PROPERTY A AS METRIC AMENABILITY AND ITS APPLICATIONS TO GEOMETRY

By

Piotr W. Nowak

Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

May, 2008

Nashville, Tennessee

Approved:

Professor Guoliang Yu

Professor Gennadi Kasparov

Professor Mark Sapir

Professor Bruce Hughes

Professor Thomas Kephart

To my closest family,

Jola, Sławek, Hania, Maria

and

my beloved wife, Ania

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor, Guoliang Yu. Without his support, constant encouragement, patience and professional advice this work would not be possible.

I would also like to thank the faculty at the department, especially Dietmar Bisch, Bruce Hughes and Mark Sapir for many helpful and inspiring conversations, not only on mathematics.

Finally my thanks go to my friends in the graduate program, in particular to Matt Calef, Iva Kozakova, Bogdan Nica and Jan Špakula, with whom we spent a lot of time discussing mathematical problems.

TABLE OF CONTENTS

Page

DEDICATION ii

ACKNOWLEDGEMENTS iii

Chapter

I. INTRODUCTION 1

II. PRELIMINARIES 3

Coarse geometry 3
Spaces and maps 3
Groups as geometric objects 5
The combinatorial model 6
Amenability 6
Property A as metric amenability 8

III. AN AVERAGING THEOREM FOR PROPERTY A 10

IV. PROPERTY A IS NOT EQUIVALENT TO EMBEDDABILITY INTO ℓ_2 13

Behavior of Følner sets in high-dimensional products 13
Constructing coarsely embeddable metric spaces without Property A 17

V. ISOPERIMETRY AND ASYMPTOTIC DIMENSION 20

Definition of the A-profile 20
Basic properties 21
Relation to isoperimetric profiles 24
Asymptotic dimension and the A-profile 27
The main estimate 29
Estimates of isoperimetric profiles 30
Applications to dimension theory 32

VI. COARSE INDEX THEORY AND THE ZERO-IN-THE-SPECTRUM CONJECTURE .. 36

The zero-in-the-spectrum problem 36
The Laplace-Beltrami operator 36
Index theory via the coarse assembly map 37
Coarse index 39
Group quotients with Property A 40
Large Riemannian manifolds 44
Index of the de Rham operator 45

BIBLIOGRAPHY 47

CHAPTER I

INTRODUCTION

Our object of study is Yu's Property A. It was introduced in [58] as a non-equivariant counterpart of amenability and turned out to be a useful tool in large-scale geometry of metric spaces and finitely generated groups.

The starting point for the work presented below is an averaging theorem for Property A. It turns out that over amenable groups Property A, even though it a priori expected to be much more flexible, in one aspect behaves exactly like amenability. This allows us to reduce some problems about the geometry of an amenable group to the equivariant setting, which is much easier to handle. We present three different applications of the averaging principle.

The first application is a construction of a metric space which does not have Property A but does admit a coarse embedding into a Hilbert space. The existence of such spaces was not settled since [58], where Property A was shown to imply coarse embeddability into the Hilbert space and the question left open was whether one can reverse this implication. The examples given here are the only known at present with the above properties. They also disprove a conjecture of Dranishnikov that Property A is equivalent to coarse embeddability into ℓ_1.

The second application is to a quasi-isometry invariant A_X related to Property A which we introduce in the fifth chapter. The definition is based on the study of the asymptotic geometry of sets arising in the definition of Property A and the averaging principle allows to reduce this geometry to the geometry of Følner sets used to define amenability. This yields a bound on our invariant in terms of isoperimetric profiles, a classical invariant studied in differential geometry and geometric group theory. On the other hand we obtain a different bound in terms of type of asymptotic dimension, which is a fine invariant related to asymptotic dimension, both were introduced by Gromov in [28]. We apply these results to reprove some estimates on isoperimetric profiles of certain amenable groups and, more importantly, construct finitely generated groups with finite asymptotic dimension which cannot have linear type. The question whether such groups exist was posed by Roe.

Our last result concerns the spectrum of the Laplace-Beltrami operator. We use coarse index theory and the averaging principle now applied to show that certain factor groups inherit Property

A from the original group. This allows to use Yu's theorem on the coarse Baum-Connes Conjecture to show that the spectrum of the Laplace-Beltrami operator acting on square-integrable p-forms on a certain Galois coverings of compact manifolds contains zero for some p. For universal covers this follows directly from Yu's theorem, however in our case it is important to consider covers which have a non-trivial, amenable fundamental group. This together with an assumption of macroscopical largeness yields the required theorem.

The material presented below is the subject of the articles [42, 41, 43].

CHAPTER II

PRELIMINARIES

Coarse geometry

Coarse geometry originates from Mostow's celebrated rigidity theorem and was later popularized by Gromov in the context of finitely generated groups, mainly through his proof of Milnor's conjecture that groups with polynomial growth are exactly the ones which are virtually nilpotent.

Consider an unbounded metric space (X, d) with the metric topology. Every such space carries another metric $\widetilde{d}$ which induces the same topology and such that $(X, \widetilde{d})$ is bounded - take for example $\widetilde{d} = \min(1, d)$. In other words, topology concentrates on local, infinitesimal phenomena, neglecting the global properties of the metric. It is however natural to expect that some of the geometric properties of a unbounded metric space should take place globally and should not depend on any local data. The idea behind coarse geometry is to formalize these intuitions.

Imagine that we're given a discrete metric space and that we start moving away from it with some speed. Because of perspective, the further away we move the closer to each other the points look. As we move away to infinity from our space, the object we observe looks more and more "dense", "continuous" - think of the integers becoming a real line when viewed from infinity or of a bounded space looking like a single point when viewed from a sufficiently large distance.

This intuition was formalized by Gromov [26, 28] and then extended by van den Dries and Wilkie [18]. It is done by taking an appropriate limit of a sequence of metric spaces $X_n = (X, \frac{1}{s_n}d)$ which are just given by the original space X with metric divided by elements of an increasing sequence s_n (which governs the speed with which we're moving away from X). See also [49] for details.

Spaces and maps

All metric spaces will be assumed to be *proper*, by which we mean that every closed ball of finite radius is compact. A metric space will be called *discrete* if there exists a constant $C > 0$ such

that $d(x,y) \geq C$ for all $x, y \in X$. A discrete metric space will be called locally finite if every ball of finite radius is finite, and will be said to be of bounded geometry if for every $R > 0$ there exists a number $N(R)$ such that $\#B(x,R) \leq N(R)$ for every $x \in X$. Bounded geometry clearly implies local finiteness. Also note that any locally finite metric space must be at most countable.

As explained earlier, we would like to identify spaces which look the same from infinity. Below we give the appropriate notions of morphisms and equivalences implementing the ideas of coarse geometry.

Definition 2.1. *Let X, Y be metric spaces. A map $f : X \to Y$ is called a coarse map if*

(1) *f is proper i.e. the preimage of a compact set is compact*

(2) *there exists a non-decreasing function $\rho_+ : [0,\infty) \to [0,\infty)$ such that*

$$d_Y(f(x), f(y)) \quad \leq \quad \rho_+(d_X(x,y)).$$

The map f is called large-scale Lipschitz if ρ_+ can be chosen to be an affine function.

Note that we do not impose any behavior of f on small distances, in particular f can be discontinuous - in that case the constant C only controls the size of discontinuities.

Definition 2.2. *Two maps $f, g : X \to Y$ are close if there exists a constant $C > 0$ such that*

$$d_Y(f(x), g(x)) \leq C$$

for every $x \in X$.

The following notion of a coarse embedding describes a controlled inclusion in large-scale geometry. It was introduced by Gromov in [28] and is of great importance for applications [58].

Definition 2.3. *A coarse map $f : X \to Y$ is a coarse embedding if there exists a nondecreasing function $\rho_- : [0,\infty) \to [0,\infty)$ such that*

$$\rho_-(d_X(x,y)) \leq d_Y(f(x), f(y))$$

and $\lim_{t\to\infty} \rho_-(t) = \infty$. *We say that* f *is a quasi-isometric embedding if both* ρ_+ *and* ρ_- *can be chosen to be affine. We call* f *a coarse equivalence (quasi-isometry) if it is a coarse embedding (quasi-isometric embedding) and there is a constant* $C < \infty$ *such that the image* $f(X)$ *is a* C*-net in* Y.

Equivalently $f : X \to Y$ is a coarse equivalence (quasi-isometric equivalence) if there is a coarse map (large-scale Lipschitz map) $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to identity maps Id_Y and Id_X respectively.

We will additionally that the metric space we deal with are quasi-geodesic.

Definition 2.4. *A metric space is uniformly quasi-geodesic if there exist constants* $C, L > 0$ *such that for any* $x, y \in X$ *there exists a sequence* $x = x_1, x_2 \ldots, x_{n-1}, x_n = y$ *of points in* X *such that* n *depends only on* $d(x, y)$ *and*

$$\sum_{i=1}^{\infty} d(x_i, x_{i+1}) \leq C d(x, y)$$

where $x_1 = x$, $x_n = y$ *and* $d(x_i, x_{i+1}) \leq L$.

Groups as geometric objects

One particular class of examples of bounded geometry metric spaces arises naturally in group theory. Let $G = \langle \Sigma \mid \mathcal{R} \rangle$ be a finitely generated group, where $\Sigma = \Sigma^{-1}$ is the generating set. Then every element $g \in G$ can be written as a word in Σ. By $|g|$ we denote the length of the shortest word representing g and we call it the *length of* g. This length function can be thought of as a norm on the group G, and just as one defines the metric on normed spaces we define the word length metric by setting

$$d_G(g, h) = |g^{-1}h|$$

for all $g, h \in G$. This metric is invariant under left translations, i.e. $d_G(\gamma g, \gamma h) = d_G(g, h)$ is satisfied for any $\gamma, g, h \in G$.

Example 2.5. Take the group of integers $\mathbb{Z} = \langle \{-1, 1\} \mid \emptyset \rangle$. Then the word length of $n \in \mathbb{Z}$ is just the absolute value $|n|$ and the word length metric is $d_{\mathbb{Z}}(m, n) = |m - n|$.

A natural question to ask is whether this metric depends on the choice of the generating set,

and it is not hard to notice it does: just compare the above example with $\mathbb{Z} = \langle \{-2, -1, 1, 2\} \mid \emptyset \rangle$. However it turns out that two metrics arising from different generating sets give coarsely equivalent metric spaces. This shows that a finitely generated group carries an intrinsic coarse geometry and every coarse-geometric invariant is in fact an invariant of the group.

The combinatorial model

We will also use a combinatorial model for our discrete metric spaces, it will become important in the last chapter in coarsening of homology. This model is known as the Rips complex and it is a certain simplicial approximation of our discrete metric space X.

Definition 2.6. *Let X be a discrete metric space. The Rips complex of X , denoted $P_d(X)$, is the polyhedron constructed in the following way: the set of vertices is the set X and points $x_1, ..., x_n$ span a simplex if $d(x_i, x_j) \leq d$.*

In other words $P_d(X)$ is the nerve of the covering $\{B(x, d)\}_{x \in X}$. We metrize the Rips complex by giving each simplex the metric that it inherits from the sphere under the projection of the standard simplex in R^n onto S^{n-1}. Note that for $d = 1$ we get simply the Cayley graph of the group G.

The Rips complex P_d encodes the process of *"killing the local topology on scale d"*, by "squeezing" everything of diameter less than d, into a single simplex. As d grows to infinity this is exactly what we are looking for in the coarse geometric setting. An example of a very intuitive observation is Rips' theorem that if Γ is a hyperbolic group then the complex $P_d(\Gamma)$ is contractible for all sufficiently large d.

Amenability

In what follows whenever A is a set, $\#A$ will denote its cardinality. Let G be a finitely generated group with a word length metric. The boundary ∂F of a set $F \subset G$ is defined as

$$\partial F = \{x \in G \setminus F \mid d(x, F) = 1\}.$$

Denote

$$\ell_1(X) = \left\{ f : X \to \mathbb{R} \mid \sum_{x \in X} |f(x)| < \infty \right\},$$

with the usual norm

$$\|f\|_{\ell_1(X)} = \sum_{x \in X} |f(x)|$$

and

$$\ell_1(X)_{1,+} = \{ f \in \ell_1(X) \mid \|f\|_1 = 1, f \geq 0 \}.$$

In other words, $\ell_1(X)_{1,+}$ is the space of positive probability measures on X. If Γ is a finitely generated group, $\gamma \in \Gamma$ and $f \in \ell_1(\Gamma)_{1,+}$ then by $\gamma \cdot f$ we denote the translation of f by element γ, i.e.

$$(\gamma \cdot f)(g) = f(\gamma^{-1} g).$$

Definition 2.7. *A finitely generated group Γ is amenable if any of the following equivalent conditions is satisfied:*

(1) ***(Invariant Mean Condition)*** *There exists a left invariant mean on $\ell_\infty(\Gamma)$, i.e. a positive, linear functional $\int \cdot \, dg$ on $\ell_\infty(\Gamma)$ such that $\int 1_G \, dg = 1$ and $\int \gamma \cdot f \, dg = \int f \, dg$ for any $\gamma \in \Gamma$;*

(2) ***(Følner condition)*** *For every $\varepsilon > 0$ there exists a finite set $F \subset G$ such that*

$$\frac{\#\partial F}{\#F} \leq \varepsilon.$$

(3) ***(Hulanicki-Reiter condition)*** *For every $\varepsilon > 0$ and $R < \infty$ there exists a function $f \in \ell_1(\Gamma)_{1,+}$ such that*

$$\|f - \gamma \cdot f\|_{\ell_1(X)} \leq \varepsilon$$

for all $|\gamma| \leq R$ and $\#\operatorname{supp} f < \infty$.

Amenability was introduced by von Neumann in his study of the Banach-Tarski paradox. It has many different equivalent definitions and a large number of application in different branches of mathematics. Standard references on amenability include [6, 24, 45].

Examples of amenable groups include finite groups, abelian groups and any group that can be obtained from these by taking extensions, subgroups or quotients. Also groups with subexponential volume growth are amenable. It is easy to show on the other hand that free groups are not amenable. Thus also any group that contains a free subgroup is not amenable.

Property A as metric amenability

Property A was introduced by Yu in [58] as a metric, "non-equivariant" version of amenability.

Definition 2.8 ([58])**.** *A discrete metric space X has Property A if for every $R > 0$ and $\varepsilon > 0$ there is a collection $\{\mathcal{A}_x\}_{x\in X}$ of finite subsets of $X \times \mathbb{N}$ and $S > 0$ such that*

(1) $\dfrac{\#\mathcal{A}_x \triangle \mathcal{A}_y}{\#\mathcal{A}_x \cap \mathcal{A}_y} \leq \varepsilon$ *when* $d(x, y) \leq R$*;*

(2) $\mathcal{A}_x \subset B(x, S) \times \mathbb{N}$

The class of finitely generated groups possessing Property A is quite large, at present the only groups known not to have Property A are Gromov's groups which contain expanders in their Cayley graphs. There are also groups for which it is not yet known whether they have Property A, e.g. Thompson's group F.

It was also shown by Guentner, Kaminker and Ozawa that a finitely generated group has Property A if and only if the reduced group C^*-algebra $C^*_r(\Gamma)$ is exact, see [31, 44], while Higson and Roe [33] proved that Property A for G is equivalent to the existence of a topologically amenable action of G on some compact space.

We recall the characterization of Property A in terms of finitely supported functions in the unit sphere of the Banach space ℓ_1. This characterization, modeled after the Hulanicki-Reiter condition, was proved in [33].

Proposition 2.9 ([33])**.** *Let X be a discrete metric space with bounded geometry. The following conditions are equivalent:*

(1) *X has property A;*

(2) *For every $R > 0$ and $\varepsilon > 0$ there exists a map $\xi : X \to \ell_1(X)_{1,+}$, $(x \mapsto \xi_x)$ and $S > 0$ such that*

(a) $\|\xi_x - \xi_y\|_1 \leq \varepsilon$ *whenever* $d(x,y) \leq R$

(b) $\operatorname{supp} \xi_x \subseteq B(x, S)$ *for every* $x \in X$.

Property A was introduced as a condition sufficient to coarsely embed a metric space into a Hilbert space.

Theorem 2.10 ([58])**.** *Let X be a discrete metric space with Property A. Then X admits a coarse embedding into the Hilbert space.*

This on the other hand, via coarse index theory has application to problems such as the Novikov Conjecture, positive scalar curvature problem, zero-in-the-spectrum problem. These applications follow from a remarkable theorem of Yu.

Theorem 2.11 ([58])**.** *Let X be a bounded geometry metric space which admits a coarse embedding into the Hilbert space. Then the Coarse Baum-Connes Conjecture holds for X*.

One of the main features of Property A is that it is satisfied by a class of groups incomparably larger than that of amenable groups. In fact the only examples of groups known not to have Property A are random groups containing expanders in their Cayley graphs constructed by Gromov [29, 30].

CHAPTER III

AN AVERAGING THEOREM FOR PROPERTY A

In this chapter we prove the theorem which will be our main tool. Our further results are derived using this theorem. In order to present it we first need to introduce some definitions which quantify Property A and amenability.

Definition 3.1. *Let X be a discrete metric space.*

(1) *For a map $\xi : X \to \ell_1(X)_{1,+}$ satisfying condition (2) in Proposition 2.9 with $\varepsilon > 0$ and $R > 0$ denote*

$$\mathcal{S}_X(\xi, R, \varepsilon) = \inf S,$$

$\mathcal{S}_X(\xi, R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$*, where the infimum is taken over all $S > 0$ satisfying* $\operatorname{supp} \xi_x \subseteq B(x, S)$ *for every $x \in X$.*

(2) *Define*

$$\operatorname{rad}_X(R, \varepsilon) = \inf \mathcal{S}_X(\xi, R, \varepsilon),$$

$\operatorname{rad}(R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$*, where the infimum is taken over all maps satisfying the conditions in (1) above for R and ε.*

(3) *If Γ is a finitely generated group then given $R > 0$, $\varepsilon > 0$ by $\operatorname{rad}_\Gamma^{eqv}(R, \varepsilon) \in \mathbb{N} \cup \{\infty\}$ we denote the smallest S for which there exists a function $f \in \ell_1(\Gamma)_{1,+}$ with* $\operatorname{supp} f \subseteq B(S)$ *satisfying condition (3) in Definition 2.7 for all $\gamma \in \Gamma$ such that $|\gamma| \leq R$.*

In other words, $\operatorname{rad}_\Gamma^{eqv}$ is the notion resulting from restricting (1) and (2) to considering only functions $\xi : \Gamma \to \ell_1(\Gamma)_{1,+}$ given by translates of a single function $f \in \ell_1(\Gamma)_{1,+}$, i.e. $\xi_\gamma = \gamma \cdot f$ for every $\gamma \in \Gamma$ and for some fixed $f \in \ell_1(X)_{1,+}$.

Theorem 3.2 (**Averaging theorem for Property A, [42]).** *Let Γ be finitely generated amenable*

group.Then for any $\varepsilon > 0$ and $R > 0$ the following equality holds,

$$\mathrm{rad}_\Gamma(R,\varepsilon) \quad = \quad \mathrm{rad}^{eqv}_\Gamma(R,\varepsilon)\,.$$

Proof. To show the inequality $\mathrm{rad}_\Gamma(R,\varepsilon) \leq \mathrm{rad}^{eqv}_\Gamma(R,\varepsilon)$, given a finitely supported function $f \in \ell_1(\Gamma)_{1,+}$ satisfying condition (3) from Definition 2.7 for $R > 0$ and $\varepsilon > 0$ and all $\gamma \in \Gamma$ such that $|g| \leq R$, consider the map $\xi : \Gamma \to \ell_1(\Gamma)_{1,+}$ defined by $\xi_\gamma = \gamma \cdot f$.

To prove the other inequality assume that Γ satisfies condition (2) of Proposition 2.9 for $R > 0$, $\varepsilon > 0$ with $S > 0$ realized by the function $\xi : \Gamma \to \ell_1(\Gamma)_{1,+}$. For every $\gamma \in \Gamma$ define

$$f(\gamma) = \int_\Gamma \xi_g(\gamma^{-1}g)\, dg.$$

This gives a well-defined function $f : \Gamma \to \mathbb{R}$, $\xi_g(\gamma^{-1}g)$ as a function of g belongs to $\ell_\infty(\Gamma)$ since $\xi_g(\gamma) \leq 1$ for all $\gamma, g \in \Gamma$.

First observe that if $|\gamma| > S$ then $\xi_g(\gamma^{-1}g) = 0$ for all $g \in \Gamma$, thus $f(\gamma) = 0$ whenever $|\gamma| > S$. Consequently,

$$\begin{aligned} \|f\|_{\ell_1(\Gamma)} \quad &= \quad \sum_{\gamma \in B(S)} f(\gamma) = \sum_{\gamma \in B(S)} \int_\Gamma \xi_g(\gamma^{-1}g)\, dg \\ &= \quad \int_\Gamma \left(\sum_{\gamma \in B(S)} \xi_g(\gamma^{-1}g) \right) dg \quad = \quad \int_\Gamma 1\, dg \quad = \quad 1. \end{aligned}$$

Thus f is an element of $\ell_1(\Gamma)_{1,+}$. If $\lambda \in \Gamma$ is such that $|\lambda| \leq R$ then

$$\begin{aligned} \|f - \lambda \cdot f\|_{\ell_1(\Gamma)} \quad &= \quad \sum_{\gamma \in \Gamma} |f(\gamma) - f(\lambda^{-1}\gamma)| \\ &= \quad \sum_{\gamma \in B(S) \cup \lambda B(S)} \left| \int_\Gamma \xi_g(\gamma^{-1}g)\, dg - \int_\Gamma \xi_g((\lambda^{-1}\gamma)^{-1}g)\, dg \right| \end{aligned}$$

$$= \sum_{\gamma \in B(S) \cup \lambda B(S)} \left| \int_\Gamma \xi_g(\gamma^{-1} g)\, dg - \int_\Gamma \xi_{\lambda^{-1} g}(\gamma^{-1} g)\, dg \right|$$

$$= \sum_{\gamma \in B(S) \cup \lambda B(S)} \left| \int_\Gamma (\xi_g(\gamma^{-1} g) - \xi_{\lambda^{-1} g}(\gamma^{-1} g))\, dg \right|$$

$$\leq \int_\Gamma \left(\sum_{\gamma \in B(S) \cup \lambda B(S)} |\xi_g(\gamma^{-1} g) - \xi_{\lambda^{-1} g}(\gamma^{-1} g)| \right) dg$$

$$\leq \int_\Gamma \varepsilon\, dg = \varepsilon,$$

since

$$\int_\Gamma \xi_g((\lambda^{-1}\gamma) g)\, dg = \int_\Gamma \lambda \cdot \left(\xi_g((\lambda^{-1}\gamma)^{-1} g) \right) dg$$

$$= \int_\Gamma \xi_{\lambda^{-1} g}(\gamma^{-1} g)\, dg,$$

this is a consequence of the invariance of the mean.

Thus for the previously chosen R and ε we have constructed a function $f \in \ell_1(\Gamma)_{1,+}$ satisfying $\| f - \gamma \cdot f \|_{\ell_1(\Gamma)} \leq \varepsilon$ whenever $1 \leq |\gamma| \leq R$ and supp $f \subseteq B(S)$ for the same S as for ξ. This proves the second inequality. □

CHAPTER IV

PROPERTY A IS NOT EQUIVALENT TO EMBEDDABILITY INTO ℓ_2

In this chapter we present the first application of the averaging theorem. As mentioned earlier the original motivation to introduce Property A in [58] was that it was a sufficient condition to embed coarsely a metric space into the Hilbert space. Since then it was an open problem whether the converse implication holds, i.e. whether Property A is equivalent to coarse embeddability into the Hilbert space. Below we present the first and at present the only known construction of a metric space which does not have Property A but does embed coarsely into any ℓ_p-space, $1 \leq p \leq \infty$, including the Hilbert space. Our construction also gives counterexamples to a conjecture by A.N. Dranishnikov, that for discrete metric spaces Property A is equivalent to coarse embeddability into ℓ_1.

Behavior of Følner sets in high-dimensional products

Let (X_1, d_{X_1}), (X_2, d_{X_2}) be metric spaces. We will consider the cartesian product $X_1 \times X_2$ with the ℓ_1-metric, i.e.

$$d_{X_1 \times X_2}(x, y) = d_{X_1}(x_1, y_1) + d(x_2, y_2),$$

for $x = (x_1, x_2)$, $y = (y_1, y_2)$, both in $X_1 \times X_2$. If Γ_1, Γ_2 are finitely generated groups such metric on $\Gamma_1 \times \Gamma_2$ is left-invariant if and only if the metrics on the factors are. In particular, if the metric on the factors is the word length metric then the ℓ_1-metric on the direct product gives the word length metric associated to the standard set generators arising from the generators on the factors.

In this section we study how does the number rad_Γ behave for cartesian powers of a fixed finitely generated amenable group Γ. Theorem 3.2 will be our main tool, allowing us to reduce questions about rad_{Γ^n} to questions about $\mathrm{rad}^{eqv}_{\Gamma^n}$. Note that if X and Y are discrete metric spaces, and for every $R > 0$ and $\varepsilon > 0$ there are maps $\xi : X \to \ell_1(X)$ and $\zeta : Y \to \ell_1(Y)$ realizing Property A for X and Y respectively, then the maps $\xi \otimes \zeta : X \times Y \to \ell_1(X \times Y)$ of the form

$$\xi \otimes \zeta_{(x,y)} = \xi_x \, \zeta_y,$$

give Property A for $X \times Y$ in the sense of Proposition 2.9 and in the particular case when $Y = X$ the diameter of the supports increases (the reader can extract precise estimates from [14]). The main result of this section shows that this is always the case.

The next theorem is the key ingredient in the construction of spaces without Property A.

Theorem 4.1. *Let* Γ *be a finitely generated amenable group. Then for any* $0 < \varepsilon < 2$,

$$\liminf_{n\to\infty} \mathrm{rad}^{eqv}_{\Gamma^n}(1, \varepsilon) = \infty.$$

Proof. Assume the contrary. Then there exists an $S \in \mathbb{N}$ such that for infinitely many $n \in \mathbb{N}$ there is a function $f_n \in \ell_1(\Gamma^n)_{1,+}$ satisfying

$$\|f_n - \gamma \cdot f_n\|_1 \leq \varepsilon,$$

supp $f_n \subset B_{\Gamma^n}(S)$ for all $\gamma \in \Gamma$ such that $|\Gamma| = 1$. Fix $\delta \leq \frac{2-\varepsilon}{2S}$ and $m \in \mathbb{N}$ and for any $n \in \mathbb{N}$ for which f_n as above exists consider the decomposition

$$\Gamma^n = \Gamma^m \times \Gamma^m \times ... \times \Gamma^m \times \Gamma^r$$

where $0 \leq r < m$. For $k = 1, ..., \frac{n-r}{m}$ denote by $\partial_k f_n$ the restriction of f_n to the set

$$\{g \in \text{supp } f_n \,:\, |g| = S,\, g_{(i)} \neq e \Leftrightarrow (k-1)m + 1 \leq i \leq mk\},$$

of those elements of supp f_n whose length in this k-th factor Γ^m is exactly S, and extend it with 0 to a function on the whole Γ^n; we denote by $g_{(i)}$ the i-th coordinate of $g \in \Gamma^n$ as an element of the cartesian product.

Since for $k \neq l$, where $km + r \leq n$ and $lm + r \leq n$, we have

$$\text{supp}\, \partial_k f_n \,\cap\, \text{supp}\, \partial_l f_n \;=\; \emptyset$$

and

$$\sum_{k=1}^{\frac{n-r}{m}} \|\partial_k f_n\|_1 \;\leq\; \|f_n\|_1 \;=\; 1,$$

we can conclude that for every $\hat{\varepsilon} > 0$, which we now choose to satisfy $\frac{\varepsilon+2\hat{\varepsilon}}{1-\hat{\varepsilon}} \leq \varepsilon + \delta$, there exists a sufficiently large $n \in \mathbb{N}$ and $i \in \mathbb{N}$ such that

$$\|\partial_i f_n\|_1 \leq \hat{\varepsilon}.$$

Denote

$$\varphi = \frac{f_n - \partial_i f_n}{\|f_n - \partial_i f_n\|_1} \in \ell_1(X)_{1,+} .$$

We have

$$\begin{aligned} \|\varphi - \gamma \cdot \varphi\|_1 &= \frac{\|(f_n - \gamma \cdot f_n) + (\gamma \cdot \partial_i f_n - \partial_i f_n)\|_1}{\|f_n - \partial_i f_n\|_1} \\ &\leq \frac{\varepsilon + 2\hat{\varepsilon}}{1 - \hat{\varepsilon}} \leq \varepsilon + \delta, \end{aligned}$$

by the previous choice of $\hat{\varepsilon}$. Now consider the decomposition $\Gamma^n = \Gamma^m \times \Gamma^{n-m}$ where Γ^m is the i-th factor in which we performed the previous operations on f_n. For every $g \in \Gamma^m$ define (we're recycling the letter f here, the "old" f's don't appear in the proof anymore)

$$f(g) = \sum_{h \in \Gamma^{n-m}} \varphi(gh),$$

where $h \in \Gamma^{n-m}$. Then $f \in \ell_1(\Gamma^m)_{1,+}$ and supp $f \subseteq B_{\Gamma^m}(S-1)$. Moreover, for an element $\gamma \in \Gamma^m$ of length 1,

$$\begin{aligned} \|f - \gamma \cdot f\|_1 &= \sum_{g \in \Gamma^m} |f(g) - f(\gamma^{-1} g)| \\ &= \sum_{g \in \Gamma^m} \Big| \sum_{h \in \Gamma^{n-m}} \varphi(gh) - \varphi(\gamma^{-1} gh) \Big| \end{aligned}$$

$$\leq \sum_{g\in\Gamma^m}\sum_{h\in\Gamma^{n-m}} |\varphi(gh)-\varphi(\gamma^{-1}gh)|$$

$$= \sum_{g\in\Gamma^n} |\varphi(g)-\varphi(\gamma^{-1}g)| \;=\; \|\varphi-\gamma\cdot\varphi\|_1 \;\leq\; \varepsilon+\delta.$$

Since $m\in\mathbb{N}$ was arbitrary we can obtain a family $\{f_m\}_{m\in\mathbb{N}}$ of functions $f_m\in\ell_1(\Gamma^m)_{1,+}$ satisfying

$$\|f_m-\gamma\cdot f_m\|_1\leq\varepsilon+\delta$$

and supp $f_m\subseteq B_{\Gamma^m}(S-1)$ where δ is independent of m. If we apply the procedure described above to this family we can again reduce the diameter of the supports of the functions f_m and obtain yet another new family $\{f_m\}_{m\in\mathbb{N}}$ of functions $f_m\in\ell_1(\Gamma^m)_{1,+}$ such that

$$\|f_m-\gamma\cdot f_m\|_1\leq\varepsilon+2\delta$$

and supp $f_m\subseteq B_{\Gamma^m}(S-2)$.

After repeating this procedure S times we obtain a family $\{f_m\}_{m\in\mathbb{N}}$ such that $f\in\ell_1(\Gamma^m)_{1,+}$ and

$$\|f_m-\gamma\cdot f_m\|_1 \;\leq\; \varepsilon+S\delta$$

$$\leq\; \varepsilon+S\,\frac{2-\varepsilon}{2S} \;<\; 2,$$

since $\delta\leq\frac{2-\varepsilon}{2S}$. However, for every $m\in\mathbb{N}$

$$f_m(g)=\begin{cases}1 & \text{when } g=e,\\ 0 & \text{otherwise.}\end{cases}$$

and

$$\|f_m-\gamma\cdot f_m\|=2$$

for every $m\in\mathbb{N}$ and every $\gamma\in\Gamma^m$, which gives a contradiction. □

Remark 4.2. In the proofs in this section we have reduced the study Property A to studying

amenability, however we expect that the above considerations can be carried out as well in a more general setting for the price of complicating the arguments and estimates.

Constructing coarsely embeddable metric spaces without Property A

In this section we construct metric spaces which do not have Property A, but which do admit a coarse embedding into the Hilbert space.

First observe that the exact values of both rad and rad^{eqv} depend on the metric, in particular in the case of a word length on the group, on the choice of the generating set. What is independent of such choices is whether rad and rad^{eqv} are finite or infinite. The following is a straight-forward consequence of Definition 3.1 and the Proposition 2.9.

Proposition 4.3. (1a) *If a discrete metric space X has Property A then* $\mathrm{rad}(R,\varepsilon) < \infty$ *for every* $R > 0$ *and* $\varepsilon > 0$.
(1b) *A discrete metric space X with bounded geometry has Property A if and only if* $\mathrm{rad}(R,\varepsilon) < \infty$ *for every* $R > 0$ *and* $\varepsilon > 0$.
(2) *A finitely generated group is amenable if and only if* $\mathrm{rad}^{eqv}_{\Gamma}(R,\varepsilon) < \infty$ *for every* $R > 0$ *and* $\varepsilon > 0$.

The idea for our construction is natural: take a disjoint union of bounded, locally finite metric spaces, for which it is known that they satisfy Property A with diameters growing to infinity, so that we violate the condition from Proposition 4.3.

On the other hand the condition $\mathrm{rad}(R,\varepsilon) = \infty$ for any $R > 0$ and $\varepsilon > 0$ does not rule out coarse embeddability into the Hilbert space, which is characterized by the existence of a c_0-type functions in the sphere of ℓ_1. This was proved by Dadarlat and Guentner [14], see also [40] for discussion and applications.

Given a sequence $\{(X_n, d_n)\}_{n=1}^{\infty}$ we will make the disjoint sum $\mathcal{X} = \coprod X_n$ into a metric space by giving it a metric $d_{\mathcal{X}}$ such that

1. $d_{\mathcal{X}}$ restricted to X_n is d_n,
2. $d_{\mathcal{X}}(X_n, X_{n+1}) \geq n + 1$,
3. if $n \leq m$ we have $d_{\mathcal{X}}(X_n, X_m) = \sum_{k=n}^{m-1} d_{\mathcal{X}}(X_k, X_{k+1})$.

Theorem 4.4. *Let Γ be a finite, group. The (locally finite) metric space $\mathcal{X}_\Gamma = \coprod_{n=1}^{\infty} \Gamma^n$ has the following properties:*

(1) *$\mathcal{X}_\Gamma$ does not have Property A*

(2) *$\mathcal{X}_\Gamma$ embeds coarsely into ℓ_p for any $1 \leq p \leq \infty$.*

Proof. To prove 1) observe that by 4.3 if $\mathcal{X}_\Gamma$ would satisfy Property A then $\mathrm{rad}_{\mathcal{X}_\Gamma}(1, \varepsilon)$ would be finite for every $0 < \varepsilon < 2$, which in turn would imply that the restriction of maps ξ realizing Property A for every ε and $R = 1$ to each $\Gamma^n \subseteq \mathcal{X}_\Gamma$ gives Property A with diameter bounded uniformly in n,

$$\sup_{n \in \mathbb{N}} \mathrm{rad}_{\Gamma^n}(1, \varepsilon) < \infty,$$

since $B_{\mathcal{X}_\Gamma}(x, R) = B_{\Gamma^n}(x, R)$ for all sufficiently large n and all $x \in \Gamma^n \subset \mathcal{X}_\Gamma$. However by theorems 4.1 and 3.2,

$$\mathrm{rad}^{eqv}_{\Gamma^n}(1, \varepsilon) \;=\; \mathrm{rad}_{\Gamma^n}(1, \varepsilon)$$

and

$$\mathrm{rad}^{eqv}_{\Gamma^n}(1, \varepsilon) \longrightarrow \infty$$

as $n \to \infty$.

To prove 2), note that since Γ is a finite metric space any one-to-one map from Γ into the space ℓ_1 is biLipschitz. Denote the biLipschitz constant by L. Then the product map

$$f^n = \underbrace{f \times f \times \ldots \times f}_{n \text{ times}} : \Gamma^n \to \left(\sum_{i=1}^{n} \ell_1 \right)_1$$

is also a biLipschitz map with the same constant L, where $\left(\sum_{i=1}^{n} \ell_1\right)_1$ denotes a direct sum of n copies of ℓ_1 with a ℓ_1-metric, which is of course isometrically isomorphic to ℓ_1. It is clear that this suffices to embed $\mathcal{X}_\Gamma$ into ℓ_1 coarsely.

In [40] the author proved that the Hilbert space embeds coarsely into any ℓ_p, $1 \leq p \leq \infty$ and that the properties of coarse embeddability into ℓ_p for $1 \leq p \leq 2$ are all equivalent. Thus $\mathcal{X}_\Gamma$ embeds coarsely into the Banach space ℓ_p for any $1 \leq p \leq \infty$. □

Note that in the simplest case $G = \mathbb{Z}_2$, the space $\mathcal{X}_{\mathbb{Z}_2}$ is a disjoint union of discrete cubes of

increasing dimensions, with the ℓ_1-metric. A *cube complex* is a metric polyhedral complex in which each cell is isometric to the Euclidean cube $[0,1]^n$, and the gluing maps are isometries. For instance the plane $\mathbb{R}^2$ is a cube complex with cubes given by $[n, n+1] \times [k, k+1]$, $k, n \in \mathbb{N}$.

Corollary 4.5. *An infinite-dimensional cube complex does not have Property A*

On the other hand it is also not hard to construct an infinite-dimensional cube complex which embeds coarsely into any ℓ_p, giving a different realization of examples discussed above.

We finally mention a conjecture formulated by Dranishnikov [15, Conjecture 4.4] that a discrete metric space X has Property A if and only if X embeds coarsely into the space ℓ_1. The examples discussed in this section are in particular counterexamples to Dranishnikov's conjecture.

CHAPTER V

ISOPERIMETRY AND ASYMPTOTIC DIMENSION

In this chapter we introduce and study a quasi-isometry invariant related to Property A and we develop some techniques to estimate it. In particular we show how to estimate it using: 1) the isoperimetric profile of a group, 2) type of asymptotic dimension. These methods allow to answer a question of Roe, who asked if there exist groups of finite asymptotic dimension but of non-linear type. We show a construction of such groups for any given prescribed value of asymptotic dimension.

Definition of the A-profile

We introduce the following function associated to any metric space with Property A.

Definition 5.1 (**Isodiametric profile of a metric space**). *Let X be a metric space with Property A. Define the function $\mathrm{A}_X : \mathbb{N} \to \mathbb{N}$ by the formula*

$$\mathrm{A}_X(n) = \mathrm{rad}_X\left(1, \frac{1}{n}\right).$$

Clearly the function A_X is well-defined and non-decreasing. We will be interested in estimating the asymptotic behavior of A_X, i.e. in the rate of divergence of A_X. We consider the following relation. We write $f \preceq g$ if there exist constants $C, K > 0$ such that $f(n) \leq Cg(Kn)$ for all $n \in \mathbb{N}$ and we write $f \simeq g$ if $f \preceq g$ and $g \preceq f$.

The asymptotic behavior of A_X does not depend on the choice of $R = 1$ and the sequence $\frac{1}{n}$ up to constants, the argument will be given further in this section.

Example 5.2. Let X be a bounded metric space. Then $\mathrm{A}_X \simeq \mathrm{const}$. In fact, $\mathrm{A}_X(n) = \frac{\mathrm{diam}\, X}{2}$ for all n large enough.

Example 5.3. Let T be any locally finite tree. Then $\mathrm{A}_T \preceq n$. Indeed, recall from [58] that for a fixed $R > 0$ and $\varepsilon > 0$ Property A for the tree is constructed by fixing a point ω on the boundary

of T and taking normalized characteristic functions of the geodesic segments of length $\frac{2R}{\varepsilon}$ on the geodesic ray starting from x in the direction of ω.

Example 5.4. Let Γ be a finitely generated group with polynomial growth. Then $A_\Gamma \leq n$. In this case the normalized characteristic functions of balls of radius n give the required estimate.

Basic properties

We now move on to prove the most natural properties - estimate for subspaces, direct products and invariance under quasi-isometries. For the first one, we will use the fact that Property A is hereditary [54].

Proposition 5.5. *Let Y have Property A and $X \subseteq Y$. Then X has Property A and for any $R > 0$, $\varepsilon > 0$*

$$\mathrm{rad}_X(R, \varepsilon) \leq 3\,\mathrm{rad}_Y(R, \varepsilon)$$

Proof. For every $y \in Y$ let $p(y) \in X$ be a point such that $d(y, p(y)) \leq 2d(y, X)$. Define an isometry $\mathcal{I} : \ell_1(Y) \to \ell_1(X \times Y)$ by the formula

$$\mathcal{I} f(x, y) = \begin{cases} f(y) & \text{if } x = p(y) \\ 0 & \text{otherwise} \end{cases}$$

Let $\varepsilon > 0$ and $R > 0$. By definition of Property A there exist a number $S < \infty$ and a map $\xi : Y \to \ell_1(Y)$ such that $\|\xi_y - \xi_{y'}\|_{\ell_1(Y)} \leq \varepsilon$ if $d(y, y') \leq R$ and $\operatorname{supp} \xi_y \subseteq B(y, S)$ for every $y \in Y$. Define $\widetilde{\xi} : X \to \ell_1(X)_{1,+}$ by the formula

$$\widetilde{\xi}_x(z) = \sum_{y \in Y} \mathcal{I}\xi_x(z, y).$$

Then it is easy to check that

$$\|\widetilde{\xi}_x - \widetilde{\xi}_{x'}\|_{\ell_1(X)} \leq \varepsilon$$

whenever $d(x, x') \leq R$ and

$$\operatorname{supp} \xi_x \subseteq B(x, 3S).$$

□

A direct consequence is the following.

Proposition 5.6. *Let $X \subseteq Y$ be a subspace. Then $\mathrm{A}_X \leq \mathrm{A}_Y$.*

Direct products

We consider direct products with the ℓ_1-metric, as explained in an earlier section.

Proposition 5.7. *Let X_1, X_2 be countable discrete metric spaces with property A. Then*

$$\mathrm{A}_{X_1 \times X_2} \simeq \max\left(\mathrm{A}_{X_1}, \mathrm{A}_{X_2}\right).$$

Proof. Let $R = 1$, $\varepsilon > 0$ and let the maps $\xi : X \to \ell_1(X)_{1,+}$, $\eta : Y \to \ell_1(Y)_{1,+}$ realize Property A for $R = 1$ and ε, for X and Y and respectively, with diameters of the supports S_X and S_Y respectively. Then the map $\xi \otimes \eta : X \times Y \to \ell_1(X \times Y)_{1,+}$ defined by

$$\xi \otimes \eta_{(x,y)}(z, w) \quad = \quad \xi_x(z)\, \eta_y(w),$$

satisfies

$$\mathrm{supp}\left(\xi \otimes \eta_{(x,y)}\right) \ \subseteq \ B((x,y), S_X + S_Y) \ \subseteq \ B((x,y), 2\max(S_X, S_Y)).$$

For $R = 1$ we also have the following estimate:

$$\begin{aligned}
\|\xi \otimes \eta_{(x,y)} - \xi \otimes \eta_{(x',y')}\|_{\ell_1(X\times Y)} \quad &= \quad \sum_{z\in X, w\in Y} |\xi_x(z)\eta_y(w) - \xi_{x'}(z)\eta_{y'}(w)| \\
&\leq \quad \sum_{z\in X, w\in Y} |\xi_x(z)\eta_y(w) - \xi_x(z)\eta_{y'}(w)| \\
&\qquad + \sum_{z\in X, w\in Y} |\xi_x(z)\eta_{y'}(w) - \xi_{x'}(z)\eta_{y'}(w)| \\
&\leq \quad \|\xi_x - \xi_{x'}\|_{\ell_1(X)} + \|\eta_y - \eta_{y'}\|_{\ell_1(Y)} \leq \varepsilon.
\end{aligned}$$

The last inequality follows from the fact that since $d((x,y)(x',y')) = R = 1$ then either $x = x'$ or $y = y'$. This proves $\mathrm{A}_{X_1\times X_2} \preceq \max(\mathrm{A}_{X_1}, \mathrm{A}_{X_2})$. The estimate "$\succeq$" follows from Proposition 5.6. □

Permanence properties of groups with Property A have been extensively studied in connection to the Novikov Conjecture, see e.g. [3], [14], [11], [54], so estimates of this sort are possible also for e.g. free products, extensions, some direct limits, groups acting on metric spaces and more.

Invariance under quasi-isometries

We devote the rest of this section to proving large-scale invariance of the asymptotics of A_X, we will in particular estimate how does the isodiametric function behave under coarse equivalences that are not necessarily quasi-isometries. Strictly for that purpose for $\kappa, R \in \mathbb{N}$ define the function $\mathrm{A}_X^{\kappa,R}(n) = \mathrm{rad}_X(R, \frac{\kappa}{n})$. With this definition $\mathrm{A}_X = \mathrm{A}_X^{1,1}$.

Lemma 5.8. *For a fixed $R > 0$ and $\kappa \in \mathbb{N}$ we have*

$$\mathrm{A}_X^{1,R} \simeq \mathrm{A}_X^{\kappa,R}. \ \square$$

Lemma 5.9. *Let X have Property A. Then for any $R, R' \in \mathbb{N}$ we have*

$$\mathrm{A}_X^{1,R} \simeq \mathrm{A}_X^{1,R'}.$$

Proof. If $R \leq R'$ then obviously $\mathrm{rad}_X(R,\varepsilon) \leq \mathrm{rad}_X(R',\varepsilon)$ for any ε and the inequality "$\preceq$" follows. Conversely, assume that $R' \leq R$. If $d(x,y) \leq R$ and that we're given the function ξ from the definition of Property A for R' and ε. Then by the uniform quasi-geodesic condition on X (Definition 2.4) with κ equal to the largest integer smaller than R/R', we have

$$\|\xi_x - \xi_y\|_{\ell_1(X)} \leq \sum_{i=0}^{\kappa-1} \|\xi_{x_i} - \xi_{x_{i+1}}\|_{\ell_1(X)} \leq \kappa\varepsilon_n,$$

where the $x = x_0, x_1, ..., x_{k-1}, x_\kappa = y$ are such that $d(x,y) \leq \sum_{i=0}^{\kappa} d(x_i, x_{i+1})$ and $d(x_i, x_{i+1}) \leq R'$. This gives the inequality $\mathcal{S}_X(\xi, R', \varepsilon) \leq \mathcal{S}_X(\xi, R, \kappa\varepsilon)$, and consequently

$$\mathrm{rad}_X(R',\varepsilon) \leq \mathrm{rad}_X(R,\kappa\varepsilon).$$

This together with the previous lemma proves the assertion. □

Having proved that the asymptotics of $\mathrm{A}_X^{\kappa,R}$ depend neither on R nor on κ, as a consequence we get the desired statement on large-scale behavior of A_X.

Theorem 5.10. *Let X, Y be metric spaces and let Y have Property A. Let $f : X \to Y$ be a coarse embedding. Then X has Property A and*

$$\mathrm{A}_X \preceq \varphi_-^{-1} \circ \mathrm{A}_Y .$$

In particular, if X and Y are quasi-isometric then $\mathrm{A}_X \simeq \mathrm{A}_Y$.

Proof. Let $f : X \to Y$ be the coarse embedding with Lipschitz constant L and distortion φ_-. Since we're only interested in the asymptotic behavior, we may assume that for large $t \in R$, $\varphi_-(t)$ is strictly increasing. Also by Proposition 5.5 without loss of generality we may assume that f is onto.

For every point $y \in Y$ choose a unique point x_y in the preimage $f^{-1}(y)$. This gives an inclusion $\ell_1(Y)_{1,+} \subseteq \ell_1(X)_{1,+}$. Since Y has Property A, for every $\varepsilon > 0$ and $R < 0$ there exists a map $\xi : Y \to \ell_1(Y)_{1,+}$ and a number $S > 0$ satisfying conditions from Proposition 2.9. Choose R large enough so that $\varphi_-(R) \geq 1$ and define a map $\eta : X \to \ell_1(Y)_{1,+} \subseteq \ell_1(X)_{1,+}$ setting

$$\eta_x(z) = \begin{cases} \xi_{f(x)}(y) & \text{if } z = x_y, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that φ satisfies the required conditions and that

$$\mathcal{S}_X(R, \varepsilon, \eta) \leq \varphi_-^{-1}(\mathcal{S}_Y(LR, \varepsilon, \xi)).$$

This, with Lemma 5.8 gives

$$\mathrm{A}_X \simeq \mathrm{A}_X^{1,R} \preceq \varphi_-^{-1} \circ \mathrm{A}_Y^{1,LR} \simeq \varphi_-^{-1} \circ \mathrm{A}_Y .$$

□

Relation to isoperimetric profiles

The Følner function measures the volume of the support of a function and rad^{eqv} measures the radius of the smallest ball in which such support is contained. Since the Følner function is defined using the number rad and the numbers rad and rad^{eqv} are equal on an amenable groups by the averaging theorem, we have

Theorem 5.11. *Let Γ be a finitely generated amenable group. Then*

$$\rho_\Gamma \circ \mathrm{A}_\Gamma \succeq \mathrm{Føl}\,.$$

Proof. Since $\mathrm{A}_G(n) = \mathrm{rad}(1, \frac{1}{n}) = \mathrm{rad}^{eqv}(1, \frac{1}{n})$, the number $\rho_\Gamma(\mathrm{A}_\Gamma(n))$ is the volume of the ball containing supp f, where f minimizes $\mathrm{rad}^{eqv}(1, \frac{1}{n})$. Thus

$$\rho_\Gamma(\mathrm{A}_\Gamma(n)) \geq \#\,\mathrm{supp}\; f \geq \mathrm{Føl}(n),$$

since Føl(n) minimizes the volume of supp f for $\varepsilon = \frac{1}{n}$. □

It follows that the function A_Γ in the case of amenable groups can have nontrivial behavior, as we now explain. Given two finitely generated groups Γ_1 and Γ_2 one defines their wreath product

$$\Gamma_1 \wr \Gamma_2 = (\oplus_{\gamma\in\Gamma_1}\,\Gamma_2) \rtimes \Gamma_2,$$

where the action of Γ_2 on $(\oplus_{\gamma\in\Gamma_1}\Gamma_2)$ is by a coordinate shift. Since the wreath product preserves amenability, one can wonder how does the function $\mathrm{Føl}_{\Gamma_1\wr\Gamma_2}$ depend on the functions $\mathrm{Føl}_{\Gamma_1}$ and $\mathrm{Føl}_{\Gamma_2}$. This was studied in [55], [46], [25] and a complete answer was given by A. Erschler in [19], where it was proved that

$$\mathrm{Føl}_{\Gamma_1\wr\Gamma_2} \simeq (\,\mathrm{Føl}_{\Gamma_1}\,)^{\,\mathrm{Føl}_{\Gamma_2}}, \tag{1}$$

provided that the following condition holds: ($\star$) *for any $C > 0$ there is a $K > 0$ such that for any $n > 0$, $\mathrm{Føl}_{\Gamma_2}(Kn) > C\,\mathrm{Føl}_{\Gamma_2}(n)$.* This last assumption will be automatically fulfilled in the cases we will consider, note however that it does not allow Γ_2 to be finite.

Now, using Theorem 5.11, we can apply this to the isodiametric function.

Proposition 5.12. *Let* Γ_1, Γ_2 *be discrete amenable groups and let* Føl_{Γ_2} *satisfy condition* $(\star)$*. Then*

$$\text{A}_{\Gamma_1 \wr \Gamma_2} \succeq \text{Føl}_{\Gamma_2}(\ln \text{Føl}_{\Gamma_1}).$$

The proof amounts to recalling the fact that growth of a finitely generated group is at most exponential. Consequently, since for $G_k = \underbrace{\mathbb{Z} \wr (\dots (\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z}) \dots)}_{k \text{ times}}$ the Følner function satisfies

$$\text{Føl}_{G_k} \simeq \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{k \text{ times}},$$

we obtain

Corollary 5.13. *Let* G_n *be as above. Then*

$$\text{A}_{G_k} \succeq \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{k-1 \text{ times}} \ln n.$$

Another example in [19] is one of a group Γ with Føl growing faster than any of the above iterated exponents. This of course gives the same conclusion for the function A_Γ.

Recall also that it is not known whether Property A is satisfied for Thompson's group F. Thompson's group F is defined by the presentation

$$\langle a, b \mid [ab^{-1}, a^{-1}ba] = [ab^{-1}, a^{-2}ba^2] = e \rangle$$

or

$$\langle x_i, i \in \mathbb{N} \mid x_j^{-1} x_i x_j = x_{i+1} \text{ for } i > j \rangle.$$

On the other hand it is known that the iterated wreath product

$$W_k = \underbrace{(\dots (\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \dots) \wr \mathbb{Z}}_{k \text{ times}}$$

is a quasi-isometrically embedded subgroup of F for every $k \in \mathbb{N}$, this was shown by S. Cleary [12], and Theorem 5.10 leads to the following statement:

Corollary 5.14. *If Thompson's group F has Property A then*

$$\mathrm{A}_F \geq n^k$$

for every $k \in \mathbb{N}$.

Asymptotic dimension and the A-profile

In this section we will show another method to estimate A_X, it is based on the connection between Property A and asymptotic dimension. In particular we show a large class of spaces for which $\mathrm{A}_X \simeq n$. These spaces will arise as spaces with finite asymptotic dimension of *linear type*, i.e. where the diameter of the elements of the covers depends linearly on disjointness.

A family $\mathcal{U}$ of subsets of a metric space will be called δ-bounded if $\operatorname{diam} U \leq \delta$ for every $U \in \mathcal{U}$. Two families $\mathcal{U}_1, \mathcal{U}_2$ are R-disjoint if $d(U_1, U_2) \geq R$ for any $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$.

Definition 5.15 ([28])**.** *We say that a metric space X has asymptotic dimension less than $k \in \mathbb{N}$, denoted* $\operatorname{asdim} X \leq k$*, if for every $R > 0$ one can find a number $\delta < \infty$ and $k+1$ R-disjoint families $\mathcal{U}_0, ..., \mathcal{U}_k$ of subsets of X such that*

$$X = \mathcal{U}_0 \cup ... \cup \mathcal{U}_k$$

and every $\mathcal{U}_i$ is δ-bounded

Asymptotic dimension is a large-scale version of the classical covering dimension in topology. It is a coarse invariant and a fundamental notion for [57], where the Novikov Conjecture for groups with finite asymptotic dimension is proved. Because of this result asymptotic dimension of groups has become a very actively studied notion, we refer the reader to the articles [4], [5] and to [49] and the references there for more on asymptotic dimension of finitely generated groups. Let us just mention here that examples of groups with finite asdim include free, hyperbolic, Coxeter groups, free products and extensions of groups with finite asdim. On the other hand it is easy to see that there are finitely generated groups which don't have finite asymptotic dimension - just take $\mathbb{Z} \wr \mathbb{Z}$ or Thompson's group F, each of which contains $\mathbb{Z}^k$ as a subgroup for every k and since such inclusion is always a coarse embedding and $\operatorname{asdim} \mathbb{Z}^n = n$, it pushes asymptotic dimension off to infinity.

The following finer invariant associated to a space with finite asymptotic dimension was also introduced by Gromov [28, p. 29], see also [49, Chapter 9], [17, Section 4]

Definition 5.16. *Let X be a metric space satisfying* $\operatorname{asdim} X \leq k$. *Define the type function $\tau_{k,X} : \mathbb{N} \to \mathbb{N}$ in the following way: $\tau_{k,X}(n)$ is the smallest $\delta \in \mathbb{N}$ for which X can be covered by $k+1$ families $\mathcal{U}_0, ..., \mathcal{U}_k$ which are all n-disjoint and δ-bounded.*

The type function is also known as *dimension function* and its linearity is often referred to as *Higson property* or *finite Assouad-Nagata dimension*, see the discussion later in this chapter. The proof of our next statement adapts an argument of Higson and Roe [33], who showed that finite asymptotic dimension implies Property A.

Theorem 5.17. *Let X be a metric space satisfying* $\operatorname{asdim} X \leq k$. *Then*

$$\mathrm{A}_X \preceq \tau_{k,X} \, .$$

Proof. By assumption, for every $n \in \mathbb{N}$, X admits a cover by $k+1$, $\tau_{k,X}(n)$-bounded, n-disjoint families $\mathcal{U}_i$, as in definition 5.15. Let $\mathcal{U}$ be a cover of X consisting of all the sets from all the families $\mathcal{U}_i$. There exists a partition of unity $\{\psi_V\}_{V \in \mathcal{U}}$ and a constant C_k depending only on k such that:

(1) each ψ is Lipschitz with constant $2/n$;

(2) $\sup \operatorname{diam}(\operatorname{supp} \psi) \leq \tau_{k,X}(n) + 4n \leq C_k \tau_{k,X}(n)$;

(3) for every $x \in X$ no more than $k+1$ of the values $\psi(x)$ are non-zero.

For every ψ choose a unique point x_ψ in the set $\operatorname{supp} \psi$ and define

$$\xi_x^n = \sum_\psi \psi(x) \, \delta_{x_\psi} .$$

Then if $d(x,y) \leq 1$ we see that

$$\|\xi_x^n - \xi_y^n\|_{\ell_1(X)} = \sum_\psi |\psi(x) - \psi(y)| \; \leq \; \frac{2}{n} C_k',$$

where C'_k is another constant depending on k only and

$$\text{supp } \xi_x^n \subseteq B(x, C_k \tau_{k,X}(C'_k n)).$$

Once again by Lemma 5.8 we are done. □

Thus spaces and groups of finite asymptotic dimension of linear type have A_X linear. The simplest examples of such are Euclidean spaces and trees, and their finite cartesian products, by an argument similar to the one in Proposition 5.7. It is also well-known that δ-hyperbolic groups are in this class, one can quickly deduce this fact either directly from [50] or from a theorem of Buyalo and Schroeder [10], which states that every hyperbolic group admits a quasi-isometric embedding into a product of a finite number of trees. In fact, Dranishnikov and Zarichnyi showed that every metric space with finite asymptotic dimension is equivalent to a subset of a product of a finite number of trees [17], however this equivalence is in general just coarse and not quasi-isometric, we will give examples illustrating this below.

The main estimate

As a corollary of the results presented in the two previous sections we get our main application, a direct relation between two of the considered large-scale invariants: Vershik's Følner function and Gromov's type of asymptotic dimension.

Theorem 5.18. *Let Γ be a finitely generated amenable group satisfying* $\operatorname{asdim}\Gamma \leq k$. *Then there exists a constant C depending only on k such that*

$$\text{Føl} \preceq \rho_\Gamma \circ C\, \tau_{k,\Gamma}.$$

Proof. The estimate follows from Theorem 5.11 and Theorem 5.17. □

A general conclusion coming from this result is that several asymptotic invariants considered in the literature, namely: decay of the heat kernel, isoperimetric profiles, Følner functions, type function of asymptotic dimension, our function A_Γ and distortion of coarse embeddings, in the case

of amenable groups all carry very similar information. We will show below how to use this fact to obtain results in various directions.

Remark 5.19. The constant C in the above formula is a technical consequence of the estimates in the proof of Theorem 5.17 and it doesn't seem that we can get rid of it a priori. We can however omit it once we know for example that τ satisfies condition $(\star)$ from Section 4: for every C there exists a number K such that $C\tau_{k,X}(n) \leq \tau_{k,X}(Kn)$ for all n. This is a very mild condition, in particular it holds for all common asymptotics. Another situation when the constant C does not play a role is when the upper estimate on the growth ρ_Γ is known. For the purposes of applications in Sections 7 and 8 we will be interested only in groups with exponential growth and we will omit the constant C from now on.

Estimates of isoperimetric profiles

We will use our main theorem and asymptotic dimension to get precise estimates of the function Føl for some groups. Although these estimates are known (see e.g. [46]), our purpose is to show that even though in Theorem 5.18 we, loosely speaking, pass between the volume of a set and the volume of the ball which contains it, which one can expect will cause some loss of information in the exponential growth case, we can in fact obtain sharp estimates on Føl. In other words, asymptotically Følner sets behave like balls. We will use the following consequence of Theorem 5.18.

Corollary 5.20. *If Γ is an amenable group with exponential growth and finite asymptotic dimension of linear type then*

$$\text{Føl} \simeq e^n .$$

The statement follows from Theorem 5.18 and a theorem of Coulhon and Saloff-Coste [13], stating that for groups of exponential growth the function Føl grows at least exponentially.

It should be also pointed out that the question of existence of amenable groups with exponential growth and at most exponential Følner function was first asked by Kaimanovich and Vershik in [36].

Example 5.21. The first example we consider are groups $\mathfrak{S}_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ satisfies

$|\operatorname{trace}(A)| > 2$, usually one takes just

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The group $\mathfrak{S}_A$ has exponential growth and it is a discrete, quasi-isometrically embedded lattice in the group Sol used by Thurston to describe one of the geometries in his geometrization conjecture. The group Sol is quasi-isometric to a undistorted horosphere in $\mathbb{H}^2 \times \mathbb{H}^2$, a product of two hyperbolic planes. The latter has finite asymptotic dimension of linear type, this can be seen directly or from the fact that the hyperbolic plane embeds quasi-isometrically into a product of trees ([9]), and so we recover (see e.g. [46, Section 3]) the estimate

$$\text{Føl}_{\mathfrak{S}_A} \simeq e^n.$$

The same strategy works for polycyclic groups which are lattices in solvable Lie groups of dimension at least 3.

Example 5.22. The solvable Baumslag-Solitar groups,

$$\mathrm{BS}(1,k) \;=\; \langle\, a, b \;:\; aba^{-1} = b^k \,\rangle,$$

where $k > 1$, constitute our second example. These groups are metabelian but not polycyclic and they act properly, cocompactly by isometries on a warped product $X_k = \mathbb{R} \times T_k$, where T_k is an infinite, oriented, $k+1$-regular tree. For every vertex v in this tree we have 1 incoming edge and k edges going out of v, and we orient the incoming edge towards the vertex v. Metrically, the set $\mathbb{R} \times r$ where r is an infinite, coherently oriented line, is an isometric copy of the hyperbolic plane, see [20] for a detailed construction of the space X_k. Since both the tree and the hyperbolic plane have finite asymptotic dimension of linear type, it is easy to check by a direct construction of coverings or of a quasi-isometric embedding into an appropriately chosen space that X_k also has finite asymptotic dimension of linear type. Thus, since by the Milnor-Švarc Lemma $\mathrm{BS}(1,k)$ is quasi-isometric to X_k, we get (see [46, Theorem 3.5])

$$\text{Føl}_{\mathrm{BS}(1,k)} \simeq e^n.$$

Example 5.23. Assume we are given two finitely generated amenable groups G and H and an exact

sequence

$$0 \longrightarrow K \longrightarrow \Gamma \longrightarrow H \longrightarrow 0, \tag{2}$$

i.e. Γ is an extension of K by H. Assume also that K is undistorted in Γ (recall that a subgroup H is *undistorted* in the ambient group Γ if the embedding of H as a subgroup is quasi-isometric) and that both K and H have finite asymptotic dimension of linear type. Under these assumptions, in [7] a Hurewicz-type theorem for asymptotic dimension of linear type is proved, which in particular implies that Γ also has finite asymptotic dimension of linear type. In our situation this yields the following

Corollary 5.24. *Let K,Γ,H be finitely generated amenable groups, sequence (2) be exact. Assume that K is undistorted in Γ and that the latter has exponential growth. If H and K have finite asymptotic dimension of linear type then*

$$\text{Føl} \simeq e^n.$$

Note however that this does not apply to the group $\mathfrak{S}_A$ considered above. In that example the fiber $\mathbb{Z}^2$ is well-known to be exponentially distorted in the ambient extension.

The above of course raises the question, which amenable groups with exponential growth have finite asymptotic dimension of linear type. The next section is devoted to building examples which fail this condition.

Applications to dimension theory

There are two questions concerning asymptotic dimension and its type function:

(Q.1) *How to build natural examples of finitely generated groups with $\tau_{k,\Gamma}$ growing faster than linearly for some k?* Most of the known examples of groups with finite asymptotic dimension have linear type and to the author's best knowledge no examples of groups with other behavior of the type function were known.

(Q.2) *Assume we have an example like in (Q.1), with* $\operatorname{asdim}\Gamma \leq k$ *and $\tau_{k,\Gamma} \succ n$. Can we find $k' > k$ such that $\tau_{k',\Gamma}$ will be linear?*

These questions, although quite natural, become even more relevant if one identifies after Dranishnikov and Zarichnyi [17, Section 4] asdim with linear type as the large-scale analog of the Assouad-Nagata dimension [1], [37], which is an invariant in the Lipschitz category of metric spaces. The precise definition in our setting is simply the following: *a metric space X has Assouad-Nagata dimension $\leq k$ if it satisfies* $\operatorname{asdim} X \leq k$ *and* $\tau_{k,X} \preceq n$. The above questions can be then rephrased in the following way: (Q.1) *How to build finitely generated groups with Assouad-Nagata dimension strictly greater than asymptotic dimension?* (Q.2) *Does finite asymptotic dimension imply finite Assouad-Nagata dimension?*

We will use Theorem 5.18 to answer both questions and build some interesting examples of groups with finite asymptotic dimension. For any non-trivial finite group H and for $k = 1, 2, 3, \ldots$ consider the group $\Gamma_k^{(1)} = H \wr \mathbb{Z}^k$. In the simplest case $H = \mathbb{Z}/2\mathbb{Z}$ the group Γ_k is a lamplighter group (see e.g. [25], [51]).

We have $\operatorname{asdim} \Gamma_k^{(1)} = k$. We will only sketch the proof. To see $\operatorname{asdim} \Gamma_k^{(1)} \leq k$ one needs to appeal to recent work of Dranishnikov and Smith [16], in which they extend the notion of asymptotic dimension to all *countable* groups. And so observe that by [16, Theorem 2.1], the infinitely generated countable group $\oplus_{z \in \mathbb{Z}^k} H$ (equipped with a proper length function inherited from $\Gamma_k^{(1)}$) has asymptotic dimension zero, since every of its finitely generated subgroups is finite. Since $\mathbb{Z}^k$ has asymptotic dimension k, the semi-direct product $(\oplus_{z \in \mathbb{Z}^k} H) \rtimes \mathbb{Z}^k$ is of asymptotic dimension at most k, by the Hurewicz-type theorem in [16]. Then the inclusion of $\mathbb{Z}^k$ in $\Gamma_k^{(1)}$ as a subgroup gives $\operatorname{asdim} \Gamma_k^{(1)} \not\leq k - 1$.

Now, by equation (1) used earlier in this chapter we have

$$\text{Føl}_{\Gamma_k^{(1)}} \simeq (\text{Føl}_H)^{\text{Føl}_{\mathbb{Z}^k}} \simeq e^{(n^k)}.$$

For any $k' \geq k$, Theorem 5.18 gives

$$e^{(n^k)} \preceq \rho_{\Gamma_k^{(1)}} \circ \tau_{k', \Gamma_k^{(1)}},$$

but this implies

$$n^k \preceq \tau_{k',\Gamma_k^{(1)}},$$

since the growth of $\Gamma_k^{(1)}$ is exponential.

Now take the group $\Gamma_k^{(2)} = H \wr \Gamma_k^{(1)}$. By the same argument as before $\operatorname{asdim} \Gamma_k^{(2)} = k$, and again by Theorem 5.18 for any k' we get

$$e^{\left(e^{(n^k)}\right)} \preceq \rho_{\Gamma_k^{(2)}} \circ \tau_{k',\Gamma_k^{(2)}},$$

which gives

$$e^{(n^k)} \preceq \tau_{k',\Gamma_k^{(2)}}.$$

Iterating this construction we get for a fixed k and $i = 1, 2, \ldots$ infinitely many (depending on different choices of H) finitely generated groups $\Gamma_k^{(i)}$ with asdim equal exactly k and type function growing at least as fast as the iterated exponential function

$$\underbrace{\exp \exp \ldots \exp}_{i-1 \text{ times}} n^k.$$

This gives the examples postulated by (Q.1) and answers (Q.2) negatively, since in particular all estimates are independent of k'.

Two comments are in order.

Remark 5.25. In the case of asymptotic dimension 1, the construction above is optimal in the following sense. Januszkiewicz and Świątkowski [35] and independently Gentimis [22] proved that if a finitely presented group G has asymptotic dimension 1 then it is virtually free, and it follows that it satisfies $\tau_{1,G} \preceq n$. So the groups $\Gamma_1^{(i)}$ for $i \geq 2$ are examples showing that results of Januszkiewicz-Świątkowski and Gentimis will not be true if one drops the requirement of finite presentation. It also follows that one cannot obtain examples with properties like $\Gamma_1^{(i)}$ and which would be finitely presented.

Remark 5.26. By [17] all the groups considered in this section embed coarsely into a product of finitely many trees. It might be interesting to note that by arguments similar to those in Theorem

5.10, any such embedding must be strongly distorting, i.e. for $\Gamma_k^{(i)}$ it must satisfy

$$\varphi_- \preceq n^{\frac{1}{k}} \qquad \text{for } i = 1$$

and

$$\varphi_- \preceq (\underbrace{\ln \ln \ldots \ln}_{i-1 \text{ times}} n)^{\frac{1}{k}} \qquad \text{for } i = 2, 3, \ldots$$

This contrasts again to the case of hyperbolic groups, which, as mentioned previously, embed quasi-isometrically into an appropriately chosen product of finitely many trees [9].

CHAPTER VI

COARSE INDEX THEORY AND THE ZERO-IN-THE-SPECTRUM CONJECTURE

The zero-in-the-spectrum problem

The zero-in-the-spectrum conjecture was first formulated by Gromov [27, 28] and asks if the spectrum Laplace-Beltrami operator acting on the square-integrable p-forms on the universal cover of a closed aspherical manifold contains zero. This fact is implied by the Strong Novikov Conjecture and thus the interest in finding a counterexample. A more general zero-in-the-spectrum conjecture on open complete manifolds was stated by Lott and it is true if there is a positive answer to the following question: *does the spectrum of the Laplace-Beltrami operator Δ_p acting on square-integrable p-forms of a complete manifold $\mathcal{M}$ contain zero for some $p = 0, 1, \ldots$?* The answer is negative in general: Farber and Weinberger [21] showed that for every $n \geq 6$ there exists a manifold $\mathcal{N}$ such that zero is not in the spectrum of Δ_p for any $p \in 0, 1 \ldots$ acting on the universal cover of $\mathcal{N}$. Later Higson, Roe and Schick [34] extended this result and gave a complete description of groups which can appear as fundamental groups of manifolds whose universal covers don't have zero in the spectrum of the Laplacian.

Because of the origins of the problem, various covering spaces are a natural environment for considering zero-in-the-spectrum questions. An early result of this type is a theorem of Brooks [8] stating that given a regular cover $\mathcal{M}$ of a compact manifold $\mathcal{N}$, 0 is in the spectrum of Δ_0 on $\mathcal{M}$ if and only if the group of deck transformations is amenable. The articles [38, 39] provide a comprehensive survey of this topic.

The Laplace-Beltrami operator

The Laplace-Beltrami operator and the Laplace-de Rham operator are generalizations of the usual Laplace operator from $\mathbb{R}^n$ to a Riemannian manifold M. Its action on a function $f : M \to \mathbb{R}$

can be expressed by the formula

$$\Delta f = \operatorname{div} \cdot \nabla f = \frac{1}{\sqrt{\det(G)}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right).$$

where: $G = J^T J = \{g_{ij}\}$ is the metric tensor (J denotes the Jacobian), $G^{-1} = \{g^{ij}\}$ is the inverse of G.

More generally we consider a complete Riemannian manifold and $\Lambda^p(M)$, the space of square-integrable p-forms on M. Thus $\Lambda^p(M)$ is a Hilbert space. Because of completeness of M we can integrate by parts on M:

$$\int_M d\omega \wedge \eta = (-1)^{\deg \omega + 1} \int_M \omega \wedge d\eta$$

where $\omega, d\omega, \eta, d\eta$ are smooth, square-integrable forms on M (see [38, Lemma 1] for a proof). Consider d^*, the adjoint operator to d. We construct a self-adjoint operator $\Delta : \Lambda^*(M) \to \Lambda^*(M)$ by setting

$$\Delta = dd^* + d^* d.$$

Then, by restricting Δ to $\Lambda^p(M)$ we obtain Δ_p. The spectrum of Δ_p is contained in $[0, \infty)$. The zero-in-the-spectrum question asks if 0 belongs to the spectrum of Δ_p for some p.

Index theory via the coarse assembly map

One way of showing that zero *is* in the spectrum of the Laplace-Beltrami operator is to use index theory. We first need to briefly recall the construction of the coarse index.

Let X be a discrete metric space. An *X-module* $\mathcal{H}_X = (\mathcal{H}, \rho)$ is a pair consisting of a separable Hilbert space $\mathcal{H}$ and a *-representation $\rho : C_0(X) \to \mathcal{B}(\mathcal{H})$ of the algebra of complex-valued functions vanishing at infinity. We will say that an X-module is *non-degenrate* if the representation ρ is non-trivial, and we will call it *standard* if no compact operators are in the image of ρ, except $\rho(0)$ of course.

Definition 6.1. *Let X, Y be discrete metric space and $\mathcal{H}_X$, $\mathcal{H}_Y$ the corresponding modules. The* support of a bounded operator $T : \mathcal{H}_X \to \mathcal{H}_Y$ *is denoted* supp T *and is defined to be the complement in $X \times Y$ of the set of all points $(x, y) \in X \times Y$ for which there exist functions $f, g \in C_0(X)$ such that*

$gTf = 0$ *and* $f(x) \neq 0 \neq g(y)$.

Definition 6.2. *Let X and $\mathcal{H}_X$ be as above and let T be a bounded operator on $\mathcal{H}_X$. Then*

(1) *The* propagation *of T is the number* $\sup_{(x,y)\in \operatorname{supp} T} d(x,y)$.

(2) *T is said to be locally compact if for any $f \in C_0(X)$ the operators fT and Tf are compact.*

The following definition introduces the Roe algebra

Definition 6.3. *Let $\mathcal{H}_X$ be a standard, non-degenerate X-module. The Roe algebra $C^*(X)$ is the C^*-algebra closure of all locally compact operators on $\mathcal{H}_X$ which have finite propagation.*

The algebra $C^*(X)$ does not depend on the choices we made along the way, namely on the choice of the X-module as long as it is standard and non-degenerate.

Take now $d \in \mathcal{N}$ and a cycle $(\mathcal{H}, \rho, F)$ in the K-homology $K_*(P_d(X)) = KK_*(C_0(P_d(X)), \mathbb{C})$ such that $\mathcal{H}$ equipped with ρ is a standard, non-degenerate $P_d(X)$-module. Choose $\mathcal{U} = \{U\}$ be a locally finite, uniformly bounded open cover of $P_d(X)$ and $\{\varphi_U\}$ be a continuous partition of unity subordinate to the cover $\mathcal{U}$. We define an operator

$$\widetilde{F} = \sum_U \rho(\varphi_U)^{1/2} F \rho(\varphi_U)^{1/2},$$

where the infinite sum converges in the strong topology because the covering is locally finite.

Lemma 6.4 ([57])**.** *For F, $\widetilde{F}$ as above we have*

(1) *$\widetilde{F}$ has finite propagation*

(2) $\|\widetilde{F}\| \leq 4\|F\|$.

Note that $(H, \rho, \widetilde{F})$ is equivalent to (H, ρ, F) in the group $K_0(P_d(X))$. This is so because the operator $\rho(f)(F - \widetilde{F})$ is compact for any $f \in C_0(P_d(X))$

We will now construct an element of K-theory of the Roe algebra that will be the "index at scale d" of the cycle above. In the description of K-theory we're using, the elements of the group are Kasparov modules over the pair $(\mathbb{C}, C^*(P_d(X)))$. Take $\mathcal{E} = C^*(P_d(X))$ as a module over itself, and the representation $\psi : \mathbb{C} \to \mathcal{B}(\mathcal{E})$ is given by the assignment $1 \mapsto \mathrm{Id}_{\mathcal{E}}$.

The last element we need in the triple is an operator on $\mathcal{E}$. It will be given by $\widetilde{F}$ acting as a multiplier on $C^*(P_d(X))$. All we need is to observe that $\widetilde{F}^2 - 1 \in C^*(P_d(X))$, and that is so because both $\widetilde{F}^2$ and 1 have finite propagation. Thus we can construct a Kasparov module

$$(C^*(P_d(X)), \psi, \widetilde{F}). \tag{3}$$

With the above notation, the coarse index at scale d is the map $\mathrm{Index}_d : K_0(P_d(X)) \to K_0(C^*(P_d(X)))$ defined by the formula

$$\mathrm{Index}_d([(H, \rho, F)]) = [(C^*(P_d(X)), \psi, \widetilde{F})].$$

Coarse index

There is a natural inclusion $i_d : P_d(X) \to P_{d+1}(X)$ given simply by the fact that the cover $\{B(x,d)\}_{x\in X}$ is inscribed in the cover $\{B(x,d+1)\}_{x\in X}$. Combining this with the index maps for $d = 1, 2, \ldots$ we get the following diagram:

$$\begin{array}{ccccccc}
K_0(P_1(X)) & \xrightarrow{i_1} & K_0(P_2(X)) & \xrightarrow[\cdots]{} & K_0(P_d(X)) & \xrightarrow{i_d} & \cdots \\
\downarrow{\scriptstyle Index_1} & & \downarrow{\scriptstyle Index_2} & & \downarrow{\scriptstyle Index_d} & & \\
K_0(C^*(P_1(X))) & \xrightarrow{\approx} & K_0(C^*(P_2(X))) & \xrightarrow[\cdots]{} & K_0(C^*(P_d(X))) & \xrightarrow{\approx} & \cdots
\end{array}$$

Note that

$$\lim_{d\to\infty} K_0(C^*(P_d(X)) = K_0(C^*(X)).$$

Definition 6.5. *We define the coarse index map*

$$\mu^c : \lim_{d\to\infty} K_0(P_d(X)) \to K_0(C^*(X))$$

as the direct limit of the maps Index_d.

The strategy for showing that 0 is not in the spectrum of the Laplace-Beltrami operator relies on the following statement.

Proposition 6.6 ([47])**.** *Let $\mathcal{M}$ be a complete Riemannian manifold and $\mathcal{D}$ be the de Rham operator.*

If $\mu^c([\mathcal{D}]) \neq 0$ *in* $K_*(C^*(\mathcal{M}))$ *then the spectrum of the Laplace-Beltrami operator acting on the square-integrable forms on* $\mathcal{M}$ *contains zero.*

We refer [38, 47] for the arguments.

Group quotients with Property A

As mentioned earlier, Property A resembles amenability and enjoys a number of the same properties as amenability e.g. inheritance by subgroups and extensions. Unlike amenability however, Property A is preserved by free products [11] and, more importantly, is not preserved under surjective homomorphisms. The latter is a consequence of Gromov's construction [30] of finitely presented groups which do not coarsely embed into the Hilbert space together with exactness of free groups. Thus we are interested in general conditions guaranteeing that in the exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1 \tag{4}$$

G/H has Property A provided that G has it. If we exclude the trivial case and assume that G/H is infinite there are two other cases in which such a statement is obviously true:

(1) If H is a finite group then G and G/H are quasi-isometric, so G is exact if and only if G/H is exact.

(2) If the exact sequence (4) splits then then G/H is embedded in G, and inherits exactness from the ambient group.

Our first step in proving Theorem 6.11 is the following

Theorem 6.7. *Let* G *be a finitely generated group satisfying Property A and let* H *be an amenable subgroup of* G*. Then the quotient* G/H *has Property A.*

This condition is sharp: as soon as we drop the amenability condition on H, we can take G in (4) to be a free group and get all finitely generated groups as quotients. We will treat the cosets of H as orbits of its action on G and denote the orbit of $x \in G$ by Hx. The quotient G/H is a metric space with the metric

$$d(Hx, Hy) = \min_{h,h' \in H} d(hx, h'y).$$

This in particular means that the quotient map $G \to G/H$ is a contraction.

The next statement is a version of the averaging theorem for Property A, only now we average over the subgroup instead of the whole group and we are not interested in quantitative statement.

Proposition 6.8. *Let G be a group with Property A and let H be an amenable subgroup of G. Then for every $\varepsilon > 0$ Property A can be realized by a map $\xi : G \to \ell_1(G)_{1,+}$ such that ξ is equivariant under the action of H, i.e.,*

$$\xi_{hx} = h \cdot \xi_x \tag{5}$$

for every $h \in H$ and $x \in G$.

Proof. Assume that G satisfies conditions of Definition 2.9 for $R < \infty$, $\varepsilon > 0$ with $S > 0$ realized by a function $\zeta : G \to \ell_1(G)_{1,+}$. For every $x \in G$ define

$$\xi_x(y) = \int_H \zeta_{hx}(hy)\, dh.$$

Since $0 \leq \zeta_x(y) \leq 1$ for all $x, y \in G$ we get a well-defined function $\xi_x : G \to \mathbb{R}$ satisfying $0 \leq \xi_x(y) \leq 1$ for all $x, y \in G$. Observe that if $d(x,y) > S$ then $\zeta_x(y) = 0$. Since H acts by isometries, $d(hx, hy) = d(x,y)$ and it follows that $\xi_x(y) = 0$ if $d(x,y) > S$. This allows to compute the norm of ξ_x:

$$\begin{aligned} \|\xi_x\|_{\ell_1(G)} &= \sum_{y \in B(x,S)} \xi_x(y) = \sum_{y \in B(x,S)} \int_H \zeta_{hx}(hy)\, dh \\ &= \int_H \left(\sum_{y \in B(x,S)} \zeta_{hx}(hy) \right) dh \;=\; \int_H 1\, dh \\ &= 1, \end{aligned}$$

which shows that ξ_x is an element of $\ell_1(G)_{1,+}$ for every $x \in G$.

Let now $x_1, x_2 \in G$ satisfy $d(x_1, x_2) = 1$. Then

$$\begin{aligned}
\|\xi_{x_1} - \xi_{x_2}\|_{\ell_1(G)} &= \sum_{y\in G} \left|\xi_{x_1}(y) - \xi_{x_2}(y)\right| \qquad (6)\\
&= \sum_{y\in G} \left|\int_H \zeta_{hx_1}(hy)\, dh - \int_H \zeta_{hx_2}(hy)\, dh\right|\\
&\leq \int_H \left(\sum_{y\in X} \left|\zeta_{hx_1}(hy) - \zeta_{hx_2}(hy)\right|\right) dh\\
&\leq \int_H \varepsilon\, dh \;=\; \varepsilon,
\end{aligned}$$

since the sum is in fact finite and the metric is left-invariant.

Finally we need to show that (5) holds. Indeed, if $\gamma \in H$ and $x \in G$ then by the invariance of the mean on H we obtain

$$\begin{aligned}
\xi_{\gamma x}(y) &= \int_H \zeta_{h\gamma x}(hy)\, dh\\
&= \int_H \zeta_{\tilde{h}x}(\tilde{h}\gamma^{-1}y)\, d\tilde{h} \;=\; \xi_x(\gamma^{-1}y)\\
&= \gamma\cdot\xi_x(y),
\end{aligned}$$

after substituting $\tilde{h} = h\gamma$. This ends the proof. □

Theorem 6.9. *Let G be a group with Property A and let H be an amenable subgroup of G. Then the quotient G/H has Property A.*

Proof. By Proposition 6.8, the function $\xi : G \to \ell_1(G)$ can be chosen to be equivariant on cosets of H. We define the map $\eta : G/H \to \ell_1(G/H)_{1,+}$ by

$$\eta_{Hx}(Hy) = \sum_{h\in H} \xi_x(hy).$$

We need to show that η is well defined. So let $Hx = Hx'$ and $Hy = Hy'$ as elements of G/H. Then there are elements $\gamma, g \in H$ such that $\gamma x = x'$ and $gy = y'$.

$$\begin{aligned} \eta_{Hx'}(Hy') &= \sum_{h\in H} \xi_{x'}(hy') = \sum_{h\in H} \xi_{\gamma x}(hgy) \\ &= \sum_{h\in H} \xi_x(\gamma^{-1}hgy) = \sum_{h\in H} \xi_x(hy) \\ &= \eta_{Hx}(Hy), \end{aligned}$$

after using Proposition 6.8 and substituting $\gamma^{-1}hg$ for g.

Now note that since the quotient map $G \to G/H$ is a contraction, if the elements Hx and Hy are more than distance S away from each other, ξ_x vanishes on the coset Hy and we have

$$\eta_{Hx}(Hy) = \sum_{h\in H} \xi_x(hy) = 0,$$

thus $\operatorname{supp} \eta_{Hx} \subseteq B(Hx, S)$ and we also have

$$\begin{aligned} \|\eta_{Hx}\|_{\ell_1(G/H)} &= \sum_{Hy\in G/H} \eta_{Hx}(Hy) \\ &= \sum_{Hy\in G/H} \sum_{h\in H} \xi_x(hy) \\ &= \|\xi_h\|_{\ell_1(G/H)} = 1. \end{aligned}$$

Suppose now that elements Hx and Hx' are distance 1 from each other. This means that the elements

x and x' can be assumed to satisfy $d(x,x') \leq 1$. Thus

$$\begin{aligned}
\|\eta_{Hx} - \eta_{Hx'}\|_{\ell_1(G/H)} &= \sum_{Hy \in G/H} \left| \eta_{Hx}(Hy) - \eta_{Hx'}(Hy) \right| \\
&= \sum_{Hy \in G/H} \left| \sum_{h \in H} \xi_x(hy) - \sum_{h \in H} \xi_{x'}(hy) \right| \\
&\leq \sum_{Hy \in G/H} \sum_{h \in H} |\xi_x(hy) - \xi_{x'}(hy)| \\
&= \|\xi_x - \xi_{x'}\|_{\ell_1(G)} \leq \varepsilon.
\end{aligned}$$

□

Large Riemannian manifolds

There are several notions of largeness of open manifolds, see e.g. [27]. One of them is uniform contractibility, which means that for every $R > 0$ there exists an $S_R > 0$ such that for any point $x \in \mathcal{M}$ the ball $B(x,R)$ is contractible inside $B(x,S_R)$. We will need a less restrictive criterion.

Let X be a metric space. An anti-Čech system is a sequence $\{U_k\}_{k \in \mathbb{N}}$ of covers such that:

(1) there exist numbers R_k, $k = 1,2,\ldots$, such that $\mathrm{diam}(U) \leq R_k$ for every $U \in U_k$;

(2) the Lebesque numbers λ_k of U_k satisfy $\lambda_k \geq R_{k-1}$;

(3) $\lambda_k \to \infty$ as $k \to \infty$.

Let H_*^{lf} be the locally finite homology theory. The coarse homology $HX_*(X)$ [47] is defined by setting

$$HX_*(X) = \lim_{k \to \infty} H_*^{lf}(|U_k|)$$

where $\{U_k\}$ is an anti-Čech system for X and $|U_k|$ denotes the nerve space of the cover U_k; see also [32]. There is a character map $c_* : H_*^{lf}(X) \to HX_*(X)$ induced by the map $c : X \to |U_1|$ defined by the formula

$$c(x) = \sum_{U \in U_1} \varphi_U(x)[U],$$

where $\{\varphi_U\}_{U_1}$ is a partition of unity subordinate to the cover U_1. The character map c_* is an isomorphism provided that X is uniformly contractible [47], see also [32].

Definition 6.10 ([23])**.** *Let $\mathcal{M}$ be a complete, oriented n-dimensional manifold. Let $[\mathcal{M}] \in H_n^{lf}(\mathcal{M})$ be the fundamental class of $\mathcal{M}$. We call $\mathcal{M}$ macroscopically large if*

$$c_*([\mathcal{M}]) \neq 0.$$

Note that the notion of macroscopical largeness for equivalent metrics depends only on the quasi-isometry class of these metrics. More precisely, take a manifold $\mathcal{M}$ which is equipped with two equivalent, quasi-isometric metrics, d_1 and d_2 and the corresponding character maps by c_*^1 and c_*^2 respectively. Then for any n the diagram

$$\begin{array}{ccc} & H_n^{lf}(\mathcal{M}) & \\ {\scriptstyle c_*^1}\swarrow & & \searrow{\scriptstyle c_*^2} \\ HX_n((\mathcal{M},d_1)) & \xrightarrow[\mathrm{id}_* \cong]{} & HX_n((\mathcal{M},d_2)) \end{array}$$

is commutative, so in particular $(\mathcal{M}, d_1)$ is large if and only if $(\mathcal{M}, d_2)$ is.

Index of the de Rham operator

Let $(\mathcal{M}, d_{\mathcal{M}})$ be an open complete Riemannian manifold and let G be a group acting freely, properly on $\mathcal{M}$ by isometries with a compact quotient $\mathcal{N} = \mathcal{M}/G$. We have the following exact sequence:

$$1 \longrightarrow \pi_1(\mathcal{M}) \longrightarrow \pi_1(\mathcal{N}) \longrightarrow G \longrightarrow 1.$$

In the above setting we will say that $\mathcal{M}$ is a *co-amenable cover* if $\pi_1(\mathcal{M})$ is amenable ($\mathcal{M}$ is often called an amenable cover when G is amenable). Obviously if $\pi_1(\mathcal{M})$ is non-trivial, the manifold $\mathcal{M}$ is not uniformly contractible.

Theorem 6.11. *Let $(\mathcal{N}, d_{\mathcal{N}})$ be a closed Riemannian manifold such that $\pi_1(\mathcal{N})$ is C^*-exact. Let $(\mathcal{M}, d_{\mathcal{M}})$ be a co-amenable cover of $\mathcal{N}$ which is large and has bounded geometry (i.e. bounded sectional curvature and positive injectivity radius). Then the zero-in-the-spectrum conjecture holds*

for $\mathcal{M}$ with any bounded geometry metric which is quasi-isometric and topologically equivalent to $d_{\mathcal{M}}$.

Proof. Let $\mathcal{M}$, $\mathcal{N}$ and G be as above. By assumptions and Theorem 6.7 we have that G has Property A. By the Švarc-Milnor lemma, G and $\mathcal{M}$ are quasi-isometric due to the fact that $\mathcal{N} = \mathcal{M}/G$ is compact. Since G has Property A and Property A is a coarse invariant, $\mathcal{M}$ equipped with any metric in the quasi-isometry class of $d_{\mathcal{M}}$ has Property A. We take the following composition

$$K_*(\mathcal{M}) \xrightarrow[c_*]{} KX_*(\mathcal{M}) \xrightarrow[\mu^c]{} K_*(C^*(\mathcal{M})).$$

Let $\mathcal{D}$ be the de Rham operator on $\mathcal{M}$. Since $\mathcal{M}$ is large, we have that $[c_*(\mathcal{D})] \neq 0$ in the coarse K-homology group $KX_*(\mathcal{M})$. By Theorems 2.2 and 1.1 in [58], Property A for $\mathcal{M}$ implies that the Coarse Baum-Connes Conjecture is true for $\mathcal{M}$, i.e. μ^c is an isomorphism and consequently $\mu^c[c_*(\mathcal{D})] \neq 0$ in $K_*(C^*(\mathcal{M}))$. By Proposition 6.6 this ends the proof. □

In the case $\pi_1(\mathcal{M}) = \{1\}$ (i.e. the cover is universal), Theorem 6.11 is due to Yu [58]. The assumption of largeness cannot be dropped as the example of Farber and Weinberger [21] shows (see also [34]), since in their construction the fundamental group of the manifold $\mathcal{N}$ is a direct product of free groups, which is exact.

BIBLIOGRAPHY

[1] P. Assouad, *Sur la distance de Nagata*, C. R. Acad. Sci. Paris Sr. I Math. 294 (1982), no. 1, 31–34.

[2] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's Property (T)*, manuscript avaliable online.

[3] G.C. Bell, *Property A for groups acting on metric spaces*, Topology Appl. 130 (2003), no.3, 239–251.

[4] G.C. Bell, A.N. Dranishnikov, *On asymptotic dimension of groups*, Algebr. Geom. Topol. 1 (2001), 57–71.

[5] G.C. Bell, A.N. Dranishnikov, *A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory*, Trans. Amer. Math. Soc. 358 (2006), no. 11, 4749–4764.

[6] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's Property (T)*, manuscript avaliable online at http://www.mmas.univ-metz.fr/ bekka/.

[7] N. Brodskiy, J. Dydak, M. Levin, A. Mitra, *Hurewicz theorem for Nagata-Assouad dimension*, preprint.

[8] R. Brooks, *The fundamental group and the Spectrum of the Laplacian*, Comment. Math. Helv. 56 (1981), no. 4, 581–598.

[9] S. Buyalo, V. Schroeder, *Embedding of hyperbolic spaces in the product of trees*, Geom. Dedicata 113 (2005), 75–93.

[10] S. Buyalo, V. Schroeder, *A product of trees as universal space for hyperbolic groups*, arXiv:math.GR/0509355.

[11] X. Chen, M. Dadarlat, E. Guentner, G. Yu, *Uniform embeddability and exactness of free products*, J. Funct. Anal. 205 (2003), no. 1, 168–179.

[12] S. Cleary, *Distortion of wreath products in some finitely presented groups*, arXiv:math.GR/0505288.

[13] T. Coulhon, L. Saloff-Coste, *Isopérimétrie pour les groupes et variétés*, Rev. Math. Iberoamericana 9 (1993), 293-314.

[14] M. Dadarlat, E. Guentner, *Constructions preserving Hilbert space uniform embeddability of discrete groups*, Vol. 355, Trans. Amer. Math. Soc. No. 8 (2003), p.3253-3275.

[15] A.N. Dranishnikov, *Anti-Ĉech approximation in coarse geometry*, IHES Preprint, 2002.

[16] A.N. Dranishnikov, J. Smith, *Asymptotic dimension of discrete groups*, Fund. Math. 189 (2006), 27-34.

[17] A.N. Dranishnikov, M. Zarichnyi, *Universal spaces for asymptotic dimension*, Top. App. 140 (2004), 203-225.

[18] L. van den Dries, A.J. Wilkie,*Gromov's theorem on groups of polynomial growth and elementary logic*, J. Algebra 89 (1984), no. 2, 349–374.

[19] A. Erschler, *On Isoperimetric Profiles of Finitely Generated Groups*, Geom. Dedicata 100: 157-171, 2003.

[20] B. Farb, L. Mosher, *A rigidity theorem for the solvable Baumslag-Solitar groups. With an appendix by Daryl Cooper*, Invent. Math. 131 (1998), no. 2, 419–451.

[21] M. Farber, S. Weinberger, *On the zero-in-the-spectrum conjecture*, Ann. of Math. (2) 154 (2001), no. 1, 139–154.

[22] T. Gentimis, *Asymptotic dimension of finitely presented groups*, preprint.

[23] G. Gong, G. Yu, *Volume growth and positive scalar curvature*, Geom. Funct. Anal. 10 (2000), no. 4, 821–828.

[24] F.P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Mathematical Studies, No. 16 Van Nostrand Reinhold Co., 1969.

[25] R.I. Grigorchuk, A. Żuk, *The Lamplighter group as a Group Generated by a 2-state Automaton, and its Spectrum*, Geom. Dedicata 87: 209-244, 2001.

[26] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Etudes Sci. Publ. Math. No. 53 (1981), 53–73.

[27] M. Gromov, *Large Riemannian manifolds*, Curvature and topology of Riemannian manifolds (Katata, 1985), 108–121, Lecture Notes in Math., 1201, Springer, Berlin, 1986.

[28] M. Gromov, *Asymptotic invariants of infinite groups*, London Mathematical Society Lecture Notes, no.182, s. 1-295, Cambridge University Press, 1993.

[29] M. Gromov, *Spaces and Questions*, GAFA 2000 (Tel Aviv 1999), Geom. Funct. Anal. 2000, Special Volume, Part I, 118-161.

[30] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. 13 (2003), no. 1, 73–146.

[31] E. Guentner, J. Kaminker, *Exactness and the Novikov conjecture*, Topology 41 (2002), no. 2, 411–418.

[32] N. Higson, J. Roe, *On the coarse Baum-Connes conjecture*, in: Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), 227–254, London Math. Soc. Lecture Note Ser., 227, Cambridge Univ. Press, Cambridge, 1995.

[33] N. Higson, J. Roe, *Amenable group actions and the Novikov conjecture*, J. Reine Angew. Math. 519 (2000), 143-153.

[34] N. Higson, J. Roe, T. Schick *Spaces with vanishing l^2-homology and their fundamental groups (after Farber and Weinberger)*, Geom. Dedicata 87 (2001), no. 1-3, 335–343.

[35] T. Januszkiewicz, J. Świątkowski, *Filling invariants in systolic complexes and groups*, preprint.

[36] V.A. Kaimanovich, A.M. Vershik, *Random walks on discrete groups: Boundary and entropy*, Ann. Probab. 11 (1983), 457-490.

[37] U. Lang, T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, Int. Math. Res. Not. 2005, no.58, 3625–3655.

[38] J. Lott, *The zero-in-the-spectrum question*. Enseign. Math. (2) 42 (1996), no. 3-4, 341–376.

[39] W. Lück, *L^2-invariants of regular coverings of compact manifolds and CW-complexes*, Handbook of geometric topology, 735–817, North-Holland, Amsterdam, 2002.

[40] P.W. Nowak, *On coarse embeddability into ℓ_p-spaces and a conjecture of Dranishnikov*, Fund. Math. 189 (2006), 111-116.

[41] P.W. Nowak, *On exactness and isoperimetric profiles of discrete groups*, J. Funct. Anal. 243 (2007), no. 1, 323-344.

[42] P.W. Nowak, *Coarsely embeddable metric spaces without Property A*, J. Funct. Anal., 252 (2007), no.1, 126-136

[43] P.W. Nowak, *Zero-in-the-spectrum conjecture on regular covers of compact manifolds*, Comment. Math. Helv., to appear

[44] N. Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Ser. I Math., 330 (2000), 691–695.

[45] J.-P. Pier, *Amenable locally compact groups*, A Wiley-Interscience Publication, 1984.

[46] C. Pittet, L. Saloff-Coste, *Amenable groups, isoperimetric profiles and random walks*, In: J. Cossey et al. (eds.), Geometric Group Theory Down Under (Canberra, Australia, July 14-19, 1996), de Gruyter, Berlin, 1999, 293-316.

[47] J. Roe, *Coarse cohomology and index theory on complete Riemannian manifolds*, Mem. Amer. Math. Soc. 104 (1993), no. 497.

[48] J. Roe, *Index theory, coarse geometry, and topology of manifolds*, CBMS Regional Conference Series in Mathematics, 90. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996.

[49] J. Roe, *Lectures on coarse geometry*, University Lecture Series, 31. American Mathematical Society, Providence, RI, 2003.

[50] J. Roe, *Hyperbolic groups have finite asymptotic dimension*, Proc. Amer. Math. Soc. 133 (2005), no. 9, 2489–2490.

[51] L. Saloff-Coste, *Probability on Groups: Random Walks and Invariant Diffusion*, Notices Amer. Math. Soc. 48 (2001), no. 9, 968–977.

[52] L. Saloff-Coste, *Analysis on Riemannian co-compact covers*, Surveys in differential geometry. Vol. IX, 351–384, Surv. Differ. Geom., IX, Int. Press, Somerville, MA, 2004.

[53] D.W. Stroock, *An introduction to Markov processes*, Graduate Texts in Mathematics, 230. Springer-Verlag, Berlin, 2005.

[54] J.L. Tu, *Remarks on Yu's "Property A" for discrete metrics spaces and groups*, Bull. Soc. Math. France 129(1), 2001, 115-139.

[55] A.M. Vershik, *Amenability and approximation of infinite groups*, Selected translations. Selecta Math. Soviet. 2 (1982), no. 4, 311–330.

[56] R. Willett, *Some notes on Property A*, arXiv:math/0612492v1.

[57] G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math. (2) 147 (1998), no. 2, 325–355.

[58] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. (1) 139 (2000), 201-240.

[59] R.J. Zimmer, *Amenable ergodic actions, hyperfinite factors, and Poincaré flows* Bull. Amer. Math. Soc. 83 (1977), no. 5, 1078–1080.