CLASSICAL TREES AND COMPACT ULTRAMETRIC SPACES

By

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To my loved ones

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CHAPTER I

INTRODUCTION

If X is a locally finite tree of minimal vertex degree three the end space of X is a perfect compact ultrametric space of diameter one. Although it is quite easy to work with trees, even in the infinite case, ultrametric spaces are not as accessible. Having seen Euclidean spaces as the main examples of metric spaces, it is not very intuitive to work in ultrametric spaces where if two balls intersect, then one contains the other, or consider balls where every point in the ball is the center of that ball.

In this paper a faithful functor between the category $\mathcal{T}$ of locally finite trees with minimal vertex degree three and equivalence classes of quasi-isometries, and the category $\mathcal{U}$ of perfect compact ultrametric spaces and bi-Hölder homeomorphisms is established. The main theorem is as follows:

Theorem 1.1. *There is a faithful functor from the category of locally finite trees of vertex degrees at least equal to three and equivalence classes of quasi-isometries to the category of perfect compact ultrametric spaces and bi-Hölder homeomorphisms.*

Ghys and de la Harpe [GH] establish that if $f : X \to Y$ is a quasi-isometry between locally finite trees of vertex degree greater than or equal to three, then the induced map ∂f between the end spaces of X and Y is a bi-Hölder quasi-conformal homeomorphism. They do not look at equivalence classes of quasi-isometries or at categories. We prove that if two quasi-isometries are in the same equivalence class, then they induce the same map on the end spaces.

Holly [Hol] provides illustrations that help us visualize ultrametric spaces and gives p-adic norms as the main examples of such spaces. We prove many interesting properties that are unique to compact ultrametric spaces some of which are well known and some of which are not.

Quasi-conformal homeomorphisms on metric spaces have not been extensively studied. If $f : X \to Y$ is a conformal homeomorphism between metric spaces, the image of circle in X is a circle in Y. If f is a quasi-conformal homeomorphism, then the image of a circle in X will lie between two circles in Y. In chapter 7 we will see that even in the special case where $\phi : X \to Y$ and $\psi : Y \to Z$ are quasi-conformal homeomorphisms and X, Y and Z are perfect compact ultrametric spaces of diameter one, the composition $\psi \circ \phi$ need not be quasi-conformal. It is shown however, that in the case where the where ϕ and ψ are the induced bi-Hölder homeomorphisms, then, not only are ϕ and ψ quasi-conformal, but the composition is a bi-Hölder quasi-conformal homeomorphism.

There are many results related to the main theorem in this paper. Bridson and Haefliger [BH] observe that if $f : X \to Y$ is a quasi-isometry, where X and Y are proper geodesic spaces, then f induces a homeomorphism on the end spaces of X and Y and prove this. They also introduce the notion of quasi-isometry classes. We have adapted their definition of quasi-isometry classes and show that the composition of maps induces a group structure on the set of equivalence classes of quasi-isometries from a metric space X to itself. Bridson and Haefliger do not look at categories.

Hughes [Hug] shows that there is an equivalence from the category of geodesically complete, rooted $\mathbb{R}$-trees and equivalence classes of isometries at infinity, to the category of complete ultrametric spaces of finite diameter and local similarity equivalences. Thus, our approach to looking at categorical equivalences is similar to the one taken in [Hug]. Even though the trees considered in the category studied in this paper are geodesically complete, rooted $\mathbb{R}$-trees, the morphisms that we consider are not the same. Section 9 of [Hug] gives an example of a bi-Hölder quasi-conformal homeomorphism between the end spaces of the Cantor tree and the Fibonacci tree. He then goes on to show that there exists no local similarity equivalence between these two spaces. In Chapter VIII of this paper we give examples showing that the morphisms in the categories studied here are different from the ones in [Hug], although any object in the categories in this paper are objects in the categories in [Hug]. Many of the relevant definitions, lemmas and theorems in [Hug] have been utilized here.

Bonk and Schramm [BS] show that there exists a functor from the category C whose objects are Gromov hyperbolic almost geodesic spaces and whose morphisms are quasi-isometries to the category D whose objects are complete and bounded B-structures and whose morphisms are power quasi-symmetries. They also go on to show that there is a morphism from the category whose objects are bounded metric spaces, and whose morphisms are quasi-symmetries to the category whose objects are visual Gromov hyperbolic spaces and morphisms are quasi-isometries. Although the objects in the category $\mathcal{T}$ considered in this paper have the property of the objects in the category C in [BS], the morphisms have more restrictions imposed on them. In Chapter VIII we compare the morphisms in the two categories.

The paper is organized as follows. Chapters II and III introduce the objects and morphisms in the first category, and Chapters IV and V introduce the objects and morphisms in the second category. The main theorem is proved in Chapter VI. Chapter VII of the paper shows, by examples, why the conditions imposed on the objects and the morphisms in the categories are necessary. Chapter VIII discusses the morphisms in [Hug] and in [BS] and shows the relationship between those morphisms and the ones studied here. Chapter IX looks at problems for future research. The proofs related to the properties of some of the maps in chapter VIII are in Appendix A.

CHAPTER II

LOCALLY FINITE TREES

In this chapter we will introduce the objects of our interest, which are geodesically complete locally finite classical trees, and establish the fact that any such object endowed with its natural metric is a 0-hyperbolic proper geodesic space. All the definitions and lemmas in this chapter are well known and are stated for clarification purposes only.

Definition 2.1. A *real tree*, or $\mathbb{R}$-*tree*, is a metric space (T,d) that is uniquely arcwise connected, and for any two points $x, y \in T$ the unique arc from x to y, denoted $[x, y]$, is isometric to the subinterval $[0, d(x, y)]$ of $\mathbb{R}$.

Definition 2.2. *Classical trees* are one-dimensional, simply connected simplicial complexes. The metric assigned to these trees is a length metric; see [BH] for the definition; such that every 1-simplex is isometric to the unit interval $[0, 1]$.

Note that classical trees are $\mathbb{R}$-trees.

Definition 2.3. A *rooted* $\mathbb{R}$ *tree* (T, v) consists of an $\mathbb{R}$-tree (T, d) and a point $v \in T$, called the *root.*

Definition 2.4. A rooted $\mathbb{R}$-tree (T, v) is *geodesically complete* if every isometric embedding $f : [0, t] \to T$, $t > 0$, with $f(0) = v$, extends to an isometric embedding $\tilde{f} : [0, \infty) \to T$. In this case, we say that $[v, f(t)]$ can be *extended to a geodesic ray.*

Lemma 2.5. *If T is an $\mathbb{R}$-tree, $\alpha, \beta : [0, \infty) \to T$ are two isometric embeddings such that $\alpha(0) = \beta(0)$ and there exist $t_0, t_1 > 0$ such that $\alpha(t_0) = \beta(t_1)$, then $t_0 = t_1$ and $\alpha(t) = \beta(t)$ whenever $0 \le t \le t_0$.*

Definition 2.6. A simplicial complex K is said to be *locally finite* if each vertex of K only belongs to finitely many simplicies of K

Some of the theorems we will be referring to in the next sections require our objects to be δ-hyperbolic proper geodesic spaces. For clarification purposes, in this section we will give the well known formal definitions of δ-hyperbolic, geodesic and proper spaces (see [GH] or [BH]). We will then verify that any locally finite tree equipped with the tree metric is a 0-hyperbolic proper geodesic space.

Definition 2.7. ([GH], 1.25) Let (X, d) be a metric space and let x_0, x_1 be points in X and $a = d(x_0, x_1)$ their distance. A *geodesic segment* in X starting at x_0 and terminating at x_1 is an isometric embedding $g : [0, a] \to X$ such that $g(0) = x_0$ and $g(a) = x_1$. We say that g is a *parameterized geodesic segment* and that the image of g is a geometric geodesic segment (or even by abuse of notation, a geodesic segment).

Definition 2.8. A metric space (X, d) is said to be a *geodesic space* if for all pairs of points x_0 and x_1 in X there exists a geodesic segment $g : [0, d(x_0, x_1)] \to X$ with endpoints x_0 and x_1.

Definition 2.9. Let X be a metric space. For $x, y, p \in X$ the *Gromov product* $(x|y)_p$ is defined by

$$2(x|y)_p = |x - p| + |y - p| - |x - y| \tag{2.1}$$

Definition 2.10. Let $\delta \geq 0$. A metric space X is *Gromov δ-hyperbolic* if

$$(x|z)_p \geq \min\{(x|y)_p, (y|z)_p\} - \delta \tag{2.2}$$

for all $x, y, z, p \in X$.

An equivalent definition of Gromov-hyperbolicity is the Rips condition in which thin triangles are considered. In many papers the Rips condition is taken as the definition of a Gromov hyperbolic space.

Definition 2.11. A *geodesic triangle* with vertices x, y, and z in X is the union of three geodesic segments joining these points two by two. We say a triangle is *degenerate* if x, y and z are not distinct or if one point lies on the geodesic segment containing the other two.

The following are well-known definitions, see for example [GH].

Definition 2.12. Let $\delta \geq 0$. A geodesic metric space X is said to satisfy the *Rips condition for the constant δ* if for any geodesic triangle Δ in X the distance of any point on a side of the triangle to the union of the other two sides is at most δ. In formulas :

$$\text{for all } \Delta = [x, y] \cup [y, z] \cup [x, z]$$
$$\text{and for all } u \in [y, z], \text{ one has } d(u, [x, y] \cup [z, x]) \leq \delta.$$

Definition 2.13. A metric space is said to be *proper* if any closed ball is compact.

Henceforth in this paper whenever we say tree we mean classical rooted trees endowed with the natural tree metric. We also require that the trees be geodesically complete.

Lemma 2.14. Every locally finite tree is a 0-hyperbolic, proper geodesic space.

Proof: Let T be a locally finite tree. Let x, y and z be three distinct points in the tree T and let $\Delta = [x, y] \cup [x, z] \cup [y, z]$. If Δ is a degenerate triangle then figure 2.1 is a sketch of Δ.

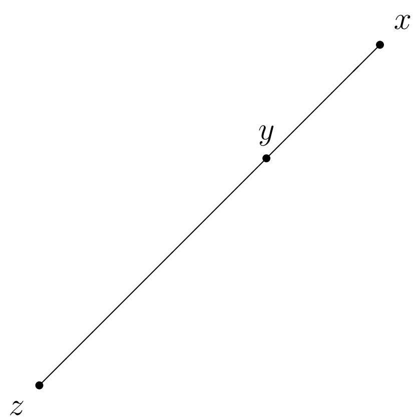


Figure 2.1: Case (i)

Observe that $[z, x] = [z, y] \cup [y, x]$. Therefore if $u \in [z, x]$ then $u \in [z, y]$ or $u \in [y, x]$. In both these cases either $d(u, [z, x] \cup [y, x]) = 0$ or $d(u, [z, y] \cup [y, x]) = 0$. If $u \in [x, y]$ or $u \in [y, z]$ then $u \in [z, x]$ and we have $d(u, [z, y] \cup [y, x]) = 0$ and $d(u, [z, x] \cup [y, x]) = 0$ and hence Rips condition is satisfied.

Suppose that Δ is not degenerate. Let u be a point in Δ. If $u \in [y, z]$, then $u \in [x, y]$ or $u \in [x, z]$ so that $d(u, [x, y] \cup [z, x]) = 0$. The proof of the other cases are similar to the above and is proved by interchanging x, y and z. Figure 2.2 is a rough sketch of this case.

The tree T is proper since any closed ball will only contain finitely many vertices and each vertex has finite degree, therefore each ball will be isometric to a finite union of compact subsets of $\mathbb{R}$, and hence compact. [Munk2] lemma 2.6 gives an alternate proof for this statement.□

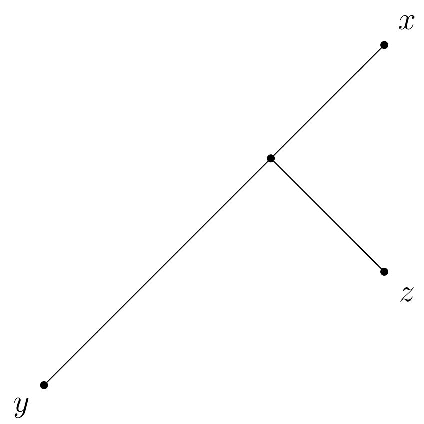


Figure 2.2: Case (ii)

CHAPTER III

QUASI-ISOMETRIES

In this section we will be looking at the category $\mathcal{M}$ whose objects are metric spaces and whose morphisms are classes of quasi-isometries.

Definition 3.1. A category, $\mathcal{C}$, consists of a collection of objects, ob($\mathcal{C}$) together with morphisms between them. Specifically, if X and Y are objects of $\mathcal{C}$ there is a set of morphisms from X to Y, denoted $Mor_{\mathcal{C}}(X, Y)$. Morphism may be composed: If Z is another object of $\mathcal{C}$, then for any $f \in Mor_{\mathcal{C}}(Y, Z)$ and any $g \in Mor_{\mathcal{C}}(X, Y)$ there is a composite morphism $f \circ g \in Mor_{\mathcal{C}}(X, Z)$. The composition rule has to satisfy two properties. First, there's an associativity law: if $h \in Mor_{\mathcal{C}}(Z, W)$, then for f and g as above, we have $h \circ (f \circ g) = (h \circ f) \circ g$ as elements of $Mor_{\mathcal{C}}(X, W)$. Second, each object has an identity morphism. We write $id_X \in Mor_{\mathcal{C}}(X, X)$ for the identity of X. The defining property of the identity morphisms is that $g \circ id_X = id_Y \circ g = g$ for all $g \in Mor_{\mathcal{C}}(X, Y)$.

Our aim in this section will be to introduce quasi-isometries on metric spaces, define an equivalence relationship on them, and prove that the resulting equivalence classes form the morphisms in the category $\mathcal{M}$.

Definition 3.2. ([BH],I.8.14) Let (X, d_X) and (X_0, d_{X_0}) be metric spaces. A map $F : X_0 \to X$ is said to be a *(λ, ϵ)-quasi-isometric embedding* if there exists $\lambda \geq 1$ and $\epsilon \geq 0$ such that for any s, t in X_0;

$$\frac{1}{\lambda}\, d_{X_0}(s,t) - \epsilon \leq d_X(F(s), F(t)) \leq \lambda d_{X_0}(s,t) + \epsilon.$$

If in addition, there exists a constant $C \geq 0$ such that every point of X lies in the C neighborhood of the image of F, F is called a (λ, ϵ)-*quasi-isometry*. In this case X_0 and X are said to be *quasi-isometric*.

Note that F need not be continuous.

Example 3.3. *Let (T, w) be a tree in which every vertex has degree two. Let w be the root of T. In this case T has two ends, which we shall call f and g. In other words, end$(T, w) = \{f, g\}$ where $f, g : [0, \infty) \to (T, w)$ with $f(0) = w$ and $g(0) = w$ are isometric embeddings. Note that Im(f) $\cup$ Im(g) $= (T, w)$.*

Figure 3.1 is a rough sketch of this tree.

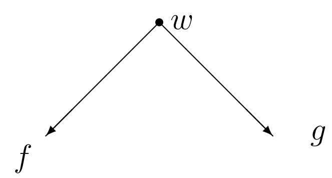


Figure 3.1: A rooted tree with two ends of distance 1

Define $\gamma : (T, w) \to (T, w)$ by

$$\gamma(x) = \begin{cases} f(2t), & \text{if } x = f(t) \text{ for some } t \in [0, \infty) \\ g(2t), & \text{if } x = g(t) \text{ for some } t \in [0, \infty) \end{cases} \tag{3.1}$$

Then γ defined above is a quasi-isometry. To see this we need to show that there exists real numbers $\lambda \geq 0$ and $c > 0$ such that for any $x, y \in T$ we have

$\frac{1}{\lambda}d(x,y) - c \leq d(\gamma(x), \gamma(y)) \leq \lambda d(x,y) + c$, and there exists a real number $D > 0$ such that for any $z \in T$ we can find $x \in T$ such that $d(z, \gamma(x)) \leq D$. To prove this we consider the following cases:

Case 1) $x, y \in Imf$. In this case there exists t_1, t_2 such that $x = f(t_1), y = f(t_2)$. Therefore, $\gamma(x) = f(2t_1)$, $\gamma(y) = f(2t_2)$. Thus, $d(\gamma(x), \gamma(y)) = d(f(2t_1), f(2t_2)) = |2t_1 - 2t_2| = 2|t_1 - t_2| = 2d(x,y)$.

Case 2) $x, y \in Img$. As in case 1) we see that $d(\gamma(x), \gamma(y)) = 2d(x,y)$.

Case 3) $x \in Imf, y \in Img$. This means that there exist t_1, t_2 such that $x = f(t_1)$ and $y = g(t_2)$. In this case $d(x,y) = d(x,w) + d(y,w) = d(f(t_1), f(0)) + d(g(t_2), g(0)) = t_1 + t_2$. We also have that $\gamma(x) = f(2t_1)$ and $\gamma(y) = g(2t_2)$, and $d(\gamma(x), \gamma(y)) = d(f(2t_1), g(2t_2)) = d(f(t_1), f(0)) + d(g(t_2), g(0)) = 2t_1 + 2t_2 = 2d(x,y)$.

By the above three cases and the fact that γ is onto, we conclude that γ is a $(2,0)$-quasi-isometry.

Example 3.4. *Let (T,w) be a tree in which every vertex has degree two except the vertex v, that has vertex degree three, and the root w, that has vertex degree one. Furthermore, $d(w,v) = 10$. In this case (T,w) has two ends $f, g : [0,\infty) \to (T,w)$ and $t_* = \sup\{t|\ f(t) = g(t)\} = 10$. Figure 3.2 gives a rough sketch of this tree.*

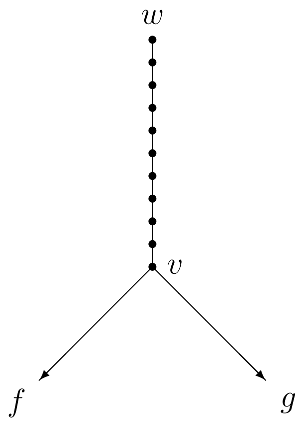


Figure 3.2: A rooted tree with two ends of distance 10

Define $\gamma : (T, w) \to (T, w)$ *as follows:*

$$\gamma(x) = \begin{cases} f(t-1), & \textit{if } x = f(t) \textit{ and } 1 < t \leq 10 \\ g(t-1), & \textit{if } x = g(t) \textit{ and } 1 < t \leq 10 \\ f(t+1), & \textit{if } x = f(t) \textit{ and } t > 10 \\ g(t+1), & \textit{if } x = g(t) \textit{ and } t > 10 \\ w, & \textit{if } x = f(t) \textit{ and } 0 \leq t \leq 1 \\ w, & \textit{if } x = g(t) \textit{ and } 0 \leq t \leq 1 \end{cases} \tag{3.2}$$

We claim that γ is a $(1, 2)$-quasi-isometry. The first thing to note is that γ is well defined. If $x = f(t_1)$ for some $t_1 \in [0, \infty]$, and $x = g(t_2)$ for some $t_2 \in [0, \infty)$, then by 2.5, $t_1 = t_2$ and for any $t \leq t_1$, $f(t) = g(t)$ and hence $t_1 \leq 10$ and therefore, $t_1 - 1 < 10$ and hence $f(t_1 - 1) = g(t_1 - 1)$, which implies that γ is well defined. Let $x, y \in (T, w)$. Suppose that $x = f(t_1)$ and $y = f(t_2)$, furthermore, suppose that $t_1 \leq 10$ and $t_2 \leq 10$. In this case $\gamma(x) = f(t_1 - 1)$ and $\gamma(y) = f(t_2 - 1)$ and hence, $d(\gamma(x), \gamma(y)) = d(f(t_1 - 1), f(t_2 - 1)) = |t_2 - 1 - t_1 + 1| = d(x, y)$. If x and y are as above but $t_1 > 10$ and $t_2 > 10$, then $\gamma(x) = f(t_1 + 1)$ and $\gamma(y) = f(t_2 + 1)$ and hence $d(\gamma(x), \gamma(y)) = d(f(t_1 + 1), f(t_2 + 1)) = |t_2 + 1 - t_1 - 1| = d(x, y)$. If x and y are as above and $t_1 \leq 10$, but $t_2 > 10$. Then $\gamma(x) = f(t_1 - 1)$ and $\gamma(y) = f(t_2 + 1)$ and hence,

$d(\gamma(x), \gamma(y)) = d(f(t_1 - 1), f(t_2 + 1)) = |t_2 - 1 - t_1 - 1| \leq d(x, y) + 2$. In the above if we replace f by g the same will hold, i.e. $d(\gamma(x), \gamma(y)) = d(g(t_1 - 1), g(t_2 + 1)) = |t_2 - 1 - t_1 - 1| \leq d(x, y) + 2$.

Now suppose that $x = f(t_1)$, and $y = g(t_2)$ and $t_1 > 10$ and $t_2 > 10$. In this case $d(x, y) = |t_2 + t_1 - 20|$. On the other hand $d(\gamma(x), \gamma(y)) = d(f(t_1 + 1), g(t_2 + 1)) = |t_1 + 1 + t_2 + 1 - 20| \geq d(x, y) - 2$.

Therefore γ is a $(1, 2)$-quasi-isometry.$\square$

In [GH] the definition of two metric spaces being quasi-isometric is stated as follows:

Definition 3.5. The metric spaces (X, d_X) and (X_0, d_{X_0}) are *quasi-isometric* if and only if there exist maps $f : X_0 \to X$ and $g : X \to X_0$ and constants $\lambda > 0$ and $\epsilon \geq 0$ such that the following hold:

i) $d_X(f(x), f(y)) \leq \lambda d_{X_0}(x, y) + \epsilon$, for any $x, y \in X_0$

ii) $d_{X_0}(g(x'), g(y')) \leq \lambda d_X(x', y') + \epsilon$, for any $x', y' \in X$

iii) $d_{X_0}(g(f(x)), x) \leq \epsilon$, for any $x \in X_0$

iv) $d_X(f(g(x')), x') \leq \epsilon$, for any $x' \in X$

The following lemma proves that definitions 3.2 and 3.5 are equivalent. We will be using both definitions depending on which is more appropriate in the text and hence the need for this lemma.

Lemma 3.6. *Definitions 3.2 and 3.5 are equivalent.*

Proof that 3.2 implies 3.5: Let f and g be as above. We need to show that there exists constants $\lambda \geq 1$ and $\epsilon \geq 0$ and a (λ, ϵ)-quasi-isometry $F : X_0 \to X$ with the properties stated in definition 3.5.

Define $F(x) = f(x)$ for any $x \in X_0$. Then by the hypothesis, $d_X(F(x), F(y)) \leq \lambda d_{X_0}(x, y) + \epsilon$, for any $x, y \in X_0$. Also $F(x), F(y) \in X$, therefore by (iii), $d_{X_0}(g(F(x)), g(F(y))) \leq \epsilon$ and by (iv), $d_X(F(g(x')), F(g(y'))) \leq \epsilon$, for any $x', y' \in X$. By the triangle inequality in (X_0, d_{X_0}) :

$$d_{X_0}(x, y) \leq d_{X_0}(g(F(x)), x) + d_{X_0}(g(F(x)), g(F(y))) + d_{X_0}(g(F(y)), y)$$

By (ii) $d_{X_0}(g(F(x)), g(F(y))) \leq \lambda d_X(F(x), F(y)) + \epsilon$, for any $x, y \in X_0$.

Therefore,

$$d_{X_0}(x, y) \leq \epsilon + \lambda d_X(F(x), F(y)) + \epsilon + \epsilon.$$

Hence,

$$\frac{1}{\lambda} d_{X_0}(x, y) - \frac{3\epsilon}{\lambda} \leq d_X(F(x), F(y))$$

Let $\epsilon' = \max\left\{\frac{3\epsilon}{\lambda}, \epsilon\right\}$ and let $\lambda' = \max\{\lambda, 1\}$, then

$$\frac{1}{\lambda'} d_{X_0}(x, y) - \epsilon' \leq \frac{1}{\lambda} d_{X_0}(x, y) - \frac{3\epsilon}{\lambda} \leq d_X(F(x), F(y)) \leq \lambda d_{X_0}(x, y) + \epsilon \leq \lambda' d_{X_0}(x, y) + \epsilon'$$

Hence, F is a (λ', ϵ')-quasi-isometric embedding from X_0 to X.

Now we need to show that there exists $C > 0$, such that $d_X(x, ImF) \leq C$ for any $x \in X$. Let $x \in X$ be given. Then $g(x) \in X_0$, and $F(g(x)) \in Imf$. By (iv) $d(x, F(g(x))) \leq \epsilon$. Letting $C = \varepsilon$ we obtain the desired result. □

Proof that 3.5 implies 3.2: Suppose that $F : X_0 \to X$ is a (λ, ϵ)-quasi-isometry, and there exists $c > 0$ such that for any $x \in X$ there exists $x_0 \in X_0$ with $d(x, F(x_0)) \leq c$. We need to find maps f and g with properties (i)-(iv) above.

Let $f(x) = F(x)$ for any $x \in X_0$. Define $g : X \to X_0$ by $g(x) = x_0$, where $d_{X_0}(F(x_0), x)$ is minimum. Note that $g(F(x)) = x$ for any $x \in X$.

Let $x', y' \in X$ and $g(x') = x_1$ and $g(y') = y_1$. Then

$d_{X_0}(g(x'), g(y'))$

$= d_{X_0}(x_1, y_1)$

$\leq \lambda d_X(F(x_1), F(y_1)) + \lambda\epsilon$

$\leq \lambda(d_X(x', F(x_1)) + d_X(F(y_1), y') + d_X(x', y')) + \lambda\epsilon$

$\leq \lambda(2c + d_X(x', y')) + \lambda\epsilon$

$\leq \lambda d_X(x', y') + \epsilon$

This proves (ii). Item (i) follows from the definition of F. Item (iii) holds trivially since $g(f(x)) = x$ for all $x \in X_0$. Item (iv) holds since $d_X(x, ImF) \leq c \leq \epsilon'$ for any $x \in X$. $\square$

Definition 3.7. The map g defined above is called the *quasi-inverse* of f.

Lemma 3.8. *The composition of any two quasi-isometries is a quasi-isometry.*

Proof: Let $f : X_0 \to X$ and $g : X \to X_1$ be (λ, ϵ) and (λ', ϵ')- quasi-isometric embeddings respectively. Then $g \circ f : X_0 \to X_1$ is a map from X_0 to X_1 and :

$\frac{1}{\lambda'}\, d_X(f(x), f(y)) - \epsilon' \leq d_{X_1}(g \circ f(x), g \circ f(y)) \leq \lambda' d_X(f(x), f(y)) + \epsilon'$.

$\frac{1}{\lambda'}(\frac{1}{\lambda}\, d_{X_0}(x, y) - \epsilon) - \epsilon' \leq d_{X_1}(g \circ f(x), g \circ f(y)) \leq \lambda'(\lambda d_{X_0}(x, y) + \epsilon) + \epsilon'$.

$\frac{1}{\lambda'\lambda}d_{X_0}(x,y) - (\frac{\epsilon}{\lambda'} + \epsilon') \leq d_{X_1}(g \circ f(x), g \circ f(y)) \leq \lambda'\lambda d_{X_0}(x,y) + (\lambda'\epsilon + \epsilon')$.

Since $\lambda' \geq 1$,

$\frac{1}{\lambda'\lambda}d_{X_0}(x,y) - (\lambda'\epsilon + \epsilon') \leq \frac{1}{\lambda'\lambda}d_{X_0}(x,y) - (\frac{\epsilon}{\lambda'} + \epsilon') \leq d_{X_1}(g \circ f(x), g \circ f(y)) \leq \lambda'\lambda d_{X_0}(x,y) + (\lambda'\epsilon + \epsilon')$.

Therefore $g \circ f$ is a $(\lambda'\lambda, \lambda'\epsilon + \epsilon')$-quasi-isometric embedding from X_0 to X_1. To prove that it is a quasi-isometry, we need to show that there exists $c \geq 0$ such that every point of X_1 lies in the c-neighborhood of the image of $g \circ f$.

Since g is a quasi-isometry there exists $c_g \geq 0$ such that for any $x_1 \in X_1$, $d_{X_1}(x_1, Img) \leq c_g$. Let $y \in X_1$. Then there exists $x \in X$ such that $d_{X_1}(y, g(x)) \leq c_g$. Since f is a quasi-isometry there exists $c_f \geq 0$ such that for any $x \in X$, $d_X(x, Imf) \leq c_f$. Therefore, there exists $x_0 \in X_0$ such that $d_X(f(x_0), x) \leq c_f$. Thus,

$d_{X_1}(g \circ f(x_0), y) \leq d_{X_1}(y, g(x)) + d_{X_1}(g(x), g \circ f(x_0)) \leq c_g + \lambda' d_X(x, f(x_0)) + \epsilon' \leq c_g + \lambda' c_f + \epsilon'$.

λ', c_g, c_f and ϵ' do not depend on the choice of y, hence $g \circ f$ is a quasi-isometry. □

Definition 3.9. Let $f, g : X_0 \to X_1$ be two quasi-isometries from the metric space (X_0, d_0) to the metric space (X, d). We say that f is *equivalent* to g and write $f \sim g$, if $\sup_{x \in X} d(f(x), g(x))$ is finite.

Lemma 3.10. *The relationship defined in 3.9 is an equivalence relation.*

Proof: The reflexive and symmetric properties of this relation are obvious. For the transitivity property suppose that $f \sim g$ and $g \sim h$, then $\sup_{x \in X} d((f(x), g(x)) < M_1$ and $\sup_{x \in X} d((g(x), h(x)) < M_2$ for some real numbers M_1 and M_2. Note that for any $x \in X$, $d(f(x), h(x)) \leq d(f(x), g(x)) + d(g(x), h(x)) \leq \sup_{x \in X} d((f(x), g(x)) + \sup_{x \in X} d((g(x), h(x)) \leq M_1 + M_2$. Therefore $f \sim h$.$\square$

Notation: Let $\mathcal{QI}(X_0, X)$ denote the set of equivalence classes of quasi-isometries from X_0 to X. We write $\mathcal{QI}(X)$ for $\mathcal{QI}(X, X)$

Definition 3.11. The composition of equivalence classes of quasi-isometries is defined as follows: let $[f] \in \mathcal{QI}(X_0, X)$ and $[g] \in \mathcal{QI}(X, X_1)$, define $[g] \circ [f]$ to be $[g \circ f]$.

The following lemmas have been stated as exercises in [BH]. Proofs have been given for the sake of completion.

Lemma 3.12. *The composition of quasi-isometry classes is well-defined.*

Proof: Let $[f] \in QI(X_0, X)$ and $[g] \in QI(X, X_1)$, where (X_0, d_0), (X, d) and (X_1, d_1) are metric spaces. Let $f_1, f_2 \in [f]$ and $g_1, g_2 \in [g]$. We need to show that $[g_1 \circ f_1] = [g_2 \circ f_2]$, which is equivalent to showing that $\sup_{x \in X_0} d_1((g_1 \circ f_1)(x), (g_2 \circ f_2)(x))$ is finite. Since f_1 and f_2 are in the same quasi-isometry class, there exists $M_1 \in \mathbb{R}$ such that $d_1(f_1(x), f_2(x)) < M_1$ for any $x \in X_0$. Using the triangle inequality in (X_1, d_1) we have

$$d_1(g_1(f_1(x)), g_2(f_2(x))) \leq d_1(g_1(f_1(x)), g_2(f_1(x))) + d_1(g_2(f_1(x)), g_2(f_2(x)))$$

Since g_1 and g_2 are in the same quasi-isometry class there exists $M_2 \in \mathbb{R}$ such that $d_1(g_1(x), g_2(x)) < M_2$ for any $x \in X$, hence $d_1(g_1(f_1(x)), g_2(f_1(x))) < M_2$. On the other hand since g_2 is a quasi-isometry from X to X_1, there exist $\lambda \geq 1$ and $\epsilon \geq 0$ such that for any s, t in X,

$$\frac{1}{\lambda}d(s,t) - \epsilon \leq d_1(g_2(s), g_2(t)) \leq \lambda d(s,t) + \epsilon.$$

and therefore, $d_1(g_2(f_1(x)), g_2(f_2(x))) \leq \lambda d_X(f_1(x), f_2(x))) + \epsilon$. Combing the above facts we see that,

$$d_1(g_1(f_1(x)), g_2(f_2(x))) \leq M_2 + \lambda M_1 + \epsilon$$

The constants M_1, M_2 and ϵ do not depend on the choice of $x \in X_0$ or f_1, f_2, g_1 org_2, therefore $g_1 \circ f_1 \sim g_2 \circ f_2$. $\square$

Lemma 3.13. *The composition of maps induces a group structure on $\mathcal{QI}(X)$ and any quasi-isometry $\phi : X_0 \to X$ induces an isomorphism $\phi_* : \mathcal{QI}(X_0) \to \mathcal{QI}(X)$.*

Proof: We have already shown that the composition of quasi-isometries is well defined, and that each object has an identity morphism. By definition 3.5 parts (iii) and (iv) for any $[f] \in \mathcal{QI}(X)$ there exists $[g] \in \mathcal{QI}(X)$ such that $[g] \circ [f] = [id_X]$ and $[f] \circ [g] = [id_X]$. Therefore $\mathcal{QI}(X)$ is a group.

Let $\phi : X_0 \to X$ be a (λ, c)-quasi-isometry and let ϕ^{-1} be its quasi-inverse. For any $[f] \in \mathcal{QI}(X_0)$ define $\phi_*([f]) = [\phi \circ f \circ \phi^{-1}]$. Since ϕ, f and ϕ^{-1} are quasi-isometries, then $\phi \circ f \circ \phi^{-1}$ is a quasi-isometry and $[\phi \circ f \circ \phi^{-1}] \in \mathcal{QI}(X)$ and the function is well defined.

$$\begin{tikzcd}
X \arrow[d, "\phi^{-1}"'] \arrow[drr, "\phi \circ f \circ \phi^{-1}"] & & \\
X_0 \arrow[r, "f"] & X_0 \arrow[r, "\phi"] & X
\end{tikzcd}$$

We claim that ϕ_* is a homomorphism. To prove this we need to show that for $f, g \in \mathcal{QI}(X_0)$, $\phi_*([f] \circ [g]) = \phi_*([f]) \circ \phi_*([g])$. By the definition of ϕ_*, $\phi_*([f] \circ [g]) = [\phi \circ f \circ g \circ \phi^{-1}]$. By definition 3.7 and the definition of quasi-isometry classes, $[\phi \circ f \circ g \circ \phi^{-1}] = [\phi \circ f \circ \phi^{-1} \circ \phi \circ g \circ \phi^{-1}]$. By definition 3.11 and the definition of ϕ_*, $[\phi \circ f \circ \phi^{-1} \circ \phi \circ g \circ \phi^{-1}] = [\phi \circ f \circ \phi^{-1}] \circ [\phi \circ g \circ \phi^{-1}] = \phi_*([f]) \circ \phi_*([g])$.

To show that ϕ_* is an isomorphism it suffices to show that ϕ_* has an inverse. We claim that $(\phi^{-1})_*$ is the inverse of ϕ_*. This is easy to see since $(\phi_*^{-1} \circ \phi_*)([f]) = [\phi^{-1} \circ \phi \circ f \circ \phi^{-1} \circ \phi] = [f]$. Similarly $(\phi_* \circ \phi_*^{-1})([f]) = [f]$ showing that ϕ_* is an isomorphism.□

Definition 3.14. Let $\mathcal{M}$ denote the category of metric spaces and classes of quasi-isometries. The objects of $\mathcal{M}$ are metric spaces and the morphisms are equivalence classes of quasi-isometries.

Lemma 3.15. *The category $\mathcal{M}$ defined above is a category.*

We only need to verify that the quasi-isometry classes form morphisms. We need to show that there is an associativity law, and each object has an identity morphism. The associativity law follows from the associativity property of composition of functions. Let X be an object in $\mathcal{M}$, the identity morphism of $\mathcal{QI}(X)$ is $[id]$. Note that, $[id] = \{f : X \to X | \sup_{x \in X}(d(x, f(x) < \infty\}$. It is clear that $[id] \circ [g] = [g]$ for any $[g] \in \mathcal{QI}(X)$.□

Definition 3.16. A morphism $f : A \to B$ in a category is called an *isomorphism* if there exists a morphism which is a left and right inverse for f.

Theorem 3.17. $\mathcal{M}$ *is a category in which every morphism is an isomorphism.*

Proof: The proof follows directly from Lemma 3.8. $\square$

CHAPTER IV

THE END SPACE OF A TREE

In this section we will take a close look at perfect compact ultrametric spaces and some of their properties. We will then go on to associate to every locally finite rooted tree of minimal vertex degree three, the space of its ends and show that this space is a perfect compact ultrametric space.

Definition 4.1. If (X, d) is a metric space and $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$, then d is an *ultrametric* and (X, d) is an *ultrametric space.*

The following proposition lists some well-known properties of ultrametric spaces which have been quoted from [Hug](Proposition 4.2).

Proposition 4.2. (Elementary properties of ultrametric spaces). The following properties hold in any ultrametric space (X, d).

1- If two open balls in X intersect, then one contains the other.

2- If two closed balls intersect, then one contains the other.

3- **(Egocentricity)**Every point in an open ball is a center of the ball.

4- **(Closed Egocentricity)**Every point in a closed ball is a center of the ball.

5- Every open ball is closed, and every closed ball is open.

6- **(ISB)** Every triangle in X is isosceles with a short base

(i.e., if x_1, x_2, $x_3 \in X$, then there exists an i such that

$d(x_j, x_k) \leq d(x_i, x_j) = d(x_i, x_k)$ whenever $j \neq i \neq k$). $\square$

Definition 4.3. Let X be a metric space. The point $x \in X$ is an *accumulation point* of X if there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that, $\lim_{n\to\infty} y_n = x$ and $y_n \neq x$ for all $n \in \mathbb{N}$. The set of all accumulation points of X is denoted by P'.

Definition 4.4. Let P' be the set of accumulation points of the topological space P. If $P = P'$ then P is a *perfect* space.

Lemma 4.5. *Let X be a metric space with metric d. Then $x \in X$ is an accumulation point of X, if and only if, there exists a sequence of positive real numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ converging to zero such that $\{y \in X | d(y, x) = \varepsilon_n\}$ is not empty.*

Proof: Let x be an accumulation point of X. Then there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = x$ and $y_n \neq x$ for any $n \in \mathbb{N}$. Let $d(x, y_n) = \varepsilon_n$, then, $\lim_{n\to\infty} \varepsilon_n = 0$. The sequences $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ satisfy the lemma.

Conversely, let $x \in X$ and let $\{\varepsilon_j\}_{j=1}^{\infty}$ be such that $\{y \in X | d(y, x) = \varepsilon_j\}$ is not empty. Let $Y_i = \{y \in X | d(x, y) = \varepsilon_i\}$ for $i \in \mathbb{N}$. By the hypothesis $Y_i \neq \emptyset$. Let $y_i \in Y_i$, then $d(x, y_i) = \varepsilon_i$ which converges to zero. This implies that x is a limit point of X. $\square$

The following is a well-known theorem about perfect compact ultrametric spaces. A proof of this theorem can be found in [Ber].

Theorem 4.6. *The set $A \subseteq \mathbb{R}$ is the set of positive distances of some perfect, compact ultrametric space if and only if A can be enumerated as a countable decreasing sequence $\{d_i | i \geq 0\}$ with $\lim_{i\to\infty} d_i = 0$.*

From here on we focus our attention to the end space of a rooted tree.

Definition 4.7. An isometric embedding $g : [0, \infty) \to T$ is called *ray* in the tree T.

Notation: We will denote the set of rays in the rooted tree (T, w) starting at w by $end(T, w)$. In other words,

$$end(T, w) = \{g : [0, \infty) \to (T, w)|\ g(0) = w \text{ and } g \text{ is a ray in } T\ \}.$$

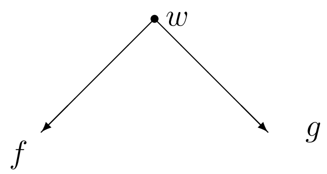
Definition 4.8. We denote the metric on $end(T, w)$ by d_w, and define it as follows: for $f, g \in end(T, w)$

$$d_w(f, g) = \begin{cases} 0, & \text{if } f = g \\ 1/e^{t_0}, & \text{if } f \neq g \text{ and } t_0 = \sup\{t \geq 0 | f(t) = g(t)\ \} \end{cases} \tag{4.1}$$

This definition is adapted from [Hug](Def 5.1). We use the notation $(f|g)_w$ to denote $\sup\{t \geq 0 | f(t) = g(t)\}$; the notation $(f|g)$ will be used when the root w is clear from the context.

Remarks 4.9. *Let (T, w) be a rooted tree and $(end(T, w), d_w)$ its end space. Then,* $\{d_w(f, g) | f, g \in (end(T, w), d_w)\} \subseteq \{\frac{1}{e^n}\}_{n=0}^{\infty}$.

Example 4.10. Let (T, w) be a rooted tree in which every vertex has degree two. (T, w) has two ends which we will denote by f and g. In other words $end(T, w) = \{f, g\}$. Observe that $t_0 = \sup\{t \geq 0 | f(t) = g(t)\} = 0$ and hence $d_w(f, g) = 1$. The figure below shows this tree.



Example 4.11. Let (T, w) be the tree in figure 4.1.

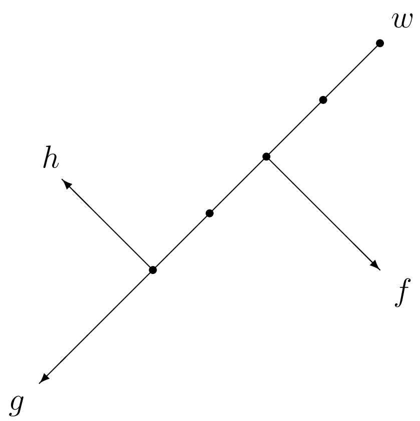


Figure 4.1: A rooted tree with three ends

Then $end(T, w) = \{f, g, h\}$, and $d_w(f, g) = \dfrac{1}{e^2}$, $d_w(f, h) = \dfrac{1}{e^2}$,and $d_w(g, h) = \dfrac{1}{e^4}$.

The fact that $d_w(f, h) = d_w(f, g)$ is not a coincidence as proposition 4.12 will show.

Proposition 4.12. *The space* $(end(T, w), d_w)$ *is an ultrametric space of diameter* ≤ 1 *which is totally bounded and complete (and therefore compact).*

Proof : A proof that the space is a complete ultrametric space of diameter ≤ 1 can be found in [Hug](Proposition 5.2). We will prove that it is totally bounded. To show that it is totally bounded, we need to show that for any given $\delta > 0$, there is a finite covering of $end(T, w)$ by δ balls.

Note that since (T, w) is a locally finite tree, then for any point $w \in T$ and any number $n \geq 0$, there are only finitely many vertices in $B(w, n)$.

Let $\delta > 0$ be given. If $\delta \geq 1$, $B(f, \delta)$ will cover $end(T, w)$ for any $f \in end(T, w)$, and the proposition is proved. Let $0 < \delta < 1$ and let $n = [|-\ln \delta|] + 1$, where $[|x|]$ is the greatest integer part of x. Let k be the number of points in $B(w, n+1)$ such that the distance between these points and w is equal to n and let $\{x_1, x_2, \ldots, x_k\}$ be this set of points. Let $f_i : [0, \infty) \to T$ be an isometric embedding such that $f_i(0) = w$ and $f_i(n) = x_i$. We claim that $\bigcup_{i=1}^{k} B(f_i, \delta)$ covers $end(T, w)$. let $f \in end(T, w)$, then $\mathrm{Im} f([0, n]) \subseteq B(w, n+1)$. f is an isometric embedding, thus $|w - f(n)| = n$. Therefore, there exists a unique i between 1 and k such that $f_i(0) = f(0)$ and $f_i(n) = f(n)$. By lemma 2.5, $f_i|_{[0,n]} = f|_{[0,n]}$. Hence, $(f_i|f)_w \geq n$, therefore $d_w(f_i, f) \leq e^{-n} \leq e^{\ln \delta} = \delta$. $\square$

Lemma 4.13. *Let (T, w) be a locally finite rooted tree of minimal vertex degree three. Let $(end(T, w), d_w)$ be its end space. Then for any $f \in (end(T, w), d_w)$ we have*

$$\{d_w(f, g) | g \in (end(T, w), d_w)\} = \left\{ \frac{1}{e^n} \right\}_{n=0}^{\infty}$$

Proof: Let $f \in (end(T, w), d_w)$ and let $n \in \mathbb{N} \cup \{0\}$. We need to show that there exists $g \in (end(T, w), d_w)$ such that for any $0 \leq t \leq n$, $f(t) = g(t)$ and if $t > n$, then $f(t) \neq g(t)$. Let $n \in \mathbb{N} \cup \{0\}$ be given, and let $x = f(n)$. Let $y \in (T, w)$ such that $d_T(x, y) = 1$ and $y \notin \mathrm{Im}(f)$. Such a y can be found because the minimal vertex degree of (T, w) is three. Since (T, w) has the natural tree metric we can find an isometry $g_1 : [n, n+1] \to (T, w)$, such that $g(n) = x$ and $g(n+1) = y$. Let

$$\tilde{g}(t) = \begin{cases} f(t), & \text{if } 0 \leq t \leq n \\ g_1(t), & \text{if } n \leq t \leq n+1 \end{cases} \tag{4.2}$$

We have that $g_1(n) = f(n)$, therefore $\tilde{g}$ is well defined and by the pasting lemma it is continuous. To see that it is an isometry, let $t_1, t_2 \in [0, n+1]$. If $t_1, t_2 \in [0, n]$, then $d_w(\tilde{g}(t_1), \tilde{g}(t_2)) = d_w(f(t_1), f(t_2)) = |t_2 - t_1|$, and similarly if $t_1, t_2 \in [n, n+1]$, then $d_w(\tilde{g}(t_1), \tilde{g}(t_2)) = d_w(g_1(t_1), g_1(t_2)) = |t_2 - t_1|$ because both f and g are isometries. If $t_1 \leq n$ and $t_2 \geq n$, then $d_w(\tilde{g}(t_1), \tilde{g}(t_2)) = d_w(f(t_1), f(n)) + d_w(g_1(n), g_1(t_2)) = (n - t_1) + (t_2 - n) = t_2 - t_1 = |t_2 - t_1|$. Hence $\tilde{g}$ is an isometry. Using the fact that (T, w) is a geodesically complete rooted tree we can extend $\tilde{g}$ to an isometric embedding $g : [0, \infty) \to (T, w)$. Therefore, $g \in (end(T, w), d_w)$ and $d_w(f, g) = \dfrac{1}{e^n}$. □

Proposition 4.14. *Let (T, w) be a rooted tree with root w, and let $(end(T, w), d_w)$ be the end space of (T, w). If the minimal vertex degree of the rooted tree (T, w) is three, then $(end(T, w), d_w)$ is a perfect compact ultrametric space of diameter one .*

Proof: Let $f \in (end(T, w), d_w)$. By lemma 4.13, the set $\{g \in (end(T, w), d_w) | d_w(f, g) = \dfrac{1}{e^n}\}$ is non-empty for every $n \in \mathbb{N} \cup \{0\}$. Hence by lemma 4.5, f is an accumulation point of $(end(T, w), d_w)$. The choice of f was arbitrary. Therefore, every point of $(end(T, w), d_w)$ is an accumulation point, hence $(end(T, w), d_w)$ is a perfect metric space.

By the above and lemma 4.12 we conclude that $(end(T, w), d_w)$ is a perfect compact ultrametric space.

By lemma 4.13 there exists $g \in (end(T, w), d_w)$ such that $d_w(f, g) = 1$, therefore the diameter of $end(T, w)$ is equal to one. □

Lemma 4.15. *Let $\phi : (U, d_U) \to (V, d_V)$ be a homeomorphism from the perfect compact ultrametric space (U, d_U) to the perfect compact ultrametric space (V, d_V). Let $\{\alpha_p\}_{p=0}^{\infty}$ be the set of distances in U. Furthermore, suppose that for $q > p$ we have $\alpha_q < \alpha_p$. Then for any $p > 0$ there exists $\delta > 0$ such that for any $x, x' \in U$ with $d_U(x, x') < \delta$ we have:*

$$\{d_V(\phi(x), \phi(y))|\ d_U(x, y) = \alpha_p, y \in U\} = \{d_V(\phi(x'), \phi(y)|\ d_U(x', y) = \alpha_p, y \in U\}.$$

Proof: Let $\delta_1 = \alpha_{p+1}$. By the hypothesis, ϕ is a homeomorphism from the perfect compact ultrametric space (U, d_U) to the perfect compact ultrametric space (V, d_V), and hence ϕ and ϕ^{-1} are both uniformly continuous. By the uniform continuity of ϕ^{-1}, there exists $\epsilon > 0$ such that for any $x, y \in U$ if $d_v(\phi(x), \phi(y)) < \epsilon$, then, $d_V(x, y) < \delta_1$. Similarly, by the uniform continuity of ϕ, there exists $\delta_2 > 0$ such that if $d_U(x, y) < \delta_2$, then, $d_V(\phi(x), \phi(y)) < \epsilon$.

Let $p > 0$ be given and let $d_U(x, y) = \alpha_p$. Then $d_U(x, y) > \delta_1$ and hence, $d(\phi(x), \phi(y)) \geq \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x' \in U$ be such that $d_U(x, x') < \delta$. Then $d_V(\phi(x), \phi(x')) < \epsilon$. In (U, d_U) every triangle is isosceles with a short base because it is an ultra-metric space. Hence in the triangle with vertices x, x' and y we must have $d_U(x', y) = d_U(x, y)$, and therefore $d_U(x', y) = \alpha_p$. Thus, $d_V(\phi(x'), \phi(y)) \geq \epsilon$. Similarly, in the ultra-metric space (V, d_V) the triangle with vertices $\phi(x), \phi(x')$ and $\phi(y)$ is isosceles with a short base and thus $d_V(\phi(x), \phi(y)) = d_V(\phi(x'), \phi(y))$. □

The above lemma does not hold in general metric spaces.

Example 4.16. *Let $X = [0, 1]$, $Y = [1, 4]$. Define $f : [0, 1] \to [1, 4]$ by $x \mapsto (x + 1)^2$.*

Then f is a homeomorphism from the compact metric space $[0,1]$ to the compact metric space $[1,4]$. Let $x = \frac{1}{2}$ and let $p = \frac{1}{4}$. We will show that for any given $\delta > 0$ we can find x' such that $d(x,x') < \delta$ but

$$\{d_Y(f(x), f(y)|\ d_X(x,y) = p, y \in X\} \neq \{d_Y(f(x'), f(y))|\ d_X(x',y) = p, y \in X\}.$$

Proof: For any given δ take $x' = \frac{1}{2} + \frac{\delta}{2}$. In this case $d_X(x,x') = \frac{\delta}{2}$. In order for $y \in X$ and $d_X(x,y) = \frac{1}{4}$ we must have $y = \frac{1}{4}$ or $y = \frac{3}{4}$. Therefore $\{d_Y(f(x), f(y)|\ d_X(x,y) = p, y \in X\} = \{\frac{9}{16}, \frac{7}{16}\}$. To have $d_X(x',y) = \frac{1}{4}$ we must have $y = \frac{3}{4} + \frac{\delta}{2}$ or $y = \frac{1}{4} + \frac{\delta}{2}$ in which case $\{d_Y(f(x'), f(y))|\ d_X(x',y) = p, y \in X\} = \{\frac{5}{16} + \frac{\delta}{4}, \frac{3}{16} + \frac{\delta}{4}\}$. These two sets are not equal for any value of δ.

Lemma 4.17. *Let the hypothesis of lemma 4.15 hold. Let $\alpha_q > \alpha_p$ and let δ be the value obtained in lemma 4.15 associated to p. Suppose that for $x, x' \in U$ with $d_U(x,x') < \delta$ then,*

$$\{d_V(\phi(x), \phi(y))|\ d_U(x,y) = \alpha_q, y \in U\} = \{d_V(\phi(x'), \phi(y)|\ d_U(x',y) = \alpha_q, y \in U\}.$$

Proof: The proof is analogous to the proof of lemma 4.15 with α_q substituted for α_p.$\square$

CHAPTER V

QUASI-RAYS IN PROPER METRIC SPACES

Definition 5.1. Let (X, d) be a metric space. Let Y and Z be subsets of X, and H a positive real number. The *H-neighborhood* of Y in X is denoted by $\mathcal{V}_H(Y)$ and is the set $\{x \in X | d(x, Y) \leq H\}$. The *Hausdorff distance* between Y and Z is denoted by $\mathcal{H}(Y, Z)$, and is defined as $\inf\{H > 0 | Y \subset \mathcal{V}_H(Z) \text{ and } Z \subset \mathcal{V}_H(Y)\}$ if this is a real number and infinity otherwise. Let A and B be two sets and $f : A \to X$ and $g : B \to X$ be two maps, note that they do not need to be continuous. The Hausdorff distance between f and g is denoted by $\mathcal{H}(f, g)$ and is defined to be the Hausdorff distance between the images $f(A)$ and $g(B)$.

Remarks 5.2. The Hausdorff distance between the images of maps on a metric space X forms an equivalence relation on the set of maps with images in X.

1) For any map $f : A \to X$ where X is a metric space $\mathcal{H}(f, f) < \infty$.

2) Let X be a metric space and A and B be sets. Let $f : A \to X$ and $g : B \to X$ be maps into X. If $\mathcal{H}(f, g) < \infty$, then $\mathcal{H}(g, f) < \infty$.

3) Let X be a metric space and A, B and C be sets. Let $f : A \to X$, $g : B \to X$ and $h : C \to X$ be maps into X. If $\mathcal{H}(f, g) < \infty$, and $\mathcal{H}(g, h) < \infty$, then $\mathcal{H}(f, h) < \infty$. Proof: Let $H_1 = \mathcal{H}(f, g)$ and $H_2 = \mathcal{H}(g, h)$. Then $f(A) \subseteq \mathcal{V}_{H_1}(g(B))$ and $g(B) \subseteq \mathcal{V}_{H_1}(f(A))$. Similarly $g(B) \subseteq \mathcal{V}_{H_2}(h(C))$ and $h(C) \subseteq \mathcal{V}_{H_2}(g(B))$. Therefore,

$h(C) \subseteq \mathcal{V}_{H_2}(\mathcal{V}_{H_1}(f(A))) \subseteq \mathcal{V}_{H_1+H_2}(f(A))$ and $f(A) \subseteq \mathcal{V}_{H_1}(\mathcal{V}_{H_2}(h(C))) \subseteq \mathcal{V}_{H_1+H_2}(h(C))$. Hence, $\mathcal{H}(f, h) < \infty$. $\square$

Lemma 5.3. *Let T be a locally finite tree, and f and f' two rays in T with the same initial point. $\mathcal{H}(f, f') < \infty$ if and only if $f = f'$.*

Proof: Recall that a ray in the tree T is defined to be an isometric embedding $g : [0, \infty) \to T$. Let $t_0 = \sup\{t | f(t) = f'(t); t \in [0, \infty)\}$. Suppose that $t_0 = \infty$. This means that $f(t) = f'(t)$ for all $t \in [0, \infty)$, hence $f = f'$. If t_0 is finite, then for any $t > t_0$ we have $f(t) \neq f'(t)$. In this case $d(f(t), Im(f')) = d(f(t), f'(t_0))$. Note that, $f'(t_0) = f(t_0)$ therefore, $d(f(t), Im(f')) = d(f(t), f(t_0))$. Let $t \to \infty$,then $d(f(t), f'(t_0)) \to \infty$. This implies that $\mathcal{H}(f, f') = \infty$. $\square$

Definition 5.4. A (λ, c)-*quasi-segment* in X is a map $g : I \to X$, where I is a bounded interval in $\mathbb{R}$ or in $\mathbb{N}$, with the property that there exists $\lambda \geq 1$ and $c > 0$ such that for any $s, t \in I$:

$$\frac{1}{\lambda}|s - t| - c \leq d(g(s), g(t)) \leq \lambda|s - t| + c.$$

Looking closely at the definition above we see that a quasi-segment is a distorted geodesic segment. The next theorem shows that every quasi-segment is in fact “close” to a geodesic segment.

Theorem 5.5. *([GH], Theorem 5.11) Let X be a δ-hyperbolic geodesic space, $I = [a, b]$ an interval in $\mathbb{R}$ or in $\mathbb{N}$, and $f : I \to X$ a (λ, c)-quasi-segment, with $\lambda \geq 1$ and $c > 0$. Let J be an interval in $\mathbb{R}$ of length $|f(a) - f(b)|$, and let $g : J \to X$ be a* *geodesic segment starting at $f(a)$ and terminating at $f(b)$. Then $\mathcal{H}(f,g) \leq H$, where H is a constant that depends only on λ, c and δ.*

The complete proof of this theorem can be found in [GH].

Definition 5.6. A (λ, c)-*quasi-ray* in X is a map $g : I \to X$, where I is an unbounded interval in $[0,\infty)$ or in $\mathbb{N}$, with the property that there exists $\lambda \geq 1$ and $c > 0$ such that for any s, t in I or in $\mathbb{N}$:

$$\frac{1}{\lambda}|s-t| - c \leq d(g(s), g(t)) \leq \lambda|s-t| + \text{c}.$$

Lemma 5.7. *([GH], Lemma 6.5(ii)) Let T be a locally finite tree. Let $f : [0,\infty) \to T$ be a quasi-ray. Then there exists a unique ray $f_0 : [0,\infty) \to T$ such that $f_0(0) = f(0)$ and the Hausdorff distance between f and f_0 is finite.*

Proof: The existence follows from [GH], Theorem 5.25. Uniqueness follows from lemma 5.3. $\square$

Theorem 5.8. *Let (T, w) and (S, v) be locally finite rooted trees. If $\gamma : (T, w) \to (S, v)$ is an isometry such that $\gamma(w) = v$, then $\hat{\gamma} : (end(T, w), d_w) \to (end(S, v), d_v)$ defined by $\hat{\gamma}(f)(t) = (\gamma \circ f)(t)$ is also an isometry.*

Proof: Since any isometry is a homeomorphism, γ is continuous and has a continuous inverse γ^{-1}. Define $\hat{\gamma}(f) = g$ where $g(t) = (\gamma \circ f)(t)$ for any $f \in end(T, w)$. We note that $\hat{\gamma}$ is well defined and $g(0) = (\gamma \circ f)(0) = \gamma(w) = v$. The composition of two isometries is an isometry therefore $g \in end(S, v)$. Let $\hat{\gamma}(f) = \hat{\gamma}(f')$, then $(\gamma \circ f)(t) = (\gamma \circ f')(t)$ for any $t \in [0,\infty)$. Since γ is one to one, this implies that

$f(t) = f'(t)$ for any $t \in [0, \infty)$. Therefore $f = f'$, so $\hat{\gamma}$ is one to one. For any $g \in end(S, v)$, $\hat{\gamma}(\gamma^{-1} \circ g)(t) = (\gamma \circ \gamma^{-1} \circ g)(t) = g(t)$ for any $t \in [0, \infty)$, thus $\hat{\gamma}$ is onto. To see that $\hat{\gamma}$ is an isometry, note that $d_v(\hat{\gamma}(f), \hat{\gamma}(f') = e^{-t_0}$ where $t_0 = sup\{t \geq 0 | \hat{\gamma}(f)(t) = \hat{\gamma}(f')(t)\}$. Since $\hat{\gamma}$ is one to one, $t_0 = \{t \geq 0 | f(t) = f'(t)\}$. Hence $d_v(\hat{\gamma}(f), \hat{\gamma}(f')) = d_w(f, f')$. Therefore $\hat{\gamma}$ is an isometry. $\square$

Theorem 5.8 and the similarities between quasi-isometries and isometries leads us to conjecture that a quasi-isometry between locally finite trees may induce some type of map from the space of ends of these trees to each other. Bridson and Haefliger; [BH]; show that the induced map is a homeomorphism. Ghys and de la Harpe; [GH]; show that if the trees have minimal vertex degree three, then the induced map is a bi-Hölder quasi-conformal homeomorphism. The definitions and proofs follow.

Definition 5.9. Let (X, d_X) and (Y, d_Y) be metric spaces and $h : X \to Y$ a homeomorphism. We say h is an (α, c)-*bi-Hölder* homeomorphism if there exist constants $\alpha > 0$ and $c > 0$ such that for any $x, y \in X$;

$$\frac{1}{c} d_X(x, y)^{\frac{1}{\alpha}} \leq d_Y(h(x), h(y)) \leq c d_X(x, y)^{\alpha}.$$

Definition 5.10. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $x \in X$ be an accumulation point of X. Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a sequence of positive real numbers tending to zero such that $\{y \in X | d(y, x) = \varepsilon_j\}$ is not empty. Such a sequence can be found by lemma 4.5. Let $\phi : X \to Y$ be a homeomorphism. Let $A_j = \{d_Y(\phi(x), \phi(y)) | d(x, y) = \varepsilon_j\}$ for $j = 1, 2, \ldots$. We say that ϕ has *finite dilation* at the point x, if $H_\phi^{d_X}(x) = \limsup_{j\to\infty} \dfrac{\sup A_j}{\inf A_j}$ is finite for any sequence $\{\varepsilon_j\}_{j=1}^\infty$ with the above properties .

Definition 5.11. Let (X, d_X) be a perfect metric space. Let (Y, d_Y) be a metric space and $\phi : (X, d_X) \to (Y, d_Y)$ be a homeomorphism between these two metric spaces. Let $H_\phi^{d_X}(x)$ be defined as in definition 5.10.

(1) If $H_\phi^d(x) = 1$ for all $x \in X$, ϕ is said to be *conformal.*

(2) If $\sup_{x\in X} H_\phi^d(x) < \infty$, ϕ is said to be *quasi-conformal.*

(3) If there exists a constant K such that $\sup_{x\in X} H_\phi^d(x) \le K$, ϕ is said to be K*-quasi-conformal.*

Definition 5.12. Let (T, w) be a locally finite rooted tree. The set

$$S_{f,p} = \{f' \in end(T, w) | (f|f')_w = p\}$$

is defined to be the *closed sphere of radius* p *and center* f. In other words, $S_{f,p}$ is the set of all rays such that for all $t \le p$ we have $f(t) = f'(t)$.

Theorem 5.13. *Let (T, w) and (S, v) be locally finite trees of vertex degree greater than or equal to three. Let $\gamma : (T, w) \to (S, v)$ be a (λ, c)- quasi-isometry such that $\gamma(w) = v$. Then, $\hat{\gamma} : (end(T, w), d_w) \to (end(S, v), d_v)$ is a quasi-conformal bi-Hölder homeomorphism where $\hat{\gamma}$ is defined as follows:*

For any $f \in (end(T, w), d_w)$ define $\hat{\gamma}(f) = (\gamma \circ f)_0$ where $(\gamma \circ f)_0 \in (end((S, v), d_v)$ and $\mathcal{H}\ (\gamma \circ f, (\gamma \circ f)_0) < \infty$.

The proof that $\hat{\gamma}$ is a homeomorphism is left as an exercise in [BH]. [GH] state that $\hat{\gamma}$ is a quasi-conformal bi-Hölder homeomorphism, but only prove that it is quasi-conformal. I found that the proof of $\hat{\gamma}$ being quasi-conformal was not clear. The proof given here is not the same as those in [BH] or in [GH]. A complete proof of the theorem follows.

We will prove the theorem in the following steps:

1. Establish that $\hat{\gamma}$ is well-defined.

2. Show that $\hat{\gamma}$ is onto.

3. Show that $\hat{\gamma}$ is one-to-one.

4. Show that $\hat{\gamma}$ is bi-Hölder and continuous, and hence with items 2 and 3 above, deduce that $\hat{\gamma}$ is a Hölder homeomorphism.

5. Show that $\hat{\gamma}$ is quasi-conformal.

Proof:

1. $\hat{\gamma}$ **is well defined**.

Proof: $\hat{\gamma}$ being well defined follows from lemma 5.3.

2. $\hat{\gamma}$ **is onto**

Proof: To prove that $\hat{\gamma}$ is onto, we need to show that for any $g \in end(S, v)$ there exists $f \in end(T, w)$ such that $\hat{\gamma}(f) = g$. We claim that $f = \hat{\gamma}^{-1} \circ g$ satisfies this condition, where $\hat{\gamma}^{-1} : end(S, v) \to end(T, w)$ is the map induced by γ^{-1} the quasi-inverse of γ. By lemma 5.4 it suffices to show that $\mathcal{H}(\hat{\gamma}(f), g) < \infty$. That is, to show that there exists a constant H, such that for any $t \in [0, \infty)$ there exists $t', t'' \in [0, \infty)$ with the property that (i) $d_S(\hat{\gamma}(f)(t), g(t')) \leq H$ and (ii) $d_S(g(t), \hat{\gamma}(f)(t'')) \leq H$. We note that

$\hat{\gamma}(f) = \hat{\gamma}(\hat{\gamma}^{-1} \circ g) = (\gamma \circ (\gamma^{-1} \circ g)_0)_0$. Let $t \in [0, \infty)$ be a given base point. By the definition of $\hat{\gamma}$ there exists $t_1 \in [0, \infty)$ such that

$d_S((\gamma \circ (\gamma^{-1} \circ g)_0)_0(t), \gamma \circ (\gamma^{-1} \circ g)_0(t_1)) \leq H_1$, where H_1 is a constant that only depends on λ and c. Let $t_2 \in [0, \infty)$ be such that $d_T((\gamma^{-1} \circ g)_0(t_1), (\gamma^{-1} \circ g)(t_2)) \leq H_2$. Since γ is a (λ, c)-quasi-isometry :

$\frac{1}{\lambda} d_T((\gamma^{-1}(g))_0(t_1)(\gamma^{-1}(g))(t_2)) - c \leq d_S(\gamma(\gamma^{-1}(g))_0(t_1), \gamma(\gamma^{-1}(g))(t_2))$
$\leq \lambda d_T((\gamma^{-1}(g))_0(t_1), (\gamma^{-1}(g))(t_2)) + c.$

Therefore $d_S((\gamma \circ (\gamma^{-1} \circ g)_0)_0(t), (\gamma \circ \gamma^{-1} \circ g)(t_2)) \leq H_1 + \lambda H_2 + c$. By lemma 3.2(iii) $d_S(\gamma \circ \gamma^{-1} \circ g(t_2), g(t_2)) \leq c_2$ for some constant c_2 which depends only on λ and c. Hence $d_S((\gamma \circ (\gamma^{-1} \circ g)_0)_0(t), g(t_2)) \leq H_1 + \lambda H_2 + c + c_1$. This proves (i). The proof of (ii) is similar.

3. $\hat{\gamma}$ **is one-to-one.**

Proof: Let $f, g \in (end(T, w), d_w)$ be such that $\hat{\gamma}(f) = \hat{\gamma}(g)$. By the definition of $\hat{\gamma}$ we conclude that $\mathcal{H}(\hat{\gamma}(f), (\gamma \circ f)_0) < \infty$ and $\mathcal{H}(\hat{\gamma}(g), (\gamma \circ g)_0) < \infty$, hence $\mathcal{H}((\gamma \circ f)_0), (\gamma \circ g)_0) < \infty$. By lemma 5.3 we must have $(\gamma \circ f)_0 = (\gamma \circ g)_0$. By remark 5.2, $\mathcal{H}((\gamma \circ f), (\gamma \circ g) < \infty$. Let $t \in [0, \infty)$ be arbitrary. Then, there exists $t' \in [0, \infty)$ such that $d_S((\gamma \circ f)(t), (\gamma \circ g)(t')) \leq H$ for some constant H. Since γ is a (λ, c)-quasi-isometry, we have

$$\frac{1}{\lambda} d_T(f(t), g(t')) - c \leq d_S((\gamma \circ f)(t), (\gamma \circ g)(t')) \leq H$$

This implies that $\text{Im}(f) \subset \mathcal{V}_{\lambda H + cH}(\text{Im } (g))$. By the same argument we see that $\text{Im}(g) \subset \mathcal{V}_{\lambda H + cH}(\text{Im}(f))$and hence f and g are a finite Hausdorff distance apart and thus by lemma 5.3, $\hat{\gamma}$ is one-to-one.

4. **Show that $\hat{\gamma}$ is bi-Hölder and continuous**

Proof: Let $f \in (end(T,w), d_w)$ and let $x \in Im(f)$. Therefore, there exists $t_0 \in [0, \infty)$ such that $f(t_0) = x$. Let $p = d_w(w, x)$. Let $S_{f,p}$ be the closed sphere of radius p and center w, i.e. $S_{f,p} = \{f' \in end(T,w) | (f|f')_w = p\}$. Let x' be the projection of $\gamma(x)$ on $\hat{\gamma}(f)$, in other words, the point on $\hat{\gamma}(f)$ which is the closest to $\gamma(x)$. Note that such a point exists because (S, v) is a tree. Let $p' = d_S(v, x')$. Let $f' \in S_{f,p}$ be a base point and x'_b be the unique point on $\hat{\gamma}(f')$ such that $(\hat{\gamma}(f)|\hat{\gamma}(f'))_v = d_S(v, x'_b)$.

Only one of the following two cases can occur.

Case 1: $|v - x'_b| < |v - (\gamma f)(t_0)|$

Case 2: $|v - x'_b| \geq |v - (\gamma f)(t_0)|$

Using the fact that γ is a (λ, c)-quasi-isometry we obtain the following inequalities:

Case 1: $|v - x'_b| < |v - (\gamma f)(t_0)| = |(\gamma f)(0) - (\gamma f)(t_0)| \leq \lambda t_0 + c = \lambda(f|f') + c$.

Case 2: $|v - x'_b| \geq |v - (\gamma f)(t_0)| = |(\gamma f)(0) - (\gamma f)(t_0)| \geq \dfrac{1}{\lambda} t_0 - c = \dfrac{1}{\lambda}(f|f') - c$

Hence from the two cases above we have that $\dfrac{1}{\lambda}(f|f') - c \leq |v - x'_b| \leq \lambda(f|f') + c$. Recall that $|v - x'_b| = (\hat{\gamma}(f)|\hat{\gamma}(f'))$. Substituting in the above and multiplying the inequality by -1 and taking exponentials of the inequality we obtain:

$e^{-\lambda(f|f')-c} \leq e^{-(\hat{\gamma}(f)|\hat{\gamma}(f'))} \leq e^{\frac{1}{\lambda}(f|f')+c}$. Let $e^c = k$ and recall that $e^{-((f)|(f'))} = d_w(f, f')$ and $e^{-(\hat{\gamma}(f)|\hat{\gamma}(f'))} = d_v(\hat{\gamma}(f), \hat{\gamma}(f'))$. Substituting we get

$$\frac{1}{k} d_w(f, f')^{\lambda} \leq d_v(\hat{\gamma}(f), \hat{\gamma}(f')) \leq k d_w(f, f')^{\frac{1}{\lambda}}$$

Therefore, $\hat{\gamma}$ is bi-Hölder and continuous with a continuous inverse. This fact in conjunction with 2 and 3 lets us deduce that $\hat{\gamma}$ is a $\left(\dfrac{1}{\lambda}, k\right)$- bi-Hölder homeomorphism.

5. $\hat{\gamma}$ **is quasi-conformal**

Before proving this point we need the following lemma.

Lemma 5.14. *Let f, f', x, t_0, x'_b and γ be as in 4 above. Then there exists a constant k, which only depends on γ, such that $d_S(\gamma(x), x'_b) \leq k$.*

Proof : There are two cases to consider.

Case 1)$d_S(v, x') \geq d_S(v, x'_b)$

We have $\gamma(x) = (\gamma \circ f)(t_0) = (\gamma \circ f')(t_0)$, and therefore by the construction of $\hat{\gamma}(f)$, $d_S(\gamma(x), x')) \leq H$. On the other hand, (S, v) is a tree therefore the projection of $\gamma(x)$ on $\hat{\gamma}(f')$ is x'_b and again by the construction of $\hat{\gamma}(f')$ we must have $d_S(\gamma(x), x'_b) \leq H$. Thus, in this case, any $k \geq H$ satisfies the conclusion of the lemma. The figure 5.1 is a rough sketch of this case.

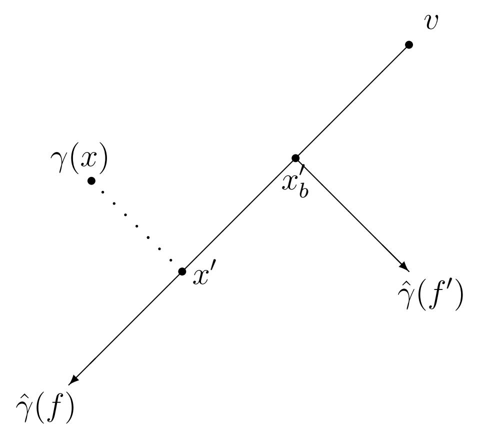


Figure 5.1: $d_S(v, x') \geq d_S(v, x'_b)$

Case 2) $d_S(v, x') \leq d_S(v, x'_b)$ To prove this case we use lemma III.1.11 of [BH] called "Taming Quasi-Geodesics".

Lemma 5.15. Let X be a geodesic space. Given any (λ, c)-quasi-segment $g : [a, b] \to X$, there exists a continuous (λ, c')- quasi-segment $g' : [a, b] \to X$ such that:

$g(a) = g'(a)$ and $g(b) = g'(b)$

$c' = 2(\lambda + c)$

$\mathcal{H}(g, g') \leq \lambda + c.$

□

The construction of g' is such that $g'(t) = g(t)$ for all integer values of t.

Let $(\gamma \circ f)' : [0, \infty) \to (S, v)$ be the continuous (λ, c)-quasi-segment obtained from the "Taming quasi-geodesics" lemma related to $\gamma \circ f$.

Since $(\gamma \circ f)' : [0, t_0] \to (S, v)$ is continuous. Then we can find $t_1, t_2, t_3 \in [0, t_0]$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_0$ and $(\gamma \circ f)'(t_1) = (\gamma \circ f)'(t_3) = x'$ and $(\gamma \circ f)'(t_2) = x'_b$. Note that $(\gamma \circ f)'$ and $\gamma \circ f$ agree on all integer values of $t \in [0, t_0]$. Figure 5.2 represents the above.

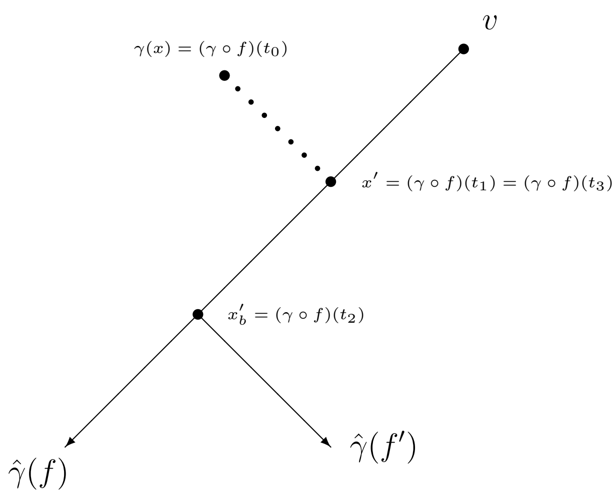


Figure 5.2: $d_S(v, x') \leq d_S(v, x'_b)$

$\gamma \circ f$ is a (λ, c)-quasi-segment, therefore

$$\frac{1}{\lambda}d_w(f(t_1), f(t_3)) - c \leq d_v((\gamma \circ f)(t_1), (\gamma \circ f)(t_3)) \leq \lambda d_w(f(t_1), f(t_3)) + c.$$

Since $d_v((\gamma \circ f)(t_1), (\gamma \circ f))(t_3) = 0$, we conclude form the above equation that $d_w(f(t_1), f(t_3)) \leq \lambda c$. On the other hand, $f \in end(T, w)$ therefore, $d_w(f(t_1), f(t_3)) = |t_1 - t_3|$. From the above conditions on t_1, t_2 and t_3 we have $|t_1 - t_2| \leq |t_1 - t_3|$, hence $|t_1 - t_2| \leq \lambda c$. By the triangle inequality for d_v we have $d_v(\gamma(x), x'_b) \leq d_v(\gamma(x), x') + d_v(x', x'_b) \leq H + d_v((\gamma \circ f)(t_2), (\gamma \circ f)(t_1)) \leq H + \lambda^2 c + c$. In this case any $k \geq H + \lambda^2 c + c$ satisfies the conclusion of the lemma.

By taking k to be the maximum of the k values found in case 1 and 2, the lemma is proved. $\square$

We now go on to show that $\hat{\gamma}$ is quasi-conformal. We have two cases: (1) x'_b lies on the geodesic segment between v and x' or (2) x' lies on the geodesic segment between x'_b and v. In the first case $d_v(v, x'_b) = d_v(v, x') - d_v(x', x'_b)$, and in the second case we have $d_v(v, x'_b) = d_v(v, x') + d_v(x', x'_b)$. We note that $d_v(\gamma(x), x') \leq H$ and $d_v(\gamma(x), x'_b) \leq H$. By the triangle inequality in the tree S, $d_v(x', x'_b) \leq 2H$. Recall that $p' = d_v(v, x'_b) = (\hat{\gamma}(f)|\hat{\gamma}(f'))_v$. Therefore $p' - 2H \leq (\hat{\gamma}(f)|\hat{\gamma}(f'))_v \leq p' + 2H$. This implies that $e^{-p'-2H} \leq d_v(\hat{\gamma}(f), \hat{\gamma}(f')) \leq e^{-p'+2H}$, and thus $H^d_{\hat{\gamma}}(f) \leq e^{4H}$, and H is independent of f. Therefore $\hat{\gamma}$ is quasi-conformal. $\square$

The proof of theorem 5.13 is now complete.

The following lemmas will be used in the proofs of theorems in future chapters.

Lemma 5.16. *: Let $\gamma, \beta : (T, w) \to (S, v)$ be (λ, c) and (λ', c')-quasi-isometries respectively which are in the same quasi-isometry class (recall definition 3.9) and*

$\gamma(w) = \beta(w)$. Then $\hat{\gamma} = \hat{\beta}$.

Proof: We need to show that for any $f \in (end(T, w), d_w)$ we have $\hat{\gamma}(f) = \hat{\beta}(f)$, which by lemma 5.4 is equivalent to showing that $\mathcal{H}((\gamma \circ f)_0, (\beta \circ f)_0)$ is finite. Let $t \in [0, \infty)$, then since the Hausdorff distance between $\gamma \circ f$ and $(\gamma \circ f)_0$ is finite, there exists $t' \in [0, \infty)$ such that $d_S((\gamma \circ f)_0(t), (\gamma \circ f)(t')) \leq H$, where H is a number that only depends on λ and c. Similarly, there exists $t'' \in [0, \infty)$ such that $d_S((\beta \circ f)_0(t), (\beta \circ f)(t'')) \leq H_2$, where H_2 is a constant that only depends on λ' and c'. By the triangle inequality in S, $d_S((\gamma \circ f)_0(t), (\beta \circ f)_0(t)) \leq H_1 + H_2$, for any $t \in [0, \infty)$. Hence $\hat{\gamma} = \hat{\beta}$. $\square$

Lemma 5.17. *Let $\phi : (U, d_U) \to (V, d_V)$ be an (α, k)-bi-Hölder homeomorphism between metric spaces of diameter less than or equal to one with at least one accumulation point. Then, $0 < \alpha \leq 1$.*

Proof: If ϕ is an (α, k)-bi-Hölder homeomorphism we must have:

$$\frac{1}{k}\{d_U(x, y)\}^{\frac{1}{\alpha}} \leq d_V(\phi(x), \phi(y)) \leq k\{d_U(x, y)\}^{\alpha}$$

for all $x, y \in U$. Hence $d_U(x, y)^{\frac{1}{\alpha} - \alpha} \leq k^2$. Taking natural logs of both sides of the inequality we have $(\dfrac{1}{\alpha} - \alpha)\ln(d_U(x, y)) \leq 2\ln k$. Since $d_U(x, y) \leq 1$ we have $\ln(d_U(x, y)) < 0$ for all $x, y \in U$. Let x be an accumulation point of U, then there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} d_U(x, y_n) = 0$. Hence, $\lim_{n\to\infty} \ln d_U(x, y_n) = -\infty$. Suppose that $\alpha > 1$, then $\dfrac{1}{\alpha} - \alpha < 0$. Let $L < \dfrac{2\ln k}{\frac{1}{\alpha} - \alpha}$. Therefore we can find $N > 0$ such that $\ln d_U(x, y_N) < L$. This implies that $(\dfrac{1}{\alpha} - \alpha)\ln(d_U(x, y)) > 2\ln k$, which is a contradiction. Hence, $0 < \alpha < 1$. $\square$

CHAPTER VI

THE FUNCTOR $\mathcal{E}$

In this section the categories $\mathcal{T}$ and $\mathcal{U}$ and a functor $\mathcal{E}$ from $\mathcal{T}$ to $\mathcal{U}$ are defined. It is also shown that the functor $\mathcal{E}$ is faithful.

Definition 6.1. Let $\mathcal{T}$ denote the category whose objects are locally finite rooted trees of minimal vertex degree three and morphisms are equivalence classes of quasi-isometries between objects in $\mathcal{T}$ that map roots to roots.

Lemma 6.2. *$\mathcal{T}$ is a category.*

Proof: Let (T, w), (S, v) and (R, x) be objects in $\mathcal{T}$. Let $[\beta]$ and $[\gamma]$ be morphisms in $\mathcal{T}$. Let $\beta \in [\beta]$ and $\gamma \in [\gamma]$ be representatives of these quasi-isometry classes such that $\gamma : (T, w) \to (S, v)$ and $\beta : (S, v) \to (R, x)$. Then $\beta \circ \gamma(w) = \beta(v) = x$. The associativity property of morphisms and the existence of an identity morphism follow from lemma 3.15. Hence, $\mathcal{T}$ is a category.□

Theorem 6.3. *Let (U, d_U), (V, d_V) and (W, d_W) be perfect compact ultrametric spaces. Let $f : (U, d_U) \to (V, d_V)$ and $g : (V, d_V) \to (W, d_W)$ be (α, k)- and (β, l)-bi-Hölder homeomorphisms, respectively. Then $g \circ f : (U, d_U) \to (W, d_W)$ is a $(\beta\alpha, kl)$-bi-Hölder homeomorphism.*

Proof: Let $x, y \in U$, since f is an (α, k)-bi-Hölder homeomorphism we have :

$$\frac{1}{k}\{d_U(x,y)\}^{\frac{1}{\alpha}} \leq d_V(f(x),f(y)) \leq k\{d_U(x,y)\}^{\alpha}$$

Since $f(x), f(y) \in (V, d_V)$ and g is a (β, l)-quasi-conformal bi-Hölder homeomorphism, we have :

$$\frac{1}{l}\{d_V(f(x),f(y))\}^{\frac{1}{\beta}} \leq d_W(g(f(x)),g(f(y))) \leq l\{d_V(f(x),f(y))\}^{\beta}$$

Combining these two inequalities we obtain :

$$\frac{1}{kl}\{d_U(x,y)\}^{\frac{1}{\alpha\beta}} \leq d_W(g(f(x)),g(f(y))) \leq kl\{d_U(x,y)\}^{\alpha\beta}.$$

The above combined with the fact that the composition of two homeomorphisms is a homeomorphism enables us to deduce that $g \circ f$ is a bi-Hölder homeomorphism.$\square$

Definition 6.4. Let $\mathcal{U}$ denote the category whose objects are perfect compact ultrametric spaces and whose morphisms are bi-Hölder homeomorphisms.

Lemma 6.5. *$\mathcal{U}$ is a category.*

Proof: For any object (U, d_U) in $\mathcal{U}$ the identity morphism is the identity map $id : (U, d_U) \to (U, d_U)$ defined by $id(x) = x$ for any $x \in U$. Lemma 6.3 shows that the composition of two morphisms is a morphism. The associativity property of morphisms follows from the associativity property of maps. All the condition stated in definition 3.1 are satisfied, hence $\mathcal{U}$ is a category. $\square$

Definition 6.6. Define $\mathcal{E} : \mathcal{T} \to \mathcal{U}$ in the following manner:

$\mathcal{E}\ ((T, w)) = (end(T, w), d_w)$, i.e., $\mathcal{E}$ takes any rooted tree to the space of its ends, and $\mathcal{E}([\gamma]) = \hat{\gamma}$ where $\hat{\gamma}$ is defined as in theorem 5.13. In other words, $\mathcal{E}$ takes any class of quasi-isometries to the quasi-conformal bi-Hölder homeomorphism induced by any representative of that class.

Theorem 6.7. *$\mathcal{E}$ is a well defined.*

Proof: In section 3 we saw how we could associate to every locally finite rooted tree the space of its ends. In Proposition 4.14 we observed that for any rooted tree which is locally finite and has minimal vertex degree three $(end(T, w), d_w)$ is a perfect compact ultrametric space. The fact that the image of any morphism in $\mathcal{T}$ is a morphism in $\mathcal{U}$ follows from theorem 5.13 and lemma 5.16.□

Definition 6.8. A *functor* is a map between categories that maps objects to objects, and morphisms to morphisms.

Theorem 6.9. *$\mathcal{E}$ is a functor.*

Proof: In Chapter V we saw how we could associate to every locally finite rooted tree the space of its ends. In Propositions 4.12 and 4.14 we observed that for any rooted tree which is locally finite and has minimal vertex degree three $(end(T, w), d_w)$ is a perfect compact ultrametric space.

The fact that $\mathcal{E}([\gamma])$ is a morphism in $\mathcal{U}$ follows from theorem 5.13 and lemma 5.16.

Lets check the functorial properties. In other words, show that:

1) For any object (T, w) in the category $\mathcal{E}$ if $[id]_{(T,w)}$ is the quasi-isometry class of the identity on (T, w), then $\mathcal{E}([id_{(T,w)}])$ is the identity map on $\mathcal{E}((T, w))$.

2) $\mathcal{E}([\gamma]\circ[\beta]) = \mathcal{E}([\gamma])\circ\mathcal{E}([\beta])$ for any morphisms $[\gamma]$ and $[\beta]$ in the category $\mathcal{T}$ when the compositions can be defined.

Proof:

1) Let (T, w) be an object in $\mathcal{T}$. Let $id_{(T,w)} : (T, w) \to (T, w)$ be the identity map from (T, w) to itself. Then, $id_{(T,w)} \in [id_{(T,w)}]$ and can be taken as a representative of this class. Recall that $\mathcal{E}([id_{(T,w)}]) : (end(T, w), d_w) \to (end(T, w), d_w)$ is defined by $\mathcal{E}([id_{(T,w)}])(f) = (id_{(T,w)} \circ f)_0$, where $(id_{(T,w)} \circ f)_0$ and $(f)_0$ are defined as in theorem 5.13, but $(id_{(T,w)} \circ f)_0 = (f)_0 = f$. By lemma 5.16, $\mathcal{E}([id_{(T,w)}])$ is independent of the representative taken in $[id_{(T,w)}]$ and hence $\mathcal{E}([id_{(T,w)}])$ is the identity map on $\mathcal{E}((T, w))$.

2) Let (T, w), (S, v) and (R, x) be objects in $\mathcal{T}$ and let $\gamma : (T, w) \to (S, v)$ and $\beta : (S, v) \to (R, x)$ be (λ, c) and (λ', c') quasi-isometries between them. Denote $\mathcal{E}([\gamma])$ by $\hat{\gamma}$ and $\mathcal{E}([\beta])$ by $\hat{\beta}$. Then, $\hat{\gamma} : (end(T, w), d_w) \to (end(S, v), d_v)$ and $\hat{\beta} : (end(S, v), d_v) \to (end(R, x), d_x)$. Recall that for any $f \in (end(T, w), d_w)$ we define $\hat{\gamma}(f) = (\gamma \circ f)_0$ where $(\gamma \circ f)_0 \in (end(S, v), d_v)$ and $\mathcal{H}((\gamma \circ f)_0, \gamma \circ f) < \infty$. We need to show that $\mathcal{E}([\beta]\circ[\gamma]) = \mathcal{E}([\beta])\circ\mathcal{E}([\gamma])$, but this is equivalent to showing that $\mathcal{E}([\beta\circ\gamma]) = \mathcal{E}([\beta])\circ\mathcal{E}([\gamma])$ by definition 3.11. Thus it suffices to show that for any $f \in (end(T, w), d_w)$ we have $\mathcal{H}((\beta\circ\gamma\circ f)_0, (\beta\circ(\gamma\circ f)_0)_0) < \infty$.

First we show that $(\beta\circ(\gamma\circ f)_0)([0,\infty)) \subseteq \mathcal{V}_{\lambda' M + c'}((\beta\circ\gamma\circ f)([0,\infty)))$, where M is the Hausdorff distance between $\gamma\circ f$ and $(\gamma\circ f)_0$. Let $t \in [0,\infty)$, then there exists $t' \in [0,\infty)$ such that $d_v((\gamma\circ f)_0(t), (\gamma\circ f)(t')) < M$. Since β is a (λ', c')-quasi-isometry, we have

$$\frac{1}{\lambda'}\, d_v((\gamma\circ f)_0(t), (\gamma\circ f)(t')) - c' \le d_x((\beta\circ(\gamma\circ f)_0)(t), (\beta\circ\gamma\circ f)(t'))$$

and

$d_x((\beta \circ (\gamma \circ f)_0)(t), (\beta \circ \gamma \circ f)(t')) \leq \lambda' d_v(\gamma \circ f)_0(t), (\gamma \circ f)(t')) + c'$.

Therefore $\frac{1}{\lambda'}M - c \leq d_x((\beta \circ (\gamma \circ f)_0)(t), (\beta \circ \gamma \circ f)(t')) \leq \lambda' M + c'$, and hence $(\beta \circ (\gamma \circ f)_0)([0, \infty)) \subseteq \mathcal{V}_{\lambda' M + c'}((\beta \circ \gamma \circ f)([0, \infty))$. Similarly, following the same steps as above, $(\beta \circ (\gamma \circ f))([0, \infty)) \subseteq \mathcal{V}_{\lambda' M + c'}((\beta \circ \gamma \circ f)_0)([0, \infty))$. Hence $\mathcal{H}(\beta \circ (\gamma \circ f), \beta \circ (\gamma \circ f)_0) < \infty$. On the other hand, $\mathcal{H}(\beta \circ (\gamma \circ f)_0)_0, \beta \circ (\gamma \circ f)_0) < \infty$. Using these facts and remark 5.2 part (3), $\mathcal{H}(\beta \circ (\gamma \circ f)_0)_0, (\beta \circ \gamma \circ f)_0) < \infty$. Thus by lemma 5.3, $(\beta \circ \gamma \circ f)_0 = (\beta \circ (\gamma \circ f)_0)_0$.□

Definition 6.10. A functor $\mathcal{F}$ is said to be *faithful* if it is injective on morphisms.

Theorem 6.11. *The functor $\mathcal{E}$ is faithful.*

Proof: Let (T, w) and (S, v) be objects in the category $\mathcal{T}$. Let $\gamma \in [\gamma]$ and $\beta \in [\beta]$ be representatives of $[\gamma]$ and $[\beta]$. Let $\gamma : (T, w) \to (S, v)$ and $\beta : (T, w) \to (S, v)$ be (λ, c) and (λ', c') quasi-isometries respectively. We need to show that if $\mathcal{E}([\gamma]) = \mathcal{E}([\beta])$, then $[\gamma] = [\beta]$. In other words, we need to prove that for any $f \in (end(T, w), d_w)$ if $(\gamma \circ f)_0(t) = (\beta \circ f)_0(t)$ for all $t \in [0, \infty)$, then $\sup_{x \in (T,w)} d_v(\gamma(x), \beta(x)) < \infty$, where d_v is the metric on $end(S, v)$.

Let γ and β be as above, and let $x \in$Im(f), and $f \in end(T, w)$. Let $f(t_x) = x$ and $t_x \in [0, \infty)$. Let us further suppose that x is a vertex of the rooted tree (T, w). Then there exists $g \in end(T, w)$ such that for any $t \leq t_x$, $f(t) = g(t)$ and for any $t > t_x$, $f(t) \neq g(t)$. By lemma 5.14, there exists k_1 such that $d_v(\gamma(x), x'_b) \leq k_1$ where x'_b is defined as in lemma 5.14. The same holds for β, i.e. there exists k_2 such that

$d_v(\beta(x), x'_b) \leq k_2$ where k_2 only depends on β. Thus, by the triangle inequality in (S, v) we have $d_v(\gamma(x), \beta(x)) \leq k_1 + k_2$.

Now suppose that x is not a vertex of (T, w). In this case let x' be a vertex of (T, w) such that $d_w(x, x') < 1$ and let $f \in end(T, w)$ have the property that there exists $t, t' \in [0, \infty)$ with $t < t'$ and $f(t) = x$ and $f(t') = x'$.

Then $d_v(\gamma(x), \gamma(x')) \leq \lambda + c$ and similarly $d_v(\beta(x), \beta(x')) \leq \lambda' + c'$and hence $d_v(\gamma(x), \beta(x)) \leq k_1 + k_2 + \lambda + c + \lambda' + c'$. Let $k = k_1 + k_2 + \lambda + c + \lambda' + c'$, k is independent of the choice of x, x', f and g and only depends on γ and β. Hence, $\sup_{x \in (T,w)} d_v(\gamma(x), \beta(x)) \leq k$.□

The following are corollaries of theorem 6.9. In the next chapter we will show that these corollaries are not true in more general cases.

Corollary 6.12. *Let (T, w) and (S, v) be objects in the category $\mathcal{T}$ and let γ be a quasi-isometry between them. Let ϕ be its induced bi-Hölder homeomorphism. Then, ϕ is quasi-conformal.*

Proof: This follows directly from theorem 6.9 and theorem 5.13 . □

Corollary 6.13. *Let (T, w), (S, v) and (R, x) be objects in the category $\mathcal{T}$ and let $\gamma : (T, w) \to (S, v)$ and $\beta : (S, v) \to (R, x)$ be quasi-isometries. Let $\phi : end(T, w) \to end(S, v)$ and $\psi : end(S, v) \to end(R, x)$ be their induced bi-Hölder homeomorphisms respectively. Then, $\psi \circ \phi : end(T, w) \to end(R, x)$ is bi-Hölder and quasi-conformal.*

Proof: By lemma 3.8, $\beta \circ \gamma$ is a quasi-isometry. By theorem 5.13 the map induced by $\beta \circ \gamma$ is a bi-Hölder and quasi-conformal homeomorphism. By theorem 6.9, $\mathcal{E}([\beta \circ \gamma]) =$

$\mathcal{E}([\beta]) \circ \mathcal{E}([\gamma]) = \psi \circ \phi$. Hence $\psi \circ \phi$ is the map induced by $\beta \circ \gamma$, therefore, it is a bi-Hölder and quasi-conformal homeomorphism. $\square$

Corollary 6.14. *Let (T, w) and (S, v) be objects in the category $\mathcal{T}$ and γ be a quasi-isometry between them. Let ϕ be its induced bi-Hölder homeomorphism. Then the inverse of ϕ is a bi-Hölder and quasi-conformal homeomorphism.*

Proof: Let γ^{-1} be the quasi-inverse of γ. Then $[\gamma \circ \gamma^{-1}] = [id_{(S,v)}]$. By theorem 6.9, $\mathcal{E}([id_{(S,v)}] = id_{end(S,v)}$, where $id_{end(S,v)}$ is the identity map on $end(S, v)$. By the same theorem; that is theorem 6.9; $\mathcal{E}([\gamma \circ \gamma^{-1}] = \mathcal{E}[\gamma] \circ \mathcal{E}[\gamma^{-1}]$. Using the fact that $\mathcal{E}([\gamma]) = \phi$ and the above, we conclude that $\mathcal{E}([\gamma^{-1}]) = \phi^{-1}$. By theorem 5.13 the map induced by γ^{-1} is bi-Hölder and quasi-conformal, hence the desired result. $\square$

CHAPTER VII

EXAMPLES AND COUNTEREXAMPLES

When we look at the categories being studied in this text, we notice that there are many conditions imposed both on the objects and the morphisms in these categories. This section will give examples showing that when the conditions are weakened, the conclusion of the main theorem does not hold.

The objects in the category $\mathcal{T}$ are locally finite trees with minimal vertex degree three. The following examples show the necessity of such a condition. We will see examples of quasi-isometries that are not in the same equivalence class, but induce the same bi-Hölder quasi-conformal homeomorphism. Therefore, if we remove the minimal vertex degree three condition from the objects in the category $\mathcal{T}$, the functor $\mathcal{E}$ will not remain faithful.

Example 7.1. *Let (T, w) and γ be the tree and the $(2, 0)$ -quasi-isometry from example 3.3 respectively.*

Let β be the quasi-isometry $\beta : (T, w) \to (T, w)$ defined by

$$\beta(x) = \begin{cases} f(4t), & \text{if } x = f(t) \text{ for some } t \in [0, \infty) \\ g(4t), & \text{if } x = g(t) \text{ for some } t \in [0, \infty) \end{cases} \tag{7.1}$$

The proof that β is a $(4, 0)$-quasi-isometry is analogous to that of γ being a $(2, 0)$-

quasi-isometry in example 3.3.

The definitions of β and γ, and the properties of supremum let us deduce that, $\sup_{x\in(T,w)} d(\gamma(x),\beta(x)) \geq \sup_{t\in[0,\infty)} d(\gamma(f(t)),\beta(f(t)) = \sup_{t\in[0,\infty)} 2t = \infty$, therefore γ and β do not belong to the same isometry class.

Let $\mathcal{E}$ be as in definition 6.6; then, $\mathcal{E}(\gamma)(f) = (\gamma\circ f)_0 = f = \mathcal{E}(\beta)(f)$ and $\mathcal{E}(\gamma)(g) = (\gamma\circ g)_0 = g = \mathcal{E}(\beta)(g)$. Hence $\mathcal{E}$ cannot be a faithful functor.

Example 7.2. *Let (T,w) be a tree for which there exists an isometric embedding $f:[0,\infty)\to(T,w)$ with $f(0)=w$, Furthermore,suppose that there exists $t_1\in(0,\infty)$ such that for any $t>t_1$, if $f(t)$ a vertex of (T,w), the degree of $f(t)$ is 2. In other words, f is an isolated end. We claim that in this case we can find two quasi-isometries γ and β such that $[\gamma]\neq[\beta]$, but $\mathcal{E}(\gamma)=\mathcal{E}(\beta)$ with $\mathcal{E}$ as in definition 6.6. This example shows that $\mathcal{E}$ is not faithful in this case.*

Define $\gamma:(T,w)\to(T,w)$ by

$$\gamma(x) = \begin{cases} x, & \text{if } x\notin Im(f) \text{ or if } x=f(t) \text{ and } t<t_1 \\ f(2t-t_1), & \text{if } x=f(t) \text{ and } t\geq t_1 \end{cases} \tag{7.2}$$

We claim that γ is a quasi-isometry.

Proof of claim: If x and y both satisfy the first condition, then $d(\gamma(x),\gamma(y)) = d(x,y)$.

If $x=f(t_x)$ and $y=f(t_y)$, and $t_x>t_1$ and $t_y>t_1$, then $\gamma(x)=f(2t_x-t_1)$ and $\gamma(y)=f(2t_y-t_1)$, thus $d(\gamma(x),\gamma(y)) = d(f(2t_x-t_1),f(t_y-t_1)) = |2t_x-t_1-2t_y+t_1| = 2|t_x-t_y| = 2d(x,y)$.

If $x \notin Im(f)$ and $y = f(t_y)$ and $t_y > t_1$, then $\gamma(y) = f(2t_y - t_1)$ and $d(\gamma(x), \gamma(y)) = d(x, f(2t_y - t_1)) = d(x, f(t_1)) + d(f(t_1), f(2t_y - t_1))$ because (T, w) is a tree and $f(t_1)$ lies on the geodesic segment between $f(2t_y - t_1)$ and x. On the other hand, f is an isometric embedding hence, $d(f(t_1), f(2t_y - t_1)) = 2t_y - t_1 - t_1 = 2(t_y - t_1)$, and for the same reason we have $2(t_y - t_1) = 2d(f(t_1), f(t_y)) = 2d(f(t_1), y)$. Therefore, $d(\gamma(x), \gamma(y)) = d(x, f(t_1)) + 2d(f(t_1), y) \leq 2(d(x, f(t_1)) + d(f(t_1), y)) = 2d(x, y)$ due to the fact that $f(t_1)$ lies on the geodesic segment between x and y.

If $x = f(t_x)$ and $y = f(t_y)$, and $t_x \leq t_1$ and $t_y > t_1$, then $d(\gamma(x), \gamma(y)) = d(f(t_x), f(t_y - t_1)) = 2t_y - t_1 - t_x \leq 2t_y - 2t_1 = 2d(x, y)$.

Therefore for any $x, y \in (t, w)$ we have $\frac{1}{2}d(x, y) \leq d(\gamma(x), \gamma(y)) \leq 2d(x, y)$.

We also need to show that there exists $D > 0$, such that for any $z \in (T, w)$ there exists $x \in (T, w)$ such that $d(z, \gamma(x)) \leq D$. The claim is that for $D = 2t_1$ this holds. If $z \notin Imf$ take $x = z$, then $d(\gamma(z), z) = 0 < D$. If $z = f(t)$ and if $t > 3t_1$, let $x = f(\frac{1}{2}(t - t_1))$. We have $\frac{1}{2}(t - t_1) > t_1$, and therefore $\gamma(x) = f(2(\frac{1}{2}(t - t_1) - t_1)) = f(t) = z$, thus $d(\gamma(x), z) = 0 < D$. If $t_1 < t < 3t_1$, let $x = f(t_1)$, then $d(z, \gamma(f(t_1)) = d(f(t), f(t_1)) = t - t_1 \leq 3t_1 - t_1 = 2t_1 = D$. Hence γ is a quasi-isometry.

Define $\beta : (T, w) \to (T, w)$ by :

$$\beta(x) = \begin{cases} x, & \text{if } x \notin Im(f) \text{ or if } x = f(t) \text{ and } t < t_1 \\ f(4t - 3t_1), & \text{if } x = f(t) \text{ and } t > t_1 \end{cases} \tag{7.3}$$

We can see, by following the same steps as we did for γ, that β is a (4,0)-quasi-isometry.

$\sup_{x\in(T,w)} d(\gamma(x),\beta(x)) \geq \sup_{t\in[t_1,\infty)} d(\gamma(f(t)),\beta(f(t))) = \sup_{t\in[t_1,\infty)} d(f(4t-3t_1), f(2t - t_1)) = \sup_{t\in[t_1,\infty)} |4t-3t_1-2t+2t_1| = \sup_{t\in[t_1,\infty)} |2t-2t_1| = \infty$, Therefore, $[\gamma] \neq [\beta]$.

On the other hand $\mathcal{E}(\gamma)(g) = g = \mathcal{E}(\beta)(g)$ for any $g \in end(T,w)$. This shows that the functor $\mathcal{E}$ is not faithful in this case.

In the category $\mathcal{U}$ we ask that our morphisms be bi-Hölder homeomorphisms. One may wonder why so many conditions are imposed on the morphisms. In Theorem 6.3 we saw that the composition of two bi-Hölder maps is bi-Hölder. We will now give examples of quasi-conformal homeomorphisms on perfect compact ultrametric spaces for which the composition is not quasi-conformal. In doing so, we give two examples of quasi-conformal homeomorphisms on perfect compact ultrametric spaces, one that is not bi-Hölder, and another one that is bi-Hölder.

The following in an example that shows that the composition of two functions with finite dilation does not necessarily have to have finite dilation. Another purpose of this example is that it will help set up the examples mentioned above.

Example 7.3. Let (T,w) be the tree with $\{f_0, f_{i,1}, f_{i,2} | i = 1,2,3,\ldots; j = 1,2,3,\ldots\}$ its set of ends for which the following relationships hold:

1- $(f_0|f_{i,j}) = 2i$ where $i = 1,2,\ldots$ and $j = 1,2$, implying that, $d_w(f_0, f_{i,j}) = \dfrac{1}{e^{2i}}$.

2- $(f_{i,1}|f_{i,2}) = 2i+1$ where $i = 1,2,\ldots$, implying that $d_w(f_{i,1}, f_{i,2}) = \dfrac{1}{e^{2i+1}}$.

3- $(f_{i,j}|f_{k,l}) = 2i$ if $i < k$ and $i,k = 1,2,\ldots$ and $j,l = 1,2$, implying that, $d_w(f_{i,j}, f_{k,l}) = \dfrac{1}{e^{2i}}$.

Figure 7.1 is a rough sketch of (T, w) and its ends.

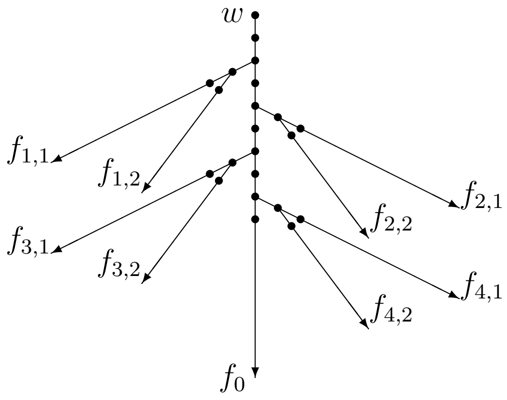


Figure 7.1: The tree (T, w)

Let (S, v) be the tree with ends $\{g_0, g_i | i = 1, 2, \dots\}$ for which the following relationships hold:

1- $(g_0|g_i) = i - 1$ where $i = 1, 2, \dots$ which implies that $d_v(g_0, g_i) = \dfrac{1}{e^i}$.

2- $(g_i|g_j) = i - 1$ with $i < j$ for $i, j = 1, 2, \dots$, hence $d_v(g_i, g_j) = \dfrac{1}{e^{i-1}}$.

Figure 7.2 rough sketch of this tree and its ends.

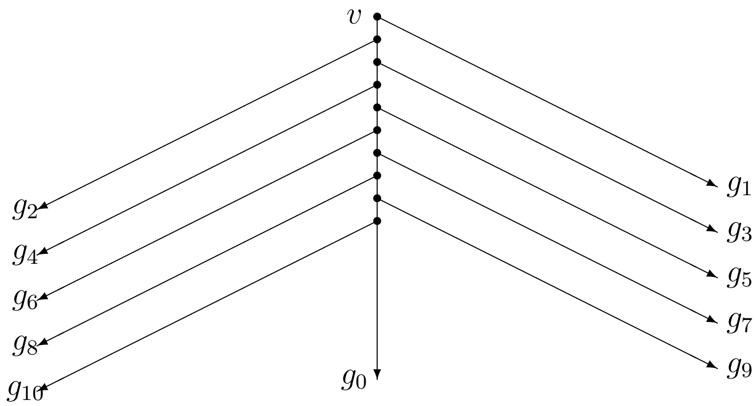


Figure 7.2: The tree (S, v)

Finally let (R, x) be the tree with ends $\{h_0, h_i | i = 1, 2, \dots\}$ for which the following relationships hold:

1- $(h_0|h_{2i}) = 2i$ for $i = 1, 2, \ldots$ and hence, $d_x(h_0, h_{2i}) = \dfrac{1}{e^{2i}}$.

2- $(h_0|h_{2i-1}) = 2i^2 + 1$ for $i = 1, 2, \ldots$ and hence, $d_x(h_0, h_{2i-1}) = \dfrac{1}{e^{2i^2+1}}$.

3- $(h_{2i}|h_{2j}) = 2i$ when $i < j$ for $i, j = 1, 2, \ldots$ and hence $d_x(h_{2i}, h_{2j}) = \dfrac{1}{e^{2i}}$.

4- $(h_{2i-1}|h_{2j-1}) = 2i^2 + 1$ when $i < j$ for $i, j = 1, 2, \ldots$ and hence $d_x(h_{2i-1}, h_{2j-1}) = \dfrac{1}{e^{2i^2+1}}$.

5- $(h_{2i}|h_{2j-1}) = 2i$ when $2i < 2j^2+1$ for $i, j = 1, 2, \ldots$ and hence $d_x(h_{2i}, h_{2j-1}) = \dfrac{1}{e^{2i}}$.

6- $(h_{2i}|h_{2j-1}) = 2j^2+1$ when $2i > 2j^2+1$ for $i, j = 1, 2, \ldots$ and hence $d_x(h_{2i}, h_{2j-1}) = \dfrac{1}{e^{2j^2+1}}$.

The following is a rough sketch of this tree and its ends.

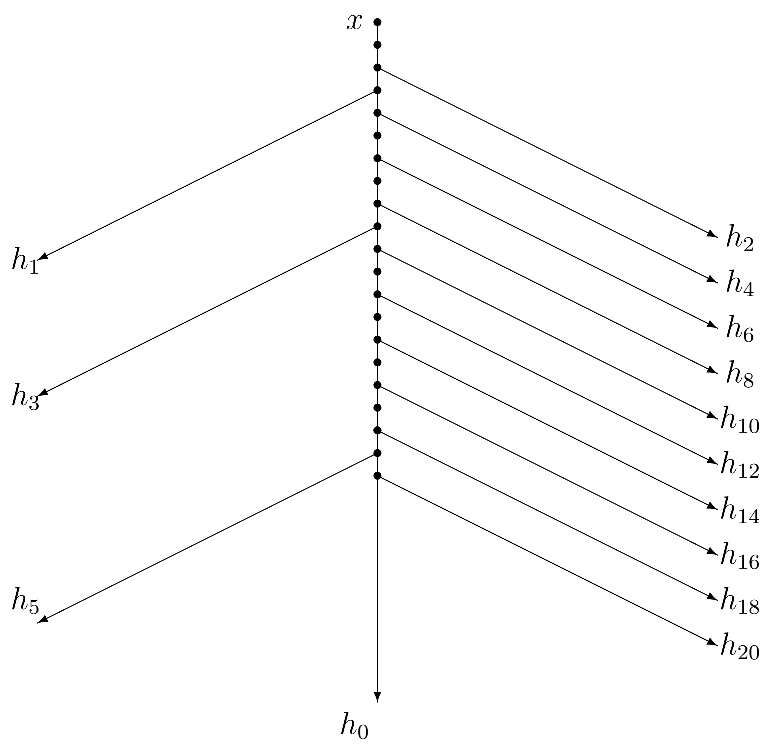


Figure 7.3: The tree(R, x)

We will show that there exist maps $\phi : end(T, w) \to end(S, v)$ and $\psi : end(S, v) \to end(R, x)$ such that $H_{\phi}^{d_w}(f_0)$ and $H_{\psi}^{d_v}(\phi(f_0))$ are finite, but $H_{\psi\circ\phi}^{d_w}(f_0)$ is infinite.

Define $\phi : end(T, w) \to end(S, v)$ as follows:

$\phi(f_0) = g_0$, $\phi(f_{i,1}) = g_{2i-1}$ and $\phi(f_{i,2}) = g_{2i}$.

Let $\mathrm{Dist}(f_0) = \{d_w(f_0, f_{i,j}) | i = 1, 2, \ldots$ and $j = 1, 2\}$. From the above construction of (T, w), $\mathrm{Dist}(f_0) = \{\frac{1}{e^{2k}} | k = 1, 2, \ldots\}$ and this sequence tends to zero.

Let $A_k = \{d_v(\phi f_0, \phi f_{i,j}) | d_w(f_0, f_{i,j}) = \frac{1}{e^{2k}}\}$. For any k, if $d_w(f_0, f_{i,j}) = \varepsilon_k$, then $f_{i,j} = f_{k,1}$ or $f_{i,j} = f_{k,2}$, and since $\phi f_{k,1} = g_{2k-1}$ and $\phi f_{k,2} = g_{2k}$, then $d_v(\phi f_0, \phi f_{k,1}) = d_v(g_0, g_{2k-1}) = \frac{1}{e^{2k-2}}$ and $d_v(\phi f_0, \phi f_{k,2}) = d_v(g_0, g_{2k}) = \frac{1}{e^{2k-1}}$, hence $\frac{\sup A_k}{\inf A_k} = e$, therefore $H_{\phi}^{d_w}(f_0) = e$.

Define $\psi : end(S, v) \to end(R, x)$ by $\psi(g_0) = h_0$ and $\psi(g_i) = h_i$. As above define $\mathrm{Dist}(g_0) = \{d_v(g_0, g_i) | i = 1, 2, \ldots\}$. By the construction of the tree (S, v) we see that $\mathrm{Dist}(g_0) = \{\frac{1}{e^{i-1}} | i = 1, 2, \ldots\}$. There is only one end satisfying the required condition,therefore for each k the set $B_k = \{d_x(\psi(g_0), g_i) | d_v(g_0, g_i) = \varepsilon_k$ and $\varepsilon_i = \frac{1}{e^i}\}$ is a singleton. Therefore $\sup B_k = \inf B_k$ and hence, $H_{\psi}^{d_v}(g_0) = 1$.

The composition $\psi\circ\phi : end(T, w) \to end(R, x)$ has the property that $\psi\circ\phi(f_0) = h_0$ and $\psi \circ \phi(f_{i,1}) = h_{2i-1}$ and $\psi \circ \phi(f_{i,2}) = h_{2i}$ where $i = 1, 2, \ldots$. If $C_k = \{d_x(\psi \circ \phi(f_0), \psi \circ \phi(f_{i,j})) | d_w(f_0, f_{i,j}) = \varepsilon_k\}$, then C_k consists of two points, one is $d_x(\psi \circ \phi(f_0), \psi \circ \phi(f_{k,1})) = \frac{1}{e^{2k^2+1}}$ and the other is $d_x(\psi \circ \phi(f_0), \psi \circ \phi(f_{k,2})) = \frac{1}{e^{2k}}$. Therefore $\frac{\sup C_k}{\inf C_k} = e^{2k+1}$, hence $H_{\psi\circ\phi}^{d_x}(f_0) = \limsup_{k\to\infty} \frac{\sup C_k}{\inf C_k} = \infty$.

This example shows that the composition of two functions that have finite dilation does not necessarily have to have finite dilation.

We use example 7.3 to construct trees (T', w'), (S', v') and (R', x') such that their end spaces are perfect compact ultrametric spaces. We will also construct quasi-conformal homeomorphisms ϕ and ψ on their end spaces such that $\psi \circ \phi$ is not quasi-conformal. Before doing so we need the following well-known definitions. The definitions stated below have been adapted from [Hug] definition 9.1 and 2.12.

Definition 7.4. The Cantor tree C and its end space $end(C)$.

The Cantor tree C, also called the infinite binary tree, is a locally finite, simply connected one dimensional simplicial complex (with the natural length metric d so that every edge is of length 1). It has a root r of valency two (i.e., there exist exactly two edges containing r) and every other vertex is of valency three. If v is a vertex different from r, then two edges that contain v and are separated from r by v are not labelled identically. Each edge is labelled 0 or 1 so that for every vertex v, at least one edge containing v is labelled 0 and at least one is labelled 1.

Let $end(C) = end(C, r)$ since the root r is understood. An element of $end(C)$, being an infinite sequence of successively adjacent edges in C beginning at r, can be labelled uniquely by an infinite sequence of 0's and 1's. Thus,

$$end(C) = \{(x_0, x_1, x_2, \dots)|x_i \in \{0, 1\} \text{ for each } i\}$$

and

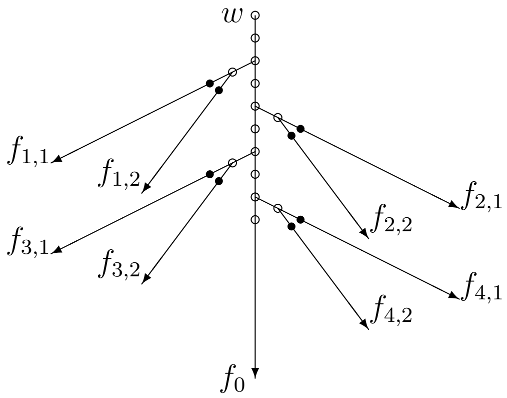
$$d_e((x_i), (y_i)) = \begin{cases} 0, & \text{if } (x_i) = (y_i) \\ \dfrac{1}{e^n}, & \text{if } (x_i) \neq (y_i) \text{and } n = \inf\{i \geq 0 | x_i \neq y_i\} \end{cases} \tag{7.4}$$

Definition 7.5. If c is any point of the rooted $\mathbb{R}$-tree (T, v), the *subtree of* (T, v) *determined by* c is

$$T_c = \{x \in T | c \in [v, x]\}.$$

Note that T_c is indeed a subtree of T (that is to say, as a metric subspace of T, T_c is a tree).

Example 7.6. Let (T, w) be the tree in example 7.3, let $c_{i,j}$ be the vertex on $f_{i,j}$ such that $d(c_{i,j}, f_{i,j}(f_{i,1}|f_{i,2})) = 1$. The figure below shows these as filled in vertices in the tree (T, w).



Define the subtree $T_{c_{i,j}}$ for $i = 1, 2, \dots$ and $j = 1, 2$ as in definition 7.5. Then $T_{c_{i,j}} \cup_{c_{i,j}} [w, c_{i,j}] = f_{i,j}$ where $[w, c_{i,j}] : [0, t] \to (T, w)$ is an isometric embedding with $[w, c_{i,j}](0) = w$ and $[w, c_{i,j}](t) = c_{i,j}$.

Let $X_{i,j}$ be a copy of the Cantor tree for any $i = 1, 2, \dots$ and any $j = 1, 2$. Let $k_{i,j} : T_{c_{i,j}} \to X_{i,j}$ be the isometry that identifies $T_{c_{i,j}}$ with the end in $X_{i,j}$ represented by the sequence $\{0\}_{i=1}^{\infty}$. Define $(T', w') = \dfrac{(T, w) \coprod \bigcup X_{i,j}}{\sim}$, where $\sim$ is the equivalence coming from $k_{i,j}$. The following is a schematic drawing of the tree (T', w').

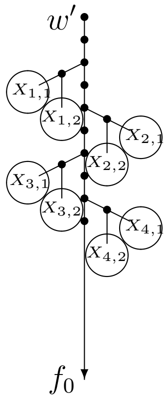


Figure 7.4: The tree (T', w')

By the same construction as above, we can obtain the trees (S', v') and (R', x'), schematic drawings of which are shown in figures 7.5 and 7.6.

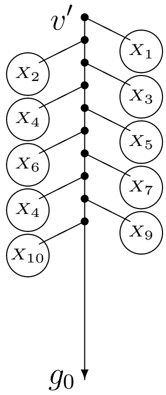


Figure 7.5: The tree (S', v')

Note that in the above constructions we have identified g_i and h_i with the sequence of zeros $\{0\}_{i=1}^{\infty}$ of the Cantor trees X_i and X_i' respectively for $i = 1, 2, \ldots$

Notation:

Let $[w', c_{i,j}] \cup (x_0, x_1, x_2, \ldots)$ denote the end in (T', w') starting at w', identifying $c_{i,j}$ with the root r in the Cantor tree $X_{i,j}$ and other vertices with the sequence $(x_0, x_1, x_2, \ldots)$ of the Cantor tree $X_{i,j}$, for $i \in \{1, 2\}$ and $j = 1, 2, 3, \ldots$. Define

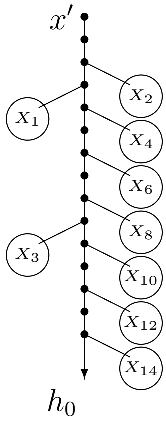


Figure 7.6: The tree (R', x')

$[v', c_i] \cup (x_0, x_1, x_2, \dots)$ and $[x', c_i'] \cup (x_0, x_1, x_2, \dots)$ analogously.

Example 7.7. *The following example is an example of a bi-Hölder quasi-conformal map between perfect compact ultra-metric spaces.*

Let (T', w') and (S', v') be as above. Then, by lemma 4.5 and proposition 4.12, $end(T', w')$ and $end(S', v')$ are perfect compact ultrametric spaces.

Define $\phi : end(T', w') \to end(S', v')$ by

$\phi(f_0) = g_0$, and

$$\phi([w', c_{i,j}] \cup (x_0, x_1, x_2, \dots)) = \begin{cases} [v', c_{2i-1}] \cup (x_0, x_1, x_2, \dots), & \text{if } j = 1 \\ [v', c_{2i}] \cup (x_0, x_1, x_2, \dots), & \text{if } j = 2 \end{cases} \tag{7.5}$$

The map ϕ defined above is a $(\frac{1}{2}, e^3)$-bi-Hölder e-quasi-conformal homeomorphism. This statement is proven is Appendix A by considering the different cases that can occur.

Example 7.8. *This is an example of a quasi-conformal homeomorphism between perfect compact ultra-metric spaces that is not bi-Hölder.*

Let (S', v') and (R', x') be defined as above. By lemma 4.5 and proposition 4.12, $end(R', x')$ is a perfect compact ultrametric space, and by the above example, $end(S', v')$ is also a perfect compact ultrametric space.

Define $\psi : end(S', v') \to (R', x')$ as follows:

$\psi(g_0) = h_0$, and

$$\psi([v', c_i] \cup (x_0, x_1, x_2, \dots)) = [x', c'_i] \cup (x_0, x_1, x_2, \dots) \tag{7.6}$$

The map ψ defined above is a conformal, i.e. 1-quasi-conformal, homeomorphism which is not bi-Hölder. The complete proof of this statement can be found in Appendix A.

Example 7.9. *An example of two quasi-conformal maps on perfect compact ultrametric spaces one of which is also bi-Hölder whose composition is not quasi-conformal.*

Let ϕ, ψ, (T', w'), (S', v') and (R', x') be as in examples 7.8 and 7.7. We will show that although ϕ and ψ are both quasi-conformal, their composition $\psi \circ \phi : end(T', w') \to end(R', x')$ is not.

$\psi \circ \phi$ is defined by $\psi \circ \phi(f_0) = h_0$, and

$$\psi \circ \phi([w', c_{i,j}] \cup (x_0, x_1, x_2, \dots)) = \begin{cases} [x', c'_{2i-1}] \cup (x_0, x_1, x_2, \dots), & \text{if } j = 1 \\ [x', c'_{2i}] \cup (x_0, x_1, x_2, \dots), & \text{if } j = 2 \end{cases} \tag{7.7}$$

.

Let $d_{w'}(f_0, x) = \dfrac{1}{e^{2i}}$. Then $x \in \{[w', c_{i,j}] \cup (x_0, x_1, \dots) | j = 1, 2 \text{ and } x_k \in \{0, 1\}$ for all $k = 0, 1, 2, \dots\}$. Hence $A_i = \{d_{x'}(\psi \circ \phi(f_0), \psi \circ \phi(x)) | d_{w'}(f_0, x) = \dfrac{1}{e^{2i}}\} =$

$\{\frac{1}{e^{2i^2+1}}, \frac{1}{e^{2i}}\}$. Therefore $\sup A_i = \frac{1}{e^{2i}}$ and $\inf A_i = \frac{1}{e^{2i^2+1}}$, thus $\limsup_{i\to\infty} \frac{\sup A_i}{\inf A_i} = \infty$.

Therefore $\psi \circ \phi$ is not quasi-conformal.

CHAPTER VIII

SIMILARITIES AND DIFFERENCES WITH RELATED PAPERS

In the paper titled "Trees and Ultrametric Spaces: A Categorical Equivalence", Hughes shows that there is an equivalence from the category of geodesically complete rooted $\mathbb{R}$-trees and equivalence classes of isometries at infinity, which we will call $\mathcal{T}'$, to the category of complete ultrametric spaces of finite diameter and local similarity equivalences, which we will call $\mathcal{U}'$. Locally finite rooted classical trees of minimal vertex degree three in the category $\mathcal{T}$ are objects in $\mathcal{T}'$, and perfect compact ultrametric spaces in the category $\mathcal{U}$ are objects in the category $\mathcal{U}'$. One may wonder about the relationship between the morphisms in these categories. We will show that any morphism in the category $\mathcal{T}'$, when defined on an object in the category $\mathcal{T}$, is a morphism in the category $\mathcal{T}$. We will also investigate the relationship between the morphisms in $\mathcal{U}$ and $\mathcal{U}'$.

The following definitions are quoted from [Hug].

Definition 8.1. A *cut set* C for a geodesically complete, rooted $\mathbb{R}$-tree (T, v) is a subset C of T such that $v \notin C$ and for every isometric embedding $\alpha : [0, \infty) \to T$ with $\alpha(0) = v$ there exists a unique $t_0 > 0$ such that $\alpha(t_0) \in C$.

In other words, to go to infinity from v you must pass through a unique point of C.

Definition 8.2. Let (T, w) and (S, v) be geodesically complete, rooted $\mathbb{R}$-trees. An *isometry at infinity* from (T, w) to (S, v) is a triple (f, C_T, C_S) where C_T and C_S are cut sets of T and S respectively, and $f : \cup\{T_C | c \in C_T\} \to \cup\{S_C | c \in C_S\}$ is a homeomorphism such that

(1) $f(C_T) = C_S$, and

(2) for every $c \in C_T$, $f| : T_c \to S_{f(c)}$ is an isometry.

Theorem 8.3. *Let (T, w) and (S, v) be locally finite rooted classical trees of minimal vertex degree three, and let (f, C_T, C_S) be an isometry at infinity between them. Then f is a quasi-isometry.*

Proof: The proof of the theorem is in two steps:

Step 1: We prove that C_T is finite.

Step 2: By considering the different cases that can arise for $x, y \in T$, we show that f is a quasi-isometry.

Step 1

Let $\alpha_0 \in \text{end}(T, w)$. Then there exists $t_0 \in [0, \infty)$ such that $\alpha_0(t_0) \in C_T$.

<u>Claim:</u> For all $\beta \in B(\alpha_0, e^{-t_0})$ we have $\beta(t_0) \in C_T$.

Proof of claim: Let $\beta \in B(\alpha_0, e^{-t_0})$ then, $d_{end(T,w)}(\alpha_0, \beta) \le e^{-t_0}$. Recall that $d_{end(T,w)}(\alpha_0, \beta) = e^{-(\alpha_0|\beta)}$ and that $(\alpha_0|\beta) = \sup\{t \in [0, \infty) | \alpha_0(t) = \beta(t)\}$. Therefore, we have $e^{-(\alpha_0|\beta)} \le e^{-t_0}$, hence $-(\alpha_0|\beta) \le -t_0$ and thus, $(\alpha_0|\beta) \ge t_0$. Hence we have $\beta(t_0) = \alpha(t_0)$, so $\beta(t_0) \in C_T$.$\square$

If for all $\alpha \in \text{end}(T, w)$, $\alpha \in B(\alpha_0, e^{-t_0})$, we are done because $C_T = \{\alpha(t_0)\}$ and is finite. If not there exists $\alpha_1 \in \text{end}(T, w)$ such that $\alpha_1 \notin B(\alpha_0, e^{-t_0})$, and there exists $t_1 \in [0, \infty)$ such that $\alpha_1(t_1) \in C_T$. Consider the open ball $B(\alpha_1, e^{-t_1})$.

If $\mathrm{end}(T,w) \subseteq B(\alpha_0, e^{-t_0}) \cup B(\alpha_1, e^{-t_1})$, then $C_T = \{\alpha_0(t_0), \alpha_1(t_1)\}$. Otherwise, we continue to find α_2. We claim that this process will stop after finitely many steps. Suppose not, then there exist infinitely many distinct isometric embeddings α_i such that $\mathrm{end}(T,w) \subseteq \cup_{i=1}^{\infty} B(\alpha_i, e^{-t_i})$ and no finite subset of these open balls covers $\mathrm{end}(T,w)$. This is contradictory to proposition 4.12 which states that $end(T,w)$ is compact. Observe that the number of points in C_T is equal to the number of distinct balls that cover $\mathrm{end}(T,w)$. Therefore C_T is finite.

Note that $C_S = \{f(c) | c \in C_T\}$. Let $D_T = \max\{d_{(T,w)}(w,c) | c \in C_T\}$ and let $D_S = \max\{d_{(S,v)}(v, f(c)) | f(c) \in C_S\}$. We will show that f is a $(1, 2D_T + 2D_S)$-quasi-isometry. Let $x, y \in (T,w)$.

Case 1: Suppose that there exists $c \in C_T$ such that $x, y \in T_C$. Then, $f(x), f(y) \in T_{f(c)}$ and $f| : T_c \to T_{f(c)}$ is an isometry, therefore $d_v(f(x), f(y)) = d_w(x,y)$ and hence f is a $(1, 2D_T + 2D_S)$-quasi-isometry.

Case 2: There does not exist any $c \in C_T$ such that $x \in T_C$ or any $c \in C_T$ such that $y \in C_T$. This implies that $f(x) \notin T_{f(c)}$ for any $f(c) \in C_S$ and $f(y) \notin T_{f(c)}$ for any $f(c) \in C_S$. In this case $d_{(T,w)}(w,x) < d_{(T,w)}(w,c)$ for all $c \in C_T$, and the same holds for y, in other words, $d_{(T,w)}(w,y) < d_{(T,w)}(w,c)$ for all $c \in C_T$. Hence $d_{(T,w)}(x,y) < 2D_T$. Similarly, we can see that $d_{(S,v)}(x,y) < 2D_S$. In this case

$$d_w(x,y) - 2D_S - 2D_T \leq d_v(f(x), f(y)) \leq d_w(x,y) + 2D_S + 2D_T.$$

Hence, f is a $(1, 2D_S + 2D_T)$-quasi-isometry.

Case 3: There exists $c \in C_T$ such that $x \in T_c$ and there exists $c' \in C_T$ with $c' \neq c$ such that $y \in T_{c'}$. Therefore $f(x) \in T_{f(c)}$ and $f(y) \in T_{f(c')}$. Let $\alpha : [0, \infty) \to (T,w)$

and

$\beta : [0, \infty) \to (T, w)$ be isometric embeddings such that $x \in \mathrm{Im}(\alpha)$ and $y \in \mathrm{Im}(\beta)$. Then $d_w(x, y) = d_w(x, c) + d_w(c, \alpha((\alpha|\beta))) + d_w(\alpha((\alpha|\beta)), c') + d_w(c', y)$. On the other hand, $d_v(f(x), f(y)) = d_v(f(x), f(c)) + d_v(f(c), f(c')) + d_v(f(c'), f(y))$. Since $f| : T_c \to T_{f(c)}$ is an isometry, we have $d_v(f(x), f(c)) = d_w(x, c)$, and similarly, $f| : T_{c'} \to T_{f(c')}$ is an isometry, therefore $d_v(f(c'), f(y)) = d_w(c', y)$. Hence

$$\begin{aligned}
d_v(f(x), f(y)) &= d_w(x, c) + d_v(f(c), f(c')) + d_w(c', y) \\
&\leq d_{(T,w)}(x, c) + d_v(v, f(c)) + d_v(v, f(c')) + d_w(c', y) \\
&\leq d_w(x, c) + 2D_S + d_w(c', y) \\
&\leq d_w(x, c) + 2D_S + d_w(c', y) + d_w(c, \alpha((\alpha|\beta))) + d_w(\alpha((\alpha|\beta)), c') \\
&\leq d_w(x, y) + 2D_S + 2D_T.
\end{aligned}$$

We will now go on to show that $d_w(x, y) - 2D_S - 2D_T \leq d_v(f(x), f(y))$. The first thing to note is that $d_w(x, w) \geq d_w(v, c)$ and $d_w(y, w) \geq d_w(v, c')$ due to the fact that $x \in T_c$ and $y \in T_{c'}$. On the other hand, $D_T \geq d_w(w, c)$ which implies that $d_w(x, w) - D_T \leq d_w(x, w) - d_w(w, c) = d_w(x, c) = d_v(f(x), f(c))$. Similarly, $d_w(y, w) - D_T \leq d_v(f(y), f(c'))$. Therefore,

$d_w(x, w) + d_w(w, y) - 2D_T \leq d_v(f(x), f(c)) + d_v(f(c'), f(y))$. We also have

$d_w(x, \alpha((\alpha|\beta))) \leq d_w(x, w)$ and

$d_w(y, \alpha((\alpha|\beta))) \leq d_w(y, w)$ and

$d_w(x, \alpha((\alpha|\beta))) + d_w(y, \alpha((\alpha|\beta))) = d_w(x, y)$,

thus $d_w(x, y) - 2D_T \leq d_v(f(x), f(y))$. This implies

$d_w(x, y) - 2D_S - 2D_T \leq d_v(f(x), f(y))$. In this case

$d_w(x, y) - 2D_S - 2D_T \leq d_v(f(x), f(y)) \leq d_w(x, y) + 2D_S + 2D_T$. So f is a $(1, 2D_S +$

$2D_T$)- quasi-isometry.

Case 4: Suppose that $x \notin T_c$ for any $c \in C_T$ and there exists $c' \in C_T$ such that $y \in T_{c'}$. This implies that $f(x) \notin T_{f(c)}$ for any $f(c) \in C_S$ and $f(y) \in C_{f(c')}$. We have

$$\begin{aligned} d_v(f(x), f(y)) &\leq d_v(f(x), v) + d_v(v, f(c')) + d_v(f(c'), f(y)) \\ &\leq 2D_S + d_v(f(c), f(y)) \\ &= 2D_S + d_w(c, y) \\ &\leq 2D_S + d_w(x, y) \\ &\leq d_w(x, y) + 2D_S + 2D_T. \end{aligned}$$

For the left hand side of the inequality we use the fact that

$d_w(y, \alpha((\alpha|\beta))) - D_T \leq d_w(y, w) - D_T \leq d_v(f(c'), f(y))$. We also have $d_w(x, w) - D_T \leq 0$. Adding the two inequalities we obtain:

$d_w(x, y) - 2D_T \leq d_v(f(x), f(y)) \leq d_w(x, y) + 2D_S$. By adding D_S to the right hand side and subtracting D_T from the left hand side we have

$d_w(x, y) - 2D_S - 2D_T \leq d_v(f(x), f(y)) \leq d_w(x, y) + 2D_S + 2D_T$. So f is a $(1, 2D_S + 2D_T)$-quasi-isometry.

The above four cases are all the cases that can occur, hence the conclusion of the theorem.$\square$

In [Hug] and the current paper the morphisms in the categories under study are equivalence classes. In what follows we give the definition of an equivalence class of an isometry at infinity and go on to show that if $[(f, C_T, C_S)] = [(g, C'_T, C'_S)]$ as equivalence classes of isometries at infinity, then $[f] = [g]$ as equivalence classes of quasi-isometries.

Definition 8.4. If C and C' are cut sets for (T, w), then C' is *larger than* C if for every $c \in C$, $[v, c] \cap C' \subseteq \{c\}$. C' is *strictly larger* than C if for every $c \in C$, $[v, c] \cap C' = \emptyset$.

Definition 8.5. Two isometries at infinity (f, C_T, C_S) and (f, C'_T, C'_S) from (T, w) to (S, v) are said to be *equivalent* if there exists a cut set C''_T for (T, w) larger than C_T and C'_T such that for every $c \in C''_T$:

(1) if T_c is not an isolated ray, then $f|T_c = f'|T_c$,

(2) if T_c is an isolated ray, then $f(T_c) \cap f'(T_c) \neq \emptyset$.

The category $\mathcal{T}$ is only concerned with trees of minimal vertex degree at least equal to three, and hence isolated rays do not exist in these trees. Therefore, we do not need to concern ourselves with case (2).

Lemma 8.6. *If two isometries at infinity (f, C_T, C_S) and (f, C'_T, C'_S) from (T, w) to (S, v) are in the same equivalence class as isometries at infinity, then f and f' are in the same equivalence class as quasi-isometries.*

Proof: Let C''_T be a cut set that is larger than C_T and C'_T.

Let $D_S = \max\{d_v(f(w), f(c)) | c \in C''_T\}$. In order to prove that f and f' are in the same equivalence class as quasi-isometries we need to show that $\sup_{x \in X} d_v((f(x), f'(x))$ is finite. Let $x \in (T, w)$.

If $x \in T_c$ for some $c \in C''_T$. Then by (1) of definition 8.5 we have $f|T_c = f'|T_c$ and hence, $f(x) = f'(x)$, therefore $d_v(f(x), f'(x)) = 0$.

Suppose that $x \notin T_c$ for any $c \in C''_T$. Then $f(x) \notin T_{f(c)}$ for any $c \in C''_T$. This implies that $d_v(f(x), w) \leq D_S$ and $d_v(f'(x), w) \leq D_S$. By the triangle inequality in the metric space (S, v) we have $d_v(f(x), f'(x)) \leq d_v(f(x), w) + d_v(f'(x), w)$ and hence, $d_v(f(x), f'(x)) \leq 2D_S$.

The two cases above are the only cases that can occur, therefore

$\sup_{x \in X} d((f(x), f'(x)) \leq 2D_S$ which is finite, thus the conclusion of the lemma. □

Corollary 8.7. *Let (T, v) and (S, w) be objects in the category $\mathcal{T}$ and let $[(f, C_T, C_S)]$ be a morphism in $\mathcal{T}'$ between them. Then $[f]$ is a morphism in $\mathcal{T}$.*

We will now go on to talk about the morphisms in the categories $\mathcal{U}$ and $\mathcal{U}'$. Before doing so, we need to lay the foundation by giving relevant definitions and lemmas from [Hug].

Definition 8.8. A function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a *similarity* if there exists $\lambda > 0$ such that $d_Y(f(x), f(y)) = \lambda d_X(x, y)$ for all $x, y \in X$. In this case λ is the *similarity constant* of f, and f is a λ-*similarity.* A *similarity equivalence* is a similarity that is also a homeomorphism.

Definition 8.9. A homeomorphism between metric spaces is a *local similarity equivalence*, if for every $x \in X$ there exists $\epsilon > 0$ and $\lambda > 0$ such that the restriction $h| : B(x, \epsilon) \to B(h(x), \lambda\epsilon)$ is a surjective λ-similarity.

Notation: Let $||x|| = d(x, r)$ for any tree T with root r and metric d.

Proposition 8.10. *(Proposition 5.6 of [Hug]) Let $(f, C_T, C_S) : (T, v) \to (S, w)$ be an isometry at infinity between geodesically complete, rooted $\mathbb{R}$-trees. Then there is* *an induced local similarity equivalence* $f_* : end(T,v) \to end(S,w)$. *Moreover, if* $(g, C'_T, C'S)$ *is another such isometry at infinity and* $[f] = [g]$, *then* $f_* = g_*$.

The complete proof can be found in [Hug]. In the proof he defines the induced map of f which he calls f_* as follows:

In order to define f_*, let $\alpha : [0,\infty) \to T$ be an element of $end(T,v)$. Since C_T is a cut set, there exists a unique $t_0 > 0$ such that $\alpha(t_0) \in C_T$. Moreover, $\alpha([t_0,\infty)) \subseteq T_{\alpha(t_0)}$. Let $\hat{\alpha} : [0, ||f\alpha(t_0)||] \to S$ be the unique isometric embedding such that $\hat{\alpha}(0) = w$ and $\hat{\alpha}(||f\alpha(t_0)||) = f\alpha(t_0)$. Define

$$f_*(\alpha)(t) = \begin{cases} \hat{\alpha}(t), & \text{if } 0 \le t \le ||f\alpha(t_0)|| \\ f\alpha(t - ||f\alpha(t_0)|| + t_0), & \text{if } ||f\alpha(t_0)|| \le t \end{cases} \tag{8.1}$$

Recall that the map induced by the quasi-isometry f, is $\hat{f} : end(T,v) \to end(S,w)$ and is defined as follows: for $\alpha : [0,\infty) \to T$ which is an element of $end(T,v)$, $\hat{f}(\alpha) = (\hat{f} \circ \alpha)_0$, where $(\hat{f} \circ \alpha)_0 \in end(S,w)$ is the unique isometric embedding which is a finite distance away from $\hat{f} \circ \alpha$. If we show that $\mathcal{H}(f_*\alpha, \hat{f} \circ \alpha)$ is finite, then we can conclude that $f_* = \hat{f}$.

Let us use the notations in lemma 8.6. If $0 \le t < ||f\alpha(t_0)||$, then $\alpha(t) \notin T_c$ for any $c \in C_T$, therefore $d_w(w, f \circ \alpha(t)) \le D_S$. On the other hand, $d_w(\hat{\alpha}(t), w) \le D_S$ and thus, $d_w(\hat{\alpha}(t), f \circ \alpha(t)) \le 2D_S$. By lemma 8.6, f is a $(1, 2D_T + 2D_S)$-quasi-isometry. If $||f\alpha(t_0)|| \le t$, then $d_w(f_*(\alpha(t)), \hat{f}(\alpha(t))) = d_w(f\alpha(t - ||f\alpha(t_0) + t_0||), f\alpha(t)) \le d_w(\alpha(t), \alpha(t - ||f\alpha(t_0) + t_0||) + 2D_T + 2D_S = |||f\alpha(t_0) - t_0|| + 2D_T + 2D_S||$. This is a fixed finite value, therefore, $\mathcal{H}(f_*\alpha, \hat{f} \circ \alpha)$ is finite. Hence, $f_* = \hat{f}$. We can summarize our results in the following theorem.

Theorem 8.11. *Let $(f, c_T, C_S) : (T, v) \to (S, w)$ be an isometry at infinity between objects in $\mathcal{T}$. Then the induced local similarity equivalence $f_* : end(T, v) \to end(S, w)$ is a bi-Hölder quasi-conformal homeomorphism.*

The following is an example of a quasi-isometry that is not an isometry at infinity.

Example 8.12. Let (T, w) be the tree defined as follows:

The root of T is w and has valency three, i.e., there exist exactly three edges containing w, and every other vertex is of valency four. If v is a vertex different from w, then the three edges that contain v, and are separated from w by v, are not labelled identically. Each edge is labelled 0, 1 or 2 so that for every vertex v, at least one edge containing v is labelled 0, at least one edge is labelled 1, and at least one is labelled 2.

An element of $end(T, w)$, being an infinite sequence of successive adjacent edges in (T, w) beginning at w, can be labelled uniquely by an infinite sequence of $0's$, $1's$ and $2's$. Thus $end(T, w) = \{(x_0, x_1, x_2, \dots)|x_i \in \{0, 1, 2\} \text{ for each } i\}$.

Let x be a vertex in (T, w). Then $d(w, x) \in \mathbb{Z}$. Let $d(w, x) = n_x$. There exists $f \in end(T, w)$ such that $x \in \mathrm{Im}(f)$. Suppose that $f = (x_0, x_1, x_2, \dots, x_{n_x}, x_{n_x+1}, \dots)$, then $x = f(x_{n_x})$. Denote the vertex x by $(x_0, x_1, x_2, \dots, x_{n_x})$. This notation is unique, for if $g \in end(T, w)$ is another end such that $x \in \mathrm{Im}(g)$, then we must have $f(t) = g(t)$ for all $0 \leq t \leq n_x$, which means that the first n_x terms in the sequence representing g is the same as that which represents f.

Define $\gamma : (T, w) \to (T, w)$ in the following manner. If $x = (x_0, x_1, x_2, \dots, x_{n_x})$, then $\gamma(x) = (x_0, x_1, x_2, \dots, x_{n_x}, x_{n_x}, x_{n_x})$. If x is not a vertex of (T, w), then there exists a vertex $v = (x_0, x_1, x_2, \dots, x_{n_x})$ of (T, w) such that x belongs to the geodesic segment $[w, v]$ and $d(x, v) < 1$. Thus, if $v \in \mathrm{Im}(f)$ for some $f \in end(T, w)$, then $x \in \mathrm{Im}(f)$. Let $\hat{f} \in end(T, w)$ be such that $\gamma(v) = (x_0, x_1, x_2, \ldots, x_{n_x}, x_{n_x}, x_{n_x}) \in \mathrm{Im}(\hat{f})$. Define $\gamma(x) = \hat{f}(n_x + 2 + d(x, v))$.

The claim is that γ is a $(1, 4)$-quasi-isometry. Let $x, y \in (T, w)$ be vertices in (T, w), and let $x = (x_0, x_1, x_2, \ldots, x_{n_x})$, $y = (y_0, y_1, y_2, \ldots, y_{n_y})$, let $n = \max\{i \geq 0 | x_i = y_i\}$, then $d(x, y) = n_x + n_y - n$. From the definition of γ, $\gamma(x) = (x_0, x_1, x_2, \ldots, x_{n_x}, x_{n_x}, x_{n_x}$ and $\gamma(y) = (y_0, y_1, y_2, \ldots, y_{n_y}, y_{n_y}, y_{n_y})$ thus, $d(\gamma(x), \gamma(y)) = (n_x + 2) + (n_y + 2) - n = d(x, y) + 4$. If x and y are not vertices of (T, w), and v_x and v_y are as above, then $d(x, y) \leq d(x, y) + 4$.

We also need to show that there exists a constant $C \geq 0$ such that every point of (T, w) lies in the C neighborhood of the image of γ. $C = 4$ satisfies the condition. Let $x \in (T, w)$. Then $x \in \mathrm{Im}(\gamma)$. If $x = (x_0, x_1, x_2, \ldots, x_{n_x})$ is a vertex of (T, w), then $\gamma(x) = (x_0, x_1, x_2, \ldots, x_{n_x}, x_{n_x}, x_{n_x})$ and we have $d(x, \gamma(x)) = 2$. Suppose that x is not a vertex in (T, w), and v is the vertex in (T, w) such that $x \in [w, v]$ and $d(x, v) < 1$. Then, $d(x, \gamma(x)) \leq d(x, v) + d(v, \gamma(v)) + d(\gamma(v), \gamma(x)) = 2d(x, v) + d(v, \gamma(v)) \leq 4$. Hence γ is a $(1, 4)$-quasi-isometry.

We will show that for any cut set C_T of (T, w), γ is not an isometry at infinity. Let C_T be an arbitrary cut set, and let $c = (c_0, c_1, \ldots, c_n) \in C_T$. If $x \in T_C$ is a vertex of C_T, then $x = (c_0, c_1, \ldots, c_n, x_{n+1}, \ldots, x_{n_x})$. Therefore the vertex $x = (c_0, c_1, \ldots, c_n, x_{n+1})$ belongs to C_T, where $x_{n+1} \neq c_n$. The image of T_c under γ is $T_{\gamma(c)}$ where $\gamma(c) = (c_0, c_1, \ldots, c_n, c_n, c_n)$. But $\gamma(x) = (c_0, c_1, \ldots, c_n, x_{n+1}, x_{n+1}, x_{n+1})$ which does not belong to $T_{\gamma(c)}$, which means that γ cannot be an isometry at infinity.

□

The following is an example of a bi-Hölder quasi-conformal homeomorphism that is not a local similarity equivalence.

Example 8.13. Let (T', w'), (S', v') and ϕ be as in example 7.7. We have seen that $\phi : end(T', w') \to (S', v')$ is a $\left(\frac{1}{2}, e^3\right)$-bi-Hölder e-quasi-conformal homeomorphism. We claim that ϕ is not a local similarity equivalence. To prove this claim, we must show that there exists $x \in end(T', w')$ such that for any and $\lambda > 0$ the restriction $\phi| : B(x, \varepsilon) \to B(\phi(x), \lambda\varepsilon)$ is not a surjective λ-similarity. In other words, for any $\varepsilon > 0$ we can find $y \in B(x, \varepsilon)$ such that $d_{v'}(\phi(x), \phi(y)) \neq \lambda d_{w'}(x, y)$ for any $\lambda > 0$.

Let $x = f_0$ and $\varepsilon > 0$ be a given arbitrary value. There exists $i > 0$ such that $\frac{1}{e^{2i}} < \varepsilon$. Let $y = [w', c_{i,1}] \cup (1, 1, 1, \dots)$. Then by case (I) in appendix A, we have $d_{w'}(x, y) = \frac{1}{e^{2i}}$ and $d_{v'}(\phi(x), \phi(y)) = \frac{1}{e^{2i-1}}$. Hence, $d_{v'}(\phi(x), \phi(y)) = ed_{w'}(x, y)$. In order for ϕ to be a local similarity equivalence we must have $\lambda = e$. We will show that this value of ϕ does not work for all $z \in B(x, \varepsilon)$. To see this let $z = [w', c_{i,2}] \cup (1, 1, 1, \dots)$. Case (II) of appendix A shows that $d_{w'}(x, z) = \frac{1}{e^{2i}}$ and $d_{v'}(\phi(x), \phi(z)) = \frac{1}{e^{2i-2}}$, and therefore $d_{v'}(\phi(x), \phi(y)) = e^2 d_{w'}(x, y) \neq \lambda d_{w'}(x, y)$. Therefore, ϕ is not a local similarity equivalence.

At this point, we focus our attention to the paper titled "Embedding of Gromov Hyperbolic Spaces" by M. Bonk and O. Schramm. The above mentioned paper is of interest because of its similarities to the current paper. In [BS] a functor is constructed from the category C_1 whose objects are Gromov k-almost geodesic metric spaces, and whose morphisms are quasi-isometries to the category D_1 whose objects are bounded and complete B-structures and morphisms are power quasi-symmetries. Concepts and definitions are given in the following paragraphs. It is shown that although the objects studied in the categories in C_1 and D_1 of [BS] contain the objects in the categories $\mathcal{T}$ and $\mathcal{U}$ of the current paper, the morphisms in D_1 are morphisms in the category $\mathcal{U}$.

Notation: Let X be a metric space, and let $x, y \in X$. Denote the geodesic segment between x and y by $[x, y]$.

Definition 8.14. A metric space X is said to be *k-almost geodesic*, if for every $x, y \in X$ and every $t \in [0, |x-y|]$, there is some $z \in X$ with $||x-z| - t| \leq k$ and $||y-z| - (|x-y| - t)| \leq k$, where $|x-y|$ denotes the distance from x to y in the space X.

Let X be an object in the category $\mathcal{T}$, and let $x, \in X$. The geodesic segment $[x, y]$ between x and y is connected, and hence, there exists z on the geodesic segment $[x, y]$ such that $d(x, z) = t$. Hence X is 0-almost geodesic. The choice of X was arbitrary, therefore we conclude that every object in the category $\mathcal{T}$ is 0-almost geodesic.

Definition 8.15. Let X and Y be two metric spaces with metrics d_1 and d_2 respectively, $f : X \to Y$ be a bijection and $\alpha > 0$, $\lambda \geq 1$ The map f is an (α, λ)-power quasi-symmetry if for all distinct points $x, y, z \in X$

$$\frac{d_2(f(x), f(z))}{d_2(f(x), f(y))} \leq \eta_{\alpha,\lambda} \left(\frac{d_1(x, z)}{d_1(x, y)} \right). \tag{8.2}$$

Here

$$\eta_{\alpha,\lambda} = \begin{cases} \lambda t^{1/\alpha}, & \text{for } 0 < t < 1 \\ \lambda t^{\alpha} & \text{for } 1 \leq t. \end{cases}$$

Note that any power quasi-symmetry is a homeomorphism. Let $x \in X$ be a given arbitrary point, and let $y \in X$ be a fixed point in X not equal to x. Given $\varepsilon > 0$ we can let $\delta < \left(\dfrac{\varepsilon}{\lambda d_2(f(x) - f(y))}\right)^{\alpha} d_1(x - y)$.

Lemma 8.16. *Let (T, w) and (S, v) be rooted classical trees, and let end(T, w) and end(S, v) be their end spaces. Let ϕ : end$(T, w) \to (S, v)$ be an (α, λ)-power quasi-symmetry. Then ϕ is a bi-Hölder homeomorphism.*

Proof: We need to show that given $f, g \in end(T, w)$, there exists $\beta > 0$and $\xi \geq 1$ such that:

$$\frac{1}{\xi} d_1(f, g)^{\frac{1}{\beta}} \leq d_2(\phi(f), \phi(g)) \leq \xi d_1(f, g)^{\beta} \tag{8.3}$$

where d_1 and d_2 are the metrics in $end(T, w)$ and $end(S, v)$ respectively .

Let $f \in end(T, w)$ be an arbitrary point which is fixed from now on. Let $h \in end(T, w)$ be such that $d_2(\phi(f), \phi(h)) = 1$, such an end exists as seen in the proof of lemma 4.14. Since ϕ is a homeomorphism on compact metric spaces it is uniformly continuous, hence for $\varepsilon = 1$ there exists $\delta > 0$, independent of the choice f' and h', such that if $d_1(f', h') < \delta$, then $d_2(\phi(f'), \phi(h')) < \varepsilon$. Since $d_2(\phi(f), \phi(h)) = 1$, then $d_1(f, h) > \delta$. The diameter of $end(T, w)$ is one by proposition 4.14, hence $\delta \leq d(f, h) \leq 1$. Using the fact that ϕ is an (α, λ)-power quasi-symmetry the following inequality holds:

$$\frac{d_2(\phi(f), \phi(g))}{d_2(\phi(f), \phi(h))} \leq \eta_{\alpha,\lambda} \left(\frac{d_1(f, g)}{d_1(f, h)}\right).$$

Suppose that $0 < \dfrac{d_1(f, g)}{d_1(f, h)} < 1$, by equation 8.2 and the hypothesis that

$d_2(\phi(f), \phi(h)) = 1$,

$$d_2(\phi(f), \phi(g)) \leq \lambda \left(\frac{d_1(f,g)}{d_1(f,h)} \right)^{\frac{1}{\alpha}} \leq \frac{\lambda}{\delta^{\frac{1}{\alpha}}} \left(d_1(f,g) \right)^{\frac{1}{\alpha}}$$

.

Similarly,

$$\frac{d_2(\phi(f), \phi(h))}{d_2(\phi(f), \phi(g))} \leq \eta_{\alpha,\lambda} \left(\frac{d_1(f,h)}{d_1(f,g)} \right).$$

and $\frac{d_1(f,h)}{d_1(f,g)} \geq 1$, hence

$$d_2(\phi(f), \phi(h)) \geq \frac{1}{\lambda} (d_1(f,g))^{\alpha}$$

.

Combining the above relationships we obtain:

$$\frac{1}{\lambda} d_1(f,g)^{\alpha} \leq d_2(\phi(f), \phi(g)) \leq \frac{\lambda}{\delta^{\frac{1}{\alpha}}} d_1(f,g)^{\frac{1}{\alpha}}$$

A proof similar to the above for the case where $\frac{d_1(f,g)}{d_1(f,h)} \geq 1$ results in the following inequality:

$$\frac{1}{\lambda} d_1(f,g)^{\frac{1}{\alpha}} \leq d_2(\phi(f), \phi(g)) \leq \frac{\lambda}{\delta^{\alpha}} d_1(f,g)^{\alpha}$$

Let $\xi = \max\{\lambda, \frac{1}{\lambda}, \frac{\lambda}{\delta^{\alpha}}, \frac{\lambda}{\delta^{\frac{1}{\alpha}}}\}$ and $\beta = \min\{\alpha, \frac{1}{\alpha}\}$ to obtain equation 3.2. $\square$

Let $\phi : end(T, w) \rightarrow end(S, v)$ and $\psi : end(S, v) \rightarrow end(R, x)$ be power quasi-symmetries. Using definition 8.15, it is easy to see that the composition $\psi \circ \phi : end(T, w) \rightarrow end(R, x)$ is a power quasi-symmetry.

In example 7.8, a quasi-conformal map is given that is not bi-Hölder and hence, cannot be a power quasi-symmetry. This shows that quasi-conformal maps are not in general power quasi-symmetries. Lemma 8.16 along with proposition 5.13, show that in the case where ϕ is a power quasi-symmetry induced by a quasi-isometry, ϕ is quasi-conformal.

The following concept is quite well-known. Let X be a metric space. Given three points $x, y, w \in X$, the *Gromov product of x and y* with respect to the base point w is defined by

$$(x|y)_w = \frac{1}{2}(|x-w| + |y-w| - |x-y|)$$

.

Definition 8.17. Let X be δ-hyperbolic and $w \in X$. A sequence of points $\{x_i\} \subset X$ is said to *converge at infinity*, if

$$\lim_{i,j\to\infty} (x_i|x_j)_w = \infty.$$

Two sequences $\{x_i\}$, $\{y_i\}$ that converge at infinity are *equivalent*, if

$$\lim_{i\to\infty} (x_i|y_j)_w = \infty.$$

This defines an equivalence relation for sequences in X converging at infinity. It is easy to see that convergence at infinity of a sequence and equivalences of two sequences does not depend on the choice of the point w. The *boundary* ∂X of X is defined as the set of equivalence classes of sequences converging at infinity.

In the special case where X is a locally finite classical rooted tree every sequence of points $\{x_i\} \subset X$ that is convergent can be identified with an isometric embedding

$f \in end(X)$ and w can be taken to be the root of the tree X. This identification is such that $\mathcal{H}(\{x_i\}, \mathrm{Im}(f)) < \infty$. Using this vocabulary, two sequences $\{x_i\}$, $\{y_i\}$ that converge at infinity are equivalent, if they are identified with the same isometric embedding $f \in end(X)$.

Bonk and Schramm give the standard construction of the metrics on ∂X, where X is a Gromov hyperbolic space. If $x, y \in \partial X$, $w \in X$, $\varepsilon > 0$, let

$$d_{\partial X,w,\varepsilon}(x,y) = d_{w,\varepsilon}(x,y) = \inf\{\sum_{i=1}^{n} e^{-\varepsilon(x_{i-1}|x_i)_w}\}, \tag{8.4}$$

where the infimum extends over all finite sequences $x = x_0, x_1, x_2, \ldots, x_n = y$ in ∂X. Here the convention $e^{-\infty} = 0$ is understood.

Let us translate the above metric into the vocabulary we have been using in this paper. If $x = (x_0, x_1, x_2, \ldots)$ is identified with $f \in end(X)$ and $y = (y_0, y_1, y_2, \ldots)$ is identified with $g \in end(X)$, then $d_{\partial X,w,\varepsilon}(x,y) = \varepsilon(f|g)$. Recall that

$(f|g) = \sup\{t \geq 0 | f(t) = g(t)\}$.

By changing the value for ε different metrics on ∂X can be defined.

Definition 8.18. The canonical gauge $\mathcal{G}(X)$ on ∂X is the set of all metrics of the form $d = d_{w,\varepsilon}$. We say that (ε_1, d_1) is *B-equivalent* to (ε_2, d_2) if there is a constant $c > 0$ such that

$$c^{-1} d_2^{\varepsilon_1} \leq d_1^{\varepsilon_2} \leq c d_2^{\varepsilon_1}.$$

A *B-structure* on ∂X is an equivalence class of this equivalence relation.

Although [BS] introduce a B-structure on ∂X they go on to work with a fixed metric on the canonical gauge and abbreviate this metric by $d_{\partial X}$. Let $\varepsilon = 1$, then the metric defined on $d_{\partial X}$ is the same as the metric that has been studied in the current paper.

The functor that Bonk and Schramm construct takes a Gromov hyperbolic almost geodesic metric space to a bounded and complete B-structure, and takes a quasi-isometry to a power quasi-symmetry. They do not show that the functor is faithful.

CHAPTER IX

PROBLEMS FOR FUTURE RESEARCH

The following is a list of conjectures that have come to mind while working on this paper.

Conjecture 9.1. *The functor $\mathcal{E} : \mathcal{T} \to \mathcal{U}$ defined in definition 6.6 is full. That is to say that if (T, w) and (S, v) are objects in the category $\mathcal{T}$, and $\phi : end(T, w) \to end(S, v)$ is a bi-Hölder homeomorphism between them, then there exists a quasi-isometry $\gamma : (T, w) \to (S, v)$ such that $\mathcal{E}([\gamma]) = \phi$.*

By definition a functor is said to be an equivalence of categories if it is faithful and full and every object in the range category of the functor is isomorphic to the image of an object under the functor. The above conjecture, if true, combined with theorem 6.11 enables us to deduce that the functor $\mathcal{E}$ is an equivalence of categories.

If the above conjecture is true, then we can prove the following conjectures.

Conjecture 9.2. *Let (T, w) and (S, v) be objects in the category $\mathcal{T}$, and let $\phi : end(T, w) \to end(S, v)$ be a bi-Hölder homeomorphism. Then ϕ is quasi-conformal.*

Proof that conjecture 9.1 implies conjecture 9.2: By conjecture 9.1 there exists a quasi-isometry $\gamma : (T, w) \to (S, v)$ such that $\mathcal{E}([\gamma]) = \phi$. By theorem 5.13, ϕ is quasi-conformal. □

Conjecture 9.3. *Let (T, w) and (S, v) and (R, x) be objects in the category $\mathcal{T}$. Let $\phi : end(T, w) \to end(S, v)$ and $\psi : end(S, v) \to end(R, x)$ be bi-Hölder homeomorphisms. Then, $\psi \circ \phi : end(T, w) \to end(R, x)$ is a bi-Hölder quasi-conformal homeomorphism.*

Proof that 9.2 implies conjecture 9.3: The composition $\psi \circ \phi : end(T, w) \to end(R, x)$ is a bi-Hölder homeomorphism by lemma 6.13. By conjecture 9.2, $\psi \circ \phi$ is quasi-conformal.$\square$

The above conjecture leads to the following conjectures.

Conjecture 9.4. *Every bi-Hölder homeomorphism on perfect compact ultrametric spaces is quasi-conformal.*

Conjecture 9.5. *The composition of two bi-Hölder homeomorphisms on perfect compact ultrametric spaces is a bi-Hölder quasi-conformal homeomorphism.*

Proof that conjecture 9.4 implies conjecture 9.5: By theorem 6.3 the composition of two bi-Hölder homeomorphisms is a bi-Hölder homeomorphism, therefore by conjecture 9.4 the composition is also quasi-conformal.

Conjecture 9.6. *The composition of two bi-Hölder quasi-conformal homeomorphisms on perfect compact ultrametric spaces is a bi-Hölder quasi-conformal homeomorphism.*

Note that conjecture 9.6 is a weaker version of conjecture 9.5. On the other hand conjecture 9.6 may be true, even if conjecture 9.5 is false.

If conjecture 9.6 is true, then we can define the category $\mathcal{U}'$ to be the category whose objects are perfect compact ultrametric spaces and whose morphisms are bi-

Hölder quasi-conformal homeomorphisms. Note that $\mathcal{U}'$ is a sub-category of $\mathcal{U}$ and contains the image of the functor $\mathcal{E}$. In this case, we can give the following version of conjecture 9.1.

Conjecture 9.7. *The functor $\mathcal{E}$ from the category $\mathcal{T}$ to the category $\mathcal{U}'$ is full. That is to say that if (T, w) and (S, v) are objects in the category $\mathcal{T}$, and $\phi : end(T, w) \to end(S, v)$ is a bi-Hölder quasi-conformal homeomorphism between them, then there exists a quasi-isometry $\gamma : (T, w) \to (S, v)$ such that $\mathcal{E}([\gamma]) = \phi$.*

Conjecture 9.7, if true, combined with theorem 6.11 enables us to deduce that the functor $\mathcal{E}$ is an equivalence between the categories $\mathcal{T}$ and $\mathcal{U}'$. If conjecture 9.1 is true, then conjecture 9.7 will be true. On the other hand conjecture 9.7 may be true without conjecture 9.1 being true. If this is the case, then there exist trees (T, w) and (S, v) in the category $\mathcal{T}$, and a bi-Hölder homeomorphism $\phi : end(T, w) \to end(S, v)$ that is not induced by any quasi-isometry between the trees (T, w) and (S, v).

If the following conjecture can be proved, the techniques used in [Hug] might be useful in proving that the functor is full.

Conjecture 9.8. *If $f : U \to V$ is a local similarity equivalence between perfect compact ultrametric spaces, then f is a bi-Hölder quasi-conformal homeomorphism.*

Appendix A

In this appendix, we present proofs of the statements made in several of the examples in Chapter VII. The example being referred to is clearly indicated by the example number. In this appendix $(x_0, x_1, x_2, \dots)$ and $(y_0, y_1, y_2, \dots)$ represent sequence of $0'$s and $1'$s; in other words, $x_i, y_i \in \{0, 1\}$ for all $i = 0, 1, 2, \dots$.

Proof of claim for example 7.7. Recall that in example 7.7 we claim that

$\phi : end(T', w') \to end(S', v')$ defined by

$\phi(f_0) = g_0$, and

$$\phi([w', c_{i,j}] \cup (x_0, x_1, x_2, \dots)) = \begin{cases} [v', c_{2i-1}] \cup (x_0, x_1, x_2, \dots), & \text{if } j = 1 \\ [v', c_{2i}] \cup (x_0, x_1, x_2, \dots), & \text{if } j = 2 \end{cases} \tag{9.1}$$

is a $\left(\frac{1}{2}, e^3\right)$- bi-Hölder e-quasi-conformal homeomorphism. What follows is the proof of this claim.

Proof of claim:

The proof that ϕ is a homeomorphism is trivial. To prove the other statements let us first look at the distances $d_{w'}(x, y)$ between the ends $x, y \in end(T', w')$. We have the following cases:

i) If $x = f_0$ and $y = [w', c_{i,j}] \cup (x_0, x_1, x_2, \dots)$, for $i = 1, 2, \dots$ and $j = 1, 2$, then $(x|y) = 2i$, implying that, $d_{w'}(x, y) = \dfrac{1}{e^{2i}}$.

ii) If $x = [w', c_{i,1}] \cup (x_0, x_1, x_2, \dots)$ and $y = [w', c_{i,2}] \cup (y_0, y_1, y_2, \dots)$, for $i = 1, 2, \dots$, then $(x|y) = 2i + 1$ and $d_{w'}(x, y) = \dfrac{1}{e^{2i+1}}$.

iii) If $x = [w', c_{i,j}] \cup (x_0, x_1, x_2, \dots)$ and $y = [w', c_{k,l}] \cup (y_0, y_1, y_2, \dots)$, for $i, k = 1, 2, \dots$

and $j,l = 1,2$ and $i < k$, then $(x|y) = 2i$ and $d_{w'}(x,y) = \dfrac{1}{e^{2i}}$.

iv) If $x = [w', c_{i,j}] \cup (x_0, x_1, x_2, \dots)$ and $y = [w', c_{i,j}] \cup (y_0, y_1, y_2, \dots)$, for $i = 1,2,\dots$ and $j = 1,2$. Then $(x|y) = d_{w'}(w', c_{i,j}) + n = 2i+1+n$ where $n = \inf\{i \geq 0 | x_i \neq y_i\}$ and $d_{w'}(x,y) = \dfrac{1}{e^{2i+1+n}}$.

The distances between the ends $x, y \in end(S', v')$. The following are the cases that can occur:

a) If $x = g_0$ and $y = [v', c_i] \cup (x_0, x_1, x_2, \dots)$, for $i = 1,2,\dots$, then $(x|y) = i-1$, implying that, $d_{v'}(x,y) = \dfrac{1}{e^{i-1}}$.

b) If $x = [v', c_i] \cup (x_0, x_1, x_2, \dots)$ and $y = [v', c_j] \cup (y_0, y_1, y_2, \dots)$, for $i,j = 1,2,\dots$ and $i < j$, then $(x|y) = i-1$ and $d_{v'}(x,y) = \dfrac{1}{e^{i-1}}$.

c) If $x = [v', c_i] \cup (x_0, x_1, x_2, \dots)$ and $y = [v', c_i] \cup (y_0, y_1, y_2, \dots)$, for $i = 1,2,\dots$, then $(x|y) = d_{v'}(v', c_i) + n = i-1+n$ where $n = \inf\{i \geq 0 | x_i \neq y_i\}$ and $d_{v'}(x,y) = \dfrac{1}{e^{i-1+n}}$.

We will show that

$$\frac{1}{e^3}\left\{d_{w'}(x,y)\right\}^2 \leq d_{v'}(\phi(x), \phi(y)) \leq e^3 \left\{d_{w'}(x,y)\right\}^{\frac{1}{2}} \tag{9.2}$$

which means that ϕ is a $\left(\dfrac{1}{2}, e^3\right)$- bi-Hölder homeomorphism.

Case I) x and y are as in (i) above and $j = 1$. In this case $\phi(x) = g_0$ and $\phi(y) = [v', c_{2i-1}] \cup (x_0, x_1, x_2, \dots)$. By (i) and (a) above, $d_{w'}(x,y) = \dfrac{1}{e^{2i}}$ and $d(\phi(x), \phi(y)) = \dfrac{1}{e^{2i-1}}$. We substitute these values in relationship 9.2 and obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i}}\right\}^2 \leq \frac{1}{e^{2i-1}} \leq e^3 \left\{\frac{1}{e^{2i}}\right\}^{\frac{1}{2}}$$

which is a true statement.

Case II) x and y are as in (ii) above and $j = 2$. In this case $\phi(x) = g_0$ and $\phi(y) = [v', c_{2i}] \cup (x_0, x_1, x_2, \dots)$. By (i) and (a) above, $d_{w'}(x, y) = \dfrac{1}{e^{2i}}$ and $d(\phi(x), \phi(y)) = \dfrac{1}{e^{2i-2}}$. We substitute these values in relationship 9.2 and obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i}}\right\}^2 \leq \frac{1}{e^{2i-2}} \leq e^3\left\{\frac{1}{e^{2i}}\right\}^{\frac{1}{2}}$$

which is always true.

Case III) x and y are as in (i) above. In this case $\phi(x) = [v', c_{2i-1}] \cup (0, 0, 0, , \dots)$ and $\phi(y) = [v', c_{2i}] \cup (0, 0, 0, \dots)$. By (i) and (b) above, $d_{w'}(x, y) = \dfrac{1}{e^{2i+1}}$ and $d(\phi(x), \phi(y)) = \dfrac{1}{e^{2i-2}}$. We substitute these values in relationship 9.2 to obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i-2}}\right\}^2 \leq \frac{1}{e^{2i+1}} \leq e^3\left\{\frac{1}{e^{2i-2}}\right\}^{\frac{1}{2}}$$

which is a true statement.

Case IV) x and y are as in (iii) above and $j = l = 1$. In this case $\phi(x) = [v', c_{2i-1}] \cup (x_0, x_1, x_2, \dots)$ and $\phi(y) = [v', c_{2k-1}] \cup (y_0, y_1, y_2, \dots)$. By (iii) and (b) above, $d_{w'}(x, y) = \dfrac{1}{e^{2i}}$ and $d(\phi(x), \phi(y)) = \dfrac{1}{e^{2i-2}}$. We substitute these values in relationship 9.2 and obtain:

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i}}\right\}^2 \leq \frac{1}{e^{2i-2}} \leq e^3\left\{\frac{1}{e^{2i}}\right\}^{\frac{1}{2}}$$

which is always true.

Case V) x and y are as in (iii) above and $j = l = 2$. In this case $\phi(x) = [v', c_{2i}] \cup (x_0, x_1, x_2, \dots)$ and $\phi(y) = [v', c_{2k}] \cup (y_0, y_1, y_2, \dots)$. By (iii) and (b) above, $d_{w'}(x, y) = \dfrac{1}{e^{2i}}$ and $d(\phi(x), \phi(y)) = \dfrac{1}{e^{2i-1}}$. We substitute these values in relationship 9.2 and obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i}}\right\}^2 \le \frac{1}{e^{2i-1}} \le e^3\left\{\frac{1}{e^{2i}}\right\}^{\frac{1}{2}}$$

which is always true.

Case VI) x and y are as in (iii) above and $j = 1$ and $l = 2$. In this case $\phi(x) = [v', c_{2i-1}] \cup (x_0, x_1, x_2, \dots)$ and $\phi(y) = [v', c_{2k}] \cup (y_0, y_1, y_2, \dots)$. By (iii) and (b) above, $d_{w'}(x, y) = \frac{1}{e^{2i}}$ and $d(\phi(x), \phi(y)) = \frac{1}{e^{2i-2}}$. We substitute these values in relationship 9.2 and obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i}}\right\}^2 \le \frac{1}{e^{2i-2}} \le e^3\left\{\frac{1}{e^{2i}}\right\}^{\frac{1}{2}}$$

which is always true.

Case VII) x and y are as in (iii) above and $j = 2$ and $l = 1$. In this case $\phi(x) = [v', c_{2i}] \cup (x_0, x_1, x_2, \dots)$ and $\phi(y) = [v', c_{2k-1}] \cup (y_0, y_1, y_2, \dots)$. By (iii) and (b) above, $d_{w'}(x, y) = \frac{1}{e^{2i}}$ and $d(\phi(x), \phi(y)) = \frac{1}{e^{2i-2}}$. We substitute these values in relationship 9.2 and obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i}}\right\}^2 \le \frac{1}{e^{2i-2}} \le e^3\left\{\frac{1}{e^{2i}}\right\}^{\frac{1}{2}}$$

which is always true.

Case VIII) x and y are as in (iv) above and $j = 1$. In this case $\phi(x) = [v', c_{2i-1}] \cup (x_0, x_1, x_2, \dots)$ and $\phi(y) = [v', c_{2i-1}] \cup (y_0, y_1, y_2, \dots)$. By (iv) and (c) above, $d_{w'}(x, y) = \frac{1}{e^{2i+1+n}}$ where n is defined in (iv) and $d(\phi(x), \phi(y)) = \frac{1}{e^{2i-1+n}}$. We substitute these values in relationship 9.2 and obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i+1+n}}\right\}^2 \le \frac{1}{e^{2i-1+n}} \le e^3\left\{\frac{1}{e^{2i+1+n}}\right\}^{\frac{1}{2}}$$

which is always true.

Case IX) x and y are as in (iv) above and $j = 2$. In this case $\phi(x) = [v', c_{2i}] \cup (x_0, x_1, x_2, \dots)$ and $\phi(y) = [v', c_{2i}] \cup (y_0, y_1, y_2, \dots)$. By (iv) and (c) above, $d_{w'}(x, y) = \frac{1}{e^{2i+1+n}}$ where n is defined in (iv) and $d(\phi(x), \phi(y)) = \frac{1}{e^{2i+n}}$. We substitute these values in relationship 9.2 and obtain :

$$\frac{1}{e^3}\left\{\frac{1}{e^{2i+1+n}}\right\}^2 \le \frac{1}{e^{2i+n}} \le e^3\left\{\frac{1}{e^{2i+1+n}}\right\}^{\frac{1}{2}}$$

which is always true.

The above nine cases exhaust all the possible cases that can occur, therefore ϕ is a $\left(\frac{1}{2}, e^3\right)$- bi-Hölder homeomorphism.

To show that ϕ is e-quasi-conformal, we need to show that $\sup_{x \in X} H_\phi^{d_{w'}}(x) \le e$, where $H_\phi^{d_{w'}}(x)$ is as in definition 5.10.

Let $x = f_0$, then $\{d_{w'}(x, y) | y \in end(T', w')\} = \left\{\frac{1}{e^{2i}}\right\}_{i=1}^{\infty}$ and this is a decreasing sequence tending to zero. Let $d_{w'}(f_0, y) = \frac{1}{e^{2i}}$, then,

$y \in \{[w', c_{i,j}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_i) \text{ and } j = 1, 2\}$.

Therefore, $\phi(y) \in \{[v', c_{2i-1}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_{2i-1})\} \cup \{[v', c_{2i}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_{2i})$ for all $i = 1, 2, \dots\}$.

Hence, $A_i = \{d_{v'}(\phi(x), \phi(y)) | d_{w'}(x, y) = \frac{1}{e^{2i}}\} = \left\{\frac{1}{e^{2i-2}}, \frac{1}{e^{2i-1}}\right\}$. Thus, $\sup A_i = \frac{1}{e^{2i-1}}$ and $\inf A_i = \frac{1}{e^{2i-2}}$ and $\frac{\sup A_i}{\inf A_i} = e$, therefore $H_\phi^{d_{w'}}(x) = e$.

Let $x = [w', c_{i,j}] \cup (x_0, x_1, \dots)$, then $\{d_{w'}(x, y) | y \in end(T', w')\} = \left\{\frac{1}{e^i}\right\}_{i=2}^{\infty}$ which tends to zero. Without loss of generality, let $k > 2i + 1$, and let $d_{w'}(x, y) = \frac{1}{e^k}$, then

$y \in \{[w', c_{i,j}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_{2i-1})$ and $y_s = x_s$ for $s \le k - 2i - 1$ for all $j = 1, 2\}$.

Therefore, $\phi(y) \in \{[v', c_{2i-1}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_{2i-1})$ and $y_s = x_s$ for $s \leq k - 2i - 1\} \cup \{[v', c_{2i}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_{2i-1})$ and $y_s = x_s$ for $s \leq k - 2i - 1\}$. Hence, $A_i = \{d_{v'}(\phi(x), \phi(y)) | d_{w'}(x, y) = \frac{1}{e^k}\} = \{\frac{1}{e^{k-4}}\}$ if $j = 1$, and $A_i = \{d_{v'}(\phi(x), \phi(y)) | d_{w'}(x, y) = \frac{1}{e^k}\} = \{\frac{1}{e^{k-3}}\}$ if $j = 2$. In both cases $\sup A_i = \inf A_i$ and hence, $H_\phi^{d_{w'}}(x) = 1$

Therefore, $\sup_{x \in X} H_\phi^{d_{w'}}(x) = \sup\{1, e\} = e.\square$

Proof of claim for example 7.8. In example 7.8 we claim that $\psi : end(S', v') \to (R', x')$ defined by :

$\psi(g_0) = h_0$, and

$$\psi([v', c_i] \cup (x_0, x_1, x_2, \dots)) = [x'c_i] \cup (x_0, x_1, x_2, \dots) \tag{9.3}$$

is a conformal homeomorphism which is not bi-Hölder. The following is a detailed proof of this statement.

First let us look at the distances in $end(R', x')$.

i) $(h_0 | [x', c_{2i}] \cup (x_0, x_1, \dots)) = 2i$ for $i = 1, 2, \dots$, therefore $d_{x'}(h_0, [x', c_{2i-1}] \cup (x_0, x_1, \dots)) = \frac{1}{e^{2i}}$.

ii) $(h_0 | [x', c_{2i-1}] \cup (x_0, x_1, \dots)) = 2i^2 + 1$ for $i = 1, 2, \dots$, therefore $d_{x'}(h_0, [x', c_2 i] \cup (x_0, x_1, \dots)) = \frac{1}{e^{2i^2+1}}$.

iii) $([x', c_{2i}] \cup (x_0, x_1, \dots)) | [x', c_{2j}] \cup (y_0, y_1, \dots)) = 2i$ for $i, j = 1, 2, \dots$ and $i < j$, therefore $d_{x'}([x', c_{2i}] \cup (x_0, x_1, \dots)), [x', c_{2j}] \cup (y_0, y_1, \dots)) = \frac{1}{e^{2i}}$.

iv) $([x', c_{2i-1}] \cup (x_0, x_1, \dots)) | [x', c_{2j-1}] \cup (y_0, y_1, \dots)) = 2i^2 + 1$ for $i, j = 1, 2, \dots$ and $i < j$, therefore $d_{x'}([x', c_{2i-1}] \cup (x_0, x_1, \dots)), [x', c_{2j-1}] \cup (y_0, y_1, \dots)) = \frac{1}{e^{2i^2+1}}$.

v) $([x', c_{2i}] \cup (x_0, x_1, \dots)) | [x', c_{2j-1}] \cup (y_0, y_1, \dots)) = 2i$ for $i, j = 1, 2, \dots$ and $2i < 2j^2 + 1$, therefore $d_{x'}([x', c_{2i}] \cup (x_0, x_1, \dots)), [x', c_{2j-1}] \cup (y_0, y_1, \dots)) = \dfrac{1}{e^{2i}}$.

vi) $([x', c_{2i}] \cup (x_0, x_1, \dots)) | [x', c_{2j-1}] \cup (y_0, y_1, \dots)) = 2i$ for $i, j = 1, 2, \dots$ and $2i > 2j^2 + 1$, therefore $d_{x'}([x', c_{2i}] \cup (x_0, x_1, \dots)), [x', c_{2j-1}] \cup (y_0, y_1, \dots)) = \dfrac{1}{e^{2j^2+1}}$.

vii) $([x', c_i] \cup (x_0, x_1, \dots)) | [x', c_i] \cup (y_0, y_1, \dots)) = d_{x'}(x', c_i) + n$ for $i, j = 1, 2, \dots$ and $n = \inf\{i \geq 0 | y_s \neq x_s\}$, therefore $d_{x'}([x', c_i] \cup (x_0, x_1, \dots)), [x', c_i] \cup (y_0, y_1, \dots)) = \dfrac{1}{e^{d_{x'}(x', c_i)+n}}$.

The fact that ψ is a homeomorphism is obvious. We will show that ψ is conformal.

Let $x = g_0$, then $\{d_{v'}(x, y) | y \in end(S', v')\} = \left\{\dfrac{1}{e^i}\right\}_{i=0}^{\infty}$ which is a decreasing sequence tending to zero. Let $d_{v'}(g_0, y) = \dfrac{1}{e^i}$, then $y \in \{[v', c_{i+1}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_{i+1})\}$. If $i+1$ is even, and $i+1 = 2k$ for some integer k, then $\psi(y) \in \{[x', c'_{2k}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X'_{2i-1})\}$. Hence, $d_{x'}(\psi(x), \psi(y)) = \dfrac{1}{e^{2i}}$. If $i+1$ is odd, and $i+1 = 2k-1$ for some integer k, then $\psi(y) \in \{[x', c'_{2k-1}] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X'_{2i-1})\}$. Hence, $d_{x'}(\psi(x), \psi(y)) = \dfrac{1}{e^{2i^2+1}}$. Let $A_i = \{d_{x'}(\psi(x), \psi(y)) | d_{w'}(x, y) = \dfrac{1}{e^i}\}$. In both cases $\sup A_i = \inf A_i$, thus, $H_{\psi}^{d_{x'}}(x) = 1$.

Let $x = [v', c_i] \cup (x_0, x_1, \dots)$, where $i = 1, 2, \dots$, then $\{d_{v'}(x, y) | y \in end(S', v')\} = \left\{\dfrac{1}{e^i}\right\}_{j=1}^{\infty}$ and this is a sequence tending to zero. Without loss of generality, let $k > i$ and let $d_{v'}(x, y) = \dfrac{1}{e^k}$, then

$y \in \{[v', c_i] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X_{2i-1})$ and $y_s = x_s$ for $s \leq k - i\}$.

Therefore,

$\psi(y) \in \{[x', c'_k] \cup (y_0, y_1, \dots) | (y_0, y_1, \dots) \in end(X'_{2i-1})$ and $x_s = y_s$ for all $s \leq k - 2i\}$

and $\psi(x) = [x', c'_k] \cup (x_0, x_1, \dots)$. Hence,

$A_k = \{d_{x'}(\psi(x),\psi(y)) | d_{w'}(x,y) = \frac{1}{e^k}\} = \left\{ e^{\frac{1}{d_{w'}(w',c_k)+k-2i}} \right\}$ Thus, $\sup A_k = \inf A_k$ for all $k > i$, therefore $H_\psi^{d_{x'}}(x) = 1$.

By the above cases we see that ψ is a conformal homeomorphism.

To see that ψ is not bi-Hölder consider the following case. Let $x = g_0$ and $y = [w', c'_{2i-1}] \cup (0,0,0,\dots)$. Then, $d_{w'}(x,y) = \frac{1}{e^{2i-2}}$. By the definition of ψ we have $\psi(g_0) = h_0$ and $\psi(y) = [x', c'_{2i-1}] \cup (0,0,0,\dots)$. Hence $d_{x'}(\psi(x),\psi(y)) = \frac{1}{e^{2i^2+1}}$

We will show that there is no $\alpha > 0$ such that the following inequality holds:

$$\frac{1}{k}\left\{\frac{1}{e^{2i-2}}\right\}^{\frac{1}{\alpha}} \leq \frac{1}{e^{2i^2+1}} \leq k\left\{\frac{1}{e^{2i-2}}\right\}^{\alpha}$$

By talking natural logarithms of both sides and rewriting the inequality we obtain:

$$\alpha(2i^2+1) - \ln(k) \leq 2i - 2 \leq \ln(k) + \frac{1}{\alpha}(2i^2+1)$$

therefore

$\alpha \leq \dfrac{2i-2+\ln(k)}{2i^2+1}$.

This relationship has to be true for all i. If we let $i \to \infty$ we see that $\alpha \to 0$. Therefore, ψ is not bi-Hölder. $\square$

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