Characterisation and Hamiltonicity of $K_{1,1,t}$-minor-free Graphs: A Fan-based Approach

By

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Chapter 1

Introduction

This dissertation presents new results in certain areas of structural graph theory. In particular, we analyse induced subgraphs called fans, and use their properties to prove structural theorems for classes of minor-free graphs. In Chapter 1, we present some definitions and known results. Chapter 2 is comprised of our investigation into fans. In Chapters 3 and 4 we prove our main results: respectively, a complete characterisation of the class of 3-connected $K_{1,1,4}$-minor-free graphs, and a Hamiltonicity result for the class of 3-connected planar $K_{1,1,5}$-minor-free graphs. Finally, Chapter 5 offers some ideas for continued research based on this work.

1.1 Concepts and Definitions

We begin by reviewing some relevant definitions and terminology.

Throughout this paper, we shall assume that we are considering only *simple* graphs, that is, graphs with no multiple edges and no loops.

A graph is said to be *planar* if it can be embedded in the plane such that any two edges meet only at a vertex; that is, edges do not cross. A graph that is not a complete graph is said to be *k-connected* if any vertex cut (set of vertices whose removal disconnects the graph) has size at least k. A complete graph K_n does not have any vertex cuts, so we define K_n to be k-connected for any $k \leq n-1$, but not k-connected for any $k \geq n$.

A *walk* in a graph G is a sequence of vertices $v_0, v_1, ... v_n$ such that (v_i, v_{i+1}) represents an edge in G, for $0 \leq i \leq n-1$. The vertices in the sequence do not necessarily have to be distinct. A *trail* in G is a walk in which no edge is repeated (so any two given vertices can appear consecutively in the sequence at most once). A *path* in G is a walk in which no edges and no vertices are repeated; that is, a vertex appears at most once in the sequence. Given a path P in a graph G, and two vertices u, v on P, we denote the subpath of P beginning at u and ending at v by $P[u,v]$. The same path minus the vertices u and v is denoted by $P(u,v)$. We denote by $P[u,v)$ the subpath of P beginning at u and ending at the vertex directly preceding v. The subpath $P(u,v]$ is defined analogously. For a path P with an assigned direction, we denote by $P[-,u]$ the subpath of P beginning at the start vertex of P and ending at u. Similarly, we denote by $P[u,-]$ the subpath of P beginning at u and ending at the last vertex of P. We use P^{-1} to denote the reverse of the path P (i.e., the same path but with the opposite direction assigned). Given two paths P and Q where the end vertex of P is the start vertex of Q, we denote the concatenation of the two paths P and Q by PQ.

A walk that has the property $v_0 = v_n$, but that does not repeat edges and repeats no vertices other than the initial (final) vertex, is called a *cycle*. For any graph G, the circumference of G is defined to be the length of a longest cycle in G. If a graph on n vertices has circumference n, then there is a cycle containing every vertex in G. A cycle of this type is called a *Hamilton cycle*, and the graph G is said to be *Hamiltonian*.

Graphs that are Hamiltonian are of particular interest. A Hamilton cycle provides a certain notion of efficient traversability - a way to 'visit' each vertex in the graph without retracing any steps. One well-known application of Hamiltonicity is the Travelling Salesman Problem: given n cities, what is the shortest route a travelling salesman can take so as to visit each city once and return to the original city?

There are many sufficient conditions for graphs to be Hamiltonian. We concern ourselves mainly with Hamiltonicity results given connectivity and planarity conditions, and later, forbidden minor restrictions.

1.2 Hamiltonicity in Planar Graphs with Connectivity Restrictions

Here we present some known results regarding Hamiltonicity for classes of planar graphs with certain connectivity conditions.

Our first theorem was proved by Whitney in 1931, and specifically talks about 4-

connected planar triangulations. A triangulation is a graph with an embedding in which every face is a triangle. Planar triangulations are maximal planar graphs, in that the addition of any any edge between existing vertices would violate planarity.

Theorem 1.1 (Whitney, [24])**.** *Every 4-connected planar triangulation G is Hamiltonian.*

Whitney's result was eventually improved by Tutte in 1956 to include *all* 4-connected planar graphs.

Theorem 1.2 (Tutte, [18])**.** *Every 4-connected planar graph is Hamiltonian.*

Tutte's result was strengthened in 1983 by Thomassen. This theorem refers to Hamilton paths, which are simply paths of length n in a graph on n vertices.

Theorem 1.3 (Thomassen, [17])**.** *Let G be a 4-connected planar graph. Then for every pair of distinct vertices x and y in G, there is a Hamilton path beginning at x and ending at y (that is, G is Hamilton-connected).*

We have now seen that all 4-connected planar graphs are Hamiltonian (in fact they are Hamilton-connected, an even stronger property). This may lead us to wonder whether we can weaken the connectivity restriction and still achieve Hamiltonicity. Unfortunately, it turns out that 3-connectivity and planarity are not sufficient conditions for Hamiltonicity; consider for example the Herschel graph, shown in Figure 1.1. It is both 3-connected and planar, however it does not admit a Hamilton cycle. As shown by the black and white vertices in the figure, the Herschel graph is *bipartite* (that is, there is a partition of the vertices into two nonempty parts such that the only edges in the graph have their end vertices in different parts). Therefore any Hamilton cycle would have to alternate white and black vertices. However, there are 5 white vertices and 6 black vertices; therefore there can be no such cycle and so the graph is not Hamiltonian. Coxeter [4] knew of this graph in 1948, and it was later shown by Barnette and Jucovič [1] and Dillencourt [5] that the Herschel graph is in fact the *smallest* example of a 3-connected planar graph that is not Hamiltonian.

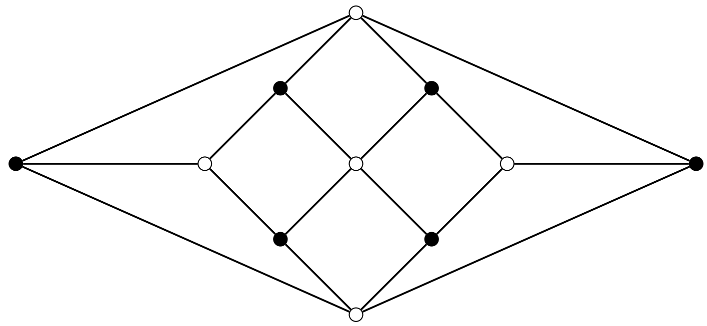

Figure 1.1: The Herschel graph

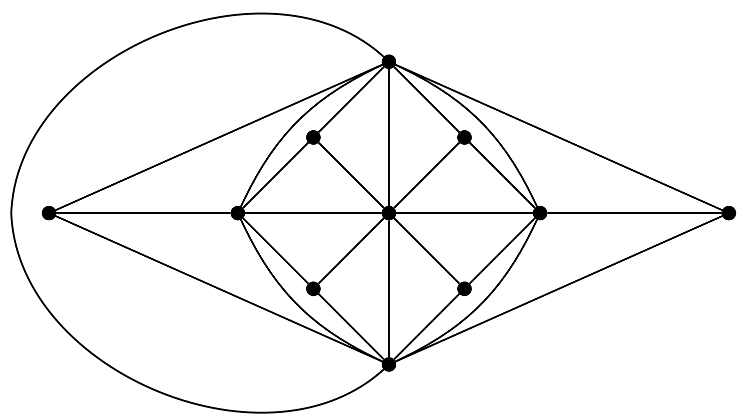

Figure 1.2: Triangulation of the Herschel graph

If we add 9 edges to the Herschel graph, we obtain a 3-connected planar triangulation that is not Hamiltonian (see Figure 1.2). Therefore, 3-connectedness is not even a sufficient condition for planar triangulations to be Hamiltonian. Whitney [24] claimed that the triangulation shown in Figure 1.2 was known to C.N. Reynolds in 1931. In fact this is the smallest 3-connected planar triangulation that is not Hamiltonian, again as was later proved in [1] and [5].

It was conjectured by Tait in 1880 [16] that 3-connected planar *cubic* graphs (that is, every vertex has degree 3) are Hamiltonian; however this conjecture was disproved in 1946 by Tutte [19], who found a counterexample (shown in Figure 1.3). Since then, infinite families of counterexamples to Tait's conjecture have been found.

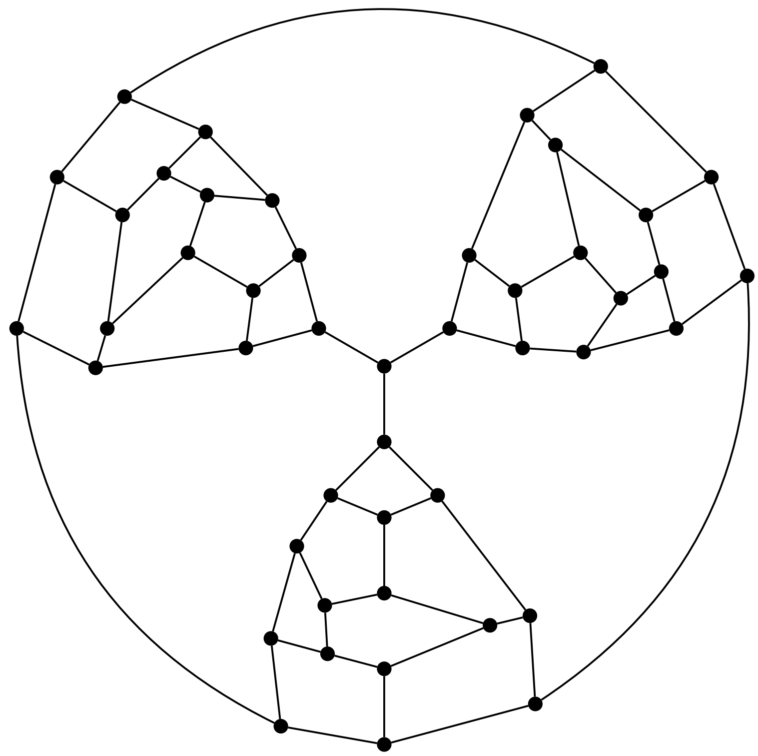

Figure 1.3: Tutte's 3-connected, cubic, planar, non-Hamiltonian graph

We have seen that it cannot be guaranteed that 3-connected planar graphs are Hamiltonian, even when we require that they are triangulations or that they are cubic. We turn our attention now to a type of graph substructure called a graph minor, and classes of graphs obtained by forbidding certain minors. As we shall see, such classes often lend themselves to characterisation, and sometimes to Hamiltonicity results.

1.3 Graph Minors

Given a graph G and a graph H, we say that H is a *minor* of G, or G has an H minor, if there is a subgraph of G to which we can apply a sequence of edge contractions and deletions to obtain a graph isomorphic to H. A graph minor is therefore in a sense a generalisation of a subgraph or a subdivision.

One way to view an H minor in a graph G is to associate a set of vertices V_i in G with each vertex v_i in H, such that the sets V_i are mutually disjoint, and whenever the edge v_iv_j

is present in H, there is some edge in G with one end vertex in V_i and the other in V_j. We also require that the graph induced on each set V_i in G is connected. This type of model is known as an *edge-based* model for a minor, and the vertex sets V_i are called *branch sets*. Another model, called a *path-based* model, for a minor allows for internally disjoint paths between the branch sets, instead of just edges. If G has an H minor, then it can be formed using either an edge-based model or a path-based model. We will mainly use the path-based model for a minor.

It turns out that a lot of useful classes of graphs can be characterised in part by forbidden or excluded minors. As an example, we have the well-known characterisation of planar graphs due to Wagner:

Theorem 1.4 (Wagner, [22]). *A graph G is planar if and only it is K_5- and $K_{3,3}$-minor-free.*

We recall that K_5 is the complete graph on 5 vertices and $K_{3,3}$ is the complete bipartite graph with both parts of size 3.

There is a special version of Wagner's Theorem (Theorem 1.4) for 3-connected graphs. It says that to determine whether a 3-connected graph (other than K_5) is planar, we need only check for $K_{3,3}$ minors, instead of both $K_{3,3}$ and K_5 minors.

Lemma 1.5 (Hall [11], Wagner [23]). *With the exception of K_5, any 3-connected graph is planar if and only if it does not contain $K_{3,3}$ as a minor.*

Theorem 1.4 actually demonstrates a very specific instance of a larger result, originally known as Wagner's Conjecture and later proved as the Robertson-Seymour Theorem [15]. Wagner conjectured in [21] that every *minor-closed* class of graphs (that is, every class of graphs that is closed under taking minors) can be characterised by excluding only finitely many minors. The original statement of the Robertson-Seymour Theorem is given below.

Theorem 1.6 (Robertson and Seymour, [15]). *Let G_i $(i = 1, 2, ...)$ be a countable sequence of graphs. Then there exist $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j.*

What the Robertson-Seymour Theorem really says is that minor relation on the set of finite graphs is a well-quasi-order. In particular, any class of graphs has finitely many minimal elements under the minor relation. This then readily implies Wagner's Conjecture.

Observe that planar graphs are a minor-closed class of graphs, hence Theorem 1.4 is indeed an example of this type of forbidden minor characterisation. We give some further examples of minor-closed classes of graphs.

Example 1.7. The class of all forests (that is, acyclic graphs) is clearly minor-closed, and can be characterised by the forbidden minor K_3, since a cycle of any length has a K_3 minor.

Example 1.8. A graph G is called *outerplanar* if G has a planar embedding with the property that all vertices lie on the boundary of the outer face. Deleting and contracting edges does not change this property, therefore the class of outerplanar graphs is minor-closed. The set of all outerplanar graphs is characterised by the forbidden minors K_4 and $K_{2,3}$. Given Wagner's Theorem, the proof of this fact is reasonably straightforward and so we present it here. The proof given is due to [13].

Proof. Suppose G is outerplanar, and consider an embedding of G in the plane such that all its vertices lie on the boundary of one face. Add a vertex v in this face and make it adjacent to all the vertices of G, thereby creating a new graph G'. Since G was planar and our addition of v did not violate planarity, G' is a planar graph. By Wagner's Theorem, G' cannot contain a K_5 or $K_{3,3}$ minor. This implies that G did not originally contain a K_4 or a $K_{2,3}$ minor.

Now suppose G is a graph that contains no K_4 or $K_{2,3}$ minor. Add a new vertex v adjacent to all vertices of G to create a new graph G'. If G' contains a K_5 or $K_{3,3}$ minor, then G would have contained either a K_4 or a $K_{2,3}$ minor, which is a contradiction. Therefore G' contains no K_5 minor and no $K_{3,3}$ minor and so is planar by Wagner's Theorem. Take a planar embedding of G'. Since all the vertices of G are adjacent to v in G', we can delete v to get an embedding of $G' - v = G$ in which all vertices of G lie on the boundary of the same face. Therefore G is an outerplanar graph. □

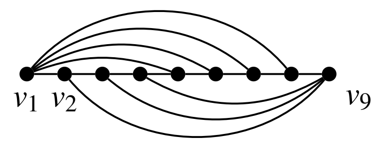

Figure 1.4: The graph $G_{9,4,3}$

We turn our attention now to graphs excluding a specific type of graph as a minor; in particular, we concern ourselves with $K_{2,t}$-minor-free graphs.

First consider $K_{2,3}$-minor-free graphs. Since $K_{2,3}$ is 2-connected, all graphs that are not 2-connected are $K_{2,3}$-minor-free. However, graphs that are both 2-connected and $K_{2,3}$-minor-free have a nice characterisation. The only 2-connected, $K_{2,3}$-minor-free graph that is not also K_4-minor-free is K_4 itself, therefore by our characterisation of outerplanar graphs (Example 1.8) all 2-connected $K_{2,3}$-minor-free graphs are either outerplanar or K_4.

Now let us look at $K_{2,4}$-minor-free graphs. In 2014, Ellingham et al. [8] gave a complete characterisation of $K_{2,4}$-minor-free graphs in terms of structure. They first defined a class of graphs $G^{(+)}_{n,r,s}$: for $n \geq 3$ and $r,s \in \{0,1,...,n-3\}$, the graph $G_{n,r,s}$ is defined to be the graph consisting of a spanning path $v_1v_2...v_n$ and edges v_1v_{n-i} for $1 \leq i \leq r$ and v_nv_{1+j} for $1 \leq j \leq s$. An example is shown in Figure 1.4. The graph $G^{+}_{n,r,s}$ is defined to be the graph $G_{n,r,s} + v_1v_n$. Now let $\tilde{\mathcal{G}}$ be the class of graphs isomorphic to a graph in the collection below:

$$\{G^{+}_{n,1,n-3}, G^{+}_{n,n-3,1} : n \geq 4\} \cup \{G^{(+)}_{n,r,s} : n \geq 5, r,s \in \{2,3,...,n-3\}, r+s = n-1 \text{ or } n-2\}.$$

Then they have the following result:

Theorem 1.9 (Ellingham et al., [8])**.** *Let G be a 3-connected graph. Then G is $K_{2,4}$-minor-free if and only if $G \in \tilde{\mathcal{G}}$ or G is isomorphic to one of nine small exceptions.*

They then characterised 2-connected $K_{2,4}$-minor-free graphs in terms of outerplanar graphs and the 3-connected $K_{2,4}$-minor-free graphs from above, and went on to characterise all $K_{2,4}$-minor-free graphs as exactly those graphs whose blocks (maximal 2-connected subgraphs) each fit their characterisation of 2-connected $K_{2,4}$-minor-free graphs.

A rough structural characterisation of $K_{2,t}$-minor-free graphs in general is given by Ding in [6]. In the 3-connected case, he proves that for any $t \geq 2$, all (3-connected) $K_{2,t}$-minor-free graphs can be obtained from a set of graphs on at most $n(t)$ vertices by adding certain structures, of arbitrary size, called fans and strips. However, $n(t)$ is a very large number, and we do not know exactly which base graphs these fans and strips can be added to. Some corollaries of the main result in Ding's paper are presented below.

Theorem 1.10 (Ding, [6])**.** *Let t be a positive integer and let G be a $K_{2,t}$-minor-free graph.*

1. *If G is 3-connected with minimum degree at least six, then the size of G is bounded.*

2. *If G is 4-connected, then the maximum degree of G is bounded.*

This dissertation's main results, presented in Chapters 3 and 4, provide details for specific cases of Ding's rough characterisation. We look at classes of 3-connected $K_{1,1,t}$-minor-free graphs, and use explicitly-found fans to prove results about the graphs. Observe that $K_{1,1,t}$ is a minor of $K_{2,t+1}$, so Ding's general description applies to our graphs.

In 2011, Chudnovsky, Reed and Seymour proved a result on the maximum number of edges a $K_{2,t}$-minor-free graph can have. This result was previously proved by Myers [14] for $t \geq 10^{29}$, but Chudnovsky, Reed and Seymour proved it for all $t \geq 2$.

Theorem 1.11 (Chudnovsky, Reed and Seymour, [3])**.** *Let $t \geq 2$, and let G be a graph with $n \geq 1$ vertices and with no $K_{2,t}$ minor. Then*

$$|E(G)| \leq \frac{1}{2}(t+1)(n-1).$$

In particular, this implies that for $t \geq 2$, any graph G on n vertices with more than $\frac{1}{2}(t+1)(n-1)$ edges must have a $K_{2,t}$ minor.

1.4 Minors and Hamiltonicity

Now that we have explored some results concerning Hamiltonicity, and some results concerning graph minors, we look at results involving both. We first present a result concerning Hamiltonicity of 3-connected planar triangulations in relation to the Herschel graph. Recall that the Herschel graph is a minimal example of a non-Hamiltonian 3-connected planar graph, and a triangulation of the Herschel graph is a minimal example of a non-Hamiltonian 3-connected planar triangulation.

Theorem 1.12 (Ding and Marshall,[7]). *Let G be a non-Hamiltonian 3-connected planar triangulation. Then G contains the Herschel graph as a minor.*

We also present some theorems regarding the Hamiltonicity of certain classes of $K_{2,t}$-minor-free graphs.

First we look at graphs with no $K_{2,3}$-minors. If a graph is not 2-connected we cannot have a Hamilton cycle, so we consider only 2-connected $K_{2,3}$-minor-free graphs. Recall that we characterised all 2-connected $K_{2,3}$-minor-free graphs as being either outerplanar or K_4. In the former case, the boundary cycle of an outerplanar embedding is a Hamilton cycle. The graph K_4 is also Hamiltonian, therefore all 2-connected $K_{2,3}$-minor-free graphs are Hamiltonian.

The natural next class of graphs to consider is the class of $K_{2,4}$-minor-free graphs. The hamiltonicity of such graphs was investigated by Ellingham et al. in their paper characterising all $K_{2,4}$-minor-free graphs [8], some results from which we have already seen. They proved the following result:

Theorem 1.13 (Ellingham et al., [8]).

(i) Every 3-connected $K_{2,4}$-minor-free graph is Hamiltonian.

(ii) There are 2-connected $K_{2,4}$-minor-free planar graphs that have no spanning closed trail and hence no Hamilton cycle.

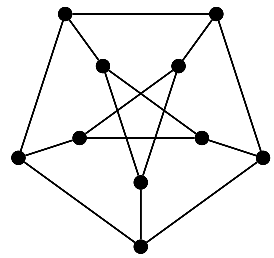

Figure 1.5: The Petersen Graph

(iii) However, every 2-connected $K_{2,4}$-minor-free graph has a Hamilton path.

Now we know that 3-connected $K_{2,4}$-minor-free graphs are Hamiltonian, we look at 3-connected $K_{2,5}$-minor-free graphs. It turns out that these are not in general Hamiltonian. One counterexample to this is the Petersen graph shown in Figure 1.5. It is 3-connected and $K_{2,5}$-minor-free, but admits no Hamilton cycle.

In fact, there are *infinitely* many non-Hamiltonian 3-connected $K_{2,5}$-minor-free graphs, as was recently found by J. Zachary Gaslowitz (personal communication).

The non-Hamiltonian 3-connected $K_{2,5}$-minor-free graphs discussed all have one thing in common: they are nonplanar. However, if we go back to considering 3-connected *planar* graphs, along with the $K_{2,5}$-minor-free restriction, we do get Hamiltonicity. This result was found by Ellingham et al. [9]:

Theorem 1.14 (Ellingham et al., [9]). *Let G be a 3-connected planar $K_{2,5}$-minor-free graph. Then G is Hamiltonian.*

The remainder of this dissertation is dedicated to proving new results, influenced by and building on the results presented in this chapter.

Chapter 2

Fans

In this chapter we define a type of induced subgraph called a *fan*, and show how certain operations on these fans preserve some useful graph properties.

Definition 2.1. We call an induced subgraph in G on vertices $r, v_1, v_2, ..., v_k$ ($k \geq 1$) a *k-fan*, or a *fan of size k*, if $v_1 v_2 ... v_k$ is a path, r is adjacent to every vertex v_i on the path, $\deg_G(r) \geq 4$, and $\deg_G(v_i) = 3$ for $1 \leq i \leq k$. We call the vertex r the *rivet* vertex of the fan, and the vertices v_i the *outer* vertices of the fan. An edge on the path $v_1 v_2 ... v_k$ is called a *collapsible edge*. When the size of a k-fan in a graph is not relevant, we shall simply say that the graph has a fan, and take this to mean a k-fan for some $k \geq 1$. We will say that a fan is *nontrivial* if $k \geq 2$.

Given this definition of a fan, we now define some operations on fans that will prove useful in induction arguments.

Definition 2.2. For a graph G with a nontrivial fan, we refer to contracting any collapsible edge of the fan as *collapsing* the fan.

Remark 2.3. Given a graph G with a k-fan, the graph obtained by contracting a collapsible edge in the fan is independent (up to isomorphism) of the choice of collapsible edge. In

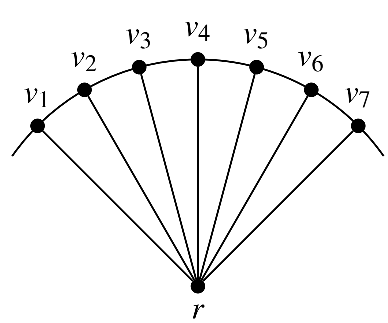

Figure 2.1: A 7-fan

particular, we get the graph obtained from G by replacing the k-fan with a $(k-1)$-fan. Because the choice of collapsible edge is irrelevant, collapsing a fan is well-defined as an operation.

Definition 2.4. Given a graph G with a fan, we define an *expansion* of an outer vertex of the fan to be the operation consisting of splitting the vertex in the following way. Fix an outer vertex v. We know v is adjacent to the rivet vertex r, and to exactly two other vertices, say u and w. We split v into the two adjacent vertices v_1 and v_2, such that both v_1 and v_2 are adjacent to r, v_1 is adjacent to u and v_2 is adjacent to w. The only neighbours of v_1 and v_2 are those specified.

Remark 2.5. Given a graph G with a k-fan, the graph obtained by expanding any outer vertex of the fan is independent (up to isomorphism) of the choice of outer vertex. In particular, we get the graph obtained from G by replacing the k-fan with a $(k+1)$-fan. Because the choice of outer vertex to expand is irrelevant, we may refer to such an operation simply as *expanding* the fan.

Remark 2.6. Collapsing and expanding fans are inverse operations. In particular, given a graph G with a nontrivial k-fan, if we collapse the k-fan and then expand the resulting $(k-1)$-fan, we recover G. If we first expand the k-fan and then collapse the resulting $(k+1)$-fan, we again recover G.

We now examine certain graph properties that are preserved by one or both of the fan operations described above.

Lemma 2.7. *Let G be a non-Hamiltonian graph with a nontrivial fan, and let e be a collapsible edge of the fan. Then G/e is non-Hamiltonian.*

Proof. In G, let r be the rivet vertex of the fan, and let u_1 and u_2 be the end vertices of e. Let w_1 be the remaining neighbour of u_1 and w_2 be the remaining neighbour of u_2. Let u be the vertex in G/e that u_1 and u_2 are contracted to. Now suppose by way of contradiction that G/e is Hamiltonian with Hamilton cycle C. If u's neighbours on C are w_1 and w_2, then the cycle formed in G by replacing w_1uw_2 in C by $w_1u_1u_2w_2$ is Hamiltonian in G, a contradiction. If u's neighbours on C are w_1 and r, then the cycle formed in G by replacing w_1ur in C by $w_1u_1u_2r$ is Hamiltonian in G. The case where u's neighbours on C are w_2 and r is symmetric to the previous case. Therefore if G/e is Hamiltonian, G must also be Hamiltonian. □

Lemma 2.8. *If G is a 3-connected graph with a nontrivial fan, then every collapsible edge in the fan is a 3-contractible edge in G.*

Proof. Let G be a 3-connected graph with a nontrivial fan, and let e be a collapsible edge in the fan, with end vertices u and v. Let r be the rivet vertex of the fan. Since $\deg_G(r) \geq 4$ (from the definition of a fan), $|V(G)| \geq 5$ and $|V(G/e)| \geq 4$. Suppose that G/e is not 3-connected. Then G/e has some 2-cut, since $|V(G/e)| \geq 4$. Since the 2-cut in G/e cannot also be a 2-cut in G, the vertex that u and v are contracted to is part of the 2-cut. Then u and v must be part of some 3-cut in G, along with some other vertex w. Since G is 3-connected, the cutset $\{u,v,w\}$ is minimal. First we show that $w \neq r$. To see this, suppose by way of contradiction that $w = r$. Since the cutset is minimal, u, v, and r must each have a neighbour in every component of $G \setminus \{u,v,r\}$. However, u and v both have degree 3 in G and therefore each have only one other neighbour outside the cutset. Therefore $\{u,v,r\}$ is not a 3-cut and so $w \neq r$.

Now consider the components of $G \setminus \{u,v,w\}$. Since r is not in the cut set, it lies in some component C_1. All of r's neighbours that are not in $\{u,v,w\}$ must also lie in C_1. Note that by definition of a fan, $\deg_G(r) \geq 4$ and so r has at least one neighbour not in $\{u,v,w\}$, therefore $|C_1| \geq 2$. Now since the cut set $\{u,v,w\}$ is minimal, each of u,v,w has a neighbour in every component of $G \setminus \{u,v,w\}$. Since u and v are degree three vertices, they each have only one neighbour other than each other and r, therefore there is only one other component (call it C_2), which contains the remaining neighbours of u and v. But now $\{w,r\}$ must be a cut set in G, since deletion of w and r separates u and v from $C_1 \setminus \{r\}$, which is nonempty. This contradicts G's 3-connectivity.

Therefore if G is 3-connected, so is G/e. □

The following is a general result for k-connected graphs, which implies a nice corollary for fans in 3-connected graphs. The case where $k = 3$ for the lemma below was proved by Tutte in [20].

Lemma 2.9. *Let G be a k-connected graph, and let v be a vertex in G. Let G' be a graph obtained by splitting v into two adjacent vertices v_1 and v_2, such that each of v_1 and v_2 has degree at least k. Then G' is also k-connected.*

Proof. Let G, G', v, v_1 and v_2 be as described in the statement of the lemma. Suppose by way of contradiction that G' is not k-connected. Then G' has some cutset S with $|S| \leq k-1$. We observe that v_1 and v_2 cannot lie in different components of $G \setminus S$, since there is an edge between them. Consider the case that both v_1 and v_2 are in the same component of $G' \setminus S$. Then if we contract the edge between v_1 and v_2 to recover G, the set S is still a cutset in G, contradicting that G is k-connected. Next consider the case that both v_1 and v_2 are in S. Then again, when we contract the edge between them to recover G, the set $S \cup \{v\} \setminus \{v_1, v_2\}$ is a cutset in G, and it has size at most $k-2$, a contradiction. Therefore it must be the case that of the vertices v_1, v_2, one of them is in S and one of them is in $G \setminus S$. Without loss of generality, assume that v_1 is in S and v_2 is in a component C of $G \setminus S$. Let us contract the edge between v_1 and v_2 to recover G. We claim that this still gives us $S \cup \{v\} \setminus \{v_1, v_2\}$ as a cutset in G. To see this, recall that in G', v_2 has degree at least k. Since $|S| \leq k-1$, v_2 must have had at least one neighbour not in S. This neighbour must have been in C, since $G' \setminus S$ was disconnected. Therefore $C \setminus \{v_2\}$ is nonempty, and it follows that after the contraction of v_1 and v_2 to v, $C \setminus \{v_2\}$ is still disconnected from the rest of the graph by $S \cup \{v\} \setminus \{v_1, v_2\}$, which has cardinality at most $k-2$. Since all cases result in a contradiction, G' must be k-connected. □

Corollary 2.10. *If G is a 3-connected graph with a fan, then the graph obtained by expanding the fan is also 3-connected.*

Our final result on fans concerns minors. Specifically, we prove that for certain graphs M, expanding a large enough fan in a graph preserves the M-minor-free property.

Lemma 2.11. *Let G be a graph with a k-fan F for some $k \geq 5$, and let M be a complete multipartite graph on at least six vertices, such that M has no degree one vertices (i.e., M is not $K_{1,t}$ for any t). Then if G has an M minor, so does the graph obtained from G by collapsing F.*

Proof. Let G be as described in the statement of the lemma, let r be the rivet vertex of F, and let v, w, x, y, z be consecutive outer vertices of F. Suppose that G has an M minor. Consider branch sets corresponding to an edge-based model of M (i.e., the paths between branch sets are just edges). Since G is connected, we may assume that every vertex lies in one of the branch sets. Now, if any two consecutive vertices in $\{v, w, x, y, z\}$ lie in the same branch set, we may collapse the edge between those two vertices and preserve the minor, and we are done.

Therefore assume that of the five vertices v, w, x, y, z, no two consecutive ones are in the same branch set. We may also assume that none of w, x, y is in the same branch set as r. To see this, suppose for example x were in the same branch set as r. Then we could move x from its current branch set to w's branch set without destroying the minor (since r is also adjacent to both of x's other neighbours). This gives the case already addressed above. A similar argument applies to the cases where w or y is in the same branch set as r. Therefore we may assume that none of w, x, y is in the same branch set as any of its neighbours. This implies that each of these three vertices entirely comprises its branch set.

Now consider M. Let x_M be the vertex in M corresponding to the branch set $\{x\}$ in G, and let w_M, y_M be defined analogously. Let X be the part (maximal independent set) of M containing x_M. If w_M and y_M are also in X, then x has at most one neighbour whose branch set corresponds to a vertex in $M \setminus X$, contradicting $\deg_M(x_M) > 1$. Therefore w_M and y_M cannot both be in X. Without loss of generality, assume that y_M is in part Y, where $Y \neq X$. Now since M is a complete multipartite graph, and x_M and y_M are in different parts, every vertex in M is adjacent to either x_M or y_M. However, $|N_G(x) \cap N_G(y)| = 5$. Therefore $|V(M)| \leq 5$. Since we assumed M had at least six vertices, this is a contradiction. □

Chapter 3

A Characterisation of 3-connected, $K_{1,1,4}$-minor-free Graphs

Recall that in Chapter 1, we discussed the characterisation of 3-connected $K_{2,4}$-minor-free graphs found by Ellingham et al. [8]. In this section, we present a complete characterisation of a superset of 3-connected $K_{2,4}$-minor-free graphs; namely, 3-connected $K_{1,1,4}$-minor-free graphs. Observe that $K_{1,1,4}$ is $K_{2,4}$ plus an edge, therefore the set of graphs obtained by forbidding $K_{2,4}$ as a minor is contained in the set of graphs obtained by forbidding $K_{1,1,4}$ as a minor.

For our characterisation, we first define a family of graphs $\mathcal{G}$, the members of which we will show to be both 3-connected and $K_{1,1,4}$-minor-free. We then prove that for $n \geq 13$, any 3-connected $K_{1,1,4}$-minor-free graph on n vertices is in our family $\mathcal{G}$. The 3-connected $K_{1,1,4}$-minor-free graphs on fewer than 13 vertices are treated separately. Finally, we use our characterisation to prove some Hamiltonicity, planarity, and counting results.

3.1 The Family

Our family $\mathcal{G}$ is comprised of three subfamilies. The first of these subfamilies is denoted by $\mathcal{F}$, and its members are formed by attaching two fans to any of the ten base graphs F_i, $1 \leq i \leq 10$, shown in Figure 3.2 below. Specifically, for any of these ten base graphs and for any two non-negative integers j and k, we add vertices to the base graph in the following

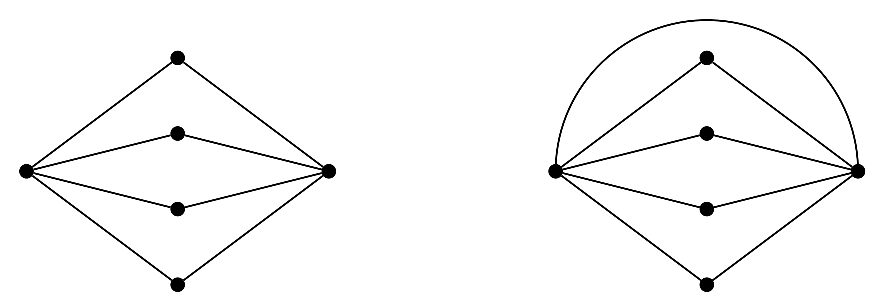

Figure 3.1: The graphs $K_{2,4}$ (left) and $K_{1,1,4}$ (right)

way. For the non-negative integer j, Let new vertices $x_0, x_1, ..., x_{j-1}$ form a path in that order, and in addition let each vertex x_i be adjacent to z_0. Also let x_{j-1} be adjacent to z_1. We attach k vertices in a similar way: let $y_0, y_1, ..., y_{k-1}$ be vertices that form a path in that order, let each y_i be adjacent to z_3, and let y_{k-1} be adjacent to z_4. Finally, let us add an edge between x_0 and y_0. If $j = 0$, then we define $x_0 := z_1$ and so the added edge is between z_1 and y_0. Similarly, if $k = 0$, then define $y_0 := z_4$ and the added edge is between x_0 and z_4. Observe that the vertices z_1 and z_4 each have degree two in every base graph F_i, so in general they will have degree three in the final graph. Notice that as long as $j \geq 2$, we have $\deg(z_0) \geq 4$ and the graph induced on the vertices $\{x_0, x_1, ..., x_{j-1}, z_1, z_0\}$ is a $(j+1)$-fan, with rivet vertex z_0. Similarly, as long as $k \geq 2$, we have $\deg(z_3) \geq 4$ and the graph induced on the vertices $\{y_0, y_1, ...y_{k-1}, z_4, z_3\}$ is a $(k+1)$-fan with rivet vertex z_3.

With some exceptions (described below), we include any graph formed this way in our family $\mathcal{F}$. In addition, for any such graph G, the graph obtained from G by adding an edge between x_0 and z_3 is also included in our family. Note that if we add this edge, then x_0 is no longer considered an outer fan vertex, as it has degree four. This means that the $(j+1)$-fan becomes a j-fan (with the exception of the case where $j = 0$, in which case the addition of the optional edge x_0z_3 destroys the fan). An example of a member of $\mathcal{F}$ is shown in Figure 3.3.

The aforementioned exceptions to inclusion in the family $\mathcal{F}$ are the cases in which the graph resulting from adding these j- and k-fans is not 3-connected, since we wish $\mathcal{F}$ to be a family of 3-connected graphs. These exceptions are those graphs that have a vertex of degree two, which occur in two situations: (1) when the degree of z_0 in the base graph is two and $j = 0$, or (2) when the degree of z_3 in the base graph is two, $k = 0$, and the edge x_0z_3 is not present. Graphs that fall into one of these categories are excluded from $\mathcal{F}$.

We observe that for a graph in $\mathcal{F}$, there are at most eight vertices that are not the outer vertex of a fan riveted at z_0 or z_3. These vertices are z_0, z_2, z_3, z_5, x_0 (if $j = 1$ or the edge x_0z_3 is present), z_1 (if $j = 1$) and y_0, z_4 (if $k = 1$). Note that the only situation in which either of z_1, z_4 may not be outer fan vertices is when one of z_0 or z_3 has degree less than four, meaning it does not fit the definition of a rivet vertex, and so there is no fan riveted at that vertex. If we have both fans, there are at most six vertices that are not outer vertices of a fan. For any graph in $\mathcal{F}$, consider the maximal extensions of the two fans described above, riveted at z_0 and z_3 respectively. By maximal extension, we mean the largest fan containing our described fan as a subfan. The graph shown in Figure 3.3 is an example of a graph where the maximal extension of one of the fans is strictly larger than the original fan. In this graph, the maximal extension of the fan riveted at z_0 includes z_2, while the original fan did not. For any graph in $\mathcal{F}$, if the maximal extension of either fan has size at least five, we refer to the extension as a *main fan* (the graph may have zero, one or two main fans). Observe that if there are two main fans, they must be vertex disjoint, since they have distinct rivet vertices and outer vertices are forced to have degree three, implying that no outer vertex can belong to both fans.

The next subfamily we define is $\mathcal{H}$. The graphs in $\mathcal{H}$ are constructed using the twelve base graphs H_i, $1 \leq i \leq 12$, shown in Figure 2. For any of the twelve base graphs, we add the j vertices and k vertices to the the graph in the same way as for $\mathcal{F}$ (including adding the edge x_0y_0), except that we require $j = 1$ (however, k is still free to be any non-negative integer). Again with some exceptions, any graph G formed in this way is a member of our family $\mathcal{H}$, as is $G + x_0z_3$. The exceptions are when the graph formed is not 3-connected, which happens in three different situations: (1) the degree of z_3 in the base graph is two, $k = 0$, and the edge x_0z_3 is absent, (2) the degree of z_3 in the base graph is one, $k = 1$, and the edge x_0z_3 is absent, or (3) the degree of z_3 in the base graph is one and $k = 0$. Any graph falling into one of these categories is excluded from $\mathcal{H}$. An example of a graph belonging to $\mathcal{H}$ is shown in Figure 3.5.

For a graph in $\mathcal{H}$, we consider its fans. Since $j = 1$, we can have a 2-fan riveted at z_0, as long as z_1 has degree two in the base graph and is adjacent to z_0, the edge x_0z_3 is absent, and the degree of z_0 in the base graph is at least three. However, it may be possible that

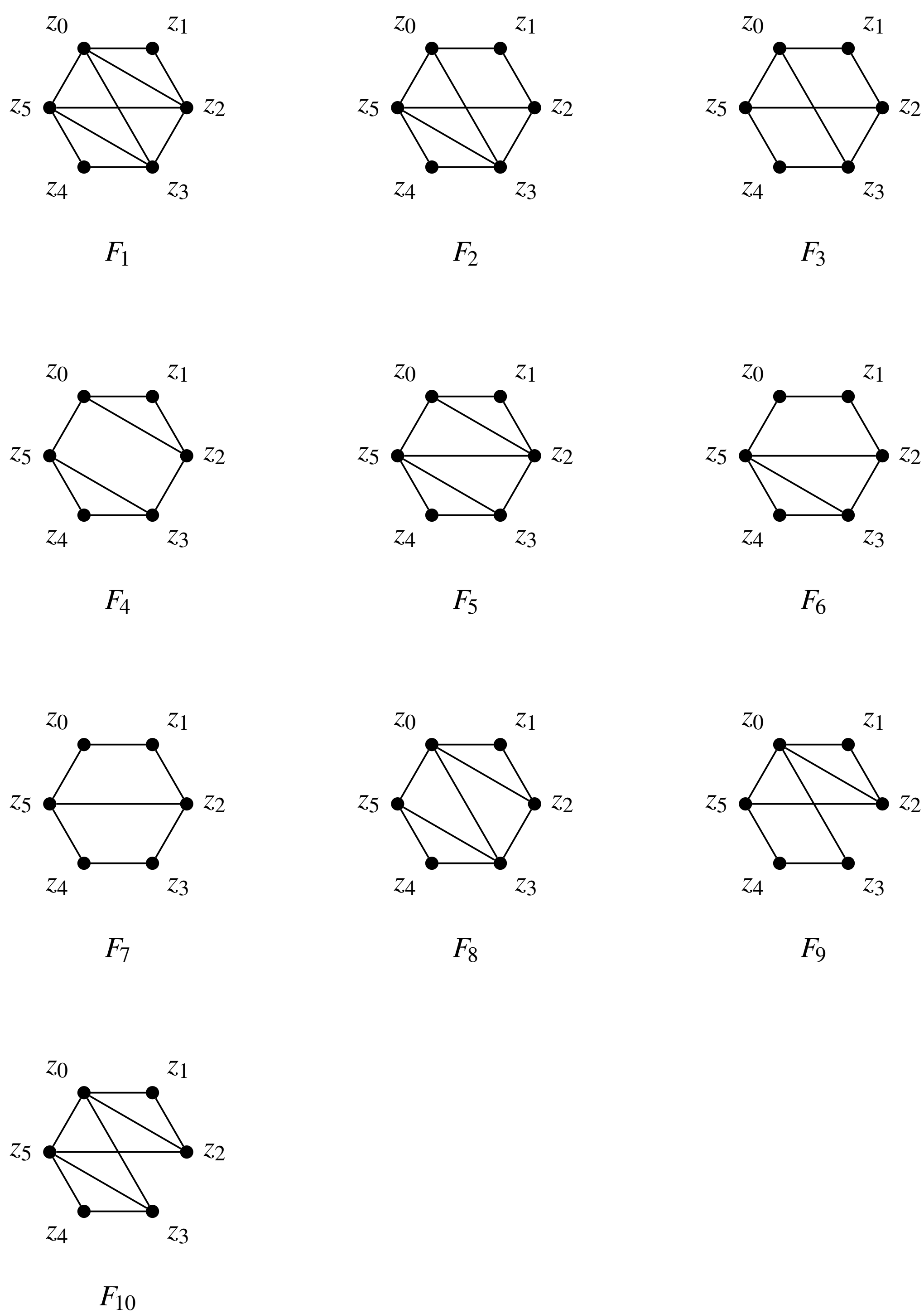

Figure 3.2: The ten base graphs used to construct $\mathcal{F}$

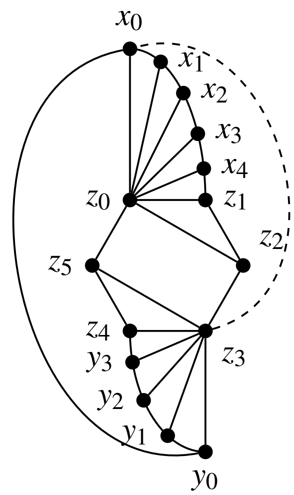

Figure 3.3: An example of a graph in $\mathcal{F}$, formed by attaching a 5-fan and a 4-fan to the base graph F_4. The optional edge x_0z_3 is shown as a dashed line.

none of these conditions hold and there is no fan riveted at z_0. There are also situations where we have a 1-fan riveted at z_0. However, since no base graph H_i has both z_0z_1 and z_0z_2 as edges and also z_2 of degree three, there is never a fan of size three or more riveted at z_0. If $k \geq 3$, then we definitely have a k-fan riveted at z_3, with outer vertices $y_0, \ldots y_{k-1}$ ($k = 2$ is not sufficient for a fan here, since we have a base graph in which $\deg(z_3) = 1$). Observe that for any graph in $\mathcal{H}$, there are at most nine vertices that are not outer vertices of a fan riveted at z_3; namely, z_i, $0 \leq i \leq 5$, x_0, y_0 and y_1 (if $k \leq 2$). For any graph in $\mathcal{H}$, if the maximal extension of the fan induced on $y_0, \ldots, y_{k-1}$ and z_3 has size at least five, we define it to be the *main fan* of the graph. A graph in $\mathcal{H}$ can have at most one main fan.

For a given $n \geq 4$, we define the *wheel graph* W_n to be the graph consisting of a cycle of length $n-1$, and a 'centre' vertex that is adjacent to all vertices on the cycle. We define $\mathcal{W}$ to be the family consisting of all wheel graphs W_n for $n \geq 4$. For any such wheel graph W_n, we let the centre vertex be denoted by w and the outer vertices denoted by $v_1, v_2, \ldots, v_{n-1}$. For $n \geq 5$, all outer vertices have degree exactly three, and the centre vertex w has degree at least four. Therefore any wheel graph on at least five vertices has an $(n-1)$-fan riveted at w. In fact, such a graph has $n-1$ different $(n-1)$-fans riveted at w, distinguished by the choice of start vertex for the path through the outer fan vertices, although all such fans

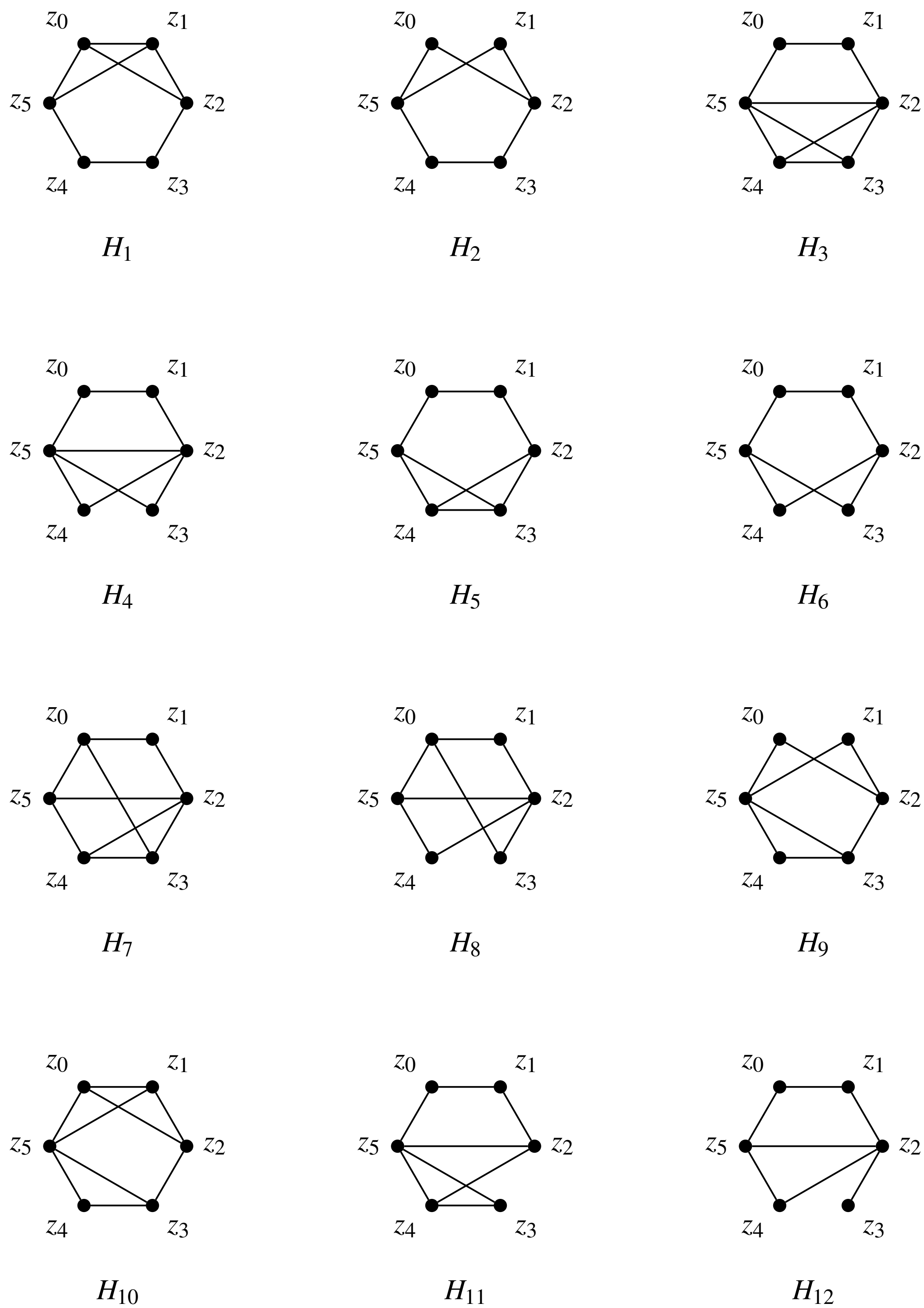

Figure 3.4: The twelve base graphs used to construct $\mathcal{H}$

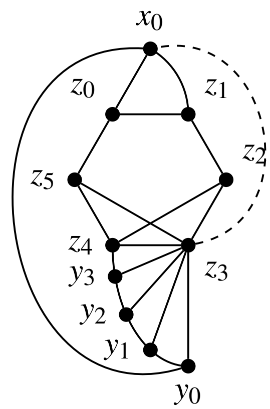

Figure 3.5: An example of a graph in $\mathcal{H}$, formed by attaching the required 1-fan and a 4-fan to the base graph H_5. The optional edge x_0z_3 is shown as a dashed line.

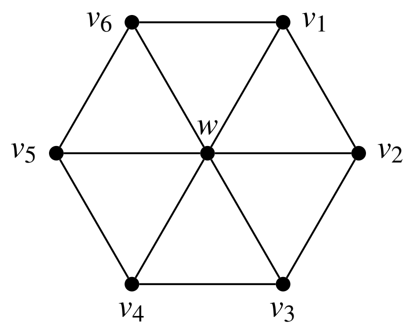

Figure 3.6: The wheel on seven vertices (W_7)

have the same vertex set. We wish to define a main fan for the graphs W_n where $n \geq 6$ (if $n < 6$ we cannot have a fan of size at least five). Since each of the $n-1$ fans has the same vertex set ($V(W_n)$) and they are all equivalent up to a relabelling of the outer vertices, we may refer to any $(n-1)$-fan of W_n as a *main fan* of W_n. Observe that expanding any $(n-1)$-fan of W_n results in the graph W_{n+1} and contracting any $(n-1)$-fan of W_n results in the graph W_{n-1}, so our choice of particular main fan is irrelevant. One wheel graph is shown in Figure 3.

We now combine these three families, by letting $\mathcal{G} := \mathcal{F} \cup \mathcal{H} \cup \mathcal{W}$. We also define $\mathcal{G}_n := \{G \in \mathcal{G} : |V(G)| = n\}$ for all $n \geq 4$ (there are no graphs in $\mathcal{G}$ with fewer than four vertices).

We now make some observations on the fans of graphs in $\mathcal{G}$, which will prove useful for our characterisation.

Remark 3.1. For any graph in $\mathcal{G}$ with a main fan, expanding or collapsing a main fan gives a graph that is also in $\mathcal{G}$, and the expanded or contracted fan is a main fan of the new graph, provided it still has size at least five. In particular, expanding or collapsing a main fan of a graph in $\mathcal{F}$ built on base graph F_i yields a graph in $\mathcal{F}$ with base graph F_i, expanding or collapsing a main fan of a graph in $\mathcal{H}$ built on base graph H_j yields a graph in $\mathcal{H}$ with base graph H_j, and expanding or collapsing a main fan of a graph in $\mathcal{W}$ yields a graph in $\mathcal{W}$. These are immediate consequences of the construction of the families.

Remark 3.2. Let G be a graph in $\mathcal{G}_n$. Then G has either a fan of size at least $n-9$, or two fans whose sizes sum to at least $n-6$. In particular, if $n \geq 12$, G has a fan of size at least $\lceil \frac{n-6}{2} \rceil$.

Remark 3.3. For any graph G in $\mathcal{G}$, the only vertices that can have degree more than four are z_0, z_3 (if G is in $\mathcal{F}$ or $\mathcal{H}$) or w (if G is in $\mathcal{W}$). Additionally, each of these three vertices has at most four neighbours that are not outer vertices of our described fan riveted at that vertex. In particular, this implies that any maximal fan of size at least five is necessarily a main fan in the graph.

Lemma 3.4. *For $n \geq 10$, the base graphs F_i, $1 \leq i \leq 10$ and H_j, $1 \leq j \leq 12$, are a minimal set of base graphs, in that we do not obtain all of the n-vertex graphs in $\mathcal{F} \cup \mathcal{H}$ from any proper subset of them. Additionally, the wheel graph on n vertices, W_n, is not isomorphic to any graph in $\mathcal{F}$ or $\mathcal{H}$.*

Proof. We verified the statement of the lemma for $10 \leq n \leq 14$ by computer, leaving out one base graph at a time, followed by the wheel graph W_n, and comparing the list of graphs generated by the remaining base graphs with $\mathcal{G}_n$ (generated previously). Fix some $m \geq 15$, and assume the statement holds for all n with $10 \leq n < m$. Let G be a graph in $\mathcal{G}_m$. Suppose that G can be constructed using two distinct base graphs, B_1 and B_2. By Remarks 3.2 and 3.3, G has a fan of size at least five, which is necessarily a main fan. Let us collapse the main fan to obtain a graph G' in $\mathcal{G}_{m-1}$. By Remark 3.1, G' is also built on both B_1 and B_2. Therefore if there exists some base graph B such that every graph in $\mathcal{G}_m$ generated by B can also be generated by some other base graph, the same base graph B is unnecessary for generating $\mathcal{G}_{m-1}$. This contradicts our inductive hypothesis. Similarly, if G is built on some base graph B_1 and G is also isomorphic to the wheel graph W_m, then collapsing a main fan of G gives us a graph G'' in $\mathcal{G}_{m-1}$ that is both built on B_1 and isomorphic to W_{m-1}, by Remark 3.1. This is also a contradiction of our inductive hypothesis. Therefore by induction, the statement of the lemma holds for $n \geq 15$. □

Remark 3.5. Although the set of base graphs is minimal for generating $\mathcal{G}_n$, $n \geq 10$, this does not imply that we do not generate some duplicate graphs. For example, for any of the symmetric $\mathcal{F}$ base graphs (namely, $F_1, F_3, F_4, F_5, F_7, F_8$), the graph obtained by adding attaching j vertices to z_0 and k vertices to z_3 is isomorphic to the graph obtained by adding k vertices to z_0 and j vertices to z_3 (as long as the optional edge x_0z_3 is absent).

Remark 3.6. Every graph in $\mathcal{G}$ has a vertex of degree three: in every graph in $\mathcal{F}$, the vertex z_4 must have degree three, by construction. For every base graph H_i, either z_1 or z_4 has degree two in the base graph and therefore degree three in any graph constructed from that base graph. For a graph in $\mathcal{W}$, every vertex except the centre vertex has degree three. In particular, this implies that no graph in $\mathcal{G}$ is 4-connected.

3.2 Computer Results

The induction arguments used in our main result rely on knowing all 3-connected $K_{1,1,4}$-minor-free graphs on up to 18 vertices. These graphs were computer-generated for us by J. Zachary Gaslowitz (personal communication). What follows is a description of how the graphs were generated and a summary of our findings. For convenience, we will denote the set of all 3-connected $K_{1,1,4}$-minor-free graphs on n vertices by $\mathcal{Z}_n$, and the overall set of all 3-connected $K_{1,1,4}$-minor-free graphs by $\mathcal{Z}$.

To justify that the process used to generate our graphs does in fact give us *all* of the 3-connected, $K_{1,1,4}$-minor-free on up to 18 vertices, we will need the lemma below.

Lemma 3.7 (Tutte, [20])**.** *Every 3-connected graph other than K_4 has a 3-contractible edge; that is, an edge e such that G/e is 3-connected.*

Lemma 3.7 implies that any 3-connected graph G on $n+1$ vertices can be obtained from a 3-connected graph G' on n vertices by 'uncontracting' a 3-contractible edge, i.e., splitting some vertex in G in a way that preserves 3-connectivity. By Lemma 2.9, splitting a vertex such that the two adjacent vertices resulting from the split have degree at least three is equivalent to splitting the vertex in a way that preserves 3-connectivity.

This lemma then implies that given $\mathcal{Z}_n$, we can generate $\mathcal{Z}_{n+1}$.

From the definition of k-connectivity, we know that K_4 is the only 3-connected graph on four vertices. K_4 is also trivially $K_{1,1,4}$-minor-free. Therefore starting with $\mathcal{Z}_4 = \{K_4\}$, it was possible to inductively generate $\mathcal{Z}_{n+1}$, for all n with $4 \leq n \leq 17$. The process used is described below.

1. For each graph G in $\mathcal{Z}_n$, and for each vertex v in G, take every possible graph obtained by splitting v into two adjacent vertices v_1 and v_2 where $\deg(v_1) \geq 3$ and $\deg(v_2) \geq 3$, and v_1 and v_2 have fewer than four mutual neighbours.

2. Remove duplicates from the resultant set of graphs.

3. Remove graphs with a $K_{1,1,4}$ minor.

In the first step, we discard the graphs where v_1 and v_2 have a least four mutual neighbours, since this gives an immediate $K_{1,1,4}$ minor (in fact, it gives $K_{1,1,4}$ as a subgraph).

In Step (2) of the process, the `nauty` graph automorphism package was used [12]. Duplicate graphs were removed by passing the list generated in Step (1) into the `labelg` feature of `nauty`. This gives every graph the canonical label of its isomorphism class, which allows us easily sort the list and remove any repeated graphs. The final step filters

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\lvert\mathcal{Z}_n\rvert$	1	3	1	33	61	81	90	88	100	110	122	133	145	156	168

Table 3.1: The counts for the generated graphs $\mathcal{Z}_n$

out graphs that have a $K_{1,1,4}$ minor. This filtering was done using a graph minor testing program written by J. Zachary Gaslowitz [10]. The number of graphs generated for each n, $4 \leq n \leq 18$, is shown in Table 3.1.

Given this set of graphs, we can prove some results that will serve as base cases in later induction arguments.

Using a basic depth first search algorithm to find Hamilton cycles in the graphs in $\mathcal{Z}_n$ for $n \leq 12$, we found the following.

Lemma 3.8. *The only 3-connected $K_{1,1,4}$-minor-free graph on at most twelve vertices that is not Hamiltonian is $K_{3,4}$.*

Using the `labelg` feature of `nauty` [12] to label graphs in our family with the canonical label of their isomorphism classes, we compared the graphs generated on n vertices with our family of graphs $\mathcal{G}_n$ for $n \leq 18$ and have the following result.

Lemma 3.9. *For $n = 4$ and $13 \leq n \leq 18$, $\mathcal{Z}_n$ is exactly equal to $\mathcal{G}_n$. For $5 \leq n \leq 12$, $\mathcal{G}_n$ is properly contained in $\mathcal{Z}_n$.*

For $5 \leq n \leq 12$, $\mathcal{Z}_n \setminus \mathcal{G}_n$ is nonempty. All graphs in $\mathcal{Z} \setminus \mathcal{G}$ are shown in Figures 3.7, 3.8, 3.9, and 3.10, grouped first by connectivity and then by planarity. The planarity of these graphs was found using the `planarg` tool of the `nauty` package [12]. The tool `planarg` takes a list of graphs as input and outputs those graphs that are planar.

There are exactly three 4-connected $K_{1,1,4}$-minor-free graphs on at most 18 vertices that are not in $\mathcal{G}$; namely, K_5, the octahedron ($K_{2,2,2}$) and $\overline{C_7}$. Observe that with one exception, every other graph on at most 18 vertices in $\mathcal{Z} \setminus \mathcal{G}$ has a vertex of degree three, automatically precluding 4-connectivity. The exception is a 4-regular graph on seven vertices, $\overline{C_4} \cup \overline{C_3}$;

however, we are able to easily find a cutset of size three in this graph. Since all graphs in $\mathcal{G}$ are 3-connected but not 4-connected (Remark 3.6), the three graphs shown in Figures 3.7 and 3.8 are the *only* 4-connected $K_{1,1,4}$-minor-free graphs on at most 18 vertices.

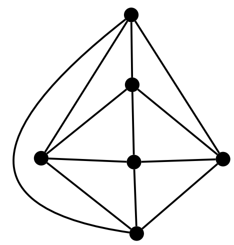

Figure 3.7: The octahedron, $K_{2,2,2}$ (the planar 4-connected $K_{1,1,4}$-minor-free graph)

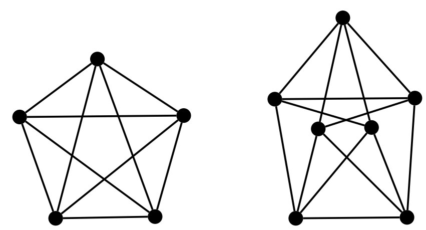

Figure 3.8: The graphs K_5 and $\overline{C_7}$ (the nonplanar 4-connected $K_{1,1,4}$-minor-free graphs)

Passing the graphs in $\mathcal{G}$ on at most 14 vertices through the `planarg` tool of `nauty`, we also found the following.

Theorem 3.10. *For $n \leq 14$, a graph in $\mathcal{G}_n$ is nonplanar if and only if it is built on one of the base graphs F_1, F_2, F_3 or $H_i, 1 \leq i \leq 10$.*

3.3 The Characterisation

To prove that for $n \geq 13$, $\mathcal{G}_n$ consists exactly of the 3-connected $K_{1,1,4}$-minor-free graphs on n vertices, we will argue by induction.

Lemma 3.11. *For all $n \geq 13$, every graph in $\mathcal{G}_n$ is 3-connected and $K_{1,1,4}$-minor-free.*

Proof. From Lemma 3.9, we know that the statement of the theorem holds for $n = 13$ and $n = 14$. Let us fix some $m > 14$ and assume the statement holds for all $n < m$. Consider a graph G in $\mathcal{G}_m$. By Remark 3.2, G has a k-fan for some $k \geq 5$, which is necessarily a main

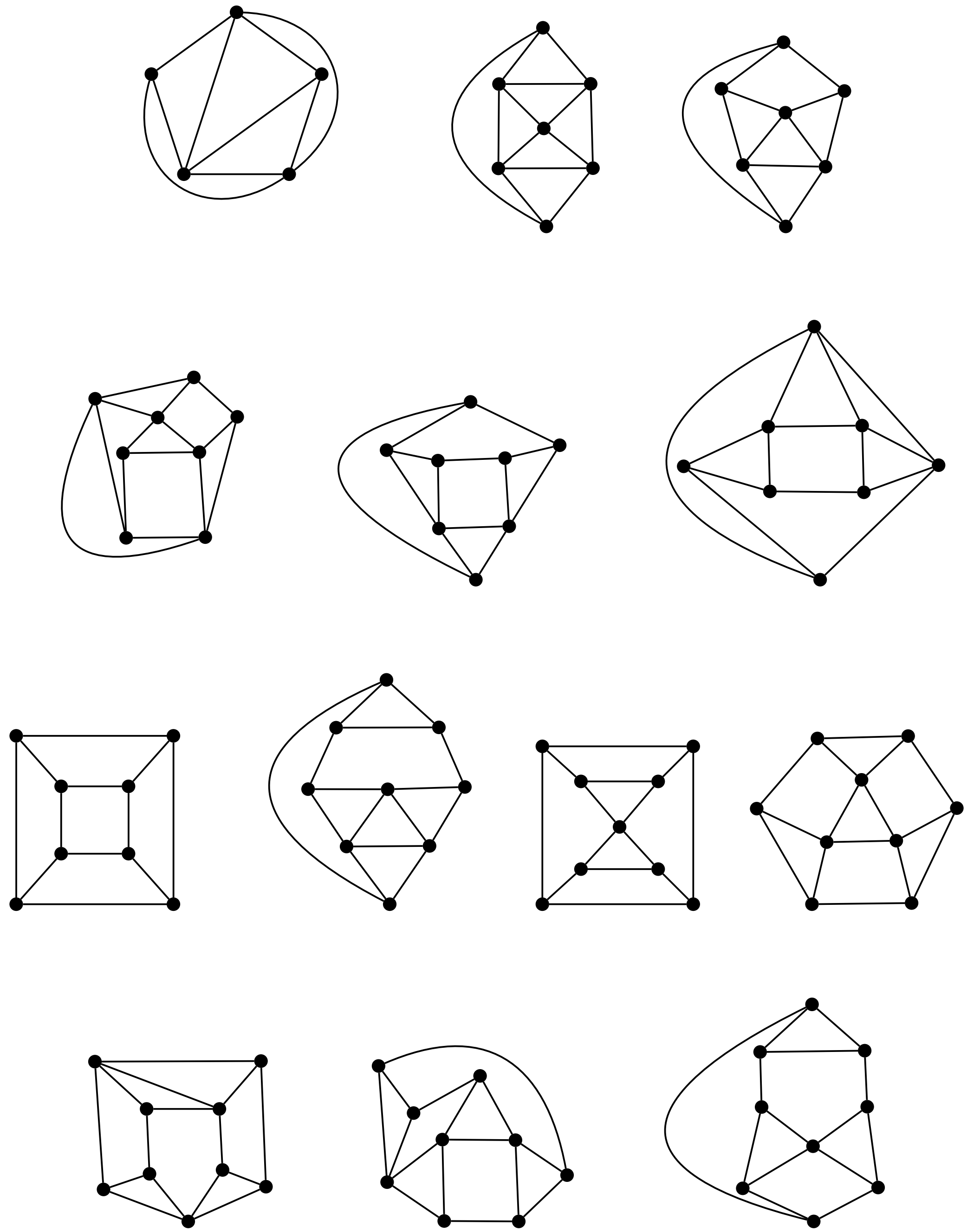

Figure 3.9: The planar 3-connected $K_{1,1,4}$-minor-free graphs not in $\mathcal{G}$

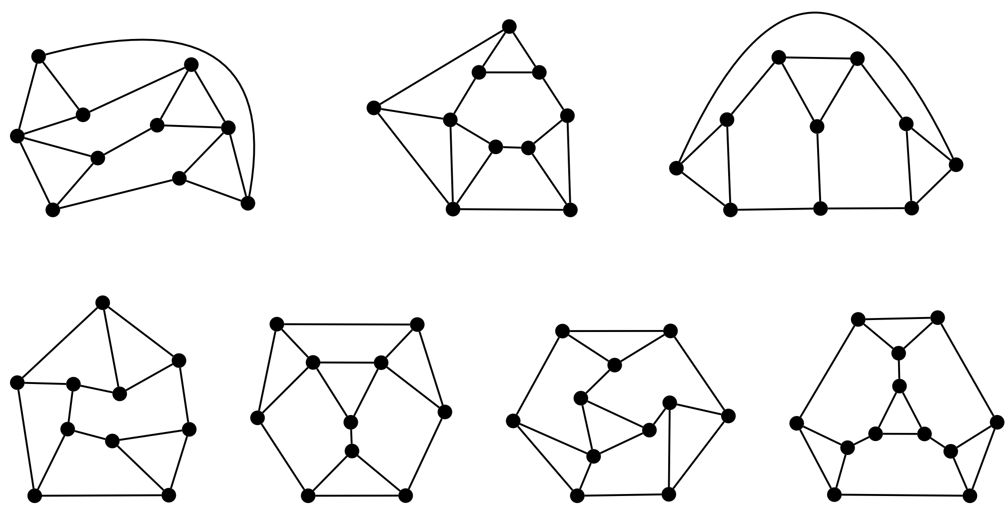

Figure 3.9: The planar 3-connected $K_{1,1,4}$-minor-free graphs not in $\mathcal{G}$

fan. Let us collapse one such fan in G to obtain a graph G' with a $(k-1)$-fan. By Remark 3.1, we know that G' is in $\mathcal{G}_{m-1}$, so by the inductive hypothesis G' is 3-connected and $K_{1,1,4}$-minor-free. Now we may expand the $(k-1)$-fan to recover G from G'. However, by Corollary 2.10 and Lemma 2.11, we know that this fan expansion preserves both 3-connectivity and the $K_{1,1,4}$-minor-free property. Therefore G is 3-connected and $K_{1,1,4}$-minor-free. By induction, we have that every graph in $\mathcal{G}_n$, for $n \geq 13$, is 3-connected and $K_{1,1,4}$-minor-free. □

Theorem 3.12. *For all $n \geq 13$, the set of 3-connected $K_{1,1,4}$-minor-free graphs on n vertices is exactly equal to $\mathcal{G}_n$.*

Proof. By Lemma 3.11, for $n \geq 13$, all graphs in $\mathcal{G}_n$ are 3-connected and $K_{1,1,4}$-minor-free. Therefore it remains only to show that every 3-connected $K_{1,1,4}$-minor-free graph on at least 13 vertices is a member of $\mathcal{G}$.

We know from the previous lemma that the statement of the theorem is true for $13 \leq n \leq 17$. To prove the theorem inductively, fix some $m > 17$ and assume the statement of the theorem holds for all n where $13 \leq n < m$. Now let G be a 3-connected $K_{1,1,4}$-minor-free

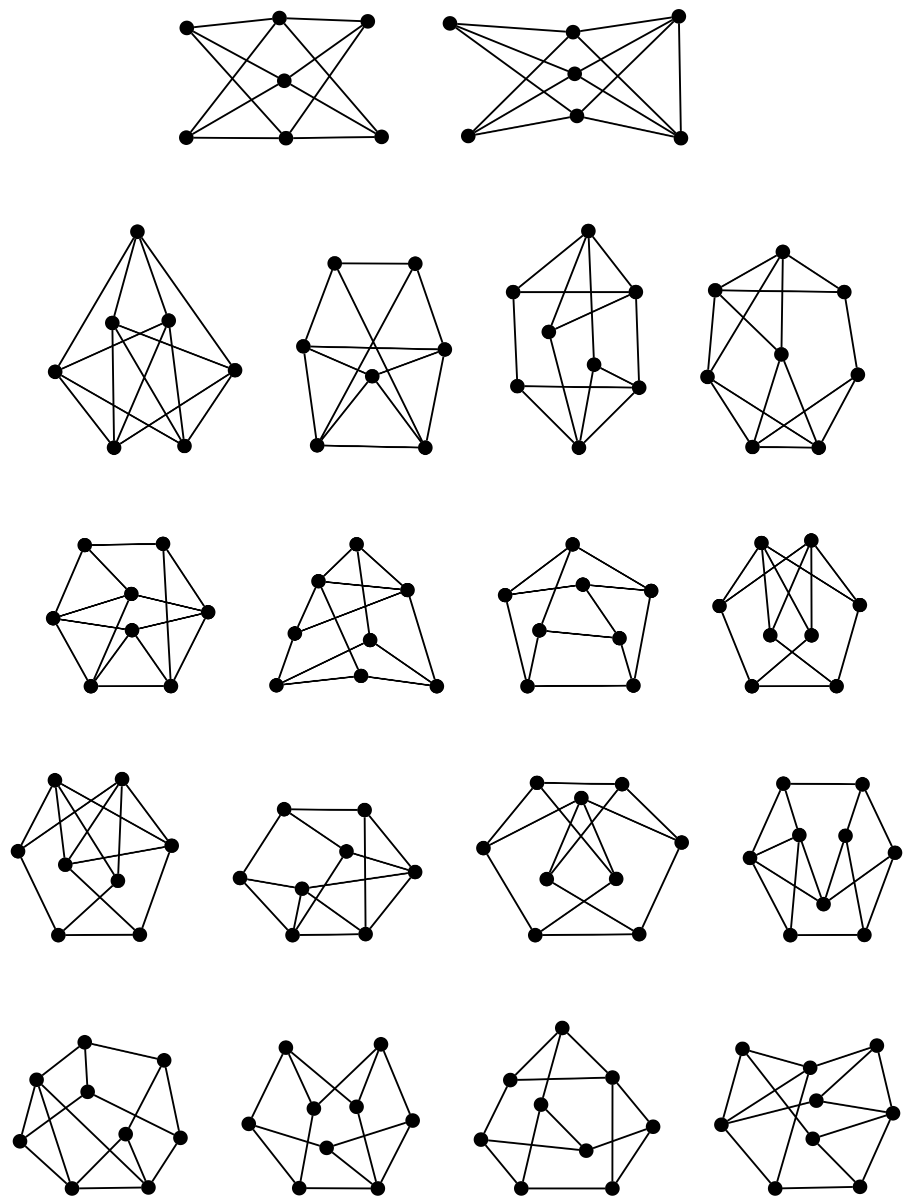

Figure 3.10: The nonplanar 3-connected $K_{1,1,4}$-minor-free graphs not in $\mathcal{G}$

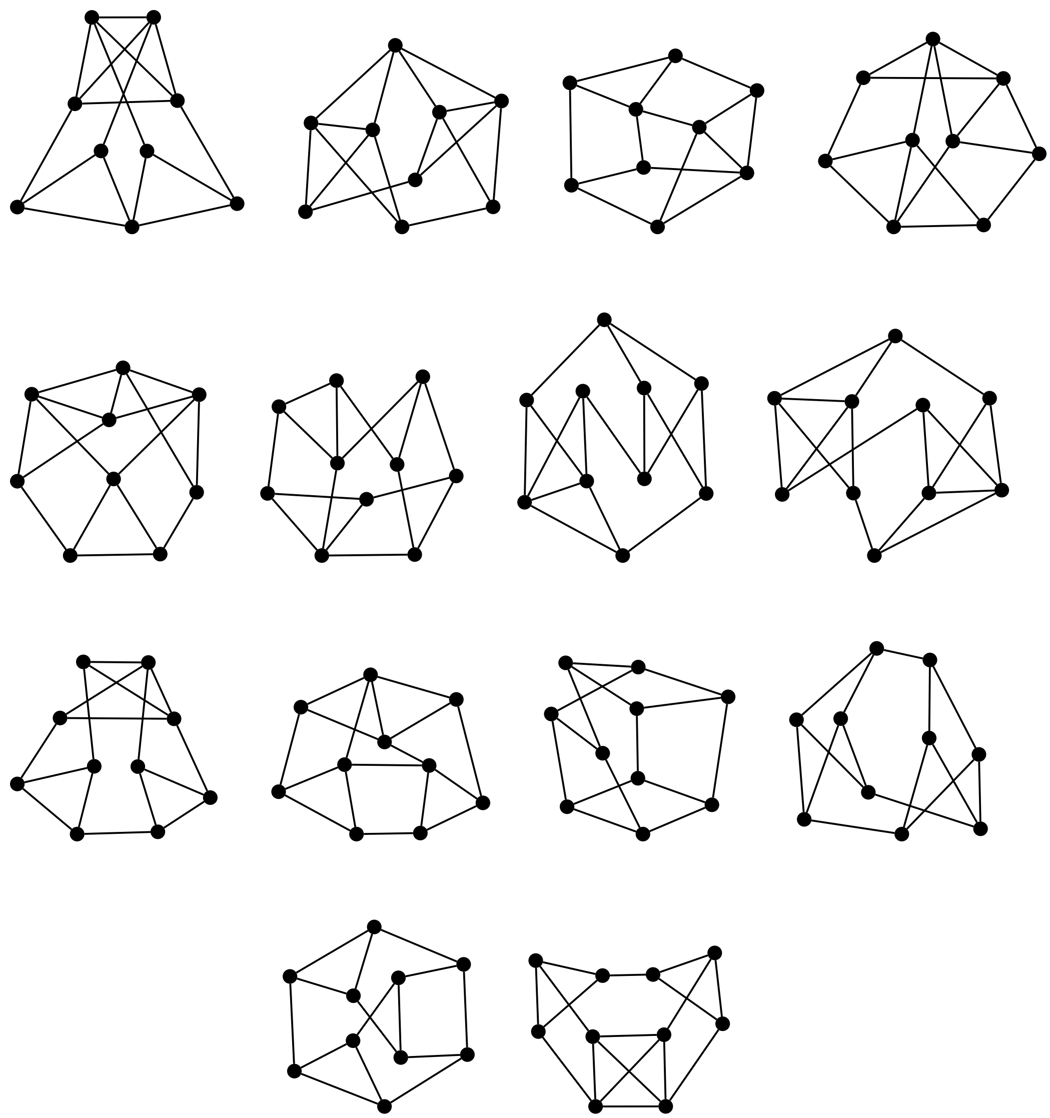

Figure 3.10: The nonplanar 3-connected $K_{1,1,4}$-minor-free graphs not in $\mathcal{G}$

graph on m vertices.

First we show that G has a nontrivial fan. Since G is 3-connected, it has a 3-contractible edge e by Lemma 3.7. Let us contract this edge to a vertex v and call the resultant graph G'. Since contracting edges does not create any new minors, G' is a 3-connected $K_{1,1,4}$-minor-free graph on $m-1$ vertices. By our inductive hypothesis, $G' \in \mathcal{G}_{m-1}$. Then Remark 3.2 implies that G' has some p-fan for $p \geq 6$, which is necessarily a main fan. Let this p-fan be riveted at a vertex r. If v is not part of this fan, or adjacent to any outer vertices of the fan, then the fan also exists in G, and we are done.

If v is not part of the fan, but is adjacent to an outer vertex of the fan, then it can only be adjacent to one or both of the end vertices of the outer path, as all other outer fan vertices have all of their neighbours in the fan. Then uncontracting v to recover G could increase the degree of these end vertices of the outer path in the fan. However, this still leaves a fan of size at least $p-2$ in G, and since $p-2 \geq 4$, this is a nontrivial fan.

So suppose that v is in the fan. First consider the case that v is an outer vertex of the fan. Since the fan is at least a 6-fan, there exist two adjacent outer vertices of the fan, x and y, such that neither x nor y is v or is adjacent to v. This implies that uncontracting v to recover G leaves the 2-fan formed by x, y and r intact, and so G has a fan.

Finally, consider the case that $v = r$, the rivet vertex of the fan in G'. If G' has another fan of size at least four that is disjoint from our p-fan, then uncontracting v leaves at least the internal portion of this fan intact, giving a nontrivial fan in G. So assume that our p-fan is the only fan in G' of size at least four. By Remark 3.2, this implies that $p \geq 8$. Now, let r_1 and r_2 be the vertices in G that were contracted to r. Let $w_1, w_2, ..., w_7, w_8$ be eight consecutive outer vertices (occurring in that order) of the fan in G'. Now consider these vertices in G. Suppose that two consecutive w_i vertices are both adjacent to r_1 in G but not r_2, or vice versa. Then this implies we have at least a 2-fan in G. To see this, suppose that two consecutive w_i vertices are adjacent to (without loss of generality) r_1 but not r_2. If $\deg_G(r_1) \geq 4$, then the two consecutive w_i vertices and r_1 make up a 2-fan in G. If $\deg_G(r_1) = 3$, then r_1's only neighbours in G are r_2 and the two consecutive w_i, so all other w_i vertices are adjacent to r_2 and not r_1. In particular, this means that there are two consecutive such vertices, which together with r_2 form a 2-fan in G. Therefore let us assume that no two consecutive w_i vertices are both adjacent to r_1 but not r_2, or vice versa. This means that for every pair of consecutive w_i vertices, at least one of them is adjacent to r_1 and at least one is adjacent to r_2. However, this allows us to construct a $K_{1,1,4}$ minor. If the graph $K_{1,1,4}$ has vertex set $\{a, b, c_1, c_2, c_3, c_4\}$ and edges $\{ab, ac_i, bc_i, 1 \leq i \leq$

$4\}$, then an edge-based model of a $K_{1,1,4}$ minor in G can be described by the branch sets ($A = \{r_1\}, B = \{r_2\}, C_i = \{w_{2i-1}, w_{2i}\}, 1 \leq i \leq 4$), where each branch set corresponds to the vertex of $K_{1,1,4}$ with its same label in lower case. Since this is a contradiction, we must conclude that G has a nontrivial fan.

Now, let q be the largest integer such that G has a q-fan, and consider one such q-fan, which we shall call F. We know from above that $q \geq 2$, which means that F has a collapsible edge. Let us collapse such an edge to obtain a graph which we shall call G''. Since collapsible edges are 3-contractible, G'' is in $\mathcal{G}_{m-1}$ by our inductive hypothesis. The fan that we collapsed is now a $(q-1)$-fan in G''; call it F'. We note that G'''s largest fan is either a $(q-1)$-fan or a q-fan. If G'' had any larger fans, these would have existed in G, contradicting our choice of F. Since $m-1 \geq 17$, and $G'' \in \mathcal{G}_{m-1}$, G'' has at least a 6-fan by Remark 3.2. This implies that $q-1 \geq 5$.

Since $q-1 \geq 5$, Remark 3.3 implies that F' is a main fan of G''. Therefore expanding F' yields a graph in $\mathcal{G}$ (Remark 3.1). Therefore, we have that G is in $\mathcal{G}$. □

This completes the characterisation of 3-connected $K_{1,1,4}$-minor-free graphs. Given this characterisation and our results on fans, we are able to prove some properties for these graphs.

Corollary 3.13. *There are exactly three 4-connected $K_{1,1,4}$-minor-free graphs; namely K_5, the octahedron ($K_{2,2,2}$) and $\overline{C}_7$ (shown in Figures 3.7 and 3.8).*

Proof. We know that these graphs are the only 4-connected $K_{1,1,4}$-minor-free graphs on at most 18 vertices, and that no graph in $\mathcal{G}$ is 4-connected (Remark 3.6). Our characterisation implies that there are no 4-connected $K_{1,1,4}$-minor-free graphs on more than 12 vertices. □

Theorem 3.14. *With the exception of $K_{3,4}$, every 3-connected $K_{1,1,4}$-minor-free graph is Hamiltonian.*

Proof. Recall that Lemma 3.8 tells us that the only non-Hamiltonian 3-connected $K_{1,1,4}$-minor-free graph on at most 12 vertices is $K_{3,4}$. In particular, every 3-connected $K_{1,1,4}$-minor-free graph on exactly 12 vertices is Hamiltonian. For some $m \geq 13$, assume that all 3-connected $K_{1,1,4}$-minor-free graphs on fewer than m vertices are Hamiltonian, and let G be a 3-connected $K_{1,1,4}$-minor-free graph on m vertices. By Theorem 3.12, G is in $\mathcal{G}$. By Remark 3.2, G has a nontrivial fan. Let us collapse one such fan to obtain a 3-connected $K_{1,1,4}$-minor-free graph G' on $m-1$ vertices. By our inductive hypothesis, G' is Hamiltonian. Now we may re-expand the collapsed fan to recover G. By Lemma 2.7, expanding the fan preserves Hamiltonicity. Therefore G is Hamiltonian. By induction, all 3-connected $K_{1,1,4}$-minor-free graphs on at least 13 vertices are Hamiltonian. □

Theorem 3.15. *A graph in $\mathcal{G}$ is nonplanar if and only if it is built on one of the base graphs F_1, F_2, F_3 or $H_i, 1 \leq i \leq 10$.*

Proof. Recall that Lemma 3.10 gives us that the statement of the theorem holds for all members of $\mathcal{G}$ on at most 14 vertices.

Now fix some $m > 14$ and assume that the statement of the theorem holds for all members of $\mathcal{G}$ on fewer than m vertices. Let $G \in \mathcal{G}$ be a graph on m vertices. By Remark 3.2, one of G's main fans is a k-fan for some $k \geq 5$. Let G' be the graph obtained from G by collapsing an edge in this main fan. We claim that G is nonplanar if and only if G' is nonplanar. To prove this, suppose first that G is nonplanar. Recall that since we are working with 3-connected graphs, Lemma 1.5 implies G must have a $K_{3,3}$ minor. Then Lemma 2.11 implies that G' also has a $K_{3,3}$ minor and thus is nonplanar. To prove the converse, suppose that G' is nonplanar. Then it must have a $K_{3,3}$ minor. Splitting vertices does not destroy minors, therefore expanding the main fan in G' to recover G preserves the $K_{3,3}$ minor and so G is also nonplanar. Therefore we have proved that G is nonplanar if and only if G' is nonplanar.

By our inductive hypothesis, G' is nonplanar if and only if it is built on one of the base graphs F_1, F_2, F_3 or $H_i, 1 \leq i \leq 10$. By Remark 3.1, we know that expanding a main fan of

G' gives a graph corresponding to the same base graph as G'. Since we collapsed a main fan in G to get G', the collapsed fan is a main fan in G' (this is also a consequence of Remark 3.1), and re-expanding the fan to recover G preserves the base graph. Since G is nonplanar if and only if G' is nonplanar, and G' is nonplanar if and only if it is built on one of the base graphs F_1, F_2, F_3 or $H_i, 1 \leq i \leq 10$, this implies that G is nonplanar if and only if its base graph is one of F_1, F_2, F_3 or $H_i, 1 \leq i \leq 10$. Then by induction, the theorem is proved for all graphs in $\mathcal{G}$. □

Corollary 3.16. *The set of nonplanar 3-connected $K_{1,1,4}$-minor-free graphs is exactly those graphs built on one of the base graphs F_1, F_2, F_3 or $H_i, 1 \leq i \leq 10$, and the graphs shown in Figure 3.8 and 3.10.*

Lemma 3.17. *For $k \geq 5$ and $n \geq 2k+7$, the number of graphs in $\mathcal{G}_n$ with a maximal k-fan is equal to the number of graphs in $\mathcal{G}_{n+1}$ with a maximal k-fan.*

Proof. First note that since $k \geq 5$, any maximal k-fan in a graph in $\mathcal{G}_n$ is necessarily a main fan, by Remark 3.3. Also note that any graph G in $\mathcal{G}_n$ with a maximal k-fan must have another main fan of size at least $k+1$, by Remark 3.2. Since $k+1 \geq 6$, Remark 3.3 implies that this second fan is also a main fan. Now let us define a bijection between the graphs with a maximal k-fan in $\mathcal{G}_n$ and the graphs with a maximal k-fan in $\mathcal{G}_{n+1}$. For any graph G with a maximal k-fan in $\mathcal{G}_n$, we can expand its larger main fan and get a graph in $\mathcal{G}_{n+1}$ which retains the maximal k-fan as the other main fan. Similarly, for a graph G' with a maximal k-fan in $\mathcal{G}_{n+1}$, we can collapse its larger main fan (which has size at least $k+2$) and get a graph in $\mathcal{G}_n$ with the other main fan still a maximal k-fan. Since $n \geq 2k+7$, the fan we are performing operations on always has size strictly greater than five, therefore these operations are well-defined. The expansion and contraction operations are also inverses of each other by Remark 2.6, therefore we have the required bijection. □

Theorem 3.18. *For $n \geq 13$, there are exactly* $\left\lfloor \frac{23n}{2} \right\rfloor - 39$ *distinct graphs in $\mathcal{G}_n$.*

Proof. First we look at $\mathcal{G}_n$ for $n \geq 15$. By Remark 3.2 $G \in \mathcal{G}_n$ has at least one main fan (necessarily of size five or more). Let us define a map $\phi_n : \mathcal{G}_n \to \mathcal{G}_{n+2}$ for all $n \geq 15$. The image of G under this map will depend on whether G has one main fan or two. If G has two main fans, we define $\phi(G)$ to be the graph obtained by expanding each of these two fans in G. If G has a unique main fan we define $\phi(G)$ to be the graph obtained from G by expanding this fan twice. Each of these operations increases the number of vertices by two, and by Remark 3.1 $\mathcal{G}$ is closed under expansion and contraction of main fans, therefore in both cases $\phi(G)$ is in $\mathcal{G}_{n+2}$. Observe that performing the first operation results in a graph with two main fans each of size at least six, and performing the second operation results in a graph with one main fan of size at least seven, and no other fans with size greater than four. Therefore any graph in $\phi(\mathcal{G}_n)$ is mapped to by only one of the operations, and we uniquely recover G from $\phi(G)$ by contracting the fan(s). Therefore ϕ is injective.

Next we consider $\mathcal{G}_{n+2} \setminus \phi(\mathcal{G}_n)$. For any graph G' in $\mathcal{G}_{n+2}$ with either two main fans of size at least six, or one main fan with size at least seven and no other fans of size greater than four, we can perform contraction operations mentioned above to obtain the pre-image $\phi^{-1}(G')$ in $\mathcal{G}_n$. Therefore any such graph G' is in $\phi(\mathcal{G}_n)$. Furthermore, we know that ϕ only maps to graphs with either two main fans of size at least six, or one main fan of size at least seven and no other fans of size greater than four. Therefore the graphs in $\mathcal{G}_{n+2} \setminus \phi(\mathcal{G}_n)$ are exactly those graphs that satisfy neither of these conditions. Since $n+2 \geq 17$, any graph in $\mathcal{G}_{n+2}$ that has only one main fan must have its main fan of size at least eight; therefore there are no graphs in $\mathcal{G}_{n+2}$ that have only one main fan whose size is six. Therefore the graphs in $\mathcal{G}_{n+2} \setminus \phi(\mathcal{G}_n)$ are exactly those graphs with a maximal 5-fan.

The rest of the proof proceeds by induction. From Table 3.1, we know that the statement of the theorem holds for $13 \leq n \leq 18$. So fix some $m \geq 19$, assume the statement holds for all $n < m$, and consider $\mathcal{G}_m$. From above, we know that $\mathcal{G}_m$ consists of the disjoint union of $\phi(\mathcal{G}_{m-2})$ and the set of graphs in $\mathcal{G}_m$ with a maximal 5-fan. We also know that the number of graphs in $\mathcal{G}_m$ with a maximal 5-fan is the same as the number of graphs in $\mathcal{G}_{m-2}$ with a maximal 5-fan, by Lemma 3.17. Furthermore, the number of graphs with a maximal 5-fan in $\mathcal{G}_{m-2}$ is exactly $|\mathcal{G}_{m-2} \setminus \phi(\mathcal{G}_{m-4})|$. Therefore,

$$
\begin{aligned}
|\mathcal{G}_m| &= |\mathcal{G}_{m-2}| + |\mathcal{G}_{m-2} \setminus \phi(\mathcal{G}_{m-4})| \\
&= |\mathcal{G}_{m-2}| + |\mathcal{G}_{m-2}| - |\mathcal{G}_{m-4}| \\
&= \left\lfloor \frac{23(m-2)}{2} \right\rfloor - 39 + \left\lfloor \frac{23(m-2)}{2} \right\rfloor - 39 - \left(\left\lfloor \frac{23(m-4)}{2} \right\rfloor - 39 \right) \quad \text{(inductive hypothesis)} \\
&= \begin{cases} 23(m-2) - \dfrac{23(m-4)}{2} - 39, & \text{for } m \text{ even} \\ 23(m-2) - 1 - \dfrac{23(m-4)-1}{2} - 39, & \text{for } m \text{ odd} \end{cases} \\
&= \begin{cases} \dfrac{23m}{2} - 39, & \text{for } m \text{ even} \\ \dfrac{23m-1}{2} - 39, & \text{for } m \text{ odd} \end{cases} \\
&= \left\lfloor \frac{23m}{2} \right\rfloor - 39.
\end{aligned}
$$

Therefore the statement of the theorem holds for $\mathcal{G}_m$. By induction, the theorem is proved. □

Chapter 4

Hamiltonicity of 3-connected Planar $K_{1,1,5}$-minor-free Graphs

This chapter presents a new Hamiltonicity result for the class of 3-connected planar $K_{1,1,5}$-minor-free graphs. Recall Theorem 1.14, proved by Ellingham et al. in [9], which states that all 3-connected planar $K_{2,5}$-minor-free graphs are Hamiltonian. Our result is a strengthening of this theorem. Specifically, we prove that every 3-connected planar $K_{1,1,5}$-minor-free graph is Hamiltonian, with one exception; namely, the Herschel graph.

4.1 Computer Results

Similarly to Chapter 3, our main argument relies on knowing all of the 3-connected planar $K_{1,1,5}$-minor-free graphs on up to 16 vertices. Again, these small graphs were generated for us by J. Zachary Gaslowitz (personal communication), using almost the same process as was used to generate the 3-connected $K_{1,1,4}$-minor-free graphs. One difference is filtering for $K_{3,3}$ minors, since we are concerned specifically with *planar* graphs. Note that by Lemma 1.5, filtering for $K_{3,3}$ minors is enough to guarantee planarity, since we are only looking at 3-connected graphs. Let $\mathcal{H}_n$ denote the set of 3-connected planar $K_{1,1,5}$-minor-free graphs on n vertices. We know that K_4 is the only 3-connected graph on four

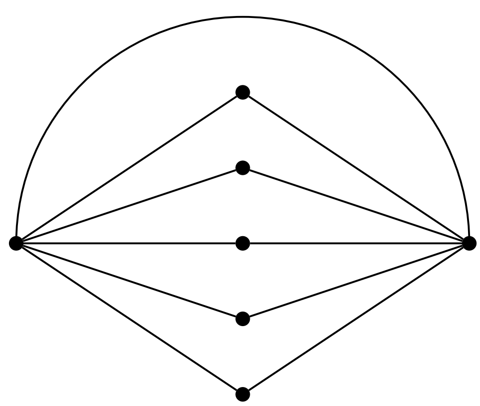

Figure 4.1: The graph $K_{1,1,5}$

vertices, and it is also planar and trivially $K_{1,1,5}$-minor-free; therefore, $\mathcal{H}_4 = \{K_4\}$. The process for generating $\mathcal{H}_{n+1}$ given $\mathcal{H}_n$, for $4 \leq n \leq 15$, is described below.

1. For each graph G in $\mathcal{H}_n$, and for each vertex v in G, take every possible graph obtained by splitting v into two adjacent vertices v_1 and v_2 where $\deg(v_1) \geq 3$ and $\deg(v_2) \geq 3$, and v_1 and v_2 have fewer than five mutual neighbours.

2. Remove duplicates from the resultant set of graphs.

3. Remove graphs with a $K_{1,1,5}$ minor or $K_{3,3}$ minor.

In the first step, we discount the graphs where v_1 and v_2 have a least five mutual neighbours, since this gives an immediate $K_{1,1,5}$ minor (in fact, it gives $K_{1,1,5}$ as a subgraph).

In Step (2) of the process, duplicate graphs were removed by passing the list generated in Step (1) into the `labelg` feature of `nauty` [12] and deleting graphs with repeated labels.

The final step filters out graphs that have a $K_{1,1,5}$ minor or a $K_{3,3}$ minor. This filtering was done using the graph minor testing program written by J. Zachary Gaslowitz [10].

Given the graphs generated for us by J. Zachary Gaslowitz, we used a basic depth first search algorithm to find Hamilton cycles in the graphs in $\mathcal{H}_n$ for $n \leq 16$, and found the following.

Lemma 4.1. *With the exception of the Herschel graph, every 3-connected planar $K_{1,1,5}$-minor-free graph on at most 16 vertices is Hamiltonian.*

Recall that the Herschel graph is a bipartite graph on eleven vertices. It is shown in Figure 4.2.

We were able to independently verify Lemma 4.1 using computer results obtained by Gordon Royle (personal communication). He generated all of the non-Hamiltonian 3-connected planar $K_{2,6}$-minor-free graphs on up to 16 vertices (the process used to do so is described below). Since $K_{1,1,5}$ is a minor of $K_{2,6}$, these graphs automatically include all non-Hamiltonian 3-connected planar $K_{1,1,5}$-minor-free graphs. He also then separated out the *minimally* 3-connected graphs from this list (meaning removal of any edge destroys 3-connectivity). We ran each of the minimally 3-connected graphs through J. Zachary Gaslowitz's minor tester to check for $K_{1,1,5}$ minors. All of the graphs had a $K_{1,1,5}$ minor, with the exception of the only minimally 3-connected graph on 11 vertices; namely, the Herschel graph. Since adding edges to a graph does not destroy any minors, this implies that all non-Hamiltonian 3-connected planar $K_{2,6}$-minor-free graphs on at least 12 and at most 16 vertices have a $K_{1,1,5}$ minor. Since the Herschel graph did not have a $K_{1,1,5}$ minor, we additionally checked all other non-Hamiltonian 3-connected (not minimally) planar $K_{2,6}$ minor-free graphs on 11 vertices. For all such graphs we found a $K_{1,1,5}$-minor; therefore, the Herschel graph is the only non-Hamiltonian 3-connected planar $K_{2,6}$-minor-free graph that is also $K_{1,1,5}$-minor-free. Thus, Lemma 4.1 has been verified twice.

We give a description of the process used by Gordon Royle to generate the non-Hamiltonian 3-connected planar $K_{2,6}$-minor-free graphs on up to 16 vertices. A program `plantri` [2], written by Gunnar Brinkman and Brendan McKay, is used. The purpose of the program `plantri` is to generate 3-connected planar triangulations, but it has a mode for generating 3-connected planar graphs.

1. With `plantri`, all 3-connected, planar graphs on up to 16 vertices are generated.

2. Using a basic Hamiltonicity checker, all Hamiltonian graphs from this list are filtered out.

3. Each graph in the list is tested for a $K_{2,6}$ minor by contracting sequences of edges to obtain an eight vertex graph, in every possible inequivalent way. These eight vertex graphs are then compared to an existing list of the eight vertex graphs with $K_{2,6}$ minors. The graphs that have a $K_{2,6}$ minor are removed from the list. This gives all non-Hamiltonian, 3-connected, planar, $K_{2,6}$-minor-free graphs on up to 16 vertices.

4. To obtain the *minimally* 3-connected graphs, edges are in turn deleted from each graph and the resultant graph is checked for 3-connectivity. If a graph is no longer

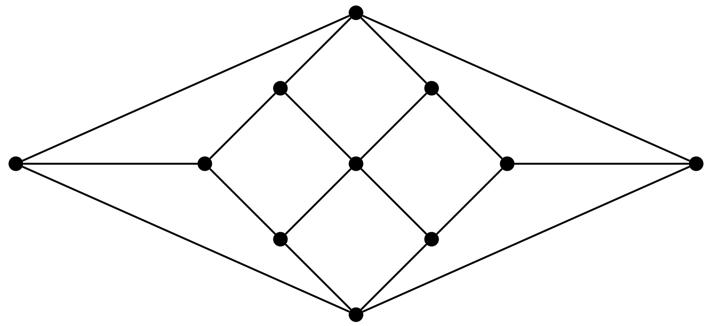

Figure 4.2: The Herschel graph

3-connected after the deletion of any edge, then it is minimally 3-connected.

4.2 Main Result

The statement of our main theorem is given below.

Theorem 4.2. *With the exception of the Herschel graph, every 3-connected planar $K_{1,1,5}$-minor-free graph is Hamiltonian.*

To prove this theorem, we need only consider graphs that are 3-connected planar and $K_{1,1,5}$-minor-free but not $K_{2,5}$-minor-free, since we already know that the 3-connected planar $K_{2,5}$-minor-free graphs are Hamiltonian, by Theorem 1.14. To this end, we consider a certain type of $K_{2,5}$ structure that must exist in a 3-connected planar $K_{1,1,5}$-minor-free graph that is not also $K_{2,5}$-minor-free, and prove a sequence of lemmas that gradually restrict what the graph looks like in relation to this structure. Eventually, we show that if our graph has enough vertices, it has a nontrivial fan, or a similar structure known as a contractible triangle, allowing us to prove our main theorem by induction.

Throughout this chapter, we will be finding contradictions in the form of $K_{1,1,5}$ minors, so we shall set up some notation to describe these minors. Let the graph $K_{1,1,5}$ have vertex set $\{a_1, a_2, b_1, b_2, b_3, b_4, b_5\}$ and edge set $\{a_1a_2\} \cup \{a_ib_j, 1 \leq i \leq 2, 1 \leq j \leq 4\}$. First we claim that for a path-based model of a $K_{1,1,5}$ minor in a graph G, we may assume that the branch sets corresponding to the vertices b_i are themselves single vertices in G. To see this, let S_i be the branch set corresponding to vertex b_i, and let R_1 and R_2 be the branch sets corresponding to a_1 and a_2 respectively. Then consider the path from R_1 to S_i, and the path from S_i to R_2. Suppose that the former path ends at vertex u in S_i and u and the latter ends at v in S_i (we assume that these paths are minimal in that they each have only one vertex in S_i). Then since S_i is connected, there is a uv path lying entirely in S_i. The concatenation of our $R_1 - S_i$ path, the uv path, and our $S_i - R_2$ path is a path from R_1 to R_2, with length at least two. We can then replace the branch set S_i with the set consisting of any vertex on the interior of this path. This remains a valid path-based model of a $K_{1,1,5}$ minor.

Therefore, assuming that the branch set corresponding to each vertex b_i in $K_{1,1,5}$ is just a single vertex, we will usually describe a path-based model of a $K_{1,1,5}$ minor in a graph G with the sets R_1, R_2, and S, where R_1 is the branch set corresponding to a_1, R_2 is the branch set corresponding to R_2, and $S = \{s_i,: 1 \leq i \leq 5\}$ is the set of five vertices respectively corresponding to the branch sets of the vertices b_i, for $1 \leq i \leq 5$. Occasionally it will be convenient to let a branch set S_i consist of more than one vertex. In this case we may let $S := \cup_{i=1}^{5} S_i$. The paths joining the branch sets will usually be apparent, but will be specified when necessary.

We define a $K_{2,5}$ *outline* to be a structure in a graph G that consists of the following. There are two disjoint connected induced subgraphs, called the *base graphs* of the outline, which are denoted by B_1 and B_2 respectively. Additionally, there are five internally disjoint paths of length at least two, each with one end vertex in B_1 and the other end vertex in B_2. None of the internal vertices of these paths are allowed to lie in B_1 or B_2. The five paths are called the *arcs* of the structure, and are denoted by A_i, $i = 1, 2, ..., 5$. We impose a direction on these arcs, and say that they start in B_1 and end in B_2. If a and b are vertices on an arc A_i, such that a precedes b on the arc, we will denote the segment (subpath) of A_i between a and b as $A_i[a,b]$. We denote the segment of A_i between the start vertex of A_i and an internal vertex a of A_i by $A_i[-,a]$, and the segment of A_i between a and the end vertex of A_i by $A_i[a,-]$.

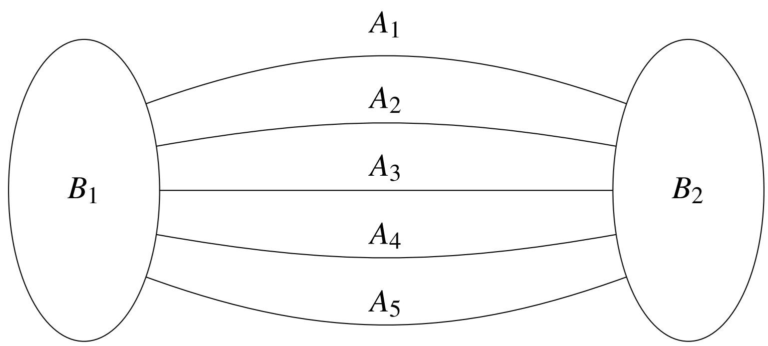

Figure 4.3: The $K_{2,5}$ outline H

Consider a graph G that is 3-connected planar and $K_{1,1,5}$-minor-free but not $K_{2,5}$-minor-free. Then G has a $K_{2,5}$ minor, which we describe by sets R_1, R_2, S as explained above. The presence of this $K_{2,5}$ minor guarantees that G contains a $K_{2,5}$ outline; for example we may take B_1 and B_2 to be the graphs induced by the vertices in R_1 and R_2 respectively, and each arc A_i to be the path consisting of the path from R_1 to s_i followed by the path from s_i to R_2, for $i = 1, 2, ..., 5$.

Now, over all $K_{2,5}$ outlines in G, fix one that first minimises the total number of vertices in the two base graphs, and then maximises the number of vertices internal to the five arcs. We will denote this structure by H. Vertices that are part of a base graph we will call *base vertices*, and vertices that are internal to an arc we will call *arc vertices* (the end vertices of the arcs are base vertices). A depiction of H is shown in Figure 4.3.

It will prove useful in later arguments to have some specified arc vertices. For each arc A_i, we will denote the second vertex on A_i (equivalently, the first *arc* vertex on A_i) by t_i.

We now prove a series of lemmas that gradually restrict the structure of H, leading eventually to the main result that if G has enough vertices, it must contain a nontrivial fan.

4.2.1 Basic Structure of the Outline

Lemma 4.3. *Each base graph has a spanning path, and each end vertex of a spanning path is an end vertex of at least two arcs.*

Proof. Without loss of generality, we will prove the lemma for B_1. Take a spanning tree of B_1. Consider the leaves of the spanning tree. If a leaf is not the end vertex of any arc, we may simply remove it from B_1, resulting in a structure with a smaller number of base vertices, contradicting our choice of H. If a leaf v is the end vertex of only one arc, we may extend the arc by a vertex, moving the leaf vertex from B_1 to the interior of the arc, so that the arc now ends at a base neighbour of v. This decreases the number of base vertices and increases the number of arc vertices, also contradicting our choice of H. Therefore each leaf vertex of a spanning tree of the base graph is the end vertex of at least two arcs. In particular, this implies that the spanning tree has at most two leaves, and is therefore a spanning path. □

Lemma 4.4. *Each base graph is an induced path.*

Proof. Again, without loss of generality, we will prove the claim for B_1. Fix a spanning path P of B_1, and suppose there is an edge between two vertices u and v in B_1 that are not adjacent on P. If w is the end vertex of P closest to u, and z is the end vertex of P closest to v (note that we may have $w = u$ or $z = v$), then define $B'_1 := P[w,u] \cup uv \cup P[v,z]$. Then $|B'_1| < |B_1|$, since any vertices in $P(u,v)$ are not included in B'_1 (and $P(u,v)$ is nonempty since u and v were not adjacent on P). Also, B'_1 contains both end vertices of P, meaning it contains the B_1 end vertices of at least four out of five arcs. If B'_1 contains the B_1 end vertices of all five arcs, then we may replace B_1 in H by B'_1, giving an outline with fewer base vertices and therefore contradicting our choice of H. If B'_1 contains only four out of the five arc end vertices, then one arc, A_i, has an end vertex in $P(u,v)$. We may extend this arc along P to end instead at v, which keeps it still internally disjoint from the other four arcs. Call this new extended arc A'_i. Then by replacing B_1 with B'_1 and A_i with A'_i, we have a new outline with fewer base vertices and more arc vertices, again contradicting our choice of H. Therefore B_1 is an induced path. □

Now that we know B_1 and B_2 are induced paths, we will name their end vertices. Let w

and z be the end vertices of B_1, and let x and y be the end vertices of B_2. We will assume a direction for each of B_1 and B_2, letting B_1 start at w and end at z, and B_2 start at x and end at y. We may also refer to B_1 and B_2 as base *paths*, instead of the more general base graph.

In the proof of the following lemmas, we will be finding contradictions in the form of $K_{1,1,5}$ minors. We will describe these minors explicitly in terms of their branch sets, and also present visual representations of many of them. In doing so, we will usually make the assumption that the arcs labelled A_1 and A_2 are incident with z in B_1, and the arcs labelled A_4 and A_5 are incident with w in B_1 (as long as this does not destroy generality). We make the disclaimer that these visual representations of the minors in G may not always be completely general, as the exact structure of the graph and the outline H has not yet been determined. For example, we will sometimes depict the $K_{1,1,5}$ minor in a graph G' that is itself a minor of G, having contracted some paths to single vertices to make it easier to show the minor. These representations are intended for clarification only, and explicit information about the minors should be taken from the branch set descriptions in the text.

Lemma 4.5. *Each arc is an induced path.*

Proof. Fix an arc A. Suppose there is an edge between two vertices a,b on A that is not part of the arc itself. Then there is some vertex c between a and b on A, and we have a $K_{1,1,5}$ minor with $R_1 = V(B_1 \cup A[-,a])$, $R_1 = V(B_2 \cup A[b,-])$, $S = \{c, t_2, t_3, t_4, t_5\}$. This minor is shown in Figure 4.4. □

Remark 4.6. There can be no edges between B_1 and B_2, since that would give an immediate $K_{1,1,5}$ minor, similar to the one above.

Lemma 4.7. *The only arc vertices that are adjacent to any base vertices are those that are 'penultimate' on the arc; that is, they are either the second vertex on the arc or the second to last vertex on the arc. In particular, only the second vertex on an arc (a t_i vertex) can be adjacent to any B_1 vertices, and only the second to last vertex on an arc can be adjacent to any B_2 vertices. Furthermore, for each base graph there is at most one penultimate arc*

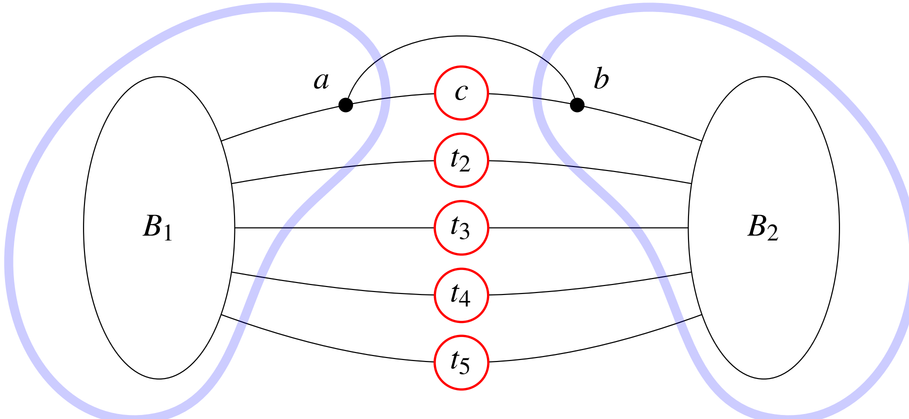

Figure 4.4

vertex that is adjacent to any base vertex other than its arc's endpoint, and each end vertex of the base graph is incident with at least two arcs that have no other edges to the base graph.

Proof. Suppose that there is an arc vertex a that is not a penultimate arc vertex, and that is adjacent to a base vertex in B_1. Then this gives us the $K_{1,1,5}$ minor with $R_1 = V(B_1), R_2 = V(B_2 \cup A[a,-])$ and $S = \{t_1,t_2,t_3,t_4,t_5\}$, shown in Figure 4.5. Therefore only penultimate arc vertices can be adjacent to base vertices.

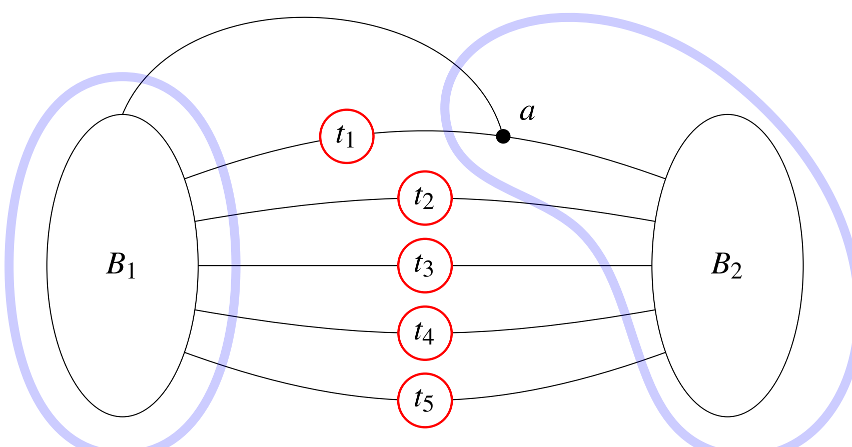

Figure 4.5

Next we consider which penultimate arc vertices can be adjacent to *multiple* base vertices. Without loss of generality, let us consider the base vertices in B_1. We know that w is the end vertex of at least two arcs. If it is the end of exactly two arcs, we claim that neither of the B_1-adjacent penultimate vertices of these two arcs can be adjacent to any vertices in B_1 other than w. To see this claim, suppose by way of contradiction that a penultimate vertex of one of these two arcs, call it a, is adjacent to some vertex v in $B_1 \setminus \{w\}$. Then we can modify the arc, replacing the edge aw with the edge av. This keeps the arc internally disjoint from the other four arcs, and does not change the number of base vertices or the number of arc vertices. But now w is the end of vertex of only one arc, and so we can shift w from the base graph to the arc, thereby decreasing the number of base vertices and increasing the number of arc vertices. This gives a contradiction. Therefore if w is the end vertex of exactly two arcs, neither of the penultimate vertices of those arcs are adjacent to any other vertices in B_1.

Now consider the case that w is the end vertex of exactly three arcs. Suppose that one of them has a w-adjacent penultimate vertex a that is also adjacent to some vertex v in $B_1 \setminus \{w\}$. Then we may modify the arc, replacing the edge aw with the edge av, which does not change the number of base vertices or arc vertices. In this new modified outline, w is the end point of only two arcs, and so the previous argument implies that neither of these arcs has a penultimate vertex adjacent to any other vertices in B_1.

If w is the end vertex of more than three arcs, then $B_1 = \{w\}$, as each end vertex of B_1 must be the end vertex of at least two arcs, and there are only five arcs in total.

Therefore the only case in which a penultimate arc vertex whose arc ends at an end vertex w of B_1 can be adjacent to a B_1 vertex other than w is if w is the end vertex of three arcs, and in this case only one of w's penultimate neighbours may be adjacent to such a vertex. Since this situation requires that w be the end vertex of three arcs, the other end vertex of B_1 must be the end vertex of exactly two arcs, and any internal vertices of B_1 are not end vertices of any arcs, therefore no other penultimate arc vertices are adjacent to multiple B_1 vertices.

If there is an arc that ends at an internal vertex u of B_1, then the penultimate vertex of the arc ending at u may be adjacent to vertices in $B_1 \setminus \{u\}$. However, in this situation, none of the penultimate vertices of the other four arcs may be adjacent to multiple vertices in B_1, since of the remaining four arcs, two must end at w and two at z.

In particular, this proves that each of w and z is the exclusive B_1 neighbour of at least two arcs. □

4.2.2 $G \setminus H$

We now consider the structure of the graph G outside of H. We will use $G \setminus H$ to mean the graph obtained from G by deleting all vertices in $V(H)$.

Lemma 4.8. *Let C be a component of $G \setminus H$, and let $c_1, c_2, ...c_n$ denote the end vertices in H of the edges between C and H. Then $c_1, c_2, ..., c_n$ all lie in one base graph - either B_1 or B_2.*

Proof. Let C be as described. We observe that by 3-connectivity, C has at least three edges to H, so $n \geq 3$. Let v be a vertex in C and consider three internally disjoint paths from v to three distinct vertices in H, without loss of generality c_1, c_2 and c_3.

First suppose that one of c_1, c_2, c_3 vertex is a base vertex and another is an arc vertex (without loss of generality say c_1 lies in R_1 and c_2 lies on arc A_1). Then we get a $K_{1,1,5}$ minor (shown in Figure 4.6), with $R_1 = V(B_1), R_2 = V(B_2 \cup A_1[c_2, -])$, and $S = \{v, t_2, t_3, t_4, t_5\}$. We get a similar minor if c_1 lies in B_1 and c_2 lies in B_2, or if both c_1 and c_2 lie on the same arc.

If all three of c_1, c_2, c_3 lie on different arcs, we get a $K_{3,3}$ minor, contradicting planarity. If $K_{3,3}$ is described by $V(K_{3,3}) = \{p_1, p_2, p_3, q_1, q_2, q_3\}$ and $E(K_{3,3}) = \{p_i q_j, 1 \leq i, j \leq 3\}$, then the $K_{3,3}$ minor in G is described by the branch sets (for a path-based model) $P_1 = V(B_1), P_2 = V(B_2), P_3 = \{v\}, Q_1 = \{c_1\}, Q_2 = \{c_2\}, Q_3 = \{c_3\}$, where each branch set corresponds to the vertex in $K_{3,3}$ with its lower case label. This $K_{3,3}$ minor in G is shown in

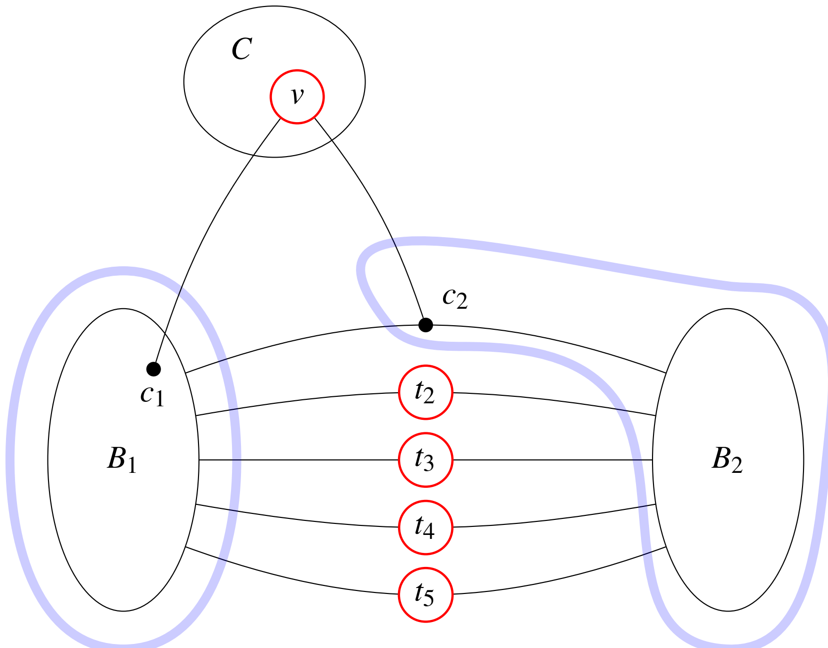

Figure 4.6

Figure 4.7.

Therefore all vertices $c_1, c_2, \ldots c_n$ lie in one base graph. □

Without loss of generality, assume that the vertices $c_1, c_2, \ldots, c_n$ occur in that order along B_1.

Lemma 4.9. *If C is a component of $G \setminus H$ with points of attachment $c_1, c_2, \ldots, c_n$ along B_1 in that order, then there is an edge from an arc vertex to a vertex in $B_1(c_1, c_n)$.*

Proof. Let C be as described, and suppose by way of contradiction that no arc vertex has a neighbour in $B_1(c_1, c_n)$. By 3-connectivity, there must be a path from $B_1(c_1, c_n)$ to $H \setminus B_1[c_1, c_n]$ that does not use c_1 or c_n (since $n \geq 3$, $B_1(c_1, c_n)$ is nonempty). We may assume that no internal vertices of the path lie in $B_1(c_1, c_n)$ or H, since we can always take a subpath with this property (in general, we shall assume that for subgraphs or sets of vertices D and E, a 'path from D to E' means a path with no internal vertices in either D or E). This path cannot be an edge, since there are no edges from $B_1(c_1, c_n)$ to any arc vertices by

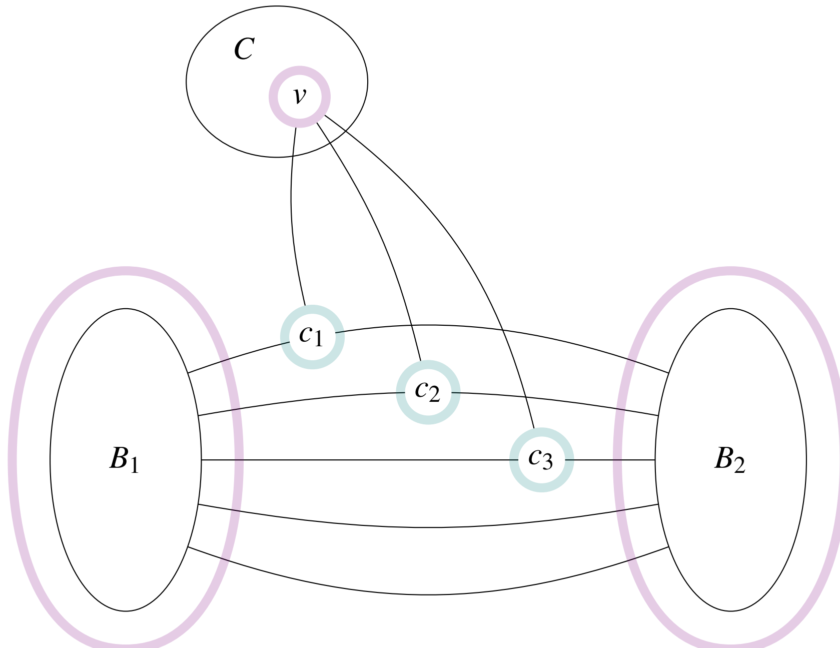

Figure 4.7

assumption, there are no edges between B_1 vertices that do not lie on the B_1 path, and there are no edges between B_1 and B_2. Therefore the path passes through some vertex u in $G \setminus H$. Also note that u is not in C, since all of C's points of attachment to H lie in $B_1[c_1, c_n]$. Since u is in some component of $G \setminus H$, and it has one point of attachment in B_1, all of its points of attachment are in B_1, by Lemma 4.8. This implies that the path out of $B_1(c_1, c_n)$ to the rest of H ends in $B_1 \setminus B_1[c_1, c_n]$ (and hence $B_1 \setminus B_1[c_1, c_n]$ is nonempty). Let us assume without loss of generality that the path ends in $B_1[w, c_1)$. Now since no arc vertices have neighbours in $B_1(c_1, c_n)$, one component of $B_1 \setminus B_1(c_1, c_n)$, i.e., $B_1[w, c_1]$ or $B_1[c_n, z]$, contains the end vertices of three arcs, and we get one of two $K_{1,1,5}$ minors; this is a contradiction. For the construction of our minors, let v be a vertex in C with two internally disjoint paths ending at c_1 and c_n respectively. For the case that $B_1[w, c_1]$ contains the end vertices of three arcs, say A_3, A_4, A_5, the $K_{1,1,5}$ minor is given by $R_1 = V(B_1[w, c_1])$, $R_2 = V(B_2 \cup A_1 \cup B_1(c_1, z])$, and $S = \{v, u, t_3, t_4, t_5\}$. This minor is shown in Figure 4.8. In the case that $B_1[c_n, z]$ contains the end vertices of three arcs, say A_1, A_2, A_3, the $K_{1,1,5}$ minor is given by $R_1 = V(B_1(c_1, z])$, $R_2 = V(B_2 \cup A_5 \cup B_1[w, c_1])$, and $S = \{v, u, t_1, t_2, t_3\}$. This minor is shown in Figure 4.9. □

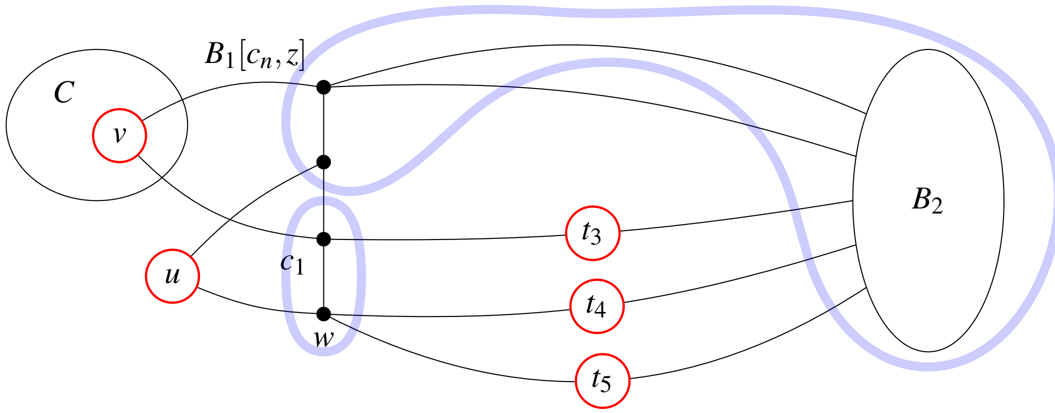

Figure 4.8

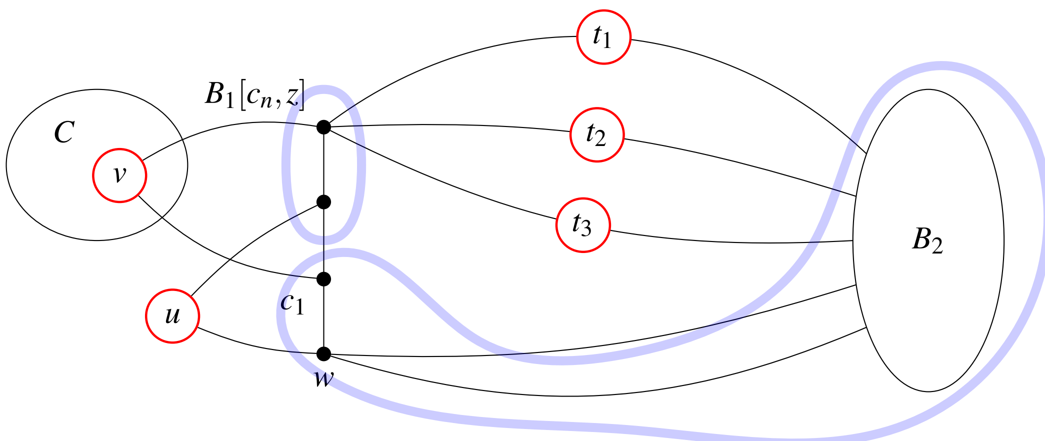

Figure 4.9

Lemma 4.10. *Any component C of $G \setminus H$ has exactly three points of attachment in H, and they occur consecutively along a base path.*

Proof. Let C be a component of $G \setminus H$ with points of attachment $c_1, c_2, \ldots, c_n$ in that order along B_1. By the above result, we know that there is an edge from an arc vertex a (we may assume A_1, A_2 end at w, a is on A_3, and A_4, A_5 end at z) to some vertex u in $B_1(c_1, c_n)$. Now if either $B_1(c_1, u)$ or $B_1(u, c_n)$ is nonempty, we get a $K_{1,1,5}$ minor. Since the two cases are symmetric, we may assume without loss of generality that $B_1(c_1, u)$ is nonempty, and contains a vertex r. Then the $K_{1,1,5}$ minor is given by $R_1 = V(B_1[w, c_1] \cup B_1[c_n, z] \cup C)$, $R_2 = V(B_2 \cup A_3) \cup \{u\}$, and $S = \{r, t_1, t_2, t_4, t_5\}$. See Figure 4.10. So there must be only one vertex in $B_1(c_1, c_n)$, meaning that $n = 3$ and c_1, c_2, c_3 are consecutive on B_1. □

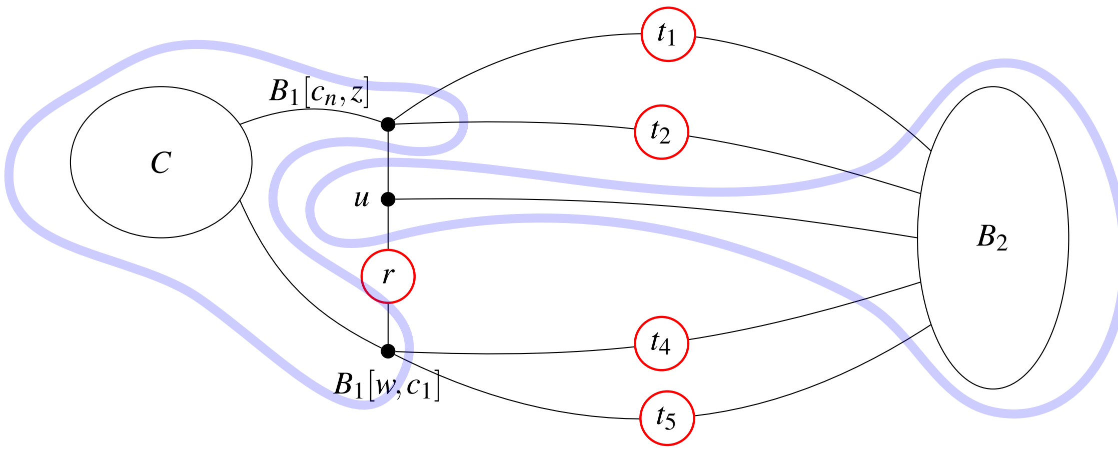

Figure 4.10

The results in the remainder of this section are concerned with a component C of $G \setminus H$, which we assume (without loss of generality) has its neighbours in B_1 By Lemma 4.9, we know there is some arc A_i with t_i adjacent to an internal vertex of B_1 (specifically, c_2 for the component C). We will assume for the rest of the section that the arc adjacent to an internal vertex of B_1 is A_3. Recall that there can be only one such arc, by Lemma 4.7. Then A_1, A_2 can be assumed to have start vertex z, and A_4, A_5 can be assumed to have start vertex w.

Lemma 4.11. *Let C be a component of $G \setminus H$, with points of attachment c_1, c_2, c_3 in that order along a base path of H. Then the vertices $V(C) \cup \{c_2\}$ induce a k-fan, with rivet vertex c_2, and $k = |V(C)|$.*

Proof. Let v be a vertex of C. By 3-connectivity, there are three internally disjoint paths from v to (without loss of generality) B_1, ending at the distinct vertices c_1, c_2, c_3 respectively. Observe that any such paths lie in the subgraph of G induced by $V(C) \cup \{c_1, c_2, c_3\}$. Let P_1, P_2, P_3 be three such paths (with P_i ending at c_i for $i = 1, 2, 3$) such that the total number of vertices on the paths is minimised. This implies that each P_i is in fact an induced path: if there were any edge between two vertices on a path P_i that did not lie on the path itself, we could use that edge to construct a shorter path.

We claim that the path P_2 is in fact just the edge vc_2. To see this claim, suppose by way of contradiction that P_2 has some internal vertex. By 3-connectivity there has to be a path from some internal vertex u on P_2 to $P_1 \cup P_3$ that does not pass through v or c_2. This path also lies entirely in the subgraph induced by $V(C) \cup \{c_1, c_2, c_3\}$. Without loss of generality, suppose the path ends on P_1. Then we have a $K_{1,1,5}$ minor, given by $R_1 = V(B_1[c_2, z])$, $R_2 = V(B_2 \cup A_5 \cup B[w, c_1] \cup P_1(v, c_1])$, and $S = \{v, u, t_1, t_2, t_3\}$. This minor is shown in Figure 4.11.

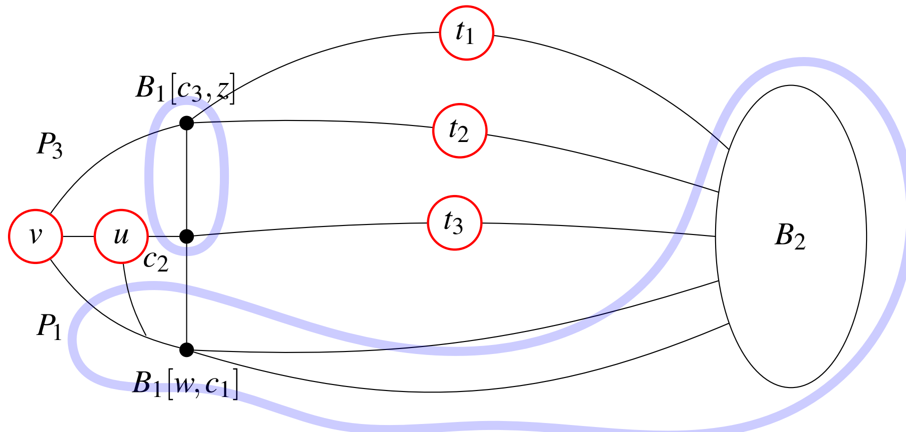

Figure 4.11

Therefore P_2 is just the edge vc_2. Additionally, there are no edges or paths between $P_1 \setminus \{v\}$ and $P_3 \setminus \{v\}$ whose internal vertices are all in $G \setminus H$. If there were such a path, call it Q, we would obtain the $K_{1,1,5}$ given by $R_1 = V(B_1[w, c_1] \cup P_1(v, c_1] \cup P_3(v, c_3] \cup Q)$, $R_2 = V(B_2 \cup A_3) \cup \{c_2\}$, and $S = \{v, t_1, t_2, t_4, t_5\}$, shown in Figure 4.12.

We now claim that there are no other vertices in C other than those on P_1, P_2, P_3. To see this, suppose by way of contradiction there is some vertex u in C that does not lie on any of P_1, P_2, P_3. Then u must have three internally disjoint paths to distinct vertices (which we shall call points of attachment) on $P_1 \cup P_2 \cup P_3$, by 3-connectivity. Since there are no such paths between vertices of $P_1 \setminus \{v\}$ and vertices of $P_3 \setminus \{v\}$, u cannot have points of attachment on both $P_1 \setminus \{v\}$ and $P_3 \setminus \{v\}$. Therefore u either has all three points of attachment on one of P_1 and P_3, or has two points of attachment on one of P_1 or P_3 and

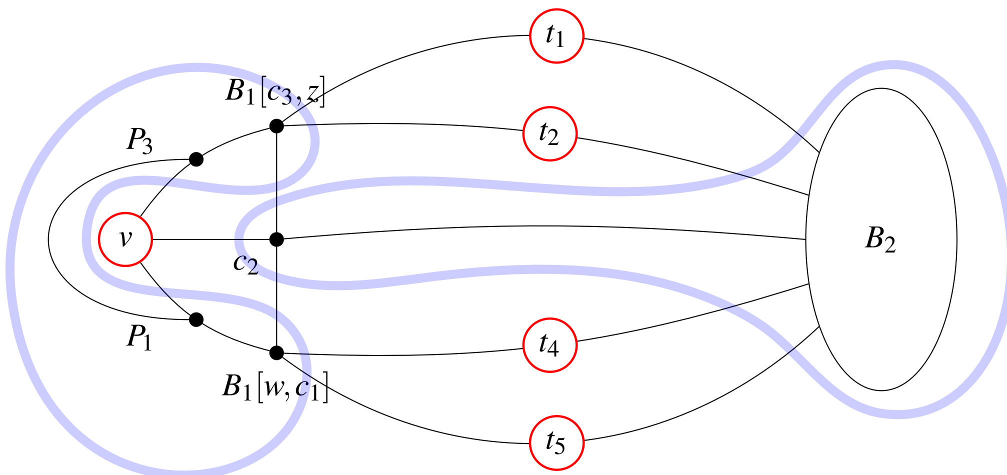

Figure 4.12

the third point is c_2 (observe that c_2 is the only vertex on P_2 other than v, and v is also part of P_1 and P_3). In the former case, assume without loss of generality that u has all points of attachment on P_1, and denote the points of attachment respectively by r_1, r_2, r_3, with r_1 closest to v and r_3 closest to c_1. Then a $K_{1,1,5}$ minor is given by $R_1 = V(B_1[w,c_2] \cup P_1[r_3,c_1]), R_2 = V(B_2 \cup A_1 \cup B_1[c_3,z] \cup P_3)$ and $S = \{u,r,t_3,t_4,t_5\}$ (see Figure 4.13). In the latter case, assume without loss of generality that u has two points of attachment on P_3. We will these two vertices r_1 and r_2, where r_1 is closest to v and r_2 is closest to c_3 (note that we could have $r_1 = v$ or $r_2 = c_3$). Then a $K_{1,1,5}$ minor (shown in Figure 4.14) is given by $R_1 = V(B_1[w,c_2]), R_2 = V(B_2 \cup A_1 \cup B_1[c_3,z] \cup P_3[r_2,c])$ and $S = \{u,v,t_3,t_4,t_5\}$.

Therefore there are no vertices in C that do not lie on one of P_1, P_2, P_3. Also, since P_2 is just the edge vc_2, and there are no edges along either P_1 or P_3 or between $P_1 \setminus \{v\}$ and $P_3 \setminus \{v\}$, C is in fact just the induced path $P_1^{-1}(c_1,v]P_3[v,c_3)$. All that remains to be shown in order to prove that that $V(C) \cup \{c_2\}$ induces a fan with rivet vertex c_2, is that every vertex in $V(C)$ is adjacent to c_2 and the degree of c_2 is greater than four. However, we proved that v is adjacent to c_2, and v was an arbitrary vertex of C. Therefore *every* vertex in C must be adjacent to c_2. In addition, c_2 is adjacent to c_1, c_3 and t_3, so c_2 must have degree at least four.

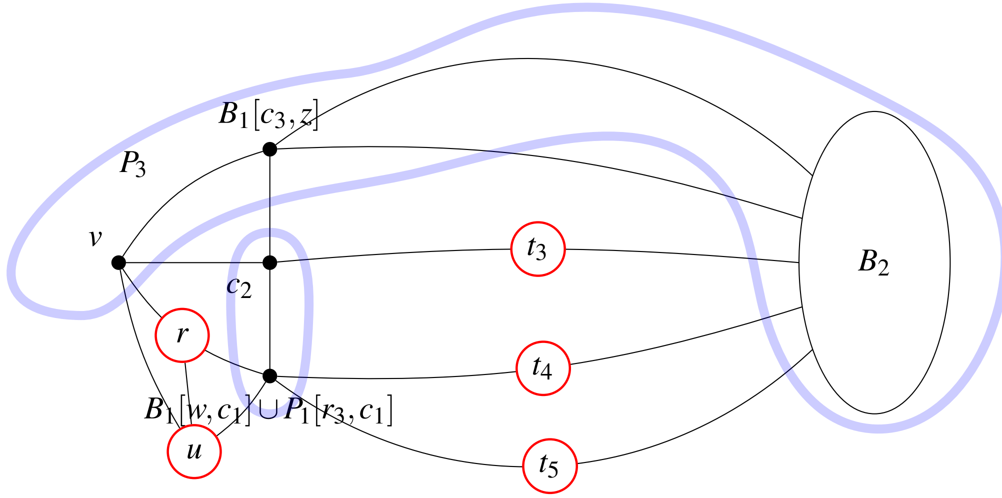

Figure 4.13

Therefore the vertices $V(C) \cup \{c_2\}$ induce a k-fan with rivet vertex c_2, and $k = |V(C)|$. □

Corollary 4.12. *If a component of $G \setminus H$ has at least two vertices, then G has a nontrivial fan.*

Lemma 4.13. *A component of $G \setminus H$ cannot have only one vertex.*

Proof. Suppose that we have a component C of $G \setminus H$ such that C has only one vertex, v. Without loss of generality, assume that v has its neighbours c_1, c_2, c_3 on B_1, occurring in that order. We know that c_2 must have an edge to an arc vertex, t_3. Now we may replace arc A_3 by $A'_3 := vc_2A_3[t_3, -]$, and B_1 by $B'_1 := B_1[w, c_1]vB_1[c_3, z]$. This results in a $K_{2,5}$ outline H' with the same number of total base vertices, but one more arc vertex; a contradiction. Therefore C cannot consist of a single vertex. □

Corollary 4.14. *If $G \setminus H$ is nonempty, then G has a nontrivial fan.*

4.2.3 Edges Between Arcs

Now consider the arcs. Since there are no edges between two vertices on an arc that are not adjacent on the arc (Lemma 4.5), each arc vertex (with the exception of possibly two

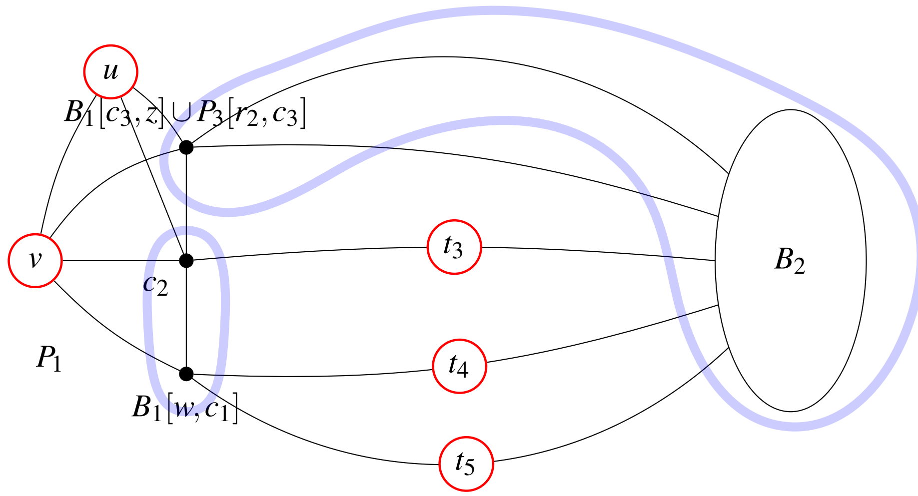

Figure 4.14

penultimate arc vertices that may have edges to base vertices) has an edge to another arc. By planarity, any vertex on an arc can only have an edge to either one of its neighbouring arcs. For the arguments that follow, we assume a certain numbering of the arcs. Given a planar embedding of G, consider the graph obtained by contracting each of B_1 and B_2 to a single vertex. Then label the arcs consecutively in clockwise order around the contracted B_1 vertex. This necessarily implies that the arcs are labelled consecutively in an anticlockwise order around the contracted B_2 vertex. We now lift this numbering of the arcs to the original graph.

Lemma 4.15. *If an arc vertex a on A_i has an edge to arc vertex b on A_j, then either A_i has length two, A_j has length two, or a and b are both penultimate on their respective arcs and are both adjacent to B_1 or both to B_2.*

Proof. Suppose that none of these three situations are happen. Then a has an arc neighbour to one side of the edge ab and b has an arc neighbour on the other side. This results in a $K_{1,1,5}$ minor. To construct an example of such a minor, let us assume that a lies on A_3, b lies on A_2, a has arc neighbour c directly following it on A_3, and b has arc neighbour d directly preceding it on A_2. Then the $K_{1,1,5}$ minor is given by $R_1 = V(B_1 \cup A_3[-,a]), R_2 = V(B_2 \cup A_2[b,-])$, and $S = \{c,d,t_1,t_4,t_5\}$. This minor is shown in Figure 4.15.

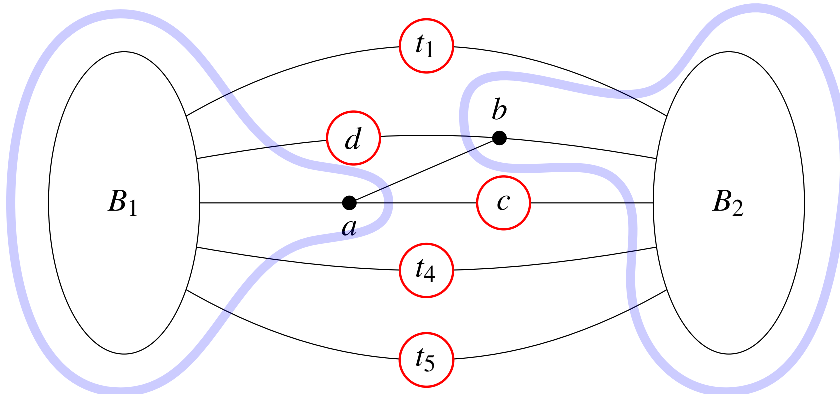

Figure 4.15

□

Corollary 4.16. *If an arc vertex a on arc A_i is not penultimate on its arc, and has an edge to arc vertex b on arc A_j, then A_j has length two.*

Lemma 4.17. *If an arc of G has at least four arc vertices, then G has a nontrivial fan.*

Proof. Suppose that G has an arc (without loss of generality, A_3) with at least four arc vertices. Let a_1, a_2, a_3, a_4 be the first four arc vertices of the arc, occurring in that order (i.e., $a_1 = t_1$). First we show that if a_1 is adjacent to more than one vertex in B_1, we have a nontrivial fan. To see this, suppose that a_1 is adjacent to two base vertices, u and v in B_1. Now consider the neighbours of a_2 and a_3. By 3-connectivity, they must both have a third neighbour not on their arc. We know that neither a_2 nor a_3 is adjacent to a base vertex since they are not penultimate on their arc, therefore they must have edges to neighbouring arcs, A_2 or A_4. Suppose that both a_2 and a_3 have edges only to one of the neighbouring arcs, say A_2. Then by Corollary 4.16, A_2 has length two and only one internal vertex, say b, and the vertices b, a_2, a_3 form a fan riveted at b with collapsible edge a_2a_3. Therefore suppose that a_2 and a_3 have edges to different arcs - without loss of generality, a_2 to vertex b on A_2 and a_3 to vertex c on A_4. We also know that a_4 has a third neighbour. Since a_4 may be a penultimate arc vertex, this third edge may be to a B_2 vertex, or it may be to A_2 or A_4. If a_4 has an edge to A_4, we get a $K_{1,1,5}$ minor (Figure 4.16). The minor is explicitly described by $R_1 = V(B_1 \cup A_4[-,c])$, $R_2 = V(B_2 \cup A_2[b,-]) \cup \{a_2\}$, and $S = \{t_1, a_1, a_3, a_4, t_5\}$.

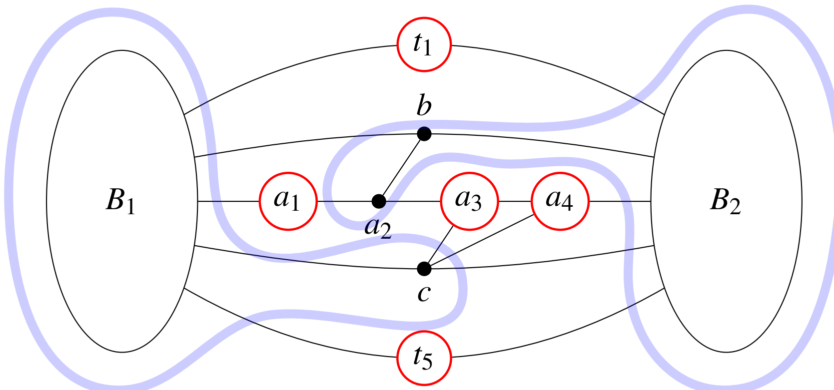

Figure 4.16

Therefore the third neighbour of a_4 is either on A_2 (i.e., is b) or B_2. Neither of these possibilities immediately gives a $K_{1,1,5}$ minor, so consider the arcs A_1 and A_5. Each of these arcs has at least one internal vertex, say d on A_1 and e on A_5. If a_4 does not have multiple neighbours in B_2, then either d or e may have another B_2 vertex as its third neighbour, but at least one of d and e must have an edge to another arc. Therefore we have at least one of the following edges: an edge from e to an arc vertex on A_4 (i.e., c), an edge from d to an arc vertex on A_2 (i.e., b), or an edge between arc vertices on A_1 and A_5 respectively. If either of the first two edges exists, we get a $K_{1,1,5}$ minor. The two situations are symmetric, so we need only construct one minor explicitly. Let us assume we have the edge from e to an arc vertex on A_4.This gives the minor described by $R_1 = \{a_2, a_3, c\}$, $R_2 = V(B_2 \cup A_1 \cup B_1[v,z])$, and $S = \{w, a_1, a_4, b, e\}$. A depiction of this minor is given in Figure 4.17.

Therefore we must have an edge between arc vertices on A_1 and A_5, which we may assume is between d and e. But now either of the two possibilities for a_4's third neighbour (b or another B_2 vertex) gives a $K_{1,1,5}$ minor. First consider the case that a_4's third neighbour is b. To describe the $K_{1,1,5}$ minor present in this situation, we have $R_1 = V(B_1[w,u] \cup A_5[-,e] \cup A_4[-,c]) \cup \{a_3\}$ and $R_2 = V(B_1[v,z] \cup A_2[-,b])$. It is simplest for this minor to allow S_1 to consist of multiple vertices, although we have until now been using the convention that each S_i branch set consists of a single vertex. We let $S_1 = V(B_2)$

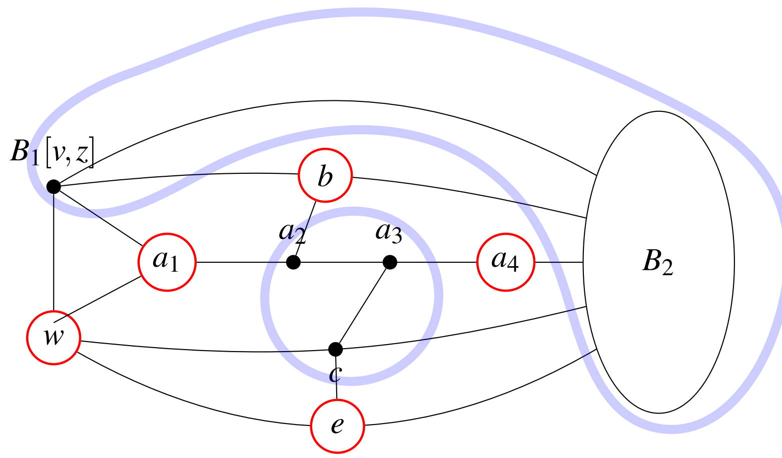

Figure 4.17

and S_2, S_3, S_4, S_5 be the usual single-vertex branch sets consisting of the vertices d, a_1, a_2, a_4 respectively. See Figure 4.18.

Next we construct the $K_{1,1,5}$ minor obtained if a_4's third neighbour is another B_2 vertex, instead of b. Let a_4's neighbours in B_2 be g and h, where g is closest to x and h is closest to y. We know that out of A_1, A_2, A_4, A_5, two arcs end at x in B_2 and two arcs end at y in B_2. Given that A_1 and A_2 start at z in B_1 and A_4 and A_5 start at w in B_1, we may assume that A_1 ends at x in B_2. We claim that this implies that A_2 also ends at x in B_2. To see this, suppose not. Then A_2 ends at y, and one of A_4, A_5 ends at x; we may assume A_4 ends at x. To construct the contradictory minor, we do not use any of the internal information of arcs A_1, A_2, A_4, A_5, therefore assuming A_4 ends at x does not lose us generality. Now we have a $K_{3,3}$ minor, contradicting planarity. The $K_{3,3,}$ minor is described by the branch sets $P_1 = \{a_1\}$, $P_2 = V(B_2[x,g])$, $P_3 = V(B_2[h,y])$, $Q_1 = \{a_4\}$, $Q_2 = V(B_1[w,u])$ and $Q_3 = V(B_1[v,z])$, and is shown in Figure 4.19. Therefore A_1 and A_2 both end at x in B_2, and A_4 and A_5 both end at y in B_2. Now we may finally construct the $K_{1,1,5}$ minor for this case. The minor is described by $R_1 = V(B_1[v,z] \cup A_2[-,b]) \cup \{a_2,a_3\}$, $R_2 = V(B_1[w,u] \cup A_5 \cup B_2[h,y])$, and $S = \{a_1,a_4,d,c,x\}$ (see Figure 4.20).

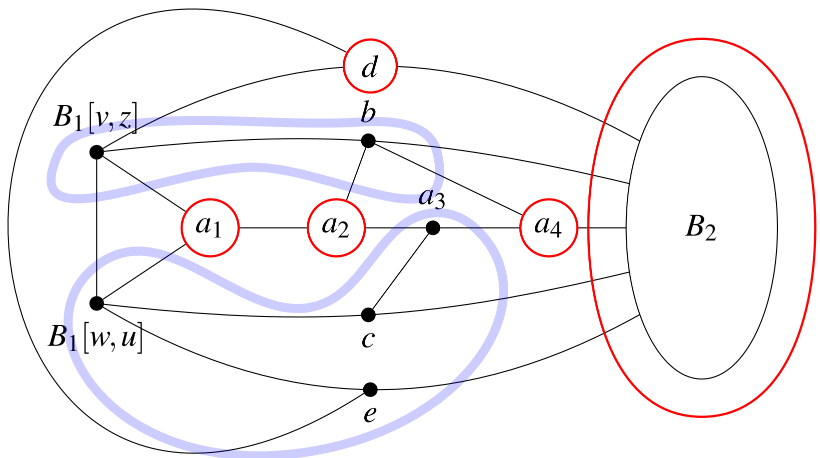

Figure 4.18

So if a_1 is adjacent to more than one B_1 vertex, we have either a nontrivial fan or a contradiction in the form of a $K_{1,1,5}$ minor.

So assume that we have an arc, A_3, with at least four arc vertices, a_1,a_2,a_3,a_4 such that neither penultimate vertex on our arc is adjacent to multiple vertices on a base path. Then each of a_1,a_2,a_3,a_4 is adjacent to a neighbouring arc (either A_2 or A_4). If two consecutive vertices in $\{a_1,a_2,a_3,a_4\}$ both only have an edge to one of the arcs, say A_2, then A_2 has only one internal vertex, say b, and the two consecutive a vertices form a nontrivial fan with b as the rivet. So assume that any two consecutive vertices in $\{a_1,a_2,a_3,a_4\}$ have edges to different arcs. In particular, we know that a_1 and a_2 have edges to different arcs. Suppose a_1 has an edge to vertex b on A_2 and a_2 has an edge to vertex c on A_4. Now, if a_3 only has an edge to A_4, then a_2 must have edges to both A_2 and A_4, and we get a $K_{1,1,5}$ minor described by $R_1 = V(B_1 \cup A_4[-,c])$, $R_2 = V(B_2 \cup A_2[b,-])$, and $S = \{t_1,a_1,a_2,a_3,t_5\}$,

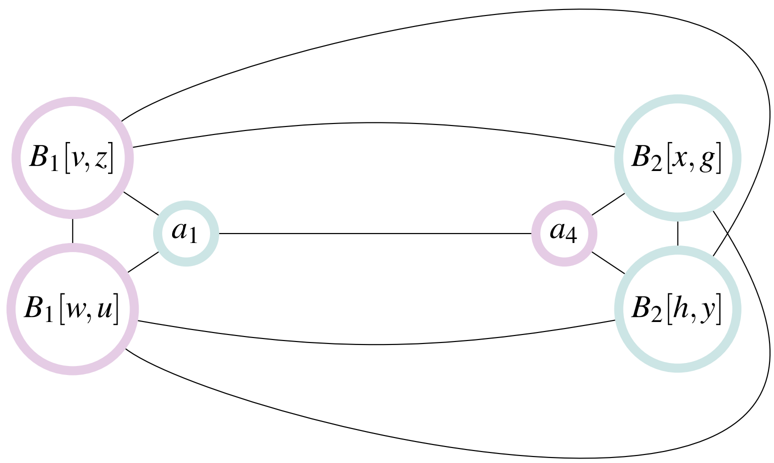

Figure 4.19

shown in Figure 4.21.

So a_3 has an edge to A_2 (i.e., to b). Now if a_4 has an edge to A_4, we get a $K_{1,1,5}$ minor almost identical to the one found above (replace $S_3 = \{a_2\}$ with $S_3 = \{a_2, a_3\}$ and replace $S_4 = \{a_3\}$ with $S_4 = \{a_4\}$). If a_4 only has an edge to A_2, then a_3 has edges to both A_2 and A_4, and we again get a similar minor to the one above.

Therefore if we have four arc vertices along any one arc, we must have a nontrivial fan. □

Lemma 4.18. *If G has more than ten total arc vertices, then G has a nontrivial fan.*

Proof. We know that no single arc of G can have four or more arc vertices without resulting in a nontrivial fan. Therefore assume we have no arcs with four or more arc vertices. We also know that if an arc has three arc vertices, then the middle arc vertex must have an edge to a neighbouring arc with length two, by Corollary 4.16. Therefore any arc with three arc vertices must have a neighbouring arc with one arc vertex. This implies that we can have at most three arcs with three arc vertices.

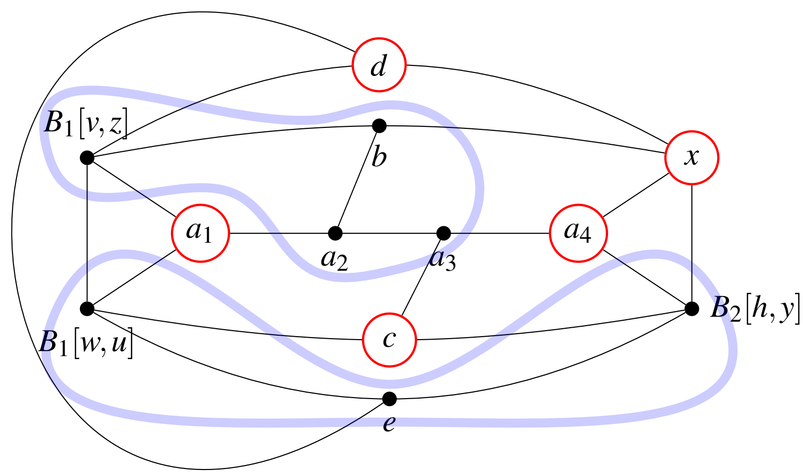

Figure 4.20

Suppose we have exactly three arcs with three arc vertices each. Then each of these three arcs must have a neighbouring arc with only one arc vertex, so the remaining two arcs must have one arc vertex each. Let us assume that A_1, A_3, A_5 each have three arc vertices (they cannot all occur consecutively since each must have a length two neighbouring arc) and A_2 and A_4 each have one arc vertex. Let A_1's arc vertices be a_1, a_2, a_3, A_2's be b, A_3's be c_1, c_2, c_3, A_4's be d, and A_5's be e_1, e_2, e_3. Then we know that we must have the edges a_2b and e_2d (these are the only possible non arc edges out of a_2 and e_2). We must also have either c_2b or c_2d. If we have both edges, we get a $K_{1,1,5}$ minor, shown in Figure 4.22 and described by $R_1 = V(B_1 \cup A_5 \cup B_2)$, $R_2 = \{a_2, b, c_2\}$ and $S = \{a_1, a_3, c_1, c_3, d\}$. Therefore we only have one of the edges c_2b or c_2d. Since the cases are symmetric, we may assume we have c_2d.

We will refer to edges between arc vertices on distinct arcs as *jumps*, and say that these two vertices *jump* to each other. Consider edges incident with a_1. Since a_2 jumps only to b, if a_1 also has b as its only non-arc neighbour we have an induced fan on a_1, a_2, b with b

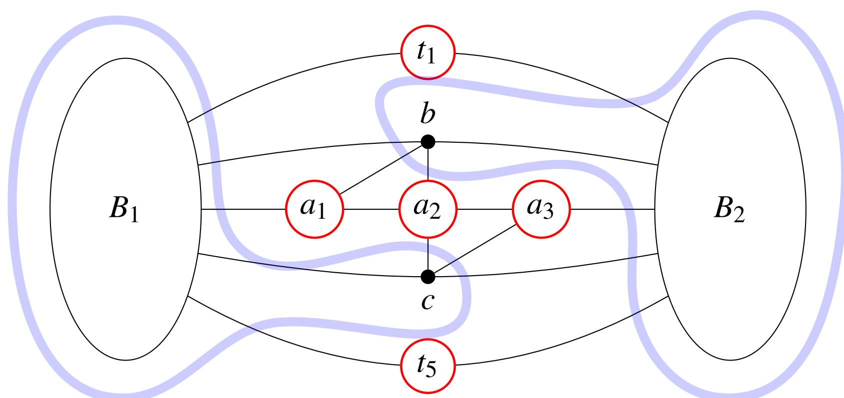

Figure 4.21

as the rivet vertex and a_1a_2 a collapsible edge. Therefore assume a_1 has a non-arc edge to somewhere other than b. It can jump to arc A_5 or to another base vertex of B_1. If a_1 jumps to arc A_5, then it can only jump to e_1 (a jump to e_2 or e_3 would give a minor, by Lemma 4.15). Similarly, e_1 must have an edge to a vertex other than d and its arc neighbours, which can be either another base vertex of B_1, or a_1. Since e_1 and a_1 cannot both be adjacent to multiple base vertices, the edge e_1a_1 must exist.

Now consider edges incident with c_1. Since c_2's only non-arc edge is to d, if c_1 only has a non-arc edge to d we have an induced fan on c_1, c_2, d with rivet vertex d and collapsible edge c_1c_2. Therefore assume c_1 has a neighbour other than d. This neighbour can only be b, or an additional base vertex. If c_1 has an edge to b, we get a $K_{1,1,5}$ minor. Again, for the construction of this minor it is simplest to allow $S_1 = V(B_1)$, instead of just a single vertex. We then have the remaining sets given by $R_1 = \{a_2, b, c_1, c_2\}$, $R_2 = V(B_2) \cup \{e_1, e_2, e_3\}$, and $S \setminus S_1 = \{a_1, a_3, c_3, d\}$. See Figure 4.23.

Therefore c_1 does not have an edge to b, and so c_1 must be adjacent to multiple base

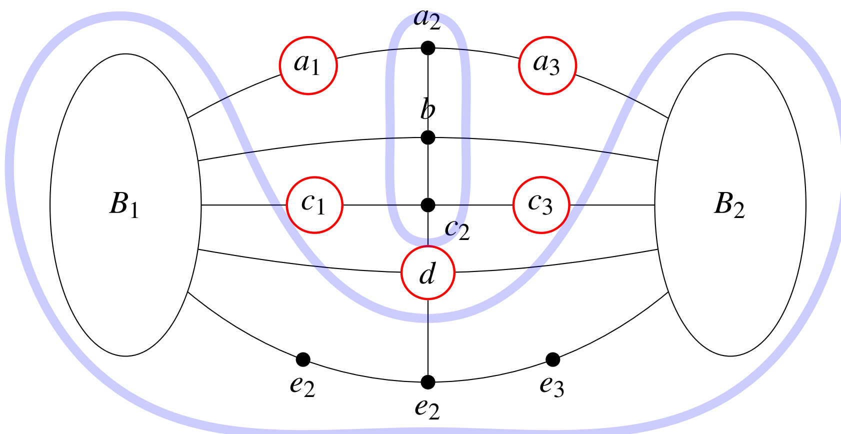

Figure 4.22

vertices in B_1, say u and v, where u is closest to w and v is closest to z. However, this also gives a $K_{1,1,5}$ minor, with $R_1 = \{e_1, e_2, d, c_2\}$, $R_2 = V(B_2 \cup A_2 \cup B_1[v,z])$, and $S = \{a_1, c_1, c_3, e_3, w\}$. See Figure 4.24.

Therefore if we have three arcs each with three arc vertices, we have a nontrivial fan.

Suppose we have exactly two arcs each with three arc vertices. At least one of the remaining arcs must have only one arc vertex. We assume exactly one arc has one arc vertex and the other two have two arc vertices each, otherwise we have at most ten total arc vertices. The arc with one arc vertex must lie in between the arcs with three arc vertices, so the order of the arcs (up to symmetry) is uniquely determined. Let us say that A_1 has two arc vertices, a_1 and a_2, A_2 has two arc vertices, b_1 and b_2, A_3 has three arc vertices, c_1, c_2, c_3, A_4 has one arc vertex, d, and A_5 has three arc vertices e_1, e_2, e_3. We know we must have the edges e_2d and c_2d, and these are the only possible non-arc edges incident with e_2 and c_2 respectively. Now consider edges incident with c_1. If c_1's only non-arc neighbour is d, then c_1, c_2, d induce a fan with rivet vertex d and collapsible edge c_1c_2. So c_1 has some

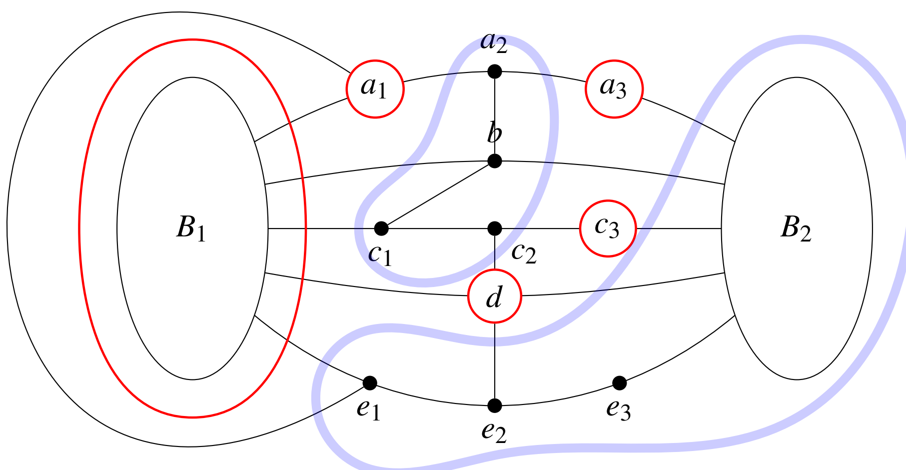

Figure 4.23

other neighbour - either an additional base vertex, or a vertex on A_2 (which must be b_1 by Lemma 4.15). Suppose that c_1 is adjacent to two base vertices u and v in B_1, and consider the neighbours of e_1. Similarly to c_1, e_1 has a neighbour that is either an additional base vertex or a_1. However, since c_1 is adjacent to multiple base vertices, e_1 cannot be adjacent to multiple base vertices, so e_1 must have an edge to a_1. Then we get a $K_{1,1,5}$ minor, given by $R_1 = \{c_2, d, e_2\}$, $R_2 = V(B_1[v,z] \cup A_1 \cup B_2)$, and $S = \{w, c_1, c_3, e_1, e_3\}$ (see Figure 4.25).

Therefore c_1 is not adjacent to multiple base vertices, so it is adjacent to b_1 on A_2. Since c_1, c_3, e_1 and e_3 are all symmetric, none of them can be adjacent to multiple base vertices, therefore the edges c_3b_2 and e_3a_2 are also forced. But this creates a $K_{1,1,5}$ minor. Again, for simplicity we set $S_1 = V(B_2)$, instead of just a single vertex, $R_1 = V(B_1) \cup \{a_1, a_2\}$, $R_2 = \{b_2, c_2, c_3, d, e_2\}$, and $S \setminus S_1 = \{b_1, c_1, e_1, e_3\}$. This minor is shown by Figure 4.26.

Therefore we have at most one arc with three arc vertices. This forces a neighbouring arc to have at most one arc vertex (by Lemma 4.15), and the three remaining arcs must each have at most two arc vertices. This gives at most ten total arc vertices. □

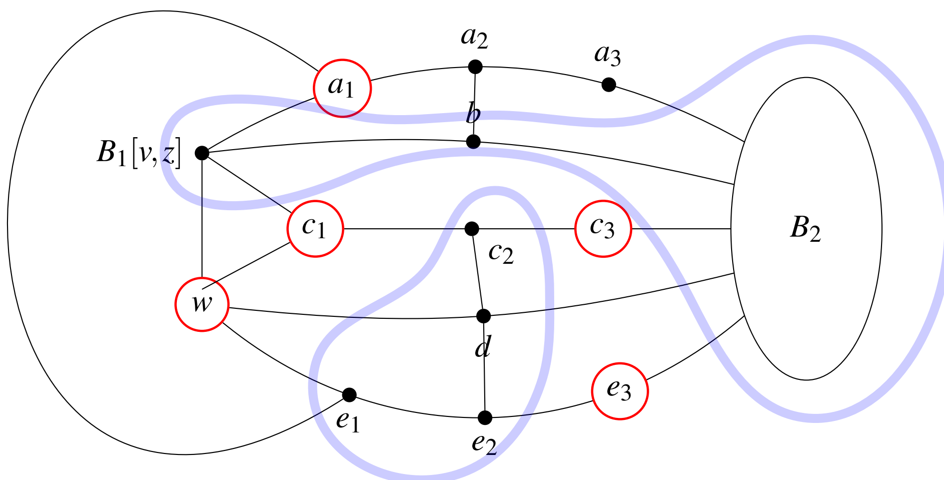

Figure 4.24

4.2.4 Proof of Main Result

Lemma 4.19. *If G has more than 18 vertices, then G has a nontrivial fan.*

Proof. Suppose that G does not have a nontrivial fan. Then by Corollary 4.14, $G = H$. Consider the base graphs, B_1 and B_2. The internal vertices of each base path must each have a neighbour other than their path neighbours, by 3-connectivity. Since there are no components outside H, all of the internal vertices in the same base path must be adjacent to the same penultimate arc vertex a, and they therefore all have degree exactly three. If a base path has at least five vertices, then it has at least three internal base vertices, which form a 3-fan with a as the rivet vertex; a contradiction. Therefore each base path can have size at most four, and G has at most eight base vertices. We observe that a base path with size exactly four is not enough to guarantee a fan, since a might then have degree only three and cannot act as the rivet vertex.

From above, we know that if G does not have a nontrivial fan, then G can have at most

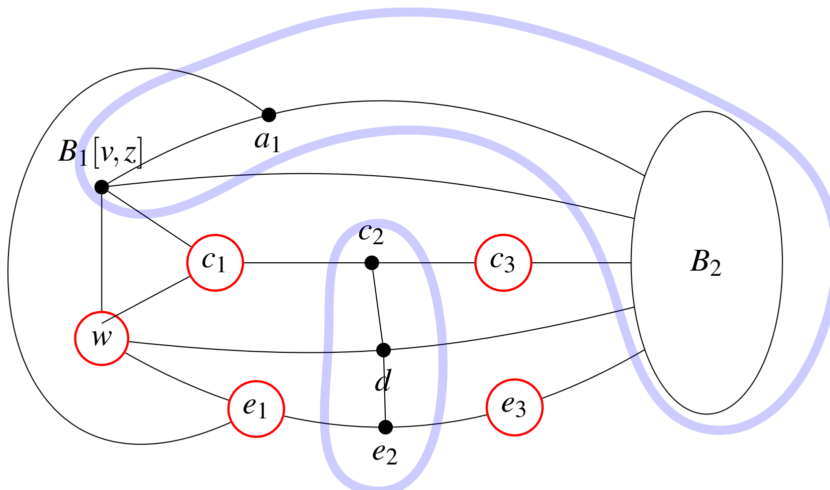

Figure 4.25

ten arc vertices. Since $G = H$, the only vertices in G are either base vertices or arc vertices, so G has at most 18 vertices in total. □

To reduce the order of the graphs in the base case of our forthcoming induction argument, we define a new type of subgraph that proves useful for Hamiltonicity.

Definition 4.20. Let a,b,c all be vertices of degree three that induce a triangle (3-cycle) in a graph G. If G is 3-connected and $|V(G)| \geq 5$, then this triangle is said be a *contractible triangle* in G. Contracting the vertices a,b,c to a single vertex t is referred to as contracting the triangle. Observe that each of a,b and c have exactly one neighbour outside the triangle in G, and each of these neighbours must be distinct; if for example a and b have a common neighbour d outside the triangle, then $\{d,c\}$ is a cutset in G, contradicting that G is 3-connected. Therefore t has degree exactly three in the graph obtained by contracting the triangle.

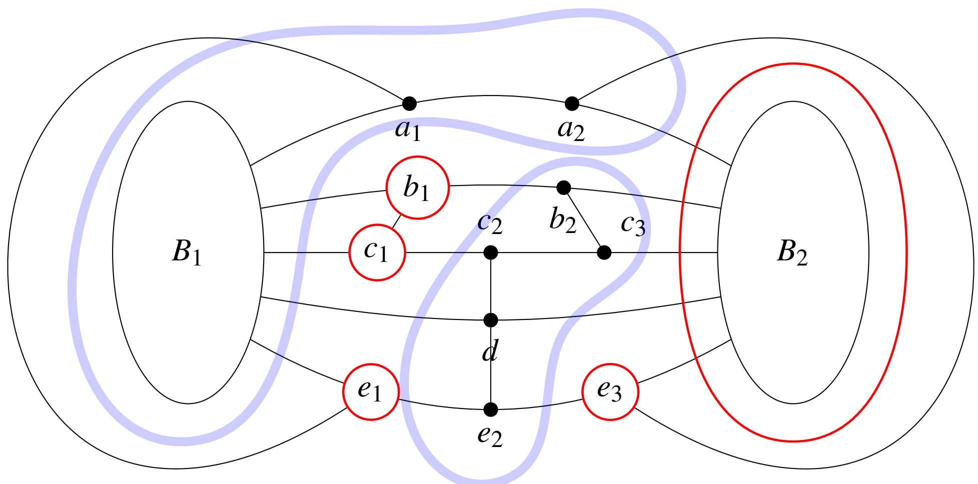

Figure 4.26

Lemma 4.21. *If G is a 3-connected graph with a contractible triangle formed by the vertices a, b, c, then the graph obtained from G by contracting this triangle is also 3-connected.*

Proof. Let G, a, b, c be as described in the statement of the lemma. Let G' be the graph obtained by contracting the contractible triangle. Denote the vertex that the triangle is contracted to by t, and let t's three neighbours in G' be denoted by d, e, f. Assume these vertices are neighbours of a, b, c respectively in G. Suppose that G' is not 3-connected. Consider some minimal cutset S in G'. We must have $|S| \leq 2$. If t is not in S, then S is also a cutset in G, a contradiction. Therefore t must be a vertex of S.

We claim that there is some component C of $G' \setminus S$ that contains only one of d, e, f. To see this, first observe that not all of d, e, f can be in one component of $G' \setminus S$, since then $S \setminus \{t\}$ would still be a cutset in G', contradicting minimality. Now suppose that one of d, e, f (without loss of generality, d) is also in S, so that $S = \{t, d\}$. Then e and f are in different components of $G' \setminus S$, and it follows that e and f are in different components of $G \setminus \{b, d\}$. In particular, $\{b, d\}$ is a 2-cut in G, a contradiction. Therefore none of d, e, f

are in S, and they are not all in the same component of $G' \setminus S$, so some component contains only one of the three vertices. Let us assume that a component C of $G' \setminus S$ contains d but neither of e and f. But then $S \setminus \{t\} \cup \{a\}$ is a 2-cut in G; again a contradiction. □

Lemma 4.22. *If G is a non-Hamiltonian, 3-connected graph with a contractible triangle, then the graph obtained from G by contracting this triangle is also non-Hamiltonian.*

Proof. Let G be as described, and let the vertices comprising its contractible triangle be a, b and c. Let G' be the graph obtained by contracting this triangle to a single vertex t. Suppose that G' is Hamiltonian, and let C' be a Hamilton cycle in G'. Consider t's neighbours in G'. We know that t has exactly three neighbours; call them d, e, f, and assume these vertices are respectively neighbours of a, b, c in G. Since all of d, e, f are symmetric, we may assume without loss of generality that d directly precedes t on C', and e directly follows t on C'. Then replacing the segment dte of C' in G' with the path $dacbe$ in G yields a Hamilton cycle in G, a contradiction. □

Lemma 4.23. *If G has 17 or 18 vertices, then G has either a nontrivial fan, or a contractible triangle.*

Proof. Suppose that G has either 17 or 18 vertices, and G does not have a nontrivial fan. Then by Corollary 4.14 and Lemma 4.18, $G = H$ and G has at most ten arc vertices. Therefore G must have at least seven base vertices. In particular, one of G's base paths, without loss of generality B_1, has size at least four, and thus at least two internal vertices. As in the proof of Lemma 4.19, we must have all internal vertices of B_1 adjacent to the same penultimate arc vertex a. Since we have assumed G has no nontrivial fan, B_1 must have exactly two internal arc vertices in B_1, and a must have degree three (otherwise the internal base path vertices would form a nontrivial fan with a as the rivet). But then the two internal vertices of B_1, together with a, form a contractible triangle. □

We are finally in a position to prove our main result.

Proof of Theorem 4.2. By Lemma 4.1, we know that the statement of the theorem holds for 3-connected planar $K_{1,1,5}$-minor-free graphs on at most 16 vertices. Let us fix some $m > 16$ and assume the statement holds for all 3-connected planar $K_{1,1,5}$-minor-free graphs on fewer than m vertices. Let G be a 3-connected planar $K_{1,1,5}$-minor-free graph on m vertices. By Lemma 4.23, we know that G has either a nontrivial fan or a contractible triangle.

First consider the case that G has a nontrivial fan. Then we may collapse the fan to obtain a graph G' on $m-1$ vertices. By Lemma 2.8, G' is still 3-connected. Since contracting edges does not create any minors, G' is also $K_{1,1,5}$-minor-free and planar. By our inductive hypothesis, G' is Hamiltonian. Now expanding the fan to recover G preserves Hamiltonicity, by Lemma 2.7. Therefore G is Hamltonian.

Now consider the case that G has a contractible triangle. Let us contract this triangle to obtain a graph G'' on $m-2$ vertices. By Lemma 4.21, G'' is 3-connected, and again since contracting edges does not create minors, G'' remains $K_{1,1,5}$-minor-free and planar. By the inductive hypothesis, G'' is Hamiltonian. Then Lemma 4.22 implies that G is Hamiltonian.

Both cases result in G being Hamiltonian, therefore by induction our result is proved.

□

Chapter 5

Future directions

There are several avenues for further research based on the results presented in this dissertation.

As mentioned in Chapter 1, there exists an infinite family of nonplanar 3-connected $K_{2,5}$-minor-free graphs that are not Hamiltonian. Characterising all non-Hamiltonian 3-connected $K_{2,5}$-minor-free graphs would be an interesting project.

Another way forward would be extending the characterisation of 3-connected $K_{1,1,4}$-minor-free graphs in Chapter 3 to *all* $K_{1,1,4}$-minor-free graphs, without connectivity restrictions. Furthermore, we would like to find the orientable and nonorientable genus of the 2-connected and 3-connected nonplanar $K_{1,1,4}$-minor-free graphs. We suspect that these may all be toroidal and projective-planar.

We also propose a strengthening of our Hamiltonicity result for 3-connected planar $K_{1,1,5}$-minor-free graphs given in Chapter 4. In particular, we would like to characterise the non-Hamiltonian 3-connected planar $K_{2,6}$-minor-free graphs. Clearly not all 3-connected, planar, $K_{2,6}$-minor-free graphs are Hamiltonian, since we have the Herschel graph as an example of a non-Hamiltonian 3-connected, planar, $K_{1,1,5}$-minor-free graph, and any $K_{1,1,5}$-minor-free graph is automatically $K_{2,6}$-minor-free. In fact, we have many counterexamples to Hamiltonicity for 3-connected, planar, $K_{2,6}$-minor-free graphs. Gordon Royle's computer results discussed in Section 4.1 give the following:

Lemma 5.1. *There are exactly 206 non-Hamiltonian, 3-connected, planar, $K_{2,6}$-minor-free graphs on fewer than 16 vertices.*

Taking it even further, Ellingham et al [9] constructed an *infinite* family of 3-connected, planar, $K_{2,6}$-minor-free graphs that are not Hamiltonian, all closely related to the Herschel graph. This family is described by Figure 5.1. In the figure, dashed edges represent edges

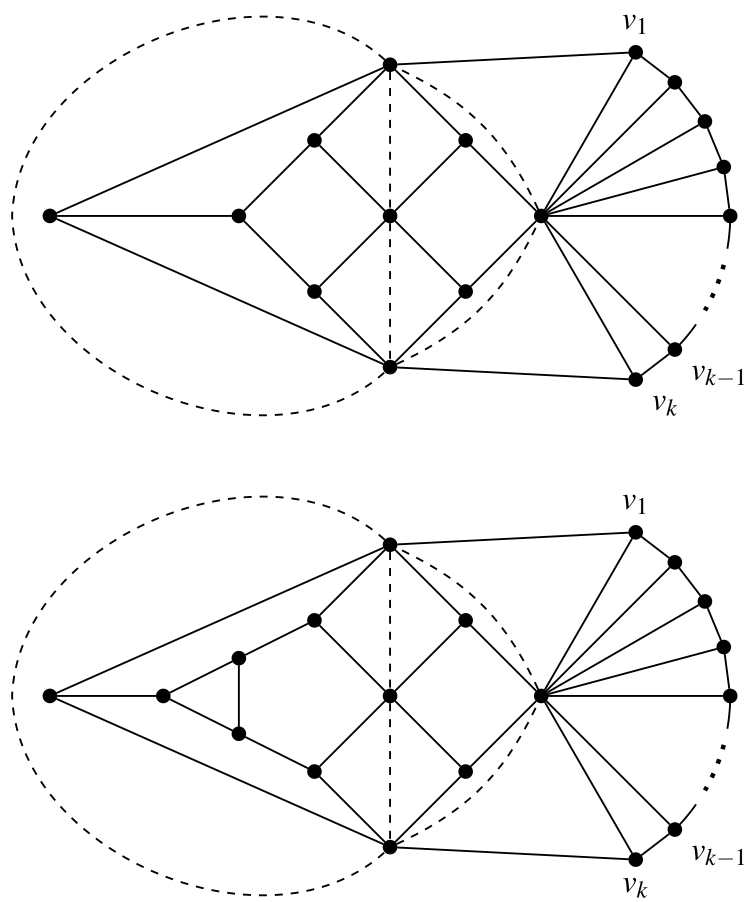

Figure 5.1: Non-Hamiltonian, 3-connected, planar, $K_{2,6}$-minor-free graphs.

that may be either absent or present, each combination giving one of 40 graphs after allowing for symmetries. They also conjectured that *every* 3-connected, planar, $K_{2,6}$-minor-free graph on at least 16 vertices is a member of this family.

Conjecture 5.2. *For each $n \geq 16$, the 3-connected planar $K_{2,6}$-minor-free graphs on n vertices that are not Hamiltonian are exactly those graphs described by Figure 5.1. There are exactly 40 such graphs for each n.*

We believe that this conjecture might be proved using methods similar to those used for the 3-connected planar $K_{1,1,5}$-minor-free case. Specifically, one may be able to consider those graphs that are 3-connected, planar, and $K_{2,6}$-minor-free but not $K_{2,5}$-minor-free or not $K_{1,1,5}$-minor-free, and gradually restrict the structure of a $K_{2,5}$ or $K_{1,1,5}$ outline, ultimately proving that if such a graph has enough vertices, it has a fan. The possible flaw with this approach, however, is that it appears the threshold number of vertices to have a fan might be significantly higher than the $K_{1,1,5}$-minor-free case. This means that the number of graphs to be checked by computer for the base case of the induction argument could be infeasibly large. Further investigation is needed.

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