Balian-Low Type Results for Gabor Schauder Bases

By

Sara Jane Mota Leshen

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Approved:

Alexander Powell, Ph.D.

Akram Aldroubi, Ph.D.

Doug Hardin, Ph.D.

Gieri Simonett, Ph.D.

David Smith, Ph.D.

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Chapter 1

Introduction

In this chapter, we introduce the Balian-Low theorem and our main results. In Section 1.1, we present some of the known versions of the Balian-Low theorem for orthonormal bases, Riesz bases, and exact systems. In Section 1.2, we state and discuss our main results.

1.1 The Balian-Low Theorem

For $x, \omega \in \mathbb{R}$, we define the *translation operator* T_x and the *modulation operator* M_ω by

$$T_x g(t) = g(t-x)$$

and

$$M_\omega g(t) = e^{2\pi i \omega t} g(t).$$

Then given a window function $g \in L^2(\mathbb{R})$ and $a, b > 0$, $g_{k,n}(t) = M_{nb} T_{ka} g(t)$ is called a *time-frequency shift* of g, and the associated *Gabor system* $\mathscr{G}(g,a,b)$ is defined by

$$\mathscr{G}(g,a,b) = \left\{ g_{k,n}(t) \right\}_{k,n \in \mathbb{Z}} = \{ M_{nb} T_{ka} g(t) \}_{k,n \in \mathbb{Z}} = \left\{ e^{2\pi i n b t} g(t-ka) \right\}_{k,n \in \mathbb{Z}}. \qquad (1.1.1)$$

Gabor systems are an important tool for providing signal expansions in the setting of time-frequency analysis. A central problem is to understand how the triple (g,a,b) determines spanning properties of $\mathscr{G}(g,a,b)$. This remains a challenging problem, and, despite a large literature, there are relatively few window functions g for which the spanning structure of $\mathscr{G}(g,a,b)$ is completely understood for general $a, b > 0$, e.g., [10, 16].

The Balian-Low theorem is a fundamental obstruction which shows that there are strong trade-offs between the spanning structure of a Gabor system and the time-frequency localization of the window function g. The Balian-Low theorem is a manifestation of the uncertainty principle which says that if $\mathscr{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$, then either g or its Fourier transform $\widehat{g}(\xi) = \int g(t)e^{-2\pi it\xi}dt$ must be poorly localized. Note that throughout this thesis, we assume unless otherwise specified that we integrate over $\mathbb{R}$.

We consider two examples to illustrate the aforementioned trade-offs before stating the theorem.

Example 1.1.1. *Recall that the Fourier series* $\left\{e^{2\pi int}\right\}_{n\in\mathbb{Z}}$ *provides an orthonormal basis for* $L^2([0,1])$. *Consider the Gabor system* $\mathscr{G}(g,1,1)$ *with the window function* $g = \chi_{[0,1]}$, *the indicator function of* $[0,1]$. *Note* $g_{k,n}(t) = e^{2\pi int}\chi_{[0,1]}(t-k)$, *and thus* $\mathscr{G}(g,1,1)$ *is an orthonormal basis for* $L^2(\mathbb{R})$. *However, g is poorly localized in frequency since* $|\widehat{g}(\xi)| = \left|\frac{\sin(\pi\xi)}{\pi\xi}\right| \sim \frac{1}{|\xi|}$.

Thus we have a Gabor system with strong spanning structure as an orthonormal basis, but poor frequency localization.

Example 1.1.2. *Let* $g(t) = e^{-\pi t^2}$. *Suppose* $0 < ab < 1$ *and consider* $\mathscr{G}(g,a,b)$. *Since* $\widehat{g}(\xi) = e^{-\pi\xi^2}$, *g is well-localized in time and frequency. However, g is not an orthonormal basis for* $L^2(\mathbb{R})$, *and provides only redundant, or non-unique, signal expansions.*

Here we have a Gabor system with weak spanning structure as the expansions are redundant, but good time and frequency localization. We elaborate on the structure of this Gabor system if $a = b = 1$ in Section 2.1.2.

The classical Balian-Low theorem dates back to [3, 23, 4], but it will be convenient to consider the following non-symetrically weighted generalization from [14].

Theorem 1.1.3. *Suppose that* $1 < p,q < \infty$ *satisfy* $\frac{1}{p}+\frac{1}{q} = 1$. *If* $g \in L^2(\mathbb{R})$ *satisfies*

$$\int |t|^p|g(t)|^2dt < \infty \quad \textit{and} \quad \int |\xi|^q|\widehat{g}(\xi)|^2d\xi < \infty, \tag{1.1.2}$$

then $\mathscr{G}(g,1,1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The classical Balian-Low theorem [3, 23, 4] addresses $(p,q)=(2,2)$ in Theorem 1.1.3, while the endpoint $(p,q)=(\infty,1)$ is addressed by the following theorem from [14], cf. [6]. In particular, the endpoint weight $|t|^p$ with $p=\infty$ is replaced by a compact support condition.

Theorem 1.1.4. *If $g \in L^2(\mathbb{R})$ is compactly supported and*

$$\int |\xi||\widehat{g}(\xi)|^2 d\xi < \infty,$$

then $\mathscr{G}(g,1,1)$ is not an orthonormal basis for $L^2(\mathbb{R})$. The same conclusion holds for g and $\widehat{g}$ interchanged.

We have stated Theorems 1.1.3 and 1.1.4 for orthonormal bases, but both results remain true for the more general class of Riesz bases [14], and both results are sharp [5]. Moreover, if one moves to the even more general class of exact systems, then a different version of the Balian-Low theorem holds. Recall that $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is *minimal* if for every N, f_N is not in the $L^2(\mathbb{R})$-closure of $span\{f_n : n \neq N\}$. The system $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is *exact* if it is both minimal and complete in $L^2(\mathbb{R})$.

The following version of the Balian-Low theorem for exact systems originates in [13] for $(p,q)=(4,4)$, and was later extended to general (p,q) in [21].

Theorem 1.1.5. *Suppose that $3 < q \leq p < \infty$ satisfy $\frac{1}{p}+\frac{3}{q}=1$. If $g \in L^2(\mathbb{R})$ satisfies*

$$\int |t|^p |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\xi|^q |\widehat{g}(\xi)|^2 d\xi < \infty, \tag{1.1.3}$$

then $\mathscr{G}(g,1,1)$ is not an exact system in $L^2(\mathbb{R})$. The same conclusion holds for p and q interchanged.

Analogous to Theorem 1.1.4, the following result from [21] addresses the endpoint $(p,q)=(\infty,3)$ in Theorem 1.1.5.

Theorem 1.1.6. *If $g \in L^2(\mathbb{R})$ is compactly supported and*

$$\int |\xi|^3 |\widehat{g}(\xi)|^2 d\xi < \infty,$$

then $\mathscr{G}(g,1,1)$ is not an exact system in $L^2(\mathbb{R})$. The same conclusion holds for g and $\widehat{g}$ interchanged.

Motivated by a conjecture in [11], the work in [20] investigated the extent to which the Balian-Low theorem holds for Schauder bases. It was constructively shown in [20] that several versions of the Balian-Low theorem fail for the class of Schauder bases. In particular, if $1 < q < 2 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then there exists $g \in L^2(\mathbb{R})$ such that $\mathscr{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ and such that (1.1.2) holds. In other words, Theorem 1.1.3 fails if "orthonormal basis" is replaced by "Schauder basis". However, the counterexamples from [20] were only valid for $(p,q) \neq (2,2)$, and it remained open whether similar examples could address the validity of the classical Balian-Low theorem with $(p,q) = (2,2)$ in the setting of Schauder bases. Moreover, while [20] showed that certain existing Balian-Low theorems do not extend to Schauder bases, it was not known if there exist distinct new Balian-Low theorems that are specific to the class of Schauder bases. The results in this thesis will resolve these issues by proving a new endpoint Balian-Low theorem for Schauder bases, and by extending the counterexamples from [20] to $(p,q) = (2,2)$.

1.2 Main Theorems

Our first main result provides a new endpoint Balian-Low theorem for Gabor systems that form a certain type of Schauder basis.

Theorem 1.2.1. *If $g \in L^2(\mathbb{R})$ is compactly supported and*

$$\int |\xi|^2 |\widehat{g}(\xi)|^2 d\xi < \infty,$$

then $\mathscr{G}(g,1,1)$ is not a Schauder basis of type Λ for $L^2(\mathbb{R})$. The same conclusion holds for g and $\widehat{g}$ interchanged.

See Section 2.1.3 for relevant technical background and the definition of Schauder basis of type Λ. For now, it suffices to note that Schauder basis expansions can be conditionally convergent, and hence the ordering of the system is important. Since $\mathscr{G}(g,1,1)$ is indexed by $\mathbb{Z}^2$, one must discuss the manner in which $\mathbb{Z}^2$ is enumerated. We focus on a class $\Lambda_{\mathscr{R}}(\mathbb{Z}^2)$ of enumerations of $\mathbb{Z}^2$ introduced by K. Moen in [24]. We say $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ if $\mathscr{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ whenever $\mathbb{Z}^2$ is ordered using an enumeration from $\Lambda_{\mathscr{R}}(\mathbb{Z}^2)$, and with uniformly bounded basis constants.

It is interesting to compare Theorem 1.2.1 with the other endpoint results in Theorems 1.1.4 and 1.1.6. Theorem 1.1.4 uses the weaker weight $|\xi|$ and involves the stronger spanning structure of orthonormal bases or Riesz bases. Theorem 1.1.6 uses the stronger weight $|\xi|^3$ and involves the weaker spanning structure of exact systems. Theorem 1.2.1 uses the intermediate weight $|\xi|^2$ and involves the intermediate spanning structure of Schauder bases. In view of this, Theorem 1.2.1 addresses a phenomenon that occurs "between" Theorems 1.1.4 and 1.1.6. These results illustrate a trade-off between the strength of the weight and the strength of the spanning structure. Further trade-offs of this type have been shown for (C_q)-systems in [28] and will be addressed in further detail in Section 2.4.

Theorem 1.2.1 is sharp in the sense that it fails if the weight $|\xi|^2$ is replaced by $|\xi|^s$ with $s<2$. In particular, Theorem 6.1 in [20] constructs a compactly supported $g\in L^2(\mathbb{R})$ such that $\mathscr{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ and such that $\int|\xi|^s|\widehat{g}(\xi)|^2d\xi<\infty$ holds whenever $s<2$, and Theorem 4.3 in [24] shows that $\mathscr{G}(g,1,1)$ is in fact a Schauder basis of type Λ.

Our second main result shows that if $(p,q)=(2,2)$, then Theorem 1.1.3 fails when "orthonormal basis" is replaced by "Schauder basis". This closes an unresolved case from [20].

Theorem 1.2.2. *For every $\varepsilon>0$, there exists $g\in L^2(\mathbb{R})$ such that $\mathscr{G}(g,1,1)$ is a Schauder* *basis of type* Λ *for* $L^2(\mathbb{R})$ *and*

$$\int |t|^{3-\varepsilon}|g(t)|^2 dt < \infty \quad \textit{and} \quad \int |\xi|^{3-\varepsilon}|\widehat{g}(\xi)|^2 d\xi < \infty. \tag{1.2.1}$$

Theorem 1.2.2 shows that the weight parameter $(p,q) = (2,2)$ is far from critical for Gabor Schauder bases, and it is reasonable to ask if $(p,q) = (3,3)$ is sharp.

Unlike the counterexamples for $(p,q) \neq (2,2)$ in Theorem 6.1 in [21], the function g in Theorem 1.2.2 cannot be compactly supported because of Theorem 1.2.1. In particular, Theorem 1.2.1 provides a theoretical explanation for why the counterexamples in Theorem 6.1 in [20] were not able to address the case $(p,q) = (2,2)$ for Schauder bases in Theorem 1.1.3. The key point is that the examples in [20] involved compactly supported window functions.

Chapter 2

Background

In this chapter, we collect definitions and background which will be used throughout this thesis. In Section 2.1, we define Schauder bases and other spanning structures, review their density properties, and introduce a class of enumerations of $\mathbb{Z}^2$. In Section 2.2, we define the Zak transform and $\mathscr{A}_{2,\mathscr{R}}$ weights, and state results connecting these objects to Schauder bases. In Section 2.3, we state several miscellaneous results that are used frequently throughout Chapter 3. In Section 2.4, we provide a brief history of the Balian-Low theorem and review some additional Balian-Low type results that are relevant to our work.

2.1 Schauder Bases

2.1.1 Schauder Bases in Relation to Other Spanning Structures

A sequence $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is a *Schauder basis* for $L^2(\mathbb{R})$ if for every $f \in L^2(\mathbb{R})$, there exist unique scalars $c_n(f)$ such that

$$f = \sum_{n=1}^{\infty} c_n(f) f_n, \qquad (2.1.1)$$

with (possibly conditional) convergence in $L^2(\mathbb{R})$. It is well-known that $c_n(f) = \langle f, g_n \rangle$ for the unique $\{g_n\}_{n=1}^{\infty}$ that is biorthogonal to $\{f_n\}_{n=1}^{\infty}$. Recall that $\{g_n\}_{n=1}^{\infty}$ is biorthogonal to $\{f_n\}_{n=1}^{\infty}$ if $\langle f_m, g_n \rangle = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta. For our purposes, it will be useful to consider the following equivalent characterization of Schauder bases, e.g., [18].

Theorem 2.1.1. *A sequence $\{f_n\}_{n=1}^{\infty}$ is a Schauder basis for $L^2(\mathbb{R})$ if and only if the following both hold:*

- *There exists a unique sequence $\{g_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ biorthogonal to $\{f_n\}_{n=1}^{\infty}$, and*

- *The partial sum operators $S_N f = \sum_{n=1}^{N} \langle f, g_n \rangle f_n$ are uniformly bounded in operator norm, i.e., $sup_N \|S_N\| < \infty$.*

For perspective, recall that $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is a *Riesz basis* if it is a bounded unconditional Schauder basis. In other words, for a Riesz basis the expansions (2.1.1) converge unconditionally and there exist $0 < A \leq B < \infty$ such that $A \leq \|f_n\|_2 \leq B$ for all n. A Riesz basis can also be defined as the image of an orthonormal basis under an invertible linear transformation. That is, there is an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $L^2(\mathbb{R})$ and an invertible transformation T such that $Te_n = f_n$ for all n. The class of Schauder bases for $L^2(\mathbb{R})$ is strictly larger than the class of Riesz bases for $L^2(\mathbb{R})$.

On the other hand, the class of Schauder bases for $L^2(\mathbb{R})$ is strictly smaller than the class of exact systems in $L^2(\mathbb{R})$. It is well-known that $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is exact if and only if there is a unique system $\{g_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ that is biorthogonal to $\{f_n\}_{n=1}^{\infty}$, e.g., [18]. Unlike a Schauder basis, an exact system does not guarantee signal expansions for the elements of $L^2(\mathbb{R})$.

We also recall the following related structure. A sequence $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R})$ is a *frame* if there exist $0 < A \leq B < \infty$ such that $A\|f\|_2^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|_2^2$ for all $f \in L^2(\mathbb{R})$. A frame has an associated operator S, defined by $Sf = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$, that yields expansions of the form $f = \sum_{n=1}^{\infty} \left\langle f, \widetilde{f_n} \right\rangle f_n$, where $\widetilde{f_n} = S^{-1} f_n$. Note that a Riesz basis can also be defined as an exact frame.

In the context of the Balian-Low theorem, a non-exact frame represents a total trade-off in the direction of localization. We can now point out that in Example 1.1.2, we have a Gabor system which is a non-exact frame. The system provides non-unique representations but good time-frequency localization.

2.1.2 Density of Gabor Schauder Bases

We now move to the specific setting of Gabor systems that form a Schauder basis for $L^2(\mathbb{R})$. Note that every Schauder basis is exact. It therefore follows from standard Gabor density theorems that $\mathscr{G}(g,a,b)$ can only be a Schauder basis for $L^2(\mathbb{R})$ when $ab = 1$.

Lemma 2.1.2. *Let $g \in L^2(\mathbb{R})$ and $a,b > 0$. If $\mathscr{G}(g,a,b)$ is exact, then $ab = 1$.*

Proof. Since $\mathscr{G}(g,a,b)$ is complete, $ab \le 1$ (see [19] for a history of this result). Since $\mathscr{G}(g,a,b)$ is minimal, it is ℓ^2-linearly independent. Thus by [8], $\mathscr{G}(g,\frac{1}{b},\frac{1}{a})$ is complete and so $\frac{1}{ab} \le 1$. □

So, Lemma 2.1.2, together with the unitary dilation $D_a f(t) = \sqrt{a} f(at)$ which maps $\mathscr{G}(g,a,\frac{1}{a})$ to $\mathscr{G}(D_a g,1,1)$, allows us to restrict our attention to Gabor systems $\mathscr{G}(g,1,1)$ with $a = b = 1$.

We also have the following density result for frames, see [2] and Chapter 7.5 of [15].

Lemma 2.1.3. *Let $a,b > 0$.*

1. *Let $g \in L^2(\mathbb{R})$. If $\mathscr{G}(g,a,b)$ is a frame, then $ab \le 1$.*

2. *Let $g(t) = e^{-\pi t^2}$. Then $\mathscr{G}(g,a,b)$ is a frame if and only if $ab < 1$.*

We again return to Example 1.1.2 of a non-exact frame. If we consider the same Gaussian window function but let $a = b = 1$, the system now has a chance to be exact due to Lemma 2.1.2. However, the system is actually not minimal. If we remove one element, we do have an exact system, see [2]. But by Lemma 2.1.3, if $a = b = 1$, the Gaussian-windowed system is no longer a frame and so the Gaussian-windowed system with a single element removed is also not a frame.

2.1.3 Enumerations and Schauder Bases of Type Λ

Schauder basis expansions may converge conditionally and are dependent on ordering. Since Gabor systems $\mathscr{G}(g,1,1)$ are indexed by $\mathbb{Z}^2$, Gabor Schauder bases are sensitive to the manner in which $\mathbb{Z}^2$ is enumerated. In other words, a Gabor system $\mathscr{G}(g,1,1)$ is a

Gabor Schauder basis for $L^2(\mathbb{R})$ if for every $f \in L^2(\mathbb{R})$, there exist unique scalars $c_{k,n}(f)$ such that

$$f = \sum_{k,n\in\mathbb{Z}} c_{k,n}(f) g_{k,n}$$

with convergence in $L^2(\mathbb{R})$ for at least one enumeration of $\mathbb{Z}^2$.

Given $g \in L^2(\mathbb{R})$ and an enumeration σ of $\mathbb{Z}^2$, we let $\{g_{\sigma(j)}\}_{j=1}^{\infty}$ denote the corresponding enumeration of $\mathscr{G}(g,1,1) = \{g_{k,n}\}_{k,n\in\mathbb{Z}}$. If $\mathscr{G}(g,1,1)$ is exact, then it has a unique biorthogonal system which is of the form $\mathscr{G}(h,1,1)$ for some $h \in L^2(\mathbb{R})$, e.g., see [13, 20]. So, if $\{g_{\sigma(j)}\}_{j=1}^{\infty}$ is a Schauder basis for $L^2(\mathbb{R})$, Theorem 2.1.1 gives that the partial sum operators

$$S_N^{\sigma} f = \sum_{j=1}^{N} \langle f, h_{\sigma(j)} \rangle g_{\sigma(j)}$$

are uniformly bounded in operator norm, i.e., $\sup_N \|S_N^{\sigma}\| < \infty$.

Let $\Lambda_{\mathscr{R}}(\mathbb{Z}^2)$ be the special class of enumerations of $\mathbb{Z}^2$ that is defined by Definition 3.4 in [24]. Roughly speaking, $\Lambda_{\mathscr{R}}(\mathbb{Z}^2)$ consists of enumerations of $\mathbb{Z}^2$ that are, in some sense, analogous to the manner in which $0,1,-1,2,-2,3,-3,\cdots$ enumerates $\mathbb{Z}$. The enumerations in $\Lambda_{\mathscr{R}}(\mathbb{Z}^2)$ are based on building up increasingly large rectangular sub-blocks of $\mathbb{Z}^2$ in a controlled manner, with technical restrictions on how elements of $\mathbb{Z}^2$ are added along the edges of a rectangular block to create larger rectangular blocks.

More precisely, we say an enumeration σ is *consecutive* to a rectangle $R \subset \mathbb{Z}^2$ if σ fills out each element of the 1-dimensional hyperplanes of R consecutively, while filling out the hyperplanes in a consecutive manner. We say σ is *adapted* to R if there exists a sequence of rectangles in $\mathbb{Z}^2$ such that $R = R^{(0)} \subset R^{(1)} \subset R^{(2)} \cdots$, $\bigcup_i R^{(i)} = \mathbb{Z}^2$, and $R^{(i+1)} \backslash R^{(i)}$ is a rectangle that σ is consecutive to. Then $\Lambda_{\mathscr{R}}(\mathbb{Z}^2)$ is the set of enumerations that are adapted to any rectangle in $\mathbb{Z}^2$.

Below we illustrate one possible consecutive enumeration σ of the horizontal hyperplanes of a rectangle $R \subset \mathbb{Z}^2$.

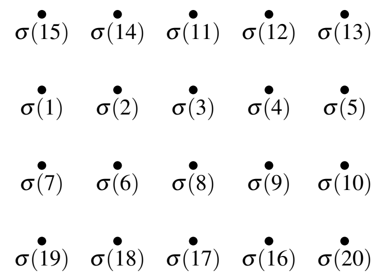

Figure 2.1: A consecutive enumeration σ of R

Definition 2.1.4. *Given $g \in L^2(\mathbb{R})$, we say that $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ if for every $\sigma \in \Lambda_{\mathscr{R}}(\mathbb{Z}^2)$, $\{g_{\sigma(k)}\}_{k=1}^{\infty}$ is a Schauder basis for $L^2(\mathbb{R})$, and the partial sum operators S_N^σ satisfy*

$$\sup_{\sigma \in \Lambda_{\mathscr{R}}(\mathbb{Z}^2)} \left(\sup_N \|S_N^\sigma\| \right) < \infty.$$

The examples in [24, 20] show that the class of Schauder bases of type Λ is strictly larger than the class of Riesz bases.

2.2 The Zak Transform and Muckenhoupt Weights

The Zak transform is an important tool for characterizing spanning properties of Gabor systems. Given $g \in L^2(\mathbb{R})$, its *Zak transform* Zg is defined by

$$\forall (t,\xi) \in \mathbb{R}^2, \quad Zg(t,\xi) = \sum_{k \in \mathbb{Z}} g(t-k) e^{2\pi i k \xi}.$$

The Zak transform is *quasiperiodic*, i.e.,

$$\forall (t,\xi) \in \mathbb{R}^2, \quad Zg(t,\xi+1) = Zg(t,\xi) \quad \text{and} \quad Zg(t+1,\xi) = e^{2\pi i \xi} Zg(t,\xi).$$

Quasiperiodicity implies that $|Zg|$ is $\mathbb{Z}^2$-periodic, and that Zg is fully determined by its values on the cube $Q = [0,1)^2$. The Zak transform is also a unitary operator, mapping $L^2(\mathbb{R})$ to $L^2(Q)$, see Theorem 8.2.3 in [15].

The following topological result plays an important role in the proof of the Balian-Low theorem. Due to its importance, we include the proof for the reader, also see Lemma 8.4.2 in [15] or Theorem 11.25 in [18].

Lemma 2.2.1. *If $G : \mathbb{R}^2 \to \mathbb{C}$ is quasiperiodic and continuous then G has a zero.*

Proof. We proceed by contradiction, and assume that $G(t,\xi) \neq 0$ for all $(t,\xi) \in \mathbb{R}^2$.

Since G is continuous and $\mathbb{R}^2$ is simply connected, we can apply general topological lifting principles to show that there exists a continuous function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$G(t,\xi) = |G(t,\xi)|\, e^{2\pi i \varphi(t,\xi)}, \qquad (t,\xi) \in \mathbb{R}^2. \tag{2.2.1}$$

By (2.2.1) and since G is quasiperiodic, for $k,n \in \mathbb{Z}$ and $t,\xi \in \mathbb{R}$, we have

$$G(t+k,\xi+n) = e^{2\pi i k\xi} G(t,\xi) = e^{2\pi i k\xi}\, |G(t,\xi)|\, e^{2\pi i \varphi(t,\xi)}$$

and

$$G(t+k,\xi+n) = |G(t+k,\xi+n)|\, e^{2\pi i \varphi(t+k,\xi+n)} = |G(t,\xi)|\, e^{2\pi i \varphi(t+k,\xi+n)}.$$

Thus there exists an integer $\kappa(k,n)$ such that

$$\varphi(t+k,\xi+n) = \varphi(t,\xi) + k\xi + \kappa(k,n).$$

Then

$$\begin{aligned}\varphi(1,1) &= \varphi(0,1) + 1 + \kappa(1,0)\\ &= \varphi(0,0) + \kappa(0,1) + 1 + \kappa(1,0),\end{aligned}$$

and

$$\varphi(1,1) = \varphi(1,0) + \kappa(0,1),$$
$$= \varphi(0,0) + \kappa(1,0) + \kappa(0,1).$$

This gives the contradiction $1 = 0$; thus G must have a zero. □

The Zak transform maps the translation and modulation operators to multiplication operators, diagonalizing time-frequency shifts, see Theorem 11.29 in [18]. Let $E_{n,k}(t,\xi) = e^{2\pi int}e^{-2\pi ik\xi}$.

Theorem 2.2.2. *Let $g \in L^2(\mathbb{R})$ and $k,n \in \mathbb{Z}$. Then for a.e. $t,\xi \in \mathbb{R}$,*

$$Z(M_nT_kg)(t,\xi) = E_{n,k}(t,\xi)Zg(t,\xi)$$

The next result uses Theorem 2.2.2 to give Zak transform characterizations of spanning properties of Gabor systems, see Theorem 3.1 in [7].

Theorem 2.2.3. *Let $g \in L^2(\mathbb{R})$.*

1. *$\mathscr{G}(g,1,1)$ is complete if and only if $Zg \neq 0$ a.e.*
2. *$\mathscr{G}(g,1,1)$ is exact if and only if $1/Zg \in L^2(Q)$.*
3. *$\mathscr{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$ if and only if there exist $0 < A \leq B < \infty$ such that $A \leq |Zg|^2 \leq B$ a.e.*
4. *$\mathscr{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|Zg|^2 = 1$ a.e.*

The work in [24] extends Theorem 2.2.3 to provide a characterization of when $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ, but the statement requires some background on Muckenhoupt weights, specifically rectangular $\mathscr{A}_2$ weights, on the d-dimensional torus $\mathbb{T}^d$.

Definition 2.2.4. *An a.e. positive function $v \in L^1(\mathbb{T}^d)$ is a rectangular $\mathscr{A}_2$ weight on $\mathbb{T}^d$, denoted $v \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^d)$, if*

$$[v]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^d)} = \sup_R \left(\frac{1}{|R|}\int_R v(t)dt\right)\left(\frac{1}{|R|}\int_R \frac{1}{v(t)}dt\right) < \infty, \tag{2.2.2}$$

where R is any rectangle contained in $\mathbb{T}^d$ with sides parallel to the axes.

In dimension $d = 1$, $\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})$ coincides with the traditional class of Muckenhoupt weights where the averages in (2.2.2) are taken over squares or balls (instead of rectangles), but this is not the case in higher dimensions.

The next result gives two alternate characterizations of $\mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$, one in terms of one-dimensional sections and one in terms of the boundedness of the rectangular partial sum operators, e.g., see Theorem 2.2 in [24]. We will use the one-dimensional characterization explicitly in the following chapters. The partial sum characterization is used to prove Theorem 2.2.6 and is thus useful to be aware of.

Given a function $v(t,\xi)$ of two variables, if ξ is fixed, $v_\xi(t) = v(t,\xi)$ denotes the associated one-dimensional section of v. Likewise, if t is fixed, $v_t(\xi) = v(t,\xi)$.

Theorem 2.2.5. *Let $v(t,\xi)$ be an a.e. positive integrable function defined for $(t,\xi) \in \mathbb{T}^2$. The following statements are equivalent:*

1. $v \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$.

2. *The functions v_t and v_ξ are uniformly in $\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})$ for a.e. $t,\xi \in \mathbb{T}$. In other words,*

$$\operatorname*{ess\,sup}_{\xi\in\mathbb{T}} [v_\xi]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} < \infty \qquad \textit{and} \qquad \operatorname*{ess\,sup}_{t\in\mathbb{T}} [v_t]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} < \infty.$$

3. *The rectangular partial sum operators defined by*

$$S_{\mathscr{R},(K,N)} f = \sum_{|k|\le K} \sum_{|n|\le N} \langle f, E_{n,k}\rangle E_{n,k}$$

are uniformly bounded on $L^2(\mathbb{T}^2, v)$. That is,

$$\sup_{K,N} \|S_{\mathscr{R},(K,N)}\|_{L^2(\mathbb{T}^2,v)} < \infty.$$

The next result characterizes when $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ, see Theorem 4.3 in [24], cf. [20, 25, 26, 27] for related work. We identify $\mathbb{T}^d$ with any translate of $[0,1)^d$ and identify $L^1(\mathbb{T}^d)$ with the integrable $\mathbb{Z}^d$-periodic functions on $\mathbb{R}^d$. With these identifications, note that by quasiperiodicity of Zg and recalling that Zg is a unitary operator, and by Theorem 2.2.3, if $g \in L^2(\mathbb{R})$ and $\mathscr{G}(g,1,1)$ is complete then $|Zg|^2$ may be identified with an a.e. positive element of $L^1(\mathbb{T}^2)$.

Theorem 2.2.6. *Let $g \in L^2(\mathbb{R})$. The Gabor system $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ if and only if $|Zg|^2 \in A_{2,\mathscr{R}}(\mathbb{T}^2)$.*

Theorem 2.2.6 can be used to give simple examples of Gabor systems $\mathscr{G}(g,1,1)$ that are Schauder bases of type Λ, but are not Riesz bases. For example, let $\chi_{[0,1]}(t)$ denote the indicator function of $[0,1]$, and let $g(t) = t^s \chi_{[0,1]}(t)$, for any fixed $-1/2 < s < 1/2$. It can be verified that $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ, but it is not a Riesz basis, cf. [24, 20].

2.3 Other Useful Results

We begin with a fact which we will apply in Chapter 3 often (and without reference), see (22) in [28].

Fact 2.3.1. *For $\beta > 0$, $x \geq 0$, and $y \geq 0$, there exist constants $c_\beta > 0$ and $C_\beta > 0$ such that*

$$c_\beta (x+y)^\beta \leq x^\beta + y^\beta \leq C_\beta (x+y)^\beta.$$

The following result was proved in Lemma 5 of [28]. Let $\Gamma_h F(t,\xi) = F(t,\xi+h) -$

$F(t,\xi)$ and $\Gamma_h^2 F = \Gamma_h \Gamma_h F$. Similarly, let $\Delta_h F(t,\xi) = F(t+h,\xi) - F(t,\xi)$ and $\Delta_h^2 F = \Delta_h \Delta_h F$.

Lemma 2.3.2. *For* $0 < \varepsilon < 4$*,*

$$\int_{\mathbb{R}} |t|^{4-\varepsilon} |g(t)|^2 dt < \infty \iff \int_{\mathbb{R}} \iint_{[0,1]^2} \frac{\left|\Gamma_h^2 Zg(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh < \infty$$

and

$$\int_{\mathbb{R}} |\xi|^{4-\varepsilon} |\hat{g}(\xi)|^2 d\xi < \infty \iff \int_{\mathbb{R}} \iint_{[0,1]^2} \frac{\left|\Delta_h^2 Zg(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh.$$

We end this section with another result from Nitzan and Olsen, see Lemma 3(c) of [28] which can be extended to the two-variable case below.

Lemma 2.3.3. *If f is a function on $\mathbb{R}^2$, $h \geq 0$ and $f \in C^k([x, x+kh] \times [y, y+kh])$, then*

$$\left|\Delta_h^k f(x,y)\right| \leq |h|^k \sup_{\beta \in [x, x+kh]} \left|\frac{\partial^k f(\beta, y)}{\partial x^k}\right|$$

and

$$\left|\Gamma_h^k f(x,y)\right| \leq |h|^k \sup_{\beta \in [y, y+kh]} \left|\frac{\partial^k f(x, \beta)}{\partial y^k}\right|.$$

For $h < 0$, the same estimates hold over the rectangle $[x+kh, x] \times [y+kh, y]$.

2.4 Balian-Low Literature

In Section 2.4.1, we state and provide a brief history of the classical Balian-Low theorem. In Section 2.4.2, we provide an overview of some relevant versions of the Balian-Low theorem. Comprehensive surveys on Balian-Low type results were published in 1995 and 2006, see [7], [9].

2.4.1 History of the Balian-Low Theorem

In general, an uncertainty principle is any result that places restrictions on the localization of a function and its Fourier transform. The classical uncertainty principle is often referred to as the Heisenberg-Pauli-Weyl inequality, see Theorem 2.2.1 in [15] for a proof.

Theorem 2.4.1. *If $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, then*

$$\left(\int (t-a)^2 |f(t)|^2 dt\right)^{\frac{1}{2}} \left(\int (\xi - b)^2 |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \geq \frac{1}{4\pi} \|f\|_2^2.$$

The classical Balian-Low theorem is an uncertainty principle that constrains the time and frequency localization of the generator of a Gabor orthonormal basis. It was stated separately by Balian in [3] and Low in [23].

Theorem 2.4.2. *If $g \in L^2(\mathbb{R})$ satisfies*

$$\int |t|^2 |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\xi|^2 |\widehat{g}(\xi)|^2 d\xi < \infty,$$

then $\mathscr{G}(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

Both Balian and Low's proofs contained the same gap, assuming that the time and frequency localization of g would give a continuous Zg. Corrected proofs were subsequently independently given by Battle in [4] and Daubechies, Coifman, and Semmes in [12]. Battle's proof was entirely new, while Daubechies, Coifman, and Semmes directly filled in the gap from Balian and Low.

2.4.2 Balian-Low Type Results

In Section 1.1, we stated versions of the Balian-Low theorem for orthonormal bases, Riesz bases, and exact systems, as well as results from [20] relating to Schauder bases. We collectively refer to these and other uncertainty principles that constrain the time and frequency localization of the generator of a Gabor structured spanning system as Balian-Low theorems or Balian-Low type results.

The classical Balian-Low theorem considers Gabor systems with a single generating function. In [33], Zibuski and Zeevi address whether a Gabor system generated by multiple functions would allow for better localization. Below we present Theorem 7 from [33].

Theorem 2.4.3. *Let $R > 0$ and $G = \{g^r\}_{r=1}^R \subset L^2(\mathbb{R})$. Let $a > 0$ and $R(ab)^{-1} = 1$. If for $1 \leq r \leq R$, g^r satisfies*

$$\int |t|^2 |g^r(t)|^2 dt < \infty \quad \textit{and} \quad \int |\xi|^2 |\widehat{g^r}(\xi)|^2 d\xi < \infty,$$

then $\mathcal{G}(G,a,b) = \left\{ g^r_{k,n}(t) \right\}_{1\leq r\leq R, k,n\in\mathbb{Z}}$ is not a Riesz basis for $L^2(\mathbb{R})$.

In other words, they find that if all the generating functions are well-localized, the corresponding Gabor system cannot be a Riesz basis.

In addition to the results on exact systems in [28], Nitzan and Olsen provide Balian-Low theorems that deal with the full range of spanning structures "between" exact systems and Riesz bases, called (C_q)-*systems*. Given $q \geq 2$, a system $\{f_n\}_{n=1}^\infty$ is a (C_q)-system for $L^2(\mathbb{R})$ if and only if for every $f \in L^2(\mathbb{R})$, $c\|f\| \leq (\sum_{n=1}^\infty |\langle f, f_n\rangle|^p)^{\frac{1}{p}}$, where $c = c(p)$ is a positive constant independent from f and $\frac{1}{p} + \frac{1}{q} = 1$, see [28]. Exact (C_q)-systems provide a continuous scale between exact systems, with $q = \infty$, and Riesz bases, with $q = 2$. We state below the symmetric Balian-Low result for (C_q)-systems, Theorem 1(a) from [28]; see Theorem 2(a) in [28] for a non-symmetric version. Further, [28] proves that both results are sharp.

Theorem 2.4.4. *Fix $q > 2$. Let $g \in L^2(\mathbb{R})$ and $r > 4(q-1)/q$. If g satisfies*

$$\int |t|^r |g(t)|^2 dt < \infty \quad \textit{and} \quad \int |\xi|^r |\widehat{g}(\xi)|^2 d\xi < \infty,$$

then $\mathcal{G}(g,1,1)$ is not an exact (C_q)-system in $L^2(\mathbb{R})$.

There have been several papers published with subspace Balian-Low type theorems, specifically in the setting of shift-invariant spaces. Let $g \in L^2(\mathbb{R}^d)$ and define the *integer translates of* g as $T(g) = \{g(t-k)\}_{k \in \mathbb{Z}^d}$. Then $V(g)$, the closed linear span of $T(g)$ in $L^2(\mathbb{R}^d)$, is said to be the *principal shift-invariant space generated by* g. We will consider shift-invariant spaces with some type of additional invariance. Let Γ be a lattice with $\mathbb{Z}^d \subset \Gamma$ and $\left[\Gamma : Z^d\right] > 1$. We say $V(g)$ is Γ-invariant if $h \in V(g)$ implies that $\{h(t-\gamma)\}_{\gamma \in \Gamma} \subset V(g)$. Note that we have stated the definition for a shift-invariant space with a single generator; one could also consider shift-invariant spaces with multiple generators.

Shift-invariant space results by Aldroubi, Sun, and Wang in [1] and by Tessera and Wang in [31] were improved upon by Hardin, Northington, and Powell in [17]. We state a simplified version of Corollary 1.4 in [17] below.

Theorem 2.4.5. *If* $g \in L^2(\mathbb{R})$ *satisfies* $V(g)$ *is* Γ*-invariant and*

$$\int |t||g(t)|^2 dt < \infty,$$

then $T(g)$ *does not form a Riesz basis for* $V(g)$.

This result is sharp.

There is a sharp version of Theorem 2.4.5 for exact systems, see Theorem 1.2.8 in [29].

Theorem 2.4.6. *If* $g \in L^2(\mathbb{R})$ *satisfies* $V(g)$ *is* Γ*-invariant and*

$$\int |t|^2 |g(t)|^2 dt < \infty,$$

then $T(g)$ *does not form an exact system for* $V(g)$.

Chapter 3

Proofs of Theorems

Throughout the following chapter, we use the notation $A \lesssim B$ to mean that there exists a constant $C > 0$ (that may vary from one usage to another) such that $A \leq CB$. We use the notation $A \asymp B$ to mean that $A \lesssim B$ and $B \lesssim A$.

3.1 Proof of Theorem 1.2.1

In this section we will prove Theorem 1.2.1. For fixed t, let $(Zg)_t$ be the function defined by $(Zg)_t(\xi) = Zg(t,\xi)$.

3.1.1 Window Function with Support on $[-1,1]$

We first prove the following intermediate theorem, which concretely shows how Theorem 1.2.1 operates on the interval $[-1,1]$ for a real-valued window function g.

Theorem 3.1.1. *If $g \in L^2(\mathbb{R})$ is real-valued, supported on $[-1,1]$, and satisfies*

$$\int |\xi|^2 |\widehat{g}(\xi)|^2 d\xi < \infty,$$

then $\mathcal{G}(g,1,1)$ is not a Schauder basis for $L^2(\mathbb{R})$.

Proof. Since g is supported on $[-1,1]$, the Zak transform Zg satisfies

$$\forall (t,\xi) \in [0,1)^2, \quad Zg(t,\xi) = g(t) + g(t-1)e^{2\pi i \xi}.$$

Since $\int |\xi|^2 |\widehat{g}(\xi)|^2 d\xi < \infty$, the Sobolev embedding theorem (see Theorem 8.4 in [22]) gives that g is continuous and

$$\forall x, y \in \mathbb{R}, \quad |g(x) - g(y)| \lesssim |x-y|^{\frac{1}{2}}. \tag{3.1.1}$$

Since g is continuous and supported on $[-1,1]$, the Zak transform Zg is continuous on $\mathbb{R}^2$. Since Zg is continuous and quasiperiodic, Lemma 2.2.1 shows that Zg has a zero in the unit square $Q = [0,1)^2$. Namely, there exists $(t_0, \xi_0) \in [0,1)^2$ such that $Zg(t_0, \xi_0) = 0$. Note that $g(t_0) \neq 0$ implies $g(t_0 - 1) \neq 0$, and $g(t_0) = 0$ implies $g(t_0 - 1) = 0$.

Case 1: Assume $g(t_0) \neq 0 \neq g(t_0 - 1)$. By Theorems 2.2.6 and 2.2.5, to show that $\mathscr{G}(g,1,1)$ is not a Schauder basis of type Λ, it suffices to show that $\operatorname{ess\,sup}_{t\in[0,1)} [|(Zg)_t|^2]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} = \infty$.

Since g is real-valued and $g(t_0 - 1) \neq 0$, $(Zg)_{t_0}(\xi_0) = 0$ implies that $\sin(2\pi\xi_0) = 0$, or that $\xi_0 = 0$ or $\xi_0 = \frac{1}{2}$. Note that

$$|(Zg)_t(\xi)|^2 = g^2(t) + g^2(t-1) + 2g(t)g(t-1)\cos(2\pi\xi). \tag{3.1.2}$$

<u>Subcase 1:</u> Assume $\xi_0 = 0$. Then $(Zg)_{t_0}(0) = 0$, so $g(t_0) = -g(t_0 - 1)$. Since g is continuous, there exists δ_1 such that $|t - t_0| < \delta_1$ implies

$$|g(t)||g(t-1)| = -g(t)g(t-1). \tag{3.1.3}$$

Then by (3.1.2) and (3.1.3), for t such that $|t - t_0| < \delta_1$,

$$|(Zg)_t(\xi)|^2 = g^2(t) + g^2(t-1) - 2|g(t)||g(t-1)|\cos(2\pi\xi). \tag{3.1.4}$$

Note that for sufficiently small u, the Taylor approximation for cosine yields

$$1 - \cos(2\pi u) \asymp u^2. \tag{3.1.5}$$

Fix $0 < \Delta < 1$ sufficiently small. Then using (3.1.4) and (3.1.5), we find

$$\begin{aligned}\frac{1}{\Delta}\int_0^\Delta |(Zg)_t(\xi)|^2\,d\xi &= \frac{1}{\Delta}\int_0^\Delta \left(g^2(t)+g^2(t-1)-2\,|g(t)|\,|g(t-1)|\cos(2\pi\xi)\right)d\xi \\ &= \frac{1}{\Delta}\int_0^\Delta \Big((|g(t)|-|g(t-1)|)^2+2\,|g(t)|\,|g(t-1)|\,(1-\cos(2\pi\xi))\Big)\,d\xi \\ &\gtrsim \frac{1}{\Delta}\int_0^\Delta \Big((|g(t)|-|g(t-1)|)^2+|g(t)|\,|g(t-1)|\,\xi^2\Big)\,d\xi \\ &= (|g(t)|-|g(t-1)|)^2+|g(t)|\,|g(t-1)|\,\Delta^2, \qquad (3.1.6)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\Delta}\int_0^\Delta \frac{1}{|(Zg)_t(\xi)|^2}d\xi = \frac{1}{\Delta}\int_0^\Delta \frac{1}{g^2(t)+g^2(t-1)-2\,|g(t)|\,|g(t-1)|\cos(2\pi\xi)}d\xi \\ &= \frac{1}{\Delta}\int_0^\Delta \frac{1}{(|g(t)|-|g(t-1)|)^2+2\,|g(t)|\,|g(t-1)|\,(1-\cos(2\pi\xi))}d\xi \\ &\gtrsim \frac{1}{\Delta}\int_0^\Delta \frac{1}{(|g(t)|-|g(t-1)|)^2+|g(t)|\,|g(t-1)|\,\xi^2}d\xi \\ &= \frac{1}{\Delta}\cdot\frac{1}{(|g(t)|-|g(t+1)|)^2}\cdot\frac{||g(t)|-|g(t-1)||}{\sqrt{|g(t)|\,|g(t-1)|}}\cdot\arctan\left(\frac{\sqrt{|g(t)|\,|g(t-1)|}}{||g(t)|-|g(t-1)||}\Delta\right). \qquad (3.1.7)\end{aligned}$$

Let $A = \frac{\sqrt{|g(t)||g(t+1)|}}{||g(t)|-|g(t+1)||}\Delta$. Then by (3.1.6) and (3.1.7),

$$\left(\frac{1}{\Delta}\int_0^\Delta |(Zg)_t(\xi)|^2\,d\xi\right)\left(\frac{1}{\Delta}\int_0^\Delta \frac{1}{|(Zg)_t(\xi)|^2}d\xi\right) \gtrsim \frac{1}{A}\arctan(A)+A\arctan(A). \qquad (3.1.8)$$

Using (3.1.8), and since $|g(t_0)| = |g(t_0+1)|$ and g is continuous,

$$\lim_{t\to t_0}\left(\frac{1}{\Delta}\int_0^\Delta |(Zg)_t(\xi)|^2\,d\xi\right)\left(\frac{1}{\Delta}\int_0^\Delta \frac{1}{|(Zg)_t(\xi)|^2}d\xi\right) = \infty.$$

Thus $\lim_{t\to t_0}[|(Zg)_t|^2]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} = \infty$, and $\mathscr{G}(g,1,1)$ is not a Schauder basis of type Λ.

Subcase 2: Assume $\xi_0 = \frac{1}{2}$. Then $(Zg)_{t_0}(\frac{1}{2}) = 0$, so $g(t_0) = g(t_0-1)$. Since g is continuous, there exists δ_2 such that $|t-t_0|<\delta_2$ implies

$$|g(t)||g(t-1)| = g(t)g(t-1). \tag{3.1.9}$$

Then by (3.1.2) and (3.1.9), for t such that $|t-t_0|<\delta_2$,

$$|(Zg)_t(\xi)|^2 = g^2(t)+g^2(t-1)+2\,|g(t)|\,|g(t-1)|\cos(2\pi\xi). \tag{3.1.10}$$

Fix $0<\Delta<1$ sufficiently small. Using (3.1.10), we find

$$\begin{aligned}\frac{1}{\Delta}\int_{\frac{1}{2}}^{\frac{1}{2}+\Delta}|(Zg)_t(\xi)|^2\,d\xi &= \frac{1}{\Delta}\int_{\frac{1}{2}}^{\frac{1}{2}+\Delta}\big(g^2(t)+g^2(t-1)+2\,|g(t)|\,|g(t-1)|\cos(2\pi\xi)\big)\,d\xi\\ &= \frac{1}{\Delta}\int_{0}^{\Delta}\big(g^2(t)+g^2(t-1)-2\,|g(t)|\,|g(t-1)|\cos(2\pi u)\big)\,du,\end{aligned} \tag{3.1.11}$$

and

$$\begin{aligned}\frac{1}{\Delta}\int_{\frac{1}{2}}^{\frac{1}{2}+\Delta}\frac{1}{|(Zg)_t(\xi)|^2}d\xi &= \frac{1}{\Delta}\int_{\frac{1}{2}}^{\frac{1}{2}+\Delta}\frac{1}{g^2(t)+g^2(t-1)+2\,|g(t)|\,|g(t-1)|\cos(2\pi\xi)}d\xi\\ &= \frac{1}{\Delta}\int_{0}^{\Delta}\frac{1}{g^2(t)+g^2(t-1)-2\,|g(t)|\,|g(t-1)|\cos(2\pi u)}du.\end{aligned} \tag{3.1.12}$$

We can now apply (3.1.5) and continue as in (3.1.6) and (3.1.7) in Subcase 1.

Case 2: Assume $g(t_0)=0=g(t_0-1)$. This implies that $Zg(t_0,\xi)=0$ for all $\xi\in\mathbb{R}$. We will show that $\mathscr{G}(g,1,1)$ is not exact in $L^2(\mathbb{R})$, by using a similar argument as for Theorem 5.1 in [21]. By (3.1.1), we have

$$|Zg(t,\xi)| = |Zg(t,\xi)-Zg(t_0,\xi)| \le |g(t)-g(t_0)|+|g(t-1)-g(t_0-1)| \lesssim |t-t_0|^{\frac{1}{2}},$$

and hence

$$\int_0^1 \int_0^1 \frac{1}{|Zg(t,\xi)|^2} dt d\xi \gtrsim \int_0^1 \int_0^1 \frac{1}{|t-t_0|} dt d\xi = \infty.$$

By Theorem 2.2.3, this shows that $\mathscr{G}(g,1,1)$ is not exact, and hence is not a Schauder basis.

□

3.1.2 Window Function with General Compact Support

We now prove Theorem 1.2.1: If $g \in L^2(\mathbb{R})$ is compactly supported and

$$\int |\xi|^2 |\widehat{g}(\xi)|^2 d\xi < \infty,$$

then $\mathscr{G}(g,1,1)$ is not a Schauder basis of type Λ for $L^2(\mathbb{R})$.

Without loss of generality, we assume that g is supported on $[-N,1]$ for some $N \in \mathbb{N}$. So, the Zak transform Zg satisfies

$$\forall (t,\xi) \in [0,1)^2, \quad Zg(t,\xi) = \sum_{n=0}^{N} g(t-n) e^{2\pi i n \xi}.$$

Since $\int |\xi|^2 |\widehat{g}(\xi)|^2 d\xi < \infty$, the Sobolev embedding theorem (see Theorem 8.4 in [22]) gives that g is continuous and

$$\forall x,y \in \mathbb{R}, \quad |g(x)-g(y)| \lesssim |x-y|^{\frac{1}{2}}. \tag{3.1.13}$$

Since g is continuous and compactly supported, the Zak transform Zg is continuous on $\mathbb{R}^2$. Since Zg is continuous and quasiperiodic, Lemma 2.2.1 shows that Zg has a zero in the unit square $Q = [0,1)^2$. Namely, there exists $(t_0,\xi_0) \in [0,1)^2$ such that $Zg(t_0,\xi_0) = 0$.

For each fixed $t \in \mathbb{R}$, it will be convenient to consider the polynomial P_t defined by

$$\forall z \in \mathbb{C}, \quad P_t(z) = \sum_{n=0}^{N} g(t-n) z^n. \tag{3.1.14}$$

Note that $P_t(e^{2\pi i \xi}) = (Zg)_t(\xi) = Zg(t,\xi)$. In particular, the polynomial $P_{t_0}(z)$ has a root at $z = e^{2\pi i \xi_0}$; let m_0 denote the multiplicity of this root.

We collect here some notation about P_t that will be used throughout the proof. For $t \in \mathbb{R}$, let $0 \leq d_t \leq N$ denote the degree of the polynomial P_t, and when $d_t \geq 1$ label the (possibly non-distinct) roots of $P_t(z)$ as $\{r_k(t)\}_{k=1}^{d_t}$. Moreover, when $t = t_0$, we shall assume that $P_{t_0}(z)$ has its roots $\{r_k(t_0)\}_{k=1}^{d_{t_0}}$ ordered so that $r_k(t_0) = e^{2\pi i \xi_0}$ for $1 \leq k \leq m_0$.

Case 1: Suppose the degree of the polynomial P_{t_0} satisfies $1 \leq d_{t_0} \leq N$ and that P_{t_0} has at least two distinct roots. By Theorems 2.2.6 and 2.2.5, to show that $\mathscr{G}(g,1,1)$ is not a Schauder basis of type Λ, it suffices to show that $\operatorname{ess\,sup}_{t \in [0,1)} [|(Zg)_t|^2]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} = \infty$. Since the coefficient functions of P_t are continuous, we may use results on the continuity of roots of polynomials to relate the roots of P_t to the roots of P_{t_0} when $|t - t_0|$ is sufficiently small.

Given arbitrary $\varepsilon > 0$, by Theorem 1 of [32] there exists $\delta = \delta_\varepsilon > 0$ such that if $|t - t_0| < \delta$, then there exists an ordering of the roots of $P_t(z)$ such that

$$\forall\ 1 \leq k \leq d_{t_0}, \quad |r_k(t) - r_k(t_0)| < \varepsilon, \tag{3.1.15}$$

and

$$\forall\ d_{t_0} < k \leq d_t, \quad |r_k(t)| > \frac{1}{\varepsilon}. \tag{3.1.16}$$

The existence of large roots of P_t as in (3.1.16) only occurs when P_t has larger degree than P_{t_0}, i.e., $d_t > d_{t_0}$. For perspective, since the leading coefficient of P_{t_0} is $g(t_0 - d_{t_0}) \neq 0$ and g is continuous, there exists $\delta_1 > 0$ such that $|t - t_0| < \delta_1$ implies $d_t \geq d_{t_0}$.

Let $\beta_0 > 0$ be the smallest distance between *distinct* roots of P_{t_0}. By (3.1.15) and (3.1.16), there exists $\delta_2 > 0$ such that $|t - t_0| < \delta_2$ implies $|r_k(t) - r_k(t_0)| < \beta_0/4$ for $1 \leq k \leq$

d_{t_0}, and $|r_k(t)| > 2$ for $d_{t_0} < k \le d_t$. We shall assume henceforth that $|t - t_0| < \min\{\delta_1, \delta_2\}$.

Fix $0 < \Delta < 1$ sufficiently small so that $\xi \in [\xi_0, \xi_0 + \Delta]$ implies $|e^{2\pi i\xi} - e^{2\pi i\xi_0}| \le \beta_0/4$. We will next estimate the integral

$$I_1 = \frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta} |(Zg)_t(\xi)|^2 d\xi = \frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta} |P_t(e^{2\pi i\xi})|^2 d\xi. \tag{3.1.17}$$

Before bounding I_1 from below, we require some preliminary estimates. It will be convenient to write the polynomial P_t in factored form, so that

$$P_t(e^{2\pi i\xi}) = g(t-d_t)\prod_{j=1}^{m_0}(e^{2\pi i\xi} - r_j(t)) \prod_{k=m_0+1}^{d_{t_0}}(e^{2\pi i\xi} - r_k(t)) \prod_{l=d_{t_0}+1}^{d_t}(e^{2\pi i\xi} - r_l(t)). \tag{3.1.18}$$

If $m_0 + 1 \le k \le d_{t_0}$ and $\xi \in [\xi_0, \xi_0 + \Delta]$, then

$$\begin{aligned} \left|e^{2\pi i\xi} - r_k(t)\right| &\ge \left|e^{2\pi i\xi_0} - r_k(t_0)\right| - \left|e^{2\pi i\xi_0} - e^{2\pi i\xi}\right| - |r_k(t_0) - r_k(t)| \\ &> \beta_0 - \frac{\beta_0}{4} - \frac{\beta_0}{4} = \frac{\beta_0}{2}. \end{aligned} \tag{3.1.19}$$

If $d_{t_0} + 1 \le l \le d_t$, then (here recall that $|r_l(t)| > 2$) one has

$$\left|e^{2\pi i\xi} - r_l(t)\right| \ge \left||e^{2\pi i\xi}| - |r_l(t)|\right| = |r_l(t)| - 1. \tag{3.1.20}$$

Combining (3.1.17), (3.1.18), (3.1.19) and (3.1.20) gives

$$I_1 \ge \frac{|g(t-d_t)|^2(\beta_0^2/4)^{(d_{t_0}-m_0)}\prod_{l=d_{t_0}+1}^{d_t}(|r_l(t)|-1)^2}{\Delta}\int_{\xi_0}^{\xi_0+\Delta}\prod_{j=1}^{m_0}\left|e^{2\pi i\xi} - r_j(t)\right|^2 d\xi. \tag{3.1.21}$$

To estimate the integral in (3.1.21), recall from (3.1.5) that $|e^{2\pi iu} - 1| \asymp |u|$ holds for sufficiently small $|u|$. So

$$\begin{aligned}\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta}\prod_{j=1}^{m_0}\left|e^{2\pi i\xi}-r_j(t)\right|^2 d\xi &\geq \frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta}\prod_{j=1}^{m_0}\left(\left|e^{2\pi i\xi}-e^{2\pi i\xi_0}\right|-\left|e^{2\pi i\xi_0}-r_j(t)\right|\right)^2 d\xi\\ &=\frac{1}{\Delta}\int_0^{\Delta}\prod_{j=1}^{m_0}\left(\left|e^{2\pi iu}-1\right|-\left|e^{2\pi i\xi_0}-r_j(t)\right|\right)^2 du\\ &\gtrsim \frac{1}{\Delta}\int_0^{\Delta}\prod_{j=1}^{m_0}\left(u-\left|e^{2\pi i\xi_0}-r_j(t)\right|\right)^2 du. \end{aligned} \tag{3.1.22}$$

Expanding the product in (3.1.22) gives the following form for suitable $A_n(t)$

$$\prod_{j=1}^{m_0}\left(u-\left|e^{2\pi i\xi_0}-r_j(t)\right|\right)^2 = u^{2m_0}-\sum_{n=1}^{2m_0}u^{2m_0-n}A_n(t). \tag{3.1.23}$$

By (3.1.15), we have $\lim_{t\to t_0}\left|e^{2\pi i\xi_0}-r_j(t)\right|=0$ for each $1\leq j\leq m_0$. Thus, taking limits of both sides of (3.1.23) implies that $\lim_{t\to t_0}A_n(t)=0$ for each $1\leq n\leq 2m_0$.

Combining (3.1.22) and (3.1.23) gives

$$\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta}\prod_{j=1}^{m_0}\left|e^{2\pi i\xi}-r_j(t)\right|^2 d\xi \gtrsim \left(\Delta^{2m_0}-\sum_{n=1}^{2m_0}\Delta^{2m_0-n}A_n(t)\right). \tag{3.1.24}$$

So, (3.1.21) together with (3.1.24), and absorbing the constant $(\beta_0^2/4)^{(d_{x_0}-m_0)}$, gives a lower bound on the integral I_1 in (3.1.17)

$$I_1 \gtrsim |g(t-d_t)|^2\left(\prod_{l=d_{t_0}+1}^{d_t}(|r_l(t)|-1)^2\right)\left(\Delta^{2m_0}-\sum_{n=1}^{2m_0}\Delta^{2m_0-n}A_n(t)\right). \tag{3.1.25}$$

We next estimate the integral

$$I_2=\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta}\frac{1}{|(Zg)_t(\xi)|^2}d\xi=\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta}\frac{1}{|P_t(e^{2\pi i\xi})|^2}d\xi. \tag{3.1.26}$$

Before bounding I_2 from below, we again require some preliminary estimates.

If $m_0+1 \le k \le d_{t_0}$ and $M_0 = \max_{m_0+1\le k\le d_{t_0}} \left|e^{2\pi i\xi_0} - r_k(t_0)\right|$ and $\xi \in [\xi_0, \xi_0+\Delta]$, then

$$\begin{aligned}\left|e^{2\pi i\xi} - r_k(t)\right| &\le \left|e^{2\pi i\xi} - e^{2\pi i\xi_0}\right| + \left|e^{2\pi i\xi_0} - r_k(t_0)\right| + |r_k(t_0) - r_k(t)| \\ &< \frac{\beta_0}{4} + M_0 + \frac{\beta_0}{4} = M_0 + \frac{\beta_0}{2}. \end{aligned} \tag{3.1.27}$$

If $d_{t_0}+1 \le l \le d_t$, we will use the bound

$$\left|e^{2\pi i\xi} - r_l(t)\right| \le \left|e^{2\pi i\xi}\right| + |r_l(t)| = |r_l(t)| + 1. \tag{3.1.28}$$

Using (3.1.18), (3.1.26), (3.1.27), (3.1.28), and proceeding similarly as for (3.1.21), we have

$$I_2 \ge \frac{(M_0+\beta_0/2)^{-2(d_{t_0}-m_0)}}{|g(t-d_t)|^2\Delta\prod_{l=d_{t_0}+1}^{d_t}(|r_l(t)|+1)^2}\int_{\xi_0}^{\xi_0+\Delta}\prod_{j=1}^{m_0}\frac{1}{\left|e^{2\pi i\xi} - r_j(t)\right|^2}d\xi. \tag{3.1.29}$$

To estimate the integral in (3.1.29), let $R(t) = \max_{1\le j\le m_0}|e^{2\pi i\xi_0} - r_j(t)|$, and note that $\lim_{t\to t_0} R(t) = 0$ because of (3.1.15). Since $0 < \Delta < 1$ is sufficiently small we have

$$\begin{aligned}\int_{\xi_0}^{\xi_0+\Delta}\prod_{j=1}^{m_0}\frac{1}{\left|e^{2\pi i\xi} - r_j(t)\right|^2}d\xi &\gtrsim \int_{\xi_0}^{\xi_0+\Delta}\prod_{j=1}^{m_0}\frac{1}{\left|e^{2\pi i\xi} - e^{2\pi i\xi_0}\right|^2 + \left|e^{2\pi i\xi_0} - r_j(t)\right|^2}d\xi \\ &\ge \int_0^{\Delta}\prod_{j=1}^{m_0}\frac{1}{|e^{2\pi iu}-1|^2 + R^2(t)}du \\ &\gtrsim \int_0^{\Delta}\frac{1}{(u^2+R^2(t))^{m_0}}du \\ &\gtrsim \int_0^{\Delta}\frac{1}{u^2+R^{2m_0}(t)}du \\ &= \frac{1}{R^{m_0}(t)}\arctan\left(\frac{\Delta}{R^{m_0}(t)}\right). \end{aligned} \tag{3.1.30}$$

Combining (3.1.29) and (3.1.30), and absorbing the constants $(M_0+\beta_0/2)^{-2(d_{t_0}-m_0)}$ and

Δ, gives a lower bound for the integral I_2

$$I_2 \gtrsim \frac{1}{|g(t-d_t)|^2 \prod_{l=d_{t_0}+1}^{d_t} (|r_l(t)|+1)^2} \left(\frac{1}{R^{m_0}(t)}\right) \arctan\left(\frac{\Delta}{R^{m_0}(t)}\right). \tag{3.1.31}$$

By (3.1.25) and (3.1.31),

$$\begin{aligned} &\left(\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta} |(Zg)_t(\xi)|^2 \, d\xi\right) \left(\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta} \frac{1}{|(Zg)_t(\xi)|^2} d\xi\right) \qquad (3.1.32)\\ &\gtrsim \left(\prod_{l=d_{t_0}+1}^{d_t} \frac{(|r_l(t)|-1)^2}{(|r_l(t)|+1)^2}\right) \left(\Delta^{2m_0} - \sum_{n=1}^{2m_0} \Delta^{2m_0-n} A_n(t)\right) \left(\frac{1}{R^{m_0}(t)}\right) \arctan\left(\frac{\Delta}{R^{m_0}(t)}\right). \end{aligned}$$

Now recall that $|r_l(t)| > 2$ when $d_{t_0}+1 \le l \le d_t$, and $d_t \le N$, and that $\lim_{t\to t_0} A_n(t) = 0$ for $1 \le n \le 2m_0$ and $\lim_{t\to t_0} R(t) = 0$. This, together with (3.1.32), implies that

$$\lim_{t\to t_0} \left(\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta} |(Zg)_t(\xi)|^2 \, d\xi\right) \left(\frac{1}{\Delta}\int_{\xi_0}^{\xi_0+\Delta} \frac{1}{|(Zg)_t(\xi)|^2} d\xi\right) = \infty. \tag{3.1.33}$$

Thus $\lim_{t\to t_0} [|(Zg)_t|^2]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} = \infty$, and $\mathscr{G}(g,1,1)$ is not a Schauder basis of type Λ.

Case 2: Suppose the degree of the polynomial P_{t_0} satisfies $1 \le d_{t_0} \le N$ and that all roots of P_{t_0} are the same. The proof of this case is similar to *Case 1*, except that we now have $m_0 = d_{t_0}$, and instead of (3.1.18) we have the simpler factorization

$$P_t(e^{2\pi i\xi}) = g(t-d_t) \prod_{j=1}^{m_0} (e^{2\pi i\xi} - r_j(t)) \prod_{l=d_{t_0}+1}^{d_t} (e^{2\pi i\xi} - r_l(t)). \tag{3.1.34}$$

Then combining (3.1.17), (3.1.34), and (3.1.20) gives

$$I_1 \ge \frac{|g(t-d_t)|^2 \prod_{l=d_{t_0}+1}^{d_t} (|r_l(t)|-1)^2}{\Delta} \int_{\xi_0}^{\xi_0+\Delta} \prod_{j=1}^{m_0} \left|e^{2\pi i\xi} - r_j(t)\right|^2 d\xi,$$

and combining (3.1.26), (3.1.34), and (3.1.28) gives

$$I_2 \geq \frac{1}{|g(t-d_t)|^2 \Delta \prod_{l=d_{t_0}+1}^{d_t} (|r_l(t)|+1)^2} \int_{\xi_0}^{\xi_0+\Delta} \prod_{j=1}^{m_0} \frac{1}{\left|e^{2\pi i \xi} - r_j(t)\right|^2} d\xi.$$

Thus the same arguments as in *Case 1* show that the estimates (3.1.32) and (3.1.33) both still hold, so that $\mathscr{G}(g,1,1)$ is not a Schauder basis of type Λ.

Case 3: Suppose that the polynomial P_{t_0} has degree $d_{t_0} = 0$. Since $P_{t_0}(e^{2\pi i \xi_0}) = 0$, this means that P_{t_0} must be identically zero. So, $g(t_0 - n) = 0$ for all $n \in \{0, 1, \cdots, N\}$. This implies that $Zg(t_0, \xi) = 0$ for all $\xi \in \mathbb{R}$.

We will show that $\mathscr{G}(g,1,1)$ is not exact in $L^2(\mathbb{R})$, by using a similar argument as for Theorem 5.1 in [21]. By (3.1.13), we have

$$|Zg(t,\xi)| = |Zg(t,\xi) - Zg(t_0,\xi)| \leq \sum_{n=0}^{N} |g(t-n) - g(t_0-n)| \lesssim |t-t_0|^{\frac{1}{2}},$$

and hence

$$\int_0^1 \int_0^1 \frac{1}{|Zg(t,\xi)|^2} dt d\xi \gtrsim \int_0^1 \int_0^1 \frac{1}{|t-t_0|} dt d\xi = \infty.$$

By Theorem 2.2.3, this shows that $\mathscr{G}(g,1,1)$ is not exact, and hence is not a Schauder basis. □

Recall that in the original statement of Theorem 1.2.1, we claimed that the conclusion held for g and $\widehat{g}$ interchanged. We have the identity $Z\widehat{g}(t,\xi) = e^{2\pi i t \xi} Zg(-\xi, t)$, see Proposition 8.2.2 in [15]. Thus we have the following extensions of the Zak transform characterizations of spanning properties of Gabor systems in Theorem 2.2.3 and Theorem 2.2.6

1. $\mathscr{G}(g,1,1)$ is exact if and only if $1/Z\widehat{g} \in L^2(Q)$.

2. $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ if and only if $|Z\widehat{g}|^2 \in A_{2,\mathscr{R}}(\mathbb{T}^2)$.

This allows for the interchange of g and $\widehat{g}$ in the statement of Theorem 1.2.1.

3.2 Proof of Theorem 1.2.2

In this section we will prove Theorem 1.2.2. We first require the following lemma.

Lemma 3.2.1. *Fix $0 < \alpha < \frac{1}{2}$ and $\gamma > 0$. If f is the $\mathbb{Z}^2$-periodic function defined by*

$$f(t,\xi) = (|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\alpha}{\gamma}})^{2\gamma}$$

for $(t,\xi) \in [-\frac{1}{2},\frac{1}{2})^2$, then $f \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$.

Proof. For fixed ξ, let f_ξ be the function defined by $f_\xi(t) = f(t,\xi)$. By Theorem 2.2.5 and by the symmetry of $f(t,\xi)$, it suffices to show that

$$\operatorname*{ess\,sup}_{\xi\in\mathbb{T}} \left[f_\xi\right]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} < \infty.$$

Given an interval $I \subset \mathbb{T}$, we must bound the quantity

$$J = \left(\frac{1}{|I|}\int_I (|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\alpha}{\gamma}})^{2\gamma} dt\right)\left(\frac{1}{|I|}\int_I \frac{1}{(|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\alpha}{\gamma}})^{2\gamma}} dt\right).$$

Since $\alpha,\gamma > 0$, we have

$$\begin{aligned} J &\lesssim \left(\frac{1}{|I|}\int_I |t|^{2\alpha} dt + |\xi|^{2\alpha}\right)\left(\frac{1}{|I|}\int_I \frac{1}{(|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\alpha}{\gamma}})^{2\gamma}} dt\right) \\ &\leq \left(\frac{1}{|I|}\int_I |t|^{2\alpha} dt\right)\left(\frac{1}{|I|}\int_I \frac{1}{|t|^{2\alpha}} dt\right) + \left(\frac{1}{|I|}\int_I \frac{|\xi|^{2\alpha}}{(|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\alpha}{\gamma}})^{2\gamma}} dt\right) \\ &\leq [|t|^{2\alpha}]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} + 1. \end{aligned}$$

Remark 3.8 and Example 3.11 from [24] show that $|t|^{2\alpha} \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T})$, since $0 < \alpha < 1/2$. It follows that $\operatorname{ess\,sup}_{\xi\in\mathbb{T}} \left[f_\xi\right]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} \lesssim [|t|^{2\alpha}]_{\mathscr{A}_{2,\mathscr{R}}(\mathbb{T})} + 1 < \infty$, as required. So $f \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$. $\square$

3.2.1 Version 1 of Proof

We are now ready to prove Theorem 1.2.2. We first provide a proof relying on the construction from Section 6.1 of [28], with some detail omitted.

Proof of Theorem 1.2.2 (Version 1) Fix $0 < \varepsilon < 1/2$, and let $g \in L^2(\mathbb{R})$ be the function given by part (b) of Theorem 1 in [28] when $r = 3-2\varepsilon$ and $q > \frac{4}{1+\varepsilon}$. Equivalently, let $g \in L^2(\mathbb{R})$ be the function given by part (b) of Theorem 2 in [28] when $r = s = 3-2\varepsilon$ and $q > \frac{4}{1+\varepsilon}$. Note that $q > \frac{4}{1+\varepsilon} > \frac{4}{1+2\varepsilon}$, and that $q > \frac{4}{1+2\varepsilon}$ is equivalent to $r < \frac{4(q-1)}{q}$. Hence, the hypotheses of Theorem 1(b) and Theorem 2(b) in [28] hold, and g satisfies

$$\int |t|^{3-2\varepsilon}|g(t)|^2 dt < \infty \quad \text{and} \quad \int |\xi|^{3-2\varepsilon}|\widehat{g}(\xi)|^2 d\xi < \infty.$$

It remains to show that $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ. Since (48) in [28] holds, Section 6.1 of [28] shows that the Zak transform $G = Zg$ satisfies:

- G is quasiperiodic. In particular, $|G|$ is $\mathbb{Z}^2$-periodic.
- $|G|$ is continuous and bounded on $\mathbb{R}^2$.
- There exist $\delta, \eta > 0$ such that if $(t,\xi) \notin (-\eta,\eta)^2 + \mathbb{Z}^2$, then $\delta < |G(t+\frac{1}{2}, \xi+\frac{1}{2})|$.
- There exist parameters $\alpha, \beta, \gamma > 0$ such that if $(t,\xi) \in (-\eta,\eta)^2$ then

$$|G(t+1/2, \xi+1/2)| = \left(|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\beta}{\gamma}}\right)^{\gamma}. \tag{3.2.1}$$

The condition (3.2.1) follows from (54) in [28], together with the expression for $\Phi(x,y)$ on page 596 in [28]. The parameters $\alpha, \beta > 0$ in (3.2.1) are defined as follows. Since $r = s = 3-2\varepsilon$, we may define $r' = s' = 3-\varepsilon$ as in equation (51) in [28], and note that (52) in [28] holds because $q > \frac{4}{1+\varepsilon}$. Now, equation (53) in [28] gives $\alpha = \beta = \frac{r'}{2}\left(1-\frac{2}{r'}\right) = \frac{1-\varepsilon}{2}$.

By Theorem 2.2.6, it suffices to show that $|G|^2 = |Zg|^2 \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$. Recall, by (3.2.1), that if $(t,\xi) \in (-\eta,\eta)^2$, then $\left|G\left(t+\frac{1}{2},\xi+\frac{1}{2}\right)\right|^2 = (|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\alpha}{\gamma}})^{2\gamma}$. Also $\left|G\left(t+\frac{1}{2},\xi+\frac{1}{2}\right)\right|^2$ is bounded away from 0 and ∞ when $(t,\xi) \notin (-\eta,\eta)^2 + \mathbb{Z}^2$. Therefore, to show that $|G|^2 \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$, it suffices to show that if f is the $\mathbb{Z}^2$-periodic function defined by $f(t,\xi) = (|t|^{\frac{\alpha}{\gamma}} + |\xi|^{\frac{\alpha}{\gamma}})^{\gamma}$ for $(t,\xi) \in [-\frac{1}{2},\frac{1}{2})^2$, then $f^2 \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$. Since $0 < \alpha < 1/2$, Lemma 3.2.1 shows that $f^2 \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$. This completes the proof of Theorem 1.2.2. $\square$

3.2.2 Version 2 of Proof

We now provide a complete proof with full detail. We recreate the construction from Section 6.1 of [28] to build a function $G = Zg$ with $|G|^2 = |Zg|^2 \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$, which by Theorem 2.2.6 implies $\mathscr{G}(g,1,1)$ is a Schauder basis of type Λ. We then determine when the window function g has the required time and frequency localization.

3.2.2.1 Construction of the Zak Transform as an $\mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$ Weight

For $(t,\xi) \in \mathbb{R}^2$, define

$$G(t,\xi) = G_a(t,\xi) = \Phi(t-\frac{1}{2},\xi-\frac{1}{2})e^{2\pi i\Psi(t-\frac{1}{2},\xi-\frac{1}{2})},$$

where Φ and Ψ are defined as follows:

1. Fix $0 < \eta < \frac{1}{4}$ and let $\rho \in C^\infty(\mathbb{R})$ be an even function that satisfies
   - $\rho(x) = 1$ for $|x| \leq \eta$,
   - $\rho(x) = 0$ for $|x| \geq 2\eta$, and
   - $0 \leq \rho(x) \leq 1$ for $\eta < |x| < 2\eta$.
2. Let $\phi(x)$ be a $C^\infty(\mathbb{R})$ function that satisfies
   - $\phi(x) = -1$ for $x \leq 0$,

- $\phi(x) = 0$ for $x \geq 1$, and

- $-1 \leq \phi(x) \leq 0$ for $0 < x < 1$.

3. Define the function $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(t,\xi) = \begin{cases} \phi\left(\frac{\xi}{t}\right), & t > 0 \text{ and } 0 \leq \xi \leq t \\ 0, & \text{else.} \end{cases}$$

4. Fix $0 < a < 1$ and define $\Phi(t,\xi)$ on $\left[-\frac{1}{2},\frac{1}{2}\right)^2$ by

$$\Phi(t,\xi) = \rho(\xi)\left(\rho(t)\left(t^2+\xi^2\right)^{\frac{a}{2}} + 1 - \rho(t)\right) + 1 - \rho(\xi).$$

Extend Φ to $\mathbb{R}^2$ as a $\mathbb{Z}$-periodic function. Note $\lim_{t \to \frac{1}{2}^-} \Phi(t,\xi) = 1 = \Phi(-\frac{1}{2},\xi)$ and $\lim_{\xi \to \frac{1}{2}^-} \Phi(t,\xi) = 1 = \Phi(t,-\frac{1}{2})$. Thus $\Phi(t,\xi)$ is continuous on $\mathbb{R}^2$, in $C^2(\mathbb{R}^2 \setminus \{(0,0)\})$, and equal to 0 only on $\mathbb{Z}^2$.

5. Define $\Psi(t,\xi)$ on $\left[-\frac{1}{2},\frac{1}{2}\right) \times [0,1)$ by

$$\Psi(t,\xi) = \begin{cases} 0, & t \in \left[-\frac{1}{2},0\right] \\ \rho(t)H(t,\xi) + (1-\rho(t))\left(\xi - \frac{1}{2}\right), & t \in \left[0,\frac{1}{2}\right). \end{cases}$$

Extend Ψ to the plane by

$$\Psi(t+1,\xi) = \Psi(t,\xi) + \xi - \frac{1}{2}, \text{ if } t \in \mathbb{R} \text{ and } \xi \in [0,1), \tag{3.2.2}$$

and

$$\Psi(t,\xi+1) = \Psi(t,\xi), \text{ if } (t,\xi) \in \mathbb{R}^2. \tag{3.2.3}$$

We verify that $e^{2\pi i\Psi(t,\xi)}$ is continuous on $\mathbb{R}^2\backslash\mathbb{Z}^2$. Since Ψ is continuous on $\left[-\frac{1}{2},\frac{1}{2}\right)\times [0,1)$, we note that by (3.2.2),

$$\begin{aligned}\lim_{t\to\frac{1}{2}^-}\Psi(t,\xi) &= \lim_{t\to\frac{1}{2}^-}\rho(t)H(t,\xi)+(1-\rho(t))\left(\xi-\frac{1}{2}\right)\\ &= \xi-\frac{1}{2}\\ &= \Psi\left(-\frac{1}{2},\xi\right)+\xi-\frac{1}{2}\\ &= \Psi\left(\frac{1}{2},\xi\right),\end{aligned}$$

and by (3.2.3),

$$\begin{aligned}\lim_{\xi\to 1^-}\Psi(t,\xi) &= \begin{cases}\lim_{\xi\to 1^-}0, & t\in\left[-\frac{1}{2},0\right]\\ \lim_{\xi\to 1^-}\rho(t)H(t,\xi)+(1-\rho(t))\left(\xi-\frac{1}{2}\right), & t\in\left(0,\frac{1}{2}\right)\end{cases}\\ &= \begin{cases}\Psi(t,0), & t\in\left[-\frac{1}{2},0\right]\\ \Psi(t,0)+1, & t\in\left(0,\frac{1}{2}\right).\end{cases}\end{aligned}$$

The fact that $e^{2\pi i\Psi(t,\xi)}$ is C^∞ on $\mathbb{R}^2\backslash\mathbb{Z}^2$ is verified in [5].

Now, consider $\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)}$. This function is in $C^2(\mathbb{R}^2\backslash\mathbb{Z}^2)$, has bounded modulus, and is equal to 0 only on $\mathbb{Z}^2$. Further, for any $(t,\xi)\in(-\eta,\eta)^2$,

$$\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)} = \left(t^2+\xi^2\right)^{\frac{a}{2}}e^{2\pi iH(t,\xi)}$$

Thus

$$G(t,\xi) = \Phi\left(t-\frac{1}{2},\xi-\frac{1}{2}\right)e^{2\pi i\Psi\left(t-\frac{1}{2},\xi-\frac{1}{2}\right)}$$

is quasiperiodic, as verified below:

$$
\begin{aligned}
G(t,\xi+1) &= \Phi\left(t-\frac{1}{2},\xi+\frac{1}{2}\right)e^{2\pi i\Psi\left(t-\frac{1}{2},\xi+\frac{1}{2}\right)} \\
&= \Phi\left(t-\frac{1}{2},\xi-\frac{1}{2}\right)e^{2\pi i\Psi\left(t-\frac{1}{2},\xi-\frac{1}{2}\right)} \\
&= G(t,\xi),
\end{aligned}
$$

and

$$
\begin{aligned}
G(t+1,\xi) &= \Phi\left(t+\frac{1}{2},\xi-\frac{1}{2}\right)e^{2\pi i\Psi\left(t+\frac{1}{2},\xi-\frac{1}{2}\right)} \\
&= \Phi\left(t-\frac{1}{2},\xi-\frac{1}{2}\right)e^{2\pi i\left(\Psi\left(t-\frac{1}{2},\xi-\frac{1}{2}\right)+\xi-\frac{1}{2}-\frac{1}{2}\right)} \\
&= G(t,\xi)e^{2\pi i\xi}.
\end{aligned}
$$

Moreover, since $G\in L^\infty(\mathbb{T}^2)\subset L^2(\mathbb{T}^2)$, and since the Zak transform $Z: L^2(\mathbb{R})\to L^2(Q)$ is a unitary operator on the cube $Q=[0,1)^2$, there exists $g=g_a\in L^2(\mathbb{R})$ such that $Zg_a = G_a = G$.

Now, we will show that $\mathscr{G}(g_a,1,1)$ is a Schauder basis of type Λ for $0<a<\frac{1}{2}$. Since $|G(t,\xi)|^2 = \left|\Phi(t-\frac{1}{2},\xi-\frac{1}{2})\right|^2$, and translating $|\Phi(t,\xi)|^2$ does not affect its $\mathscr{A}_{2,\mathscr{R}}$ weight properties, it suffices to show that $|\Phi(t,\xi)|^2\in\mathscr{A}_{2,\mathscr{R}}(\mathbb{T}^2)$ for $0<a<\frac{1}{2}$. That is, given a rectangle $R\subset\mathbb{T}^2$ and letting $R_1 = R\cap(-\eta,\eta)^2$ and $R_2 = R\backslash R_1$, we must uniformly bound the quantity

$$
H = \left(\frac{1}{|R|}\iint_R |\Phi(t,\xi)|^2 dt d\xi\right)\left(\frac{1}{|R|}\iint_R \frac{1}{|\Phi(t,\xi)|^2} dt d\xi\right) = H_1+H_2+H_3+H_4,
$$

where

$$
H_1 = \left(\frac{1}{|R|}\iint_{R_1} |\Phi(t,\xi)|^2 dt d\xi\right)\left(\frac{1}{|R|}\iint_{R_1} \frac{1}{|\Phi(t,\xi)|^2} dt d\xi\right)
$$

$$H_2 = \left(\frac{1}{|R|}\iint_{R_1} |\Phi(t,\xi)|^2 dtd\xi\right)\left(\frac{1}{|R|}\iint_{R_2} \frac{1}{|\Phi(t,\xi)|^2} dtd\xi\right),$$

$$H_3 = \left(\frac{1}{|R|}\iint_{R_2} |\Phi(t,\xi)|^2 dtd\xi\right)\left(\frac{1}{|R|}\iint_{R_1} \frac{1}{|\Phi(t,\xi)|^2} dtd\xi\right),$$

and

$$H_4 = \left(\frac{1}{|R|}\iint_{R_2} |\Phi(t,\xi)|^2 dtd\xi\right)\left(\frac{1}{|R|}\iint_{R_2} \frac{1}{|\Phi(t,\xi)|^2} dtd\xi\right).$$

We first consider H_1. If $(t,\xi) \in (-\eta,\eta)^2$, then $\rho(t) = \rho(\xi) = 1$, so

$$\Phi(t,\xi) = \left(t^2 + \xi^2\right)^{\frac{a}{2}}. \tag{3.2.4}$$

So since $R_1 \subseteq (-\eta,\eta)^2$, we refer to Lemma 3.2.1 with $\gamma = \frac{a}{2}$ and $\alpha = a$ to see that H_1 is uniformly bounded above.

We next consider H_4. If $(t,\xi) \in \mathbb{T}^2 \backslash (-\eta,\eta)^2$, then since $\Phi = 0$ only on $\mathbb{Z}^2$ and Φ is continuous on $\mathbb{R}^2$, there are positive constants C_1 and C_2 such that

$$\frac{1}{C_1} \leq |\Phi(t,\xi)|^2 \leq C_2. \tag{3.2.5}$$

Thus since $R_2 \subset \mathbb{T}^2 \backslash (-\eta,\eta)^2$, H_4 is uniformly bounded above.

We proceed to H_2. Note that by (3.2.4) and (3.2.5),

$$\begin{aligned} H_2 &\leq \left(\frac{1}{|R|}\iint_{R_1} (t^2+\xi^2)^a dtd\xi\right)\left(\frac{1}{|R|}\iint_{R_2} C_1 dtd\xi\right) \\ &\lesssim \left(\frac{1}{|R|}\iint_{R} \eta^{2a} dtd\xi\right)\left(\frac{1}{|R|}\iint_{R} C_1 dtd\xi\right) = C_1\eta^{2a}. \end{aligned}$$

We finish with H_3. Note that since $\rho(t) \in [0,1]$,

$$|\Phi(t,\xi)|^2 = |\rho(t)(t^2+\xi^2)^{\frac{a}{2}} + 1 - \rho(t)|^2 \leq \left((t^2+\xi^2)^{\frac{a}{2}} + 1\right)^2. \tag{3.2.6}$$

Also note that if $\eta \leq |t|$, $\eta^a \leq (t^2+\xi^2)^{\frac{a}{2}}$, so

$$1 \leq \frac{(t^2+\xi^2)^{\frac{a}{2}}}{\eta^a}. \tag{3.2.7}$$

Thus by (3.2.6) and (3.2.7),

$$\begin{aligned} H_3 &\leq \left(\frac{1}{|R|}\iint_{R_2} (t^2+\xi^2)^a \left(1+\frac{1}{\eta^a}\right)^2 dt d\xi\right)\left(\frac{1}{|R|}\iint_{R_1} \frac{1}{(t^2+\xi^2)^a} dt d\xi\right) \\ &\lesssim \left(\frac{1}{|R|}\iint_{R} (t^2+\xi^2)^a dt d\xi\right)\left(\frac{1}{|R|}\iint_{R} \frac{1}{(t^2+\xi^2)^a} dt d\xi\right), \end{aligned}$$

which is again uniformly bounded above by Lemma 3.2.1 with $\gamma = \frac{a}{2}$ and $\alpha = a$.

Thus H is uniformly bounded above for any R, and $\mathscr{G}(g_a, 1, 1)$ is a Schauder basis of type Λ.

3.2.2.2 Localization of the Window Function

It remains to determine for which values of a the function g_a has the required time and frequency localization. We will prove the following result:

Theorem 3.2.2. *The function $g_a(t)$ has*

$$\int_{\mathbb{R}} |t|^{4-\varepsilon}|g_a(t)|^2 dt < \infty \textit{ and } \int_{\mathbb{R}} |\xi|^{4-\varepsilon}|\widehat{g_a}(\xi)|^2 d\xi < \infty$$

for $\varepsilon \in (0,2]$ and $1-\frac{\varepsilon}{2} < a < 1$.

We first prove two intermediate results, recalling Lemma 2.3.2.

Lemma 3.2.3. *If f is the $\mathbb{Z}^2$-periodic function defined by*

$$f(t,\xi) = (t^2+\xi^2)^{\frac{a}{2}}$$

for $(t,\xi) \in [-\frac{1}{2},\frac{1}{2})^2$, then

$$\int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Gamma_h^2 f(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh < \infty$$

and

$$\int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 f(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh < \infty$$

for $\varepsilon \in (0,2]$ and $1-\frac{\varepsilon}{2} < a < 1$.

Proof. By symmetry of f, it suffices to consider only

$$I = \int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 f(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh = I_1 + I_2, \tag{3.2.8}$$

where

$$I_1 = \int_{\mathbb{R}\setminus[-\frac{1}{6},\frac{1}{6}]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 f(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh, \tag{3.2.9}$$

and

$$I_2 = \int_{[-\frac{1}{6},\frac{1}{6}]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 f(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh. \tag{3.2.10}$$

We begin with I_1. Note that

$$\Delta_h^2 f(t,\xi) = \left((t+2h)^2+\xi^2\right)^{\frac{a}{2}} - 2\left((t+h)^2+\xi^2\right)^{\frac{a}{2}} + \left(t^2+\xi^2\right)^{\frac{a}{2}}, \tag{3.2.11}$$

so

$$\left|\Delta_h^2 f(t,\xi)\right|^2 \lesssim \left((t+2h)^2+\xi^2\right)^a + 2\left((t+h)^2+\xi^2\right)^a + \left(t^2+\xi^2\right)^a \lesssim t^{2a}+h^{2a}+\xi^{2a}. \tag{3.2.12}$$

Thus

$$I_1 \lesssim \int_{\mathbb{R}\setminus[-\frac{1}{6},\frac{1}{6}]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{t^{2a}+h^{2a}+\xi^{2a}}{|h|^{5-\varepsilon}} dt d\xi dh$$

which is finite if and only if $-\frac{1}{2} < a < 2-\frac{\varepsilon}{2}$, which is satisfied.

By symmetry of ξ in (3.2.11), in order to bound I_2 it suffices to show that

$$J = \int_0^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\Delta_h^2 f(t,\xi)\right|^2 dt d\xi \lesssim h^{2a+2}, \tag{3.2.13}$$

since $2a+2-5+\varepsilon > 2\left(1-\dfrac{\varepsilon}{2}\right)-3+\varepsilon = -1$. Without loss of generality, it suffices to consider only $h>0$.

We can partition $[-\frac{1}{2},\frac{1}{2}]\times[0,\frac{1}{2}]$ into $V_1\cup V_2\cup V_3\cup V_4$, where

$$\begin{aligned}
V_1 &= [-3h,3h]\times[0,3h], \\
V_2 &= [-3h,3h]\times\left[3h,\frac{1}{2}\right], \\
V_3 &= \left[3h,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right], \quad \text{and} \\
V_4 &= \left[-\frac{1}{2},-3h\right]\times\left[0,\frac{1}{2}\right].
\end{aligned}$$

Then $J = J_1+J_2+J_3+J_4$, where

$$J_i = \iint_{V_i} \left|\Delta_h^2 f(t,\xi)\right|^2 dt d\xi$$

for $i\in\{1,2,3,4\}$.

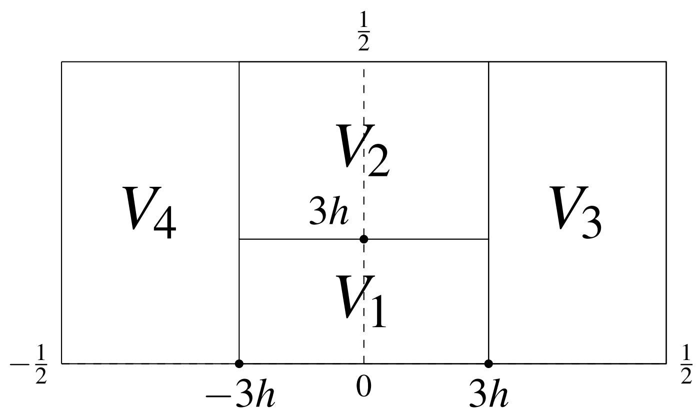

Figure 3.1: Partition of $[-\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{2}]$ into $V_1 \cup V_2 \cup V_3 \cup V_4$

Using (3.2.12),

$$J_1 \lesssim \iint_{V_1} t^{2a} + h^{2a} + \xi^{2a} dt d\xi \lesssim \iint_{V_1} h^{2a} dt d\xi \lesssim h^{2a+2}.$$

To estimate J_2, we note that $f \in C^2(\mathbb{R}^2 \setminus \{(0,0)\})$ and apply Lemma 2.3.3 to get

$$\left|\Delta_h^2 f(t,\xi)\right| \leq h^2 \sup_{\beta \in [t,t+2h]} \left|\frac{\partial^2 f(\beta,\xi)}{\partial t^2}\right|. \tag{3.2.14}$$

On V_2, $\beta \in [-3h, 5h]$ and $\xi \in \left[3h, \frac{1}{2}\right]$ so $\beta \lesssim \xi$, and since $a < 1$,

$$\left|\frac{\partial^2 f(\beta,\xi)}{\partial t^2}\right| = \left|a\left(\beta^2+\xi^2\right)^{\frac{a}{2}-1} + a(a-2)\beta^2\left(\beta^2+\xi^2\right)^{\frac{a}{2}-2}\right| \lesssim \xi^{a-2}. \tag{3.2.15}$$

Thus by (3.2.14) and (3.2.15), and since $2a-3 < -1$ implies $\frac{1}{2a-3} < 0$,

$$J_2 \lesssim \iint_{V_2} h^4 \xi^{2a-4} dt d\xi \lesssim h^5 \left(h^{2a-3} - \frac{1}{2^{2a-3}}\right) \leq h^{2a+2}.$$

To estimate J_3, we use the same technique as in evaluating J_2 and note that $t + 2h <$

$t+3h \leq 2t$. Thus using Lemma 2.3.3,

$$\left|\Delta_h^2 f(t,\xi)\right| \leq h^2 \sup_{\beta\in[t,2t]} \left|\frac{\partial^2 f(\beta,\xi)}{\partial t^2}\right|. \tag{3.2.16}$$

Since $\beta \in [t,2t]$ and $a<1$,

$$\begin{aligned}
\left|\frac{\partial^2 f(\beta,\xi)}{\partial t^2}\right| &= \left|a\left(\beta^2+\xi^2\right)^{\frac{a}{2}-1} + a(a-2)\beta^2\left(\beta^2+\xi^2\right)^{\frac{a}{2}-2}\right| \\
&\lesssim \left(t^2+\xi^2\right)^{\frac{a}{2}-1} + t^2\left(t^2+\xi^2\right)^{\frac{a}{2}-2} \\
&\lesssim (t+\xi)^{a-2} + t^2(t+\xi)^{a-4}. \qquad (3.2.17)
\end{aligned}$$

Then by (3.2.16) and (3.2.17), and since $\frac{1}{2a-7}<0$,

$$\begin{aligned}
J_3 &\lesssim \iint_{V_3} h^4\left((t+\xi)^{2a-4} + t^4(t+\xi)^{2a-8}\right) d\xi\, dt \qquad (3.2.18) \\
&\lesssim \int_{3h}^{\frac{1}{2}} h^4\left(t^{2a-3} + t^4 x^{2a-7} - \left(t+\frac{1}{2}\right)^{2a-3} - t^4\left(t+\frac{1}{2}\right)^{2a-7}\right) dt \\
&\lesssim h^4\left(h^{2a-2} - \frac{1}{2^{2a-2}}\right) \leq h^{2a+2}.
\end{aligned}$$

Lastly, we estimate J_4 in the exact same manner as J_3, except that $t+2h \leq t-\frac{2}{3}t = \frac{t}{3}$. Thus by Lemma 2.3.3,

$$\left|\Delta_h^2 f(t,\xi)\right| \leq h^2 \sup_{\beta\in[t,\frac{t}{3}]} \left|\frac{\partial^2 f(\beta,\xi)}{\partial t^2}\right|. \tag{3.2.19}$$

Since $\beta \in \left[t,\frac{t}{3}\right]$ and $a<1$,

$$\begin{aligned}
\left|\frac{\partial^2 f(\beta,\xi)}{\partial t^2}\right| &= \left|a\left(\beta^2+\xi^2\right)^{\frac{a}{2}-1} + a(a-2)\beta^2\left(\beta^2+\xi^2\right)^{\frac{a}{2}-2}\right| \\
&\lesssim \left(t^2+\xi^2\right)^{\frac{a}{2}-1} + t^2\left(t^2+\xi^2\right)^{\frac{a}{2}-2} \\
&\lesssim (|t|+|\xi|)^{a-2} + t^2(|t|+|\xi|)^{a-4}.
\end{aligned}$$

Thus by a simple change of variable, $J_4 \lesssim$ (3.2.18), and thus $J_4 \lesssim h^{2a+2}$.

$\square$

Lemma 3.2.4. *Let f be defined as in Lemma 3.2.3. Then the function $F(t,\xi) = f(t,\xi)e^{2\pi iH(x,\xi)}$ has*

$$\int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Gamma_h^2 F(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh < \infty$$

and

$$\int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 F(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh < \infty$$

for $\varepsilon \in (0,2]$ and $1 - \frac{\varepsilon}{2} < a < 1$.

Proof. First we consider

$$\int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 F(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh. \tag{3.2.20}$$

Note that

$$\Delta_h^2 F(t,\xi) = e^{2\pi iH(t,\xi)} \Delta_h^2 f(t,\xi) + 2\Delta_h^1 e^{2\pi iH(t,\xi)} \Delta_h^1 f_h(t,\xi) + \Delta_h^2 e^{2\pi iH(t,\xi)} f_{2h}(t,\xi),$$

where $f_{nh}(t,\xi) = f(t+nh,\xi)$. Thus

$$\left|\Delta_h^2 F(t,\xi)\right|^2 \lesssim \left|e^{2\pi iH(t,\xi)} \Delta_h^2 f(t,\xi)\right|^2 + \left|\Delta_h^1 e^{2\pi iH(t,\xi)} \Delta_h^1 f_h(t,\xi)\right|^2 + \left|\Delta_h^2 e^{2\pi iH(t,\xi)} f_{2h}(t,\xi)\right|^2.$$

By Lemma 3.2.3, to show that (3.2.20) is bounded above, it suffices to consider only

$$K^1 = \int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^1 e^{2\pi iH(t,\xi)} \Delta_h^1 f_h(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh = K_1^1 + K_2^1 \tag{3.2.21}$$

and

$$K^2 = \int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh = K_1^2 + K_2^2, \tag{3.2.22}$$

where

$$K_1^1 = \int_{\mathbb{R}\setminus[-\frac{1}{10},\frac{1}{10}]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^1 e^{2\pi i H(t,\xi)} \Delta_h^1 f_h(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh, \tag{3.2.23}$$

$$K_2^1 = \int_{[-\frac{1}{10},\frac{1}{10}]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^1 e^{2\pi i H(t,\xi)} \Delta_h^1 f_h(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh, \tag{3.2.24}$$

$$K_1^2 = \int_{\mathbb{R}\setminus[-\frac{1}{10},\frac{1}{10}]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh, \tag{3.2.25}$$

and

$$K_2^2 = \int_{[-\frac{1}{10},\frac{1}{10}]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh. \tag{3.2.26}$$

We bound K_1^1 and K_1^2 as we did I_1 in Lemma 3.2.3. Note that

$$\begin{aligned}\left|\Delta_h^1 e^{2\pi i H(t,\xi)} \Delta_h^1 f_h(t,\xi)\right|^2 &\lesssim \left(\left|e^{2\pi i H(t+h,\xi)}\right|^2 + \left|e^{2\pi i H(t,\xi)}\right|^2\right)\left(|f(t+2h,\xi)|^2 + |f(t+h,\xi)|^2\right) \\ &\lesssim t^{2a} + h^{2a} + \xi^{2a} \end{aligned} \tag{3.2.27}$$

and

$$\left|\Delta_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2 \lesssim \left(\left|e^{2\pi i H(t+2h,\xi)}\right|^2 + \left|e^{2\pi i H(t+h,\xi)}\right|^2 + \left|e^{2\pi i H(t,\xi)}\right|^2\right) |f(t+2h,\xi)|^2$$
$$\lesssim t^{2a} + 2h^{2a} + \xi^{2a} \tag{3.2.28}$$

so we proceed with the same argument that follows (3.2.12).

In order to bound K_2^1 and K_2^2, it suffices to show that

$$L^1 = \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \left|\Delta_h^1 e^{2\pi i H(t,\xi)} \Delta_h^1 f_h(t,\xi)\right|^2 dt d\xi \lesssim h^{2a+2} \tag{3.2.29}$$

and

$$L^2 = \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \left|\Delta_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2 dt d\xi \lesssim h^{2a+2}. \tag{3.2.30}$$

Without loss of generality, it suffices to consider only $h > 0$. Then we can partition $[-\frac{1}{2},\frac{1}{2}] \times [-\frac{1}{2},\frac{1}{2}]$ into $W_1 \cup W_2 \cup W_3$, where

$$\begin{aligned} W_1 &= [-3h, 3h] \times [0, 5h], \\ W_2 &= \left[3h, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right], \quad \text{and} \\ W_3 &= [-\frac{1}{2}, \frac{1}{2}]^2 \setminus (W_1 \cup W_2). \end{aligned}$$

Then $L^1 = L_1^1 + L_2^1 + L_3^1$ and $L^2 = L_1^2 + L_2^2 + L_3^2$, where

$$L_i^1 = \iint_{W_i} \left|\Delta_h^1 e^{2\pi i H(t,\xi)} \Delta_h^1 f_h(t,\xi)\right|^2 dt d\xi,$$

and

$$L_i^2 = \iint_{W_i} \left|\Delta_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2 dt d\xi$$

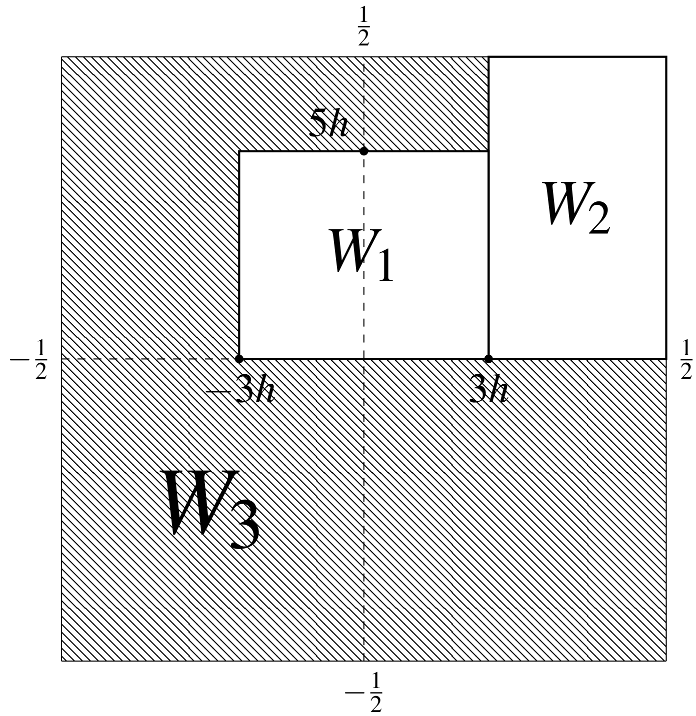

Figure 3.2: Partition of $[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]$ into $W_1 \cup W_2 \cup W_3$

for $i \in \{1,2,3\}$.

Using (3.2.27) and (3.2.28), we find that

$$L_1^1 \lesssim \iint_{W_1} t^{2a} + h^{2a} + \xi^{2a} dt d\xi \lesssim \iint_{W_1} h^{2a} dt d\xi \lesssim h^{2a+2},$$

and

$$L_1^2 \lesssim \iint_{W_1} t^{2a} + h^{2a} + \xi^{2a} dt d\xi \lesssim \iint_{W_1} h^{2a} dt d\xi \lesssim h^{2a+2}.$$

To estimate L_3^1 and L_3^2, note that if $(t,\xi) \in W_3$ and $t' \in [t, t+2h]$, then either $t' \leq 0$, $\xi < 0$, or $t' < \xi$. So $H(t',\xi) = 0$ and $e^{2\pi i H(t',\xi)} = 1$. Thus, $\Delta_h^n e^{2\pi i H(t,\xi)} = 0$ for $n \in \{1,2\}$ and $L_3^1 = L_3^2 = 0$.

We will now estimate L_2^1 and L_2^2. First, consider when $(t,\xi) \in W_2$ and $\xi > 2t$. Since $t > 3h$, for $t' \in [t, t+2h]$, $\xi > t'$. So $H(t',\xi) = 0$, $e^{2\pi i H(t',\xi)} = 1$, and $\Delta_h^n e^{2\pi i H(t,\xi)} = 0$ for

$n \in \{1,2\}$.

Now assume that $\xi \leq 2t$. Recall that $e^{2\pi i H(t,\xi)} \in C^\infty(\mathbb{R}^2 \backslash \mathbb{Z}^2)$ and $f(t,\xi) \in C^2(\mathbb{R}^2 \backslash \{(0,0)\})$. So by applying Lemma 2.3.3, we see that on W_2, since $h < t$,

$$\left|\Delta_h^1 e^{2\pi i H(t,\xi)} \Delta_h^1 f_h(t,\xi)\right| \leq h^2 \left(\sup_{\beta \in [t,2t]} \left|\frac{\partial e^{2\pi i H(\beta,\xi)}}{\partial t}\right|\right) \left(\sup_{\beta \in [t,2t]} \left|\frac{\partial f(\beta,\xi)}{\partial t}\right|\right). \tag{3.2.31}$$

Since $\phi \in C^\infty(\mathbb{R})$,

$$\left|\frac{\partial e^{2\pi i H(\beta,\xi)}}{\partial t}\right| \leq \left|-2\pi i \phi'\left(\frac{\xi}{\beta}\right) \frac{\xi}{\beta^2} e^{2\pi i H(\beta,\xi)}\right| \lesssim \frac{\xi}{t^2}. \tag{3.2.32}$$

Since $a < 1$,

$$\left|\frac{\partial f(\beta,\xi)}{\partial t}\right| = \left|a\beta(\beta^2+\xi^2)^{\frac{a}{2}-1}\right| \lesssim t^{a-1}. \tag{3.2.33}$$

Thus by (3.2.31), (3.2.32), and (3.2.33), and since $\frac{1}{2a-2} < 0$,

$$L_2^1 \lesssim \int_{3h}^{\frac{1}{2}} \int_0^{2t} h^4 \xi^2 t^{2a-6} d\xi dt \lesssim \int_{3h}^{\frac{1}{2}} h^4 t^{2a-3} dt \lesssim h^4 \left(h^{2a-2} - \frac{1}{2^{2a-2}}\right) \leq h^{2a+2}.$$

We again apply Lemma 2.3.3 to see that on W_2, for $\xi \leq 2t$ and since $2h < t$,

$$\left|\Delta_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right| \leq h^2 \left(\sup_{\beta \in [t,2t]} \left|\frac{\partial^2 e^{2\pi i H(\beta,\xi)}}{\partial t^2}\right|\right) f(t+2h,\xi). \tag{3.2.34}$$

Since $\phi \in C^\infty(\mathbb{R})$ and $\xi \leq 2t$,

$$\begin{aligned}\left|\frac{\partial^2 e^{2\pi i H(\beta,\xi)}}{\partial t^2}\right| &\leq \left|2\pi i e^{2\pi i H(\beta,\xi)} \left(\frac{2\pi i \xi^2}{\beta^4}\left(\phi'\left(\frac{\xi}{\beta}\right)\right)^2 + \frac{2\xi}{\beta^3}\phi'\left(\frac{\xi}{\beta}\right) + \frac{\xi^2}{\beta^4}\phi''\left(\frac{\xi}{\beta}\right)\right)\right| \\ &\lesssim \frac{\xi}{t^3}.\end{aligned} \tag{3.2.35}$$

Since $\xi \leq 2t$, and since $3h < t$ on W_2,

$$f(t+2h,\xi) = \left((t+2h)^2 + \xi^2\right)^{\frac{a}{2}} \lesssim t^a. \tag{3.2.36}$$

Thus by (3.2.34), (3.2.35), and (3.2.36), and since $\frac{1}{2a-2} < 0$,

$$L_2^2 \lesssim \int_{3h}^{\frac{1}{2}} \int_0^{2t} h^4 \xi^2 t^{2a-6} d\xi dt \lesssim h^{2a+2}.$$

Now consider

$$\int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Gamma_h^2 F(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh. \tag{3.2.37}$$

We show (3.2.37) is bounded above in a completely analogous manner as we showed that (3.2.20) is bounded above.

Note that

$$\Gamma_h^2 F(t,\xi) = e^{2\pi i H(t,\xi)} \Gamma_h^2 f(t,\xi) + 2\Gamma_h^1 e^{2\pi i H(t,\xi)} \Gamma_h^1 f_h(t,\xi) + \Gamma_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi),$$

now letting $f_{nh}(t,\xi) = f(t,\xi+nh)$. Thus

$$\left|\Gamma_h^2 F(t,\xi)\right|^2 \lesssim \left|e^{2\pi i H(t,\xi)} \Gamma_h^2 f(t,\xi)\right|^2 + \left|\Gamma_h^1 e^{2\pi i H(t,\xi)} \Gamma_h^1 f_h(t,\xi)\right|^2 + \left|\Gamma_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2.$$

Again by Lemma 3.2.3, it suffices to consider only

$$M^1 = \int_{\mathbb{R}} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Gamma_h^1 e^{2\pi i H(t,\xi)} \Gamma_h^1 f_h(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh = M_1^1 + M_2^1 \tag{3.2.38}$$

and

$$M^2=\int_{\mathbb{R}}\iint_{[-\frac{1}{2},\frac{1}{2}]^2}\frac{\left|\Gamma_h^2 e^{2\pi iH(t,\xi)}f_{2h}(t,\xi)\right|^2}{|h|^{5-\varepsilon}}dtd\xi dh=M_1^2+M_2^2, \tag{3.2.39}$$

where

$$M_1^1=\int_{\mathbb{R}\setminus[-\frac{1}{10},\frac{1}{10}]}\iint_{[-\frac{1}{2},\frac{1}{2}]^2}\frac{\left|\Gamma_h^1 e^{2\pi iH(t,\xi)}\Gamma_h^1 f_h(t,\xi)\right|^2}{|h|^{5-\varepsilon}}dtd\xi dh, \tag{3.2.40}$$

$$M_2^1=\int_{[-\frac{1}{10},\frac{1}{10}]}\iint_{[-\frac{1}{2},\frac{1}{2}]^2}\frac{\left|\Gamma_h^1 e^{2\pi iH(t,\xi)}\Gamma_h^1 f_h(t,\xi)\right|^2}{|h|^{5-\varepsilon}}dtd\xi dh, \tag{3.2.41}$$

$$M_1^2=\int_{\mathbb{R}\setminus[-\frac{1}{10},\frac{1}{10}]}\iint_{[-\frac{1}{2},\frac{1}{2}]^2}\frac{\left|\Gamma_h^2 e^{2\pi iH(t,\xi)}f_{2h}(t,\xi)\right|^2}{|h|^{5-\varepsilon}}dtd\xi dh, \tag{3.2.42}$$

and

$$M_2^2=\int_{[-\frac{1}{10},\frac{1}{10}]}\iint_{[-\frac{1}{2},\frac{1}{2}]^2}\frac{\left|\Gamma_h^2 e^{2\pi iH(t,\xi)}f_{2h}(t,\xi)\right|^2}{|h|^{5-\varepsilon}}dtd\xi dh. \tag{3.2.43}$$

We bound M_1^1 and M_1^2 as we did K_1^1 and K_1^2, as well as I_1 in Lemma 3.2.3. Note that

$$\begin{aligned}\left|\Gamma_h^1 e^{2\pi iH(t,\xi)}\Gamma_h^1 f_h(t,\xi)\right|^2 &\lesssim \left(\left|e^{2\pi iH(t,\xi+h)}\right|^2+\left|e^{2\pi iH(t,\xi)}\right|^2\right)\left(|f(t,\xi+2h)|^2+|f(t,\xi+h)|^2\right)\\ &\lesssim t^{2a}+h^{2a}+\xi^{2a}\end{aligned} \tag{3.2.44}$$

and

$$\left|\Gamma_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2 \lesssim \left(\left|e^{2\pi i H(t,\xi+2h)}\right|^2 + \left|e^{2\pi i H(t,\xi+h)}\right|^2 + \left|e^{2\pi i H(t,\xi)}\right|^2\right) |f(t,\xi+2h)|^2$$
$$\lesssim t^{2a} + 2h^{2a} + \xi^{2a} \tag{3.2.45}$$

so we proceed with the same argument that follows (3.2.12).

In order to bound M_2^1 and M_2^2, it suffices to show that

$$N^1 = \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \left|\Gamma_h^1 e^{2\pi i H(t,\xi)} \Gamma_h^1 f_h(t,\xi)\right|^2 dt d\xi \lesssim h^{2a+2} \tag{3.2.46}$$

and

$$N^2 = \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \left|\Gamma_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2 dt d\xi \lesssim h^{2a+2}. \tag{3.2.47}$$

Without loss of generality, it suffices to consider only $h > 0$. Then we can partition $[-\frac{1}{2},\frac{1}{2}] \times [-\frac{1}{2},\frac{1}{2}]$ into $T_1 \cup T_2 \cup T_3$, where

$$\begin{aligned} T_1 &= [0,h] \times [-3h,3h], \\ T_2 &= \left[h,\frac{1}{2}\right] \times \left[-3h,\frac{1}{2}\right], \quad \text{and} \\ T_3 &= [-\frac{1}{2},\frac{1}{2}]^2 \backslash (T_1 \cup T_2). \end{aligned}$$

Then $N^1 = N_1^1 + N_2^1 + N_3^1$ and $N^2 = N_1^2 + N_2^2 + N_3^2$, where

$$N_i^1 = \iint_{T_i} \left|\Gamma_h^1 e^{2\pi i H(t,\xi)} \Gamma_h^1 f_h(t,\xi)\right|^2 dt d\xi,$$

and

$$N_i^2 = \iint_{T_i} \left|\Gamma_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right|^2 dt d\xi$$

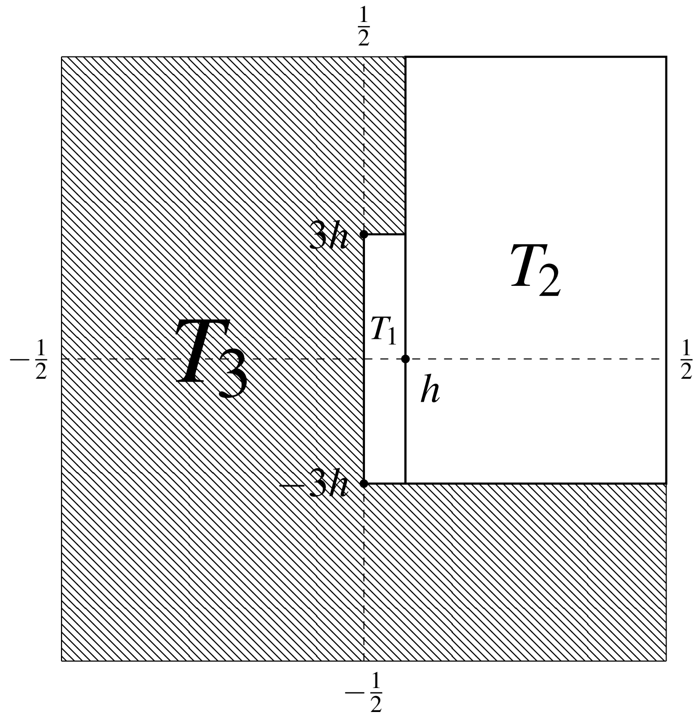

Figure 3.3: Partition of $[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]$ into $T_1 \cup T_2 \cup T_3$

for $i \in \{1,2,3\}$.

Using (3.2.44) and (3.2.45), we find that

$$N_1^1 \lesssim \iint_{T_1} t^{2a} + h^{2a} + \xi^{2a} dt d\xi \lesssim \iint_{T_1} h^{2a} dt d\xi \lesssim h^{2a+2},$$

and

$$N_1^2 \lesssim \iint_{T_1} t^{2a} + h^{2a} + \xi^{2a} dt d\xi \lesssim \iint_{T_1} h^{2a} dt d\xi \lesssim h^{2a+2}.$$

To estimate N_3^1 and N_3^2, note that if $(t,\xi) \in T_3$ and $\xi' \in [\xi, \xi+2h]$, then either $t \leq 0$, $\xi' < 0$, or $t < \xi'$. So $H(t,\xi') = 0$ and $e^{2\pi i H(t,\xi')} = 1$. Thus, $\Gamma_h^n e^{2\pi i H(t,\xi)} = 0$ for $n \in \{1,2\}$ and $N_3^1 = N_3^2 = 0$.

We will now estimate N_2^1 and N_2^2. First, consider when $(t,\xi) \in T_2$ and $\xi > t$. Then for $\xi' \in [\xi, \xi+2h]$, $\xi' > t$. So $H(t,\xi') = 0$, $e^{2\pi i H(t,\xi')} = 1$, and $\Gamma_h^n e^{2\pi i H(t,\xi)} = 0$ for $n \in \{1,2\}$.

Now assume that $\xi \le t$. Recall that $e^{2\pi i H(t,\xi)} \in C^\infty(\mathbb{R}^2 \backslash \mathbb{Z}^2)$ and $f(t,\xi) \in C^2(\mathbb{R}^2 \backslash \{(0,0)\})$. So by applying Lemma 2.3.3, we see that on T_2, since $h < t$,

$$\left|\Gamma_h^1 e^{2\pi i H(t,\xi)} \Gamma_h^1 f_h(t,\xi)\right| \le h^2 \left(\sup_{\beta\in[\xi,2t]} \left|\frac{\partial e^{2\pi i H(t,\beta)}}{\partial \xi}\right|\right) \left(\sup_{\beta\in[\xi,2t]} \left|\frac{\partial f(t,\beta)}{\partial \xi}\right|\right). \quad (3.2.48)$$

Since $\phi \in C^\infty(\mathbb{R})$,

$$\left|\frac{\partial e^{2\pi i H(t,\beta)}}{\partial \xi}\right| \le \left|2\pi i \phi'\left(\frac{\beta}{t}\right)\frac{1}{t} e^{2\pi i H(t,\beta)}\right| \lesssim \frac{1}{t}. \quad (3.2.49)$$

Since $a < 1$,

$$\left|\frac{\partial f(t,\beta)}{\partial \xi}\right| = \left|a\beta(t^2+\beta^2)^{\frac{a}{2}-1}\right| \lesssim t^{a-1}. \quad (3.2.50)$$

Thus by (3.2.48), (3.2.49), and (3.2.50), and since $\frac{1}{2a-3} < 0$,

$$\begin{aligned} N_2^1 &\lesssim \int_h^{\frac{1}{2}} \int_{-3h}^{t} h^4 t^{2a-4} d\xi dt \\ &\lesssim \int_{3h}^{\frac{1}{2}} h^4 t^{2a-3} + h^5 t^{2a-4} dt \\ &\lesssim h^4\left(h^{2a-2} - \frac{1}{2^{2a-2}}\right) + h^5\left(h^{2a-3} - \frac{1}{2^{2a-3}}\right) \le h^{2a+2}. \end{aligned}$$

We again apply Lemma 2.3.3 to see that on T_2, for $\xi \le t$,

$$\left|\Gamma_h^2 e^{2\pi i H(t,\xi)} f_{2h}(t,\xi)\right| \le h^2 \left(\sup_{\beta\in[\xi,\xi+2h]} \left|\frac{\partial^2 e^{2\pi i H(t,\beta)}}{\partial \xi^2}\right|\right) f(t,\xi+2h). \quad (3.2.51)$$

Since $\phi \in C^\infty(\mathbb{R})$,

$$\left|\frac{\partial^2 e^{2\pi i H(t,\beta)}}{\partial \xi^2}\right| \le \left|2\pi i e^{2\pi i H(t,\beta)}\left(\frac{2\pi i}{t^2}\left(\phi'\left(\frac{\beta}{t}\right)\right)^2 + \frac{1}{t^2}\phi''\left(\frac{\beta}{t}\right)\right)\right| \lesssim \frac{1}{t^2}. \quad (3.2.52)$$

Since $\xi \leq t$, and since $h < t$ on T_2,

$$f(t,\xi+2h) = \left(t^2+(\xi+2h)^2\right)^{\frac{a}{2}} \lesssim t^a. \tag{3.2.53}$$

Thus by (3.2.51), (3.2.52), and (3.2.53), and since $\frac{1}{2a-3} < 0$,

$$L_2^2 \lesssim \int_h^{\frac{1}{2}} \int_{-3h}^{t} h^4 t^{2a-4} d\xi dt \lesssim h^{2a+2}.$$

This completes the proof.

□

We are now ready to prove Theorem 3.2.2:

The function $g_a(t)$ has

$$\int_{\mathbb{R}} |t|^{4-\varepsilon} |g_a(t)|^2 dt < \infty \text{ and } \int_{\mathbb{R}} |\xi|^{4-\varepsilon} |\widehat{g_a}(\xi)|^2 d\xi < \infty$$

for $\varepsilon \in (0,2]$ and $1-\frac{\varepsilon}{2} < a < 1$.

Proof. By Lemma 2.3.2, it suffices to consider

$$\int_{\mathbb{R}} \iint_{[0,1]^2} \frac{\left|\Delta_h^2 G_a(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh = \int_{\mathbb{R}} \iint_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \frac{\left|\Delta_h^2(\Phi(t,\xi)e^{2\pi i \Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh \tag{3.2.54}$$

and

$$\int_{\mathbb{R}} \iint_{[0,1]^2} \frac{\left|\Gamma_h^2 G_a(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh = \int_{\mathbb{R}} \iint_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \frac{\left|\Gamma_h^2(\Phi(t,\xi)e^{2\pi i \Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh. \tag{3.2.55}$$

Choose $\delta > 0$ such that $[-5\delta, 5\delta] \subset (-\eta, \eta)$. Note that $\Phi(t,\xi)e^{2\pi i \Psi(t,\xi)}$ has bounded modulus, so $\left|\Delta_h^2(\Phi(t,\xi)e^{2\pi i \Psi(t,\xi)})\right|^2$ and $\left|\Gamma_h^2(\Phi(t,\xi)e^{2\pi i \Psi(t,\xi)})\right|^2$ are bounded above. Thus, since $\varepsilon < 2$,

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Delta_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh < \infty$$

and

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \frac{\left|\Gamma_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh < \infty.$$

Let $S_1 = [-3\delta, 3\delta]^2$ and $S_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \setminus S_1$.

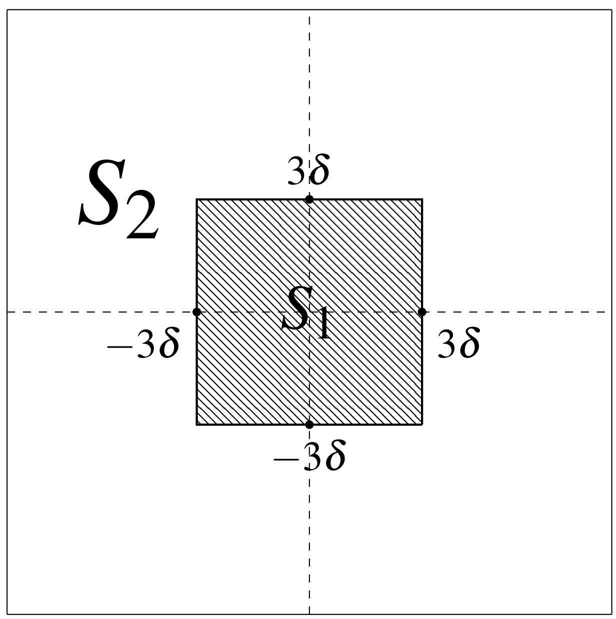

Figure 3.4: Partition of $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ into $S_1 \cup S_2$

It remains to consider

$$D_1 = \int_{-\delta}^{\delta} \iint_{S_1} \frac{\left|\Delta_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh, \qquad (3.2.56)$$

$$D_2 = \int_{-\delta}^{\delta} \iint_{S_2} \frac{\left|\Delta_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh, \tag{3.2.57}$$

$$G_1 = \int_{-\delta}^{\delta} \iint_{S_1} \frac{\left|\Gamma_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh, \tag{3.2.58}$$

and

$$G_2 = \int_{-\delta}^{\delta} \iint_{S_2} \frac{\left|\Gamma_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh. \tag{3.2.59}$$

Without loss of generality, let $h > 0$.

Since for $h < \delta$, $[-3\delta, 3\delta + 2h] \subset [-5\delta, 5\delta] \subset (-\eta, \eta)$,

$$D_1 = \int_{-\delta}^{\delta} \iint_{S_1} \frac{\left|\Delta_h^2 F(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh$$

and

$$G_1 = \int_{-\delta}^{\delta} \iint_{S_1} \frac{\left|\Gamma_h^2 F(t,\xi)\right|^2}{|h|^{5-\varepsilon}} dt d\xi dh,$$

which are finite by Lemma 3.2.4.

To show that D_2 and G_2 are finite, it suffices to show that

$$D_2' = \iint_{S_2} \left|\Delta_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2 dt d\xi \lesssim h^4,$$

and

$$G_2' = \iint_{S_2} \left|\Gamma_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right|^2 dt d\xi \lesssim h^4,$$

since $\varepsilon > 0$ implies $4 - 5 + \varepsilon = \varepsilon - 1 > -1$.

Since $\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)}$ is in $C^2(\mathbb{R}^2\backslash\mathbb{Z}^2)$, we find that on S_2 and for $h<\delta$, by applying Lemma 2.3.3,

$$\begin{aligned}\left|\Delta_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right| &\leq h^2 \sup_{\beta\in[t,t+2\delta]}\left|\frac{\partial^2(\Phi(\beta,\xi)e^{2\pi i\Psi(\beta,\xi)})}{\partial t^2}\right| \\ &\leq h^2 \sup_{(t,\xi)\in S_2}\sup_{\beta\in[t,t+2\delta]}\left|\frac{\partial^2(\Phi(\beta,\xi)e^{2\pi i\Psi(\beta,\xi)})}{\partial t^2}\right|,\end{aligned}$$

and

$$\begin{aligned}\left|\Gamma_h^2(\Phi(t,\xi)e^{2\pi i\Psi(t,\xi)})\right| &\leq h^2 \sup_{\beta\in[\xi,\xi+2\delta]}\left|\frac{\partial^2(\Phi(t,\beta)e^{2\pi i\Psi(t,\beta)})}{\partial \xi^2}\right| \\ &\leq h^2 \sup_{(t,\xi)\in S_2}\sup_{\beta\in[\xi,\xi+2\delta]}\left|\frac{\partial^2(\Phi(t,\beta)e^{2\pi i\Psi(t,\beta)})}{\partial \xi^2}\right|,\end{aligned}$$

where the suprema are finite and independent of h.

Thus

$$D_2' \leq \iint_{S_2} h^4\left(\sup_{(t,\xi)\in S_2}\sup_{\beta\in[t,t+2\delta]}\left|\frac{\partial^2(\Phi(\beta,\xi)e^{2\pi i\Psi(\beta,\xi)})}{\partial t^2}\right|\right)^2 dtd\xi \lesssim h^4,$$

and

$$G_2' \leq \iint_{S_2} h^4\left(\sup_{(t,\xi)\in S_2}\sup_{\beta\in[\xi,\xi+2\delta]}\left|\frac{\partial^2(\Phi(t,\beta)e^{2\pi i\Psi(t,\beta)})}{\partial \xi^2}\right|\right)^2 dtd\xi \lesssim h^4.$$

□

3.2.2.3 The Counterexamples

Proof of Theorem 1.2.2 (Version 2) In Section 3.2.2.1, we construct $G=G_a=Zg_a$ and show that $\mathscr{G}(g_a,1,1)$ is a Schauder basis of type Λ for $0<a<\frac{1}{2}$. In Theorem 3.2.2, we show that

$$\int_{\mathbb{R}} |t|^{4-\varepsilon}|g_a(t)|^2 dt < \infty \text{ and } \int_{\mathbb{R}} |\xi|^{4-\varepsilon}|\widehat{g_a}(\xi)|^2 d\xi < \infty$$

for $\varepsilon \in (0,2]$ and $1 - \frac{\varepsilon}{2} < a < 1$.

Thus for $\varepsilon \in (0,1]$, there exists a value of a such that $\mathscr{G}(g_a, 1, 1)$ is a Schauder basis of type Λ,

$$\int_{\mathbb{R}} |t|^{3-\varepsilon}|g_a(t)|^2 dt < \infty, \text{ and } \int_{\mathbb{R}} |\xi|^{3-\varepsilon}|\widehat{g_a}(\xi)|^2 d\xi < \infty,$$

as required. □

Chapter 4

Questions

In this chapter, we discuss some questions for further work. First, it would be interesting to understand if the compact support condition in Theorem 1.2.1 can be relaxed. For perspective, Theorem 1.1.4 remains true if the assumption that g is compactly supported is replaced by the assumption that g is in the Wiener amalgam space $W(\ell^1, L^\infty)$, see Section 5 in [14]. Here, $g \in W(\ell^1, L^\infty)$ if $\sum_{n\in\mathbb{Z}} \|\chi_{[n,n+1]} g\|_\infty < \infty$.

Question 4.0.1. *Suppose that $g \in W(\ell^1, L^\infty)$ and $\int |\xi|^2 |\widehat{g}(\xi)|^2 d\xi < \infty$. Is it possible for $\mathscr{G}(g,1,1)$ to be a Schauder basis of type Λ?*

We would also like to understand if one can relax the assumption $\varepsilon > 0$ in Theorem 1.2.2. If Theorem 1.2.2 fails when $\varepsilon = 0$, then this would provide a new symmetrically weighted $(3,3)$ version of the Balian-Low theorem for Schauder bases of type Λ, and would complement the endpoint result in Theorem 1.2.1.

Question 4.0.2. *Suppose that $g \in L^2(\mathbb{R})$ and*

$$\int |t|^3 |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\xi|^3 |\widehat{g}(\xi)|^2 d\xi < \infty.$$

Is it possible for $\mathscr{G}(g,1,1)$ to be a Schauder basis of type Λ?

Finally, we would like to relate our results to some of the existing literature discussed in Section 2.4. It would be interesting to use matrix valued Muckenhoupt weights to look at the multi-generated case, where a Gabor system is generated by multiple window functions instead of a single window function, as in [33].

Question 4.0.3. *Let $R > 0$ and $G = \{g^r\}_{r=1}^R \subset L^2(\mathbb{R})$. Let $a > 0$ and $R(ab)^{-1} = 1$. Suppose that for $1 \le r \le R$, g^r is compactly supported and $\int |\xi|^2 |\widehat{g^r}(\xi)|^2 d\xi < \infty$. Is it possible for $\mathscr{G}(G,a,b) = \left\{g^r_{k,n}(t)\right\}_{1\le r\le R, k,n\in\mathbb{Z}}$ to be a Schauder basis of type Λ?*

As Schauder bases are also an intermediate spanning structure between Riesz bases and exact systems, it is natural to wonder what the connection is between Gabor Schauder bases and (C_q)-systems. It is known that every Schauder basis is a (C_q)-system for some q, see Theorem 2.4 of [30]. However, there is no known equivalence between Schauder bases and any particular q, making our results independent and unique. It is possible that the fact that the dual of a Schauder basis is also a Schauder basis may illuminate the relationship between Schauder bases and C_q-systems, see Corollary 5.22 in [18].

We also consider the possibility of Schauder basis results for shift-invariant spaces with extra-invariance. For $d = 1$, we consider the examples on p. 72 and Lemma 6.1.1 in [29] which show that Theorem 2.4.6 is sharp. Since Example 3.11 from [24] shows that $|x|^\alpha \in \mathscr{A}_{2,\mathscr{R}}(\mathbb{T})$ for $0 < \alpha < 1$, we can use an analogous result to Theorem 2.2.6 for shift-invariant spaces, see Theorem 4.2 in [24], to show that the examples are Schauder bases of type Λ.

Thus we have no distinction between Schauder bases and exact systems in the singly-generated shift-invariant space case for $d = 1$. It is not yet known whether for higher dimensions, or for multiply generated shift-invariant spaces, there is a distinction between Schauder bases and exact systems.

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