous stock holders would invest all the capital gains and the dividends in stocks, while the previous depositors (risk-free asset holders) would also invest their deposits.

This implies

$$SP_t = 1 - \frac{\mu_{t,t}^*}{b_t(1 - \mu_{t,t}^*)} + \frac{P_t + s_t}{P_{t-1}} \left[ 1 - \frac{\mu_{t-1,t-1}^*}{b_{t-1}(1 - \mu_{t-1,t-1}^*)} \right] + R \left[ \frac{\mu_{t-1,t-1}^*}{b_{t-1}(1 - \mu_{t-1,t-1}^*)} - \frac{\mu_{t-1,t}^*}{b_t(1 - \mu_{t-1,t}^*)} \right].$$

Otherwise, we have only a portion of previous stock holders investing in the stock market again, which means

$$SP_t = 1 - \frac{\mu_{t,t}^*}{b_t(1 - \mu_{t,t}^*)} + \frac{P_t + s_t}{P_{t-1}} \left[ 1 - \frac{\mu_{t-1,t}^*}{b_t(1 - \mu_{t-1,t}^*)} \right]$$

Obviously, the model is much more complicated.

## B. Proofs and Derivations

*Proof of Lemma 1:* When there is an encouraging (discouraging) signal, every one would be more (less) optimistic than before. First, there would be more (less) demand if the stock price were the same. As a result, the stock prices would be higher (lower) and there is a positive (negative) capital gain. Due to the assumption on capital endowment, higher (lower) stock price implies more (less) stock investors. Note that the previous marginal agents are indifferent between buying stocks and investing in the risk-free asset. Since the stock price would be higher (lower) after the arrival of new signal, the rate of return from stock investment would be smaller (larger). Therefore, agents with the same posterior beliefs as the previous marginal guys would invest in the risk-free asset (buy stock), and the new cutoff beliefs would be higher (lower) than before. Mathematically, the first result implies  $1 - F(\mu_{t+1}^*; \Omega_{t+1}, s_{t+1}) > (<) 1 - F(\mu_t^*; \Omega_t, s_t)$ , and the second one means  $\mu_{t+1}^* > (<) \mu_t^*$ . Due to the belief updating mechanism describe in equation (3.2), we have  $F(\mu_t^*; \Omega_t, s_t) = F\left(\frac{m_{t+1}\mu_t^*}{1+(m_{t+1}-1)\mu_t^*}; \Omega_{t+1}, s_{t+1}\right)$ , since the  $t$ -generation agent with a posterior belief of  $\mu_t^*$  would have the same prior belief as the  $t+1$ -generation agent with a posterior

belief of  $\frac{m_{t+1}\mu_t^*}{1+(m_{t+1}-1)\mu_t^*}$ . With the above knowledge, we can derive inequality (3.13) easily.

**Q.E.D.**

*The brute-force (iteration) method to solve the BLEP process:*

Consider the mapping  $T$  from bounded continuous function to itself satisfying

$$(TP^k)(b_t) = \frac{[\mu_t^*\theta_H + (1 - \mu_t^*)\theta_L] + \mu_t^*E[P^k(b_{t+1})|b_t, \theta_H] + (1 - \mu_t^*)E[P^k(b_{t+1})|b_t, \theta_L]}{R}$$

where

$$\mu_t^* = \frac{b_t[1 - S(TP^k)(b_t)]}{1 + b_t[1 - S(TP^k)(b_t)]}.$$

We choose  $p^0$  as the EPV price function (3.23), and compute  $p^k$  recursively. While the convergence rate for the sequence  $\{p^k\}$  is quite fast, the numerical solution for the BLEP is easy to obtain.

*The derivation of equation (3.23):*

From equations (3.22) and (3.14), we have

$$1 - \frac{\mu_t^*}{b_t(1-\mu_t^*)} = S[\mu_t^*V_H + (1 - \mu_t^*)V_L]$$

For computational convenience, we can define

$$a_t^* = \frac{\mu_t^*}{1-\mu_t^*}$$

and get

$$1 - \frac{a_t^*}{b_t} = S\left[\frac{a_t^*V_H}{1+a_t^*} + \frac{V_L}{1+a_t^*}\right]$$

or

$$\begin{aligned} 0 &= S[a_t^*V_H + V_L] - \left(1 - \frac{a_t^*}{b_t}\right)(1 + a_t^*) \\ &= \frac{(a_t^*)^2}{b_t} - \left(1 - \frac{1}{b_t} - SV_H\right)a_t^* + SV_L - 1 \end{aligned}$$

Note that  $S < 1$ , and  $V_L < 1$ , the quadratic equation has two real roots with different

signs. While only the positive root is meaningful here, we have

$$a_t^* = \frac{1}{2}(b_t - 1 - SV_H b_t) + \frac{1}{2}\sqrt{(b_t - 1 - SV_H b_t)^2 + 4b_t(1 - SV_L)}$$

Putting it into (3.6) yields

$$\begin{aligned} SP_t &= 1 - \frac{a_t^*}{b_t} = \frac{1}{2} \left[ 1 + SV_H + \frac{1}{b_t} - \sqrt{\left(1 - \frac{1}{b_t} - SV_H\right)^2 + \frac{4}{b_t}(1 - SV_L)} \right] \\ &= \frac{1}{2} \left[ 1 + SV_H + \frac{1}{b_t} - \sqrt{\left(1 + \frac{1}{b_t} - SV_H\right)^2 + \frac{4S}{b_t}(V_H - V_L)} \right] \end{aligned}$$

with

$$\lim_{b_t \rightarrow \infty} SP_t = \min\{1, SV_H\}.$$

We can rewrite the expression as

$$SP_t = \frac{2b_t SV_H + 2SV_L}{b_t + b_t SV_H + 1 + \sqrt{(1 + b_t - b_t SV_H)^2 + 4b_t S(V_H - V_L)}},$$

and thus obtain

$$\lim_{b_t \rightarrow 0} SP_t = SV_L.$$

Defining

$$x_t = \frac{1}{b_t},$$

we have

$$\frac{d(SP_t)}{dx_t} = 1 - \frac{2(x-1+SV_H)+4(1-SV_L)}{2\sqrt{(1-x-SV_H)^2+4x(1-SV_L)}} < 0$$

because

$$\begin{aligned} &(1 - x - SV_H)^2 + 4x(1 - SV_L) - [(x - 1 + SV_H) + 2(1 - SV_L)]^2 \\ &= 4x(1 - SV_L) - 4(x - 1 + SV_H)(1 - SV_L) - 4(1 - SV_L)^2 \\ &= 4x(1 - SV_L)(SV_L - SV_H) < 0. \end{aligned}$$

Hence  $P_t$  is strictly increasing in  $b_t$ . When  $b_t$  tends to infinity,  $SP_t$  goes to  $\min\{1, SV_H\}$ ; when  $b_t$  tends to zero,  $SP_t$  goes to  $SV_L$ .

In addition, we can compute the belief of a marginal investor

$$\begin{aligned}
\mu_t^* &= \frac{a_t^*}{1+a_t^*} = 1 - \frac{1}{1+a_t^*} \\
&= 1 - \frac{2}{(b_t+1-SV_H b_t) + \sqrt{(b_t-1-SV_H b_t)^2 + 4b_t(1-SV_L)}} \\
&= 1 - \frac{2}{(b_t+1-SV_H b_t) + \sqrt{(b_t+1-SV_H b_t)^2 + 4b_t S(V_H - V_L)}} \\
&= 1 - \frac{\sqrt{(b_t+1-SV_H b_t)^2 + 4b_t S(V_H - V_L)} - (b_t+1-SV_H b_t)}{2b_t S(V_H - V_L)}
\end{aligned}$$

with

$$\lim_{b_t \rightarrow 0} \mu_t^* = 0; \quad \lim_{b_t \rightarrow \infty} \mu_t^* = \begin{cases} 1 & \text{if } SV_H < 1 \\ \frac{1-SV_L}{S(V_H - V_L)} & \text{if } SV_H \geq 1 \end{cases}$$

and  $\mu_t^*$  strictly increasing in  $b_t$ .

*A numerical example*

Let

$$P_{t+1}^L = \int_{-\infty}^{+\infty} P(b_{t+1,L}) \phi(s_{t+1}; \theta_L, \sigma^2) ds_{t+1}, \quad P_{t+1}^H = \int_{-\infty}^{+\infty} P(b_{t+1,H}) \phi(s_{t+1}; \theta_H, \sigma^2) ds_{t+1}$$

we have

$$\begin{aligned}
&RP(b_t)\{1 + b_t[1 - SP(b_t)]\} - b_t[1 - SP(b_t)]\theta_H - \theta_L \\
&= P_{t+1}^L + b_t[1 - SP(b_t)]P_{t+1}^H.
\end{aligned}$$

Then

$$Rb_t SP_t^2 - P_t(R + Rb_t + b_t S \theta_H + b_t SP_{t+1}^H) + \theta_L + P_{t+1}^L + b_t P_{t+1}^H + b_t \theta_H = 0$$

When  $P_t = 0$ , the left hand side is positive. When  $P_t = 1/S$ , the left hand side is

$$\begin{aligned}
&Rb_t - (R + Rb_t + b_t \theta_H S + b_t P_{t+1}^H S) + S \theta_L + SP_{t+1}^L + b_t SP_{t+1}^H + b_t S \theta_H \\
&= -R + S \theta_L + SP_{t+1}^L < -R + \theta_L + 1 < 0
\end{aligned}$$

since  $S < 1$ ,  $SP_{t+1}^L < 1$ , and  $\theta_L < R - 1$ . Hence there is one root in  $(0, 1/S)$  with



another in  $(1/S, \infty)$ . Obviously only the smaller one is reasonable, since the other leads to  $SP_t > 1$ .

To figure out the number of shares,  $S$ , just set  $P(b_0) = 1$ , which results in

$$\begin{aligned} & R\{1 + b_0[1 - S]\} - b_0[1 - S]\theta_H - \theta_L \\ = & P_1^L + b_0[1 - S]P_1^H. \end{aligned}$$

$$S = 1 - \frac{R - \theta_L - P_1^L}{b_0(\theta_H + P_1^H - R)}.$$

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## CHAPTER IV

# A WAVELET ANALYSIS ON PATTERNS OF STOCK MARKET COMOVEMENTS

### 4.1 Introduction

The excess comovement between international stock markets is among the intriguing puzzles in finance. Based on the seminal work of Lucas (1978), stock prices are believed to be determined by fundamental factors. However the empirical evidence suggests the existence of non-fundamental factors in stock pricing.

Among all the fundamental factors, the discounted present value of the dividend flow may be the most important one. Shiller (1989) constructs the fundamental value of a stock accordingly and discovers that the (detrended) US and UK stock indices exhibit excess comovement, by comparing their covariance with that of the fundamental values. A bigger set of fundamental factors, including interest rates, exchange rates, inflation, GNP or industrial production, etc., is adopted later. Among them, Pindyck and Rotemberg (1993) find excess comovements in the OLS residuals for firms with unrelated earnings, even after controlling the expectations of GNP growth and inflation. King, Sentana and Wadhvani (1994) employ a GARCH-based multifactor model on stock markets in 16 developed economies. Only a small proportion of the stock market comovements can be accounted for macroeconomic indicators.

Ideally, we can isolate the non-fundamental factors by controlling other fundamental factors, such as tax rates of dividend income and capital gains, the creation of new investment tools, etc. However, the increase in the number of those factors would undermine statistical significance of the results. Moreover, some of the effects may be non-linear and will consume more data to control it correctly. In contrast to the *direct* study mentioned above, we propose an *indirect* approach based on wavelet analysis.

Wavelet filters enables us to decompose the original data across *time scales*, such as bi-daily, weekly, monthly, quarterly, semiannual, annual, etc., with a nice energy preservation property such that the sum of the variances of component series equals to that of the original data. The stock price changes due to transitory shocks will be contained in the high-frequency components (*i.e.* bi-daily or weekly), while those caused by permanent shocks are captured in the low-frequency components (*i.e.* quarterly, semiannual or annual). As a result, we can employ wavelet analysis to study the pattern of the comovements across the scales, and obtain some idea about what kind of shocks are the main driving forces of stock market comovements.

This paper is among the first few papers that employ wavelet analysis to investigate both the existence and the patterns of excess comovements across time scales. The basic idea comes from the observation that the fundamental changes usually have more sustained effects while the transitory shocks would decay much faster. The major components of non-fundamental factors, such as *signaling noises* and *expectation errors* usually disappear after the related true information is revealed publicly.<sup>1</sup> For instance, after the firms announce their earnings, or the Fed decides the primary interest rate, the related rumors would have no effect on stock prices any more.

The recent development in wavelet analysis, a useful tool in signal process, helps us to disentangle transitory shocks (high frequency components) from sustained effects (low frequency ingredients).<sup>2</sup> By making use of econometric knowledge in both time and frequency domain, wavelet analysis enables us to decompose a time series over different time scales. The component time series at each time scale level captures the changes in a specific frequency range, based on the spectral features of wavelet filters. Meanwhile, we can still use time series techniques to each component series. Percival and Walden (2000) provide a good introduction to wavelet analysis, while Gençay, Selçuk and Whitcher

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<sup>1</sup>Before the announcement of true information, we may have some signals about it, and the signaling error would lead to excessive price fluctuation. In addition, investors would observe signal related actions (or stock price changes) to update their beliefs. We regard the inefficiency in discovering the signals as expectation errors.

<sup>2</sup>In this paper, the sustained (long term) effect focuses on a period longer than a quarter, since most of macroeconomic indicators and corporate earnings are announced quarterly. The transitory (short-run) shocks capture the fluctuation in a period less than one month, because the most frequent announcement is made monthly.

(2002) and Ramsey (2000) exhibit some of its applications in finance and economics. Basically, we focus on the correlation coefficients at each scale level, but the results from measures of nonlinear interdependence, such as Kendall's tau and Spearman's rho, are qualitatively similar.

Empirical results on the US and Eastern Asian markets show that the correlation coefficients do vary across time scales, and short-run comovements are all significantly positive except for the closed markets in Mainland China, which are literally inert to foreign stock markets at most time scales. Long-term correlation is more remarkable than short-run ones in domestic markets with similar sectoral structure, while monthly and quarterly changes in Japanese, Hong Kong and Taiwan markets are more sensitive to the US counterparts than other scale levels. The degree of comovements is increasing in the openness of the markets.

Another appealing objective is the comovements of Chinese B-share markets with other markets in Greater China, because they are not officially open to Chinese mainland investors until 2001 while other Greater Chinese investors can buy B shares all the times. Hence, it provides a good sample to demonstrate our observation on the relationship between the level of comovements and the market openness.

Other empirical studies short-run comovements (daily or intraday) support our findings. Becker, Finnerty and Gupta (1990) find that the intraday returns in S&P500 can affect the performance of Nikkei index in the following trading period, but the other direction is much weaker and less significant. Controlling a bunch of information variables in their GARCH model, Karolyi and Stulz (1996) demonstrate that neither macroeconomic announcements nor interest shocks can be accounted for comovements between US and Japanese stock returns, while previous market performance has significant explanatory power to stock comovements. Among the few papers introducing wavelet analysis to the studies on stock market comovements, Lee (2001) employ wavelet regression to show the spillover from developed stock markets in US, Japan, German to the emerging markets in Turkey and Egypt in the short run (less than a week).

This paper contributes to the related literature by illustrating the *whole pattern* of stock market comovements over time scales, which generates a more vivid picture about the interaction among the international stock markets. As to the technical comparison

with Lee (2001), we focus on wavelet correlations and some measures of nonlinear interdependence, while Lee (2001) employs wavelet OLS regression. In addition, we employ MODWT filters rather than DWT ones, because the former is shown to insensitive to circular shifts.<sup>3</sup>

The paper is organized as follows. Section 4.2 introduces the MODWT method, and while the empirical results is presented in Section 4.3. Section 4.4 concludes. The sources and manipulations of the data can be found in Appendix.

## 4.2 Discrete Wavelet Transform as a Filtering Method

A general introduction of continuous and discrete wavelet analysis can be found in Percival and Walden (2000). In this paper, we employ the *maximal overlap discrete wavelet transform* (MODWT) as a filtering method. Hence a brief introduction of MODWT is provided in this section. Compared with other detrending methods, MODWT filter has additional features, such as scale-by-scale energy-preserving decomposition and better interpretation in the language of spectral analysis.

### 4.2.1 MODWT Filters

Moving average and differencing are among the commonly used detrending or filtering methods. The first order differencing is identical to the filtering by  $\{-1, 1\}$ , while the filter for the second order differencing is  $\{1, -2, 1\}$ . In general, all the differencing filters with a length of  $L$ ,  $\{h_l\}_{l=0}^{L-1}$ , satisfy

$$\sum_{l=0}^{L-1} h_l = 0. \tag{4.1}$$

---

<sup>3</sup>Another approach similar to wavelet analysis is the band spectrum regression proposed by Engle (1974). While the band spectrum regression is based on Fourier transform, we use the whole data set to compute each element in the filtered data. In contrast, wavelet analysis has better localization property since we only need the data in a small neighborhood to calculate each wavelet (or scaling) coefficient. This property enables us to associate the changes in the filtered data with the original one. Moreover, wavelet filtering method only requires local stationarity, while Fourier transform demands a global one.



Similarly, the general (weighted) moving average filter with a length of  $L$ ,  $\{g_l\}_{l=1}^L$ , satisfy

$$\sum_{l=0}^{L-1} g_l = 1. \quad (4.2)$$

Accordingly, a MODWT *wavelet filter* is a general differencing filter satisfying the half-energy condition

$$\sum_{l=0}^{L-1} h_l^2 = \frac{1}{2}, \quad (4.3)$$

and the orthogonality condition for even shifts

$$\sum_{l=0}^{L-1} h_l h_{l+2n} = \sum_{l=-\infty}^{\infty} h_l h_{l+2n} = 0, \text{ for any non-zero integer } n. \quad (4.4)$$

The corresponding MODWT *scaling filter* is generated from the wavelet filter by

$$g_l = (-1)^{l+1} h_{L-1-l}. \quad (4.5)$$

Percival and Walden (2000) demonstrated that this scaling filter is essentially a general moving average filter satisfying the half-energy condition

$$\sum_{l=0}^{L-1} g_l^2 = \frac{1}{2}, \quad (4.6)$$

and the orthogonality condition for even shifts

$$\sum_{l=0}^{L-1} g_l g_{l+2n} = \sum_{l=-\infty}^{\infty} g_l g_{l+2n} = 0, \text{ for any non-zero integer } n. \quad (4.7)$$

## 4.2.2 Pyramid Algorithm

With MODWT filters, we can generate MODWT *wavelet and scaling coefficients* at scale 1,  $\{W_{1,t}\}_{t=0}^{N-1}$  and  $\{V_{1,t}\}_{t=0}^{N-1}$ , from the original series,  $\{X_t\}_{t=0}^{N-1}$

$$W_{1,t} = \sum_{l=0}^{L-1} h_l X_{t-l \bmod N}, \text{ for } t = 0, \dots, N-1, \quad (4.8)$$

$$V_{1,t} = \sum_{l=1}^L g_l X_{t-l \bmod N}, \text{ for } t = 0, \dots, N-1. \quad (4.9)$$

Note that MODWT employs circular convolution. The notation  $a \bmod N$  means that, when the integer  $a \in [kN, kN + N - 1]$ ,  $a \bmod N$  equals  $a - kN$ , for any integer  $k$ . Obviously, MODWT coefficients generated by both beginning and ending components could be spurious. Hence, we will adjust the boundary-affected coefficients later.

Analogously, we can decompose the scaling coefficients to obtain MODWT coefficients at higher scales,

$$W_{j,t} = \sum_{l=0}^{L-1} h_l V_{j-1,t-2^{j-1}l \bmod N}, \text{ for } t = 0, \dots, N-1, \quad (4.10)$$

$$V_{j,t} = \sum_{l=0}^{L-1} g_l V_{j-1,t-2^{j-1}l \bmod N}, \text{ for } t = 0, \dots, N-1. \quad (4.11)$$

Consequently, we will have all the MODWT coefficients,  $\{\mathbf{W}_1, \dots, \mathbf{W}_J, \mathbf{V}_J\}$ <sup>4</sup>, up to scale  $J$ . We can easily reconstruct the original series from the MODWT coefficients by pyramid algorithm again, but from high scale to low scale

$$V_{j-1,t} = \sum_{l=0}^{L-1} h_l W_{j,t+2^{j-1}l \bmod N} + \sum_{l=0}^{L-1} g_l V_{j,t+2^{j-1}l \bmod N}, \text{ for } t = 0, \dots, N-1, \quad (4.12)$$

while  $\{X_t\}_{t=0}^{N-1}$  is regarded as  $\{V_{0,t}\}_{t=0}^{N-1}$ .

In addition, we can focus on one specific scale level and synthesize MODWT *detail* at scale  $j$ ,  $\mathcal{D}_j$ , from the MODWT coefficients  $\{\mathbf{0}, \dots, \mathbf{0}, \mathbf{W}_j, \mathbf{0}, \dots, \mathbf{0}\}$ , or MODWT *smooth* at scale  $J$ ,  $\mathcal{S}_J$ , from  $\{\mathbf{0}, \dots, \mathbf{0}, \mathbf{V}_J\}$ . Since MODWT filtering is a linear operator, it is easy to have the so-called *multiresolution analysis* (MRA)

$$\mathbf{X} = \sum_{j=1}^J \mathcal{D}_j + \mathcal{S}_J. \quad (4.13)$$

---

<sup>4</sup>The bold or calligraphic variables indicate the vectors of related series.

### 4.2.3 Features of MODWT

#### Energy-Preserving Multi-Scale Decomposition

The half-energy condition and orthogonality condition for even shifts provide additional features of MODWT over the general moving average and differencing method. The above multi-scale decomposition via pyramid algorithm has a beneficial energy-preserving property

$$\|\mathbf{X}\|^2 = \sum_{j=1}^J \|\mathbf{W}_j\|^2 + \|\mathbf{V}_J\|^2, \quad (4.14)$$

or

$$\frac{\|\mathbf{X}\|^2}{N} - \bar{X}^2 = \frac{1}{N} \sum_{j=1}^J \|\mathbf{W}_j\|^2 + \left( \frac{\|\mathbf{V}_J\|^2}{N} - \bar{X}^2 \right).$$

Note that wavelet coefficients are mean zero, while the mean of scaling coefficients is the same as that of the original series. Therefore, equation (4.14) indicates that the variance of the original series is just the sum of *wavelet variances* for MODWT wavelet and scaling coefficients at different scales.

#### Spectral Interpretation

In the language of spectral analysis, the MODWT wavelet filter  $\{h_l\}_{l=0}^{L-1}$  belongs to high pass filter focused on frequencies in the interval of  $[-2^{-1}, -2^{-2}]$  and  $[2^{-2}, 2^{-1}]$ , while the scaling filter  $\{g_l\}_{l=0}^{L-1}$  is a low pass filter for frequencies between  $-2^{-2}$  and  $2^{-2}$ . With  $\tau_j = 2^{j-1}$  denoting the scale at  $j^{\text{th}}$  level, we know that the  $\tau_j$  scale wavelet coefficients ( $\mathbf{W}_j$ ) captures most energy with frequencies in the interval of  $[-2^{-j}, -2^{-j-1}]$  and  $[2^{-j}, 2^{-j-1}]$ .

As a result, we can find the approximate relationship between wavelet variance and spectral density  $S_{\mathbf{X}}(f)$  of the original series

$$\text{var}\{\mathbf{W}_j\} \approx 2 \int_{2^{-j-1}}^{2^{-j}} S_{\mathbf{X}}(f) df \quad (4.15)$$

The use of approximation is due to the existence of leakage for MODWT filters. In application, there is no high (low) pass *only* discrete filter. Leakage is inevitable, more or less.

## MODWT vs DWT

MODWT can be regarded as a modified version of discrete wavelet transform (DWT). The DWT filters have unit energy, instead of half energy in the MODWT case. Hence the DWT filter is simply  $\sqrt{2}$  times the MODWT filters. To preserve the energy of the original series, we choose *every other* of the MODWT wavelet and scaling coefficients at each step of the pyramid algorithm. This approach is called *downsampling* by two.

In DWT analysis, the energy of details is the same as that of wavelet coefficients at the same scale level,

$$\|\mathcal{D}_j\|^2 = \|\mathbf{W}_j\|^2, \text{ for } j = 1, \dots, J. \quad (4.16)$$

The cost is that the DWT analysis might be sensitive to circular shifts due to the downsampling approach. The discarded components by the downsampling approach could contain different information from the remainder. An illustrative example is provided by Percival and Walden (2000, page 161 and 181). Moreover, as we will see later, MODWT provides a larger sample size in the wavelet variance, covariance, and correlation analysis. As a consequence, this paper focuses on MODWT.

### 4.2.4 Practical Considerations

#### Boundary-Affected Coefficients

Recall that the MODWT coefficients are generated by circular convolution. The coefficients computed from both beginning and ending data are likely to be spurious. Percival and Walden (2000) show that, if the length of filter is  $L$ , there are  $(2^j - 1)(L - 1)$  coefficients affected for  $\tau_j$ -scale wavelet and scaling coefficients, while  $(2^j - 1)(L - 1) - 1$  beginning and  $(2^j - 1)(L - 1)$  ending components in  $\tau_j$ -scale details and smooths would be affected.

There is a noteworthy remark. After deleting boundary-affected coefficients, the energy-preserving equation (4.14) does not hold any more. However, the simulations of white noise and random walk series shows that the summation of wavelet variance and the variance of scaling coefficients still accounts for about 90-110% of the variance in the original series.

## Choice of MODWT filters

There are two considerations about the filter choice: the type and the length. Percival and Walden (2000, page 196) demonstrated the artifacts in some of the DWT filters, but the problem is much mitigated in MODWT case. Daubechies *least asymmetric* (LA) MODWT filters, which are also called as *symlets*, are among the popular choices, because LA filters provide most accurate synchronization between wavelet coefficients and the original series.

The choice of filter's length is based on the trade-off between leakage and the number of boundary affected coefficients. If the length ( $L$ ) is larger, the filters are much closer to the ideal high (low) pass only filters. However, the number of boundary affected coefficients will increase, reducing the size of unaffected coefficients. In this paper, we choose LA(8) filters.

### 4.2.5 Wavelet Variance, Covariance, and Correlation

Theoretically, the *wavelet variance*,  $\nu_{j,X}^2$ , is just the variance of wavelet coefficients at scale  $\tau_j$ , as we have shown in equation (4.14). However, we know that the boundary-affected coefficients could generate spurious results. Hence the estimator of a  $\tau_j$ -scale wavelet variance is the variance estimator,  $\hat{\nu}_{j,X}^2(\tau_j)$ , for the boundary-*unaffected* coefficients conditional on mean zero.<sup>5</sup> Assuming the true series of wavelet coefficients at scale  $\tau_j$  is a Gaussian stationary process with mean zero and a spectral density function (SDF)  $S_j$ , Percival and Walden (2000) provided the asymptotic distribution for the wavelet variance estimator

$$\hat{\nu}_{j,X}^2(\tau_j) \overset{asy}{\sim} N(\nu_{j,X}^2, \frac{2A_j}{M_j}), \quad (4.17)$$

where  $M_j = N - (2^j - 1)(L - 1)$  is the number of unaffected coefficients, and

$$A_j = \int_{-1/2}^{1/2} S_j^2(f) df < \infty.$$

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<sup>5</sup>It is the so called biased variance estimator. Since the true mean of the series is known to be zero, we can save one degree of freedom.

In application,  $A_j$  is estimated by the sum of squared autocovariance sequence (ACVS). We can construct confidence interval accordingly. Recognizing that the confidence interval based on asymptotic normal distribution may contain negative values, Percival and Walden (2000) also provided an asymptotic  $\chi^2$  distribution, but it seems difficult to find the best choice of equivalent degrees of freedom. Serroukh *et al* (2000) relaxed the Gaussian condition and proposed another estimator of wavelet variance, along with a corresponding asymptotic normal distribution, by employing multitaper spectrum analysis with Slepian tapers.

As for a nonstationary series with its  $d^{\text{th}}$  order difference strictly stationary (or  $d$ -stationary for shorthand notation), Serroukh *et al* (2000) demonstrated that as long as the length of the Daubechies filter ( $L$ ) is larger than  $2d$ , the series of MODWT wavelet coefficients is also strictly stationary at each scale level. If the  $d$ -stationarity only holds locally, we can only this result at the corresponding time periods.

Analogously, Whitcher *et al* (1999, 2000) constructed wavelet covariance

$$\gamma_{j,XY} = \text{cov}\{\mathbf{W}_j^{(\mathbf{X})}, \mathbf{W}_j^{(\mathbf{Y})}\}$$

and wavelet correlation  $\rho_{j,XY}$  based on the wavelet coefficients. They demonstrated scale-by-scale covariance decomposition similar to equation (4.14) for two stationary time series  $\mathbf{X}$  and  $\mathbf{Y}$

$$\text{cov}\{\mathbf{X}, \mathbf{Y}\} = \text{cov}\{\mathbf{V}_J^{(\mathbf{X})}, \mathbf{V}_J^{(\mathbf{Y})}\} + \sum_{j=1}^J \gamma_{j,XY}. \quad (4.18)$$

The correlation is simply

$$\rho_{j,XY} = \frac{\gamma_{j,XY}}{\nu_{j,\mathbf{X}}\nu_{j,\mathbf{Y}}}. \quad (4.19)$$

The estimators are just the covariance and correlation of the boundary-unaffected wavelet coefficients. If the both wavelet coefficients are Gaussian stationary process with square integrable autospectra, Whitcher *et al* (2000) showed the

$$\hat{\gamma}_{j,XY} \stackrel{asy}{\sim} N(\gamma_{j,XY}, \frac{\mathcal{V}_j}{M_j}) \quad (4.20)$$

and

$$\hat{\rho}_{j,\mathbf{XY}} \overset{asy}{\sim} N\left(\rho_{j,\mathbf{XY}}, \frac{R_j}{M_j}\right) \quad (4.21)$$

where

$$\mathcal{V}_j = \int_{-1/2}^{1/2} S_{j,\mathbf{X}}(f)S_{j,\mathbf{Y}}(f)df + \int_{-1/2}^{1/2} S_{j,\mathbf{XY}}^2(f)df.$$

We skip the tedious expression of  $R_j$ , which can be found in Whitcher *et al* (1999). Moreover, the confidence In application, Whitcher *et al* (2000) developed a cleaner asymptotic distribution for  $\tanh^{-1}(\hat{\rho}_{j,\mathbf{XY}})$  via Fisher's  $z$  transformation

$$\tanh^{-1}(\hat{\rho}_{j,\mathbf{XY}}) \overset{asy}{\sim} N\left(\tanh^{-1}(\rho_{j,\mathbf{XY}}), \frac{1}{\tilde{M}_j - 3}\right), \quad (4.22)$$

where  $\tilde{M}_j = \lceil 2^{-j}N - (L-2)(1-2^{-j}) \rceil$  is a proxy for the number of independent samples, and the operator  $\lceil \cdot \rceil$  takes the integer part of the real number inside.

However, since  $\tilde{M}_j$  is the number of DWT coefficients and that of MODWT is close to  $2^j \tilde{M}_j$ , the confidence interval based on (4.22) is quite conservative for high scale levels. In this paper, we also compute more aggressive confidence interbals based on the number of MODWT  $M_j = N - (2^j - 1)(L - 1)$ , and report the results. The true confidence interval should lie between the two approaches. As a part of our future research, we will investigate which one is closer to the truth by simulations.

## 4.2.6 Investigation of Nonlinear Dependence

While correlation coefficients can only detect linear dependence, it may not reveal the entire dependence structure. As a result, we also calculate Kendall's tau and Spearman's rho, two of the popular measure for nonlinear dependence.

According to Nelsen (1999), Kendall's tau for the pared random variable  $(X, Y)$  with a realization of  $\{(x_i, y_i)\}_{i=1}^n$  is defined as

$$\tau_{X,Y} = \Pr[(X_1 - X_2)(Y_1 - Y_2) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_2) < 0] \quad (4.23)$$

while the estimator is

$$\hat{\tau}_{X,Y} = \frac{2}{n(n-1)} \{ \#[(x_i - x_j)(y_i - y_j) > 0] - \#[(x_i - x_j)(y_i - y_j) < 0] \}. \quad (4.24)$$

Spearman's rho is

$$\rho_{X,Y} = 3 \{ \Pr[(X_1 - X_2)(Y_1 - Y_3) > 0] - \Pr[(X_1 - X_2)(Y_1 - Y_3) < 0] \}. \quad (4.25)$$

While the number of computations is  $(n-2)/3$  times that of  $\hat{\tau}$ , it is burdensome to calculate the estimator similar to equation (4.24). Nelsen (1999) shows that Spearman's rho is essentially the "grade" correlation coefficient

$$\rho_{X,Y} = \frac{E[F(X)G(Y)] - E[F(X)]E[G(Y)]}{\sqrt{\text{var}[F(X)] \text{var}[G(Y)]}}, \quad (4.26)$$

where  $F(\cdot)$  and  $G(\cdot)$  are the cumulative density functions (CDF) for  $X$  and  $Y$  respectively. Hence we can estimate  $\rho_{X,Y}$  with empirical CDFs. The results, which are qualitatively similar to those based on correlation coefficients, are listed in Table 4.3 at the end of this chapter.<sup>6</sup>

## 4.2.7 Performance in Heuristic Examples

Since MODWT analysis is still a new tool, we simulate some examples to explore its empirical performance. First, we construct a series with a single transitory shock and another one with a single sustained shock. The lengths are both 4096, while the shocks happen at 2049. The lengths are close to the size of our data set, and the impact from deleting boundary affected coefficients is mitigated as well. We employ the LA(8) filter to obtain wavelet coefficients up to scale  $\tau_6$ . In the transitory shock case,  $\mathbf{W}_1$  accounts for 50.34% of the variance, while  $\mathbf{V}_6$  captures only 2.44% of it. By contrast,  $\mathbf{V}_6$  of sustained shock series captures 94.02% variance of the original series. Actually it is the simplest regime switching series, which is non-stationary. Hence,  $\mathbf{V}_6$  of sustained shock

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<sup>6</sup>The nonlinear interdependence measures do *not* general similar results as correlation coefficients in all cases. Counter-examples can be found in Nelsen (1999).



series is also nonstationary. In this regard, we focus on the composition of the variances among wavelet coefficients only. The results are listed in Table 4.1. Although leakage is inevitable, it decays at a fast rate (about half to the adjacent scale).

Another pair of examples is Gaussian white noise and random walk. We generate both series from the same set of random data drawn from the standard normal distribution. We simulate twice to have a better idea about the truth, and also figure out the energy in the corresponding spectral density given by equation (4.15). Note that the spectral density function (SDF) for Gaussian white noise with variance one is  $S_{wn}(f) = 1$ , and SDF for the random walk we generate is  $S_{rw}(f) = (1/4) \sin^{-2}(\pi f)$ , where  $f \in [-1/2, 1/2]$ . The results are given in Table 4.1. Actually, it is difficult, if not impossible, to differentiate white noise series from the series with single transitory shock, or random walk series from the sustained shock series. We can interpret the phenomenon by the similar properties of the shocks. Moreover, MODWT coefficients seem to seize as much energy as their counterparts in spectrum analysis, except for the **W1** in the random walk case. It means that the leakage of the LA(8) filter is negligible.

We also check the variance composition for the stock indices we have. The empirical results suggest that the levels of (log) indices behave almost the same as the sustained stock series or the random walk series, while the daily returns (*i.e.* the first difference of the log indices) perform like transitory shock series. It suggests that the (log) indices are close to the series with unit root.

By investigating a correlated bivariate stationary Gaussian series and a corresponding Gaussian random walk series with a correlation coefficient of 0.5,<sup>7</sup> we find that the performance of wavelet correlation estimator is quite reasonable. The true correlation coefficient lies within all of the 95% confidence intervals, while the wavelet correlations are alike for stationary Gaussian series and Gaussian random walk series generated by the same set of random data. In fact, we repeat it three times, and the wavelet correlation estimator at the first scale level ranges from 0.4573 to 0.5469, while all the 95% confidence intervals are contained in [0.3140, 0.6579]. The confidence intervals become

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<sup>7</sup>The correlated series are generated from two standard normal series, say  $X$  and  $Y$ . Let  $A = X + rY$ , and  $B = X - rY$ . We have  $\text{var}(A) = \text{var}(B) = 1 + r^2$ , and  $\text{cov}(A, B) = 1 - r^2$ . Hence  $\rho_{A,B} = \frac{1 - r^2}{1 + r^2}$ . Given the level of the correlation coefficient, we can find the value of  $r$  to construct two correlated series.

<b>Series</b>	<b>W1</b>	<b>W2</b>	<b>W3</b>	<b>W4</b>	<b>W5</b>	<b>W6</b>
<b>Transitory Shocks</b>						
Single Transitory Shock	50.42%	25.30%	12.74%	6.46%	3.32%	1.76%
White Noise (Median)	51.01%	25.72%	12.70%	6.26%	3.23%	1.56%
10 percentile	49.02%	24.85%	12.37%	5.70%	2.73%	1.38%
90 percentile	51.70%	26.22%	13.52%	6.81%	3.34%	2.07%
Spectral Approximation	50.79%	25.40%	12.70%	6.35%	3.17%	1.59%
<b>Sustained Shocks</b>						
Single Sustained Shock	2.30%	3.23%	6.01%	11.97%	24.51%	51.98%
Random Walk (Median)	2.61%	3.63%	6.75%	12.80%	25.90%	49.50%
10 percentile	2.10%	3.06%	5.52%	10.95%	23.70%	44.72%
90 percentile	2.87%	4.04%	7.50%	14.62%	26.95%	53.86%
Spectral Approximation	2.45%	3.47%	6.41%	12.58%	25.05%	50.03%

Table 4.1: Energy Composition for Simulated Series

larger at higher scale level, and the estimator’s performance is getting worse accordingly.

### 4.3 Empirical Results

We focus on the stock markets in US, Japan, Hong Kong, Taiwan, and Mainland China. The physical trades are active among these economies. These economies also have some similarity in sectoral structures, in cultures as well as in psychological features. On the other hand, these markets have different degree of global integration, which is believed to be important in comovements of international stock markets. Japanese and Hong Kong markets are open and highly integrated with other advanced markets, while Taiwan stock market can be regarded as semi-open due to its qualified foreign institutional investors (QFII) regulation, which sets an upper limits for the total amount of investments and the proportion of shares in a specific firm. The stock markets in Mainland China are virtually closed to foreigners.

We investigate two types of stock market comovements. Firstly, we focus on the stock markets with the same economy, in an effort to find Long-term comovements driven by fundamentals. Secondly, we consider to what extent the price changes in the US markets would affect the Eastern Asian markets in the subsequent trading periods.

<b>Series</b>	<b>W1</b>	<b>W2</b>	<b>W3</b>	<b>W4</b>	<b>W5</b>	<b>W6</b>
<b>Level of Indices</b>						
NASDAQ	2.32%	3.59%	6.66%	9.75%	20.89%	56.79%
S&P500	3.14%	4.80%	8.78%	13.47%	28.82%	40.99%
Nikkei (Japan)	3.05%	4.14%	7.30%	12.64%	24.21%	48.66%
Hang Seng (Hong Kong)	2.05%	2.80%	5.52%	10.01%	22.08%	57.53%
TWII (Taiwan)	1.96%	2.84%	6.72%	10.83%	25.32%	52.33%
SSEC (Shanghai)	2.20%	3.16%	6.80%	16.09%	27.74%	44.01%
SZSC1 (Shenzhen)	1.91%	2.72%	5.78%	13.43%	24.91%	51.25%
<b>Daily Returns</b>						
NASDAQ	48.95%	26.82%	14.36%	5.61%	2.54%	1.70%
S&P500	49.42%	26.85%	14.27%	5.51%	2.87%	1.07%
Nikkei (Japan)	51.87%	26.45%	12.38%	5.41%	2.55%	1.34%
Hang Seng (Hong Kong)	50.98%	24.02%	13.83%	5.81%	3.50%	1.86%
TWII (Taiwan)	48.74%	23.20%	15.74%	6.72%	3.74%	1.86%
SSEC (Shanghai)	48.21%	24.93%	12.90%	8.70%	3.71%	1.54%
SZSC1 (Shenzhen)	48.36%	24.49%	12.97%	8.40%	3.77%	2.02%

Table 4.2: Energy Composition of Stock Indices

Observe that the trading time of Eastern Asian markets are close to each other, while none of them share the same trading time with the US markets. The news from the US markets arrives to the Eastern Asian markets almost the same time. Hence the above Eastern Asian economies provide a good set of samples for the comovement study on international markets.

The sources and manipulations of data are explained in Appendix. Actually, we have to delete some data to match the trading days. Moreover, we use the US indices in the preceding days to study the contagion from US markets to other markets.<sup>8</sup>

### 4.3.1 Stock Markets in the Same Economy

Before we go internationally, we will study the domestic markets first. In order to have some idea about the domestic market comovements in general, we study two advanced stock markets in the US, and two closed emerging markets in Mainland China.

The first finding is that the wavelet correlation coefficients do vary over time scales. We can easily find two non-overlapping 95% confidence intervals. This feature is shared

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<sup>8</sup>One of the next scheduled tasks is to check the feedbacks from other markets to US markets. Empirical studies by other economists suggest that the feedback would be quite weak.

by all the wavelet correlation analysis we have conducted on the above stock markets. While it is difficult to obtain similar results, the employment of wavelet analysis does provide special information from the time series.

Secondly, the wavelet correlation for domestic markets seems to increase over the length of time scales. In the case of US markets from February 5, 1971 to July 31, 2003 (Figure 4.1a), the wavelet correlations are 0.7537 and 0.7913 at the first two scale levels, while that of the 8th level record as high as 0.9015. Note that the  $k$ th scale level focuses mainly on changes over  $2^k$  days in the LA(8) case. The 8th level wavelet coefficients approximately capture the changes of annual returns, since the number of trading days lies between 230 and 250 in most stock markets. Hence the comovements of NASDAQ and S&P500 are stronger in the long run than in the short run. It could be attributed to the announcement of macroeconomic indicators and/or the learning about sectoral performance from quarterly-reported corporate earnings. We find similar results for Shanghai and Shenzhen stock markets (June 1, 1992 to July 31, 2003) in Mainland China, where the wavelet correlations are above 0.9 for the 6th (quarterly) and 7th (semiannual) scale levels, and below 0.83 otherwise (Figure 4.2a, 4.2b).

In contrast, the long-term correlations are significantly lower than their short-run counterparts for NASDAQ and S&P500 from 1992 to 2003 (Figure 4.1b). It makes sense if we consider the sectoral structure of the two markets. In 1990's, more and more IT and internet companies are listed in NASDAQ, making its share of companies in traditional sectors much lower than that of S&P500. The different levels of exposure to sectoral shocks offset some of the synchronization driven by the macroeconomic environment. Observe that the corporate earnings are reported quarterly, which is in line with the dive of wavelet correlation at the 6th level.

In both cases of domestic comovements, short-run correlations are still above 0.6, which supports common empirical results in the study of excess comovements over the fundamentals-driven prices. The driving forces may include the infection of expectation errors and the common signaling noises. Some of the short-term comovements are driven by the arrival of common signals. Moreover, since many investors (institutional and individual) invest in both domestic markets, the learning process of signals is the same for them before they make decision on either market, so is the expectation error generated

from the learning. Psychological factors could be nested in the learning process. So far we fail to disentangle the learning errors from the signaling noises. However, the study on international stock market comovements sheds some light on this issue.

### 4.3.2 International Comovements

As mentioned above, we investigate the impact from US market performance to Eastern Asian markets. Now the signaling noises are under reasonable control thanks to the non-overlapping trading time and similar sectoral structures in Japan and Great China markets.

With no surprise, the correlation increases in the degree of global integration, with the Chinese mainland markets uncorrelated with most of the other markets. A market with higher degree of global integration implies larger proportions of multi-national investors, and hence more sensitive to US markets due to our within-group infection conjecture.

However, the cross-scale behaviors of wavelet correlations suggest more complexity. As shown in Figure 4.3 and 4.4, Hong Kong and Japanese markets are more sensitive to US market changes at the fifth (approximately 1.5 month) and sixth (quarterly) scale than either the seventh (semiannual) scale or the first few ones. For Taiwan market, the third (8 days, or 1.6 week) and fourth (16 days) scales provide the highest correlation (Figure 4.5). The drop of correlation coefficients at seventh (semiannual) scale excludes the fundamental factors as the only driving forces of comovements at shorter time scales. However, we also find low correlation at first few scale levels. The reason may lie in the review and decision mechanism within multi-national institutional investors. The execution group may focus more on local events, while the decision-making committee would review the signals more globally on a monthly or quarterly basis.

The interaction among Japanese, Hong Kong and Taiwan markets, illustrated in Figure 4.8, are of the same sizes as the sensitivity of Taiwan market to US markets. All the correlations are below 0.4 except for two cases. The seventh-scale (semiannual) correlation between Japanese and Taiwan markets is 0.5195, while Hong Kong and Taiwan markets comove with a correlation of 0.5432 at the sixth level (quarterly).

Chinese Mainland markets are quite closed, so their interactions with other markets

are modest. With most wavelet correlations very small (between -0.2 to 0.2), we fail to reject the hypothesis of zero correlation at 95% level in most cases (Figure 4.6, 4.7, 4.9, 4.10, 4.11). However, there do exist two exceptions. One is the seventh-level (semianual) correlations between Mainland markets and Taiwan market at 0.2636 and 0.3188, respectively (Figure 4.11). The other is the comovement at the seventh level (semianually, *again*) between Chinese Mainland markets and Japanese Nikkei Index at 0.2027 and 0.2267 respectively (Figure 4.9). Actually, in the subsequent subsection, we show that neither case is robust under the conservative approach.

Owing to the closedness of Chinese mainland markets, there is no institutional investor across the Taiwan Strait. Hence, it could only be driven by fundamental factors, such as similar structure of their exports to US. This finding sheds some lights on the explanation for the other anomalies within the Eastern Asian markets.

Compared with institutional investors, individual investors trading in two markets are much less influential. We can look at the results between Chinese Mainland markets and Hang Seng Index. Actually, Shenzhen is a city adjacent to Hong Kong, and there are several individual investors trading on both markets. However, the stock market there is only more sensitive than Shanghai market at the first level (less than 2 days), with a correlation at 0.0921 against 0.0335. The difference is marginally significant since the corresponding 95% confidence intervals overlap each other, but none contains both estimated values.

### 4.3.3 Robustness under the Conservative Approach

We also compute the confidence intervals proposed by Whitcher *et al* (2000), and find that most of our findings are still robust. By comparing the results, we find 120 correlations coefficients, out of 162, are significant from zero under both approaches. For the remaining, only 9 cases in the first four scale levels, while the infected correlation coefficients are all below 0.08. In contrast, their counterparts in the fifth, sixth and seventh scale level account for 9, 14, and 10 cases respectively, with highest infected correlation coefficients at 0.1984, 0.3060, and 0.4539. In addition, we find 31 infected case associated with Chinese Mainland markets. According to the conservative approach, Chinese

Mainland market has only one significant interaction with the others, which is between Shenzhen and Hong Kong at the first level. This is in line with the closedness of these markets.

## 4.4 Conclusion

In this paper, we investigate wavelet correlations of domestic and international stock markets, as well as two measures of nonlinear dependence. Not surprisingly, fundamentals (factors with sustained effects) play an important role in the domestic market comovements given similar sectoral structure for the listing companies. On the other hand, we do find significant short-run comovements, which are not incorporated in the prevailing asset pricing models. Monthly and quarterly comovements among open markets are relatively more significant. This implies that the investors with monthly or quarterly adjustment of their portfolios are more sensitive to the preceding performance in the US markets. Our results suggest that the degree of stock market comovements is increasing in the openness of the markets.<sup>9</sup>

One concern in our study is the nonstationarity in daily returns. Although the LA(8) filter guarantees the wavelet coefficients are all mean zero, it fails to remove the heteroscedasticity in the data. Possible approaches to deal with this issue include investigating some subperiods or some proxies of daily correlation.

Tail dependence is another interesting feature for stock market comovements. It focuses on the level of contagion when one of the markets experiences a sharp drop or a big jump. We expect that the tail dependence would also differ across the time scales. This exercise, along with further investigation of confidence intervals for wavelet correlations and other measure of dependence, is among our future research.

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<sup>9</sup>As mentioned above, the study on the comovements between Chinese B-share markets with other Greater Chinese counterparts would shed more light on the relationship between market openness and the level of comovements. So far the results are too preliminary to report, but it would be completed soon.

# Appendix

## **The Source of the Data**

Basically the stock indices are collected from Yahoo! Finance. The symbols of the indices we use in this paper are listed in Table 4.3.

The earlier NASDAQ and S&P500 indices are collected from [www.economagic.com](http://www.economagic.com), and the stock indices for Taiwan, Shanghai, and Shenzhen stock exchanges in the early 1990's are obtained from their websites: [www.tse.com.tw](http://www.tse.com.tw), [www.sse.com.cn](http://www.sse.com.cn) and [www.sse.org.cn](http://www.sse.org.cn), respectively.

## **Manipulations of the Data**

There are 8202 stock index data for NASDAQ and S&P500 from February 5, 1971 to July 31, 2003, and 2698 for Chinese Mainland markets (Shanghai and Shenzhen) from June 1, 1992 to July 31, 2003. These data are used to obtain Figure 1a and 2a.

Since we have different holidays in there markets, we have to delete the unmatched trading dates. After this operation, we have 2330 data for all the stock indices. The remaining figures are computed based on these adjusted data. Of course it will distort the result to some extent. Nonetheless we can find that the deletion of the unmatched dates does not change much of the results by comparing Figure 2a and 2b.

As per Percival and Walden (2000), we also remove the boundary affected wavelet coefficients. The numbers of the removed ones are 7, 21, 49, 105, 217, 441, 889, and 1785, respectively from the first to eighth scale level. To obtain more than 1000 data in each wavelet coefficient series, we generate the wavelet coefficients up to the seventh level, except for Figure 1a, where we have 8202 data points.

In this paper we use the level data to obtain the results, which is unlike the usual treatment for unit root series. Discussion is provided in the subsequent subsection.

## **Level or Differenced Data**

There is an issue about the choice of level data or the differenced data (daily returns). We usually difference the series with unit root before further study. However, it is not necessary for wavelet analysis, since the wavelet filters do the job by its general differencing operations.



Location of Stock Exchange	Stock Index	Yahoo Symbol
New York, US	NASDAQ	^IXIC
New York, US	S&P500	^SPC, or ^GSPC
Tokyo, Japan	Nikkei 225 Index	^N225
Hong Kong	Hang Seng Index	^HSI
Taipei, Taiwan	Taiwan Weighted Index	^TWII
Shanghai, P. R. China	SSE Composite Index	^SSEC
Shenzhen, P. R. China	SZSE Composite Subindex	^SZSC1

Table 4.3: List of Symbols in Yahoo! Finance

With some algebraic manipulation, equation (4.10) and (4.11) can be rewritten as

$$W_{j,t} = \left( \sum_{l=0}^{L-1} h_l B^{2^{j-1}l} \right) V_{j-1,t} = \left[ \sum_{l=0}^{L-2} \eta_l B^{2^{j-1}l} \right] (1 - B^{2^{j-1}}) V_{j-1,t}, \quad (\text{A1})$$

$$V_{j,t} = \left( \sum_{l=0}^{L-1} g_l B^{2^{j-1}l} \right) V_{j-1,t} = \left[ \sum_{l=0}^{L-2} \eta_l (-B^{2^{j-1}})^l \right] (1 + B^{2^{j-1}}) V_{j-1,t}, \quad (\text{A2})$$

for  $t = 0, \dots, N - 1$ , and  $j = 1, 2, \dots, J$ , where  $B$  is the lag operator, and

$$\eta_l = \sum_{k=0}^l h_k. \quad (\text{A3})$$

Note that the weights in (A1) have a sum of zero, the wavelet filter at  $j$ th level (generally) differences the first-order difference of lower level scaling coefficients with a scale of  $2^{j-1}$ . Analogously, the scaling coefficients are the general moving average of the sum of "adjacent"<sup>10</sup> lower level scaling coefficients. Observe that the differencing scale is the same as the moving average scale in the lower level scaling coefficients, so the wavelet coefficients are stationary for the unit root series. Hence, there is no need to difference the series before wavelet analysis. The only consideration for the choice between level and differenced data is just the economic interpretation of the corresponding wavelet coefficients.

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<sup>10</sup>The "adjacent" scaling coefficients means that the set of covered data are just near each other, but there is no common data between them. In fact, this pair of scaling coefficients is adjacent in the DWT scenario.

In the LA(8) case,  $\eta_3$  (-0.2491) and  $\eta_4$  (0.3193) are bigger in size than others. Hence, we approximate the wavelet coefficients:

$$W_{1,t} \approx (0.3193B - 0.2419)B^3(1 - B)X_t,$$

$$W_{2,t} \approx (0.4127B + 0.4742B^2 + 0.1804B^3 + 0.1190) \\ \times (0.5651B - 0.4918)B^9(1 - B^2).$$

Observe that  $W_{1,t}$  is roughly proportional to the negative of the changes in daily return, and  $W_{2,t}$  is approximately the moving average of changes in two-day return. If we use the differenced data, the wavelet coefficients are simply the first difference of their counterpart from level data. Now the interpretation is less straightforward.

Nonetheless, the correlations would be the same for both cases, if the related pair wavelet coefficients form a serially uncorrelated bivariate sequence with covariance stationarity and mean  $\mathbf{0}$ . The justification comes from the fact that

$$\text{cov}(x_t - x_{t-1}, y_t - y_{t-1}) = Ex_t y_t - Ex_{t-1} y_t - Ex_t y_{t-1} + Ex_{t-1} y_{t-1} = 2 \text{cov}(x_t, y_t).$$

and analogous results for the variances. Hence we should find identical correlation coefficients for the two cases. Note that it is only a *sufficient* condition.

In our study, the wavelet correlations are essentially unchanged in 91 out of 162 cases as per our 10% change criterion. For the remaining 70 cases, 40 are associated with correlation coefficients insignificant from zero, 20 from low correlation ( $<0.25$ ) cases, mostly associated with the two Chinese Mainland markets. It seems that Japanese and Taiwan markets are more vulnerable because all the 10 significant cases are related with at least one of them. The immediate implication is that the corresponding wavelet coefficients are serially correlated to some extent.

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**Table 4.4: Wavelet Correlation and other Measures of Interdependence for the Target Stock Markets**

**NASDAQ and S&P500  
(02/05/1971-07/31/2003)**

	W1	W2	W3	W4	W5	W6	W7	W8
Correlation	<b>0.7537</b>	<b>0.7913</b>	<b>0.8167</b>	<b>0.8361</b>	<b>0.8554</b>	<b>0.8525</b>	<b>0.8788</b>	<b>0.9015</b>
<i>Lower bound</i>	0.7402	0.7796	0.8063	0.8267	0.8469	0.8439	0.8717	0.8956
<i>Upper bound</i>	0.7666	0.8025	0.8267	0.8451	0.8634	0.8606	0.8856	0.9071
Kendall' s tau	<b>0.6009</b>	<b>0.6283</b>	<b>0.6589</b>	<b>0.6713</b>	<b>0.6858</b>	<b>0.6843</b>	<b>0.7218</b>	<b>0.7257</b>
Spearman' s rho	<b>0.7874</b>	<b>0.8168</b>	<b>0.8432</b>	<b>0.8520</b>	<b>0.8614</b>	<b>0.8569</b>	<b>0.8909</b>	<b>0.8874</b>

**NASDAQ and S&P500  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.8271</b>	<b>0.8268</b>	<b>0.8337</b>	<b>0.8299</b>	<b>0.7964</b>	<b>0.7229</b>	<b>0.7378</b>
<i>Lower bound</i>	0.8138	0.8135	0.8207	0.8165	0.7802	0.7007	0.7133
<i>Upper bound</i>	0.8395	0.8393	0.8458	0.8424	0.8115	0.7438	0.7605
Kendall' s tau	<b>0.6251</b>	<b>0.6387</b>	<b>0.6469</b>	<b>0.6291</b>	<b>0.6385</b>	<b>0.5629</b>	<b>0.5558</b>
Spearman' s rho	<b>0.8041</b>	<b>0.8192</b>	<b>0.8288</b>	<b>0.8100</b>	<b>0.8084</b>	<b>0.7355</b>	<b>0.7526</b>

Note: For the wavelet correlations we provide the lower and upper bounds of the 95% confidence interval.

**Table 4.4 (continued)**

**Chinese Mainland Markets  
(Shanghai SSEC and Shenzhen SZSC1)  
(06/01/1992-07/31/2003)**

	W1	W2	W3	W4	W5	W6	W7
<b>Correlation</b>	<b>0.6660</b>	<b>0.7119</b>	<b>0.7517</b>	<b>0.8103</b>	<b>0.7984</b>	<b>0.9013</b>	<b>0.9440</b>
<i>Lower bound</i>	0.6352	0.6845	0.7275	0.7911	0.7782	0.8907	0.9379
<i>Upper bound</i>	0.6947	0.7372	0.7740	0.8279	0.8170	0.9108	0.9495
<b>Kendall' s tau</b>	<b>0.5721</b>	<b>0.5627</b>	<b>0.6128</b>	<b>0.6543</b>	<b>0.6898</b>	<b>0.6808</b>	<b>0.8156</b>
<b>Spearman' s rho</b>	<b>0.7108</b>	<b>0.7123</b>	<b>0.7687</b>	<b>0.8200</b>	<b>0.8574</b>	<b>0.8529</b>	<b>0.9527</b>

**Chinese Mainland Markets  
(Shanghai SSEC and Shenzhen SZSC1)  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
<b>Correlation</b>	<b>0.7033</b>	<b>0.6799</b>	<b>0.7608</b>	<b>0.7989</b>	<b>0.8216</b>	<b>0.9076</b>	<b>0.9345</b>
<i>Lower bound</i>	0.6821	0.6573	0.7429	0.7833	0.8072	0.8993	0.9276
<i>Upper bound</i>	0.7233	0.7012	0.7775	0.8134	0.8349	0.9152	0.9407
<b>Kendall' s tau</b>	<b>0.5830</b>	<b>0.5638</b>	<b>0.6234</b>	<b>0.6693</b>	<b>0.6784</b>	<b>0.7245</b>	<b>0.7977</b>
<b>Spearman' s rho</b>	<b>0.7258</b>	<b>0.7109</b>	<b>0.7847</b>	<b>0.8284</b>	<b>0.8471</b>	<b>0.8907</b>	<b>0.9429</b>

**Table 4.4 (continued)**

**NASDAQ and Japan Nikkei  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
<b>Correlation</b>	<b>0.2734</b>	<b>0.3750</b>	<b>0.4222</b>	<b>0.3843</b>	<b>0.4321</b>	<b>0.4625</b>	<b>0.2604</b>
<i>Lower bound</i>	0.2353	0.3394	0.3878	0.3483	0.3968	0.4263	0.2116
<i>Upper bound</i>	0.3106	0.4095	0.4553	0.4191	0.4661	0.4972	0.3079
<b>Kendall' s tau</b>	<b>0.1712</b>	<b>0.2417</b>	<b>0.2738</b>	<b>0.2522</b>	<b>0.3021</b>	<b>0.2950</b>	<b>0.1835</b>
<b>Spearman' s rho</b>	<b>0.2509</b>	<b>0.3522</b>	<b>0.3985</b>	<b>0.3691</b>	<b>0.4349</b>	<b>0.4283</b>	<b>0.2723</b>

**S&P500 and Japan Nikkei  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
<b>Correlation</b>	<b>0.3059</b>	<b>0.3809</b>	<b>0.4552</b>	<b>0.4496</b>	<b>0.4705</b>	<b>0.4380</b>	<b>0.2791</b>
<i>Lower bound</i>	0.2685	0.3455	0.4221	0.4158	0.4366	0.4008	0.2308
<i>Upper bound</i>	0.3423	0.4153	0.4872	0.4821	0.5030	0.4737	0.3261
<b>Kendall' s tau</b>	<b>0.1727</b>	<b>0.2446</b>	<b>0.3183</b>	<b>0.3244</b>	<b>0.3264</b>	<b>0.2900</b>	<b>0.1997</b>
<b>Spearman' s rho</b>	<b>0.2528</b>	<b>0.3562</b>	<b>0.4575</b>	<b>0.4635</b>	<b>0.4753</b>	<b>0.4156</b>	<b>0.2861</b>

Table 4.4 (continued)

**NASDAQ and Hong Kong HSI**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.3255</b>	<b>0.3961</b>	<b>0.4678</b>	<b>0.4072</b>	<b>0.5839</b>	<b>0.5293</b>	<b>0.4539</b>
<i>Lower bound</i>	0.2886	0.3611	0.4351	0.3719	0.5551	0.4961	0.4119
<i>Upper bound</i>	0.3614	0.4299	0.4992	0.4413	0.6113	0.5610	0.4940
Kendall' s tau	<b>0.2002</b>	<b>0.2665</b>	<b>0.3345</b>	<b>0.3077</b>	<b>0.4190</b>	<b>0.3829</b>	<b>0.3354</b>
Spearman' s rho	<b>0.2923</b>	<b>0.3866</b>	<b>0.4844</b>	<b>0.4421</b>	<b>0.5928</b>	<b>0.5348</b>	<b>0.4954</b>

**S&P500 and Hong Kong HSI**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.3947</b>	<b>0.4549</b>	<b>0.5015</b>	<b>0.4824</b>	<b>0.5945</b>	<b>0.6009</b>	<b>0.5997</b>
<i>Lower bound</i>	0.3598	0.4219	0.4702	0.4499	0.5662	0.5712	0.5656
<i>Upper bound</i>	0.4285	0.4866	0.5316	0.5137	0.6214	0.6289	0.6318
Kendall' s tau	<b>0.2488</b>	<b>0.3132</b>	<b>0.3666</b>	<b>0.3522</b>	<b>0.4344</b>	<b>0.4622</b>	<b>0.4588</b>
Spearman' s rho	<b>0.3589</b>	<b>0.4495</b>	<b>0.5241</b>	<b>0.5015</b>	<b>0.6128</b>	<b>0.6443</b>	<b>0.6305</b>



**Table 4.4 (continued)**

**NASDAQ and Taiwan TWII  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	<b>W1</b>	<b>W2</b>	<b>W3</b>	<b>W4</b>	<b>W5</b>	<b>W6</b>	<b>W7</b>
<b>Correlation</b>	<b>0.1273</b>	<b>0.3211</b>	<b>0.3732</b>	<b>0.3902</b>	<b>0.1984</b>	<b>0.3060</b>	<b>0.0233</b>
<i>Lower bound</i>	0.0871	0.2840	0.3374	0.3544	0.1571	0.2645	-0.0284
<i>Upper bound</i>	0.1671	0.3572	0.4080	0.4249	0.2390	0.3463	0.0748
<b>Kendall' s tau</b>	<b>0.0790</b>	<b>0.2095</b>	<b>0.2418</b>	<b>0.2027</b>	<b>0.0958</b>	<b>0.2222</b>	<b>0.0005</b>
<b>Spearman' s rho</b>	<b>0.1162</b>	<b>0.3047</b>	<b>0.3515</b>	<b>0.2951</b>	<b>0.1403</b>	<b>0.3295</b>	<b>0.0034</b>

**S&P500 and Taiwan TWII  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	<b>W1</b>	<b>W2</b>	<b>W3</b>	<b>W4</b>	<b>W5</b>	<b>W6</b>	<b>W7</b>
<b>Correlation</b>	<b>0.1721</b>	<b>0.2824</b>	<b>0.3348</b>	<b>0.3749</b>	<b>0.2882</b>	<b>0.2821</b>	<b>0.1732</b>
<i>Lower bound</i>	0.1323	0.2444	0.2979	0.3386	0.2486	0.2401	0.1227
<i>Upper bound</i>	0.2113	0.3195	0.3708	0.4100	0.3268	0.3231	0.2229
<b>Kendall' s tau</b>	<b>0.1095</b>	<b>0.1796</b>	<b>0.2134</b>	<b>0.2023</b>	<b>0.1751</b>	<b>0.2113</b>	<b>0.1072</b>
<b>Spearman' s rho</b>	<b>0.1614</b>	<b>0.2606</b>	<b>0.3121</b>	<b>0.2959</b>	<b>0.2585</b>	<b>0.3151</b>	<b>0.1671</b>

Table 4.4 (continued)

NASDAQ and Shanghai SSEC  
(06/01/1992-07/31/2003, unmatched dates deleted)

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0477</b>	<b>-0.0059</b>	<b>0.0334</b>	<b>-0.0032</b>	<b>0.0048</b>	<b>0.1277</b>	<b>-0.1436</b>
<i>Lower bound</i>	0.0071	-0.0467	-0.0076	-0.0447	-0.0378	0.0831	-0.1938
<i>Upper bound</i>	0.0882	0.0349	0.0744	0.0384	0.0474	0.1718	-0.0926
Kendall' s tau	<b>0.0423</b>	<b>-0.0176</b>	<b>0.0323</b>	<b>-0.0135</b>	<b>-0.0093</b>	<b>0.0956</b>	<b>-0.0888</b>
Spearman' s rho	<b>0.0649</b>	<b>-0.0290</b>	<b>0.0513</b>	<b>-0.0199</b>	<b>-0.0166</b>	<b>0.1407</b>	<b>-0.1334</b>

S&P500 and Shanghai SSEC  
(06/01/1992-07/31/2003, unmatched dates deleted)

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0456</b>	<b>-0.0219</b>	<b>-0.0073</b>	<b>-0.0386</b>	<b>0.0056</b>	<b>0.0907</b>	<b>-0.0738</b>
<i>Lower bound</i>	0.0049	-0.0626	-0.0483	-0.0801	-0.0371	0.0458	-0.1250
<i>Upper bound</i>	0.0861	0.0189	0.0337	0.0029	0.0482	0.1353	-0.0223
Kendall' s tau	<b>0.0448</b>	<b>-0.0159</b>	<b>-0.0114</b>	<b>-0.0536</b>	<b>-0.0244</b>	<b>0.0672</b>	<b>-0.0628</b>
Spearman' s rho	<b>0.0685</b>	<b>-0.0248</b>	<b>-0.0169</b>	<b>-0.0843</b>	<b>-0.0421</b>	<b>0.1021</b>	<b>-0.0962</b>

Table 4.4 (continued)

**NASDAQ and Shenzhen SZSC1**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0561</b>	<b>0.0166</b>	<b>0.0528</b>	<b>-0.0127</b>	<b>0.0451</b>	<b>0.1401</b>	<b>-0.1930</b>
<i>Lower bound</i>	0.0155	-0.0242	0.0117	-0.0543	0.0024	0.0956	-0.2423
<i>Upper bound</i>	0.0966	0.0573	0.0936	0.0288	0.0876	0.1840	-0.1428
Kendall' s tau	<b>0.0394</b>	<b>0.0048</b>	<b>0.0351</b>	<b>-0.0206</b>	<b>0.0277</b>	<b>0.1100</b>	<b>-0.1360</b>
Spearman' s rho	<b>0.0598</b>	<b>0.0070</b>	<b>0.0532</b>	<b>-0.0325</b>	<b>0.0422</b>	<b>0.1625</b>	<b>-0.2054</b>

**S&P500 and Shenzhen SZSC1**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0498</b>	<b>-0.0182</b>	<b>-0.0304</b>	<b>-0.0513</b>	<b>0.0543</b>	<b>0.0658</b>	<b>-0.1798</b>
<i>Lower bound</i>	0.0091	-0.0589	-0.0713	-0.0927	0.0117	0.0208	-0.2293
<i>Upper bound</i>	0.0903	0.0226	0.0107	-0.0098	0.0967	0.1106	-0.1293
Kendall' s tau	<b>0.0371</b>	<b>-0.0126</b>	<b>-0.0283</b>	<b>-0.0535</b>	<b>0.0161</b>	<b>0.0605</b>	<b>-0.1394</b>
Spearman' s rho	<b>0.0565</b>	<b>-0.0196</b>	<b>-0.0431</b>	<b>-0.0824</b>	<b>0.0214</b>	<b>0.0942</b>	<b>-0.2093</b>

Table 4.4 (continued)

<b>Japan Nikkei and Hong Kong HSI</b>							
<b>(06/01/1992-07/31/2003, unmatched dates deleted)</b>							
	<b>W1</b>	<b>W2</b>	<b>W3</b>	<b>W4</b>	<b>W5</b>	<b>W6</b>	<b>W7</b>
<b>Correlation</b>	<b>0.3690</b>	<b>0.3843</b>	<b>0.3528</b>	<b>0.3011</b>	<b>0.3651</b>	<b>0.2534</b>	<b>0.3354</b>
<i>Lower bound</i>	0.3334	0.3489	0.3163	0.2629	0.3275	0.2107	0.2888
<i>Upper bound</i>	0.4036	0.4185	0.3882	0.3385	0.4014	0.2951	0.3805
<b>Kendall' s tau</b>	<b>0.2294</b>	<b>0.2650</b>	<b>0.2481</b>	<b>0.2054</b>	<b>0.2467</b>	<b>0.1736</b>	<b>0.2334</b>
<b>Spearman' s rho</b>	<b>0.3319</b>	<b>0.3836</b>	<b>0.3594</b>	<b>0.3004</b>	<b>0.3630</b>	<b>0.2557</b>	<b>0.3450</b>
<b>Japan Nikkei and Taiwan TWII</b>							
<b>(06/01/1992-07/31/2003, unmatched dates deleted)</b>							
	<b>W1</b>	<b>W2</b>	<b>W3</b>	<b>W4</b>	<b>W5</b>	<b>W6</b>	<b>W7</b>
<b>Correlation</b>	<b>0.1503</b>	<b>0.2531</b>	<b>0.2800</b>	<b>0.3396</b>	<b>0.2987</b>	<b>0.1854</b>	<b>0.5195</b>
<i>Lower bound</i>	0.1103	0.2145	0.2417	0.3023	0.2593	0.1414	0.4807
<i>Upper bound</i>	0.1898	0.2909	0.3174	0.3759	0.3370	0.2286	0.5562
<b>Kendall' s tau</b>	<b>0.1016</b>	<b>0.1623</b>	<b>0.1552</b>	<b>0.1982</b>	<b>0.1917</b>	<b>0.1501</b>	<b>0.3893</b>
<b>Spearman' s rho</b>	<b>0.1494</b>	<b>0.2378</b>	<b>0.2295</b>	<b>0.2917</b>	<b>0.2851</b>	<b>0.2234</b>	<b>0.5659</b>
<b>Hong Kong HSI and Taiwan Market</b>							
<b>(06/01/1992-07/31/2003, unmatched dates deleted)</b>							
	<b>W1</b>	<b>W2</b>	<b>W3</b>	<b>W4</b>	<b>W5</b>	<b>W6</b>	<b>W7</b>
<b>Correlation</b>	<b>0.2432</b>	<b>0.2660</b>	<b>0.3496</b>	<b>0.3759</b>	<b>0.3666</b>	<b>0.5432</b>	<b>0.3366</b>
<i>Lower bound</i>	0.2046	0.2277	0.3130	0.3396	0.3291	0.5106	0.2900
<i>Upper bound</i>	0.2811	0.3035	0.3851	0.4110	0.4030	0.5743	0.3816
<b>Kendall' s tau</b>	<b>0.1362</b>	<b>0.1768</b>	<b>0.2068</b>	<b>0.2513</b>	<b>0.2422</b>	<b>0.3436</b>	<b>0.1932</b>
<b>Spearman' s rho</b>	<b>0.1997</b>	<b>0.2592</b>	<b>0.3034</b>	<b>0.3667</b>	<b>0.3491</b>	<b>0.4872</b>	<b>0.2876</b>

Table 4.4 (continued)

**Japan Nikkei and Shanghai SSEC**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0320</b>	<b>0.0138</b>	<b>0.0784</b>	<b>0.0172</b>	<b>0.0881</b>	<b>0.0954</b>	<b>0.2027</b>
<i>Lower bound</i>	-0.0087	-0.0270	0.0375	-0.0244	0.0456	0.0506	0.1527
<i>Upper bound</i>	0.0726	0.0545	0.1191	0.0587	0.1302	0.1399	0.2517
Kendall' s tau	<b>0.0046</b>	<b>0.0107</b>	<b>0.0472</b>	<b>0.0006</b>	<b>0.0459</b>	<b>0.1076</b>	<b>0.1565</b>
Spearman' s rho	<b>0.0067</b>	<b>0.0158</b>	<b>0.0708</b>	<b>0.0011</b>	<b>0.0695</b>	<b>0.1561</b>	<b>0.2316</b>

**Japan Nikkei and Shenzhen SZSC1**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0402</b>	<b>0.0373</b>	<b>0.0556</b>	<b>-0.0142</b>	<b>0.1325</b>	<b>0.1855</b>	<b>0.2267</b>
<i>Lower bound</i>	-0.0005	-0.0035	0.0146	-0.0557	0.0904	0.1416	0.1771
<i>Upper bound</i>	0.0807	0.0780	0.0964	0.0274	0.1742	0.2287	0.2751
Kendall' s tau	<b>0.0179</b>	<b>0.0310</b>	<b>0.0380</b>	<b>-0.0091</b>	<b>0.0431</b>	<b>0.1329</b>	<b>0.1933</b>
Spearman' s rho	<b>0.0265</b>	<b>0.0475</b>	<b>0.0576</b>	<b>-0.0127</b>	<b>0.0640</b>	<b>0.2004</b>	<b>0.2889</b>

Table 4.4 (continued)

**Hong Kong HSI and Shanghai SSEC**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0335</b>	<b>0.0340</b>	<b>0.0322</b>	<b>-0.0832</b>	<b>0.0664</b>	<b>0.1798</b>	<b>-0.0054</b>
<i>Lower bound</i>	-0.0071	-0.0068	-0.0088	-0.1243	0.0238	0.1358	-0.0571
<i>Upper bound</i>	0.0741	0.0747	0.0732	-0.0417	0.1088	0.2231	0.0462
Kendall' s tau	<b>0.0419</b>	<b>0.0329</b>	<b>0.0487</b>	<b>-0.0406</b>	<b>0.0261</b>	<b>0.1198</b>	<b>-0.0260</b>
Spearman' s rho	<b>0.0628</b>	<b>0.0501</b>	<b>0.0724</b>	<b>-0.0606</b>	<b>0.0365</b>	<b>0.1803</b>	<b>-0.0364</b>

**Hong Kong HSI and Shenzhen SZSC1**  
**(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
Correlation	<b>0.0921</b>	<b>0.0369</b>	<b>0.0648</b>	<b>-0.0148</b>	<b>0.0903</b>	<b>0.1024</b>	<b>-0.0886</b>
<i>Lower bound</i>	0.0516	-0.0039	0.0239	-0.0563	0.0478	0.0575	-0.1396
<i>Upper bound</i>	0.1323	0.0776	0.1056	0.0267	0.1324	0.1468	-0.0371
Kendall' s tau	<b>0.0468</b>	<b>0.0362</b>	<b>0.0393</b>	<b>-0.0068</b>	<b>0.0535</b>	<b>0.0714</b>	<b>-0.0673</b>
Spearman' s rho	<b>0.0703</b>	<b>0.0548</b>	<b>0.0591</b>	<b>-0.0096</b>	<b>0.0833</b>	<b>0.1005</b>	<b>-0.0933</b>

Table 4.4 (continued)

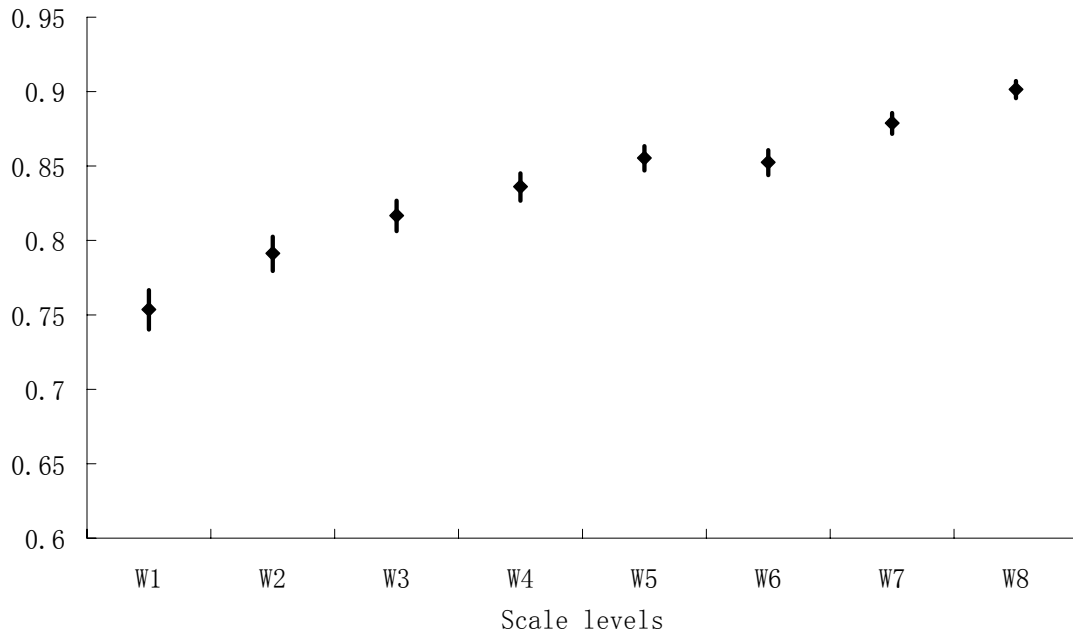
**Taiwan TWII and Shanghai SSEC  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
<b>Correlation</b>	<b>0.0295</b>	<b>-0.0097</b>	<b>-0.0417</b>	<b>0.0262</b>	<b>0.0639</b>	<b>0.0670</b>	<b>0.2636</b>
<i>Lower bound</i>	<i>-0.0111</i>	<i>-0.0505</i>	<i>-0.0826</i>	<i>-0.0154</i>	<i>0.0214</i>	<i>0.0220</i>	<i>0.2149</i>
<i>Upper bound</i>	<i>0.0701</i>	<i>0.0311</i>	<i>-0.0007</i>	<i>0.0677</i>	<i>0.1063</i>	<i>0.1118</i>	<i>0.3110</i>
<b>Kendall' s tau</b>	<b>0.0244</b>	<b>0.0010</b>	<b>-0.0230</b>	<b>0.0200</b>	<b>0.0969</b>	<b>0.0470</b>	<b>0.1789</b>
<b>Spearman' s rho</b>	<b>0.0364</b>	<b>0.0011</b>	<b>-0.0349</b>	<b>0.0309</b>	<b>0.1456</b>	<b>0.0698</b>	<b>0.2862</b>

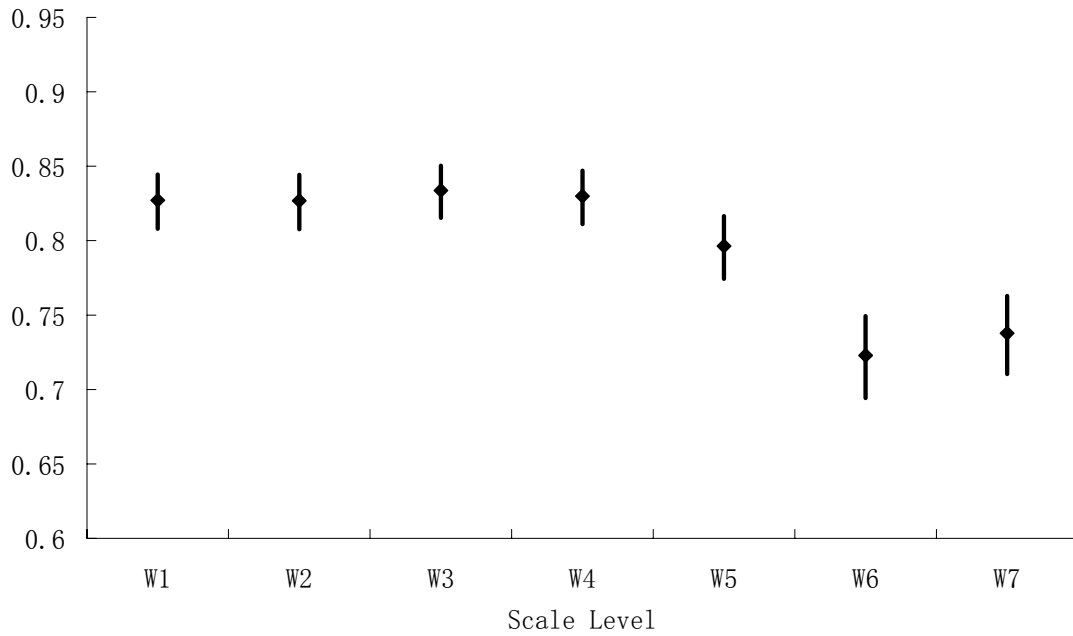
**Taiwan TWII and Shenzhen Markets SZSC1  
(06/01/1992-07/31/2003, unmatched dates deleted)**

	W1	W2	W3	W4	W5	W6	W7
<b>Correlation</b>	<b>0.0458</b>	<b>0.0396</b>	<b>-0.0063</b>	<b>0.0322</b>	<b>0.1425</b>	<b>0.0967</b>	<b>0.3182</b>
<i>Lower bound</i>	<i>0.0052</i>	<i>-0.0012</i>	<i>-0.0474</i>	<i>-0.0094</i>	<i>0.1005</i>	<i>0.0518</i>	<i>0.2710</i>
<i>Upper bound</i>	<i>0.0864</i>	<i>0.0802</i>	<i>0.0347</i>	<i>0.0736</i>	<i>0.1840</i>	<i>0.1411</i>	<i>0.3638</i>
<b>Kendall' s tau</b>	<b>0.0283</b>	<b>0.0302</b>	<b>-0.0135</b>	<b>0.0140</b>	<b>0.0980</b>	<b>0.0685</b>	<b>0.2198</b>
<b>Spearman' s rho</b>	<b>0.0423</b>	<b>0.0457</b>	<b>-0.0210</b>	<b>0.0209</b>	<b>0.1471</b>	<b>0.1016</b>	<b>0.3397</b>

**Figure 4.1a: Wavelet Correlation for NASDAQ and S&P500  
(02/05/1971-07/31/2003)**

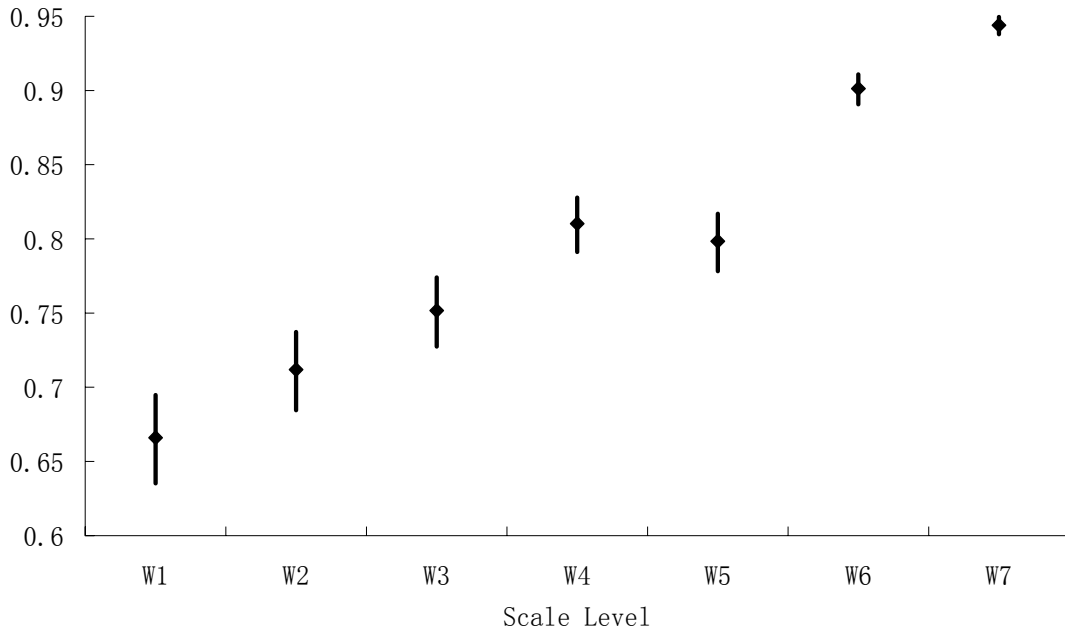


**Figure 4.1b: Wavelet Correlation for NASDAQ and S&P500  
(06/01/1992-07/31/2003, unmatched dates deleted)**

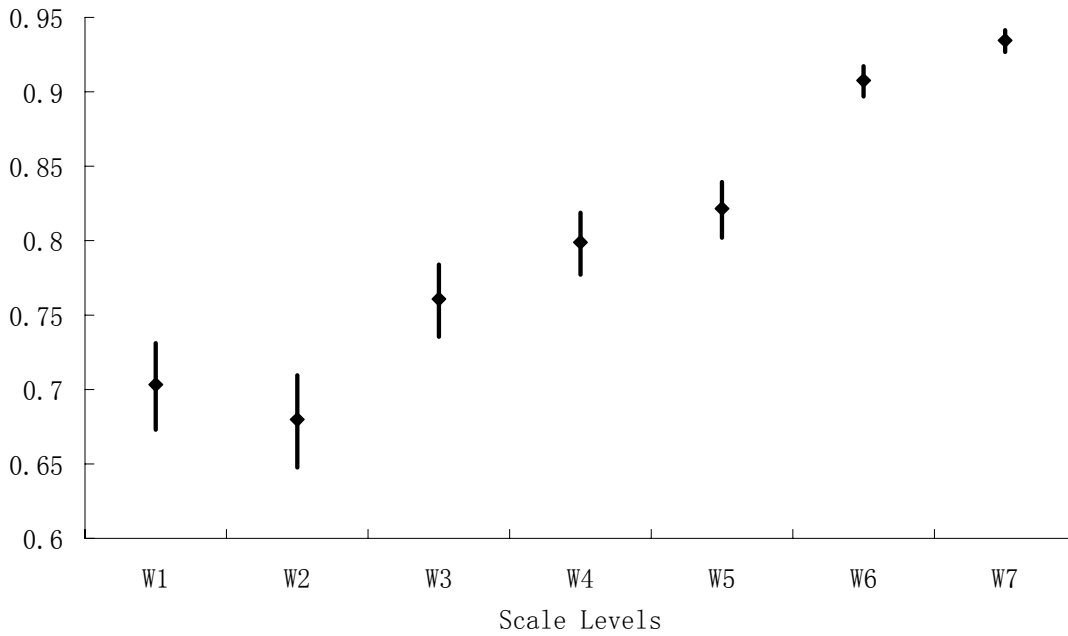




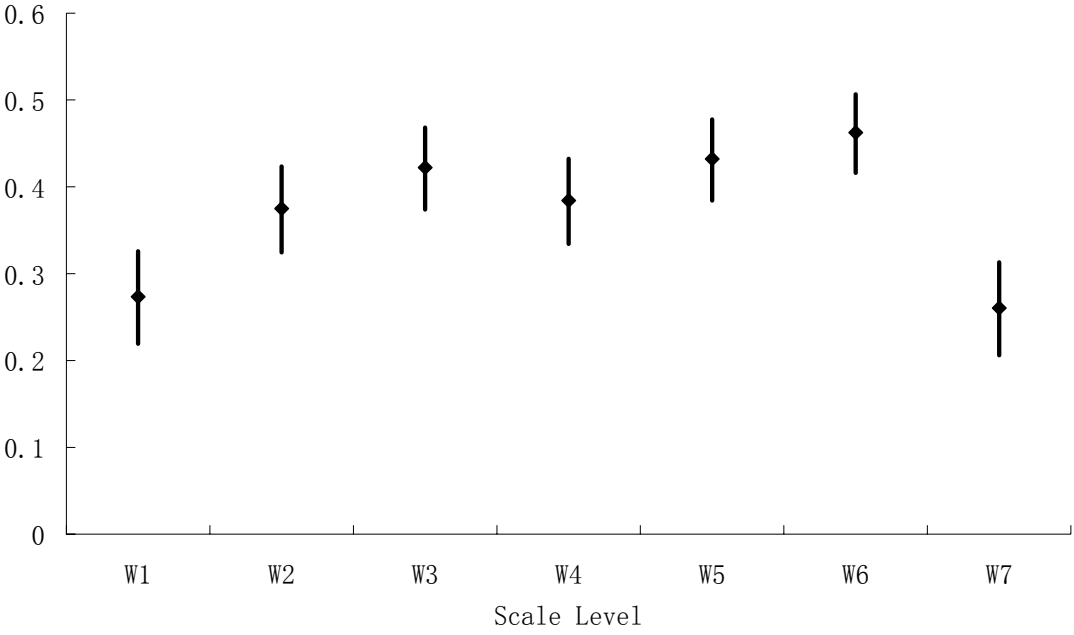
**Figure 4.2a: Wavelet Correlation for Chinese Mainland Markets  
(06/01/1992-07/31/2003)**



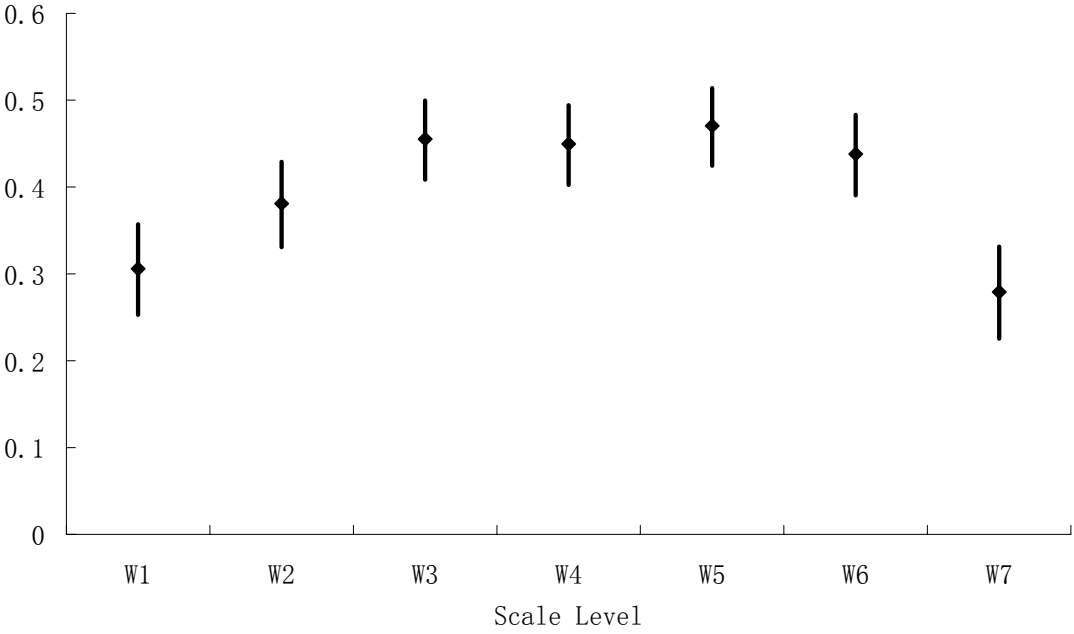
**Figure 4.2b: Wavelet Correlation for Chinese Mainland Markets  
(06/01/1992-07/31/2003, unmatched dates deleted)**



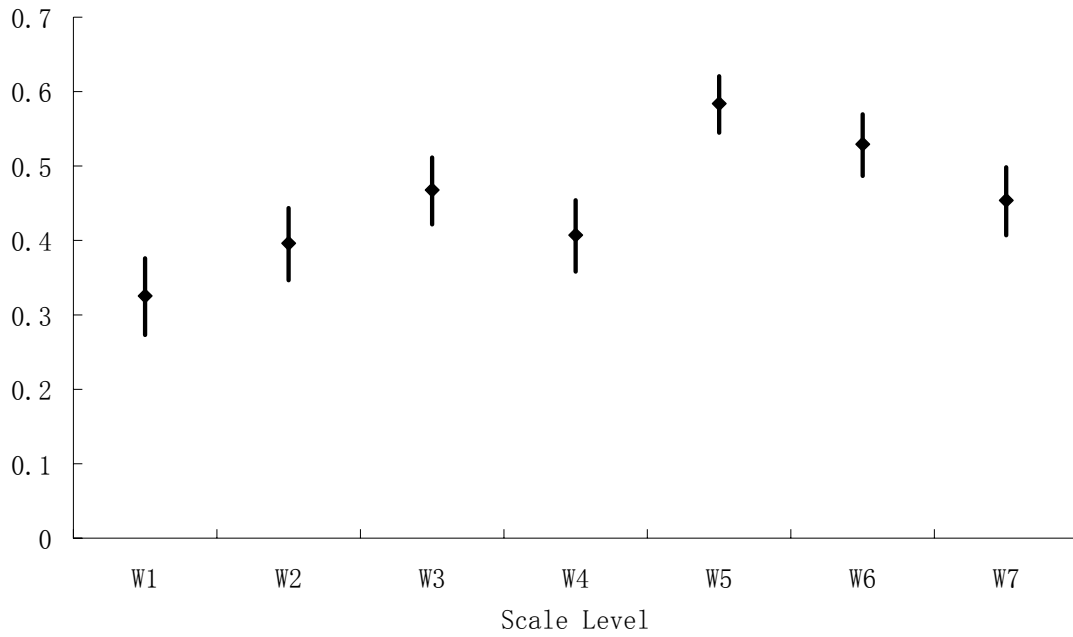
**Figure 4.3a: Wavelet Correlation for NASDAQ and Nikkei**  
(06/01/1992-07/31/2003, unmatched dates deleted)



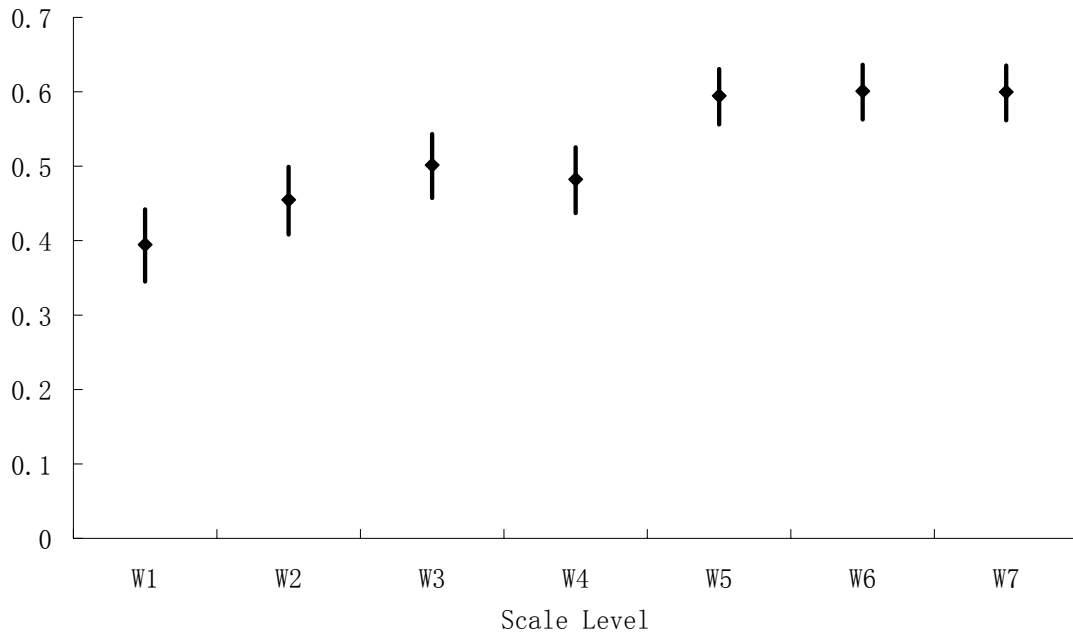
**Figure 4.3b: Wavelet Correlation for S&P500 and Nikkei**  
(06/01/1992-07/31/2003, unmatched dates deleted)



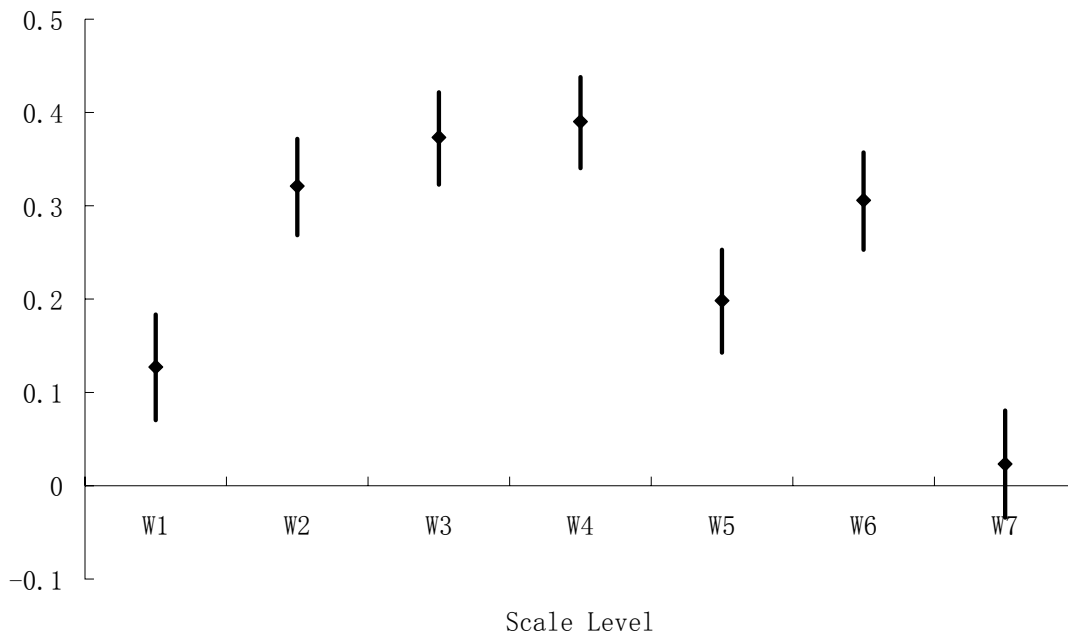
**Figure 4.4a: Wavelet Correlation for NASDAQ and Hang Seng**  
(06/01/1992-07/31/2003, unmatched dates deleted)



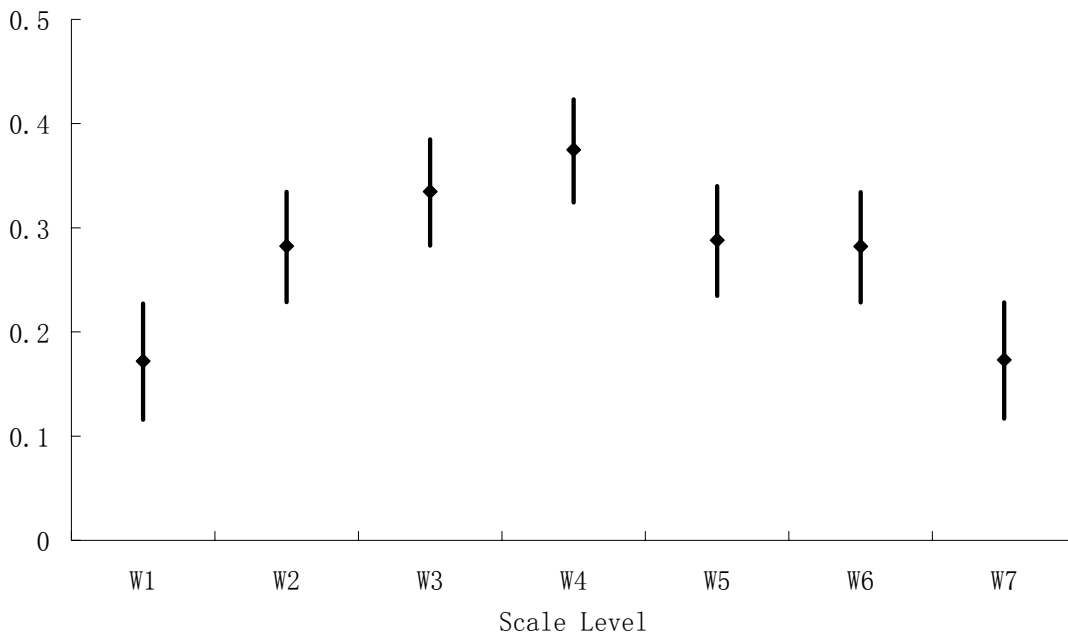
**Figure 4.4b: Wavelet Correlation for S&P500 and Hang Seng**  
(06/01/1992-07/31/2003, unmatched dates deleted)



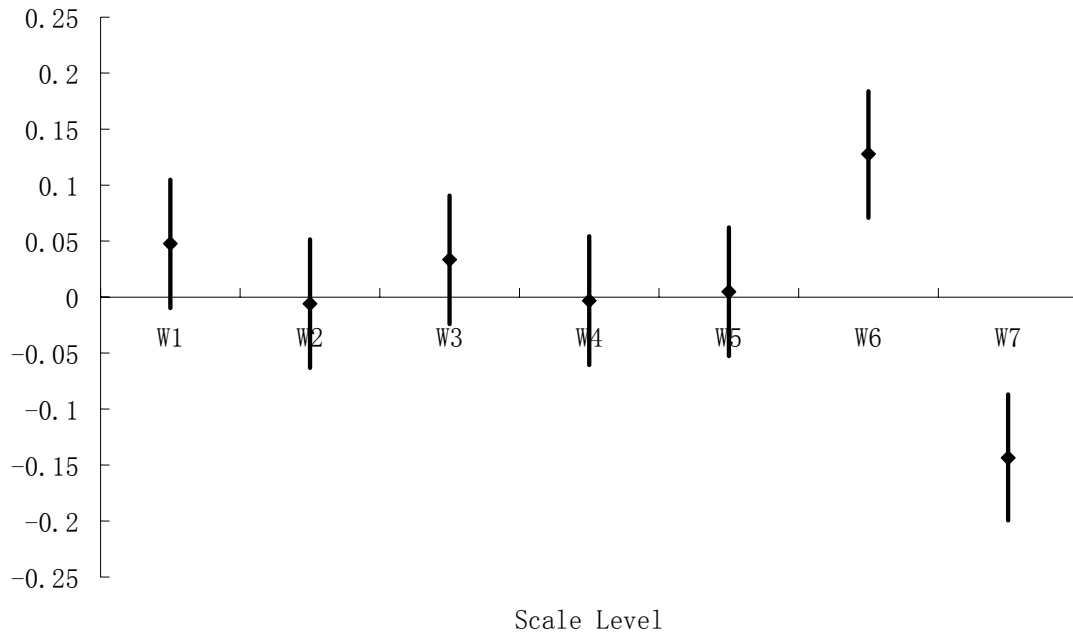
**Figure 4.5a: Wavelet Correlation for NASDAQ and Taiwan Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)



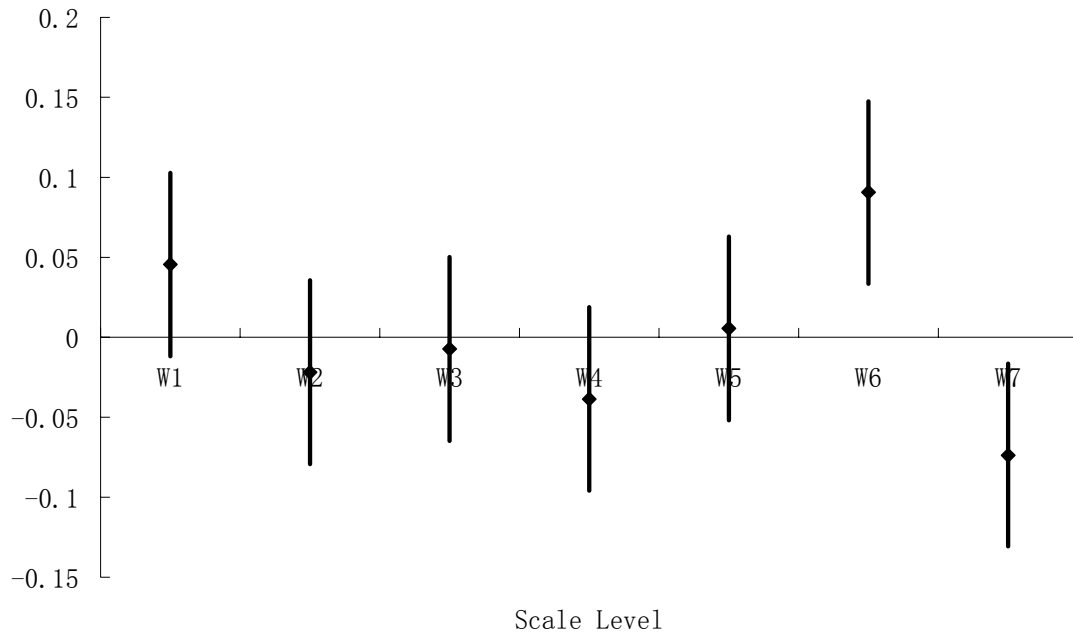
**Figure 4.5b: Wavelet Correlation for S&P500 and Taiwan Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)



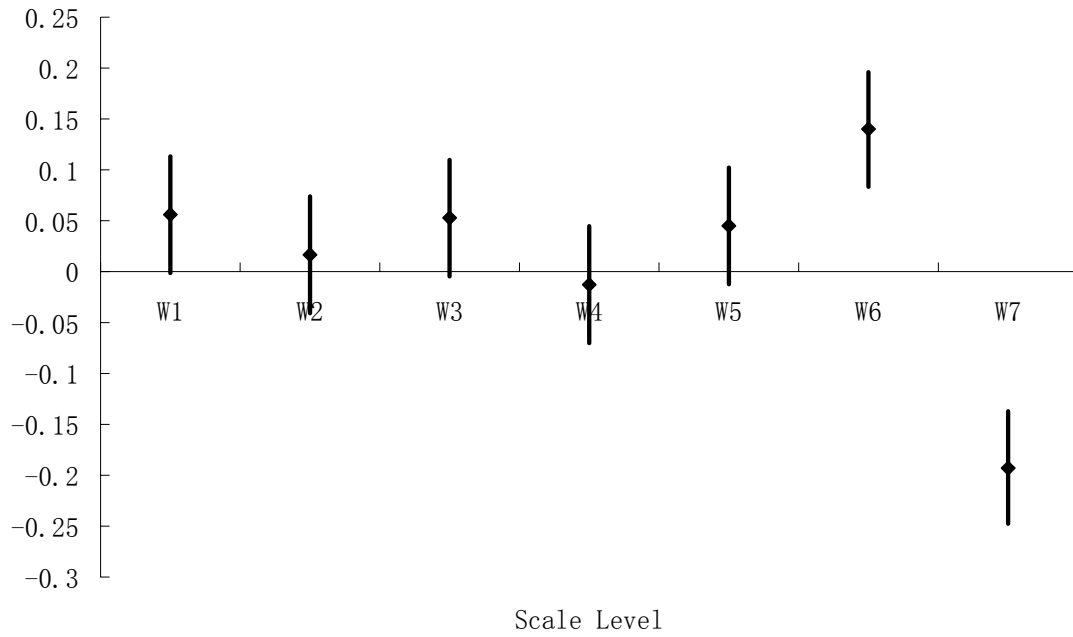
**Figure 4.6a: Wavelet Correlation for NASDAQ and Shanghai Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)



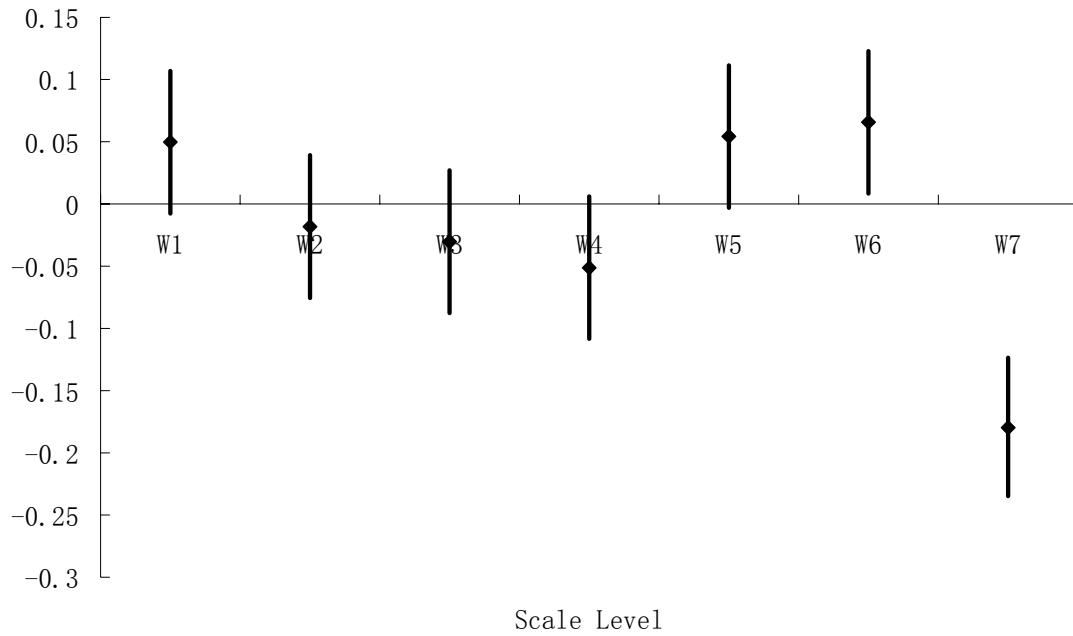
**Figure 4.6b: Wavelet Correlation for S&P500 and Shanghai Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)



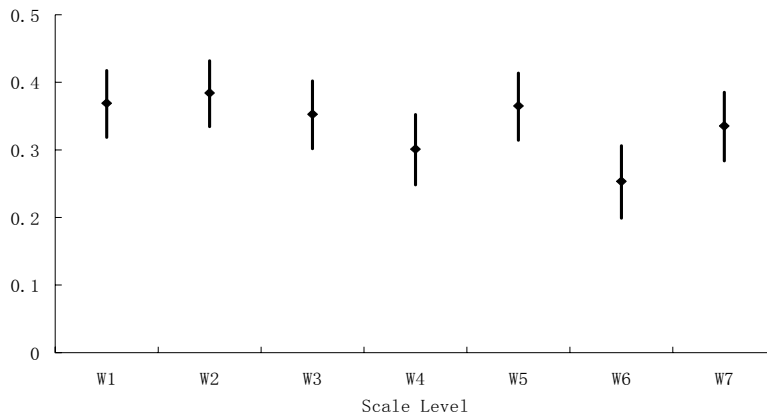
**Figure 4.7a: Wavelet Correlation for NASDAQ and Shenzhen Market  
(06/01/1992-07/31/2003, unmatched dates deleted)**



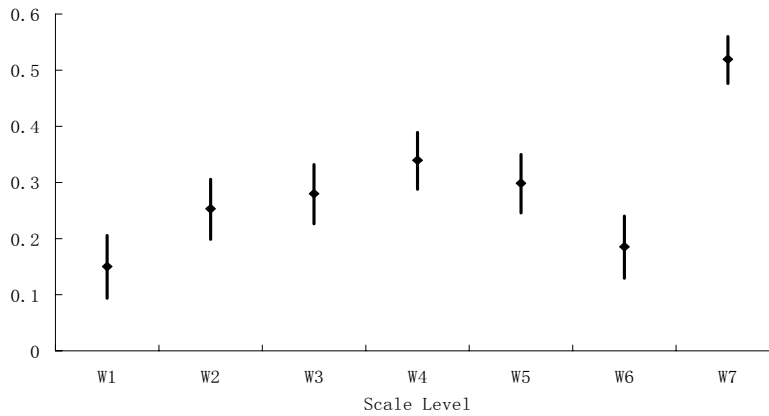
**Figure 4.7b: Wavelet Correlation for S&P500 and Shenzhen Market  
(06/01/1992-07/31/2003, unmatched dates deleted)**



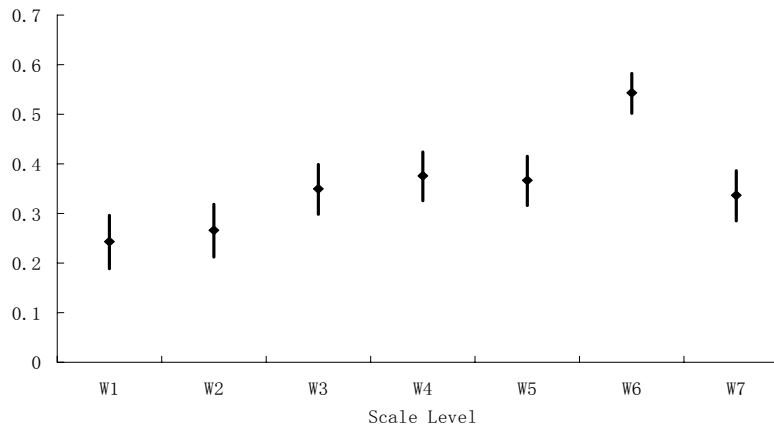
**Figure 4.8a: Wavelet Correlation for Nikkei and Hang Seng  
(06/01/1992-07/31/2003, unmatched dates deleted)**



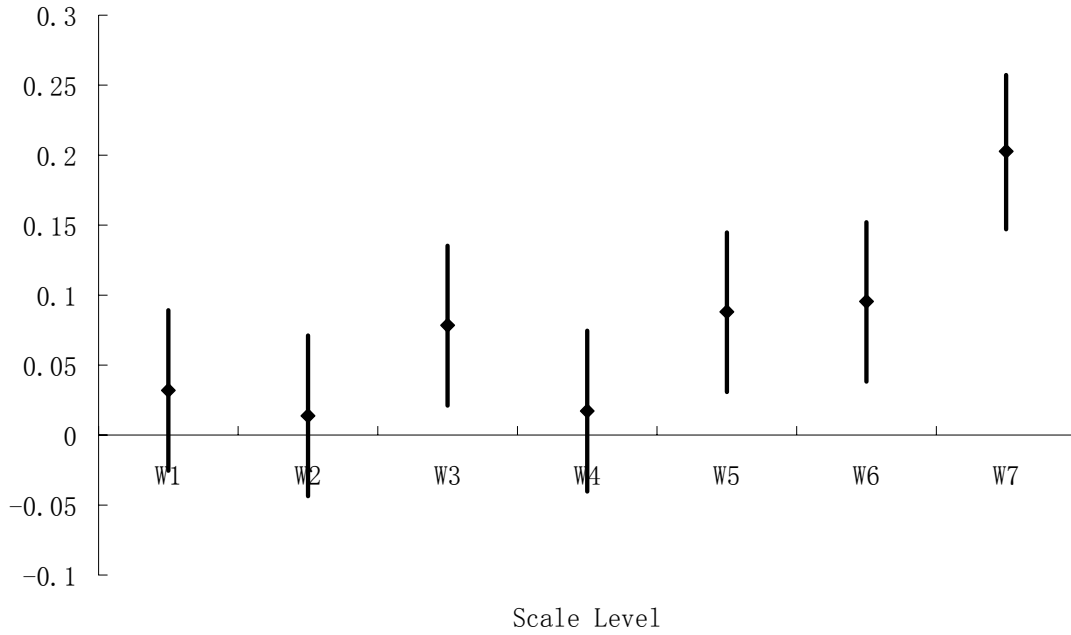
**Figure 4.8b: Wavelet Correlation for Nikkei and Taiwan Market  
(06/01/1992-07/31/2003, unmatched dates deleted)**



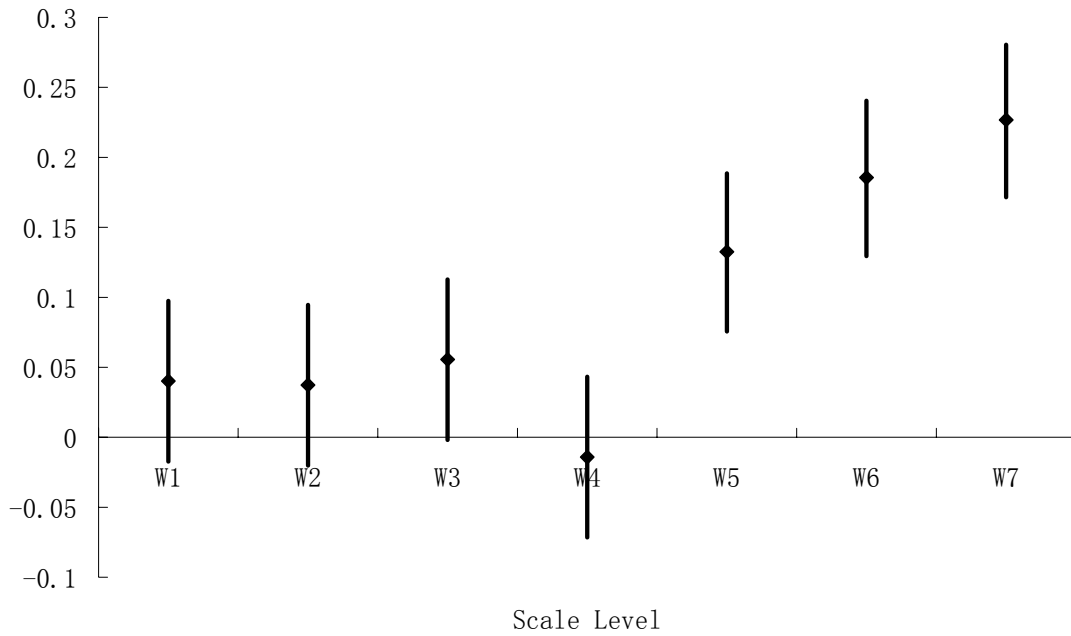
**Figure 4.8c: Wavelet Correlation for Hong Kong and Taiwan Market  
(06/01/1992-07/31/2003, unmatched dates deleted)**



**Figure 4.9a: Wavelet Correlation for Nikkei and Shanghai Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)

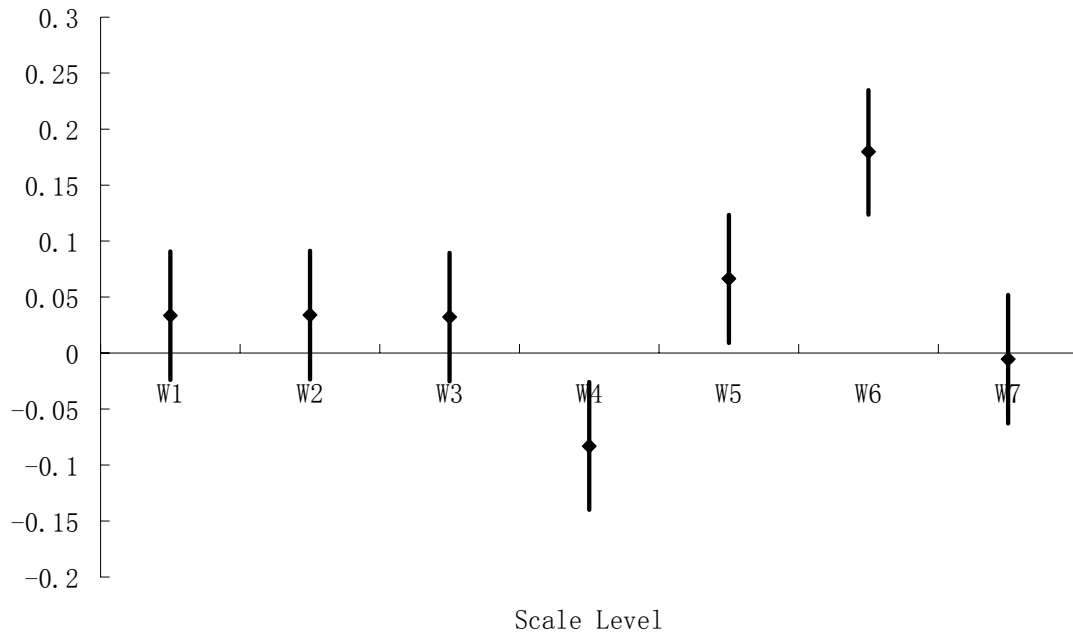


**Figure 4.9b: Wavelet Correlation for Nikkei and Shenzhen Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)

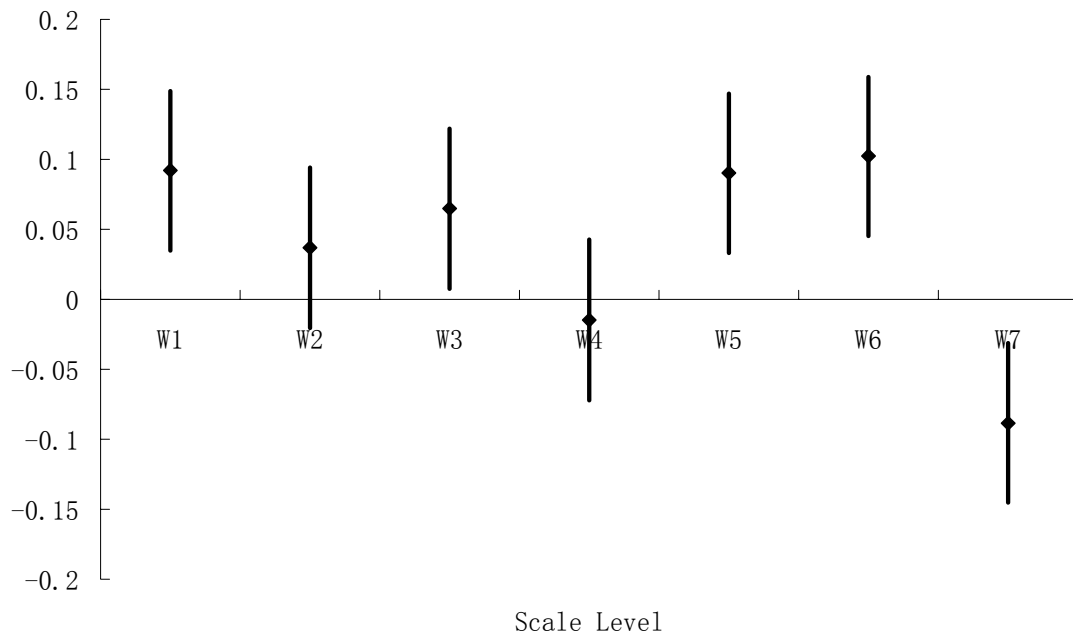




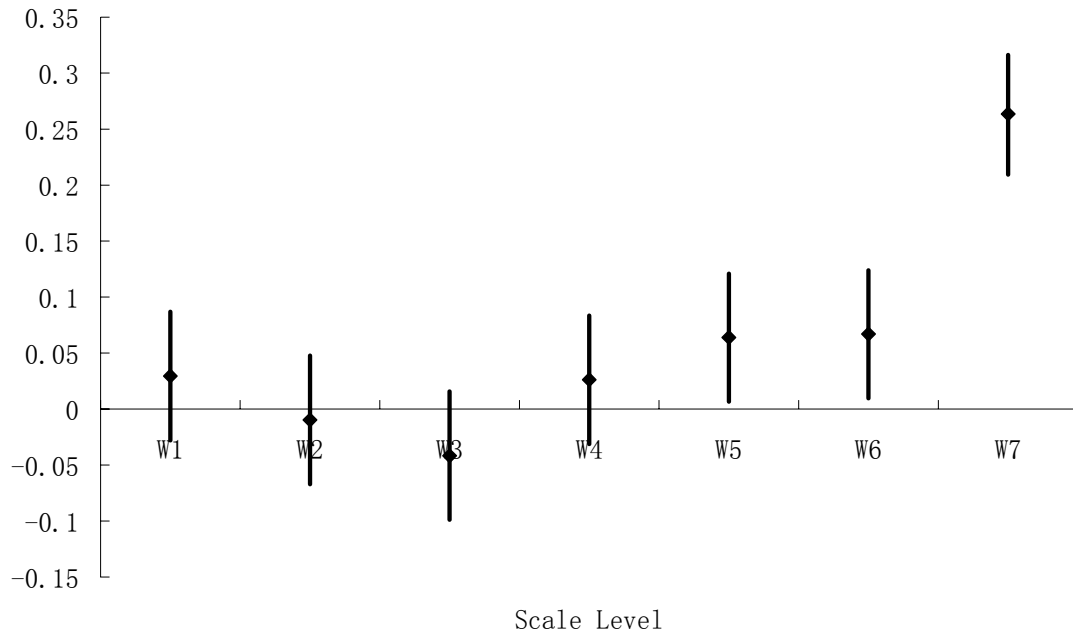
**Figure 4.10a: Wavelet Correlation for Hang Seng and Shanghai Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)



**Figure 4.10b: Wavelet Correlation for Hang Seng and Shenzhen Market**  
(06/01/1992-07/31/2003, unmatched dates deleted)



**Figure 4.11a: Wavelet Correlation for Taiwan and Shanghai Markets**  
(06/01/1992-07/31/2003, unmatched dates deleted)



**Figure 4.11b: Wavelet Correlation for Taiwan and Shenzhen Markets**  
(06/01/1992-07/31/2003, unmatched dates deleted)

