RECONSTRUCTION FROM ERROR-AFFECTED SAMPLED DATA IN SHIFT-INVARIANT SPACES

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CHAPTER I

INTRODUCTION

Sampling and reconstruction have been widely studied in recent decades, especially within the setting of shift-invariant spaces (see [1] - [8], [14], [17]). Signal reconstruction from data affected by error has received less attention. In the following chapters we provide error estimates for signals reconstructed from corrupt data. Two different types of error are considered, and the questions answered in each case are different. In Chapter II, we assume the data has additive noise with expected value zero and variance σ^2. We calculate $\mathrm{var}(f_\varepsilon(x) - f(x))$, where f_ε is the reconstructed function and f is the original signal from which the data originates, and we show that oversampling leads to reduced variance of error. In Chapter III, we consider the issue of jitter error, which results from not knowing precisely the sampling set. In this case, we answer two questions. First, under what conditions on the jitter error is our signal still uniquely and stably determined by the data? Second, how well does our reconstructed function approximate the original function?

In this chapter, we begin with an overview of some of the major tools we use to arrive at our results. In particular, Fourier analysis and frame theory play a significant role in our ability to reconstruct approximations of signals from a countable collection of data. Also included in this chapter is some background on sampling and reconstruction. We provide a brief review of sampling theory, where we include some of the important definitions and main ideas. We discuss shift-invariant spaces and provide a characterization of the shift-invariant spaces from which our continuous signals originate. We conclude this chapter with an explanation of weighted-average sampling, an extension of the classical sampling setting.

The following two chapters present the work done with error as mentioned earlier.

Chapter II consists of results by the author and collaborators Akram Aldroubi and Qiyu Sun. Chapter III is a paper by the author and Akram Aldroubi that is scheduled to appear in *Sampling Theory in Signal and Image Processing*. Finally, Chapter IV includes some results relating shift-invariant spaces to general reproducing kernel Hilbert spaces.

I.1 Fourier Analysis

Our analysis will heavily rely on the Fourier transform and its properties. We denote the Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ by $\widehat{f}$ and define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i2\pi x\cdot\xi}\, dx \qquad \text{a.e. } \xi \in \mathbb{R}^d.$$

The function $\widehat{f}$ is also in $L^2(\mathbb{R}^d)$, and $\|f\|_{L^2(\mathbb{R}^d)} = \|\widehat{f}\|_{L^2(\mathbb{R}^d)}$. Similarly, we denote the Fourier series of a sequence $c \in l^2(\mathbb{Z}^d)$ by $\widehat{c}$ and define

$$\widehat{c}(\xi) = \sum_{k\in\mathbb{Z}^d} c(k) e^{-i2\pi k\cdot\xi} \qquad \text{a.e. } \xi \in [0,1]^d.$$

The function $\widehat{c}$ is in $L^2([0,1]^d)$, and $\|c\|_{l^2(\mathbb{Z}^d)} = \|\widehat{c}\|_{L^2([0,1]^d)}$. The following properties of the Fourier transform will frequently be used.

(i) $\widehat{\tau_y f}(\xi) = e^{-i2\pi y\cdot\xi}\widehat{f}(\xi)$ where $\tau_y f = f(\cdot - y)$

(ii) $\widehat{\widehat{f}} = f^\vee$ where $f^\vee(x) = f(-x)$

(iii) $\widehat{f^\vee} = \overline{\widehat{f}}$ if f is real-valued

(iv) $\widehat{f * g} = \widehat{f}\, \widehat{g}$

For vector functions $F = (f^1, \ldots, f^n)^T$, the notation $\widehat{F}$ will represent the vector $(\widehat{f^1}, \ldots, \widehat{f^n})^T$.

Another valuable tool from Fourier analysis is the Poisson Summation Formula. If $\sum_{k\in\mathbb{Z}^d} f(x+k) \in L^2([0,1]^d)$, and if $\sum_{k\in\mathbb{Z}^d} \left|\widehat{f}(k)\right|^2 < \infty$, then

$$\sum_{k\in\mathbb{Z}^d} f(x+k) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{i2\pi k\cdot x} \qquad \text{a.e. } x \in \mathbb{R}^d. \tag{I.1.1}$$

More often we will use the equivalent version

$$\sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi+k) = \sum_{k\in\mathbb{Z}^d} f(k)e^{-i2\pi k\cdot \xi} \qquad \text{a.e. } \xi. \tag{I.1.2}$$

Notice the right-hand side of the equation is the Fourier series of the sequence whose terms are samples of f on the integer lattice. A simple exercise shows also that (I.1.1) implies

$$\sum_{k\in\mathbb{Z}^d} f(k-x)e^{i2\pi\xi\cdot(k-x)} = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k-\xi)e^{-i2\pi x\cdot k} \tag{I.1.3}$$

Please see [15] for an extensive review of the Fourier transform and its properties.

I.2 Frame Theory

Frames can be thought of as generalized orthonormal bases, in the sense that a frame for a Hilbert space $\mathcal{H}$ is a spanning set for $\mathcal{H}$. However the frame elements in general are neither orthogonal to each other nor linearly independent. The properties of frames illustrated below allow for the reconstruction of a function in a given Hilbert space from a countable collection of coefficients. We begin with the definition of a frame.

Definition I.2.1. *A countable collection* $\{f_j\}_{j\in J}$ *of elements in a Hilbert space* $\mathcal{H}$ *is called a* frame *for* $\mathcal{H}$ *if there exist positive constants* α *and* β *such that*

$$\alpha \|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le \beta \|f\|^2 \qquad \text{for all } f \in \mathcal{H}. \tag{I.2.4}$$

For a given frame $\{f_j\}_{j\in J}$ for a Hilbert space $\mathcal{H}$, the coefficient operator $C : \mathcal{H} \to l^2(J)$ given by

$$Cf = \{\langle f, f_j\rangle : j \in J\}$$

is bounded with closed range. The reconstruction operator $D : l^2(J) \to \mathcal{H}$ given by

$$Dc = \sum_{j\in J} c_j f_j$$

is well-defined and bounded with $\|D\| \le \sqrt{\beta}$. The operators C and D are adjoint to each other; that is $D = C^*$. The frame operator $S : \mathcal{H} \to \mathcal{H}$ given by

$$Sf = \sum_{j\in J} \langle f, f_j\rangle f_j$$

is a positive invertible operator satisfying $\alpha I_{\mathcal{H}} \le S \le \beta I_{\mathcal{H}}$ and $\frac{1}{\beta} I_{\mathcal{H}} \le S^{-1} \le \frac{1}{\alpha} I_{\mathcal{H}}$. Notice that $S = C^*C = DD^*$.

For a given frame $\{f_j : j \in J\}$ in a Hilbert space $\mathcal{H}$, the collection $\{S^{-1} f_j : j \in J\}$ also forms a frame for $\mathcal{H}$. We denote this so-called *canonical dual frame* with a tilde (i.e. $\widetilde{f_j} = S^{-1} f_j$ for all $j \in J$), and it satisfies the reconstruction formulas

$$f = \sum_{j\in J} \langle f, f_j\rangle \widetilde{f_j} = \sum_{j\in J} \left\langle f, \widetilde{f_j}\right\rangle f_j \qquad \text{for all } \ f \in \mathcal{H}.$$

These consequences from frame theory provide much of the foundation for sampling and reconstruction in shift-invariant spaces. Please see [9] and [15] for an extensive review of frames and their properties.

I.3 Sampling and Reconstruction

In the classical sampling problem, the objective is to recover a function f on $\mathbb{R}^d$ from its samples $\{f(x_j) : j \in J\}$, where J is a countable indexing set. This situation arises when dealing with a function (e.g. a signal or image) stored on a computer or in any digital format, which can be done only in a discretized form. For any given sampling set $X = \{x_j \in \mathbb{R}^d : j \in J\}$, where J is countable, there can be infinitely many functions on $\mathbb{R}^d$ which have the same sample values on X. Therefore, the problem of recovering f from its sampled values makes sense only if we assume some *a priori* conditions on f. We can reformulate the problem as follows: Given a class of functions V on $\mathbb{R}^d$, find conditions on sampling sets $X = \{x_j \in \mathbb{R}^d : j \in J\}$, where J is a countable index set, under which a function $f \in V$ can be reconstructed uniquely and stably from its samples $\{f(x_j) : j \in J\}$ [3], and then recover the function f from its samples at X. This problem has many applications, including medical imaging and communication.

I.3.1 Some History

The most classical sampling theorem is due to J.M. Whittaker [29]: Let $f \in L^2(\mathbb{R})$ be such that $\operatorname{supp} \hat{f} \subset [-\frac{1}{2}, \frac{1}{2}]$. Then f can be recovered exactly from its samples $\{f(k) : k \in \mathbb{Z}\}$ by the formula

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) \operatorname{sinc}(x - k), \tag{I.3.5}$$

where $\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}$. This result is often referred to as Shannon's Sampling Theorem because of Shannon's well-known work building upon this result [24].

To understand why this theorem holds, we begin with the Poisson summation formula in equation (I.1.2). For $\xi \in [-\frac{1}{2}, \frac{1}{2}]$, we have

$$\widehat{f}(\xi) = \sum_{k\in\mathbb{Z}} \widehat{f}(\xi + k) = \sum_{k\in\mathbb{Z}} f(k)e^{-i2\pi k\cdot\xi} = \sum_{k\in\mathbb{Z}} f(k)e^{-i2\pi k\cdot\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$$

for all $f \in L^2(\mathbb{R})$ satisfying $\operatorname{supp}\hat{f} \subset [-\frac{1}{2}, \frac{1}{2}]$. Recall that the Fourier transform of the sinc function is the characteristic function on $[-\frac{1}{2}, \frac{1}{2}]$, which we denoted above by $\chi_{[-\frac{1}{2},\frac{1}{2}]}$. By property (i) of the Fourier transform, we have

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\ \mathrm{sinc}(x - k).$$

This theorem is easily extended to sampling sets $T\mathbb{Z}$ for functions $f \in L^2(\mathbb{R})$ such that $\mathrm{supp}\hat{f} \subset [-\frac{\omega}{2}, \frac{\omega}{2}]$, where $\omega = \frac{1}{T}$. Notice that these sampling sets are *uniform*, i.e. the sampling sets form a d-dimensional Cartesian grid, where in this case $d = 1$. Function reconstruction from uniform sets of sampling is practical in many applications. For example, a digital image is often acquired by sampling light intensities on a uniform grid [3]. However, in many realistic situations, such as medical imaging (CT and MRI), the samples do not lie on a uniform grid. Therefore, the need also arises to reconstruct a function f from its samples $\{f(x_j) : j \in J\}$, where J is not necessarily uniformly distributed.

We call functions whose Fourier transforms have compact support *bandlimited* functions, and a great deal of work was done in recent decades by Beurling, Landau, and others extending the above theorem to the reconstruction of bandlimited functions sampled on nonuniform sets in $\mathbb{R}$. Specifically, for the exact and stable reconstruction of a one-dimensional bandlimited function from its samples $\{f(x_j) : x_j \in X\}$, it is sufficient that the *Beurling density*

$$D(X) = \lim_{r\to\infty} \inf_{y\in\mathbb{R}} \frac{\#X \cap (y + [0, r])}{r}$$

satisfies $D(X) > 1$. Conversely, if f is uniquely and stably determined by its samples on $X \subset \mathbb{R}$, then $D(X) \geq 1$.

Much is known about sampling and function reconstruction for classes of bandlimited functions. However, because all bandlimited functions are analytic, they have infinite support. This leads to inefficiency in numerical implementations. For instance, in the pointwise evaluation

$$f \mapsto f(x_0) = \sum_{k\in\mathbb{Z}} c_k \operatorname{sinc}(x_0 - k),$$

for any $x_0 \notin \mathbb{Z}$, many coefficients c_k will contribute to the value of $f(x_0)$ because of the slow decay of the sinc function. Additionally, classes of images or signals may be better modeled by other types of non-bandlimited function spaces. We come to the conclusion that it would be advantageous to consider classes of functions which are not bandlimited, to allow us to model more classes of signals and so that numerical implementation becomes practical, yet which still retain some of the simplicity and structure of bandlimited models [3].

I.4 Sampling in Shift-Invariant Spaces

A *shift-invariant space* is a space V of functions on $\mathbb{R}^d$ such that if $f \in V$, then $f(\cdot - k) \in V$ for all $k \in \mathbb{Z}^d$. In particular, as is common in much of the current research $\big($see [1]-[7],[14], [17]$\big)$, our underlying space will be a shift-invariant space of the form

$$V^2(\Phi) = \left\{ \sum_{k\in\mathbb{Z}^d} C(k)^T \Phi(\cdot - k) : C \in (l^2)^{(r)} \right\} \tag{I.4.6}$$

for some real-valued vector function $\Phi = (\phi^1, \ldots, \phi^r)^T \in \left(L^2(\mathbb{R}^d)\right)^{(r)}$, where $C = (c^1, \ldots, c^r)^T$ is a real-valued vector sequence such that $c^i := \{c^i(k)\}_{k\in\mathbb{Z}^d} \in l^2$, i.e.,

$C \in (l^2)^{(r)}$. Thus $\sum_{k\in\mathbb{Z}^d} C(k)^T\Phi(\cdot - k) = \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} c^i(k)\phi^i(\cdot - k)$.

Notice that the space of functions $\{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subset [-\frac{1}{2}, \frac{1}{2}]\}$ is the shift-invariant space

$$V^2(\mathrm{sinc}) = \left\{ \sum_{j\in\mathbb{Z}} c(j)\mathrm{sinc}(\cdot - j) : c \in l^2(\mathbb{Z}) \right\}.$$

Also notice that a shift invariant space will be a space of bandlimited functions only if its generators ϕ^i are bandlimited. Sampling in shift-invariant spaces whose generators are not bandlimited works well in many applications, especially with an appropriate choice of functions ϕ^i [3].

Toward the goal of recovering a function from its samples, we begin by defining our underlying space $V^2(\Phi)$ more precisely. As mentioned before, shift-invariant spaces are commonly used in sampling models. Moreover, it is common to consider continuous shift-invariant spaces that are subspaces of $L^2(\mathbb{R}^d)$ in order to take advantage of reproducing kernel Hilbert space properties.

Let $\Phi = (\phi^1, \ldots, \phi^r)^T$, where $\phi^i : \mathbb{R}^d \to \mathbb{R}$ is a function in $L^2(\mathbb{R}^d)$, and assume Φ is such that

$$G_\Phi(\xi) := \sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi + k)\overline{\widehat{\Phi}(\xi + k)}^T = I, \quad \text{a.e. } \xi \in \mathbb{R}^d, \tag{I.4.7}$$

where I is the $r \times r$ identity matrix. Define the shift-invariant space

$$V^2(\Phi) := \left\{ \sum_{k\in\mathbb{Z}^d} C(k)^T\Phi(\cdot - k) : C \in (l^2)^{(r)} \right\}.$$

Then $V^2(\Phi)$ is a Hilbert space, $V^2(\Phi)$ is a subspace of $L^2(\mathbb{R}^d)$, and $\{\phi^i(\cdot - k) : 1 \le i \le r, k \in \mathbb{Z}^d\}$ forms an orthonormal basis for $V^2(\Phi)$ [1, 3]. Also assume $\phi^i \in W_0^1 := W^1 \cap C^0$, where C^0 is the set of continuous functions, and

$$W^1 = \left\{ f : \sum_{k\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d} \{|f(x + k)|\} < \infty \right\}.$$

Under this assumption, $V^2(\Phi)$ is a space of continuous functions [3]. Furthermore, with this assumption, for each x in $\mathbb{R}^d$, the point evaluation map $f \mapsto f(x)$, from $V^2(\Phi)$ to $\mathbb{R}$, is bounded. To see this, denote the sequence $a_x^i(k) := \phi^i(x-k)$, and notice that for every $x \in \mathbb{R}^d$, $\|a_x^i\|_{l^1(\mathbb{Z}^d)} \leq \|\phi^i\|_{W^1}$. Let $f = \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} c^i(k)\phi^i(\cdot - k) \in V^2(\Phi)$. Then

$$
\begin{aligned}
|f(x)| &\leq \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} \left|c^i(k)\right| \left|\phi^i(x-k)\right| = \sum_{i=1}^r \left\langle \left|c^i\right|, \left|a_x^i\right| \right\rangle_{l^2} \\
&\leq \sum_{i=1}^r \left\|c^i\right\|_{l^2} \left\|a_x^i\right\|_{l^2} \leq \sum_{i=1}^r \left\|c^i\right\|_{l^2} \left\|a_x^i\right\|_{l^1} \leq \sum_{i=1}^r \left\|c^i\right\|_{l^2} \left\|\phi^i\right\|_{W^1} \\
&\leq \left(\max_{1\leq i\leq r} \left\|\phi^i\right\|_{W^1}\right) \sum_{i=1}^r \left\|c^i\right\|_{l^2} = \left(\max_{1\leq i\leq r} \left\|\phi^i\right\|_{W^1}\right) \|f\|_{L^2}
\end{aligned}
$$

We conclude that point evaluation is a bounded linear functional on $V^2(\Phi)$. Therefore, by the Riesz Representation Theorem, for every $x \in \mathbb{R}^d$, there exists a reproducing kernel $K_x \in V^2(\Phi)$ satisfying $\langle f, K_x\rangle = f(x)$ for all $f \in V^2(\Phi)$.

Remark I.4.1. If Φ satisfies (I.4.7), and if $\phi^i \in W_0^1$ for $1 \leq i \leq r$, then the unique reproducing kernels $\{K_x : x \in \mathbb{R}^d\}$ of the reproducing kernel Hilbert space $V^2(\Phi)$ are of the form

$$
K_x(y) = \sum_{i=1}^r \sum_{l\in\mathbb{Z}^d} \phi^i(x-l)\phi^i(y-l) \tag{I.4.8}
$$

where $x, y \in \mathbb{R}^d$.

Proof: The equation (I.4.7) implies $\{\phi^i(\cdot - k) : 1 \leq i \leq r, k \in \mathbb{Z}^d\}$ forms an orthonormal basis for $V^2(\Phi)$. Let $f = \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} c_k^i \phi^i(\cdot - k) \in V^2(\Phi)$. For $x \in \mathbb{R}^d$, let $K_x = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} \phi^i(x-j)\phi^i(\cdot - j)$. Then

$$
\begin{aligned}
\langle f, K_x\rangle &= \left\langle \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} c_k^i \phi^i(\cdot - k), \sum_{i'=1}^r \sum_{j\in\mathbb{Z}^d} \phi^{i'}(x-j)\phi^{i'}(\cdot - j) \right\rangle \\
&= \sum_{i=1}^r \sum_{i'=1}^r \sum_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} c_k^i \phi^{i'}(x-j) \left\langle \phi^i(\cdot - k), \phi^{i'}(\cdot - j) \right\rangle \\
&= \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} c_k^i \phi^i(x-k) = f(x).
\end{aligned}
$$

□

Once the underlying space $V^2(\Phi)$ is fixed, the ability to recover a function f in $V^2(\Phi)$ from its samples, $\{f(x_j)\}_{j\in J}$, depends on the sampling set $X := \{x_j : j \in J\}$. Let X be a countable subset of $\mathbb{R}^d$.

Definition I.4.2. *We say that $X := \{x_j : j \in J\}$ is a* set of sampling *for $V^2(\Phi)$ if there exist positive constants α and β such that*

$$\alpha||f||_{L^2}^2 \leq ||\{f(x_j)\}_{j\in J}||_{l^2(J)}^2 \leq \beta||f||_{L^2}^2 \qquad \text{for all } f \in V^2(\Phi). \tag{I.4.9}$$

Notice that if X is a set of sampling for the reproducing kernel Hilbert space $V^2(\Phi)$, then the collection $\{K_{x_j}\}_{j\in J}$ forms a frame for $V^2(\Phi)$, which gives us the following stable reconstruction formula for $f \in V^2(\Phi)$:

$$f = \sum_{j\in J} \left\langle f, K_{x_j} \right\rangle \widetilde{K_{x_j}}, \tag{I.4.10}$$

where $\{\widetilde{K_{x_j}}\}_{j\in J}$ is the canonical dual frame associated to $\{K_{x_j}\}_{j\in J}$. Namely, $\widetilde{K_{x_j}} := S^{-1}K_{x_j}$, where S is the frame operator on $V^2(\Phi)$ associated to the frame $\{K_{x_j}\}_{j\in J}$, i.e.

$$Sf = \sum_{j\in J} \left\langle f, K_{x_j} \right\rangle K_{x_j}. \tag{I.4.11}$$

I.5 Weighted-Average Sampling

In practice, the assumption that the samples $\{f(x_j) : j \in J\}$ can be measured exactly is not realistic. A better assumption is that the sampled data are of the form

$$g^i_{x_j} = \left\langle f, \psi^i_{x_j} \right\rangle = \int_{\mathbb{R}^d} f(x)\psi^i_{x_j}(x)dx,$$

where $\{\psi^i_{x_j} : 1 \le i \le s, j \in J\}$ is a set of functionals that act on the function f to produce the data $\{g^i_{x_j} : 1 \le i \le s, J \in J\}$. The functionals $\{\psi^i_{x_j} : 1 \le i \le s, j \in J\}$ may reflect the characteristics of the sampling devices [3].

Throughout this paper, in the case of weighted-average sampling, we assume the averaging kernels $\psi^i_{x_j}$ are shifts of the functions ψ^i, i.e., we assume $\psi^i_{x_j} = \psi^i(\cdot - x_j)$ for some real-valued vector function $\Psi = (\psi^1, \ldots, \psi^s)^T$ in $\left(L^2(\mathbb{R}^d)\right)^{(s)}$. We also require that the Gramian

$$G_\Psi(\xi) := \sum_{k \in \mathbb{Z}^d} \widehat{\Psi}(\xi + k)\overline{\widehat{\Psi}(\xi + k)}^T$$

be bounded, i.e. there exists a positive number η such that $G_\Psi(\xi) \le \eta I$, a.e. $\xi \in \mathbb{R}^d$ [5].

We still assume our sampled function f comes from the shift-invariant space $V^2(\Phi)$, with $\Phi \in (W^1_0)^{(r)}$ satisfying (I.4.7). In the case of weighted-average data, the problem of recovering a function $f \in V^2(\Phi)$ from the countable collection of data $\{g^i_{x_j} : 1 \le i \le s, j \in J\}$ is well-posed if

$$\alpha \|f\|^2_{L^2} \le \sum_{i=1}^{s} \sum_{j \in J} \left\| \left\langle f, \psi^i_{x_j} \right\rangle \right\|^2 \le \beta \|f\|^2_{L^2} \quad \text{for all } f \in V^2(\Phi), \tag{I.5.12}$$

where α and β are positive constants independent of f.

Notice that (I.5.12) appears to satisfy a frame condition. However, our functions $\{\psi^i : 1 \le i \le s\}$ are not necessarily in the space $V^2(\Phi)$. As in [1], consider the orthogonal projection P from $L^2(\mathbb{R}^d)$ onto $V^2(\Phi)$, and define $\theta^i_{x_j} := P\psi^i_{x_j}$. Then for all $f \in V^2(\Phi)$,

$$\left\langle f, \theta^i_{x_j} \right\rangle = \left\langle f, P\psi^i_{x_j} \right\rangle = \left\langle Pf, \psi^i_{x_j} \right\rangle = \left\langle f, \psi^i_{x_j} \right\rangle.$$

Thus condition (I.5.12) implies that $\{\theta^i_{x_j} : 1 \le i \le s, j \in J\}$ forms a frame for $V^2(\Phi)$. Furthermore, using the orthonormality of $\{\phi^l(\cdot - k)\}$, we can write

$$\begin{aligned}
\theta^i_{x_j}(x) &= \sum_{l=1}^{r} \sum_{n \in \mathbb{Z}^d} \left\langle \theta^i_{x_j}, \phi^l(\cdot - n) \right\rangle \phi^l(x-n) \\
&= \sum_{l=1}^{r} \sum_{n \in \mathbb{Z}^d} \left\langle \psi^i_{x_j}, \phi^l(\cdot - n) \right\rangle \phi^l(x-n).
\end{aligned}$$

There exists a dual frame $\{\widetilde{\theta^i_{x_j}} : 1 \le i \le s, j \in J\}$, defined by

$$\widetilde{\theta^i_{x_j}} := S^{-1}\theta^i_{x_j},$$

where S is the frame operator on $V^2(\Phi)$ corresponding to the frame $\{\theta^i_{x_j}\}$, i.e.

$$Sf = \sum_{i=1}^{s} \sum_{j \in \Omega^d_m} \sum_{k \in \mathbb{Z}^d} \left\langle f, \theta^i_{x_j} \right\rangle \theta^i_{x_j}. \tag{I.5.13}$$

We have the following reconstruction formula for any function $f \in V^2(\Phi)$:

$$f = \sum_{i=1}^{s} \sum_{j \in \Omega^d_m} \sum_{k \in \mathbb{Z}^d} \left\langle f, \psi^i_{x_j} \right\rangle \widetilde{\theta^i_{x_j}} \tag{I.5.14}$$

CHAPTER II

ERROR ANALYSIS OF FRAME RECONSTRUCTION FROM NOISY SAMPLES

Sampling and function reconstruction have been widely studied in recent decades, particularly within the setting of shift-invariant spaces (see [1] - [8], [14], [17]). However, the problem of reconstructing a function from data corrupted by noise has not been given as much attention. In [12], Eldar and Unser provide optimal results for filtering noisy samples of signals from shift-invariant and bandlimited spaces. Smale and Zhou reconstruct signals from noisy data in [25] and give error estimates for the reconstructed signal. In [23], Rohde et al. show that reconstruction from noisy data introduces spatial dependent artifacts that are undesirable for sub-pixel signal processing. In this chapter, we provide error estimates for frame reconstruction of a continuous function from a countable collection of sampled data that is corrupted by noise. We show that oversampling reduces the variance of the error of the reconstructed signal at each point $x \in \mathbb{R}^d$, and we give an exact formula for the variance as a function of the position x, of the oversampling factor m, and of the signal and sampling models.

In particular, given data $Y = \{y_j\}_{j \in J}$ of the form $y_j = f(x_j) + \varepsilon_j$, we analyze the frame reconstruction algorithm that produces a continuous function f_ε from the noisy samples $Y = \{y_j\}_{j \in J}$ of a function f in a shift invariant space. We assume the noise sequence $\{\varepsilon_j\}_{j \in J}$ to be a collection of i.i.d. random variables with $\mathrm{E}(\varepsilon_j) = 0$ and $\mathrm{var}(\varepsilon_j) = \sigma^2$. We consider uniform sets of sampling of the form $\frac{1}{m}\mathbb{Z}^d$, where m is a positive integer, and find precise estimates of $\mathrm{var}(f_\varepsilon(x) - f(x))$ which is a function of x.

We address this problem not only for exact sampling, but also for weighted average sampling as in [1] and [5]. Specifically, instead of assuming the data $\{y_j\}_{j\in J}$ arise from exact samples of f, we assume the data are of the form $y_j = \langle f, \psi(\cdot - x_j)\rangle + \varepsilon_j$, or even $y_j^i = \langle f, \psi^i(\cdot - x_j)\rangle + \varepsilon_j^i$, $1 \le i \le s$, for some vector function $\Psi = (\psi^1, \ldots, \psi^s)^T$. In this case, the uncorrupted data can be interpreted as weighted averages of f at x_j.

We begin this chapter with the case of exact sampling, and the main theorem is stated. While the complete proof is saved for section II.3, the main ideas behind the proof are illustrated by looking at the simpler case in section II.1.1. Then in section II.2, we address the weighted-average sampling problem and state the main result there. Once again, the complete proof is saved for section II.3, while we illustrate the ideas through a simpler setting in II.2.1.

II.1 Exact Sampling

Here we sample on the lattice $\frac{1}{m}\mathbb{Z}^d$, i.e., we assume our data is of the form

$$\left\{ y_{k+j/m} = f(k + \frac{1}{m}j) + \varepsilon_{k+j/m} : k \in \mathbb{Z}^d, j \in \mathbb{Z}^d \cap [0, m-1]^d \right\}$$

for some function $f \in V^2(\Phi)$. For the sake of simplicity, we denote the finite set $\Omega_m^d := \mathbb{Z}^d \cap [0, m-1]^d$, and we use the notation j/m for $\frac{1}{m}j$, where m is a positive integer and j is a vector in Ω_m^d. We also assume that for $m \ge 1$, the lattice $\frac{1}{m}\mathbb{Z}^d$ is a set of sampling for $V^2(\Phi)$, i.e., there exist positive constants α_m and β_m satisfying

$$\alpha_m \|f\|_{L^2}^2 \le \sum_{j\in\Omega_m^d} \sum_{k\in\mathbb{Z}^d} |f(k + j/m)|^2 \le \beta_m \|f\|_{L^2}^2 \quad \text{for all } f \in V^2(\Phi) \tag{II.1.1}$$

Thus the collection of reproducing kernels $\{K_{k+j/m} : k \in \mathbb{Z}^d, j \in \Omega_m^d\}$ forms a frame for $V^2(\Phi)$, and $f \in V^2(\Phi)$ is uniquely determined by its samples $\{f(k + \frac{1}{m}j) : k \in \mathbb{Z}^d, j \in \Omega_m^d\}$.

Remark II.1.1. It is reasonable to make the assumption that (II.1.1) holds. From the results in [5], we know that there exists an $M \in \mathbb{N}$ such that positive α_m and β_m satisfying (II.1.1) exist for all $m \geq M$. Moreover, if positive α_1 and β_1 exist (i.e., if $\mathbb{Z}^d$ is a set of sampling for $V^2(\Phi)$), then positive α_m and β_m exist for all $m \in \mathbb{N}$.

Recall from the previous chapter that f can be recovered from its samples as follows:

$$f = \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} \left\langle f, K_{k+j/m} \right\rangle \widetilde{K}_{k+j/m} = \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} f(k + j/m) \widetilde{K}_{k+j/m}.$$

Given data $\{y_{k+j/m} = f(k + \frac{1}{m}j) + \varepsilon_{k+j/m}\}$, we define

$$f_\varepsilon := \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} y_{k+j/m} \widetilde{K}_{k+j/m}.$$

The expected value and variance of the error between the frame reconstruction f_ε and the exact function f is a function of the position x, the oversampling factor m^d, and the noise variance σ^2. The precise estimates and best constants are given by the following theorem.

Theorem II.1.2. *Let $\Phi = (\phi^1, \dots, \phi^r)^T$ satisfy $G_\Phi(\xi) = I$ a.e. ξ, and $\phi^i \in W^1 \cap C^0$, $1 \leq i \leq r$. For $m \in \mathbb{N}$, let $\alpha_m, \beta_m > 0$ satisfy* (II.1.1). *Assume, for all $k \in \mathbb{Z}^d$ and $j \in \Omega_m^d$, that $y_{k+j/m} = f(k + j/m) + \varepsilon_{k+j/m}$ for some $f \in V^2(\Phi)$, where $\{\varepsilon_{k+j/m}\}$ is a collection of i.i.d. random variables satisfying $E(\varepsilon_{k+j/m}) = 0$ and $\operatorname{var}(\varepsilon_{k+j/m}) = \sigma^2$. Then $E(f_\varepsilon(x) - f(x)) = 0$, and*

$$\operatorname{var}(f_\varepsilon(x) - f(x)) = \frac{\sigma^2}{m^d} C_x(m),$$

where $C_x(m)$ *is given by* (II.3.22)*, and we have*

$$C_x(m) \overset{m\to\infty}{\longrightarrow} \sum_{i=1}^{r} \int_{[0,1]^d} \left| \sum_{k\in\mathbb{Z}^d} e^{i2\pi k\cdot x} \widehat{\phi^i}(k-\xi) \right|^2 d\xi.$$

Remark II.1.3. In section II.1.1 we show that we can also obtain slightly suboptimal estimates that are independent of m or x. In particular, for any $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m \geq M$

$$\mathrm{var}(f_\varepsilon(x) - f(x)) \leq \frac{(1+\epsilon)\sigma^2}{m^d} \left(\sum_{i=1}^{r} \left\| \phi^i \right\|_{W^1}^2 \right) \qquad \text{for all } x \in \mathbb{R}^d.$$

Remark II.1.4. In the case of uniform exact sampling, we see that it is possible to reduce the variance of the error of our reconstructed function simply by increasing the rate at which we sample. Later, we see the result holds in the case of average sampling as well, given certain conditions on the averaging functions.

II.1.1 Exact Sampling in $V^2(\phi)$

Before presenting the proof of the theorem above, we illustrate the simpler case where $r = 1$. In other words, our underlying shift-invariant space has only one generator, ϕ. This will also serve to lay the groundwork for the proof of Theorem II.1.2.

Recall from Chapter I that the inequality (II.1.1) implies that the collection of reproducing kernels $\{K_{k+j/m} : k \in \mathbb{Z}^d, j \in \Omega_m^d\}$ is a frame for $V^2(\phi)$, where

$$K_{k+j/m}(x) = \sum_{l\in\mathbb{Z}^d} \phi(k + j/m - l)\phi(x - l), \tag{II.1.2}$$

and f can be reconstructed from its samples on the lattice $\frac{1}{m}\mathbb{Z}^d$ as shown.

$$f = \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d}\left\langle f, K_{k+j/m}\right\rangle \widetilde{K}_{k+j/m} = \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d} f(k+j/m)\widetilde{K}_{k+j/m} \tag{II.1.3}$$

Because our sampling set is uniform, we can find $\widetilde{K}_{k+j/m} = S_m^{-1}K_{k+j/m}$ explicitly. Recall, for any $f \in V^2(\phi)$,

$$(S_m f)(x) = \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d}\left\langle f, K_{k+j/m}\right\rangle K_{k+j/m}(x). \tag{II.1.4}$$

Notice that

$$K_{k+j/m} = K_{j/m}(\cdot - k) \qquad \text{for all } k \in \mathbb{Z}^d.$$

We then apply the Fourier transform to (II.1.4), and get

$$\widehat{(S_m f)}(\xi) = \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d}\left(f * K_{j/m}^{\vee}\right)(k)\; e^{-i2\pi k\cdot\xi}\,\widehat{K_{j/m}}(\xi),$$

where $K_{j/m}^{\vee}(x) = K_{j/m}(-x)$. Notice $\sum_{k\in\mathbb{Z}^d}\left(f * K_{j/m}^{\vee}\right)(k)\; e^{-i2\pi k\cdot\xi}$ is the Fourier series of the sequence whose terms are samples of the function $f * K_{j/m}^{\vee}$ on the integer lattice. Thus, by (I.1.2) and properties (iii) and (iv) of the Fourier transform, we have

$$\widehat{(S_m f)}(\xi) = \sum_{j\in\Omega_m^d}\left(\sum_{k\in\mathbb{Z}^d}\widehat{f}(\xi+k)\overline{\widehat{K_{j/m}}(\xi+k)}\right)\widehat{K_{j/m}}(\xi).$$

For any $f = \sum_{l\in\mathbb{Z}^d} c(l)\phi(\cdot - l)$ in $V^2(\phi)$, we can use the fact that convolution becomes multiplication in the Fourier domain to express $\widehat{f}(\xi) = \widehat{c}(\xi)\widehat{\phi}(\xi)$. Then we use (II.1.2) and the fact that the Fourier series of a sequence is periodic with period 1 (i.e.,

$\widehat{c}(\xi + k) = \widehat{c}(\xi)$ for $k \in \mathbb{Z}^d$) to write

$$\widehat{(S_m f)}(\xi) = \sum_{j\in\Omega_m^d} \widehat{c}(\xi)\left(\sum_{k\in\mathbb{Z}^d}\widehat{\phi}(\xi+k)\overline{\widehat{\phi}(\xi+k)}\right)\left|\widehat{p_{j/m}}(\xi)\right|^2\widehat{\phi}(\xi)$$
$$= \left(\sum_{j\in\Omega_m^d}\left|\widehat{p_{j/m}}(\xi)\right|^2\right)\widehat{f}(\xi) \qquad \text{a.e. } \xi,$$

where $p_{j/m}$ is the sequence whose lth term is $p_{j/m}(l) = \phi(j/m - l)$. Thus, for any $f \in V^2(\phi)$, we have

$$\widehat{(S_m^{-1} f)}(\xi) = \left(\sum_{j\in\Omega_m^d}\left|\widehat{p_{j/m}}(\xi)\right|^2\right)^{-1}\widehat{f}(\xi) \tag{II.1.5}$$

Specifically, for fixed $j \in \Omega_m^d$,

$$(\widehat{S_m^{-1} K_{j/m}})(\xi) = \left(\sum_{j'\in\Omega_m^d}\left|\widehat{p_{j'/m}}(\xi)\right|^2\right)^{-1}\widehat{p_{j/m}}(\xi)\widehat{\phi}(\xi). \tag{II.1.6}$$

Using (II.1.5) and the fact that translation corresponds to modulation in the Fourier domain, it can easily be verified that $S_m^{-1}K_{k+j/m} = (S_m^{-1}K_{j/m})(\cdot - k)$.

Remark II.1.5. Using equation (II.1.1), one can verify that $0 < \alpha_m \leq \sum_{j\in\Omega_m^d}\left|\widehat{p_{j/m}}(\xi)\right|^2$ for all ξ, and hence that the formulas (II.1.5) and (II.1.6) are well defined. In the proof of Theorem II.1.2, we will prove the stronger result that when m is large, there is a positive lower bound for $\sum_{j\in\Omega_m^d}\left|\widehat{p_{j/m}}(\xi)\right|^2$ that does not depend on m.

Given data $\{y_{k+j/m} = f(k + \frac{1}{m}j) + \varepsilon_{k+j/m} : k \in \mathbb{Z}^d, j \in \Omega_m^d\}$, we define

$$f_\varepsilon := \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d} y_{k+j/m} S_m^{-1}K_{k+j/m} = \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d} y_{k+j/m}(S_m^{-1}K_{j/m})(\cdot - k).$$

We assume that the error $\{\varepsilon_{k+j/m}\}$ is a collection of i.i.d. random variables with mean zero and variance σ^2. A simple calculation shows that

$$E(f_\varepsilon(x) - f(x)) = \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d} E(\varepsilon_{k+j/m}) S_m^{-1} K_{k+j/m}(x) = 0.$$

We can compute $\operatorname{var}(f_\varepsilon(x) - f(x))$.

$$\begin{aligned}
\operatorname{var}(f_\varepsilon(x) - f(x)) &= \operatorname{var}\left(\sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d} \varepsilon_{k+j/m} S_m^{-1} K_{k+j/m}(x)\right)\\
&= \sigma^2 \sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d} \left|S_m^{-1} K_{j/m}(x-k)\right|^2\\
&= \sigma^2 \sum_{j\in\Omega_m^d} \int_{[0,1]^d} \left| e^{-i2\pi x\cdot\xi} \sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k}\widehat{S_m^{-1}K_{j/m}}(k-\xi)\right|^2 \, d\xi\\
&= \sigma^2 \sum_{j\in\Omega_m^d} \int_{[0,1]^d} \left| \sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k}\widehat{S_m^{-1}K_{j/m}}(k-\xi)\right|^2 \, d\xi\\
&= \sigma^2 \sum_{j\in\Omega_m^d} \int_{[0,1]^d} \left| \sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k} \frac{\widehat{p_{j/m}}(-\xi)\widehat{\phi}(k-\xi)}{\sum_{j'\in\Omega_m^d}\left|\widehat{p_{j'/m}}(-\xi)\right|^2}\right|^2 \, d\xi\\
&= \sigma^2 \int_{[0,1]^d} \frac{\sum_{j\in\Omega_m^d}\left|\widehat{p_{j/m}}(-\xi)\right|^2 \left|\sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k}\widehat{\phi}(k-\xi)\right|^2}{\left|\sum_{j'\in\Omega_m^d}\left|\widehat{p_{j'/m}}(-\xi)\right|^2\right|^2} \, d\xi\\
&= \frac{\sigma^2}{m^d}\int_{[0,1]^d} \frac{\left|\sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k}\widehat{\phi}(k-\xi)\right|^2}{\frac{1}{m^d}\sum_{j\in\Omega_m^d}\left|\sum_{l\in\mathbb{Z}^d}\phi(j/m-l)e^{i2\pi l\cdot\xi}\right|^2} \, d\xi\\
&= \frac{\sigma^2}{m^d} C_x(m).
\end{aligned}$$

Consider the Zak transform of ϕ,

$$Z\phi(t,\xi) = \sum_{l\in\mathbb{Z}^d} \phi(t-l)e^{i2\pi l\cdot\xi}. \tag{II.1.7}$$

Because $\phi \in W_0^1$, $Z\phi$ is a well-defined, continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ [18]. Focusing on the denominator in the above calculation, we notice

$$\frac{1}{m^d} \sum_{j \in \Omega_m^d} \left| \sum_{l \in \mathbb{Z}^d} \phi(j/m - l) e^{i2\pi l \cdot \xi} \right|^2 \overset{m \to \infty}{\longrightarrow} \int_{[0,1]^d} |Z\phi(t, \xi)|^2 \; dt$$

for all $\xi \in [0,1]^d$.

Lemma II.1.6. *For every $\xi \in [0,1]^d$, $\int_{[0,1]^d} |Z\phi(t,\xi)|^2 \; dt = 1$.*

Now, for each positive integer m, define the function

$$g_m(\xi) := \frac{1}{m^d} \sum_{j \in \Omega_m^d} \left| \sum_{l \in \mathbb{Z}^d} \phi(j/m - l) e^{i2\pi l \cdot \xi} \right|^2 \qquad \xi \in [0,1]^d.$$

Lemma II.1.6 tells us that $g_m(\xi) \to 1$ pointwise. In fact, it will be shown in the proof of Theorem II.1.2 that g_m converges uniformly to the constant function 1 on the unit cube $[0,1]^d$.

Therefore, for any $\epsilon > 0$, there exists a number $M \in \mathbb{N}$ such that for all $m \geq M$, sampling on the lattice $\frac{1}{m}\mathbb{Z}^d$ gives the estimate

$$\mathrm{var}(f_\varepsilon(x) - f(x)) \leq \frac{(1+\epsilon)\sigma^2}{m^d} \int_{[0,1]^d} \left| \sum_{k \in \mathbb{Z}^d} e^{i2\pi x \cdot k} \widehat{\phi}(k - \xi) \right|^2 \; d\xi. \tag{II.1.8}$$

Notice, by (I.1.3), that

$$\sum_{k \in \mathbb{Z}^d} e^{i2\pi x \cdot k} \widehat{\phi}(k - \xi) = \sum_{k \in \mathbb{Z}^d} \phi(x + k) e^{i2\pi \xi \cdot (x+k)},$$

which implies that

$$\left| \sum_{k \in \mathbb{Z}^d} e^{i2\pi x \cdot k} \widehat{\phi}(k - \xi) \right|^2 = \left| \sum_{k \in \mathbb{Z}^d} \phi(x + k) e^{i2\pi \xi \cdot k} \right|^2$$

Then the integral in equation (II.1.8) represents the square of the L^2 norm of the Fourier series of the sequence whose terms are $\{\phi(x+k)\}_{k\in\mathbb{Z}^d}$. By Plancherel, we have

$$\int_{[0,1]^d} \left| \sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k} \widehat{\phi}(k-\xi) \right|^2 d\xi = \sum_{k\in\mathbb{Z}^d} |\phi(x+k)|^2 \tag{II.1.9}$$

Therefore, for large enough m, we have

$$\operatorname{var}(f_\varepsilon(x) - f(x)) \leq \frac{(1+\epsilon)\sigma^2}{m^d} \sum_{k\in\mathbb{Z}^d} |\phi(x+k)|^2 \qquad \text{for every } x \in \mathbb{R}^d.$$

In other words, we obtain the slightly suboptimal estimate that depends on x but does not depend on m for large m. Also notice that $\operatorname{var}(f_\varepsilon(x) - f(x))$ is periodic with period 1. Then for any $x \in [0,1]^d$,

$$\sum_{k\in\mathbb{Z}^d} |\phi(x+k)|^2 \leq \left(\sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} |\phi(x+k)| \right)^2 = \|\phi\|_{W^1}^2 \,, \tag{II.1.10}$$

giving a coarser estimate that does not depend on x or on m.

II.2 Average Sampling

Here we assume our data is of the form

$$\left\{ \left\langle f, \psi^i(\cdot - (k + j/m)) \right\rangle + \varepsilon^i_{k+j/m} : k \in \mathbb{Z}^d,\ j \in \Omega^d_m, 1 \leq i \leq s \right\}$$

for some $f \in V^2(\Phi)$ and some real-valued vector function $\Psi = (\psi^1, \ldots, \psi^s)^T$, where $\Psi \in \left[L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \right]^{(s)}$. We use the notation $\psi^i_{k+j/m}$ to denote $\psi^i(\cdot - (k+j/m))$. We continue to assume $\Phi \in \left(L^2(\mathbb{R}^d) \right)^{(r)}$ satisfies (I.4.7) and that $\Phi \in \left(W^1_0 \right)^{(r)}$.

In order to recover a function f in $V^2(\Phi)$ from its weighted averages using shifts of the functions ψ^i, Ψ must satisfy certain conditions. We require that the Gramian

$$G_\Psi(\xi) := \sum_{k\in\mathbb{Z}^d} \widehat{\Psi}(\xi+k)\overline{\widehat{\Psi}(\xi+k)}^T$$

be bounded, i.e. there exists a number η such that $G_\Psi(\xi) \le \eta I$, a.e. $\xi \in \mathbb{R}^d$ [5]. Furthermore, we assume Ψ is such that, for each $m \in \mathbb{N}$, there exist positive constants α_m and β_m satisfying

$$\alpha_m \|f\|_2^2 \le \sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \sum_{k\in\mathbb{Z}^d} \left|\left\langle f, \psi^i_{k+j/m}\right\rangle\right|^2 \le \beta_m \|f\|_2^2 \qquad \text{(II.2.11)}$$

for all f in $V^2(\Phi)$. Finally, we also assume

$$\lim_{N\to\infty} \sup_{\xi\in[0,1]^d} \sum_{i=1}^{s} \sum_{|k|\ge N} \left|\widehat{\psi^i}(\xi+k)\right|^2 = 0 \qquad \text{(II.2.12)}$$

Condition (II.2.12) comes from [5] and serves to exclude pathological examples. Because condition (II.2.11) is satisfied, $f \in V^2(\Phi)$ is uniquely determined by, and can be stably reconstructed from, the collection

$$\{\left\langle f, \psi^i(\cdot - (k+j/m))\right\rangle : k \in \mathbb{Z}^d,\ j \in \Omega_m^d, 1 \le i \le s\}.$$

Recall that ψ^i is not necessarily in $V^2(\Phi)$, so although (II.2.11) is satisfied, the collection $\{\psi^i_{k+j/m}\}$ does not constitute a frame for $V^2(\Phi)$. As in [1], consider the orthogonal projection P from $L^2(\mathbb{R}^d)$ onto $V^2(\Phi)$, and define

$$\theta^i_{k+j/m} := P\psi^i_{k+j/m}.$$

Then for all $f \in V^2(\Phi)$,

$$\left\langle f, \theta^i_{k+j/m} \right\rangle = \left\langle f, P\psi^i_{k+j/m} \right\rangle = \left\langle Pf, \psi^i_{k+j/m} \right\rangle = \left\langle f, \psi^i_{k+j/m} \right\rangle .$$

Thus condition (II.2.11) implies that $\{\theta^i_{k+j/m} : k \in \mathbb{Z}^d, j \in \Omega^d_m, 1 \leq i \leq s\}$ forms a frame for $V^2(\Phi)$. Furthermore, using the orthonormality of $\{\phi^l(\cdot - k)\}$, we can write

$$\begin{aligned} \theta^i_{k+j/m}(x) &= \sum_{l=1}^{r} \sum_{n \in \mathbb{Z}^d} \left\langle \theta^i_{k+j/m}, \phi^l(\cdot - n) \right\rangle \phi^l(x-n) \\ &= \sum_{l=1}^{r} \sum_{n \in \mathbb{Z}^d} \left\langle \psi^i_{k+j/m}, \phi^l(\cdot - n) \right\rangle \phi^l(x-n), \end{aligned}$$

and we see that $\theta^i_{k+j/m} = \theta^i_{j/m}(\cdot - k)$. There exists a dual frame $\{\widetilde{\theta}^i_{k+j/m} : k \in \mathbb{Z}^d, j \in \Omega^d_m 1 \leq i \leq s\}$, defined by

$$\widetilde{\theta}^i_{k+j/m} := S_m^{-1} \theta^i_{k+j/m},$$

where S_m is the frame operator on $V^2(\Phi)$ corresponding to the frame $\{\theta^i_{k+j/m}\}$, i.e.

$$S_m f = \sum_{i=1}^{s} \sum_{j \in \Omega^d_m} \sum_{k \in \mathbb{Z}^d} \left\langle f, \theta^i_{k+j/m} \right\rangle \theta^i_{k+j/m}.$$

Then for any scalar-valued sequence $\{a^i_{k+j/m} : k \in \mathbb{Z}^d, j \in \Omega^d_m, 1 \leq i \leq s\}$ satisfying

$$\sum_{i=1}^{s} \sum_{j \in \Omega^d_m} \sum_{k \in \mathbb{Z}^d} \left| a^i_{k+j/m} \right|^2 < \infty,$$

the function defined by

$$\sum_{i=1}^{s} \sum_{j \in \Omega^d_m} \sum_{k \in \mathbb{Z}^d} a^i_{k+j/m} \, \widetilde{\theta}^i_{k+j/m}$$

is in $V^2(\Phi)$ [15]. Furthermore, we have the following reconstruction formula for any function $f \in V^2(\Phi)$:

$$f = \sum_{i=1}^{s} \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} \left\langle f, \psi_{k+j/m}^i \right\rangle \widetilde{\theta}_{k+j/m}^i \tag{II.2.13}$$

Given data

$$\left\{ y_{k+j/m}^i = \left\langle f, \psi_{k+j/m}^i \right\rangle + \varepsilon_{k+j/m}^i : k \in \mathbb{Z}^d, j \in \Omega_m^d, 1 \le i \le s \right\}, \tag{II.2.14}$$

we define

$$f_\varepsilon := \sum_{i=1}^{s} \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} y_{k+j/m}^i \; \widetilde{\theta}_{k+j/m}^i. \tag{II.2.15}$$

In the case of average sampling, we arrive at results for $\text{var}(f_\varepsilon(x) - f(x))$ similar to those of Theorem II.1.2. For $\xi \in [0,1]^d$, define the self-adjoint matrix

$$G_\Phi^\Psi(\xi) := \sum_{i=1}^{s} \sum_{k \in \mathbb{Z}^d} \widehat{\Phi}(\xi + k) \overline{\widehat{\Phi}(\xi + k)}^T \left| \widehat{\psi^i}(\xi + k) \right|^2 .$$

and denote the $r \times 1$ vector

$$Z\widehat{\Phi}(-\xi, -x) = \sum_{k \in \mathbb{Z}^d} \widehat{\Phi}(k - \xi) e^{i2\pi k \cdot x}.$$

As in Theorem II.1.2, the expected value and variance of the error between the frame reconstruction f_ε and the exact function f is a function of the position x, the oversampling factor m^d, and the noise variance σ^2. The precise estimates and best constants are given by the following theorem.

Theorem II.2.1. *Let $\Phi = (\phi^1, \ldots, \phi^r)^T$ satisfy $G_\Phi(\xi) = I$ a.e. ξ, and $\phi^i \in W^1 \cap C^0$, $1 \le i \le r$. Assume $G_\Psi(\xi) \le \eta I$, a.e. $\xi \in \mathbb{R}^d$ and also that equations* (II.2.11) *and* (II.2.12) *are satisfied. Assume, for all $k \in \mathbb{Z}^d$, $j \in \Omega_m^d$, and $1 \le i \le s$, the data $\{y_{k+j/m}^i\}$ are of the form* (II.2.14) *for some $f \in V^2(\Phi)$, where $\{\varepsilon_{k+j/m}^i\}$ is a collection of i.i.d. random variables satisfying $E(\varepsilon_{k+j/m}^i) = 0$ and $\text{var}(\varepsilon_{k+j/m}^i) = \sigma^2$. Then*

$E(f_\varepsilon(x) - f(x)) = 0,$ *and*

$$\operatorname{var}(f_\varepsilon(x) - f(x)) = \frac{\sigma^2}{m^d} D_x(m),$$

where $D_x(m)$ *is given by* (II.3.26), *and*

$$D_x(m) \overset{m\to\infty}{\longrightarrow} \int_{[0,1]^d} \overline{Z\widehat{\Phi}(-\xi,-x)}^T \left(G_\Phi^\Psi(\xi)\right)^{-1} Z\widehat{\Phi}(-\xi,-x)\, d\xi.$$

Remark II.2.2. In [5] it is shown that (II.2.11) and (II.2.12) imply that there exists a positive number δ_0 such that $\delta_0 I \leq G_\Phi^\Psi(\xi)$ for all ξ. It follows from (II.1.10) that there exists a number $\delta > 0$ and a number $M \in \mathbb{N}$ such that for every $m \geq M$, we obtain the suboptimal but uniform estimate:

$$\operatorname{var}(f_\varepsilon(x) - f(x)) \leq \frac{\sigma^2}{m^d}\left(\frac{1}{\delta}\right)\left(\sum_{n=1}^{r} \|\phi^n\|_{W^1}^2\right) \qquad \text{for all } x \in \mathbb{R}^d.$$

II.2.1 Average Sampling in $V^2(\phi)$

Once again, before presenting the proof of the theorem above, we will lay the groundwork for that proof by illustrating the simpler case where $r = 1$. In other words, our underlying shift-invariant space has only one generator, ϕ.

As we did in the example in the previous section, in this uniform case, we can find $\widetilde{\theta}^i_{k+j/m} = S_m^{-1}\theta^i_{k+j/m}$, or at least its Fourier transform, explicitly. Let S_m be the frame operator on $V^2(\phi)$ associated to the frame $\left\{\theta^i_{k+j/m} : k \in \mathbb{Z}^d, j \in \Omega_m^d, 1 \leq i \leq s\right\}$. Recall that

$$(S_m f)(x) = \sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \sum_{k\in\mathbb{Z}^d} \left\langle f, \psi^i_{k+j/m}\right\rangle \theta^i_{k+j/m}(x), \tag{II.2.16}$$

$$\theta^i_{k+j/m}(x) = \sum_{l\in\mathbb{Z}^d} \left\langle \psi^i_{k+j/m}, \phi(\cdot - l)\right\rangle \phi(x-l), \tag{II.2.17}$$

and also that $\theta^i_{k+j/m} = \theta^i_{j/m}(\cdot - k)$. For any $f \in V^2(\phi)$, we apply the Fourier transform to (II.2.16) and rewrite the inner product as convolution, to get

$$\widehat{(S_m f)}(\xi) = \sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \sum_{k\in\mathbb{Z}^d} \left(f * (\psi^i_{j/m})^\vee\right)(k)\ e^{-i2\pi k\cdot\xi} \widehat{\theta^i_{j/m}}(\xi),$$

where $(\psi^i_{j/m})^\vee(x) = \psi^i_{j/m}(-x)$. Notice $\sum_{k\in\mathbb{Z}^d} \left(f * (\psi^i_{j/m})^\vee\right)(k)\ e^{-i2\pi k\cdot\xi}$ is the Fourier series of the sequence whose terms are samples of the function $f * (\psi^i_{j/m})^\vee$ on the integer lattice. Thus, by (I.1.2) and properties (iii) and (iv) of the Fourier transform, we have

$$\widehat{(S_m f)}(\xi) = \sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \left(\sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi+k)\overline{\widehat{\psi^i_{j/m}}(\xi+k)}\right) \widehat{\theta^i_{j/m}}(\xi).$$

Similarly, we can use (II.2.17) to show that

$$\widehat{\theta^i_{j/m}}(\xi) = \left(\sum_{l\in\mathbb{Z}^d} \widehat{\psi^i_{j/m}}(\xi+l)\overline{\widehat{\phi}(\xi+l)}\right) \widehat{\phi}(\xi).$$

Thus for any $f = \sum_{l\in\mathbb{Z}^d} c(l)\phi(\cdot - l)$ in $V^2(\phi)$, we have

$$\begin{aligned}
\widehat{(S_m f)}(\xi) &= \sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \widehat{c}(\xi) \left(\sum_{l\in\mathbb{Z}^d} \widehat{\phi}(\xi+l)\overline{\widehat{\psi^i_{j/m}}(\xi+l)}\right) \left(\sum_{l'\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l')}\widehat{\psi^i_{j/m}}(\xi+l')\right) \widehat{\phi}(\xi) \\
&= \left(\sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \left|\sum_{l\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l)}\widehat{\psi^i_{j/m}}(\xi+l)\right|^2\right) \widehat{f}(\xi),
\end{aligned}$$

and therefore

$$\widehat{(S_m^{-1} f)}(\xi) = \frac{\widehat{f}(\xi)}{\sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \left|\sum_{l\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l)}\widehat{\psi^i_{j/m}}(\xi+l)\right|^2}, \qquad \text{(II.2.18)}$$

provided that the denominator is nonzero. Then for fixed i and j,

$$(\widehat{S_m^{-1}\theta_{j/m}^i})(\xi) = \frac{\left(\sum_{l'\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l')}\widehat{\psi_{j/m}^i}(\xi+l')\right)\widehat{\phi}(\xi)}{\sum_{i'=1}^s \sum_{j'\in\Omega_m^d} \left|\sum_{l\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l)}\widehat{\psi_{j'/m}^{i'}}(\xi+l)\right|^2}.$$

Using (II.2.18) and property (i) of the Fourier transform, it can be verified that $S_m^{-1}\theta_{k+j/m}^i = \left(S_m^{-1}\theta_{j/m}^i\right)(\cdot - k)$. Now we can use (II.2.13) and (II.2.15) to begin computing $\mathrm{var}(f_\varepsilon(x) - f(x))$.

$\mathrm{var}(f_\varepsilon(x) - f(x))$

$$\begin{aligned}
&= \mathrm{var}\left(\sum_{i=1}^s \sum_{j\in\Omega_m^d} \sum_{k\in\mathbb{Z}^d} \varepsilon_{k+j/m}^i S_m^{-1}\theta_{k+j/m}^i(x)\right) \\
&= \sigma^2 \sum_{i=1}^s \sum_{j\in\Omega_m^d} \sum_{k\in\mathbb{Z}^d} \left|S_m^{-1}\theta_{j/m}^i(x-k)\right|^2 \\
&= \sigma^2 \sum_{i=1}^s \sum_{j\in\Omega_m^d} \int_{[0,1]^d} \left|e^{i2\pi x\cdot\xi} \sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k} \widehat{S_m^{-1}\theta_{j/m}^i}(\xi+k)\right|^2 \, d\xi \\
&= \sigma^2 \sum_{i=1}^s \sum_{j\in\Omega_m^d} \int_{[0,1]^d} \left|\sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k} \widehat{S_m^{-1}\theta_{j/m}^i}(\xi+k)\right|^2 \, d\xi \\
&= \sigma^2 \sum_{i=1}^s \sum_{j\in\Omega_m^d} \int_{[0,1]^d} \left|\sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k} \frac{\left(\sum_{l'\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l')}\widehat{\psi_{j/m}^i}(\xi+l')\right)\widehat{\phi}(\xi+k)}{\sum_{i'=1}^s \sum_{j'\in\Omega_m^d} \left|\sum_{l\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l)}\widehat{\psi_{j'/m}^{i'}}(\xi+l)\right|^2}\right|^2 \, d\xi \\
&= \sigma^2 \int_{[0,1]^d} \frac{\sum_{i=1}^s \sum_{j\in\Omega_m^d} \left|\sum_{l'\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l')}\widehat{\psi_{j/m}^i}(\xi+l')\right|^2}{\left(\sum_{i'=1}^s \sum_{j'\in\Omega_m^d} \left|\sum_{l\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l)}\widehat{\psi_{j'/m}^{i'}}(\xi+l)\right|^2\right)^2} \left|\sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k}\widehat{\phi}(\xi+k)\right|^2 \, d\xi \\
&= \frac{\sigma^2}{m^d} \int_{[0,1]^d} \frac{\left|\sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k}\widehat{\phi}(\xi+k)\right|^2}{\sum_{i=1}^s \frac{1}{m^d} \sum_{j\in\Omega_m^d} \left|\sum_{l\in\mathbb{Z}^d} \overline{\widehat{\phi}(\xi+l)}\widehat{\psi_{j/m}^i}(\xi+l)\right|^2} \, d\xi \\
&= \frac{\sigma^2}{m^d} D_x(m),
\end{aligned}$$

where

$$D_x(m) = \int_{[0,1]^d} \frac{\left|\sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k}\widehat{\phi}(\xi+k)\right|^2}{\sum_{i=1}^s \frac{1}{m^d}\sum_{j\in\Omega_m^d}\left|\sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\xi+l)}\widehat{\psi^i_{j/m}}(\xi+l)\right|^2}\, d\xi.$$

Notice from the denominator that

$$\begin{aligned}
\sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\xi+l)}\widehat{\psi^i_{j/m}}(\xi+l) &= \sum_{l\in\mathbb{Z}^d}(\phi^\vee * \psi^i_{j/m})(l)e^{-i2\pi l\cdot\xi} && \big(\text{by (I.1.2)}\big)\\
&= \sum_{l\in\mathbb{Z}^d}(\phi^\vee * \psi^i)(l-j/m)e^{-i2\pi l\cdot\xi}\\
&= \sum_{l\in\mathbb{Z}^d}(\phi * \psi^{i\vee})(j/m-l)e^{-i2\pi l\cdot\xi}.
\end{aligned}$$

Then we can see that

$$\frac{1}{m^d}\sum_{j\in\Omega_m^d}\left|\sum_{l\in\mathbb{Z}^d}\overline{\widehat{\phi}(\xi+l)}\widehat{\psi^i_{j/m}}(\xi+l)\right|^2 \overset{m\to\infty}{\longrightarrow} \int_{[0,1]^d}\left|Z(\phi*\psi^{i\vee})(t,-\xi)\right|^2\, dt$$

for each ξ, where Z represents the Zak transform as defined in (II.1.7). In the proof of Theorem II.2.1, we will see that this convergence is uniform on $[0,1]^d$.

Lemma II.2.3. *For every* $\xi \in [0,1]^d$,

$$\sum_{i=1}^s \int_{[0,1]^d}\left|Z(\phi*\psi^{i\vee})(t,-\xi)\right|^2\, dt \geq \delta > 0.$$

Therefore, using this lemma and (II.1.10), we see that for any $\epsilon > 0$ there exists a number $M \in \mathbb{N}$ such that for all $m \geq M$, average sampling on $\frac{1}{m}\mathbb{Z}^d$ gives

$$\operatorname{var}(f_\varepsilon(x) - f(x)) \leq \frac{\sigma^2}{m^d}\left(\frac{1+\epsilon}{\delta}\right)\|\phi\|^2_{W^1} \qquad \text{for all } x \in \mathbb{R}^d.$$

II.3 Proofs

II.3.1 Proof of Theorem II.1.2

We wish to compute the variance of the error as in Section II.1.1. First we must find $S_m^{-1}K_{j/m}$ explicitly. In section II.1.1 we showed that

$$\widehat{(S_m f)}(\xi) = \sum_{j\in\Omega_m^d} \left(\sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi+k)\overline{\widehat{K_{j/m}}(\xi+k)} \right) \widehat{K_{j/m}}(\xi).$$

For any $f = \sum_{l\in\mathbb{Z}^d} C(l)^T\Phi(\cdot - l)$ in $V^2(\Phi)$, we then get

$$\begin{aligned}\widehat{(S_m f)}(\xi) &= \widehat{C}(\xi)^T \left(\sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi+k)\overline{\widehat{\Phi}(\xi+k)}^T \right) \left(\sum_{j\in\Omega_m^d} \overline{\widehat{P_{j/m}}(\xi)}\widehat{P_{j/m}}(\xi)^T \right) \widehat{\Phi}(\xi) \\ &= \widehat{C}(\xi)^T \left(\sum_{j\in\Omega_m^d} \overline{\widehat{P_{j/m}}(\xi)}\widehat{P_{j/m}}(\xi)^T \right) \widehat{\Phi}(\xi) \qquad \text{a.e. } \xi\end{aligned}$$

where $P_{j/m}$ is defined as the vector sequence with terms $P_{j/m}(l) = \Phi(j/m - l)$ for $l \in \mathbb{Z}^d$, and therefore $\widehat{P_{j/m}}(\xi) = \sum_{l\in\mathbb{Z}^d}\Phi(j/m-l)e^{-i2\pi l\cdot\xi}$. Notice in the equation above that $\sum_{k\in\mathbb{Z}^d} \widehat{\Phi}(\xi+k)\overline{\widehat{\Phi}(\xi+k)}^T = I$ for almost every ξ, and that $\sum_{j\in\Omega_m^d} \overline{\widehat{P_{j/m}}(\xi)}\widehat{P_{j/m}}(\xi)^T$ is a self-adjoint $r\times r$ matrix. Define the matrix

$$A_m(\xi) := \sum_{j\in\Omega_m^d} \overline{\widehat{P_{j/m}}(\xi)}\widehat{P_{j/m}}(\xi)^T.$$

Remark II.3.1. It can be shown that $\alpha_m I \le A_m(\xi)$ for all ξ, and hence the matrix $A_m(\xi)$ is invertible. Instead, for large m we provide a stronger result in Lemma II.3.2 below. Still, it should be noted that the following formulas (II.3.19) and (II.3.20) make sense as long as (II.1.1) holds.

Therefore, we have

$$(\widehat{S_m^{-1} f})(\xi) = \widehat{C}(\xi)^T \left(A_m(\xi)\right)^{-1} \widehat{\Phi}(\xi) \tag{II.3.19}$$

Finally, using (II.1.2) and (II.3.19), for any fixed $j \in \Omega_m^d$ we have

$$(\widehat{S_m^{-1} K_{j/m}})(\xi) = \widehat{P_{j/m}}(\xi)^T \left(A_m(\xi)\right)^{-1} \widehat{\Phi}(\xi) \tag{II.3.20}$$

Using (II.3.19) and the fact that translation corresponds to modulation in the Fourier domain, it can easily be verified that $S_m^{-1} K_{k+j/m} = (S_m^{-1} K_{j/m})(\cdot - k)$.

We are now ready to compute the expected value and the variance of the error $(f_\varepsilon(x) - f(x))$. A simple calculation shows that

$E(f_\varepsilon(x) - f(x)) = \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} E(\varepsilon_{k+j/m}) S_m^{-1} K_{k+j/m} = 0.$

Also, we have

$\mathrm{var}(f_\varepsilon(x) - f(x))$

$$\begin{aligned}
&= \mathrm{var}\left(\sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} \varepsilon_{k+j/m} S_m^{-1} K_{k+j/m}(x) \right) \\
&= \sigma^2 \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} \left| S_m^{-1} K_{j/m}(x-k) \right|^2 \\
&= \sigma^2 \sum_{j \in \Omega_m^d} \int_{[0,1]^d} \left| e^{-i2\pi x \cdot \xi} \sum_{k \in \mathbb{Z}^d} \widehat{S_m^{-1} K_{j/m}}(k-\xi) e^{i2\pi k \cdot x} \right|^2 \, d\xi \\
&= \sigma^2 \sum_{j \in \Omega_m^d} \int_{[0,1]^d} \left| \sum_{k \in \mathbb{Z}^d} \widehat{S_m^{-1} K_{j/m}}(k-\xi) e^{i2\pi k \cdot x} \right|^2 \, d\xi \\
&= \sigma^2 \int_{[0,1]^d} \sum_{j \in \Omega_m^d} \left| \widehat{P_{j/m}}(-\xi)^T \left(A_m(\xi)\right)^{-1} \left(\sum_{k \in \mathbb{Z}^d} e^{i2\pi k \cdot x} \widehat{\Phi}(k-\xi) \right) \right|^2 \, d\xi
\end{aligned}$$

The matrix $\left(A_m(\xi)\right)^{-1}$ is self-adjoint because it is the inverse of a self-adjoint matrix. Next we use the fact that $a^T A b = b^T \overline{A} a$ for any vectors a and b and any self-adjoint matrix A, and hence

$$\left|a^T A b\right|^2 = \overline{a^T A b} a^T A b = \overline{b}^T A \overline{a} a^T A b. \tag{II.3.21}$$

If $\left(\overline{a} a^T\right)^{-1} = A$, then we have $\overline{b}^T A b$. Now we have

$\mathrm{var}(f_\varepsilon(x) - f(x))$

$$= \sigma^2 \int_{[0,1]^d} \left(\sum_{k \in \mathbb{Z}^d} e^{-i2\pi k \cdot x} \overline{\widehat{\Phi}(k-\xi)} \right)^T (A_m(\xi))^{-1} \left(\sum_{k' \in \mathbb{Z}^d} e^{i2\pi k' \cdot x} \widehat{\Phi}(k'-\xi) \right) d\xi$$
$$= \frac{\sigma^2}{m^d} \int_{[0,1]^d} \left(\sum_{k \in \mathbb{Z}^d} e^{-i2\pi k \cdot x} \overline{\widehat{\Phi}(k-\xi)} \right)^T \left(\frac{1}{m^d} A_m(\xi) \right)^{-1} \left(\sum_{k' \in \mathbb{Z}^d} e^{i2\pi k' \cdot x} \widehat{\Phi}(k'-\xi) \right) d\xi$$

Thus we have shown that

$$\begin{aligned} \mathrm{var}(f_\varepsilon(x) - f(x)) &= \frac{\sigma^2}{m^d} \int_{[0,1]^d} \overline{Z\widehat{\Phi}(-\xi,-x)}^T \left(\frac{1}{m^d} A_m(\xi) \right)^{-1} Z\widehat{\Phi}(-\xi,-x)\, d\xi \\ &= \frac{\sigma^2}{m^d} C_x(m), \end{aligned} \tag{II.3.22}$$

where $Z\widehat{\Phi}(-\xi,-x) = \sum_{k \in \mathbb{Z}^d} e^{i2\pi k \cdot x} \widehat{\Phi}(k-\xi)$.

Lemma II.3.2. *For every $\epsilon > 0$ there is a number $M \in \mathbb{N}$ such that for every $m \geq M$*

$$(1-\epsilon) I \leq \frac{1}{m^d} A_m(\xi) \qquad \textit{for all } \ \xi \in [0,1]^d.$$

Using Lemma II.3.2, we conclude that there is a number $M \in \mathbb{N}$ such that for all $m \geq M$, sampling on the set $\frac{1}{m}\mathbb{Z}^d$ gives

$\mathrm{var}(f_\epsilon(x) - f(x))$

$$\leq \frac{(1+\epsilon)\sigma^2}{m^d} \int_{[0,1]^d} \left(\sum_{k \in \mathbb{Z}^d} e^{-i2\pi k \cdot x} \overline{\widehat{\Phi}(k-\xi)} \right)^T \left(\sum_{k' \in \mathbb{Z}^d} e^{i2\pi k' \cdot x} \widehat{\Phi}(k'-\xi) \right) d\xi.$$
$$= \frac{(1+\epsilon)\sigma^2}{m^d} \int_{[0,1]^d} \sum_{i=1}^{r} \left| \sum_{k \in \mathbb{Z}^d} e^{i2\pi k \cdot x} \widehat{\phi^i}(k-\xi) \right|^2 d\xi$$

In section II.1.1, we saw that

$$\int_{[0,1]^d} \left| \sum_{k\in\mathbb{Z}^d} e^{i2\pi x\cdot k} \widehat{\phi^i}(k-\xi) \right|^2 \, d\xi = \sum_{k\in\mathbb{Z}^d} \left|\phi^i(x+k)\right|^2 \leq \left\|\phi^i\right\|_{W^1}^2$$

for all $x \in \mathbb{R}^d$. Thus when m is large enough,

$$\begin{aligned} \operatorname{var}(f_\varepsilon(x) - f(x)) &\leq \frac{(1+\epsilon)\sigma^2}{m^d} \left(\sum_{i=1}^r \left(\sum_{k\in\mathbb{Z}^d} \left|\phi^i(x+k)\right|^2 \right) \right) \\ &\leq \frac{(1+\epsilon)\sigma^2}{m^d} \left(\sum_{i=1}^r \left\|\phi^i\right\|_{W^1}^2 \right) \qquad \text{for all } x \in \mathbb{R}^d. \end{aligned}$$

□

II.3.2 Proof of Lemma II.3.2

Notice that, for $1 \leq n, n' \leq r$, the (n, n')-entry of $\frac{1}{m^d} A_m(\xi)$ is

$$\begin{aligned} &\left[\frac{1}{m^d} A_m(\xi)\right]_{(n,n')} \\ &= \frac{1}{m^d} \sum_{j\in\Omega_m^d} \left(\sum_{l\in\mathbb{Z}^d} \phi^n(j/m - l) e^{i2\pi l\cdot\xi} \right) \left(\sum_{l'\in\mathbb{Z}^d} \phi^{n'}(j/m - l') e^{-i2\pi l'\cdot\xi} \right) \\ &= \frac{1}{m^d} \sum_{j\in\Omega_m^d} \left(\sum_{l\in\mathbb{Z}^d} \sum_{l'\in\mathbb{Z}^d} \phi^n(j/m - l) \phi^{n'}\Big(j/m - l - (l' - l)\Big) e^{-i2\pi(l'-l)\cdot\xi} \right) \\ &= \frac{1}{m^d} \sum_{j\in\Omega_m^d} \left(\sum_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \phi^n(j/m - l) \phi^{n'}(j/m - l - k) e^{-i2\pi k\cdot\xi} \right) \\ &= \sum_{k\in\mathbb{Z}^d} e^{-i2\pi k\cdot\xi} \frac{1}{m^d} \sum_{j\in\Omega_m^d} \left(\sum_{l\in\mathbb{Z}^d} \phi^n(j/m - l) \phi^{n'}(j/m - l - k) \right) \end{aligned}$$

Taking the limit as m goes to infinity, we have

$$\lim_{m\to\infty}\left[\frac{1}{m^d}A_m(\xi)\right]_{(n,n')} = \sum_{k\in\mathbb{Z}^d} e^{-i2\pi k\cdot\xi}\int_{\mathbb{R}^d}\phi^n(x)\phi^{n'}(x-k)\ dx = \delta_{n,n'}$$

Thus the diagonal entries of the matrix converge to 1 and the off-diagonal entries of the matrix converge to 0 for each ξ.

Now we will show the collection $\left\{\left[\frac{1}{m^d}A_m(\cdot)\right]_{(n,n')} : m\in\mathbb{N}\right\}$ is equicontinuous and conclude that convergence is uniform on the unit cube $[0,1]^d$. Recall that a collection $\mathcal{G}$ of continuous functions on $[0,1]^d$ is equicontinuous if for every $\epsilon>0$ there is a $\delta>0$ such that for all $g\in\mathcal{G}$, $|g(\xi_1)-g(\xi_2)|<\epsilon$ for all $\xi_1,\xi_2\in[0,1]^d$ satisfying $|\xi_1-\xi_2|<\delta$.

Let $1\le n,n'\le r$. Let $\epsilon>0$. There exists a number $N\in\mathbb{N}$ such that

$$\sum_{|l|>N}\sup_{x\in[0,1]^d}|\phi^n(x-l)| < \frac{\epsilon}{6\left\|\phi^{n'}\right\|_{W^1}}.$$

Then there exists a number $N'\in\mathbb{N}$ such that

$$\sum_{|k|>N'}\sup_{x\in[0,1]^d}\left|\phi^{n'}(x-l-k)\right| < \frac{\epsilon}{6\left\|\phi^{n}\right\|_{W^1}} \qquad \text{for all } l \text{ s.t. } |l|\le N.$$

Then there exists a number $\delta>0$ such that whenever $|\xi_1-\xi_2|<\delta$,

$$\left|e^{-i2\pi k\cdot\xi_1}-e^{-i2\pi k\cdot\xi_2}\right| < \frac{\epsilon}{3\left\|\phi^n\right\|_{W^1}\left\|\phi^{n'}\right\|_{W^1}} \qquad \text{for every } \ k \ \text{ s.t. } |k|\le N'.$$

Notice

$$\left|\left[\tfrac{1}{m^d}A_m(\xi_1)\right]_{(n,n')}-\left[\tfrac{1}{m^d}A_m(\xi_2)\right]_{(n,n')}\right|$$
$$= \left|\tfrac{1}{m^d}\textstyle\sum_{j\in\Omega_m^d}\sum_{l\in\mathbb{Z}^d}\phi^n(j/m-l)\sum_{k\in\mathbb{Z}^d}\phi^{n'}(j/m-l-k)\left(e^{-i2\pi k\cdot\xi_1}-e^{-i2\pi k\cdot\xi_2}\right)\right|$$

$$\le \frac{1}{m^d}\sum_{j\in\Omega_m^d}\left[\sum_{|l|\le N}|\phi^n(j/m-l)|\left(\sum_{|k|\le N'}\left|\phi^{n'}(j/m-l-k)\right|\left|e^{-i2\pi k\cdot\xi_1}-e^{-i2\pi k\cdot\xi_2}\right|\right.\right.$$

$$+ \sum_{|k|>N'} \left|\phi^{n'}(j/m-l-k)\right| \left|e^{-i2\pi k\cdot\xi_1} - e^{-i2\pi k\cdot\xi_2}\right|\Bigg)$$

$$+ \sum_{|l|>N} |\phi^n(j/m-l)| \sum_{k\in\mathbb{Z}^d} \left|\phi^{n'}(j/m-l-k)\right| \left|e^{-i2\pi k\cdot\xi_1} - e^{-i2\pi k\cdot\xi_2}\right|\Bigg]$$

$< \frac{1}{m^d}\sum_{j\in\Omega_m^d}\left(\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\right) \ = \ \epsilon$

Thus the collection $\left\{\left[\frac{1}{m^d}A_m(\cdot)\right]_{(n,n')} : m\in\mathbb{N}\right\}$ is equicontinuous, and hence for each pair (n,n'), $\left[\frac{1}{m^d}A_m(\cdot)\right]_{(n,n')} \to \delta_{n,n'}$ uniformly on $[0,1]^d$.

Therefore, for any $\epsilon>0$, there is a number $M\in\mathbb{N}$ such that for all $m\geq M$

$$\left\|\frac{1}{m^d}A_m(\xi) - I\right\| < \epsilon \qquad \text{for all } \ \xi\in[0,1]^d.$$

Hence our lemma is proved.

□

II.3.3 Proof of Lemma II.1.6

Our objective is to show that $\int_{[0,1]^d} |Z\phi(t,\xi)|^2 \ dt = 1$ for every $\xi\in[0,1]^d$.

$$\begin{aligned}
\int_{[0,1]^d} |Z\phi(t,\xi)|^2 \ dt &= \int_{[0,1]^d} \left|\sum_{l\in\mathbb{Z}^d}\phi(t-l)e^{i2\pi l\cdot\xi}\right|^2 \ dt \\
&= \int_{[0,1]^d} \left|\sum_{l\in\mathbb{Z}^d}\widehat{\phi}(\xi+l)e^{i2\pi t\cdot(\xi+l)}\right|^2 \ dt \qquad \big(\text{by (I.1.3)}\big) \\
&= \int_{[0,1]^d} \left|e^{i2\pi t\cdot\xi}\sum_{l\in\mathbb{Z}^d}\widehat{\phi}(\xi+l)e^{i2\pi t\cdot l}\right|^2 \ dt \\
&= \int_{[0,1]^d} \left|\sum_{l\in\mathbb{Z}^d}\widehat{\phi}(\xi+l)e^{i2\pi t\cdot l}\right|^2 \ dt \\
&= \sum_{l\in\mathbb{Z}^d}\left|\widehat{\phi}(\xi+l)\right|^2 \\
&= \ 1 \qquad \text{a.e. } \xi \qquad \big(\text{by (I.4.7)}\big)
\end{aligned}$$

Because $Z\phi$ is continuous, $\int_{[0,1]^d} |Z\phi(t,\xi)|^2 \ dt$ is a continuous function of ξ. Therefore $\int_{[0,1]^d} |Z\phi(t,\xi)|^2 \ dt = 1$ for every $\xi \in [0,1]^d$.

□

II.3.4 Proof of Theorem II.2.1

Once again, our objective is to compute the expected value and the variance of $\Big(f_\varepsilon(x) - f(x)\Big)$, where in this case

$$f = \sum_{i=1}^{s} \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} \left\langle f, \psi^i_{k+j/m} \right\rangle \widetilde{\theta}^i_{k+j/m}$$

and

$$f_\varepsilon := \sum_{i=1}^{s} \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} y^i_{k+j/m} \ \widetilde{\theta}^i_{k+j/m}.$$

A simple calculation shows

$$E\Big(f_\varepsilon(x) - f(x)\Big) = \sum_{i=1}^{s} \sum_{j \in \Omega_m^d} \sum_{k \in \mathbb{Z}^d} E\left(\varepsilon^i_{k+j/m}\right) \ \widetilde{\theta}^i_{k+j/m} = 0.$$

To compute the variance, we first need to compute $\widetilde{\theta}^i_{k+j/m} = S_m^{-1}\theta^i_{k+j/m}$ explicitly. In section II.2.1, we showed that

$$\widehat{(S_m f)}(\xi) = \sum_{i=1}^{s} \sum_{j \in \Omega_m^d} \left(\sum_{k \in \mathbb{Z}^d} \widehat{f}(\xi + k) \overline{\widehat{\psi^i_{j/m}}(\xi + k)} \right) \widehat{\theta^i_{j/m}}(\xi), \tag{II.3.23}$$

and

$$\widehat{\theta^i_{j/m}}(\xi) = \left(\sum_{l \in \mathbb{Z}^d} \widehat{\psi^i_{j/m}}(\xi + l) \overline{\widehat{\Phi}(\xi + l)} \right)^T \widehat{\Phi}(\xi). \tag{II.3.24}$$

Define the self-adjoint matrix

$$[A_m]_\Phi^\Psi(\xi) := \sum_{i=1}^{s} \sum_{j\in\Omega_m^d} \left(\sum_{l\in\mathbb{Z}^d} \widehat{\Phi}(\xi+l)\overline{\widehat{\psi_{j/m}^i}(\xi+l)} \right) \left(\sum_{l'\in\mathbb{Z}^d} \overline{\widehat{\Phi}(\xi+l')}\widehat{\psi_{j/m}^i}(\xi+l') \right)^T$$

For any $f = \sum_{l\in\mathbb{Z}^d} C(l)^T \Phi(\cdot - l)$ in $V^2(\Phi)$, we see from (II.3.23) and (II.3.24) that

$$\widehat{(S_m f)}(\xi) = \widehat{C}(\xi)^T \left([A_m]_\Phi^\Psi(\xi) \right) \widehat{\Phi}(\xi)$$

Define B_j^i to be the coefficient vector sequence for the function $\theta_{j/m}^i$, i.e., $B_j^i = \left((b_j^i)^1, \ldots, (b_j^i)^r \right)^T$, where $(b_j^i)^n(l) = \left\langle \theta_{j/m}^i, \phi^n(\cdot - l) \right\rangle = \left\langle \psi_{j/m}^i, \phi^n(\cdot - l) \right\rangle$. Then

$$\widehat{(S_m \theta_{j/m}^i)}(\xi) = \widehat{B_j^i}(\xi)^T \left([A_m]_\Phi^\Psi(\xi) \right) \widehat{\Phi}(\xi) \tag{II.3.25}$$

If $[A_m]_\Phi^\Psi(\xi)$ is invertible, then

$$\widehat{S_m^{-1}\theta_{j/m}^i}(\xi) = \widehat{B_j^i}(\xi)^T \left([A_m]_\Phi^\Psi(\xi) \right)^{-1} \widehat{\Phi}(\xi).$$

Using property (i) of the Fourier transform, it can easily be verified that

$$S_m^{-1}\theta_{k+j/m}^i = (S_m^{-1}\theta_{j/m}^i)(\cdot - k).$$

Remark II.3.3. It can be shown that $\alpha_m I \leq [A_m]_\Phi^\Psi(\xi)$ for almost every ξ, where α_m is the positive lower bound in (II.2.11). Thus, for almost every ξ, $[A_m]_\Phi^\Psi(\xi)$ is invertible for every $m \geq 1$. However, for large enough m, we will show a stronger result below, namely that there is a positive number δ (that does not depend on m) such that $\delta I \leq \frac{1}{m^d}[A_m]_\Phi^\Psi(\xi)$ for every ξ.

We are now ready to compute $\mathrm{var}(f_\varepsilon(x) - f(x))$.

$\mathrm{var}(f_\varepsilon(x) - f(x))$

$$
\begin{aligned}
&= \mathrm{var}\left(\sum_{i=1}^{s}\sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d}\varepsilon^i_{k+j/m}S_m^{-1}\theta^i_{k+j/m}(x)\right)\\
&= \sigma^2\sum_{i=1}^{s}\sum_{j\in\Omega_m^d}\sum_{k\in\mathbb{Z}^d}\left|S_m^{-1}\theta^i_{j/m}(x-k)\right|^2\\
&= \sigma^2\sum_{i=1}^{s}\sum_{j\in\Omega_m^d}\int_{[0,1]^d}\left|\sum_{k\in\mathbb{Z}^d}\widehat{S_m^{-1}\theta^i_{j/m}}(k-\xi)e^{i2\pi x\cdot(k-\xi)}\right|^2\,d\xi\\
&= \sigma^2\sum_{i=1}^{s}\sum_{j\in\Omega_m^d}\int_{[0,1]^d}\left|\sum_{k\in\mathbb{Z}^d}\widehat{S_m^{-1}\theta^i_{j/m}}(k-\xi)e^{i2\pi x\cdot k}\right|^2\,d\xi\\
&= \sigma^2\int_{[0,1]^d}\sum_{i=1}^{s}\sum_{j\in\Omega_m^d}\left|\widehat{B^i_j}(-\xi)^T\left([A_m]^\Psi_\Phi(-\xi)\right)^{-1}\left(\sum_{k\in\mathbb{Z}^d}\widehat{\Phi}(k-\xi)e^{i2\pi k\cdot x}\right)\right|^2\,d\xi
\end{aligned}
$$

We notice that the matrix $\left([A_m]^\Psi_\Phi(-\xi)\right)^{-1}$ is self-adjoint, and use the argument (II.3.21) from the proof of Theorem II.1.2, along with the fact that

$$
\sum_{i=1}^{s}\sum_{j\in\Omega_m^d}\overline{\widehat{B^i_j}(-\xi)}\,\widehat{B^i_j}(-\xi)^T = [A_m]^\Psi_\Phi(-\xi),
$$

to get

$\mathrm{var}\big(f_\varepsilon(x) - f(x)\big)$

$$
\begin{aligned}
&= \sigma^2\int_{[0,1]^d}\left(\sum_{k\in\mathbb{Z}^d}\overline{\widehat{\Phi}(k-\xi)}e^{-i2\pi k\cdot x}\right)^T\left([A_m]^\Psi_\Phi(-\xi)\right)^{-1}\left(\sum_{k'\in\mathbb{Z}^d}\widehat{\Phi}(k'-\xi)e^{i2\pi k'\cdot x}\right)\,d\xi\\
&= \frac{\sigma^2}{m^d}\int_{[0,1]^d}\left(\sum_{k\in\mathbb{Z}^d}\overline{\widehat{\Phi}(k-\xi)}e^{-i2\pi k\cdot x}\right)^T\left(\frac{1}{m^d}[A_m]^\Psi_\Phi(-\xi)\right)^{-1}\left(\sum_{k'\in\mathbb{Z}^d}\widehat{\Phi}(k'-\xi)e^{i2\pi k'\cdot x}\right)\,d\xi
\end{aligned}
$$

Thus we have shown that

$$
\begin{aligned}
\mathrm{var}(f_\varepsilon(x) - f(x)) &= \frac{\sigma^2}{m^d}\int_{[0,1]^d}\overline{Z\widehat{\Phi}(-\xi,x)}^T\left(\frac{1}{m^d}[A_m]^\Psi_\Phi(-\xi)\right)^{-1}Z\widehat{\Phi}(-\xi,x)\,d\xi\\
&= \frac{\sigma^2}{m^d}D_x(m)
\end{aligned}
\tag{II.3.26}
$$

Lemma II.3.4. *There exist a number $\delta > 0$ and a number $M \in \mathbb{N}$ such that for every $m \geq M$,*

$$\delta I \leq \frac{1}{m^d} \left[A_m\right]_\Phi^\Psi(\xi) \qquad \textit{for all } \xi \in [0,1]^d.$$

We will show that

$$D_x(m) \overset{m\to\infty}{\longrightarrow} \int_{[0,1]^d} \overline{Z\widehat{\Phi}(-\xi,-x)}^T \left(G_\Phi^\Psi(\xi)\right)^{-1} Z\widehat{\Phi}(-\xi,-x)\, d\xi$$

in the proof of Lemma II.3.4 below. Furthermore, notice that for large enough m, we have

$$\begin{aligned}
&\operatorname{var}\Big(f_\varepsilon(x) - f(x)\Big)\\
&\leq \frac{\sigma^2}{m^d}\left(\frac{1}{\delta}\right)\int_{[0,1]^d}\left(\sum_{k\in\mathbb{Z}^d}\overline{\widehat{\Phi}(k-\xi)}e^{-i2\pi k\cdot x}\right)^T\left(\sum_{k'\in\mathbb{Z}^d}\widehat{\Phi}(k'-\xi)e^{i2\pi k'\cdot x}\right)\, d\xi\\
&\leq \frac{\sigma^2}{m^d}\left(\frac{1}{\delta}\right)\int_{[0,1]^d}\sum_{n=1}^{r}\left|\sum_{k\in\mathbb{Z}^d}\widehat{\phi^n}(k-\xi)e^{i2\pi k\cdot x}\right|^2\, d\xi\\
&= \frac{\sigma^2}{m^d}\left(\frac{1}{\delta}\right)\left(\sum_{i=1}^{r}\left(\sum_{k\in\mathbb{Z}^d}\left|\phi^i(x+k)\right|^2\right)\right)\\
&\leq \frac{\sigma^2}{m^d}\left(\frac{1}{\delta}\right)\left(\sum_{n=1}^{r}\|\phi^n\|_{W^1}^2\right) \qquad \text{for all } x \in \mathbb{R}^d.
\end{aligned}$$

□

II.3.5 Proof of Lemma II.3.4

First, for $\xi \in [0,1]^d$, define the self-adjoint matrix

$$G_\Phi^\Psi(\xi) := \sum_{i=1}^{s}\sum_{k\in\mathbb{Z}^d}\widehat{\Phi}(\xi+k)\overline{\widehat{\Phi}(\xi+k)}^T\left|\widehat{\psi^i}(\xi+k)\right|^2.$$

We will now show that

$$\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\xi) \quad \overset{m\to\infty}{\longrightarrow} \quad G_\Phi^\Psi(\xi) \qquad \text{for every } \xi \in [0,1]^d,$$

i.e., for each $\xi \in [0,1]^d$, each entry of the matrix $\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\xi)$ converges to the corresponding entry of the matrix $G_\Phi^\Psi(\xi)$. For $1 \le n, n' \le r$, we look at the (n,n')-entry of the matrix $\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\xi)$.

$$\left(\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\xi)\right)_{(n,n')}$$

$$\begin{aligned}
&= \sum_{i=1}^{s} \frac{1}{m^d} \sum_{j\in\Omega_m^d} \left(\sum_{k\in\mathbb{Z}^d} \widehat{\phi^n}(\xi+k)\overline{\widehat{\psi^i_{j/m}}(\xi+k)}\right)\left(\sum_{k'\in\mathbb{Z}^d} \overline{\widehat{\phi^{n'}}(\xi+k')}\widehat{\psi^i_{j/m}}(\xi+k')\right) \\
&= \sum_{i=1}^{s} \frac{1}{m^d} \sum_{j\in\Omega_m^d} \left(\sum_{k\in\mathbb{Z}^d} \widehat{q_\xi^{n,i}}(k) e^{i2\pi(j/m)\cdot k}\right)\left(\sum_{k'\in\mathbb{Z}^d} \overline{\widehat{q_\xi^{n',i}}(k')} e^{-i2\pi(j/m)\cdot k'}\right) \\
&\overset{m\to\infty}{\longrightarrow} \sum_{i=1}^{s} \int_{[0,1]^d} \left(\sum_{k\in\mathbb{Z}^d} \widehat{q_\xi^{n,i}}(k) e^{i2\pi x\cdot k}\right)\left(\sum_{k'\in\mathbb{Z}^d} \overline{\widehat{q_\xi^{n',i}}(k')} e^{-i2\pi x\cdot k'}\right)\, dx \\
&= \sum_{i=1}^{s} \left\langle q_\xi^{n,i}, q_\xi^{n',i}\right\rangle_{L^2\left([0,1]^d\right)}
\end{aligned}$$

where, for $1 \le l \le r$, $q_\xi^{l,i}$ is the function on $[0,1]^d$ whose Fourier coefficients $\widehat{q_\xi^{l,i}}(k)$ are given by

$$\widehat{q_\xi^{l,i}}(k) = \widehat{\phi^l}(\xi+k)\overline{\widehat{\psi^i}(\xi+k)}.$$

Invoking Plancherel, we have

$$\begin{aligned}
\sum_{i=1}^{s} \left\langle q_\xi^{n,i}, q_\xi^{n',i}\right\rangle_{L^2\left([0,1]^d\right)} &= \sum_{i=1}^{s} \left\langle \widehat{q_\xi^{n,i}}, \widehat{q_\xi^{n',i}}\right\rangle_{l^2(\mathbb{Z}^d)} \\
&= \sum_{i=1}^{s} \sum_{k\in\mathbb{Z}^d} \widehat{\phi^n}(\xi+k)\overline{\widehat{\phi^{n'}}(\xi+k)}\left|\widehat{\psi^i}(\xi+k)\right|^2 \\
&= \left[G_\Phi^\Psi(\xi)\right]_{(n,n')}
\end{aligned}$$

Thus $\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\xi) \overset{m\to\infty}{\longrightarrow} G_\Phi^\Psi(\xi)$ for each $\xi \in [0,1]^d$. Now we claim, for fixed (n,n'), the collection $\left\{\left(\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\cdot)\right)_{(n,n')} : m \in \mathbb{N}\right\}$ is equicontinuous, which implies that $\left(\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\cdot)\right)_{(n,n')}$ converges uniformly to $\left[G_\Phi^\Psi(\cdot)\right]_{(n,n')}$ on $[0,1]^d$.

In a manner similar to that in the proof of Lemma II.3.2, it can be verified that

$$\left(\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\xi)\right)_{(n,n')}$$

$$= \sum_{i=1}^{s} \frac{1}{m^d} \sum_{j\in\Omega_m^d} \sum_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (\phi^n * \psi^{i\vee})(j/m+l)(\phi^{n'} * \psi^{i\vee})(j/m+l+k)e^{-i2\pi\xi\cdot k}$$

Because $W^1 * L^1 \subset W^1$, we know that $\left(\phi^n * \psi^{i\vee}\right) \in W^1$, and therefore, the argument from Lemma II.3.2 can be used to show the collection is equicontinuous.

In [5] it is shown that (II.2.11) and (II.2.12) imply that there exists a positive number δ_0 such that $\delta_0 I \leq G_\Phi^\Psi(\xi)$ for all ξ. Let $\delta = \frac{\delta_0}{2}$. Because $\left(\frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\cdot)\right)_{(n,n')}$ converges uniformly to $\left[G_\Phi^\Psi(\cdot)\right]_{(n,n')}$ on $[0,1]^d$, there exists a number $M \in \mathbb{N}$ such that for all $m \geq M$

$$\delta I \leq \frac{1}{m^d}\left[A_m\right]_\Phi^\Psi(\xi) \qquad \text{for all } \xi.$$

□

II.3.6 Proof of Lemma II.2.3

Our objective is to show that for every $\xi \in [0,1]^d$,

$$\sum_{i=1}^{s} \int_{[0,1]^d} \left|Z(\phi * \psi^{i\vee})(t,-\xi)\right|^2 dt \geq \delta > 0.$$

Notice that

$$
\begin{aligned}
\sum_{i=1}^{s}\int_{[0,1]^d}\left|Z(\phi*\psi^{i^\vee})(t,-\xi)\right|^2\ dt &= \sum_{i=1}^{s}\int_{[0,1]^d}\left|\sum_{l\in\mathbb{Z}^d}(\phi*\psi^{i^\vee})(t-l)e^{-i2\pi l\cdot\xi}\right|^2\ dt \\
&\overset{\text{(I.1.3)}}{=} \sum_{i=1}^{s}\int_{[0,1]^d}\left|\sum_{l\in\mathbb{Z}^d}\widehat{\phi}(l-\xi)\widehat{\psi^i}(l-\xi)e^{i2\pi t\cdot(l-\xi)}\right|^2\ dt \\
&= \sum_{i=1}^{s}\int_{[0,1]^d}\left|e^{-i2\pi t\cdot\xi}\sum_{l\in\mathbb{Z}^d}\widehat{\phi}(l-\xi)\widehat{\psi^i}(l-\xi)e^{i2\pi t\cdot l}\right|^2\ dt \\
&= \sum_{i=1}^{s}\int_{[0,1]^d}\left|\sum_{l\in\mathbb{Z}^d}\widehat{\phi}(l-\xi)\widehat{\psi^i}(l-\xi)e^{i2\pi t\cdot l}\right|^2\ dt \\
&= \sum_{i=1}^{s}\sum_{l\in\mathbb{Z}^d}\left|\widehat{\phi}(l-\xi)\widehat{\psi^i}(l-\xi)\right|^2 \\
&= \sum_{i=1}^{s}\sum_{l\in\mathbb{Z}^d}\left|\widehat{\phi}(l-\xi)\right|^2\left|\widehat{\psi^i}(l-\xi)\right|^2
\end{aligned}
$$

This is equal to the 1×1 matrix $G_\Phi^\Psi(\xi)$, and thus as stated in the proof of Lemma II.3.4, the lemma holds.

□

CHAPTER III

RECONSTRUCTION FROM SAMPLING SETS WITH UNKNOWN JITTER

Here we return to the original problem of sampling and function reconstruction. Instead of additive noise, a different kind of error is considered. In practice the sampling locations x_j are not known precisely. Real-world sampling devices give data of the form $\{f(x_j+\delta_j)\}_{j\in J}$, where each δ_j represents some unknown perturbation from the point x_j [8, 21]. We refer to this as *sampling jitter*, and it occurs in applications related to digital data processing of signals [20].

The issue of jitter error gives rise to two main questions. First, if $X := \{x_j\}_{j\in J}$ is a set of sampling for $V^2(\phi)$, under what conditions is the set $X + \Delta := \{x_j + \delta_j\}_{j\in J}$ also a set of sampling for $V^2(\phi)$? In other words, under what conditions is (I.4.9) still satisfied if we replace X with $X + \Delta$? The second question arises as we attempt to recover f. In general, each δ_j is unknown. Possibly our samples are affected by jitter error without our knowledge, or, even if we know that our samples are affected by jitter error, the precise amount of perturbation at each sampling point x_j is unknown. If we attempt to recover f under the assumption that our data are samples of f at X, when in actuality our data are samples of f at $X + \Delta$, is the recovered function a good approximation of f, and how does the error relate to the sequence $\Delta := \{\delta_j\}_{j\in J}$? In this chapter, we provide answers to both these questions. First, we address these questions more precisely.

III.1 Notation and preliminaries

We begin with our underlying function space $V^2(\phi)$. In this chapter, we assume the space has only one generator, ϕ. However, we relax the assumptions on ϕ by requiring only that ϕ and its shifts form a Riesz basis for the space $V^2(\phi)$, and not necessarily an orthonormal basis.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function in $L^2(\mathbb{R})$, and suppose there exist constants m and M such that

$$0 < m \leq \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + k)|^2 \leq M < \infty \quad \text{a.e. } \xi. \tag{III.1.1}$$

Define the shift-invariant space

$$V^2(\phi) := \left\{ \sum_{k \in \mathbb{Z}} c_k \phi(\cdot - k) \ : \ c \in l^2(\mathbb{Z}) \right\}.$$

Then $V^2(\phi)$ is a Hilbert space, $V^2(\phi)$ is a subspace of $L^2(\mathbb{R})$, and $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ forms a Riesz basis for $V^2(\phi)$ [1, 3]. Also assume $\phi \in W_0^1 := W^1 \cap C^0$, where C^0 is the set of continuous functions, and

$$W^1 = \left\{ f : \sum_{k \in \mathbb{Z}} \operatorname{ess}\sup_{x \in [0,1]} \{|f(x+k)|\} < \infty \right\}.$$

Under this assumption, $V^2(\phi)$ is a space of continuous functions [3].

In this chapter, we also consider a more general set of sampling X. In Chapter II, we required that X be uniform. In this chapter, we allow the countable set X to be non-uniform, and only require that X satisfy (I.4.9). Our final theorem also requires X to be a separated subset of $\mathbb{R}$.

III.2 Results

Because (III.1.1) holds, every f in $V^2(\phi)$ corresponds to a sequence c in $l^2(\mathbb{Z})$ so that $f = \sum_{k\in\mathbb{Z}} c_k\phi(\cdot - k)$. In order to see how c (and hence f) can be recovered from samples, we will look at sampling operators. We define the sampling operator U on $l^2(\mathbb{Z})$ that corresponds to the set X [14]. Let U be the linear operator on $l^2(\mathbb{Z})$ so that $Uc = f|_X = (f(x_j))_{j\in J}$. We can think of U as an infinite matrix whose j,k entry, $(U)_{j,k}$, is $\phi(x_j - k)$, where $j \in J$ and $k \in \mathbb{Z}$. Notice then that X is a set of sampling for $V^2(\phi)$ if and only if there exist positive constants α and β such that

$$\alpha \|c\|_{l^2(\mathbb{Z})} \leq \|Uc\|_{l^2(J)} \leq \beta \|c\|_{l^2(\mathbb{Z})} \quad \text{for all } c \in l^2(\mathbb{Z}). \tag{III.2.2}$$

We can also define the sampling operator on $l^2(\mathbb{Z})$ that corresponds to the set $X+\Delta$. Let U_Δ be the linear operator on $l^2(\mathbb{Z})$ so that $U_\Delta c = f|_{X+\Delta} = (f(x_j + \delta_j))_{j\in J}$. We can think of U_Δ as the infinite matrix whose j,k entry, $(U_\Delta)_{j,k}$, is $\phi(x_j + \delta_j - k)$, where $j \in J$ and $k \in \mathbb{Z}$.

We now return to the first of our original questions. If X is a set of sampling for $V^2(\phi)$, under what conditions is the perturbed set $X + \Delta$ also a set of sampling for $V^2(\phi)$? We begin with the following lemma

Lemma III.2.1. *Let X be a set of sampling for $V^2(\phi)$, and let α and β be the positive constants satisfying* (III.2.2)*. If $\|U - U_\Delta\| < \alpha$, then $X + \Delta$ is a set of sampling for $V^2(\phi)$.*

Proof: Let $c \in l^2(\mathbb{Z})$. First, we show the upper bound.

$$\begin{aligned}\|U_\Delta c\|_{l^2(J)} &\leq \|(U_\Delta - U)c\|_{l^2(J)} + \|Uc\|_{l^2(J)} \\ &< (\alpha + \beta) \|c\|_{l^2(\mathbb{Z})}\end{aligned}$$

To show the lower bound, we begin with the lower bound of (III.2.2).

$$\alpha \|c\|_{l^2(\mathbb{Z})} \leq \|Uc\|_{l^2(J)} \leq \|U - U_\Delta\| \, \|c\|_{l^2(\mathbb{Z})} + \|U_\Delta c\|_{l^2(J)}$$

Thus

$$(\alpha - \|U - U_\Delta\|) \, \|c\|_{l^2(\mathbb{Z})} \leq \|U_\Delta c\|_{l^2(J)} ,$$

and $\alpha - \|U - U_\Delta\| > 0$ if $\|U - U_\Delta\| < \alpha$.

Therefore

$$(\alpha - \|U - U_\Delta\|) \, \|c\|_{l^2(\mathbb{Z})} \leq \|U_\Delta c\|_{l^2(J)} \leq (\alpha + \beta) \, \|c\|_{l^2(\mathbb{Z})} ,$$

and hence $X + \Delta$ is a set of sampling for $V^2(\phi)$ if $\|U - U_\Delta\| < \alpha$.

□

Remark III.2.2. From the definitions of U and U_Δ it is clear that $\|U - U_\Delta\|$ depends on both the sequence Δ and the function ϕ. The final theorem of the chapter provides conditions on ϕ under which $\|U - U_\Delta\| \to 0$ as $\|\Delta\|_\infty \to 0$. Thus, under certain conditions on ϕ, for any $\alpha > 0$ (i.e., for any set of sampling X), there exists a positive number $\gamma_0 > 0$ such that $X + \Delta$ is a set of sampling whenever $\|\Delta\|_\infty \leq \gamma_0$.

Now, let $b := Uc = f|_X$ represent the samples of f at X, and let $b_\Delta := U_\Delta c = f|_{X+\Delta}$ represent the samples of f at $X + \Delta$. Notice that

$$c = (U^* U)^{-1} U^* b \qquad \text{and} \tag{III.2.3}$$

$$c = (U_\Delta^* U_\Delta)^{-1} U_\Delta^* b_\Delta \tag{III.2.4}$$

provided that the inverses exist. If X is a set of sampling for $V^2(\phi)$, then the operator $(U^* U)^{-1}$ exists and is bounded, and c can be recovered as in (III.2.3).

We return to the second of our original questions. Suppose we have b_Δ as our data, but we think we have b. If we reconstruct the function f using $(U^*U)^{-1} U^*$, do we have a good approximation of our original function? Certainly this too would require the set $X + \Delta$ to be only a small perturbation of the set X. Our goal is to determine under what conditions we have

$$\left\| (U^*U)^{-1} U^* - (U_\Delta^* U_\Delta)^{-1} U_\Delta^* \right\| \to 0 \quad \text{as} \quad \|\Delta\|_\infty \to 0 \tag{III.2.5}$$

and to give estimates for $\left\| (U^*U)^{-1} U^* - (U_\Delta^* U_\Delta)^{-1} U_\Delta^* \right\|$, where the norm is the operator norm.

Throughout the rest of this chapter, assume X is a set of sampling for $V^2(\phi)$, and let α and β be the positive constants that satisfy (III.2.2).

Theorem III.2.3. *Let $0 < \epsilon < -\beta + \sqrt{\beta^2 + \alpha^2}$, where α and β are the positive constants satisfying* (III.2.2)*. Assume there exists a number $\gamma_0 > 0$ such that $\|U - U_\Delta\| < \epsilon$ whenever $\|\Delta\|_\infty \leq \gamma_0$, and define $\eta := \alpha^{-2}\epsilon(2\beta + \epsilon)$. Then $\eta < 1$, and*

$$\left\| (U^*U)^{-1} U^* - (U_\Delta^* U_\Delta)^{-1} U_\Delta^* \right\| < \frac{1}{\alpha^2} \cdot \left(\epsilon + \frac{\eta(\beta + \epsilon)}{1 - \eta} \right)$$

whenever $\|\Delta\|_\infty \leq \gamma_0$.

From the theorem, we see that (III.2.5) is satisfied as long as $\|U - U_\Delta\| \to 0$ as $\|\Delta\|_\infty \to 0$. In other words, the reconstruction of f given data sampled with jitter error is a good approximation of the original $f \in V^2(\phi)$. To prove the theorem, we need the next two lemmas.

Lemma III.2.4. *Let $\epsilon > 0$. Assume there exists a number $\gamma_0 > 0$ such that $\|U - U_\Delta\| < \epsilon$ whenever $\|\Delta\|_\infty \leq \gamma_0$. Then $\|U^*U - U_\Delta^* U_\Delta\| < \epsilon(2\beta + \epsilon)$ whenever $\|\Delta\|_\infty \leq \gamma_0$.*

Proof: Notice $\|U\| = \|U^*\|$ and $\|U^* - U_\Delta^*\| = \|U - U_\Delta\|$. Let $\epsilon > 0$. Assume $\|\Delta\|_\infty \leq \gamma_0$. Then

$$\begin{aligned}\|U^*U - U_\Delta^* U_\Delta\| &= \|U^*(U - U_\Delta) + (U^* - U_\Delta^*)U_\Delta\| \\ &\leq \|U - U_\Delta\| \left(\|U\| + \|U_\Delta\|\right) \\ &< \epsilon \cdot (2\beta + \epsilon)\end{aligned}$$

□

Lemma III.2.5. *Let* $0 < \epsilon < -\beta + \sqrt{\beta^2 + \alpha^2}$, *where* α *and* β *are the positive constants satisfying* (III.2.2). *Assume there exists a number* $\gamma_0 > 0$ *such that* $\|U - U_\Delta\| < \epsilon$ *whenever* $\|\Delta\|_\infty \leq \gamma_0$, *and define* $\eta := \alpha^{-2}\epsilon(2\beta + \epsilon)$. *Then* $\eta < 1$, $(U_\Delta^* U_\Delta)^{-1}$ *exists, and*

$$\left\|(U^*U)^{-1} - (U_\Delta^* U_\Delta)^{-1}\right\| < \frac{\eta}{\alpha^2(1 - \eta)}$$

whenever $\|\Delta\|_\infty \leq \gamma_0$.

Proof: Recall that $(U^*U)^{-1}$ exists because X is a set of sampling for $V^2(\phi)$. Then

$$U_\Delta^* U_\Delta = U^*U\left(I + (U^*U)^{-1}\left(U_\Delta^* U_\Delta - U^*U\right)\right). \tag{III.2.6}$$

Notice

$$\frac{1}{\beta^2}\|c\| \leq \left\|(U^*U)^{-1} c\right\| \leq \frac{1}{\alpha^2}\|c\| \quad \text{for all } c \in l^2(\mathbb{Z}).$$

Let $\|\Delta\|_\infty \leq \gamma_0$. Then using Lemma III.2.4, $\left\|(U^*U)^{-1}\left(U_\Delta^* U_\Delta - U^*U\right)\right\| < 1$. For the sake of simplicity, define

$$A := U^*U, \quad A_\Delta := U_\Delta^* U_\Delta, \quad \text{and} \quad T := (U^*U)^{-1}\left(U_\Delta^* U_\Delta - U^*U\right).$$

Then $(I + T)^{-1}$ exists, since $\|T\| < 1$, and is given by the Neuman series

$$(I + T)^{-1} = I - T + T^2 - T^3 + \ldots. \tag{III.2.7}$$

From (III.2.6) we get

$$\begin{aligned} A_\Delta^{-1} = &\quad [A\,(I+T)]^{-1} \\ = &\quad (I+T)^{-1}\,A^{-1} \end{aligned} \tag{III.2.8}$$

Hence, $A_\Delta^{-1} = (U_\Delta^* U_\Delta)^{-1}$ exists whenever $\|\Delta\|_\infty \le \gamma_0$.

Now we need to give the upper bound for $\left\|A^{-1} - A_\Delta^{-1}\right\|$. Assume $\|\Delta\|_\infty \le \gamma_0$. Using (III.2.8) we get

$$A^{-1} - A_\Delta^{-1} = T(I+T)^{-1}A^{-1}. \tag{III.2.9}$$

Then

$$\begin{aligned} \left\|A^{-1} - A_\Delta^{-1}\right\| &\le \|T\| \left\|(I+T)^{-1}\right\| \left\|A^{-1}\right\| \\ &\le \frac{\|T\|}{1-\|T\|} \cdot \frac{1}{\alpha^2} \\ &< \frac{\eta}{\alpha^2(1-\eta)} \end{aligned}$$

□

III.2.1 Proof of Theorem III.2.3

Let $\|\Delta\|_\infty \le \gamma_0$. Using our notation from Lemmas III.2.4 and III.2.5 and the previous proof,

$$\begin{aligned} \left\|(U^*U)^{-1}\,U^* - (U_\Delta^* U_\Delta)^{-1}\,U_\Delta^*\right\| &= \left\|A^{-1}U^* - A^{-1}U_\Delta^* + A^{-1}U_\Delta^* - A_\Delta^{-1}U_\Delta^*\right\| \\ &= \left\|A^{-1}(U^* - U_\Delta^*) + (A^{-1} - A_\Delta^{-1})U_\Delta^*\right\| \\ &\le \left\|A^{-1}\right\| \|U^* - U_\Delta^*\| + \left\|A^{-1} - A_\Delta^{-1}\right\| \|U_\Delta^*\| \\ &< \frac{1}{\alpha^2}\left(\epsilon + \frac{\eta(\beta+\epsilon)}{1-\eta}\right), \end{aligned}$$

where η is as defined in Theorem III.2.3.

□

III.2.2 Concluding Results

In Remark III.2.2 it was mentioned that $\|U - U_\Delta\|$ will depend on ϕ, so we then ask for what functions ϕ do we have $\|U - U_\Delta\| \to 0$ as $\|\Delta\|_\infty \to 0$?

Theorem III.2.6. *Let $\phi \in W_0^1$. Suppose X is a set of sampling for $V^2(\phi)$, with ϕ satisfying* (III.1.1)*, and suppose X is separated, with $\inf_{x_i \neq x_j} |x_i - x_j| = \lambda > 0$. Then $\|U - U_\Delta\| \to 0$ as $\|\Delta\|_\infty \to 0$.*

We now have answers, stated below in the corollary, to the two main questions discussed in the introduction to jitter error. Define the reconstruction operator $R : l^2(J) \to V^2(\phi)$ so that

$$R: \ d \quad \mapsto \quad \sum_{k \in \mathbb{Z}} \left[(U^*U)^{-1} U^* d \right]_k \phi(\cdot - k).$$

Corollary III.2.7. *Let $\phi \in W_0^1$. Suppose X is a set of sampling for $V^2(\phi)$, with ϕ satisfying* (III.1.1)*, and suppose X is separated, with $\inf_{x_i \neq x_j} |x_i - x_j| = \lambda > 0$. Then*

(i) there exists a $\gamma_0 > 0$ such that $X + \Delta$ is a set of sampling whenever $\|\Delta\|_\infty \leq \gamma_0$, and

(ii) $\|Rf|_{X+\Delta} - f\|_{L^2} \to 0$ as $\|\Delta\|_\infty \to 0$.

III.2.3 Proof of Theorem III.2.6

First, for any number $\gamma > 0$, define the function $\text{osc}_\gamma \phi$ on $\mathbb{R}$ by

$$\text{osc}_\gamma \phi(x) = \sup_{|\Delta x| < \gamma} |\phi(x + \Delta x) - \phi(x)| .$$

The W^1-norm of a function f is given by

$$\|f\|_{W^1} = \sum_{k\in\mathbb{Z}} \operatorname*{ess\ sup}_{x\in[0,1]} \{|f(x+k)|\},$$

and from [3] we know that $\| \operatorname{osc}_\gamma \phi \|_{W^1} \to 0$ as $\gamma \to 0$.

Define $N := \lceil 1/\lambda \rceil + 1$. Then for any $l \in \mathbb{Z}$, there are at most N elements of X in the interval $I_l := [l, l+1)$. Define the sequence p indexed by the integers so that

$$p(l) := \operatorname*{ess\ sup}_{x\in[0,1]} \left\{ \left| \operatorname{osc}_{\|\Delta\|_\infty} \phi(x+l) \right| \right\}, \quad l \in \mathbb{Z}.$$

Then $\|p\|_{l^1(\mathbb{Z})} = \| \operatorname{osc}_\gamma \phi \|_{W^1}$. Now we will use the facts above to show that $\|U - U_\Delta\| \to 0$ as $\|\Delta\|_\infty \to 0$. Let $c \in l^2(\mathbb{Z})$, and define $X_l := X \cap I_l$.

$$\begin{aligned}
\|(U - U_\Delta)c\|^2_{l^2(J)} &= \sum_{x_j\in X} \left| \sum_{k\in\mathbb{Z}} c_k \left(\phi(x_j - k) - \phi(x_j + \delta_j - k)\right) \right|^2 \\
&\leq \sum_{x_j\in X} \left| \sum_{k\in\mathbb{Z}} |c_k| \ \left| \operatorname{osc}_{\|\Delta\|_\infty} \phi(x_j - k) \right| \right|^2 \\
&= \sum_{l\in\mathbb{Z}} \sum_{x_j\in X_l} \left| \sum_{k\in\mathbb{Z}} |c_k| \ \left| \operatorname{osc}_{\|\Delta\|_\infty} \phi(x_j - k) \right| \right|^2 \\
&\leq \sum_{l\in\mathbb{Z}} \sum_{x_j\in X_l} \left| \sum_{k\in\mathbb{Z}} |c_k| \ |p(l-k)| \right|^2 \\
&\leq N \sum_{l\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} |c_k| \ |p(l-k)| \right|^2 \\
&= N \left\| (|c| * p) \right\|_{l^2(\mathbb{Z})} \\
&\leq N \|c\|_{l^2(\mathbb{Z})} \|p\|_{l^1(\mathbb{Z})} \\
&= N \|c\|_{l^2(\mathbb{Z})} \left\| \operatorname{osc}_{\|\Delta\|_\infty} \phi \right\|_{W^1}.
\end{aligned}$$

Therefore $\|U - U_\Delta\| \leq N \left\| \operatorname{osc}_{\|\Delta\|_\infty} \phi \right\|_{W^1} \to 0$ as $\|\Delta\|_\infty \to 0$.

□

CHAPTER IV

CONSTRUCTING SHIFT-INVARIANT REPRODUCING KERNEL HILBERT SPACES

In [25], Smale and Zhou construct reproducing kernel Hilbert spaces to serve as the underlying signal space for sampling and reconstruction. They show how their construction of an RKHS generalizes the setting of bandlimited functions in the classic Shannon theorem. In this chapter, we show in fact that the construction in [25] can be used to form the shift-invariant space $V^2(\phi)$. In [25], several hypotheses must be satisfied in order for the theorems to hold. The results of this chapter give conditions under which these hypotheses are satisfied. In other words, we remove the necessary assumptions from [25] and give conditions under which they are true. We begin with the construction of the reproducing kernel Hilbert spaces as illustrated in [25]. Then we state our theorems and prove them.

IV.1 Construction of Reproducing Kernel Hilbert Spaces

Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a continuous, symmetric, positive semidefinite map. (We call a symmetric map $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *positive semidefinite* if for any finite set of distinct points $\{x_1, \ldots, x_m\} \subset \mathbb{R}^d$, the matrix $M = \Big(G(x_i, x_j)\Big)_{i,j=1}^{m}$ is positive semidefinite, i.e. $a^T M a \geq 0$ for all column vectors $a \in \mathbb{R}^m$.) For $x \in \mathbb{R}^d$, we define $K_x : \mathbb{R}^d \to \mathbb{R}$ to be the continuous function on $\mathbb{R}^d$ given by $K_x = K(x, \cdot)$.

Next we define a Hilbert space which will act as our representation space. Consider the linear space of finite linear combinations of K_x, $x \in \mathbb{R}^d$, denoted span$\{K_x : x \in$

$\mathbb{R}^d\}$. An inner product on this space is defined by linear extension from

$$\langle K_x, K_y \rangle_K = K(x, y).$$

The RKHS H_K associated to K is the completion of the linear space in the norm induced by the inner product, i.e. $H_K = \overline{\text{span}\{K_x : x \in \mathbb{R}^d\}}^K$. Now consider the closed subspace $H_{K,\mathbb{Z}^d}$ of H_K generated by $\{K_t : t \in \mathbb{Z}^d\}$. The space $H_{K,\mathbb{Z}^d} = \overline{\text{span}\{K_t : t \in \mathbb{Z}^d\}}^K$ will serve as our representation space.

Example IV.1.1. Let $d = 1$. Define $\phi : \mathbb{R} \to \mathbb{R}$ to be the sinc function, i.e. $\phi(x) = \text{sinc}(x) = \frac{\sin \pi x}{\pi x}$. It is well known that $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ is an orthonormal basis for $V^2(\phi) \subset L^2(\mathbb{R})$. Then we define, for $x, y \in \mathbb{R}$,

$$K(x, y) = \sum_{j \in \mathbb{Z}} \phi(x - j)\phi(y - j),$$

and notice that since $\text{sinc}(k) = \delta_{0,k}$ for all integers k, we have, for $t \in \mathbb{Z}$,

$$\begin{aligned} K_t &= \sum_{j \in \mathbb{Z}} \phi(t - j)\phi(\cdot - j) \\ &= \phi(\cdot - t) \end{aligned}$$

In this case, our set of generators $\{K_t : t \in \mathbb{Z}\}$ for $H_{K,\mathbb{Z}^d}$ is the *same* set as our orthonormal basis $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ for $V^2(\phi)$.

We also show in this case that the inner product $\langle \cdot, \cdot \rangle_K$ on $H_{K,\mathbb{Z}^d}$ is equal to the standard L^2 inner product which $V^2(\phi)$ inherits. For $s, t \in \mathbb{Z}^d$, we have

$$\begin{aligned}
\langle K_s, K_t \rangle_{L^2} &= \left\langle \widehat{K_s}, \widehat{K_t} \right\rangle_{L^2} \\
&= \int_{\mathbb{R}} \widehat{K_s}(x) \overline{\widehat{K_t}(x)}\ dx \\
&= \int_{\mathbb{R}} \chi_{[-\frac{1}{2},\frac{1}{2}]} e^{-i2\pi sx} \chi_{[-\frac{1}{2},\frac{1}{2}]} e^{i2\pi tx}\ dx \\
&= \int_{\mathbb{R}} \chi_{[-\frac{1}{2},\frac{1}{2}]} e^{-i2\pi(s-t)x}\ dx \\
&= \widehat{\chi_{[-\frac{1}{2},\frac{1}{2}]}}(s-t) \\
&= \phi(s-t) = K(s,t) = \langle K_s, K_t \rangle_K .
\end{aligned}$$

Therefore we have $H_{K,\mathbb{Z}^d} = V^2(\phi)$. Recall also, for this particular ϕ, that $V^2(\phi)$ is the representation space $\{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\frac{1}{2}, \frac{1}{2}]\}$ from the classical Shannon Theorem due to Whittaker.

The above example was provided to show that there is overlap between the reproducing kernel Hilbert spaces in [25] and the setting of shift-invariant spaces. While the construction in [25] is more general, we will see more practical results in the specific setting of a shift-invariant space.

We next use the kernel K to define a linear operator $K_{\mathbb{Z}^d}$ on $l^2(\mathbb{Z}^d)$ as follows:

$$\left(K_{\mathbb{Z}^d}\ a\right)_s = \sum_{t \in \mathbb{Z}^d} K(s,t) a_t, \qquad s \in \mathbb{Z}^d,\ a \in l^2(\mathbb{Z}^d).$$

Notice for $a \in l^2(\mathbb{Z}^d)$ that $[K_{\mathbb{Z}^d}\ a]$ is also a sequence indexed by $\mathbb{Z}^d$. For now, as in [25], we assume that $K_{\mathbb{Z}^d}$ is well-defined, bounded and positive with positive inverse. In section IV.2 we give conditions on K under which $K_{\mathbb{Z}^d}$ satisfies this assumption.

Notice, if $\mathbb{Z}^d$ is our sampling set as in our classic Shannon example, i.e. our data are indexed by $\mathbb{Z}^d$, then $K_{\mathbb{Z}^d}$ takes the place of our sampling operator S_X (where $X = \mathbb{Z}^d$ in this case). Recall S_X maps a function in our representation Hilbert space to its sampled values, a sequence indexed by X. If $f \in H_{K,\mathbb{Z}^d}$ can be expressed as $f = \sum_{t\in\mathbb{Z}^d} a_t K_t$, then for $s \in \mathbb{Z}^d$,

$$f(s) = \sum_{t\in\mathbb{Z}^d} a_t K_t(s) = \sum_{t\in\mathbb{Z}^d} a_t K(s,t) = \Big(K_{\mathbb{Z}^d}\ a\Big)_s.$$

We have $S_X f = K_{\mathbb{Z}^d}\ a$ for $f = \sum_{t\in\mathbb{Z}^d} a_t K_t$.

Often we cannot assume that $\mathbb{Z}^d$ is our sampling set. We call our sampling set X, and we assume $X = \{x_j \in \mathbb{R}^d : j \in J\}$, where J is a countable index set. We now define the operator K_X on $l^2(\mathbb{Z}^d)$ whose image is a sequence indexed by X. For $a \in l^2(\mathbb{Z}^d)$ and $x_j \in X$, define

$$\Big(K_X a\Big)_{x_j} = \sum_{t\in\mathbb{Z}^d} K(x_j,t)a_t.$$

In this section (as in [25]) we assume that K_X is bounded. In section IV.2 we give conditions on K and X under which K_X satisfies this assumption.

Once again, this time for a general sampling set X, K_X takes the place of our sampling operator S_X. If $f \in H_{K,\mathbb{Z}^d}$ can be expressed as $f = \sum_{t\in\mathbb{Z}^d} a_t K_t$, then for $x_j \in X$,

$$f(x_j) = \sum_{t\in\mathbb{Z}^d} a_t K_t(x_j) = \sum_{t\in\mathbb{Z}^d} a_t K(x_j,t) = \Big(K_X a\Big)_{x_j}.$$

We have $S_X f = K_X a$ for $f = \sum_{t\in\mathbb{Z}^d} a_t K_t$.

We denote by K_X^* the adjoint of K_X. How does the adjoint K_X^* act on a sequence in $l^2(X)$? Let $c \in l^2(X)$. Then for all $a \in l^2(\mathbb{Z}^d)$, we have

$$\begin{aligned}\langle a, K_X^* c\rangle_{l^2(\mathbb{Z}^d)} = \langle K_X a, c\rangle_{l^2(X)} &= \sum_{x_j\in X}\left(\sum_{t\in\mathbb{Z}^d} a_t K(t,x_j)\right) c_{x_j} \\ &= \sum_{t\in\mathbb{Z}^d} a_t \sum_{x_j\in X} c_{x_j} K(x_j,t) = \left\langle a, \left(\sum_{x_j\in X} c_{x_j} K_{x_j}(t)\right)_t\right\rangle_{l^2(\mathbb{Z}^d)}.\end{aligned}$$

Thus for $t \in \mathbb{Z}^d$, $\Big(K_X^* c\Big)_t = \sum_{x_j\in X} c_{x_j} K_{x_j}(t)$.

IV.2 Results

Recall that in order to define H_K and $H_{K,\mathbb{Z}^d}$, we require that $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous, symmetric, and positive semidefinite. Our first theorem constructs such a K.

Theorem IV.2.1. *Let ϕ be a continuous, real-valued function on $\mathbb{R}^d$ which satisfies*

(i) $\phi \in W^1(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \;\middle|\; \sum_{k\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d}\{|f(x+k)|\} < \infty \right\}$ *and*

(ii) $\sum_{j\in\mathbb{Z}^d} |\hat{\phi}(\xi + j)|^2 = 1$ *for almost every* $\xi \in \mathbb{R}^d$.

For $x, y \in \mathbb{R}^d$, define $K(x,y) = \sum_{j\in\mathbb{Z}^d} \phi(x-j)\phi(y-j)$. Then $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is well-defined, continuous, symmetric, and positive semidefinite, and $H_K \subset V^2(\phi) \subset L^2(\mathbb{R}^d)$.

Proof: We begin by showing that K, defined as above, is in fact a well-defined, continuous, symmetric and positive semidefinite map. K is clearly symmetric. Notice that for any fixed $x, y \in \mathbb{R}^d$,

$$K(x,y) = \sum_{j\in\mathbb{Z}^d} \phi(x-j)\phi(y-j) = \left\langle \Big(\phi(x-j)\Big)_j, \Big(\phi(y-j)\Big)_j \right\rangle_{l^2(\mathbb{Z}^d)}$$

is convergent because condition *(i)* and the fact that $l^1 \subset l^2$ imply that $\{\phi(x - j)\}_{j\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$ for all $x \in \mathbb{R}^d$.

Claim: $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous. First, define the function $\mathrm{osc}_\gamma\phi$ on $\mathbb{R}^d$ by

$$\mathrm{osc}_\gamma\phi(x) = \sup_{|\Delta x|<\delta} |\phi(x+\Delta x) - \phi(x)|\,.$$

Define the W^1 norm of a function f by

$$\|f\|_{W^1} = \sum_{k\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d}\{|f(x+k)|\}.$$

Claim: $\| \operatorname{osc}_\gamma \phi \|_{W^1} \to 0$ as $\delta \to 0$.

Let $\varepsilon > 0$. Because $\phi \in W^1$, there exists a number $N \in \mathbb{N}$ such that

$$\sum_{|k|>N-1} \operatorname*{ess\,sup}_{x\in[0,1]^d} \{|\phi(x+k)|\} < \frac{\varepsilon}{4 \cdot 3^d}.$$

Because ϕ is continuous on $\mathbb{R}^d$, ϕ is uniformly continuous on compact sets. Therefore there exists $\delta > 0$ such that if $|\Delta x| < \delta$ then

$$|\phi(x+\Delta x) - \phi(x)| < \frac{\varepsilon}{2(2N+1)^d} \quad \text{for all } x \in [-N,N]^d. \tag{IV.2.1}$$

Then

$$\begin{aligned}
\| \operatorname{osc}_\gamma \phi \|_{W^1} = & \sum_{k\leq N} \operatorname*{ess\,sup}_{x\in[0,1]^d} \left\{ \sup_{|\Delta x|<\delta} \{|\phi(x+k+\Delta x) - \phi(x+k)|\} \right\} \\
& + \sum_{k>N} \operatorname*{ess\,sup}_{x\in[0,1]^d} \left\{ \sup_{|\Delta x|<\delta} \{|\phi(x+k+\Delta x) - \phi(x+k)|\} \right\},
\end{aligned}$$

and by (IV.2.1) the left-hand summand is less than $\frac{\varepsilon}{2}$. We now deal with the right-hand summand. Without loss of generality, $\delta < 1$. For any fixed k,

$$\operatorname*{ess\,sup}_{x\in[0,1]^d} \left\{ \sup_{|\Delta x|<\delta} \{|\phi(x+k+\Delta x) - \phi(x+k)|\} \right\} \leq 2 \cdot \operatorname*{ess\,sup}_{x\in[-1,2]^d} \{|\phi(x+k)|\}.$$

Notice that $[-1,2]^d$ consists of 3^d unit intervals. Therefore we have

$$\begin{aligned}
\sum_{k>N} \operatorname*{ess\,sup}_{x\in[0,1]^d} & \left\{ \sup_{|\Delta x|<\delta} \{|\phi(x+k+\Delta x) - \phi(x+k)|\} \right\} \\
& \leq 2 \cdot 3^d \sum_{k>N-1} \operatorname*{ess\,sup}_{x\in[0,1]^d} \{|\phi(x+k)|\} \\
& < \frac{\varepsilon}{2}.
\end{aligned}$$

Thus $\| \operatorname{osc}_\gamma \phi \|_{W^1} \to 0$ as $\delta \to 0$.

We now define the W^2 norm by

$$\|f\|_{W^2}^2 = \sum_{k\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d}\{|f(x+k)|^2\}$$

for all functions f on $\mathbb{R}^d$ such that $\|f\|_{W^2} < \infty$. We remark that $l^1 \subset l^2$ and $W^1 \subset W^2$. Notice that if $\|\operatorname{osc}_\gamma\phi\|_{W^1} \to 0$ as $\delta \to 0$, then $\|\operatorname{osc}_\gamma\phi\|_{W^2} \to 0$ as $\delta \to 0$. We are ready to show that K is continuous.

Let $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R}^d$ be a sequence such that (x_n, y_n) converges to (x, y) (hence $x_n \to x$ and $y_n \to y$ in $\mathbb{R}^d$). For convenience we denote $\phi(\cdot - j)$ by ϕ_j. For $n \in \mathbb{N}$, we have

$$|K(x_n, y_n) - K(x, y)| = \left|\sum_{j\in\mathbb{Z}^d} \phi_j(x_n)\phi_j(y_n) - \sum_{j\in\mathbb{Z}^d} \phi_j(x)\phi_j(y)\right|,$$

and because each sum is convergent, we have

$$|K(x_n, y_n) - K(x, y)| = \left|\sum_{j\in\mathbb{Z}^d} \phi_j(x_n)\phi_j(y_n) - \phi_j(x)\phi_j(y)\right|.$$

Fix $\delta \in (0, 1)$. Let $M \in \mathbb{N}$ be such that $|x - x_n| < \delta$ and $|y - y_n| < \delta$ for all $n \geq M$. We then have, for $n \geq M$,

$$|K(x_n, y_n) - K(x, y)|$$

$$\begin{aligned}
&\leq \sum_{j\in\mathbb{Z}^d} |\phi_j(x_n)\phi_j(y_n) - \phi_j(x)\phi_j(y)| \\
&= \sum_{j\in\mathbb{Z}^d} |\phi_j(x_n)\phi_j(y_n) - \phi_j(x_n)\phi_j(y) + \phi_j(x_n)\phi_j(y) - \phi_j(x)\phi_j(y)| \\
&= \sum_{j\in\mathbb{Z}^d} \Big|\phi_j(x_n)\Big(\phi_j(y_n) - \phi_j(y)\Big) + \phi_j(y)\Big(\phi_j(x_n) - \phi_j(x)\Big)\Big| \\
&\leq \sum_{j\in\mathbb{Z}^d} |\phi_j(x_n)|\,|\phi_j(y_n) - \phi_j(y)| + |\phi_j(y)|\,|\phi_j(x_n) - \phi_j(x)| \\
&= \sum_{j\in\mathbb{Z}^d} |\phi(x_n - j)|\,|\phi(y_n - j) - \phi(y - j)| + |\phi(y - j)|\,|\phi(x_n - j) - \phi(x - j)| \\
&\leq \sum_{j\in\mathbb{Z}^d} |\phi(x_n - j)|\,|\mathrm{osc}_\gamma\phi(y - j)| + |\phi(y - j)|\,|\mathrm{osc}_\gamma\phi(x - j)|
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j\in\mathbb{Z}^d} |\phi(x_n - j)|\,|\mathrm{osc}_\gamma\phi(y - j)| + \sum_{j\in\mathbb{Z}^d} |\phi(y - j)|\,|\mathrm{osc}_\gamma\phi(x - j)| \\
&= \Big\langle \Big(|\phi(x_n - j)|\Big)_j, \Big(|\mathrm{osc}_\gamma\phi(y - j)|\Big)_j \Big\rangle_{l^2} + \Big\langle \Big(|\phi(y - j)|\Big)_j, \Big(|\mathrm{osc}_\gamma\phi(x - j)|\Big)_j \Big\rangle_{l^2} \\
&\leq \|\Big(\phi(x_n - j)\Big)_j\|_{l^2} \cdot \|\Big(\mathrm{osc}_\gamma\phi(y - j)\Big)_j\|_{l^2} + \|\Big(\phi(y - j)\Big)_j\|_{l^2} \cdot \|\Big(\mathrm{osc}_\gamma\phi(x - j)\Big)_j\|_{l^2} \\
&\leq 2\|\phi\|_{W^2} \cdot \|\,\mathrm{osc}_\gamma\phi\,\|_{W^2} \quad \to 0 \ \text{ as } \ \delta \to 0.
\end{aligned}$$

Therefore K is continuous and hence $K(x, y)$ makes sense for all pairs $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Our next goal is to show that K is positive semidefinite. First, notice that for any fixed $x \in \mathbb{R}^d$, $K_x = \sum_{j\in\mathbb{Z}^d} \phi(x - j)\phi(\cdot - j)$ is a function in $V^2(\phi)$ because condition *(i)* and the fact that $l^1 \subset l^2$ imply that $\{\phi(x - j)\}_{j\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$ for all $x \in \mathbb{R}^d$. Condition *(ii)* implies that $V^2(\phi)$ is a Hilbert space which is a subspace of $L^2(\mathbb{R}^d)$, and that $\{\phi_j\}_{j\in\mathbb{Z}^d}$ is an orthonormal basis for $V^2(\phi)$. We remark that $V^2(\phi)$ inherits the standard L^2 inner product. Then it makes sense to consider $\langle K_x, K_y\rangle_{L^2}$ for fixed $x, y \in \mathbb{R}^d$.

Let $x, y \in \mathbb{R}^d$. Then

$$\begin{aligned}
\langle K_x, K_y \rangle_{L^2} &= \left\langle \sum_{j \in \mathbb{Z}^d} \phi_j(x)\phi_j, \sum_{k \in \mathbb{Z}^d} \phi_k(y)\phi_k \right\rangle_{L^2} \\
&= \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \phi_j(x)\phi_k(y) \langle \phi_j, \phi_k \rangle_{L^2} \\
&= \sum_{j \in \mathbb{Z}^d} \phi_j(x)\phi_j(y) \\
&= K(x, y).
\end{aligned} \tag{IV.2.2}$$

Now that we have $K(x, y) = \langle K_x, K_y \rangle_{L^2}$ for all $x, y \in \mathbb{R}^d$, we can show that K is positive semidefinite. Let $\{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^d$ such that $x_i \neq x_l$ for $i \neq l$. Let $a = (a_1, a_2, \ldots, a_m)$ be a nonzero vector in $\mathbb{R}^m$. Define M to be the $m \times m$ matrix whose $i, l-$entry is $K(x_i, x_l)$. We need to show that $a^T M a > 0$.

$$\begin{aligned}
a^T M a &= \sum_{i=1}^m \sum_{l=1}^m a_i a_l K(x_i, x_l) \\
&= \sum_{i=1}^m \sum_{l=1}^m a_i a_l \langle K_{x_i}, K_{x_l} \rangle_{L^2} \\
&= \left\langle \sum_{i=1}^m a_i K_{x_i}, \sum_{l=1}^m a_l K_{x_l} \right\rangle_{L^2} \\
&= \left\| \sum_{i=1}^m a_i K_{x_i} \right\|_{L^2}^2 \geq 0.
\end{aligned}$$

Therefore K is positive semidefinite. Now that we know K is continuous, symmetric, and positive semidefinite, we can define H_K as before. Consider the inner product $\langle \cdot, \cdot \rangle_K$ on H_K. Fix $x, y \in \mathbb{R}^d$. Then we have

$$\langle K_x, K_y \rangle_K = K(x, y) \overset{\text{(IV.2.2)}}{=} \langle K_x, K_y \rangle_{L^2}.$$

Notice now that H_K is clearly a subspace of $V^2(\phi)$ because each of its generators $K_x \in V^2(\phi)$ and $\langle \cdot, \cdot \rangle_K = \langle \cdot, \cdot \rangle_{L^2}$.

□

Our next goal is to give conditions under which $H_{K,\mathbb{Z}^d} = V^2(\phi)$. First, we need the following lemma.

Lemma IV.2.2. *If $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ is an orthonormal basis for a Hilbert space V of functions on $\mathbb{R}^d$, and ψ is defined on $\mathbb{R}^d$ by*

$$\psi(x) := \sum_{j \in \mathbb{Z}^d} p(j)\phi(x - j),$$

then $\{\psi(\cdot - k) : k \in \mathbb{Z}^d\}$ is a Riesz basis for V if there exist constants m and M such that

$$0 < m \leq |\hat{p}(\xi)|^2 \leq M < \infty \qquad \text{a.e. } \xi.$$

Proof: Recall that a system is a Riesz basis of a Hilbert space V if it is the image of an orthonormal basis of V under a bounded, invertible operator. Let $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ be an orthonormal basis for a Hilbert space V. For convenience we denote $\phi(\cdot - k)$ by ϕ_k. We define the operator T on the generators of the space V by $T\phi_k = \sum_{j \in \mathbb{Z}^d} p(j)\phi_k(\cdot - j) = \sum_{j \in \mathbb{Z}^d} p(j)\phi(\cdot - k - j)$ and extend T linearly to $\text{span}\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ (finite linear combinations of the generators). Notice that $T\phi = T\phi_0 = \sum_{j \in \mathbb{Z}^d} p(j)\phi(\cdot - j)$. Furthermore, notice that $\{\psi(\cdot - k) : k \in \mathbb{Z}^d\}$, where $\psi(x) := \sum_{j \in \mathbb{Z}^d} p(j)\phi(x - j)$, is the image of the orthonormal basis $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ under the operator T, i.e., $T\phi_k = \psi_k$.

We must now show that T is bounded and invertible if there exist constants m and M such that

$$0 < m \leq |\hat{p}(\xi)|^2 \leq M < \infty \qquad \text{a.e. } \xi. \tag{IV.2.3}$$

Let $f \in V$. Then there exists a sequence $c \in l^2(\mathbb{Z}^d)$ such that $f = \sum_{j\in\mathbb{Z}^d} c_j \phi_j$, and $\|f\|_V = \|c\|_{l^2(\mathbb{Z}^d)}$. Then

$$\begin{aligned}
\|Tf\|_V^2 &= \left\| \sum_{j\in\mathbb{Z}^d} c_j T\phi_j \right\|_V^2 = \left\| \sum_{j\in\mathbb{Z}^d} c_j \sum_{k\in\mathbb{Z}^d} p(k)\phi(\cdot - j - k) \right\|_V^2 \\
&= \left\| \sum_{j\in\mathbb{Z}^d} c_j \sum_{m\in\mathbb{Z}^d} p(m-j)\phi(\cdot - m) \right\|_V^2 \\
&= \left\| \sum_{m\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} c_j p(m-j)\phi(\cdot - m) \right\|_V^2 \\
&= \left\| \left(\sum_{j\in\mathbb{Z}^d} c_j p(m-j) \right)_{m\in\mathbb{Z}^d} \right\|_{l^2}^2
\end{aligned}$$

For each sequence $a \in l^2(\mathbb{Z}^d)$, there corresponds a function $\hat{a}$ in $L^2([0,1]^d)$ defined by $\hat{a}(\xi) = \sum_{k\in\mathbb{Z}^d} a(k)e^{-i2\pi k\cdot\xi}$, and $\|a\|_{l^2(\mathbb{Z}^d)} = \|\hat{a}\|_{L^2([0,1]^d}$. Therefore

$$\begin{aligned}
\|Tf\|_V^2 &= \int_{[0,1]^d} \left| \sum_{m\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} c_j p(m-j) e^{-i2\pi m\cdot\xi} \right|^2 d\xi \\
&= \int_{[0,1]^d} \left| \sum_{j\in\mathbb{Z}^d} c_j e^{-i2\pi j\cdot\xi} \sum_{m\in\mathbb{Z}^d} p(m-j) e^{-i2\pi(m-j)\cdot\xi} \right|^2 d\xi \\
&= \int_{[0,1]^d} \left| \sum_{j\in\mathbb{Z}^d} c_j e^{-i2\pi j\cdot\xi} \right|^2 |\hat{p}(\xi)|^2 \, d\xi
\end{aligned}$$

Then we have

$$m \int_{[0,1]^d} \left| \sum_{j\in\mathbb{Z}^d} c_j e^{-i2\pi j\cdot\xi} \right|^2 d\xi \le \|Tf\|_V^2 \le M \int_{[0,1]^d} \left| \sum_{j\in\mathbb{Z}^d} c_j e^{-i2\pi j\cdot\xi} \right|^2 d\xi$$

and therefore

$$m\|c\|_{l^2}^2 \le \|Tf\|_V^2 \le M\|c\|_{l^2}^2$$

if there exist constants m and M such that

$$0 < m \leq |\hat{p}(\xi)|^2 \leq M < \infty \qquad \text{a.e. } \xi.$$

Because $\|c\|_{l^2} = \|f\|_V$, we have

$$m\|f\|_V^2 \leq \|Tf\|_V^2 \leq M\|f\|_V^2 \quad \text{for all } f \in V$$

if (IV.2.3) holds. Hence T is bounded and invertible if (IV.2.3) holds.

□

Theorem IV.2.3. *Let* $K(x,y) = \sum_{j\in\mathbb{Z}^d} \phi(x-j)\phi(y-j)$, *where* ϕ *satisfies the conditions in Theorem IV.2.1. For* $k \in \mathbb{Z}^d$, *define* $p(k) := \phi(-k)$. *If there is no* $\xi \in [0,1]$ *such that* $\hat{p}(\xi) = 0$ *then* $H_{K,\mathbb{Z}^d} = V^2(\phi)$.

Proof: Assume the hypotheses above, and assume there is no $\xi \in [0,1]$ such that $\hat{p}(\xi) = 0$. *Claim:* There exist constants m and M such that

$$0 < m \leq |\hat{p}(\xi)|^2 \leq M < \infty \qquad \text{for all } \xi \in [0,1]. \tag{IV.2.4}$$

First, $\phi \in W^1$ implies that $p \in l^1(\mathbb{Z}^d)$. Then we know that $\hat{p}$ is continuous and $\hat{p} \in L^\infty[0,1]$. Hence there exists M such that $|\hat{p}(\xi)|^2 \leq M < \infty$ for all $\xi \in [0,1]$. Furthermore, because $\hat{p}$, and therefore $|\hat{p}|$, is continuous on a compact set, it must attain its minimum value, and since there is no $\xi \in [0,1]$ such that $|\hat{p}(\xi)| = 0$, its minimum value must be $m > 0$. Thus we have our claim.

Next, notice that, for $t \in \mathbb{Z}^d$, $K_t = K_0(\cdot - t)$:

Let $t \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$.

$$\begin{aligned}
K_t(x) &= \sum_{j\in\mathbb{Z}^d} \phi(t-j)\phi(x-j) \\
&= \sum_{j\in\mathbb{Z}^d} \phi(0-(j-t))\phi(x-t-(j-t)) \\
&= \sum_{k\in\mathbb{Z}^d} \phi(-k)\phi(x-t-k) \\
&= K_0(x-t).
\end{aligned}$$

So we have $K_0 : \mathbb{R}^d \to \mathbb{R}$ defined by

$$K_0(x) = \sum_{j\in\mathbb{Z}^d} \phi(-j)\phi(x-j).$$

For $j \in \mathbb{Z}^d$, define $p(j) = \phi(-j)$. Then

$$K_0(x) = \sum_{j\in\mathbb{Z}^d} p(j)\phi(x-j).$$

Because (IV.2.4) holds, Lemma IV.2.2 tells us that $\{K_0(\cdot - t) : t \in \mathbb{Z}^d\} = \{K_t : t \in \mathbb{Z}^d\}$ forms a Riesz basis for $V^2(\phi)$. Since we know from Theorem IV.2.1 that $H_{K,\mathbb{Z}^d} \subset V^2(\phi)$, and we now know that the generators of $H_{K,\mathbb{Z}^d}$ form a Riesz basis for $V^2(\phi)$, we conclude that $H_{K,\mathbb{Z}^d} = V^2(\phi)$.

□

In [25], several assumptions were made in order to obtain the results of the theorems. Specifically, boundedness, positivity and invertibility of the operator $K_{\mathbb{Z}^d}$ and boundedness of the operator K_X were assumed. We present the following results to give specific conditions on K and X under which these assumptions hold. In section

8 of [25], examples of such K and X are provided. The following results are more general.

Theorem IV.2.4. *Let* $K(x,y) = \sum_{j\in\mathbb{Z}^d} \phi(x-j)\phi(y-j)$, *where* ϕ *satisfies the conditions in Theorem IV.2.1. For* $k \in \mathbb{Z}^d$, *define* $p(k) := \phi(-k)$. *If there is no* $\xi \in [0,1]$ *such that* $\hat{p}(\xi) = 0$, *then the operator* $K_{\mathbb{Z}^d}$ *as defined above is bounded, positive and invertible.*

Proof: We first remark that under the hypotheses of this theorem, by the proof of Theorem IV.2.3, there exist constants m and M such that

$$0 < m \leq |\hat{p}(\xi)|^2 \leq M < \infty \qquad \text{for all } \xi \in [0,1].$$

Let $a \in l^2(\mathbb{Z}^d)$. We use the fact that any sequence $c \in l^2(\mathbb{Z}^d)$ corresponds to a function $\hat{c}$ in $L^2([0,1]^d)$ defined by $\hat{c}(\xi) = \sum_{s\in\mathbb{Z}^d} c_s e^{-i2\pi s\cdot\xi}$, and that $\|c\|_{l^2(\mathbb{Z}^d)} = \|\hat{c}\|_{L^2([0,1]^d}$.

$$\begin{aligned}
\|K_{\mathbb{Z}^d}\ a\|^2_{l^2(\mathbb{Z}^d)} &= \int_{[0,1]^d} \left| \sum_{s\in\mathbb{Z}^d} \left(K_{\mathbb{Z}^d}\ a\right)_s e^{-i2\pi s\cdot\xi} \right|^2 \, d\xi \\
&= \int_{[0,1]^d} \left| \sum_{s\in\mathbb{Z}^d} \left(\sum_{t\in\mathbb{Z}^d} a_t K_t(s) \right) e^{-i2\pi s\cdot\xi} \right|^2 \, d\xi \\
&= \int_{[0,1]^d} \left| \sum_{t\in\mathbb{Z}^d} a_t \left(\sum_{s\in\mathbb{Z}^d} K_t(s) e^{-i2\pi s\cdot\xi} \right) \right|^2 \, d\xi.
\end{aligned}$$

For the moment we consider only $\left|\sum_{t\in\mathbb{Z}^d} a_t \left(\sum_{s\in\mathbb{Z}^d} K_t(s) e^{-i2\pi s\cdot\xi}\right)\right|$.

$$
\begin{aligned}
\left|\sum_{t\in\mathbb{Z}^d} a_t\left(\sum_{s\in\mathbb{Z}^d} K_t(s)e^{-i2\pi s\cdot\xi}\right)\right| &= \left|\sum_{t\in\mathbb{Z}^d} a_t\left(\sum_{s\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}\phi(t-j)\phi(s-j)e^{-i2\pi s\cdot\xi}\right)\right| \\
&= \left|\sum_{t\in\mathbb{Z}^d} a_t\left(\sum_{j\in\mathbb{Z}^d}\phi(t-j)\sum_{s\in\mathbb{Z}^d}\phi(s-j)e^{-i2\pi(s-j)\cdot\xi}e^{-i2\pi j\cdot\xi}\right)\right| \\
&= \left|\sum_{t\in\mathbb{Z}^d} a_t\left(\sum_{j\in\mathbb{Z}^d}\phi(t-j)e^{-i2\pi j\cdot\xi}\sum_{r\in\mathbb{Z}^d}\phi(-r)e^{-i2\pi r\cdot(-\xi)}\right)\right| \\
&= \left|\sum_{t\in\mathbb{Z}^d} a_t\left(\sum_{j\in\mathbb{Z}^d}\phi(t-j)e^{-i2\pi(j-t)\cdot\xi}e^{-i2\pi t\cdot\xi}\ \hat{p}(-\xi)\right)\right| \\
&= \left|\sum_{t\in\mathbb{Z}^d} a_t e^{-i2\pi t\cdot\xi}\ \Big(\hat{p}(\xi)\Big)\Big(\hat{p}(-\xi)\Big)\right| \\
&= \left|\sum_{t\in\mathbb{Z}^d} a_t e^{-i2\pi t\cdot\xi}\right| \left|\hat{p}(\xi)\right| \left|\hat{p}(-\xi)\right|.
\end{aligned}
$$

Then we have

$$
\|K_{\mathbb{Z}^d}\ a\|^2_{l^2(\mathbb{Z}^d)} = \int_{[0,1]^d}\left|\sum_{t\in\mathbb{Z}^d} a_t e^{-i2\pi t\cdot\xi}\right|^2 |\hat{p}(\xi)|^2\,|\hat{p}(-\xi)|^2\ d\xi,
$$

and therefore

$$
\int_{[0,1]^d}\left|\sum_{t\in\mathbb{Z}^d} a_t e^{-i2\pi t\cdot\xi}\right|^2\cdot m^2\ d\xi \le \|K_{\mathbb{Z}^d}\ a\|^2_{l^2(\mathbb{Z}^d)} \le \int_{[0,1]^d}\left|\sum_{t\in\mathbb{Z}^d} a_t e^{-i2\pi t\cdot\xi}\right|^2\cdot M^2\ d\xi,
$$

which gives

$$
m^2\int_{[0,1]^d}\left|\sum_{t\in\mathbb{Z}^d} a_t e^{-i2\pi t\cdot\xi}\right|^2 d\xi \le \|K_{\mathbb{Z}^d}\ a\|^2_{l^2(\mathbb{Z}^d)} \le M^2\int_{[0,1]^d}\left|\sum_{t\in\mathbb{Z}^d} a_t e^{-i2\pi t\cdot\xi}\right|^2 d\xi.
$$

Finally we see that

$$
m^2\|a\|^2_{l^2(\mathbb{Z}^d)} \le \|K_{\mathbb{Z}^d}\ a\|^2_{l^2(\mathbb{Z}^d)} \le M^2\|a\|^2_{l^2(\mathbb{Z}^d)}.
$$

Next we show that $K_{\mathbb{Z}^d}$ is positive. First, let $a \in l^2(\mathbb{Z}^d)$.

$$\begin{aligned}
\langle a, K_{\mathbb{Z}^d}\ a\rangle_{l^2(\mathbb{Z}^d)} &= \sum_{s\in\mathbb{Z}^d} a_s \Big(K_{\mathbb{Z}^d}\ a\Big)_s \\
&= \sum_{s\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^d} a_s a_t K(s,t) \\
&= \sum_{s\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^d} a_s a_t \langle K_s, K_t\rangle_K \\
&= \sum_{s\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^d} a_s a_t \langle K_s, K_t\rangle_{L^2} \\
&= \left\langle \sum_{s\in\mathbb{Z}^d} a_s K_s, \sum_{t\in\mathbb{Z}^d} a_t K_t \right\rangle_{L^2}.
\end{aligned}$$

Notice that for any sequence $a \in l^2(\mathbb{Z}^d)$, we have

$$\langle a, K_{\mathbb{Z}^d}\ a\rangle_{l^2(\mathbb{Z}^d)} = \left\langle \sum_{s\in\mathbb{Z}^d} a_s K_s, \sum_{t\in\mathbb{Z}^d} a_t K_t \right\rangle_{L^2} = \left\| \sum_{t\in\mathbb{Z}^d} a_t K_t \right\|_{L^2}^2 \geq 0,$$

implying that $K_{\mathbb{Z}^d}$ is a positive operator.

□

In Chapter I we said $X = \{x_j : j \in J\} \subset \mathbb{R}^d$ is a set of sampling for a Hilbert space $H \subset L^2(\mathbb{R}^d)$ if there exist constants c_1 and c_2 such that

$$c_1\|f\|_{L^2} \leq \left(\sum_{x_j\in X} |f(x_j)|^2\right)^{1/2} \leq c_2\|f\|_{L^2} \qquad \text{for all } f \in H. \tag{IV.2.5}$$

Recall that if X is a set of sampling for a Hilbert space $H \subset L^2(\mathbb{R}^d)$, then the sampling operator S_X is bounded and has a bounded inverse. Earlier in this chapter, we saw that the operator K_X plays the role of S_X in the sense that $S_X f = K_X a = \{f(x_j)\}_{x_j\in X}$ for $f = \sum_{t\in\mathbb{Z}^d} a_t K_t$. Under what conditions is the operator K_X bounded with a bounded inverse? In other words, when do there exist constants d_1 and d_2 such that

$$d_1\|a\|_{l^2(\mathbb{Z}^d)} \leq \|K_X a\|_{l^2(X)} \leq d_2\|a\|_{l^2(\mathbb{Z}^d)} \tag{IV.2.6}$$

for all $a \in l^2(\mathbb{Z}^d)$? Below we give the result that if $\{K_t : t \in \mathbb{Z}^d\}$ forms a Riesz basis for $V^2(\phi)$, then (IV.2.5) holds if and only if (IV.2.6) holds.

Corollary IV.2.5. *Let $K(x,y) = \sum_{j\in\mathbb{Z}^d} \phi(x-j)\phi(y-j)$, where ϕ satisfies the conditions in Theorem IV.2.1. For $k \in \mathbb{Z}^d$, define $p(k) = \phi(-k)$. Assume there is no $\xi \in [0,1]$ such that $\hat{p}(\xi) = 0$. Then a given set $X = \{x_j : j \in J\} \subset \mathbb{R}^d$ is a set of sampling for $V^2(\phi) = H_{K,\mathbb{Z}^d}$ if and only if there exist constants d_1 and d_2 such that*

$$d_1\|a\|_{l^2(\mathbb{Z}^d)} \le \|K_X a\|_{l^2(X)} \le d_2\|a\|_{l^2(\mathbb{Z}^d)} \qquad \textit{for all} \;\; a \in l^2(\mathbb{Z}^d).$$

Proof: Once again we remark that under the hypotheses of this theorem, by the proof of Theorem IV.2.3, there exist constants m and M such that

$$0 < m \le |\hat{p}(\xi)|^2 \le M < \infty \qquad \text{for all } \xi \in [0,1].$$

We need to show

$$c_1\|f\|_{L^2} \le \left(\sum_{x_j\in X} |f(x_j)|^2\right)^{1/2} \le c_2\|f\|_{L^2} \quad \text{for all } f \in V^2(\phi) \tag{IV.2.7}$$

$$\Longleftrightarrow$$

$$d_1\|a\|_{l^2(\mathbb{Z}^d)} \le \|K_X a\|_{l^2(X)} \le d_2\|a\|_{l^2(\mathbb{Z}^d)} \quad \text{for all} \;\; a \in l^2(\mathbb{Z}^d). \tag{IV.2.8}$$

(Notice that $\left(\sum_{x_j\in X} |f(x_j)|^2\right)^{1/2} = \|K_X a\|_{l^2(X)}$ for $f = \sum_{t\in\mathbb{Z}^d} a_t K_t$.)

First, let $f \in V^2(\phi) = H_{K,\mathbb{Z}^d}$. Then we can write f as $f = \sum_{t\in\mathbb{Z}^d} a_t K_t$ for some sequence a. We compute the norm of f:

$$
\begin{aligned}
\|f\|_{L^2}^2 = \langle f, f\rangle &= \left\langle \sum_{s\in\mathbb{Z}^d} a_s K_s, \sum_{t\in\mathbb{Z}^d} a_t K_t \right\rangle_{L^2} \\
&= \sum_{s\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^d} a_s a_t \langle K_s, K_t\rangle_{L^2} \\
&= \sum_{s\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^d} a_s a_t \langle K_s, K_t\rangle_{K} \\
&= \sum_{s\in\mathbb{Z}^d}\sum_{t\in\mathbb{Z}^d} a_s a_t K(s,t) \\
&= \sum_{s\in\mathbb{Z}^d} a_s \Big(K_{\mathbb{Z}^d}\ a \Big)_s \\
&= \langle a, K_{\mathbb{Z}^d}\ a\rangle_{l^2(\mathbb{Z}^d)}\,.
\end{aligned}
$$

Since $K_{\mathbb{Z}^d}$ (which is bounded by the previous theorem) is a positive operator, we know that its square root $K_{\mathbb{Z}^d}^{1/2}$ exists as a positive, self-adjoint, bounded operator. Hence

$$
\|f\|_{L^2}^2 = \left\langle K_{\mathbb{Z}^d}^{1/2} a, K_{\mathbb{Z}^d}^{1/2} a\right\rangle_{l^2(\mathbb{Z}^d)} = \|K_{\mathbb{Z}^d}^{1/2} a\|_{l^2(\mathbb{Z}^d)}^2 \le \|K_{\mathbb{Z}^d}^{1/2}\|^2 \|a\|_{l^2(\mathbb{Z}^d)}^2.
$$

Also, using the results from the previous theorem, we have

$$
\begin{aligned}
m\|a\|_{l^2(\mathbb{Z}^d)} \le \|K_{\mathbb{Z}^d}\ a\|_{l^2(\mathbb{Z}^d)} &= \|K_{\mathbb{Z}^d}^{1/2} K_{\mathbb{Z}^d}^{1/2} a\|_{l^2(\mathbb{Z}^d)} \\
&\le \|K_{\mathbb{Z}^d}^{1/2}\|\ \|K_{\mathbb{Z}^d}^{1/2} a\|_{l^2(\mathbb{Z}^d)} = \|K_{\mathbb{Z}^d}^{1/2}\|\ \|f\|_{L^2}.
\end{aligned}
$$

Therefore we have

$$
\frac{m}{\|K_{\mathbb{Z}^d}^{1/2}\|}\ \|a\|_{l^2(\mathbb{Z}^d)} \le \|f\|_{L^2} \le \|K_{\mathbb{Z}^d}^{1/2}\|\ \|a\|_{l^2(\mathbb{Z}^d)}
$$

and

$$
\frac{1}{\|K_{\mathbb{Z}^d}^{1/2}\|}\ \|f\|_{L^2} \le \|a\|_{l^2(\mathbb{Z}^d)} \le \frac{\|K_{\mathbb{Z}^d}^{1/2}\|}{m}\ \|f\|_{L^2},
$$

and we can easily get (IV.2.7) if and only if (IV.2.8).

□

In [25], it is shown that if $K(x,y) = \text{sinc}(x-y)$, then $\langle \cdot, \cdot \rangle_K = \langle \cdot, \cdot \rangle_{L^2}$ and $H_{K,\mathbb{Z}^d} = \{f : \text{supp}\, \hat{f} \subset [-\frac{1}{2}, \frac{1}{2}]\}$. Furthermore, in [25], Smale and Zhou extend this idea, and present results similar to Theorem IV.2.4. However, they define $K(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ so that $K(x,y) = \psi(x-y)$ for some continuous, even function $\psi \in L^2(\mathbb{R}^d)$. Defining K as we have done in this section is not only more general, but also is still a generalization of the classic Shannon example (see example IV.1.1). As we have just seen, defining K in this more general manner still yields the desired boundedness of operators and leads to a correspondence between the RKHS H_K and a shift-invariant space.

BIBLIOGRAPHY

[1] A. Aldroubi, Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces, *Appl. Comput. Harmon. Anal.* **13** (2002) 151-161.

[2] A. Aldroubi, K. Gröchenig, Beurling-Landau-type theorems for non-uniform sampling in shift invariant spline spaces, *J. Fourier Anal. Appl.* **6** (2000) 93-103.

[3] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev. 43* **4** (2001) 585-620.

[4] A. Aldroubi, I. Krishtal, Robustness of sampling and reconstruction and Beurling-Landau-type theorems for shift invariant spaces, *Appl. Comput. Harmon. Anal.*, to appear.

[5] A. Aldroubi, Q. Sun, W. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier Anal. Appl.* **22** (2005) 215-244.

[6] J.J. Benedetto, Irregular sampling and frames, in *Wavelets: A Tutorial in Theory and Applications*, C.K. Chui, ed., Academic Press, Boston, 1992.

[7] C. Blanco, C. Cabrelli, S. Heineken, Functions in sampling spaces, pre-print.

[8] P.L. Butzer, G. Schmeisser, R.L. Stens, An introduction to sampling analysis, in *Nonuniform Sampling Theory and Practice*, F. Marvasti, ed., Kluwer/Plenum, New York, 2001.

[9] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[10] F. Cucker, S. Smale, On the Mathematical Foundations of Learning, *Bull. Amer. Math. Soc.* **39** (1) (2001) 1-49.

[11] Z. Cvetkovic, M. Vetterli, Oversampled filter banks, *IEEE Trans. Signal Process* **46** (5) (1998) 1245-1255.

[12] Y. Eldar, M. Unser, Nonideal sampling and interpolation from noisy observations in shift-invariant spaces, *IEEE Trans. Signal Process* **54** (7) (2006) 2636-2651.

[13] H.G. Feichtinger, K. Gröchenig, Error analysis in regular and irregular sampling theory, *Appl. Anal.* **50** (1993) 167-189.

[14] K. Gröchenig, H. Schwab, Fast local reconstruction methods for nonuniform sampling in shift-invariant spaces, *SIAM J. Matrix Anal. Appl.* **24** (4) (2003) 899-913.

[15] K. Gröchenig, *Foundations of Time Frequency Analysis*, Birkhäuser, Boston, 2001.

[16] C. Heil, D. Walnut, Continuous and discrete wavelet transforms, *SIAM Review*, **31** (1989) 628-666.

[17] J. Hogan, J. Lakey, Periodic nonuniform sampling in shift-invariant spaces, in *Harmonic Analysis and Applications*, C. Heil, ed., Birkhäuser, Boston, 2006.

[18] A.J.E.M. Janssen, The Zak transform: a signal transform for sampled time-continuous signals, *Philips J. Res.* **39** (1998), 23-69.

[19] A. Jerri, The Shannon sampling theorem - its various extensions and applications: a tutorial review, *Proc. IEEE* **65** (1977) 1565-1596.

[20] B. Lacaze, Reconstruction of stationary processes sampled at random times, in *Nonuniform Sampling Theory and Practice*, F. Marvasti, ed., Kluwer/Plenum, New York, 2001.

[21] F. Marvasti, Random topics in nonuniform sampling, in *Nonuniform Sampling Theory and Practice*, F. Marvasti, ed., Kluwer/Plenum, New York, 2001.

[22] T. Poggio, S. Smale, The Mathematics of Learning: Dealing with Data, *Notices Amer. Math. Soc.* **50** (5) (2003) 537-544.

[23] G.K. Rohde, C.A. Berenstein, D.M. Healy, Jr., Measuring image similarity in the presence of noise, *Proceedings of the SPIE, Medical Imaging 2005: Image Processing* **5747** (2005) 132-143.

[24] C.E. Shannon, Communications in the Presence of Noise, *Proc. IRE*, **37** (1949) 10-21.

[25] S. Smale, D.X. Zhou, Shannon Sampling and Function Reconstruction from Point Values, *Bull. Amer. Math. Soc.* **41** (3) (2004) 279-305.

[26] S. Smale, D.X. Zhou, Shannon Sampling II. Connections to Learning Theory, *Appl. Comput. Harmon. Anal.* **19** (3) (2005) 285-302.

[27] T. Strohmer, Numerical analysis of the non-uniform sampling problem, *J. Comp. Appl. Math.* **122** (2000) 297-316.

[28] G.G. Walter, A sampling theorem for wavelet subspaces, *IEEE Trans. Inform. Theory* **38** (2) (1992) 881-884.

[29] J.M. Whittaker, *Interpolatory Function Theory*, Cambridge University Press, London, 1935.

[30] X. Zhou, W. Sun, On the sampling theorem for wavelet subspaces, *J. Fourier Anal. Appl.* **5** (4) (1999) 347-354.