Skein theory for subfactor planar algebras

By

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CHAPTER I

INTRODUCTION

Modern subfactor theory was initiated by Jones. Subfactors generalize the symmetries of groups. The index of a subfactor is analogous to the order of a group. All possible indices of subfactors,

$$\{4\cos^2\frac{\pi}{n}, n=3,4,\cdots\}\cup[4,\infty],$$

were found by Jones in his remarkable rigidity result [Jon83]. The index, principal graphs, standard invariants are important invariants of subfactors. A deep theorem of Popa [Pop94] showed that the standard invariant is a complete invariant of strongly amenable subfactors of the hyperfinite factor of type II_1. There are three axiomatizations of standard invariants: Ocneanu's paragroups [Ocn88]; Popa's standard λ-lattices [Pop95]; Jones' subfactor planar algebras [Jon98].

Planar algebras provide new perspectives to study subfactors by skein theory. One can present planar algebras by generators and relations. The simplest planar algebra of all is the one with no generators nor relations, which is a sub planar algebra of any planar algebra, known as the Temperley-Lieb-Jones algebra [Jon83]. One can also construct subfactor planar algebras by defining a partition function globally, such as the diagonal subfactor ones. With the knowledge of geometric group theory, we construct a subfactor planar algebra with undetermined dimensions and a subfactor planar algebra which is not finitely generated.

Planar algebras extremely simplify the standard invariants based on the knowledge of the Temperley-Lieb-Jones algebra. Thus we expect to provide simpler constructions for known subfactors which were first done for E_6, E_8, D_{2n}, Haagerup subfactors [Jon01; MPS10; Pet10]. A great achievement is the construction of the extended Haagerup subfactor [Big+12] which is the only known construction so far. A powerful skein theory was discovered from the construction, namely the Jellyfish algorithm. It is a universal skein theory for subfactor planar algebras, since any subfactor planar algebra has a Jellyfish relation if enough generators are added.

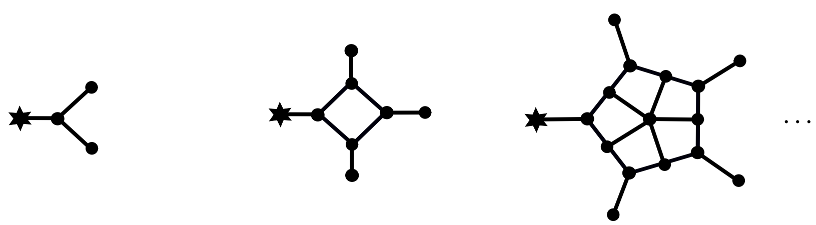
The construction of subfactor planar algebras by generators and relations will encounter three fundamental problems: Evaluation; Consistency; Positivity. Theoretically Evaluation can be provided by the Jellyfish algorithm. Consistency and Positivity are ensured by the embedding theorem [JP11; MW10]. A direct proof of the consistency and positivity for Jellyfish relations will appear in a forthcoming paper. This strategy is very efficient when the principal graph is small. However, solving the Jellyfish relation is akin to solving the connection (or 6j-symbols) (modulo the knowledge of Temperley-Lieb-Jones planar algebras) which is impossible in general. While dealing with a sequence of principal graphs simultaneously, other methods are required, e.g. [MPS10; Liua]. When the principal graph is unknown, it is much harder to construct a subfactor planar algebra by skein theory.

On the other hand, we expect to classify subfactor planar algebras with a good skin theory. Planar algebras generated by 1-boxes were completely analyzed in [Jon98]. Subfactor planar algebras generated by a single 2-box was initiated in [BJ97b]. Motivated by BMW [BW89; Mur87], we expected to classify planar algebras generated by a single 2-box with a Yang-Baxter relation which is a deformation of the Yang-Baxter equation of the generator of BMW. In the classification, a surprising one parameter family of planar algebras appeared after Temperley-Lieb-Jones [Jon83], HOMFLYPT [Fre+85; PT88], BMW [BW89; Mur87], the Potts model [Jon93], Bisch-Jones [BJ97a] planar algebras. This q-parameterized planar algebra contains both the Jones Projection and two Drinfeld-Jimbo R matrixes. Thus it has one Temperley-Lieb-Jones subalgebra and two Hecke subalgebras of type A. An algebraic presentation of this planar algebra is given in the Appendix.

We are going to overcome the three fundamental problems and construct the q-parameterized planar algebra by skein theory. The generator and relations are derived from the classification result (Chapter IV). The evaluation is given by Yang-Baxter relation (Chapter III). The consistency is proved by an oriented version of Kauffman's argument for the Kauffman polynomial [Kau90] with the knowledge of the HOMFLYPT invariant (Chapter V.3). To obtain subfactor planar algebras, we prove the positivity in three steps: constructing matrix units; computing the trace formula; taking the quotient.

The matrix units of the planar algebra are constructed by the matrix units of Hecke algebra of type A and the basic construction (Section V.4). The trace formula is computed via the q-Murphy operator (Section V.5). The positivity can only be achieved when $q = e^{\frac{i\pi}{2N+2}}, N = 1, 2, \cdots$ (Section V.6). When $q = e^{\frac{i\pi}{2N+2}}$, the planar algebra is not semisimple. The quotient by the kernel of the partition function is a subfactor planar algebra. However, it is not easy to figure out the kernel even for Temperley-Lieb-Jones planar algebras [GHJ89]. We provide a strategy to show that the kernel is the ideal generated by certain trace zero idempotents with the help of string algebras. This method also works for general cases, such as Temperley-Lieb-Jones, Bisch-Jones, BMW planar algebras.

When $q = e^{\frac{i\pi}{2N+2}}$, (the quotient of) the q-parameterized planar algebra is a subfactor planar algebra, denoted by $\mathscr{E}_{N+2}$. Its principal graph is the sublattice of the Young lattice consisting of Young diagrams whose $(1,1)$ cell has hook length at most N. For $N = 2, 3, 4, ...$, we have



Moreover, we have the following classification result,

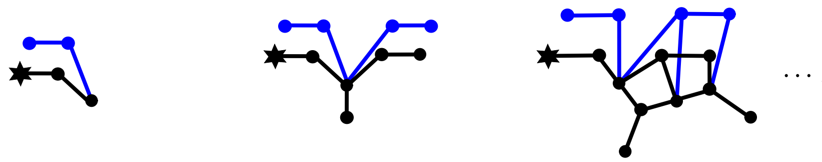
Theorem I.0.1. *Any singly generated Yang-Baxter relation planar algebra is one of the following:*

(1) Bisch-Jones;

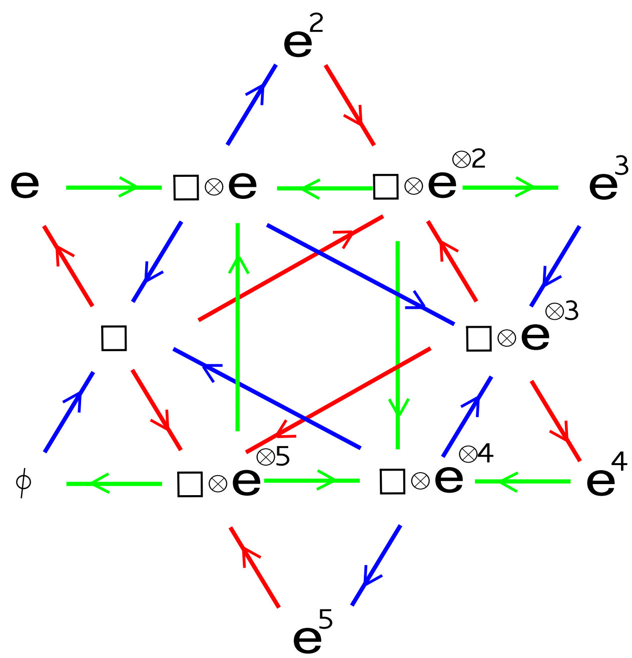
(2) BMW;

(3) $\mathscr{E}_{N+2}$.

We also construct some other subfactor planar algebras and fusion categories from $\mathscr{E}_{N+2}$. The planar algebra $\mathscr{E}_{N+2}$ admits a $\mathbb{Z}_2$ automorphism. The fixed point algebra provides a sequence of subfactor planar algebras which is an extension of the near group subfactor planar algebra for $\mathbb{Z}_4$ [Izu93]. The principal graphs for $N = 2,3,4,\ldots$ are given by



Modulo different *grading operators*, we obtain some graded (unitary, pivotal, spherical) fusion categories. In particular, two of them can be thought as the representation category of exceptional subgroups of quantum $SU(N)$ at level $N+2$ and of quantum $SU(N+2)$ at level N which are related to conformal inclusions $SU(N)_{N+2} \subset SU(N(N+1)/2)_1$ and $SU(N+2)_N \subset SU((N+2)(N+1)/2)_1$ respectively. The branching rule is also derived for all N. In particular, the one for $SU(3)_5$ is



which has appeared in many other places, e.g. in [Xu98] for conformal inclusions, in [Ocn00] for quantum subgroups. The one for $SU(5)_3$ was known in [Xu98]. The one for $SU(4)_6$ was known in [Ocn00]. We also obtain (non-unitary, pivotal, spherical) fusion categories at other roots of unity.

CHAPTER II

PRELIMINARIES

We refer the reader to [JS97], [Jon98; Jon12], [ENO05] for the definition and properties of subfactors, planar algebras and fusion categories. For convenience, we briefly recall some basic results.

II.1 Principal graphs

Suppose $\mathscr{N} \subset \mathscr{M}$ is an irreducible subfactor of type II_1 with finite index. Then $L^2(\mathscr{M})$ forms an irreducible $(\mathscr{N},\mathscr{M})$ bimodule, denoted by X. Its contragredient $\overline{X}$ is an $(\mathscr{M},\mathscr{N})$ bimodule. The tensor products $X \otimes \overline{X} \otimes \cdots \otimes \overline{X}$, $X \otimes \overline{X} \otimes \cdots \otimes X$, $\overline{X} \otimes X \otimes \cdots \otimes X$ and $\overline{X} \otimes X \otimes \cdots \otimes \overline{X}$ are decomposed into irreducible bimodules over $(\mathscr{N},\mathscr{N})$, $(\mathscr{N},\mathscr{M})$, $(\mathscr{M},\mathscr{N})$ and $(\mathscr{M},\mathscr{M})$ respectively, where $\otimes$ is the Connes fusion of bimodules.

Definition II.1.1. The principal graph of a subfactor $\mathscr{N} \subset \mathscr{M}$ is a bipartite graph. Its vertices are equivalence classes of irreducible bimodules over $(\mathscr{N},\mathscr{N})$ and $(\mathscr{N},\mathscr{M})$ in the above decomposition. The number of edges connecting two vertices, a $(\mathscr{N},\mathscr{N})$ bimodule Y and a $(\mathscr{N},\mathscr{M})$ bimodule Z, is the multiplicity of the equivalence class of Z as a sub bimodule of $Y \otimes X$. The vertex corresponding to the $(\mathscr{N},\mathscr{N})$ bimodule $L^2(\mathscr{N})$ is marked by a star sign. The dimension of a vertex λ is defined to be the dimension of the corresponding bimodule, denoted by $<\lambda>$.

II.2 Standard invariants

For an irreducible subfactor $\mathscr{N} \subset \mathscr{M}$ of type II_1 with finite index $[\mathscr{M}:\mathscr{N}]$, we have a left multiplication of $\mathscr{N}$ and $\mathscr{M}$ on the Hilbert space $L^2(\mathscr{M})$ obtained from the GNS construction. The projection onto the subspace $L^2(\mathscr{N})$ of $L^2(\mathscr{M})$ is called the Jones projection, denoted by $e_{\mathscr{N}}$. Then $\mathscr{M}_1 = <\mathscr{M}, e_{\mathscr{N}}>''$ is a factor of type II_1 and $[\mathscr{M}_1:\mathscr{M}] = [\mathscr{M}:\mathscr{N}]$. The process is known as the basic construction of a subfactor. The Jones tower is a sequence of factors $\mathscr{N} \subset \mathscr{M} \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \cdots$ obtained by repeating the basic construction [Jon83]. The system of higher relative commutants

$$\begin{array}{ccccccccc}
\mathbb{C} = \mathscr{N}' \cap \mathscr{N} & \subset & \mathscr{N}' \cap \mathscr{M} & \subset & \mathscr{N}' \cap \mathscr{M}_1 & \subset & \mathscr{N}' \cap \mathscr{M}_2 & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & \mathbb{C} = \mathscr{M}' \cap \mathscr{M} & \subset & \mathscr{M}' \cap \mathscr{M}_1 & \subset & \mathscr{M}' \cap \mathscr{M}_2 & \subset & \cdots
\end{array}$$

is called the standard invariant of the subfactor [GHJ89; Pop90].

There is a natural isomorphism between homomorphisms of bimodules $X \otimes \overline{X} \otimes \cdots \otimes \overline{X}$, $X \otimes \overline{X} \otimes \cdots \otimes X$, $\overline{X} \otimes X \otimes \cdots \otimes X$ and $\overline{X} \otimes X \otimes \cdots \otimes \overline{X}$ and the standard invariant of the subfactor [Bis97]. The equivalence class of a minimal projection corresponds to an irreducible bimodule. So the principal graph tells how minimal projections are decomposed after the inclusion.

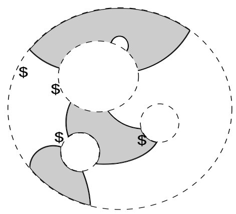
II.3 Planar Algebras

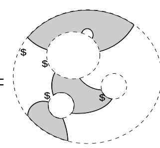
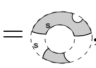
Planar algebras were introduce by Jones in [Jon98] as an axiomatization of the standard invariants.

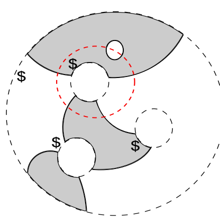
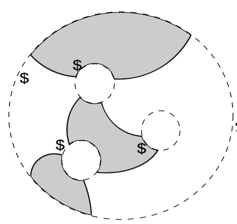
Definition II.3.1 (Planar tangles)**.** A (shaded) planar tangle has

- finite "input" discs
- an "output" disc
- non-intersecting strings
- a distinguished interval of each disc marked by \$

Example II.3.2.



Definition II.3.3 (Composition of tangles)**.** For $T =$  , $S =$ ,

$T \circ_1 S =$  $= \delta$ ,

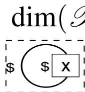
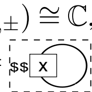
where 1 indicate the top input disc of T.

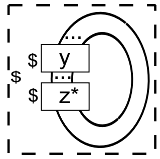
Definition II.3.4. A (shaded) planar algebra $\mathscr{P}_\bullet$ is a family of $\mathbb{Z}_2$ graded vector spaces $\{\mathscr{P}_{n,\pm}\}_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0,1,2,\cdots\}$, with multilinear maps of $\mathscr{P}_\bullet$ indexed by (shaded) planar tangles subject to

- Isotopy invariance
- Naturality

$$\begin{array}{ccc} \mathscr{P}_{2,-} \otimes \mathscr{P}_{2,+} \otimes \mathscr{P}_{1,-} & \xrightarrow{S} & \mathscr{P}_{3,+} \otimes \mathscr{P}_{2,+} \otimes \mathscr{P}_{1,-} \\ & \searrow^{T\circ_1 S} & \downarrow T \\ & & \mathscr{P}_{2,+} \end{array}$$

Definition II.3.5. A subfactor planar algebra is an evaluable spherical planar *-algebra over $\mathbb{C}$ with a positive definite Markov trace.

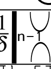
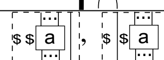
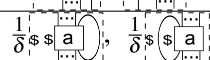
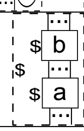
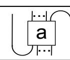
- Evaluable: $\dim(\mathscr{P}_{0,\pm}) \cong \mathbb{C}$, $\dim(\mathscr{P}_{n,\pm}) < \infty$
- Spherical:  $=$ 

• Markov trace: $tr(z^*y) =$ 

Definition II.3.6. A spherical planar algebra is nondegenerate if and only if the Markov trace defines a nondegenerate bilinear form on $\mathscr{P}_{k,\pm}$ for each k.

Theorem II.3.7 ([Jon98]). *The standard invariant of a finite index extremal subfactor is a spherical subfactor planar algebra. The converse statement is also true.*

The correspondence between subfactor planar algebras and the standard invariant is as follows,

Subfactor planar algebras	Standard invariants
$\mathscr{P}_{n,+}$	$\mathscr{N}' \cap \mathscr{M}_{n-1}$
$\mathscr{P}_{n,-}$	$\mathscr{M}' \cap \mathscr{M}_n$
$\delta =$ 	$[\mathscr{M} : \mathscr{N}]^{\frac{1}{2}}$
$\frac{1}{\delta}$ 	Jones projection
	inclusions
	conditional expectations
	multiplication
vertical reflection	adjoint operation
1-click rotation 	Fourier transform

where the thick string labeled by $n-1$ is a convention to indicate $n-1$ parallel strings. Any planar tangle is a composition of above elementary planar tangles.

Notation II.3.8. *The n-click rotation is of an n-box x is called the contragredient of x, denoted by* $\overline{x}$.

Notation II.3.9. *We write a labeled 2-box as a crossing with the label located at the position of the* \$.

CHAPTER III

SKEIN THEORY

III.1 Global and local evaluations

Planar algebras provide a way to study the standard invariant of subfactors by skein theory. For a fixed generating set, the universal planar algebra consists of linear sums of labeled tangles labeled by elements in the generating set. The partition function is a homomorphism from the 0-box space to the ground field. Modulo the kernel of the partition function, the action of planar tangles is well defined on the quotient of the universal planar algebra. We expect to obtain a subfactor planar algebra if the partition function is positive semidefinite with respect to a convolution and the quotient is finite dimensional. Then the relations of the generators are given by the elements in the kernel of the partition function. We think of the partition function as a global evaluation. The spin model [Jon00] can be realized in this way. Another kind of example comes from groups.

Let us start with a group G presented by a finite generating set Gen and relations Rel. The group G has an outer action on the type II_1 (hyperfinite) factor $\mathscr{R}$. (In general, the action of G can be twisted by a 3-cocycle, e.g. [Jon80].) Then we obtain the diagonal subfactor $\mathscr{R} \subset \oplus_{g \in Gen} \mathscr{R}$, where the inclusion is $x \to \oplus_{g \in Gen} g(x)$. Let $\mathscr{P}^{Gen}$ be the planar algebra of the diagonal subfactor. Then the vector space $\mathscr{P}^{Gen}_{n,+}$ has a basis

$$\{g_1^{-1} g_2 g_3^{-1} g_4 \cdots g_{2n-1}^{-1} g_{2n} = 1 \mid g_i \in Gen, \ \forall 1 \leq i \leq 2n\}.$$

The convolution of the planar algebra is induced by reversing the alternating word r. The action of a planar tangle is induced by summing over all possible assignment of G to (oriented) strings of the tangle, see [BDG08] for details.

Remark III.1.1. Take $H(S) = < a,b,c,d \mid a^{-i} b a^i = c^{-i} d c^i, i \in S >$ where S is a recursively enumerable set which is not recursive. It is shown in [Hig61] that the word problem [Deh11] of the group G is insoluble. Thus we cannot determine the dimension of the finite dimensional vector space $\mathscr{P}^{\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}}_{n,+}$ for all n.

Let us answer a question asked by Jesse Peterson in the Vanderbilt Subfactor seminar. Is there a planar algebra which cannot be presented by a finite generating set? (It is easy to show that any finitely generated planar algebra is singly generated.)

Theorem III.1.2. *Let G be the Lamplighter group, $G = < a,t \mid (a t^n a t^{-n})^2, \ n \in \mathbb{Z} >$. Then the associated subfactor planar algebra $\mathscr{P}^{\{a^{\pm 1}, t^{\pm 1}\}}$ is not finitely generated.*

Proof. If $\mathscr{P}^{\{a^{\pm 1}, t^{\pm 1}\}}$ is finitely generated, then for each relation $r = (a t^n a t^{-n})^2$, the associated vector b_r is a finite sum of tangles labeled by these generators. Each n-box generator contributes relations of G as length $2n$ alternating words which are only finitely many. Thus G is finitely presented. However, it was proved in [Bau61] that the Lamplighter group G is not finitely presented, a contradiction. Therefore $G^{\{a,t,a^{-1},t^{-1}\}}$ is not finitely generated. □

In general, it is hard to find a positive semidefinite partition function for the universal planar algebra directly. Another strategy is to find a proper set of generators and relations for a planar algebra. In this case, we will encounter three fundamental problems:

(1) Evaluation: enough relations are required to reduce any n-box labeled tangle to a linear sum of finitely many fixed n-boxes and the 0-box space reduced to the ground field.

(2) Consistency: different processes of evaluating a closed diagram contribute the same value.

(3) Positivity: the partition function derived from the evaluation is positive semidefinite.

When the principal graph of a subfactor planar algebra is simple, we may find out proper generators and relations for a evaluation algorithm, e.g. [MPS10; Pet10]. The Consistency and Positivity can be guaranteed by the embedding theorem [JP11; MW10]. A great achievement is the construction of the extended Haagerup planar algebra [Big+12] where a powerful skein theory was discovered, namely Jellyfish algorithm. Furthermore, the lowest weight uncappable generators have a (2-string) Jellyfish relation if and only if the principal graph of a subfactor planar algebra is a spoke graph [BP14]. In general, the Jellyfish relation provides a universal skein theory for subfactor planar algebras with known Ocneanu 4-partite principal graph [Ocn88]. If the principal graph of a subfactor planar algebra is known, then we can apply Jellyfish algorithm and the embedding theorem to overcome the three fundamental problems. However, solving the Jellyfish relation is akin to solving the connection (or 6j-symbols) modulo the knowledge of Temperley-Lieb-Jones planar algebras which is impossible in general.

A planar algebra is determined by its generators and relations, once an evaluation is known. Thus we expect to classify planar algebras with a good skein theory. Most times, the least complex labeled planar tangles form a basis for planar algebras. Thus we also expect to classify planar algebras by the restriction of dimensions instead of relations apart from a non generic condition.

III.2 Skein theory from quantum groups

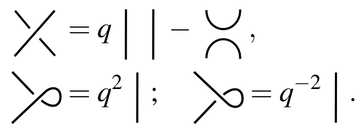
One kind of skein theory is motivated by the isotopy of links in the three dimensional space. We can construct planar algebras from the representation category of Drinfeld-Jimbo quantum groups [Dri86; Jim85]. The generator and relations of such a planar algebra are derived from a braid with type I, II, III Reidemeister moves. Precisely the braid is the universal R matrix and its type III Reidemeister move is the Yang-Baxter equation. The evaluation is known as the Jones polynomial [Jon85] for quantum $SU(2)$; the HOMFLYPT polynomial [Fre+85; PT88] for quantum $SU(N)$; the Kauffman polynomial [Kau90] for quantum $SO(N)$ and $Sp(N)$. These polynomials are invariants of links by identifying the braid in three dimensional space. They are also invariants of 3-manifolds, pointed out by Witten [Wit88] and constructed by Reshetikhin-Turaev [RT91].

Let V be the standard representation of a quantum group. The corresponding planar algebra consists of the intertwiners on the alternating tensor power of $V, \overline{V}$, where $\overline{V}$ is the contragredient of the representation V, and the Jones projection appears in $\hom(V \otimes \overline{V}, V \otimes \overline{V})$. The representation category of the quantum group consists of the intertwiners on the tensor power of V and the universal R matrix appears in $\hom(V \otimes V, V \otimes V)$.

For quantum $SU(2)$, we have $V = \overline{V}$ and the intertwiner space of $V \otimes V$ is two dimensional. Thus the identity, the Jones projection and the universal R matrix are linear dependent. The planar algebra is Temperley-Lieb-Jones which has no generators nor relations. Moreover, the universal R matrix is an unoriented braid  satisfying type II, III Reidemeister moves and the following relations,

the Jones relation:

Reidemeister moves I:

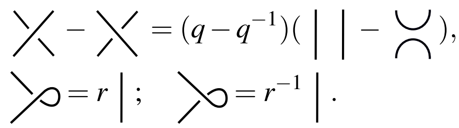


In this case, the statistical dimension is  $= q + q^{-1}$.

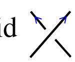
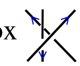
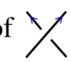
For quantum $SO(N)$ and $Sp(N)$, we have $V = \overline{V}$. Thus the universal R matrix is in the planar algebra. Moreover, the planar algebra is generated by the universal R matrix, called BMW planar algebras [BW89; Mur87]. The universal R matrix is an unoriented braid  satisfying type II, III Reidemeister moves and the following relations,

the BMW relation:

Reidemeister moves I:

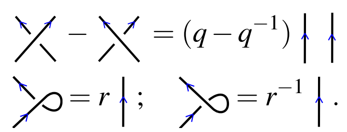


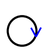
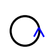
In this case, the statistical dimension is  $= \frac{r - r^{-1}}{q - q^{-1}} + 1$.

For quantum $SU(N)$, $N \geq 3$, we have $V \neq \overline{V}$. Thus the universal R matrix is an oriented braid  which is not in the planar algebra. The planar is generated by the 3-box  instead. The evaluation can be derived from the type II, III Reidemeister moves and the following relations of .

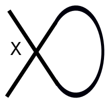
the Hecke relation:

Reidemeister moves I:

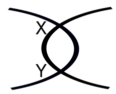


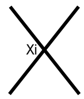
In this case, the statistical dimension is  $=$  $= \frac{r - r^{-1}}{q - q^{-1}}$.

Observe that the evaluation highly depends on the shape of diagrams in the relations, rather than the labels and the coefficients. Therefore we can modify relations by changing the coefficients, labels and adding terms with lower complexity to get a new set of relations which should be still enough for an evaluation. For example, we can modify the type I, II, III Reidemeister moves to the Yang-Baxter relation defined as follows.

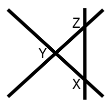
Definition III.2.1. Given a finite set of 2-box generators containing Temperley-Lieb-Jones 2-boxes, which is invariant under the 1-click rotation and the adjoint operation, a Yang-Baxter relation of the generators is a set of relations consisting of Reidemeister moves I, II, III. More precisely, for any generators X, Y, Z with the same shading, we have the following relations, Move I: 

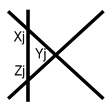
$= c$, for some constant c;

Move II: 

$= \sum_i c_i$ 

, for some generators X_i and constants c_i;

Move III: 

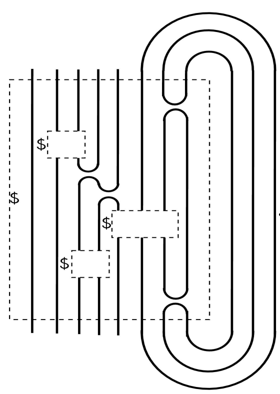
$= \sum_j c_j$ 

, for some generators X_j, Y_j, Z_j and constants c_j.

Remark III.2.2. There are two different kinds of Move I, II, III due to the shading.

Before showing the evaluation for a Yang-Baxter relation, let us define a standard multiplication form for planar tangles to describe the complexity.

Definition III.2.3. We draw the input and output discs of planar tangles as rectangles with the same number of boundary points on the top and the bottom, and the $ sign on the left. Take some annular tangles by adding through strings to the left and the right of the input (rectangle) disc, such that there are n boundary points on the top and n boundary points on the bottom. We will then take the multiplication of such tangles and n-box Tempeley-Lieb diagrams. Then add caps to the right. The final tangle is called a standard multiplication form, e.g.



.

Proposition III.2.4. *Any planar tangle is isotopic to a standard multiplication form by adding some closed circles.*

Proof. A planar tangle is isotopic to a standard multiplication form by the following process.

(1) Draw the output disc and input discs as rectangles with the same number of boundary points on the top and the bottom, and a $ sign on the left.

(2) Cut the tangle into pieces by pairs of ”horizontal” lines around input discs, such that the left and right side of the input discs are just through strings in each piece.

(3) Add circles on these lines to make sure all the lines pass through the same (large enough) number of points.

(4) Make up ”cups” on the right top and ”caps” on the right bottom to make sure the top/bottom boundary of the output disc also pass through the same number of points.

(5) Note that the numbers of ”cups” and ”caps” are the same. Add double caps on the right. Then these ”cups”, ”caps” and right caps form circles.

The final tangle is a standard multiplication form and it is isotopic to the original tangle with some extra closed circles. □

Theorem III.2.5. *The planar algebra generated by a finite set of 2-box generators with a Yang-Baxter relation is evaluable.*

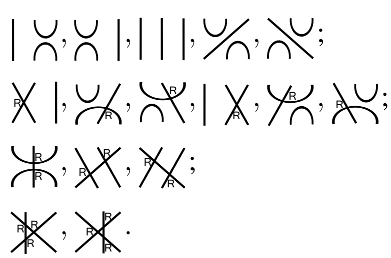
Proof. Note that any vector is a finite linear sum of labeled tangles. By Proposition III.2.4, we may assume that these tangles are standard multiplication forms. For each diagram, when we ignore the right caps and view the Temperley-Lieb-Jones 2-boxes as generators, it is a multiplication of shifts of the generators. Similar to the algebraic structure of Hecke algebra of type A, applying Reidemeister moves II and III, the multiplication part could be replaced by a linear sum of multiplications of shifts of generators with only one generator on the right most. If there is a cap on the right in the standard multiplication form, then it acts on the rightmost generator. By Reidemeister move I, the cap is reduced. Continuing this process, we reduce all the right caps. Therefore the vector is reduced to a linear sum of multiplications of shifts of generators. Consequently the planar algebra is evaluable. □

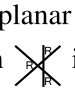
From the above proof, we have

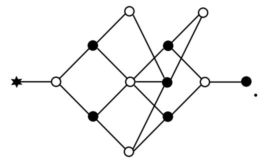
Proposition III.2.6. *The planar algebra generated by a finite set of 2-box generators with a Yang-Baxter relation is algebraically generated by 2-boxes.*

Definition III.2.7. A Yang-Baxter relation planar algebra is a subfactor planar algebra generated by 2-boxes with a Yang-Baxter relation.

From the proof of Theorem III.2.5, we see that a Yang-Baxter relation planar algebra generated by a non-Temperley-Lieb-Jones 2-box has 3 dimensional 2-boxes and at most 15 dimensional 3-boxes. Conversely for planar algebras generated a 2-box R, the complexity of 3-box labeled tangles can be considered as the number of labels. Due to this complexity, the first 16 least complex 3-box tangles can be listed as



If the 3-box space is at most 15 dimensional, then the above 16 diagrams are linear dependent which almost ensure the Yang-Baxter relation. More precisely, when the 3-box space is at most 14 dimensional, the Yang-Baxter relation holds. When the 3-box space is at most 15 dimensional, the Yang-Baxter relation holds for unshaded planar algebras, but not always for shaded planar algebras. One known example is the group subgroup subfactor planar algebra $S_2 \times S_3 \subset S_5$ in which  is not a linear sum of the 15 diagrams. The classification for the non-Yang-Baxter case is hard, because no skin theory is provided so far. The principal graph for $S_2 \times S_3 \subset S_5$ is as follows,



The above observation leads us to the classification of singly generated Yang-Baxter relation planar algebras. Note that BMW subfactor planar algebras are Yang-Baxter relation planar algebras generated by the universal R matrix. It was thought that all Yang-Baxter relation planar algebras are BMW. One hint is the following result [TW05], if a modular tensor category is generated by the braid $C_{X,X}$ for a self-dual object X, then it is BMW.

However, a surprising Yang-Baxter relation planar algebra appeared in the ongoing program of classifying small index subfactors [JMS14] and constructed in [LMP]. It is not BMW, because the generator is not self-contragredient. We shall expect that the extra Yang-Baxter relation planar algebra belongs to a one-parameterized family of planar algebras. This expectation is confirmed from the classification of singly generated Yang-Baxter relation planar algebras, but how to construct it?

We are going to derive the generator and relations of this q-parameterized family of planar algebras in Chapter IV and construct it by skein theory in Chapter V which overcome the three fundamental problems: Evaluation, Consistency, Positivity. When $q = e^{\frac{i\pi}{2N+2}}$, (the quotient of) the planar algebra forms a subfactor planar algebra, denoted by $\mathscr{E}_{N+2}$. Its principal graph is the sublattice of the Young lattice consisting of Young diagrams whose $(1,1)$ cell has hook length at most N. Moreover, we have a complete classification of singly generated Yang-Baxter relation planar algebra, i.e. Theorem I.0.1. (This result will eventually be published in [Liuc].)

Recall that the planar algebra for quantum $SU(N)$ is generated by a 3-box. The H-I relation for the 3-box generators [Thu] provides a direct evaluation for these planar algebras without the help of the HOMFLY relation of the universal R matrix.

III.3 Some other skein relations

Skein theory can also be considered from the view of graph theory. By Euler's formula, it is easy to show that a 3,4,6-valent planar graph has a face with at most 5,3,2 edges respectively. If we have enough relations to reduce these faces, then we will have an evaluation of closed diagrams. Some better algorithms are known for the three type of planar graphs.

(1) By the discharging method, any 3-valent planar graph has a pentagon adjacent to a hexagon or simpler subgraphs. Based on it, the classification of categories generated by a trivalent vertex is discussed in [MPS15].

(2) The Yang-Baxter relation provides an evaluation for 4-valent planar graphs. Singly generated Yang-Baxter relation planar algebras is classified in [BJ97b; BJ03; BJL; Liuc].

(3) The H-I relation for 3-boxes [Thu] provides an evaluation for 6-valent planar graphs. Subfactor planar algebras generated by a single 3-box with H-I relation is classified in [JLR].

For 4-valent planar graphs, some other skein relations naturally appeared in the classification project, such as exchange relations [Lan02], vanishing triangle relations [BJL]. Exchange relation planar algebras with 4 dimensional 2-boxes were classified in [Liub].

We hope to discover interesting skin theory from known subfactors and other areas. For example, the group subgroup subfactor planar algebra are realized inside the spin model. It has a global evaluation. It is an interesting question to find out a local evaluation based on proper generators and relations.

For the classification of subfactor planar algebras with a given skein theory, one need to solve the parameters in the relations by the consistency and positivity. The difficulty mostly appears in the evaluation of highly symmetric graphs. For example, the connection is related to the tetrahedron; the Yang-Baxter relation is related to the octahedron. The complexity will increase quite a lot if more generators are considered. For example, near group subfactor planar algebras have good skein relations, but it is a challenge to construct an infinite family.

CHAPTER IV

CLASSIFICATION

Suppose $\mathscr{P}_\bullet$ is a unital non-degenerate planar algebra generated by a 2-box with a Yang-Baxter relation, $\dim(\mathscr{P}_{2,\pm}) = 3$ and $\dim(\mathscr{P}_{2,\pm}) = 15$. Then $\delta \neq 0, \pm 1$, otherwise the 5 Temperley-Lieb-Jones 3-boxes are linear dependent and $\dim(\mathscr{P}_{2,\pm}) < 15$. Let $e = \frac{1}{\delta}$ , P, Q be the three minimal idempotents of $\mathscr{P}_{2,+}$. Let x, y be the solution of

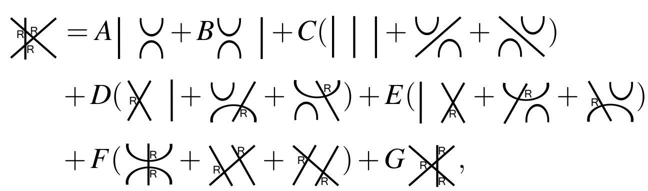
$$\begin{cases} xtr(P) + ytr(Q) = 0 \\ xy = -1 \end{cases}$$

Take $R = xP + yQ$. Then R is uncappable and $R^2 = aR + id - e$, where $a = x + y$. Note that R is determined up to a $\pm$ sign. By isotopy, we have $tr(\mathscr{F}(R)\mathscr{F}(R)^3) = tr(R^2)$. Note that $tr(R^2) = tr(id - e) = \delta^2 - 1 \neq 0$, so $\mathscr{F}(R)\mathscr{F}^3(R) = a'\mathscr{F}(R) + id - e$, for some $a' \in \mathbb{C}$. We will deal with the two cases for $\overline{R} = \pm R$.

IV.1 The generator is self-contragredient

When $R = \overline{R}$, we have $\mathscr{F}(R)^2 = a'\mathscr{F}(R) + id - e$. So $R * R = a'R + \delta e - \frac{1}{\delta} id$.

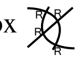
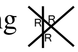
Lemma IV.1.1. *Suppose $\mathscr{P}_\bullet$ is a non-degenerate planar algebra generated by $R =$*  *$\in \mathscr{P}_{2,+}$ with a Yang-Baxter relation, such that $\dim(\mathscr{P}_{3,\pm}) = 15$, R is uncappable, $R = \overline{R}$, $R^2 = aR + id - e$, $\mathscr{F}(R)^2 = a'\mathscr{F}(R) + id - e$, and*

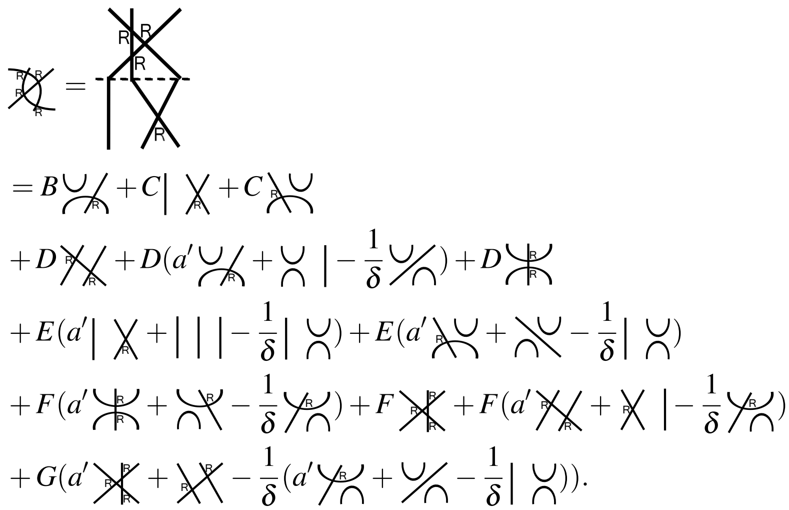


then

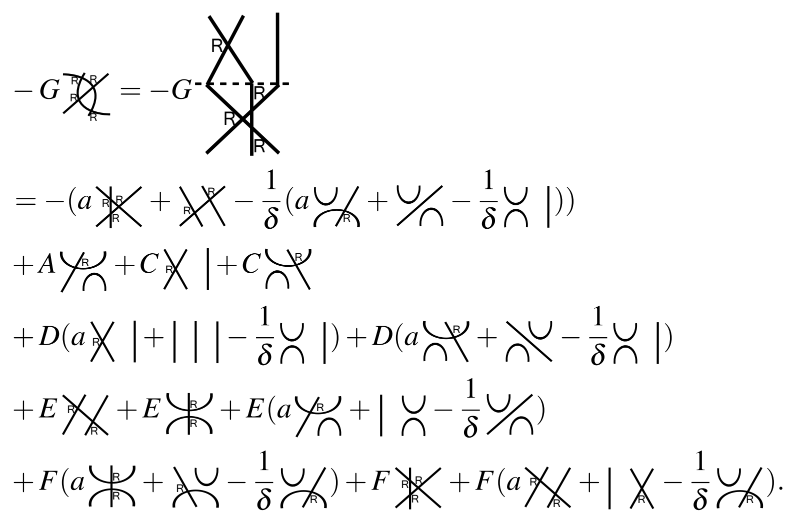
$$\begin{cases} G = \pm 1 \\ A = G\dfrac{a}{\delta} \\ B = -A \\ C = 0 \\ (G\delta^2 - 2\delta)D = 1 - Ga^2\delta \\ E = -GD \\ F = 0 \\ a' = Ga \end{cases}$$

Note that $\mathscr{P}_\bullet$ is a Yang-Baxter relation planar algebra implies $G \neq 0$.

Proof. There are two different ways to evaluate the 3-box  as a linear sum over the basis. Replacing  by  and lower terms, we have



Replacing  by  and lower terms, we have



Therefore

$$
\begin{aligned}
&(a-F) \\
&= (E-2GE\frac{1}{\delta}+G^2\frac{1}{\delta^2}) + (-\frac{1}{\delta^2}-2D\frac{1}{\delta}+GD) \\
&+ (D+GE)(+) + (\frac{1}{\delta}-E\frac{1}{\delta}-GD\frac{1}{\delta}-G^2\frac{1}{\delta}) \\
&+ (C+Da+GF)(+) + (a\frac{1}{\delta}-2F\frac{1}{\delta}+GB+GDa') \\
&+ (F+GC+GEa')(+) + (A+Ea-2GF\frac{1}{\delta}-G^2a'\frac{1}{\delta}) \\
&+ (E+Fa+GD+GFa')(+) + (-1+G^2) + (GF+G^2a').
\end{aligned}
$$

Comparing the coefficients of the basis, we have the following equations.

$$(a-F)G = GF + G^2a' \tag{IV.1}$$

$$(a-F)F = E+Fa+GD+GFa' = (-1+G^2) \tag{IV.2}$$

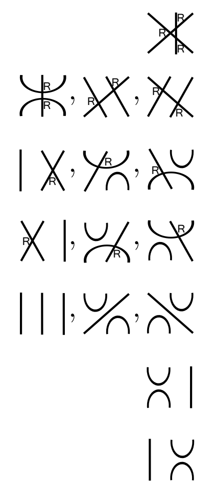
$$(a-F)E = F+GC+GEa' = A+Ea-2GF\frac{1}{\delta}-G^2a'\frac{1}{\delta} \tag{IV.3}$$

$$(a-F)D = C+Da+GF = a\frac{1}{\delta}-2F\frac{1}{\delta}+GB+GDa' \tag{IV.4}$$

$$(a-F)C = D+GE = \frac{1}{\delta}-E\frac{1}{\delta}-GD\frac{1}{\delta}-G^2\frac{1}{\delta} \tag{IV.5}$$

$$(a-F)B = -\frac{1}{\delta^2}-2D\frac{1}{\delta}+GD \tag{IV.6}$$

$$(a-F)A = (E-2GE\frac{1}{\delta}+G^2\frac{1}{\delta^2}) \tag{IV.7}$$



Case 1: If $F=0$, then equation (IV.2) implies

$$G^2 = 1, E+GD = 0.$$

By equation (IV.1), we have

$$a' = Ga.$$

Applying $F=0, a'=Ga$ to the first equality of equation (IV.3), we have

$$C = 0.$$

Applying $F=0, a'=Ga, G^2=1$ to the second equality of equation (IV.3), we have

$$A = \frac{Ga}{\delta}.$$

Applying $F=0, a'=Ga, G^2=1$ to the second equality of equation (IV.4), we have

$$B=-\frac{Ga}{\delta}.$$

Applying $B=-\frac{Ga}{\delta}$ to equation (IV.6), we have

$$(G\delta^2-2\delta)D=1-Ga^2\delta.$$

We have solved A,B,C,D,E,F,G in term of a and δ (and D).

Case 2: If $F\neq 0$, then equation (IV.2) implies

$$a=F+\frac{G^2-1}{F}.$$

Substituting a in equation (IV.1), we have

$$a'=\frac{\frac{G^2-1}{F}-F}{G}.$$

Substituting a,a' in the first equalities of equation (IV.2), (IV.3), (IV.4), we have

$$\begin{cases} GC & & -FE & = & -F \\ C & +(F+\frac{G^2-1}{F})D & & = & \frac{G^2-1}{F}-FG \\ & GD & +E & = & 1-G^2 \end{cases}$$

Let us consider F,G as constants and C,D,E as variables, then the determinant of the coefficient matrix on the left side is

$$\begin{vmatrix} G & 0 & -F \\ 1 & F+\frac{G^2-1}{F} & 0 \\ 0 & G & 1 \end{vmatrix} = \frac{G^2-1}{F}.$$

If $G^2-1\neq 0$, then we have the unique solution

$$\begin{cases} C & = & -F-FG \\ F & = & 1 \\ E & = & 1-G-G^2 \end{cases}$$

Plugging the solution into the second equality of equation (IV.5), we have

$$1+(1-G-G^2)G=\frac{1}{\delta}(1-(1-G-G^2)-G-G^2).$$

This implies $(1+G)^2(1-G)=0$. So $G=\pm 1$, and $G^2-1=0$, contradicting to the assumption.

If $G^2-1=0$, then $G=\pm 1$, $a=F$, $a'=-GF$, and

$$\begin{cases} GC & & -FE & = & -F \\ C & +FD & & = & -FG \\ & GD & +E & = & 0 \end{cases}$$

So

$$E=-GD, C=-F(G+D).$$

By equation $IV.6$, we have $(G\delta^2-2\delta)D=1$. So $G\delta^2-2\delta\neq 0$ and

$$D=\frac{1}{G\delta^2-2\delta}.$$

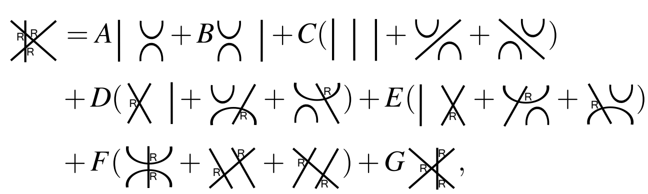
From the second equality of equation (IV.3), (IV.4), we have

$$A=B=(\frac{1}{\delta}+D)GF.$$

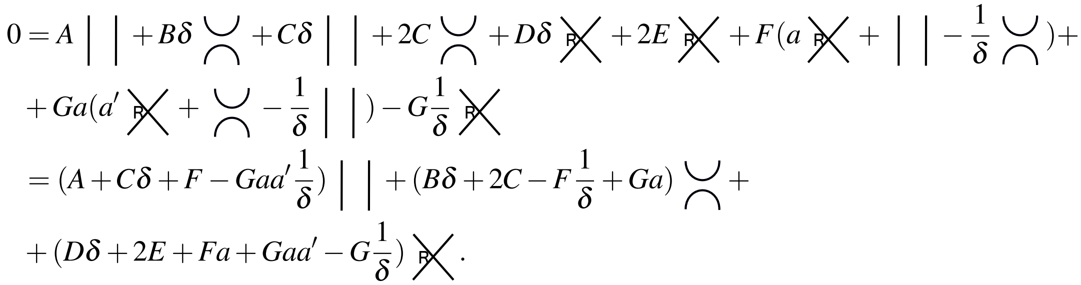
We have solved A,B,C,D,E,F,G in term of a and δ.

$$*\begin{cases} G=\pm 1 \\ A=B=Ga(\frac{1}{\delta}+D) \\ C=-a(G+D) \\ D=-GE=\frac{1}{G\delta^2-2\delta} \\ F=a \\ a'=-Ga \end{cases}$$

Adding a cap to the right of the following equation



we get



Therefore

$$\begin{aligned} 0 &= A + C\delta + F - Gaa'\frac{1}{\delta} \\ &= Ga(\frac{1}{\delta} + D) - a(G+D)\delta + a + a^2\frac{1}{\delta}. \qquad \text{by the above solution (*).} \end{aligned}$$

Recall that $a = F \neq 0$, so

$$a = -G(1+\delta D) + (G+D)\delta^2 - 1$$

If we replace the generator R by $-R$ and repeat the above arguments, then $a, \delta, A, B, C, D, E, F, G$ are replaced by $-a, \delta, -A, -B, -C, D, E, -F, G$. So we have

$$-a = -G(1+\delta D) + (G+D)\delta^2 - 1$$

Thus $a = 0$, contradicting to $a = F \neq 0$.

□

Corollary IV.1.2. *The planar algebra $\mathscr{P}_\bullet$ is unshaded.*

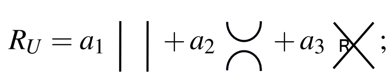
Proof. It is easy to check that $G\mathscr{F}(R)$ in $\mathscr{P}_{2,-}$ satisfies the same Yang-Baxter relation as R. Therefore $\mathscr{P}_\bullet$ is unshaded by identifying $G\mathscr{F}(R)$ as R. □

Note that BMW is a 2-parameter family of planar algebras generated by a self-contragredient braid satisfying type I, II, III Reidemester moves and the BMW relation. Let us solve such a braid with its relations in $\mathscr{P}_\bullet$. Then $\mathscr{P}_\bullet$ is BMW.

Let z_1, z_2 be the solution of

$$\begin{cases} z_1 + z_2 G = -a \\ z_1 z_2 G = -E \end{cases} \tag{IV.8}$$

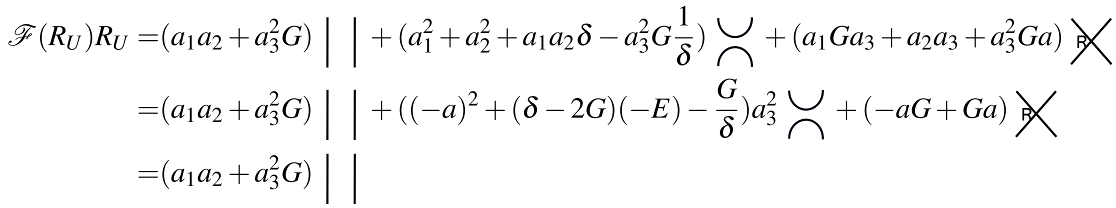
For $a_3 \neq 0$, take $a_1 = z_1 a_3$, $a_2 = z_2 a_3$;



Lemma IV.1.3 (bi-invertible)**.**

$$\mathscr{F}(R_U)R_U = G(1-E)a_3^2 \,\big|\;\big|\,.$$

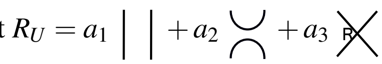
Proof. By Equation (IV.8), $E = -GD$ and $(G\delta^2 - 2\delta)D = 1 - Ga^2\delta$, we have



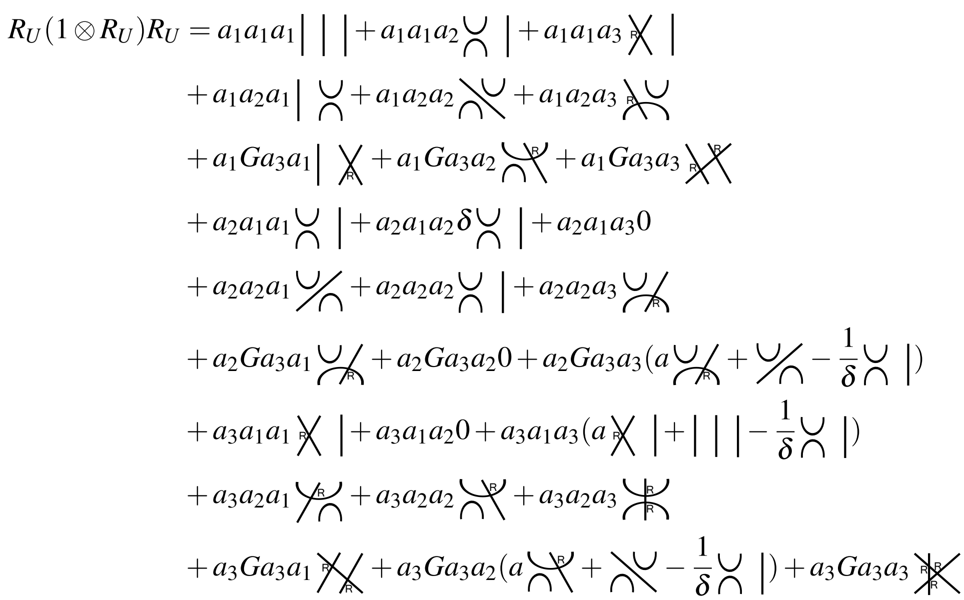
□

Lemma IV.1.4 (YBE)**.**

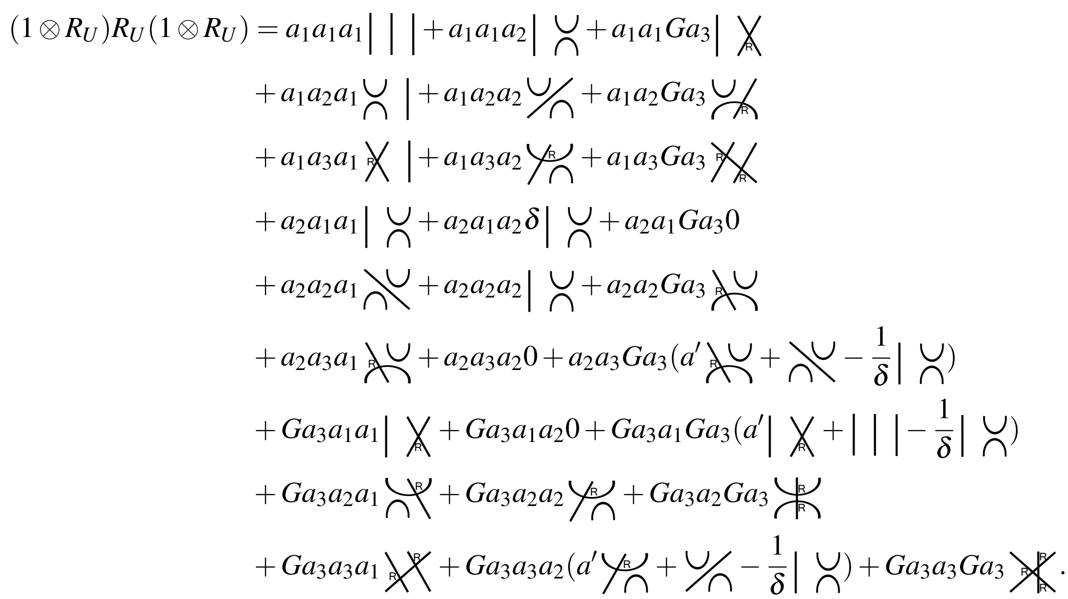
$$R_U(1 \otimes R_U)R_U = (1 \otimes R_U)R_U(1 \otimes R_U).$$

Proof. Recall that 

, $a_3 \neq 0$, we have



and



Replacing by and lower terms, then comparing the coefficients, we have

$$R_U(1\otimes R_U)R_U = (1\otimes R_U)R_U(1\otimes R_U) \iff$$

$$a_3Ga_3a_3G = Ga_3a_3Ga_3 \quad \text{(IV.9)}$$

$$a_3Ga_3a_3F + a_3Ga_3a_1 = a_1a_3Ga_3 \quad \text{(IV.10)}$$

$$a_3Ga_3a_3F + a_1Ga_3a_3 = Ga_3a_3a_1 \quad \text{(IV.11)}$$

$$a_3Ga_3a_3F + a_3a_2a_3 = Ga_3a_2Ga_3 \quad \text{(IV.12)}$$

$$a_3Ga_3a_3E + a_1a_2a_3 = a_2a_2Ga_3 + a_2a_3a_1 + a_2a_3Ga_3a' \quad \text{(IV.13)}$$

$$a_3Ga_3a_3E + a_3a_2a_1 = a_1a_3a_2 + Ga_3a_2a_2 + Ga_3a_3a_2a' \quad \text{(IV.14)}$$

$$a_3Ga_3a_3E + a_1Ga_3a_1 = a_1a_1Ga_3 + Ga_3a_1a_1 + Ga_3a_1Ga_3a' \quad \text{(IV.15)}$$

$$a_3Ga_3a_3D + a_1Ga_3a_2 + a_3a_2a_2 + a_3Ga_3a_2a = Ga_3a_2a_1 \quad \text{(IV.16)}$$

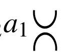
$$a_3Ga_3a_3D + a_2a_2a_3 + a_2Ga_3a_1 + a_2Ga_3a_3a = a_1a_2Ga_3 \quad \text{(IV.17)}$$

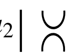
$$a_3Ga_3a_3D + a_1a_1a_3 + a_3a_1a_1 + a_3a_1a_3a = a_1a_3a_1 \quad \text{(IV.18)}$$

$$a_3Ga_3a_3C + a_1a_2a_2 + a_3Ga_3a_2 = a_2a_2a_1 + a_2a_3Ga_3 \quad \text{(IV.19)}$$

$$a_3Ga_3a_3C + a_2a_2a_1 + a_2Ga_3a_3 = a_1a_2a_2 + Ga_3a_3a_2 \quad \text{(IV.20)}$$

$$a_3Ga_3a_3C + a_1a_1a_1 + a_3a_1a_3 = a_1a_1a_1 + Ga_3a_1Ga_3 \quad \text{(IV.21)}$$

$$a_3Ga_3a_3B+a_1a_1a_2+a_2a_1a_1+a_2a_1a_2\delta+a_2a_2a_2-\frac{1}{\delta}a_2Ga_3a_3-\frac{1}{\delta}a_3a_1a_3-\frac{1}{\delta}a_3Ga_3a_2=a_1a_2a_1$$  (IV.22)

$$a_3Ga_3a_3A+a_1a_2a_1=a_1a_1a_2+a_2a_1a_1+a_2a_1a_2\delta+a_2a_2a_2-\frac{1}{\delta}a_2a_3Ga_3-\frac{1}{\delta}Ga_3a_1Ga_3-\frac{1}{\delta}Ga_3a_3a_2$$  (IV.23)

Note that $(IV.10) \iff (IV.11)$; $(IV.13) \iff (IV.14)$; $(IV.16) \iff (IV.17)$; $(IV.19) \iff (IV.20)$.

Equation (IV.9) always holds.

Since $F=0$, Equation (IV.10), (IV.12) hold.

By Equation (IV.8), z_1 and z_2G are solutions of $z^2+az-E=0$. Since $a_1/a_3=z_1$, $a_2/a_3=z_2$, we have

$$(a_2G)^2+aa_2a_3G-a_3^2E=0$$
$$a_1^2+aa_1a_3-a_3^2E=0$$

Moreover, $a'=Ga$, so Equation (IV.13), (IV.15) hold.

Since $E=-GD$, Equation (IV.16), (IV.18) follow from (IV.13), (IV.15).

Since $C=0$, Equation (IV.19), (IV.21) hold.

Note that

$$\begin{aligned}
&GB+z_1^2z_2+z_1z_2^2\delta+z_2^3-\frac{1}{\delta}z_2G-\frac{1}{\delta}z_1-\frac{1}{\delta}z_2G \\
=&-G\frac{a}{\delta}+z_1^2z_2+z_1z_2^2\delta+z_2^3-\frac{1}{\delta}z_2G-\frac{1}{\delta}z_1-\frac{1}{\delta}z_2G && (B=-A=-G\frac{a}{\delta}) \\
=&z_1^2z_2+z_1z_2^2\delta+z_2^3-\frac{1}{\delta}z_2G && \text{By Equation } (IV.8) \\
=&(-a)^2+(\delta-2G)(-E)-\frac{G}{\delta} && \text{By Equation } (IV.8) \\
=&0 && (E=-GD,\ (G\delta^2-2\delta)D=1-Ga^2\delta).
\end{aligned}$$

So Equation (IV.22) holds

Since $B=-A$, Equation (IV.23) follows from Equation (IV.22).

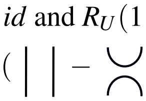
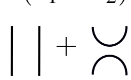
Therefore

$$R_U(1\otimes R_U)R_U=(1\otimes R_U)R_U(1\otimes R_U)$$

□

Theorem IV.1.5. *The Yang-Baxter relation in Lemma IV.1.1 is consistent. The planar algebra given by this generator and relation is BMW when $E\neq 1$; Bisch-Jones when $E=1$.*

Remark IV.1.6. The dimension of 3-boxes of Bisch-Jones planar algebras is at most 12.

Proof. When $E\neq 1$, let us take a_3 to be a square root of $\frac{1}{G(1-E)}$. Then $\mathscr{F}(R_U)R_U=id$ and $R_U(1\otimes R_U)R_U=(1\otimes R_U)R_U(1\otimes R_U)$. Moreover, when $G=1$, we have $R_U-\mathscr{F}(R_U)=(a_1-a_2)($  $)$, so $\mathscr{P}_\bullet$ is BMW from $O(N)$. When $G=-1$, we have $R_U+\mathscr{F}(R_U)=(a_1+a_2)($  $)$, so $\mathscr{P}_\bullet$ is BMW from $Sp(N)$. Consequently the Yang-Baxter relation for R is consistent.

When $E=1$, recall that $(G\delta^2-2\delta)D=1-Ga^2\delta$ and $E=-GD$, we have

$$\delta^2-(2+a^2)\delta G+1=0.$$

Recall that (at the beginning of Chapter V) $R=xP+yQ$ and

$$\begin{cases} x+y=a \\ xy=-1, \end{cases}$$

so

$$(\delta-x^2G)(\delta-y^2G)=0.$$

Without loss of generality, we assume that $y^2=G\delta$. Then $xG\delta=-y$. Note that

$$\begin{cases} xtr(P)+ytr(Q)=0 \\ tr(P)+tr(Q)=\delta^2-1, \end{cases} \tag{IV.24}$$

so

$$\begin{cases} tr(P)=G\delta-1 \\ tr(Q)=\delta^2-G\delta \end{cases}$$

Recall that z_1, z_2 are the solution of

$$\begin{cases} z_1+z_2G=-a \\ z_1z_2G=-1 \end{cases}.$$

Let us take $z_1=-x$, $z_2=-Gy$. Then

$$\begin{aligned} R_U =&(-x-Gy\delta)e+(-x+x)P+(-x+y)Q \\ =&(y-x)Q. \end{aligned}$$

Note that $R_U\neq 0$, so $y-x\neq 0$. By Lemma IV.1.3, we have

$$F(Q)Q=0 \tag{IV.25}$$

By Lemma IV.1.4, we have

$$Q(1\otimes Q)Q=(1\otimes Q)Q(1\otimes Q). \tag{IV.26}$$

Observe that the Yang-Baxter relation of Q is determined by Equation (IV.24), (IV.25), (IV.26). Moreover, the relation is the same as that of the 2-box $id\otimes(id-e)$ in the Bisch-Jones planar algebra with parameters (δ_a,δ_b), where $\delta_a=\delta_b$ is a square root of δ. Therefore $\mathscr{P}_\bullet$ is Bisch-Jones and $\dim(\mathscr{P}_3)\leq 12$. □

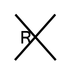
Recall that R is determined up to a $\pm$ sign. However, the coefficients D, E and G in the relation are independent of the choice of $\pm$. So they are invariants of the planar algebra. Moreover, the condition $E = 1$ distinguishes BMW and Bisch-Jones planar algebras. Furthermore, the value of $G = \pm 1$ distinguishes $O(N)$ and $Sp(N)$ for BMW; distinguishes the two unshaded Bisch-Jones planar algebras.

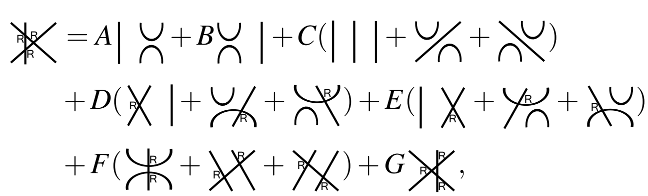
When $\delta \neq 2G$, we have $E = \dfrac{a^2\delta - 1}{G\delta^2 - 2\delta}$. Then the planar algebra $\mathscr{P}_\bullet$ is uniquely determined by a, δ, G. Note that a, δ are derived from the traces of the one 1-box and two 2-box minimal idempotents. Thus we can distinguish BMW and Bisch-Jones by the trace.

When $\delta = 2G$, we have $a^2 = \frac{1}{2}$. Up to the choice of $\pm R$, a is unique. In this case E is a free parameter. When $\delta = 2$, it is BMW for $r = q$. We cannot distinguish BMW and Bisch-Jones by δ and a in this case. The extended D subfactor planar algebra is both BMW and Bisch-Jones. The case $\delta = -2$ reduces to the case $\delta = 2$ by the following fact. For a planar algebra, we can switch the Jones idempotent to its negative, then the traces of odd boxes switch to its negative and the traces of even boxes do not change. In particular, we can change δ, a to $-\delta, a$.

IV.2 The generator is non-self-contragredient

When $R = -\overline{R}$, we have $R^2 = \overline{R^2} = a\overline{R} + id - e = -aR + id - e$. So $a = 0$ and $R^2 = id - e$. Similarly we have $a' = 0$ and $\mathscr{F}(R)^2 = -id + e$. So $R * R = -\delta e + \dfrac{1}{\delta} id$.

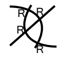
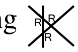
Lemma IV.2.1. *Suppose $\mathscr{P}_\bullet$ is a non-degenerate planar algebra generated by $R =$*  *in $\mathscr{P}_{2,+}$ with a Yang-Baxter relation, such that $\dim(\mathscr{P}_{3,\pm}) = 15$, R is uncappable, $R = -\overline{R}$ $R^2 = id - e$, $\mathscr{F}(R)^2 = -id + e$, and*

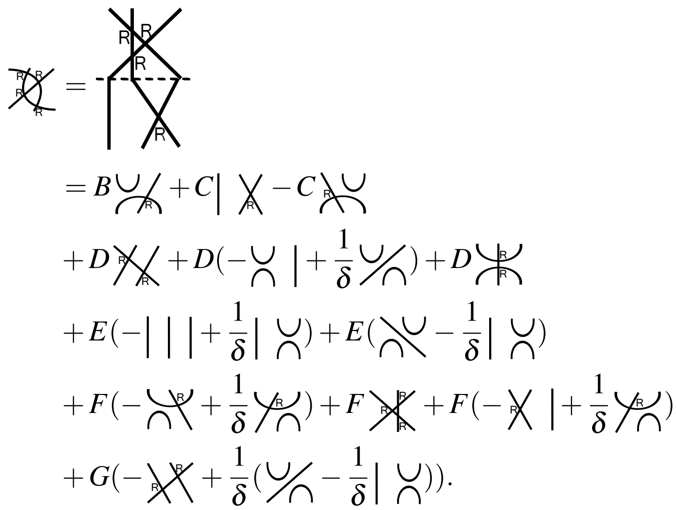


then

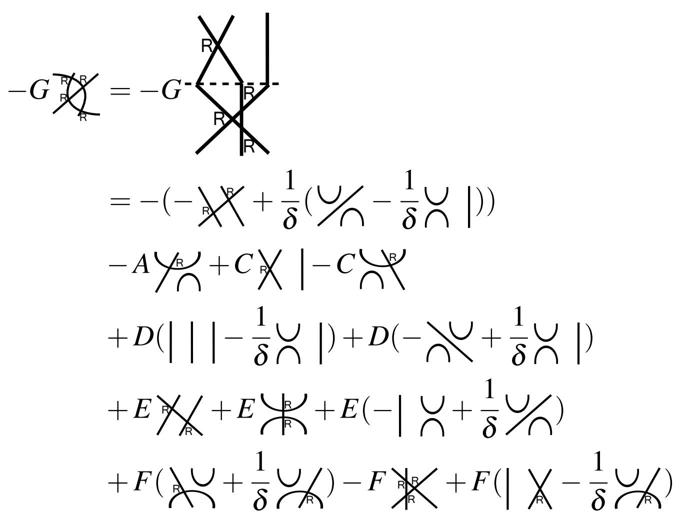
$$\begin{cases} G = \pm i \\ A = 0 \\ B = 0 \\ C = 0 \\ D = -\dfrac{1}{G\delta^2} \\ E = -\dfrac{1}{\delta^2} \\ F = 0 \end{cases}$$

Up to the complex conjugate, we only need to consider the case for $G = i$.

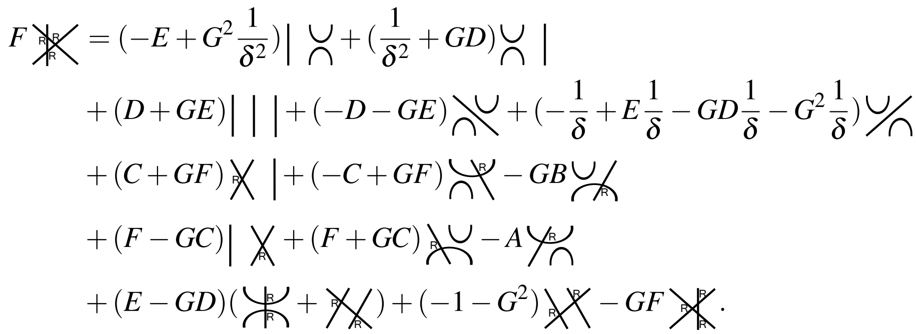
Proof. There are two different ways to evaluate the 3-box  as a linear sum over the basis. Replacing  by  and lower terms, we have



Replacing  by  and lower terms, we have



Therefore



Comparing the coefficients of , we have

$$FG = -GF.$$

Note that $\mathscr{P}_\bullet$ is a Yang-Baxter relation planar algebra, so $G \neq 0$. Then

$$F = 0.$$

Comparing the coefficients of other diagrams, we have

$$G^2 = -1, A = 0, B = 0, C = 0, D = -\frac{1}{G\delta^2}, E = -\frac{1}{\delta^2}.$$

Then $G = \pm i$. $\square$

Corollary IV.2.2. *The planar algebra $\mathscr{P}_\bullet$ is unshaded.*

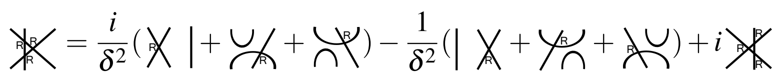
Proof. It is easy to check that $G\mathscr{F}(R)$ in $\mathscr{P}_{2,-}$ satisfies the same Yang-Baxter relation as R. Therefore $\mathscr{P}_\bullet$ is unshaded by identifying $G\mathscr{F}(R)$ as R. $\square$

CHAPTER V

CONSTRUCTION

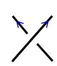
In this section, we are going to construct the one parameter family of unshaded planar algebras whose generator and relations are given in Lemma IV.2.1 (for $G = i$).

Definition V.0.3. Let us define $\mathscr{P}_\bullet$ to be the unshaded planar algebra generated by a 2-box $R =$  with the following Yang-Baxter relation: $\mathscr{F}(R) = -iR$; R is uncappable; $R^2 = id - e$; and



First let us recall some basic results of Hecke algebras and the HOMFLYPT polynomial. Then we solve the Yang-Baxter equation whose solution generates a HOMFLY subcategory. Based on the knowledge of the HOMFLYPT polynomial, we prove the consistency by an oriented version of Kauffman's arguments for Kauffman polynomial [Kau90]. With the help of the matrix units of Hecke algebra of type A, we construct the matrix units of $\mathscr{P}_\bullet$; compute the trace formula via the q-Murphy operator; prove the positivity at certain roots of unity. Then we obtain a sequence of subfactor planar algebras $\mathscr{E}_N$ and complete the classification, i.e. Theorem I.0.1. Furthermore, we prove some properties of this planar algebra and derive some other planar algebras and fusion categories. One family of them is an extension of the near group subfactor planar algebra for $\mathbb{Z}_4$. Another two families of them can be thought as the representation category of the subgroup $E_{N\pm2}$ of quantum $SU(N)$.

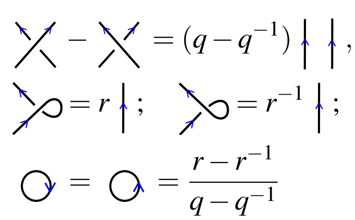
V.1 Hecke algebra of type A and HOMFLYPT polynomial

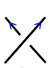
The HOMFLYPT polynomial is a knot invariant given by a braid  satisfying Reidemeister moves I, II, III and the Hecke relation

the Hecke relation:

Reidemeister moves I:

statistical dimension:



Let σ_i, $i \geq 1$, be the diagram by adding $i-1$ oriented (from bottom to top) through strings on the left of . The Hecke algebra of type A is a (unital) filtered algebra $H_\bullet$. The algebra H_n is generated by σ_i, $1 \leq i \leq n-1$ and H_n is identified as a subalgebra of H_{n+1} by adding an oriented through string on the right. Over the field $\mathbb{C}(r,q)$, rational functions over r and q, the matrix units of $H_\bullet$ were constructed in [Yok97; AM98]. A skein theoretic proof of the trace formula via the q-Murphy operator was given in [Ais97].

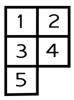
For reader's convenience, let us sketch the construction of the matrix units in [Yok97] with slightly different notations. The (l-box) symmetrizer $f^{(l)}$ and antisymmetrizer $g^{(l)}$, for $l \geq 1$, are constructed inductively as follows,

$$f^{(l)} = f^{(l-1)} - \frac{[l-1]}{[l]} f^{(l-1)}(q - \sigma_i) f^{(l-1)}; \tag{V.1}$$

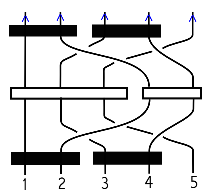
$$g^{(l)} = g^{(l-1)} - \frac{[l-1]}{[l]} g^{(l-1)}(q^{-1} + \sigma_i) g^{(l-1)}, \tag{V.2}$$

where $f^{(1)} = g^{(1)} = 1$.

Given a Young diagram λ, we can construct an idempotent by inserting the symmetrizers in each row on the top and the bottom and the antisymmetrizers in each column in the middle as follows. For example,

$\lambda =$ 

, take

$\dot{y}_\lambda =$ 

,

where the black boxes and white boxes indicate symmetrizers and antisymmetrizers respectively. Then $\dot{y}_\lambda^2 = m_\lambda \dot{y}_\lambda$. The coefficent m_λ was computed in Proposition 2.2 in [Yok97]. Over $\mathbb{C}(q,r)$, m_λ is non-zero. We can renormalize $\dot{y}_\lambda$ to y_λ by $y_\lambda = \frac{1}{m_\lambda}\dot{y}_\lambda$. Then y_λ is an idempotent. Moreover, $\{y_\lambda \mid |\lambda| = n\}$ are inequivalent minimal idempotents in $\mathscr{H}_n$.

For $\lambda > \mu$, the morphisms $\dot{\rho}_{\mu<\lambda}$ from $y_\mu \otimes 1$ to y_λ and $\dot{\rho}_{\lambda>\rho}$ from y_λ to $y_\mu \otimes 1$ were constructed in Lemma 2.10 in [Yok97]. Moreover, $(\dot{\rho}_{\mu<\lambda}\dot{\rho}_{\lambda>\rho})^2 = m_{[\mu|\lambda|\mu]}\dot{\rho}_{\mu<\lambda}\dot{\rho}_{\lambda>\rho}$ and the coefficient $m_{[\mu|\lambda|\mu]}$ was also computed there. Over $\mathbb{C}(q,r)$, $m_{[\mu|\lambda|\mu]}$ is non-zero. We renormalize $\dot{\rho}_{\mu<\lambda}$ and $\dot{\rho}_{\lambda>\rho}$ by $\rho_{\mu<\lambda} = \frac{1}{m_{[\mu|\lambda|\mu]}}\dot{\rho}_{\mu<\lambda}$ and $\rho_{\lambda>\rho} = \dot{\rho}_{\lambda>\rho}$. Then $\rho_{\mu<\lambda}\rho_{\lambda>\rho}$ is an idempotent and $\rho_{\lambda>\rho}\rho_{\mu<\lambda} = y_\lambda$. The branching formula is proved in Proposition 2.11 in [Yok97],

$$y_\mu \otimes 1 = \sum_{\lambda>\mu} \rho_{\mu<\lambda}\rho_{\lambda>\mu}. \tag{V.3}$$

Therefore the Bratteli diagram of $H_\bullet$ over $\mathbb{C}_{q,r}$ is the Young lattice, denoted by YL.

For each length n path t in YL from $\emptyset$ to λ, $|\lambda| = n$, $n \geq 1$, i.e., a standard tableau t of the Young diagram λ, take t' to be the first length $(n-1)$ path of t from $\emptyset$ to μ. There are two elements P_t^+, P_t^- in H_n defined by the following inductive process,

$$P_\emptyset^\pm = \emptyset,$$
$$P_t^+ = (P_{t'}^+ \otimes 1)\rho_{\mu<\lambda},$$
$$P_t^- = \rho_{\lambda>\mu}(P_{t'}^- \otimes 1).$$

The matrix units of H_n are given by $P_t^+ P_\tau^-$, for all Young diagrams λ, $|\lambda| = n$, and all pairs of length n paths (t, τ) in YL from $\emptyset$ to λ. Moreover, the multiplication of these matrix units coincides with the multiplication of loops, i.e.,

$$P_t^+ P_\tau^- P_s^+ P_\sigma^- = \delta_{\tau s} P_t^+ P_\sigma^-,$$

where $\delta_{\tau s}$ is the Kronecker delta.

Furthermore, when $|q| = |r| = 1$, $H_\bullet$ admits a convolution, denoted by $*$, which is a complex conjugate anti-isomorphism mapping to , (q to q^{-1} and r^{-1} to r^{-1}) over the field $\mathbb{C}$. The symmetrizer $f^{(l)}$ and antisymmetrizer $g^{(l)}$ can be constructed by Equation (V.1) and (V.2) inductively whenever $[l] \neq 0$. Note that $[l]^* = [l]$. By the Hecke relation of , we have $(q - \sigma_i)^* = q - \sigma_i$. So $(f^{(l)})^* = f^{(l)}$ and $(g^{(l)})^* = g^{(l)}$ by the inductive construction. Then y_λ can be constructed if the required symmetrizers and antisymmetrizers are well-defined and $m_\lambda \neq 0$. For $\lambda > \mu$, $\dot{\rho}_{\lambda > \rho}$ and $\dot{\rho}_{\mu < \lambda}$ can be constructed if y_λ and y_μ are well-defined. If $m_{[\mu|\lambda|\mu]} > 0$, then we have a (different) renormalization $\rho'_{\mu<\lambda} = \sqrt{\frac{1}{m_{[\mu|\lambda|\mu]}}} \dot{\rho}_{\mu<\lambda}$ and $\rho'_{\lambda>\rho} = \sqrt{\frac{1}{m_{[\mu|\lambda|\mu]}}} \dot{\rho}_{\lambda>\rho}$. By this renormalization (which is permitted over $\mathbb{C}$, but not over $\mathbb{C}(q,r)$), we have $(\rho'_{\mu<\lambda})^* = \rho'_{\lambda>\rho}$. Similarly we can define the matrix unit $P_t^+ P_\tau^-$ for a loop $t\tau^{-1}$ when the morphisms along the paths t and τ are defined. Then $(P_t^+ P_\tau^-)^* = P_\tau^+ P_t^-$.

We will consider $q = e^{\frac{i\pi}{2N+2}}, r = q^N$. For all Young diagrams whose (1,1) cell has hook length at most $N+1$, it is easy to check that all the corresponding coefficients $[l]$, m_λ, $m_{[\mu|\lambda|\mu]}$ are positive. So all the minimal idempotents y_λ and morphisms $\rho_{\mu<\lambda}$, $\rho_{\lambda>\rho}$ are well defined. We will use these matrix units to construct the matrix units of a q-parameterized planar algebra for $q = e^{\frac{i\pi}{2N+2}}$ in Section V.6. Then we obtain a sequence of subfactor planar algebras which completes our classification.

V.2 Solutions of the Yang-Baxter equation

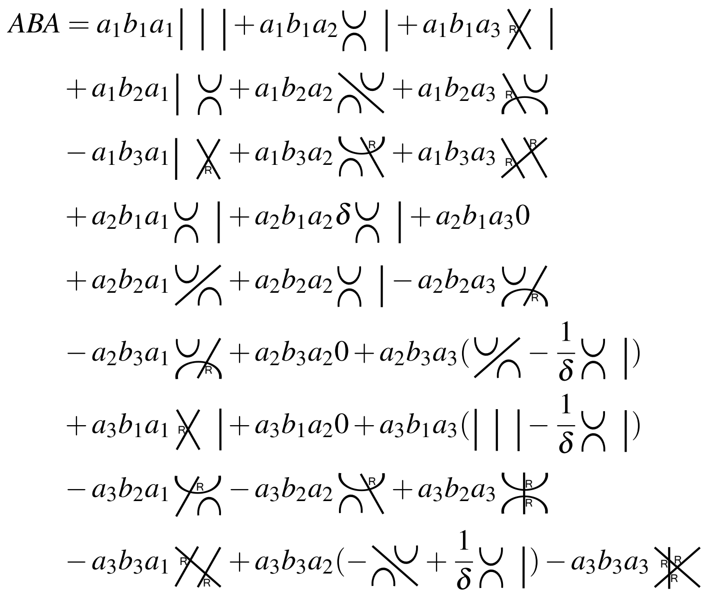
Lemma V.2.1. *Take* $\tilde{A} \in \mathscr{P}_2, \tilde{B} \in \mathscr{P}_2$,

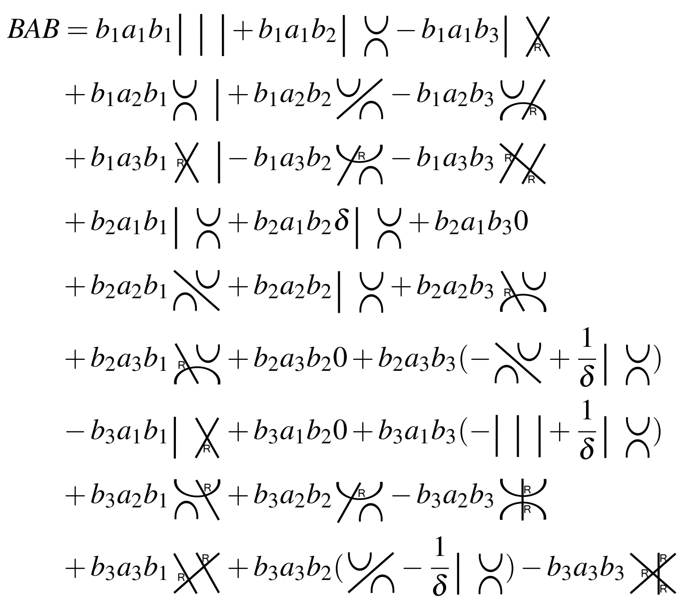
$$\tilde{A} = a_1 \;|\;| + a_2 \asymp + a_3 \times, a_3 \neq 0; \tilde{B} = b_1 \;|\;| + b_2 \asymp + b_3 \mathscr{F}(\times), b_3 \neq 0.$$

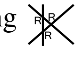
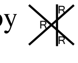
Let A and B be the 3-boxes by adding one string to the right of $\tilde{A}$ *and to the left of* $\tilde{B}$ *respectively. If* $\dim(\mathscr{P}_3) = 15$*, then* $ABA = BAB$ *if and only if*

$$a_1 = b_1, a_2 = b_2, b_3 = ia_3, a_1^2 = -\frac{a_3^2}{\delta^2}, a_2^2 = \frac{a_3^2}{\delta^2}.$$

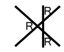
Proof.

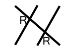


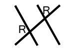


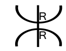
If $\dim(\mathscr{P}_3) = 15$, then the 15 diagrams excluding  forms a basis. Replacing  by  and lower terms and comparing the coefficients of the basis, we have

$$ABA = BAB \iff$$

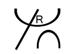
$$a_3b_3a_3i = b_3a_3b_3 \tag{V.4}$$


$$a_3b_3a_1 = b_1a_3b_3 \tag{V.5}$$


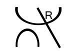
$$a_1b_3a_3 = b_3a_3b_1 \tag{V.6}$$


$$-a_3b_2a_3 = b_3a_2b_3 \tag{V.7}$$


$$a_3b_3a_3\frac{1}{\delta^2} + a_1b_2a_3 = b_2a_2b_3 + b_2a_3b_1 \tag{V.8}$$


$$a_3b_3a_3\frac{1}{\delta^2} - a_3b_2a_1 = -b_1a_3b_2 + b_3a_2b_2 \tag{V.9}$$


$$a_3b_3a_3\frac{1}{\delta^2} - a_1b_3a_1 = -b_1a_1b_3 - b_3a_1b_1 \tag{V.10}$$


$$a_3b_3a_3\frac{-i}{\delta^2} + a_1b_3a_2 - a_3b_2a_2 = b_3a_2b_1 \tag{V.11}$$


$$a_3b_3a_3\frac{-i}{\delta^2} - a_2b_2a_3 - a_2b_3a_1 = -b_1a_2b_3 \tag{V.12}$$


$$a_3b_3a_3\frac{-i}{\delta^2} + a_1b_1a_3 + a_3b_1a_1 = b_1a_3b_1 \tag{V.13}$$


$$a_1b_2a_2 - a_3b_3a_2 = b_2a_2b_1 - b_2a_3b_3 \tag{V.14}$$


$$a_2b_2a_1 + a_2b_3a_3 = b_1a_2b_2 + b_3a_3b_2 \tag{V.15}$$


$$a_1b_1a_1 + a_3b_1a_3 = b_1a_1b_1 - b_3a_1b_3 \tag{V.16}$$


$$a_1b_1a_2 + a_2b_1a_1 + a_2b_1a_2\delta + a_2b_2a_2 - \frac{1}{\delta}a_2b_3a_3 - \frac{1}{\delta}a_3b_1a_3 + \frac{1}{\delta}a_3b_3a_2 = b_1a_2b_1 \tag{V.17}$$


$$a_1b_2a_1 = b_1a_1b_2 + b_2a_1b_1 + b_2a_1b_2\delta + b_2a_2b_2 + \frac{1}{\delta}b_2a_3b_3 + \frac{1}{\delta}b_3a_1b_3 - \frac{1}{\delta}b_3a_3b_2 \tag{V.18}$$


Note that $a_3 \neq 0$, $b_3 \neq 0$, by equation (V.4), (V.5), (V.7), we have

$$b_3 = ia_3, a_1 = b_1, a_2 = b_2.$$

Then by equation (V.8), (V.10), we have

$$a_2^2 = \frac{a_3^2}{\delta^2}, a_1^2 = -\frac{a_3^2}{\delta^2}.$$

It is easy to check that the rest of the equations hold under these conditions. □

V.3 Consistency

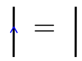
We are going to show the Yang-Baxter relation of $\mathscr{P}_\bullet$ is consistent. The idea is similar to the proof of the consistency of the Kauffman polynomial [Kau90]. Note that the Yang-Baxter relation is evaluable. To show the consistency, it is enough to find a partition function of the universal planar algebra generated by a 2-box R, such that the Yang-Baxter relation is in the kernel of the partition function. However, the consistency becomes more complicated, since the braid is oriented. Worse still, the braid and the Jones projection cannot be interpreted as diagrams simultaneously due to the orientation. Fortunately, we can simplify the argument by the knowledge of the HOMFLYPT polynomial.

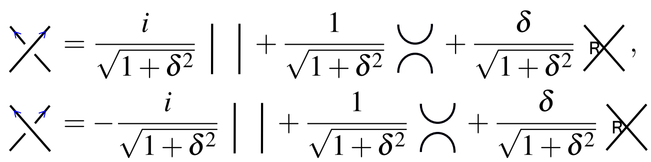
Definition V.3.1. Let $\mathscr{P}'_\bullet$ be the universal planar algebra generated by a single 2-box R.

Definition V.3.2. Let $Ann_i^j(n)$ be the set of annular tangles labeled by n copies of R from $\mathscr{P}'_i$ to $\mathscr{P}'_j$.

Definition V.3.3. Let $\mathscr{P}''_\bullet$ be the planar algebra generated by a single 2-box R such that

$$\bigcirc = \delta, \quad \mathscr{F}(R) = -iR.$$

Definition V.3.4. Let us define ,

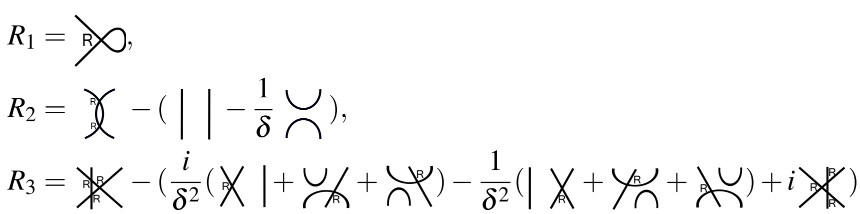


(V.19)

(V.20)

Notation V.3.5. *Take* $\mathscr{D} = \frac{\delta}{\sqrt{1+\delta^2}}$, $r = \frac{\delta i + 1}{\sqrt{1+\delta^2}}$, $q = \frac{i+\delta}{\sqrt{1+\delta^2}}$, *we have* $|r| = |q| = 1$.

Definition V.3.6. Let us define



then $\mathscr{F}(R_3) = -R_3$ in $\mathscr{P}''_\bullet$.

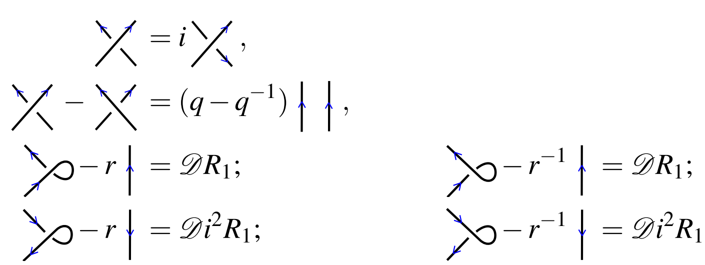
Definition V.3.7. Let us define $\mathscr{P}'''_\bullet = \mathscr{P}''_\bullet/\{R_1\}$, $\mathscr{P}''''_\bullet = \mathscr{P}'''_\bullet/\{R_2\}$. Then $\mathscr{P}_\bullet = \mathscr{P}''''_\bullet/\{R_3\}$.

Lemma V.3.8. *The following relations hold in* $\mathscr{P}''_\bullet$*:*

the Fourier relation:

the Hecke relation:

Reidemeister moves I:



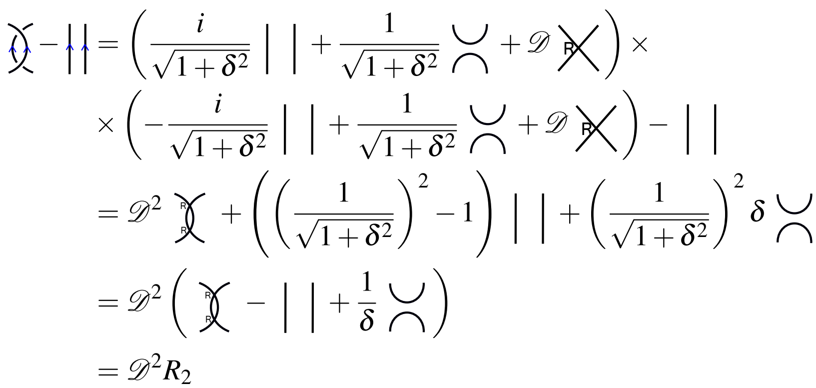
Proof. Follows from the definitions. □

Lemma V.3.9. *The following relations hold in $\mathscr{P}_\bullet'''$:*

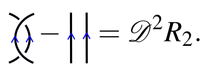
Reidemeister moves II:

The other four Reidemeister moves II can be obtained by a 2-click rotation.

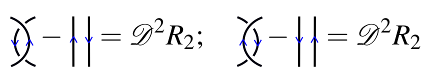
Proof.



Taking the complex conjugate of the above equation, we have



Applying the Fourier relation in Lemma V.3.8, we have



□

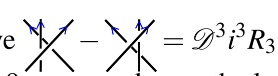
Lemma V.3.10. *The following relations hold in $\mathscr{P}_\bullet'''$:*

Reidemeister moves III:

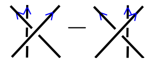
The other 10 Reidemeister moves III with different layers of strings also hold.

Note that $\mathscr{F}(R_3) = -R_3$, the other Reidemeister moves III with different orientations can be derived by applying rotations.

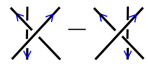
Remark V.3.11. There are 8 different orientations of the three strings, but only 2 up to rotations. For each orientation, there are 8 choices of the three braids, but only 6 of them admit a Reidemeister move III. So we have 48 Reidemeister moves III in total.

Proof. By the computation in Lemma V.2.1, we have  By the Hecke relation in Lemma V.3.8 and the Reidemeister moves II in Lemma V.3.9, we can change the layer of strings and obtain the other

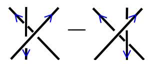
5 Reidemeister moves III with the same boundary orientation, such as

 $= \mathscr{D}^3 i^3 R_3.$

Applying the Fourier relation in Lemma V.3.8, we can switch the orientation of the string at the bottom of a Reidemeister moves III, such as

 $= \mathscr{D}^3 i^3 R_3.$

Once again applying the Hecke relation in Lemma V.3.8 and the Reidemeister moves II in Lemma V.3.9, we obtain the other 5 Reidemeister moves III with the same boundary orientation but different layers of strings, such as

 $= \mathscr{D}^3 i^3 R_3.$

The other Reidemeister moves III with different orientations can be derived by applying rotations. □

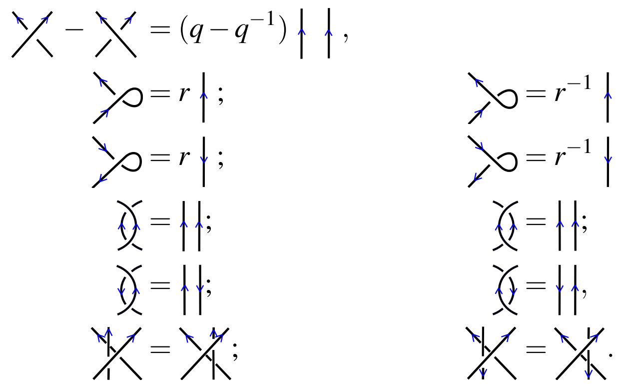
Proposition V.3.12. *The following relations hold in $\mathscr{P}_\bullet$.*

the Hecke relation:

Reidemeister moves I:

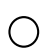
Reidemeister moves II:

Reidemeister moves III:

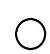


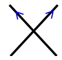
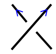
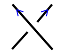
Other Reidemeister moves II, III with different layers and orientations of strings also hold.

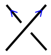
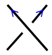
Proof. Follow from Lemma V.3.8, V.3.9, V.3.10. □

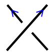
Our purpose is to construct a partition function of $\mathscr{P}'_\bullet$, such that it is well defined on the quotient $\mathscr{P}_\bullet$. By Proposition V.3.8, the restriction of the partition function on link diagrams in $\mathscr{P}'_\bullet$ has to be the HOMFLYPT polynomial. Due to the relations  $= \delta, \mathscr{F}(R) = -iR$ and linearity, the partition function is uniquely determined by these values. Motivated by this observation, we can define the partition function inductively. By linearity, we only need to define the partition function on closed diagrams labeled by R.

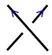
Now let us construct a partition function ζ of $\mathscr{P}'_\bullet$.

Set up ζ on closed Templey-Lieb digrams to be the evaluation map with respect to the relation  $= \delta$. Suppose ζ is defined on any closed diagram with at most $n-1$ copies of R, for $n = 1, 2, \cdots$. Let us define $\zeta(T)$ for a closed diagram T with n copies of R by the following process.

Considering R in the diagram T as , a crossing with a label R indicating the position of \$. Then T consists of k immersed circles intersecting at R's. Let $\pm(T)$ be the set of 2^k choices of orientations of the k circles. For an orientation $\sigma \in \pm(T)$, let T_σ be the corresponding oriented diagram. Let $\pm(\sigma)$ be the set of 2^n choices of replacing the n copies of the oriented crossing  of $T(\sigma)$ by a braid  or . For a choice $\gamma \in \pm(\sigma)$, we obtain an oriented link $T_{\sigma,\gamma}$ by replacing the crossings.

Substituting  and  of $T_{\sigma,\gamma}$ by Equation (V.19) and (V.20), i.e.,

 $= \frac{i}{\sqrt{1+\delta^2}}$ $+ \frac{1}{\sqrt{1+\delta^2}}$  $+ \mathscr{D}$ ;

 $= -\frac{i}{\sqrt{1+\delta^2}}$ $+ \frac{1}{\sqrt{1+\delta^2}}$  $+ \mathscr{D}$ ,

we have a decomposition of $T_{\sigma,\gamma}$ as

$$T_{\sigma,\gamma} = \sum_{j=1}^{3^n} T_{\sigma,\gamma}(j),$$

such that each $T_{\sigma,\gamma}(j)$, $2 \leq j \leq 3^n$, is a scalar multiple of a diagram with at most $n-1$ copies of R, and $T_{\sigma,\gamma}(1)$ is $\mathscr{D}^n$ times a diagram with n copies of R. Moreover, we can apply the Fourier transform to the n copies of R of this diagram W_σ times in total, such that this diagram becomes T. Note that W_σ mod 4 only depends on σ.

Recall that $Z(T_{\sigma,\gamma}(j))$, for $2 \leq j \leq 3^n$, are defined by induction. Let us define $\zeta_{\sigma,\gamma}(T)$ by the following equality,

$$\text{HOMFLY}_{q,r}(T_{\sigma,\gamma}) = \mathscr{D}^n i^{W_\sigma} \zeta_{\sigma,\gamma}(T) + \sum_{j=2}^{3^n} \zeta(T_{\sigma,\gamma}(j)). \tag{V.21}$$

Let us define $\zeta(T)$ as

$$\zeta(T) = \frac{1}{2^n 2^k} \sum_{\sigma \in \pm(T)} \sum_{\gamma \in \pm(\sigma)} \zeta_{\sigma,\gamma}(T). \tag{V.22}$$

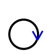
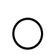
By induction and a linear extension, we obtain a function ζ on $\mathscr{P}'_0$.

Lemma V.3.13. *The function ζ is a partition function of $\mathscr{P}'_\bullet$. Consequently*  $-\delta \in Ker(\zeta)$*, the kernel of* ζ.

Proof. Let T be a disjoint union of two closed diagram T^1 and T^2.

Case 1: T^1 and T^2 are Temperley-Lieb. Obviously $\zeta(T) = \zeta(T^1)\zeta(T^2)$.

Case 2: T^1 (or T^2) is Templey-Lieb. Note that

HOMFLY$_{q,r}$() = HOMFLY$_{q,r}$() $= \frac{r-r^{-1}}{q-q^{-1}} = \delta = \zeta($  $)$,

so HOMFLY$_{q,r}$ coincide with ζ on closed Temperley-Lieb-Jones diagrams. By an induction on the number of R's in T_2, it is easy to show that $\zeta(T) = \zeta(T^1)\zeta(T^2)$.

The general case: Note that the choices of orientations and braids in the definition of ζ are independent on disjoint components. Moreover, the value of the HOMFLYPT polynomial of the union of two disjoint links is the multiplication of that of the two links. By an induction on the number of R's in T_1 and T_2, it is easy to show that $\zeta(T) = \zeta(T^1)\zeta(T^2)$.

Therefore ζ is a partition function of $\mathscr{P}'_\bullet$.

Recall that $\zeta(\bigcirc) = \delta$, so $\bigcirc - \delta \in \mathrm{Ker}(\zeta)$. $\square$

Lemma V.3.14. *The element $R - i\mathscr{F}(R)$ is in Ker(ζ). Therefore ζ passes to the quotient $\mathscr{P}''_{\bullet}$.*

Proof. For an annular tangle $\Psi \in Ann_2^0(n)$, take $T^0 = \Psi(R)$, $T^1 = \Psi(\mathscr{F}(R))$. Then the choices of orientations and braids of T^0 coincide with those of T^1. For any $\sigma \in \pm(T^0)(= \pm(T^1))$ and $\gamma \in \pm(\sigma)$, by Equation (V.21), we have

$$\mathrm{HOMFLY}_{q,r}(T^m_{\sigma,\gamma}) = \mathscr{D}^{n+1} i^{W^m_\sigma} \zeta_{\sigma,\gamma}(T^m) + \sum_{j=2}^{3^n} \zeta(T^m_{\sigma,\gamma}(j)),$$

for some elements $T^m_{\sigma,\gamma}(j)$ with at most $n-1$ copies of R, $2 \le j \le 3^n$, $m = 0,1$. Note that

$$T^0_{\sigma,\gamma} = T^1_{\sigma,\gamma}, \quad T^0_{\sigma,\gamma}(j) = T^1_{\sigma,\gamma}(j),\ \forall\, 2 \le j \le 3^n, \quad W^0_\sigma + 1 = W^1_\sigma,$$

so

$$\zeta_{\sigma,\gamma}(T^0) = i\zeta_{\sigma,\gamma}(T^1).$$

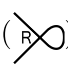
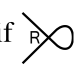
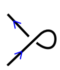
By Equation (V.22), we have

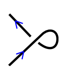
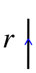
$$\zeta(T^0) = i\zeta(T^1),\ i.e.,\ \zeta(\Psi(R - i\mathscr{F}(R))) = 0.$$

So $R - i\mathscr{F}(R) \in \mathrm{Ker}(\zeta)$. $\square$

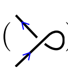
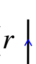
Lemma V.3.15. *The element R_1 is in Ker(ζ). Therefore ζ passes to the quotient $\mathscr{P}'''_{\bullet}$.*

Proof. Let us prove $R_1 \in \mathrm{Ker}(\zeta)$ by an inductive argument.

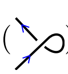
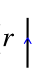
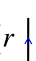
For an annular tangle $\Psi^0 \in Ann_1^0(0)$, take $T^0 = \Psi^0($$)$. For any $\sigma \in \pm(T^0)$ and $\gamma \in \pm(\sigma)$, if  is replaced by  in $T^0_{\sigma,\gamma}$, then by Equation (V.21) and the Reidemester Move I

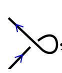
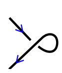
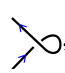
 $- r$  $= \mathscr{D}R_1$ (V.23)

in Lemma V.3.8, we have

$\mathrm{HOMFLY}_{q,r}(\Psi^0($$)) = \mathscr{D}\zeta_{\sigma,\gamma}(T^0) + \zeta(\Psi^0(r$ $))$.

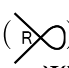
Note that

$\mathrm{HOMFLY}_{q,r}(\Psi^0($$)) = \mathrm{HOMFLY}_{q,r}(\Psi^0(r$ $)) = \zeta(\Psi^0(r$ $))$.

so $\zeta_{\sigma,\gamma}(T^0) = 0$. If  is replaced by ,  or , then we still have $\zeta_{\sigma,\gamma}(T) = 0$ by applying the corresponding Reidemester Move I in Lemma V.3.8 to a similar argument. Therefore $\zeta(T^0) = 0$, i.e., $\zeta(\Psi^0(R_1)) = 0$ by Equation V.22.

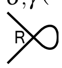
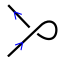
Suppose

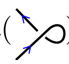
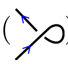
$$\zeta(\Psi^k(R_1)) = 0,\ \forall\, \Psi^k \in Ann_1^0(k),\ k < n,$$

for some $n > 0$. For an annular tangle $\Psi^n \in Ann_1^0(n)$, take $T = \Psi^n($  $)$. For any $\sigma \in \pm(T)$ and $\gamma \in \pm(\sigma)$, let us define the annular tangle $\Psi^n_{\sigma,\gamma}$ to be the restriction of $T_{\sigma,\gamma}$ on Ψ^n. Replacing the braids of $\Psi^n_{\sigma,\gamma}$ by Equation (V.19), (V.20), we have a decomposition of $\Psi^n_{\sigma,\gamma}$ as

$$\Psi^n_{\sigma,\gamma} = \sum_{j=1}^{3^n} \Psi^n_{\sigma,\gamma}(j),$$

such that each $\Psi^n_{\sigma,\gamma}(j)$, $2 \leq j \leq 3^n$, is a scalar multiple of an annular tangle with at most $n-1$ copies of R, and $\Psi^n_{\sigma,\gamma}(1)$ is $\mathscr{D}^n$ times an annular tangle with n copies of R.

If  is replaced by  in $T_{\sigma,\gamma}$, then by Equation (V.21) and the Reidemester Move I (V.23), we have

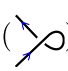
$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}($  $)) = \mathscr{D}^n i^{W_\sigma} \left(\mathscr{D}\zeta_{\sigma,\gamma}(T) + \zeta(\Psi^n(r \uparrow)) \right) + \sum_{j=2}^{3^n} \zeta(\Psi^n_{\sigma,\gamma}(j)($  $)).$ (V.24)

On the other hand

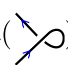
$$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\bar{\sigma},\bar{\gamma}}(\uparrow)) = \mathscr{D}^n i^{W_\sigma} (\zeta_{\bar{\sigma},\bar{\gamma}}(\Psi^n(\uparrow))) + \sum_{j=2}^{3^n} \zeta(\Psi^n_{\bar{\sigma},\bar{\gamma}}(j)(\uparrow)), \qquad \text{(V.25)}$$

where $\bar{\sigma}, \bar{\gamma}$ are the corresponding choices of orientations and braids of $\Psi^n(\,\big|\,)$.

By induction and the Reidemester Move I (V.23), we have

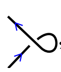
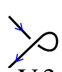
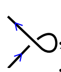
$\zeta(\Psi^n_{\sigma,\gamma}(j)($  $)) - r\zeta(\Psi^n_{\sigma,\gamma}(j)(\uparrow)) = \mathscr{D}\Psi^n_{\sigma,\gamma}(j)(R_1) = 0$

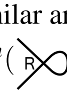
for $2 \leq j \leq 3^n$. Moreover,

$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}($  $)) = \mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}(\uparrow)).$

So Equation (V.24)-r(V.25) implies

$$\zeta_{\sigma,\gamma}(T) + r\left(\zeta(\Psi^n(\,\big|\,)) - \zeta_{\bar{\sigma},\bar{\gamma}}(\Psi^n(\,\big|\,)) \right) = 0. \qquad \text{(V.26)}$$

If  is replaced by ,  or , then we still have Equation (V.26) by applying the corresponding Reidemester Move I in Lemma V.3.8 to a similar argument.

Note that $\sigma \to \bar{\sigma}$ is a bijection from $\pm(\Psi^n($  $))$ to $\pm(\Psi^n(\,\big|\,))$, and $\gamma \to \bar{\gamma}$ is a double cover from $\pm(\sigma)$ to $\pm(\bar{\sigma})$. Summing over all σ, γ for Equation (V.26), we have $\zeta(T) = 0$, i.e., $\zeta(\Psi^n(R_1)) = 0$ by Equation (V.22).

By induction, we have $\zeta(\Psi(R_1)) = 0$, for any annular tangle Ψ. So $R_1 \in \mathrm{Ker}(\zeta)$ and ζ passes to the quotient $\mathscr{P}_\bullet'''$. □

Lemma V.3.16. *The element R_2 is in Ker(ζ). Therefore ζ passes to the quotient $\mathscr{P}_\bullet''''$.*

Proof. The proof is a similar inductive argument as in the proof of Lemma V.3.15.

For an annular tangle $\Psi^0 \in Ann_2^0(0)$, take $T^0 = \Psi^0(\, \asymp_R \,)$. For any $\sigma \in \pm(T^0)$ and $\gamma \in \pm(\sigma)$, if $\asymp_R$ is replaced by $\asymp$ in $T^0_{\sigma,\gamma}$, then by Equation (V.21) and the Reidemester Move II

$$\asymp - \uparrow\uparrow = \mathscr{D}^2 R_2 \tag{V.27}$$

in Lemma V.3.9, we have

$$\mathrm{HOMFLY}_{q,r}(\Psi^0(\asymp)) = \mathscr{D}^2(\zeta_{\sigma,\gamma}(T^0) + \zeta(\Psi^0(R_2 - \asymp_R\,)) + \zeta(\Psi^0(\uparrow\uparrow)).$$

Note that

$$\mathrm{HOMFLY}_{q,r}(\Psi^0(\asymp)) = \mathrm{HOMFLY}_{q,r}(\Psi^0(\uparrow\uparrow)) = \zeta(\Psi^0(\uparrow\uparrow)),$$

so

$$\zeta_{\sigma,\gamma}(T^0) + \zeta(\Psi^0(R_2 - \asymp_R\,)) = 0. \tag{V.28}$$

If $\asymp_R$ is replaced by the other 7 possibilities, then we still have $\zeta(\Psi^0(R_2)) = 0$ by applying the corresponding Reidemester Move II in Lemma V.3.9 to a similar argument.

Summing over all σ,γ, we have $\zeta(\Psi^0(R_2)) = 0$.

Suppose

$$\zeta(\Psi^k(R_2)) = 0,\ \forall\ \Psi^k \in Ann_2^0(k),\ k < n,$$

for some $n > 0$. For an annular tangle $\Psi^n \in Ann_2^0(0)$, take $T = \Psi^n(\, \asymp_R \,)$. For any $\sigma \in \pm(T)$ and $\gamma \in \pm(\sigma)$, let

$$\Psi^n_{\sigma,\gamma} = \sum_{j=1}^{3^n} \Psi^n_{\sigma,\gamma}(j),$$

be the same decomposition as the one in the proof of Lemma V.3.15.

If $\asymp_R$ is replaced by $\asymp$ in $T_{\sigma,\gamma}$, then by Equation (V.21), we have

$$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}(\asymp)) = \mathscr{D}^n i^{W_\sigma}\left(\mathscr{D}^2\zeta_{\sigma,\gamma}(T) + \zeta(\Psi^n(\asymp - \mathscr{D}^2 \asymp_R\,))\right) + \sum_{j=2}^{3^n} \zeta(\Psi^n_{\sigma,\gamma}(j)(\asymp)). \tag{V.29}$$

On the other hand

$$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\bar\sigma,\bar\gamma}(\uparrow\uparrow)) = \mathscr{D}^n i^{W_\sigma}(\zeta_{\bar\sigma,\bar\gamma}(\Psi^n(\uparrow\uparrow))) + \sum_{j=2}^{3^n} \Psi^n_{\bar\sigma,\bar\gamma}(j)(\uparrow\uparrow), \tag{V.30}$$

where $\bar\sigma, \bar\gamma$ are the corresponding choices of orientations and braids of $\Psi^n(\uparrow\uparrow)$. By induction and the Reidemester Move II (V.27), we have

$$\Psi^n_{\sigma,\gamma}(j)(\asymp) - \Psi^n_{\sigma,\gamma}(j)(\uparrow\uparrow) = \mathscr{D}^2\Psi^n_{\sigma,\gamma}(j)(R_2) = 0$$

for $2 \leq j \leq 3^n$. Moreover,

$$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}(\;)) = \mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}(\;)).$$

So Equation (V.29)-(V.30) implies

$$\mathscr{D}^2 \zeta_{\sigma,\gamma}(T) + \zeta(\Psi^n(\; - \mathscr{D}^2 \;)) - \zeta_{\bar{\sigma},\bar{\gamma}}(\Psi^n(\;)) = 0. \tag{V.31}$$

By the Reidemester Move II (V.27), we have

$$\mathscr{D}^2 \left(\zeta_{\sigma,\gamma}(T) - \zeta(\Psi^n(\;)) \right) + \left(\zeta(\Psi^n(\;)) - \zeta_{\bar{\sigma},\bar{\gamma}}(\Psi^n(\;)) \right) + \mathscr{D}^2 \zeta(\Psi^n(R_2)) = 0. \tag{V.32}$$

If is replaced by the other 7 possibilities, then we still have Equation (V.32) by applying the corresponding Reidemester Move II in Lemma V.3.9 to a similar argument.

Note that $\sigma \to \bar{\sigma}$ is a bijection from $\pm(\Psi^n(\;))$ to $\pm(\Psi^n(\;))$, and $\gamma \to \bar{\gamma}$ is a double cover from $\pm(\sigma)$ to $\pm(\bar{\sigma})$. Recall that $T = \Psi^n(\;)$. Summing over all σ, γ for Equation (V.32), we have $\zeta(\Psi^n(R_2)) = 0$ by Equation (V.22).

By induction, we have $\zeta(\Psi(R_2)) = 0$, for any annular tangle Ψ. So $R_2 \in \mathrm{Ker}(\zeta)$ and ζ passes to the quotient $\mathscr{P}_{\bullet}''''$. $\square$

Lemma V.3.17. *The element R_3 is in Ker(ζ). Therefore ζ passes to the quotient $\mathscr{P}_{\bullet}$.*

Proof. The proof is a similar inductive argument as in the proof of Lemma V.3.15, V.3.16.

For an annular tangle $\Psi^0 \in Ann_3^0(0)$, take $T^0 = \Psi^0(\;)$. For any $\sigma \in \pm(T^0)$ and $\gamma \in \pm(\sigma)$, if is replaced by in $T^0_{\sigma,\gamma}$, then by Equation (V.21), we have

$$\mathrm{HOMFLY}_{q,r}(\Psi^0(\;)) = \mathscr{D}^3 i^3 \zeta_{\sigma,\gamma}(T^0) + \zeta(\Psi^0(\; - \mathscr{D}^3 i^3 \;)).$$

On the other hand, take $S^0 = \Psi^0(\;)$ and $\bar{\sigma} \in \pm(S^0), \bar{\gamma} \in \pm(\bar{\sigma})$ such that $S_{\bar{\sigma},\bar{\gamma}}$ is isotopic to $T_{\sigma,\gamma}$ by a Reidemester move III. Then by Equation (V.21), we have

$$\mathrm{HOMFLY}_{q,r}(\Psi^0(\;)) = \mathscr{D}^3 \zeta_{\bar{\sigma},\bar{\gamma}}(S^0) + \zeta(\Psi^0(\; - \mathscr{D}^3 \;)).$$

Note that $\mathrm{HOMFLY}_{q,r}(\Psi^0(\;)) = \mathrm{HOMFLY}_{q,r}(\Psi^0(\;))$. By the Reidemester Move III

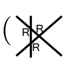
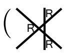
$$\; - \; = \mathscr{D}^3 i^3 R_3 \tag{V.33}$$

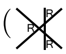
in Lemma V.3.10, we have

$$i^3 \left(\zeta_{\sigma,\gamma}(T^0) - \zeta(\Psi^0(\;)) \right) - \left(\zeta_{\bar{\sigma},\bar{\gamma}}(S^0) - \zeta(\Psi^0(\;)) \right) + i^3 \zeta(\Psi^0(R_3)) = 0. \tag{V.34}$$

If  is replaced other 47 possibilities, then we still have Equation (V.34) by applying the corresponding Reidemester Move III in Lemma V.3.10 to a similar argument.

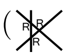
Note that $\sigma \to \bar{\sigma}$ is a bijection from $\pm(T^0)$ to $\pm(S^0)$, and $\gamma \to \bar{\gamma}$ is a bijection from $\pm(\sigma)$ to $\pm(\bar{\sigma})$. Summing over all σ, γ, we have

$$i^3 \left(\zeta(T^0) - \zeta(\Psi^0(\right.$$  $$\left.)) \right) - \left(\zeta(S^0) - \zeta(\Psi^0(\right.$$  $$\left.)) \right) + i^3 \zeta(\Psi^0(R_3)) = 0.$$

Recall that $T^0 = \Psi^0($$)$, $S^0 = \Psi^0($$)$, so $\zeta(\Psi^0(R_3)) = 0$.

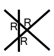
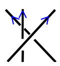
Suppose

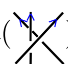
$$\zeta(\Psi^k(R_3)) = 0,\ \forall\ \Psi^k \in Ann_3^0(k),\ k < n,$$

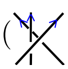
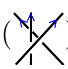
for some $n > 0$. For an annular tangle $\Psi^n \in Ann_3^0(0)$, take $T = \Psi^n($$)$. For any $\sigma \in \pm(T)$ and $\gamma \in \pm(\sigma)$, let

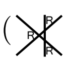
$$\Psi^n_{\sigma,\gamma} = \sum_{j=1}^{3^n} \Psi^n_{\sigma,\gamma}(j),$$

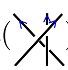
be the same decomposition as the one in the proof of Lemma V.3.15.

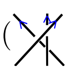
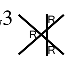
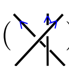
If  is replaced by  in $T_{\sigma,\gamma}$, then by Equation (V.21), we have

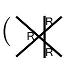
$$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}($$  $$))$$

$$= \mathscr{D}^n i^{W_\sigma} \left(\mathscr{D}^3 i^3 \zeta_{\sigma,\gamma}(T) + \zeta(\Psi^n(\right.$$  $$- \mathscr{D}^3 i^3$$  $$\left.)) \right) + \sum_{j=2}^{3^n} \zeta(\Psi^n_{\sigma,\gamma}(j)($$  $$)). \quad \text{(V.35)}$$

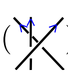
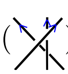
On the other hand, take $S = \Psi^n($$)$, we have

$$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\bar{\sigma},\bar{\gamma}}($$  $$))$$

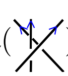
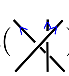
$$= \mathscr{D}^n i^{W_\sigma} \left(\mathscr{D}^3 \zeta_{\bar{\sigma},\bar{\gamma}}(S) + \zeta(\Psi^n(\right.$$  $$- \mathscr{D}^3$$  $$\left.)) \right) + \sum_{j=2}^{3^n} \zeta(\Psi^n_{\bar{\sigma},\bar{\gamma}}(j)($$  $$)). \quad \text{(V.36)}$$

where $\bar{\sigma}, \bar{\gamma}$ are the corresponding choices of orientations and braids of $\Psi^n($$)$, such that $\Psi^n_{\sigma,\gamma} = \Psi^n_{\bar{\sigma},\bar{\gamma}}$.

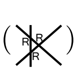
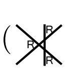
By induction and the Reidemester Move III (V.33), we have

$$\zeta(\Psi^n_{\sigma,\gamma}(j)($$  $$)) - \zeta(\Psi^n_{\bar{\sigma},\bar{\gamma}}(j)($$  $$)) = \mathscr{D}^3 i^3 \zeta(\Psi^n_{\sigma,\gamma}(j)(R_3)) = 0,$$

for $2 \le j \le 3^n$. Moreover,

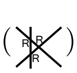
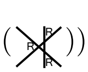
$$\mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}($$  $$)) = \mathrm{HOMFLY}_{q,r}(\Psi^n_{\sigma,\gamma}($$  $$)).$$

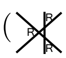
Applying the Reidemester Move III (V.33) to Equation (V.35)-(V.36), we have

$i^3 \left(\zeta_{\sigma,\gamma}(T) - \zeta(\Psi^n($  $))\right) - \left(\zeta_{\bar{\sigma},\bar{\gamma}}(S) - \zeta(\Psi^n($  $))\right) + i^3\zeta(\Psi^n(R_3)) = 0.$ (V.37)

If  is replaced other 47 possibilities, then we still have Equation (V.37) by applying the corresponding Reidemester Move III in Lemma V.3.10 to a similar argument.

Note that $\sigma \to \bar{\sigma}$ is a bijection from $\pm(T^0)$ to $\pm(S^0)$, and $\gamma \to \bar{\gamma}$ is a bijection from $\pm(\sigma)$ to $\pm(\bar{\sigma})$. Summing over all σ, γ, we have

$i^3 \left(\zeta(T) - \zeta(\Psi^n($  $))\right) - \left(\zeta(S) - \zeta(\Psi^n($  $))\right) + i^3\zeta(\Psi^n(R_3)) = 0.$

Recall that $T = \Psi^n($  $)$, $S = \Psi^n($  $)$, so $\zeta(\Psi^0(R_3)) = 0$.

By induction, we have $\zeta(\Psi(R_1)) = 0$, for any annular tangle Ψ. So $R_2 \in \mathrm{Ker}(\zeta)$ and ζ passes to the quotient $\mathscr{P}_\bullet$. □

Theorem V.3.18. *The Yang-Baxter relation of $\mathscr{P}_\bullet$ is consistent over $\mathbb{C}$ for any $\delta \in \mathbb{R}$.*

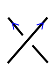
Proof. The Yang-Baxter relation of $\mathscr{P}_\bullet$ is evaluable by Theorem III.2.5. By Lemma V.3.17, the partition function ζ passes to the quotient $\mathscr{P}$. So any evaluation of a closed diagram T has to be $\zeta(T)$. □

Recall that $q = \dfrac{i+\delta}{\sqrt{1+\delta^2}}$, so $\delta = \dfrac{i(q+q^{-1})}{q-q^{-1}}$. Therefore the Yang-Baxter relation for $\mathscr{P}_\bullet$ is also a relation over the field $\mathbb{C}(q)$, rational functions of q.

Corollary V.3.19. *The Yang-Baxter relation of $\mathscr{P}_\bullet$ is consistent over $\mathbb{C}(q)$.*

Proof. Over the field $\mathbb{C}(q)$, any two evaluations of a closed diagram in $\mathscr{P}_\bullet$ are two rational functions over q. Moreover, the two rational functions have the same value for $q = \frac{i+\delta}{\sqrt{1+\delta^2}}$, $\delta \in \mathbb{R}$ by Theorem V.3.18. So they are the same. Therefore the Yang-Baxter relation is consistent over $\mathbb{C}(q)$. □

V.4 Matrix units

Recall that the braid  satisfies the Hecke relation, so $\mathscr{P}_\bullet$ has a subalgebra $H_\bullet$, the Hecke algebra of type A with parameters q, r. Moreover $\mathscr{P}_n/\mathscr{I}_n \cong H_n$, where $\mathscr{I}_n$ is the two sided ideal of $\mathscr{P}_n$ generated by the Jones projection e_{n-1}, called the basic construction ideal. The Bratteli diagram of $H_\bullet$ is Young's Lattice, denoted by YL, so the principal graph of (a proper quotient of) $\mathscr{P}_\bullet$ is a subgraph of Young's Lattice. To construct the matrix units of $\mathscr{P}_\bullet$, we need to decompose minimal idempotents of $\mathscr{P}_n$ in $\mathscr{P}_{n+1}$. This decomposition can be derived from Wenzl's formula for the basic construction $\mathscr{P}_{n-1} \subset \mathscr{P}_n \subset \mathscr{I}_{n+1}$ and Branching formula for $H_\bullet$. The basic construction and Wenzl's formula will work, if $\mathscr{P}_n$ is semisimple and the trace of the idempotent is non-zero. To ensure the two conditions, let us take the ground field to be $\mathbb{C}(q)$. We are going to prove that $\mathscr{P}_\bullet$ over the field $\mathbb{C}(q)$ is isomorphic to the string algebra of the Young's Lattice starting from the empty Young diagram.

Definition V.4.1. The string algebra $YL_\bullet$ of YL over the field $\mathbb{C}(q)$ is an inclusion of matrix algebras YL_n, $n = 0, 1, \cdots$. Moreover, the basis of YL_n consists of all length $2n$ loops of YL starting from $\emptyset$. The multiplication of YL_n is a linear extension of the multiplication of length $2n$ loops. The inclusion $\iota : YL_n \to YL_{n+1}$ is a linear extension of

$$\iota(t\tau^{-1}) = \sum_{s(e)=v} tee^{-1}\tau^{-1},$$

where t and τ are length n paths from $\emptyset$ to some vertex v, and $s(e)$ is the source vertex of the edge e.

Definition V.4.2. For $n \geq 1$, the vertices of YL whose distance to $\emptyset$ are at most $n-1$ and the edges between these vertices form a subgraph of YL, denoted by YL^{n-1}. Let IYL_n to be the subspace of YL_n with a basis consisting of all length $2n$ loops of YL^{n-1} starting from $\emptyset$. Let HYL_n to be the subspace of YL_n with a basis consisting of all length $2n$ loops passing a vertex in $YL \setminus YL^{n-1}$ starting from $\emptyset$.

Lemma V.4.3. *The subspace IYL_n is a two sided ideal of YL_n, $YL_n = IYL_n \oplus HYL_n$, and $HYL_n \simeq H_n$, for $n \geq 1$.*

Proof. Follows from the definitions. □

Notation V.4.4. *The elements $x \otimes 1$, $x \otimes \cap$, $x \otimes \cup$, are adding a string, a cap $\cap$, a cup $\cup$ to the right of x respectively.*

Theorem V.4.5 (matrix units)**.** *Over the field $\mathbb{C}(q)$, $\mathscr{P}_\bullet \cong YL_\bullet$ as a filtered algebra.*

A trace of a finite dimensional matrix algebra is non-degenerate if and only if the trace of any minimal idempotent of the matrix algebra is non-zero.

Proof. Note that TL_0 and $\mathscr{P}_0$ are isomorphic to the ground field $\mathbb{C}(q)$, set up $\omega_0 : YL_0 \to \mathscr{P}_0$ to be the isomorphism. Moreover, the trace of the empty diagram $\emptyset$ is 1.

We are going to prove the following properties of $\mathscr{P}_m$ inductively for $m \geq 1$.

(1) $\mathscr{P}_m$ is a matrix algebra and its trace is non-degenerated.

(A trace of a finite dimensional matrix algebra is non-degenerate if and only if the trace of any minimal idempotent of the matrix algebra is non-zero.)

Then the two sided ideal $\mathscr{I}_m$ is a finite dimensional matrix algebra, so it has a unique maximal idempotent, called the support of $\mathscr{I}_m$. Moreover, its support is central in $\mathscr{P}_m$. Let s_m be the complement of the support of $\mathscr{I}_m$.

(2) $\mathscr{P}_m = \mathscr{I}_m \oplus s_m\mathscr{P}_m$, for some central idempotent $s_m \in \mathscr{P}_m$. Note that $\mathscr{P}_m$ has a subalgebra H_m generated by the braid . Moreover, s_m is central and $s_m e_i = 0$, for any $1 \leq i \leq m-1$, so $s_m\mathscr{P}_m = s_m H_m$ by Proposition III.2.6. For each equivalent class of minimal idempotents of H_m corresponding to the Young diagram λ, $|\lambda| = m$, we have a minimal idempotent y_λ in H_m. Thus $s_m y_\lambda$ is either a minimal idempotent of $s_m H_m$ or zero.

(3) For any $|\lambda| = m$, $\tilde{y}_\lambda = s_m y_\lambda$ is a minimal idempotent in $\mathscr{P}_m$ with a non-zero trace $<\lambda>$.

For a length m path t in YL from $\emptyset$ to λ, take t' to be the first length $(m-1)$ path of t from $\emptyset$ to μ. Let us define $\tilde{P}_t^{\pm}$ by induction as follows,

$$\begin{aligned}
&P_\emptyset^{\pm} = \emptyset \\
&\tilde{P}_t^{+} = (\tilde{P}_{t'}^{+} \otimes 1)\rho_{\mu<\lambda}\tilde{y}_\lambda, && \text{when } \mu < \lambda \\
&\tilde{P}_t^{+} = \frac{<\lambda>}{<\mu>}(\tilde{P}_{t'}^{+} \otimes 1)(\rho_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap), && \text{when } \mu > \lambda \\
&\tilde{P}_t^{-} = (\tilde{P}_{t'}^{-} \otimes \cup)(\rho_{\mu<\lambda} \otimes 1)(\tilde{y}_\lambda \otimes 1), && \text{when } \mu < \lambda \\
&\tilde{P}_t^{-} = \tilde{P}_{t'}^{+}\rho_{\mu>\lambda}(\tilde{y}_\lambda \otimes 1), && \text{when } \mu > \lambda
\end{aligned}$$

(4) The map $\omega_m : YL_m \to \mathscr{P}_m$ as a linear extension of

$$\omega_m(t\tau^{-1}) = \tilde{P}_t^{+}\tilde{P}_\tau^{-}$$

is an algebraic isomorphism.

(5) $\omega_m(\iota(x)) = \omega_{m-1}(x) \otimes 1, \forall\, x \in TL_{m-1}$.

When $m = 1$, it is easy to check Property (1)-(5). Suppose Property (1)-(5) hold for $m = 1, 2, \cdots, n$, $n \geq 1$, let us prove them for $m = n+1$.

By Property (4),(5), we have an isomorphism $\omega_n : YL_n \to \mathscr{P}_n$, such that $\omega_n(\iota(x)) = \omega_{n-1}(x) \otimes 1$, for any $x \in YL_{n-1}$. So $\mathscr{P}_{n-1} \subset \mathscr{P}_n \cong YL_{n-1} \subset YL_n$ is an inclusion of finite dimensional matrix algebras.

By Property (1), $\mathscr{P}_{n-1} \subset \mathscr{P}_n$ is an inclusion of finite dimensional matrix algebras with a non-degenerate trace. So we have the basic construction $\mathscr{P}_{n-1} \subset \mathscr{P}_n \subset \mathscr{I}_{n+1}$ by [GHJ89], and $\mathscr{I}_{n+1}$ is a finite dimensional matrix algebra. Therefore we can define s_{n+1} to be the complement of the support of $\mathscr{I}_{n+1}$, and $\mathscr{P}_{n+1} = \mathscr{I}_{n+1} \oplus s_{n+1}\mathscr{P}_{n+1}$. Property (2) holds for $m = n+1$.

Moreover, we have $s_{n+1}\mathscr{P}_{n+1} = s_{n+1}H_{n+1}$. For any $|\lambda| = n+1$, the minimal idempotent $\tilde{y}_\lambda = s_{n+1}y_\lambda$ in $\mathscr{P}_{n+1}$ has a non-zero trace $<\lambda>$ by Theorem V.5.14. (The proof of Theorem V.5.14 only needed the matrix units of $\mathscr{P}_k$, $k \leq n+1$.) Property (3) holds for $m = n+1$.

Furthermore, $s_{n+1}H_{n+1} \cong H_{n+1}$ is a finite dimensional matrix algebra. Therefore $\mathscr{P}_{n+1}$ is a finite dimensional matrix algebra. By the basic construction, the traces of minimal idempotents in $\mathscr{I}_{n+1}$ are given by the traces of minimal idempotents in $\mathscr{P}_{n-1}$, and they are non-zero by Property (1). So the trace of $\mathscr{P}_{n+1}$ is non-degenerated. Property (1) holds for $m = n+1$.

By Property (4),(5), $\mathscr{P}_{n-1} \subset \mathscr{P}_n \cong YL_{n-1} \subset YL_n$ is an inclusion of finite dimensional matrix algebras. By the basic construction, we can define an isomorphism $\omega_m : IYL_{n+1} \to \mathscr{I}_{n+1}$ with Property (4). Note that $HYL_{n+1} \cong H_{n+1} \cong s_nH_n = s_n\mathscr{P}_n$, $YL_n = HYL_{n+1} \oplus HYL_{n+1}$ and $\mathscr{P}_n = \mathscr{I}_n \oplus s_n\mathscr{P}$, we can extend the isomorphism to $\omega_m : YL_{n+1} \to \mathscr{P}_n$ with Property (4).

Property (5) for $m = n+1$ follows from Wenzl's formula:

$$\tilde{y}_\mu \otimes 1 = \sum_{\lambda<\mu} \frac{<\lambda>}{<\mu>} (\tilde{y}_\mu \otimes 1)(\rho_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap)(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)$$
$$+ \sum_{\lambda>\mu} (\tilde{y}_\mu \otimes 1)\rho_{\mu<\lambda}\tilde{y}_\lambda \rho_{\lambda>\mu}(\tilde{y}_\mu \otimes 1), \qquad \forall |\mu| \leq n-1. \quad \text{(V.38)}$$

Proof of Wenzl's formula:

Take

$$x = \tilde{y}_\mu \otimes 1 - \sum_{\lambda<\mu} \frac{<\lambda>}{<\mu>} (\tilde{y}_\mu \otimes 1)(\rho_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap)(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1) \quad \text{(V.39)}$$

and a length $(|\mu|+1)$ path t from $\emptyset$ to λ', $|\lambda'| < |\mu|$.

If $\lambda' < \mu$ does not hold, then

$$(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)\tilde{P}^+(t) = 0, \ \forall\ \lambda < \mu,$$

since it is a morphism from $\tilde{y}_\lambda$ to $\tilde{y}'_\lambda$. By the Frobenius reciprocity, $(\tilde{y}_\mu \otimes 1)\tilde{P}^+(t) = 0$, since it is a morphism from $\tilde{y}_\lambda \otimes 1$ to $\tilde{y}'_\lambda$. Therefore $x\tilde{P}^+(t) = 0$.

If $\lambda' < \mu$, then

$$(\tilde{y}_\mu \otimes 1)\tilde{P}^+(t) = c(\tilde{y}_\mu \otimes 1)(\rho_{\mu\to\lambda'} \otimes 1)(y_{\lambda'} \otimes \cap),$$

for some constant c, since it is a morphism from $\tilde{y}_\lambda \otimes 1$ to $\tilde{\lambda}'$. Thus

$$\begin{aligned}
&(\tilde{y}_{\lambda'} \otimes \cup)(\rho_{\lambda'<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)\tilde{P}^+(t) \\
=&c(\tilde{y}_{\lambda'} \otimes \cup)(\rho_{\lambda'<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)(\rho_{\mu\to\lambda'} \otimes 1)(y_{\lambda'} \otimes \cap) \\
=&\frac{c<\mu>}{<\lambda'>}\tilde{y}_{\lambda'}.
\end{aligned}$$

Moreover,

$$(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)\tilde{P}^+(t) = 0, \text{ when } \lambda \neq \lambda',$$

since it is a morphism from $\tilde{y}_\lambda$ to $\tilde{\lambda}'$. Therefore

$$x\tilde{P}^+(t) = c(\tilde{y}_\mu \otimes 1)(\rho_{\mu\to\lambda'} \otimes 1)(y_{\lambda'} \otimes \cap) - c(\tilde{y}_\mu \otimes 1)(\rho_{\mu\to\lambda'} \otimes 1)(y_{\lambda'} \otimes \cap) = 0.$$

Recall that $IYL_{|\mu|+1} \cong \mathscr{I}_{|\mu|+1}$, so $xz = 0$, for any $z \in \mathscr{I}_{|\mu|+1}$. Thus $xs_{|\mu|+1} = x$. Note that $s_{|\mu|+1}$ is central and $(\tilde{y}_\lambda \otimes \cup)s_{|\mu|+1} = 0$, by Equation (V.39), we have

$$x = xs_{|\mu|+1} = (\tilde{y}_\mu \otimes 1)s_{|\mu|+1}. \quad \text{(V.40)}$$

On the other hand,

$$
\begin{aligned}
&(\tilde{y}_\mu \otimes 1)s_{|\mu|+1} \\
=&(y_\mu s_{|\mu|} \otimes 1)s_{|\mu|+1} \\
=&(y_\mu \otimes 1)s_{|\mu|+1} \\
=&\sum_{\lambda>\mu} (y_\mu \otimes 1)\rho_{\mu\to\lambda} y_\lambda \rho_{\lambda\to\mu}(y_\mu \otimes 1)s_{|\mu|+1} && \text{Branching formula (V.3)} \\
=&\sum_{\lambda>\mu} (\tilde{y}_\mu \otimes 1)\rho_{\mu<\lambda} \tilde{y}_\lambda \rho_{\lambda>\mu}(\tilde{y}_\mu \otimes 1) && \text{(V.41)}
\end{aligned}
$$

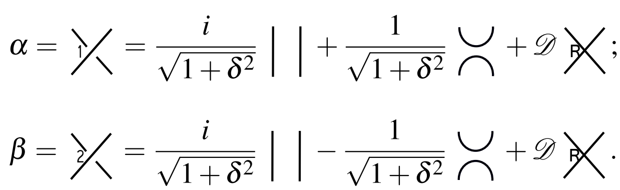
By Equation (V.39), (V.40), (V.41), we obtain Wenzl's formula.

Therefore Property (1)-(5) hold for all m by induction, and $\mathscr{P}_\bullet \cong YL_\bullet$ as a filtered algebra □

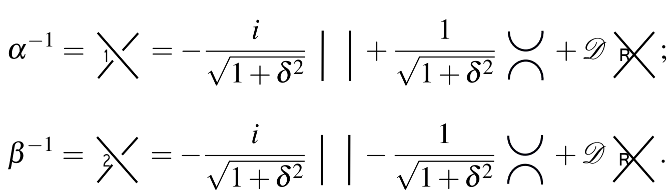
V.5 Trace formula

Recall that $\mathscr{D} = \frac{\delta}{\sqrt{1+\delta^2}}$, $r = \frac{\delta i+1}{\sqrt{1+\delta^2}}$, $q = \frac{i+\delta}{\sqrt{1+\delta^2}}$, and $|r| = |q| = 1$.

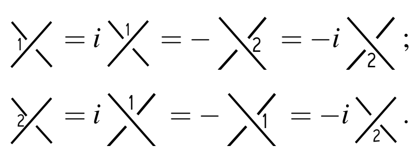
Notation V.5.1. *Let us define*



Note that α, β *are unitary. Let us define*

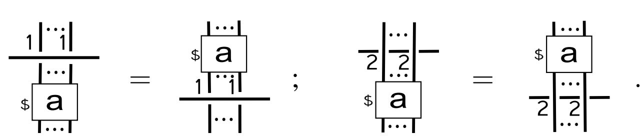


Proposition V.5.2.

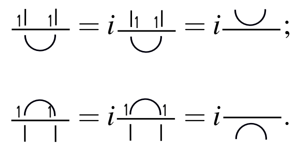


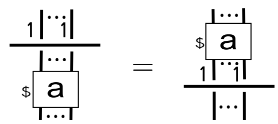
Proof. Follows from the definitions and the fact that $\mathscr{F}(R) = -iR$. □

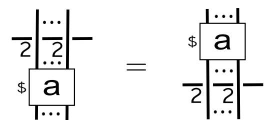
Proposition V.5.3. *For any element* $a \in \mathscr{P}_\bullet$, *we have*

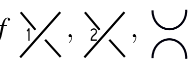


Proof. By Proposition V.5.2, we have

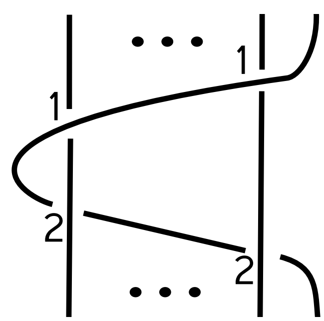


So the first equation  holds for $a =$ . By Lemma V.2.1, it also holds for $a =$ . Recall that $\mathscr{P}_\bullet$ is a Yang-Baxter relation planar algebra, so the first equation holds for any element a by Proposition III.2.6.

The equation  can be proved in a similar way. □

Notation V.5.4. *Let α_n, β_n, h_n be the diagrams by adding $n-1$ through strings to the left of*  *respectively.*

Definition V.5.5. Let us define τ_n, $n \geq 1$, to be the n-box



Remark V.5.6. This operator is known as the q-Murphy operator. The computation of the trace formula of $\mathscr{P}_\bullet$ via the q-Murphy operator is similar to the computation for BMW planar algebras given by Beliakova and Blanchet in [BB01] which was inspired by the work of Nazarov in [Naz96].

Recall that for $|\mu| = n$, $\lambda > \mu$, $\rho_{\lambda>\mu}$ is an intertwiner from λ to $\mu \otimes 1$, and y_μ, y_λ are the minimal idempotents corresponding to μ and λ respectively. So $y_\lambda = y_\lambda \rho_{\lambda>\mu}(y_\mu \otimes 1)$. Then $y_\lambda \tau_{n+1} = y_\lambda \rho_{\lambda>\mu}(y_\mu \otimes 1)\tau_{n+1} = y_\lambda \rho_{\lambda>\mu}\tau_{n+1}(y_\mu \otimes 1)$. It is also an intertwiner from λ to $\mu \otimes 1$. The intertwiner space in the Hecke algebra $H_\bullet$ is one dimensional, so $y_\lambda \tau_{n+1}$ is a multiple of y_λ. The coefficient was known ([Bla00], Prop. 1.11) as

Proposition V.5.7. *For $|\mu| = n$, $n \geq 0$, $\lambda > \mu$,*

$$\rho_{\lambda>\mu}\tau_{n+1} = b_{\lambda-\mu}\rho_{\lambda>\mu},$$

where $b_{\lambda-\mu} = q^{2cn(\lambda-\mu)}$, and $cn(\lambda-\mu) = j - i$ is the content of the cell $\lambda - \mu$ which is in the i-th row and j-th column of λ.

Proposition V.5.8. *For $|\mu| = n$, $n \geq 0$, we have*

$$\tilde{y}_\lambda \rho_{\lambda>\mu}(\tilde{y}_\mu \otimes 1)\tau_{n+1} = b_{\lambda-\mu}\tilde{y}_\lambda \rho_{\lambda>\mu}(\tilde{y}_\mu \otimes 1), \qquad \text{for } \lambda > \mu; \tag{V.42}$$

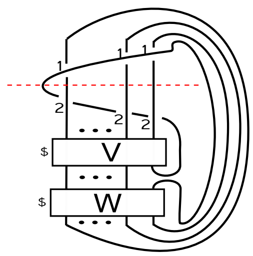
$$(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)\tau_{n+1} = -b_{\mu-\lambda}(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1), \qquad \text{for } \lambda < \mu. \tag{V.43}$$

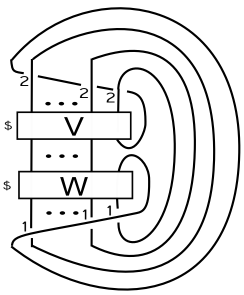
Proof. Since $(\tilde{y}_\mu \otimes 1)\tau_{n+1} = \tau_{n+1}(\tilde{y}_\mu \otimes 1)$, Equation (V.42) follows from Proposition V.5.7.

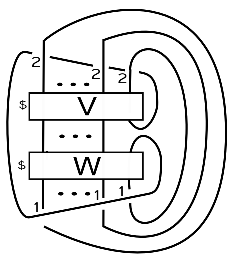
Note that $(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)\tau_{n+1} = (\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)\tau_n(\tilde{y}_\mu \otimes 1)$ is an intertwiner from λ to $\mu \otimes 1$. Moreover, the intertwiner space in $\mathscr{P}_\bullet$ is one dimensional. So Equation (V.43) holds for some coefficient. Furthermore, the coefficient $-b_{\mu-\lambda}$ is determined by computing the inner product as follows.

Take $V = (\tilde{y}_\lambda \otimes 1)\rho_{\lambda<\mu}\tilde{y}_\mu$, $W = \tilde{y}_\mu \rho_{\mu>\lambda}(\tilde{y}_\lambda \otimes 1)$. Then

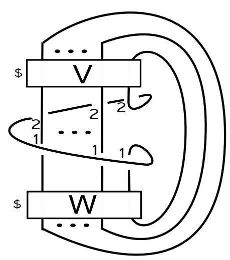
$$
\begin{aligned}
& tr_{n+1}\left((\tilde{y}_\mu \otimes 1)(\rho_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap)(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)\tau_{n+1}\right) \\
=& tr_{n+1}(W h_n V \tau_{n+1})
\end{aligned}
$$

$=$ 

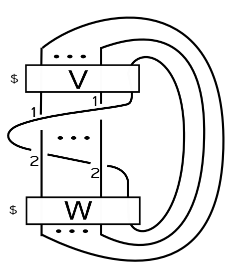
$=$ 

$=$ 

Spherical

$=$ 

Proposition V.5.3

$=-$ 

Proposition V.5.2

$= -b_{\mu-\lambda} tr_n(WV)$ Equation (V.42)

$= -b_{\mu-\lambda} tr_{n+1}(W h_n V)$

$= -b_{\mu-\lambda} tr_{n+1}\left((\tilde{y}_\mu \otimes 1)(\rho_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap)(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)\right)$

□

Let $\Phi_{n+1} : \mathscr{P}_{n+1} \to \mathscr{P}_n$ be the trace preserving conditional expectation, i.e. adding a cap on the right of an $(n+1)$-box. Then $\Phi_{n+1}(\tau_{n+1}^i) = Z_{n+1}^{(i)}$ defines a central element $Z_{n+1}^{(i)}$ in $\mathscr{P}_n$. We consider the formal power series in u^{-1},

$$Z_{n+1}(u) = \sum_{i\geq 0} Z_{n+1}^{(i)} u^{-i}.$$

Then

$$Z_{n+1}(u) = \Phi_{n+1}(\frac{u}{u-\tau_{n+1}}). \tag{V.44}$$

By Theorem V.4.5, each simple components of $\mathscr{P}_n$ is indexed by a Young diagram μ, $|\mu| = n$. Moreover, $\tilde{y}_\mu$ is a minimal idempotent in this component. Since $Z_{n+1}^{(i)}$ is central in $\mathscr{P}_n$, it is a scalar on the simple component of $\mathscr{P}_n$. Let us define $Z(\mu,u)$ to be the formal power series in u^{-1} by

$$Z_{n+1}(u)\tilde{y}_\mu = Z(\mu,u)\tilde{y}_\mu.$$

The relation between Z_{n+1} and the trace formula is given by

Lemma V.5.9. *For $|\mu| = n$, $n \geq 0$, $\lambda > \mu$,*

$$\frac{<\lambda>}{<\mu>} = res_{u=b_{\lambda-\mu}} \frac{Z(\mu,u)}{u}.$$

Proof. By Equation (V.38), we have

$$\tilde{y}_\mu \otimes 1 = \sum_{\lambda<\mu,\lambda>\mu} p_\lambda,$$

where

$$p_\lambda = \begin{cases} \frac{<\lambda>}{<\mu>}(\tilde{y}_\mu \otimes 1)(\rho_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap)(\tilde{y}_\lambda \otimes \cup)(\rho_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1), & \lambda < \mu; \\ (\tilde{y}_\mu \otimes 1)\rho_{\mu<\lambda}\tilde{y}_\lambda\rho_{\lambda>\mu}(\tilde{y}_\mu \otimes 1), & \lambda > \mu. \end{cases}$$

Then p_λ is an idempotent in $\mathscr{P}_{n+1}$ with trace $<\lambda>$. Moreover, by Proposition V.5.8,

$$\tau_{n+1}p_\lambda = \begin{cases} -b_{\mu-\lambda}p_\lambda & \lambda < \mu; \\ b_{\lambda-\mu}p_\lambda & \lambda > \mu. \end{cases}$$

By definitions, we have

$$
\begin{aligned}
Z(\mu,u)\tilde{y}_\mu =&Z_{n+1}(u)\tilde{y}_\mu \\
=&\sum_{i\geq 0} Z_{n+1}^{(i)}\tilde{y}_\mu u^{-i} \\
=&\sum_{i\geq 0} \Phi_{n+1}(\tau_{n+1}^i)\tilde{y}_\mu u^{-i} \\
=&\sum_{i\geq 0} \Phi_{n+1}(\tau_{n+1}^i(\tilde{y}_\mu \otimes 1))u^{-i} \\
=&\sum_{i\geq 0} \Phi_{n+1}(\tau_{n+1}^i(\sum_{\lambda<\mu,\lambda>\mu} p_\lambda))u^{-i} \\
=&\sum_{i\geq 0} \Phi_{n+1}(\sum_{\lambda<\mu}(-b_{\mu-\lambda})^i p_\lambda + \sum_{\lambda>\mu} b_{\lambda-\mu}^i p_\lambda)u^{-i} \\
=&\sum_{i\geq 0} \left(\sum_{\lambda<\mu}(-b_{\mu-\lambda})^i \frac{<\lambda>}{<\mu>}\tilde{y}_\mu + \sum_{\lambda>\mu} b_{\lambda-\mu}^i \frac{<\lambda>}{<\mu>}\tilde{y}_\mu\right) u^{-i} \\
=&\left(\sum_{\lambda<\mu} \frac{u}{u+b_{\mu-\lambda}}\frac{<\lambda>}{<\mu>} + \sum_{\lambda>\mu} \frac{u}{u-b_{\lambda-\mu}}\frac{<\lambda>}{<\mu>}\right)\tilde{y}_\mu \qquad \text{Fubini's theorem}
\end{aligned}
$$

Therefore

$$\frac{Z(\mu,u)}{u} = \sum_{\lambda<\mu} \frac{1}{u+b_{\mu-\lambda}}\frac{<\lambda>}{<\mu>} + \sum_{\lambda>\mu} \frac{1}{u-b_{\lambda-\mu}}\frac{<\lambda>}{<\mu>}.$$

Recall that $b_c = q^{2\mathrm{cn}(c)}$, so $\{-b_{\mu-\lambda}\}_{\lambda<\mu}$ and $\{b_{\lambda-\mu}\}_{\lambda>\mu}$ are distinct. Therefore

$$\frac{<\lambda>}{<\mu>} = \mathrm{res}_{u=b_{\lambda-\mu}} \frac{Z(\mu,u)}{u}, \text{ for } \lambda>\mu$$

and

$$\frac{<\lambda>}{<\mu>} = \mathrm{res}_{u=-b_{\mu-\lambda}} \frac{Z(\mu,u)}{u}, \text{ for } \lambda<\mu.$$

□

Before computing $Z(\mu,u)$, let us prove some basic results as follows.

Lemma V.5.10. *For $n \geq 1$, we have*

$$\beta_n^{-1}\tau_{n+1} = \tau_n \alpha_n \quad \text{(V.45)}$$

$$\tau_{n+1}\alpha_n^{-1} = \beta_n \tau_n \quad \text{(V.46)}$$

$$h_n \tau_{n+1} = -h_n \tau_n \quad \text{(V.47)}$$

$$\tau_{n+1} h_n = -\tau_n h_n \quad \text{(V.48)}$$

$$\tau_n \tau_{n+1} = \tau_{n+1} \tau_n \quad \text{(V.49)}$$

$$h_n(u - \tau_{n+1})^{-1} = h_n(u + \tau_n)^{-1} \quad \text{(V.50)}$$

$$(u - \tau_{n+1})^{-1} h_n = (u + \tau_n)^{-1} h_n \quad \text{(V.51)}$$

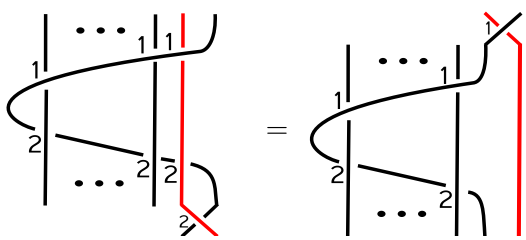
$\beta^{-1} - \alpha = -(q - q^{-1})$  $+ i(q - q^{-1})$  (V.52)

$\beta - \alpha^{-1} = (q - q^{-1})$  $+ i(q - q^{-1})$  (V.53)

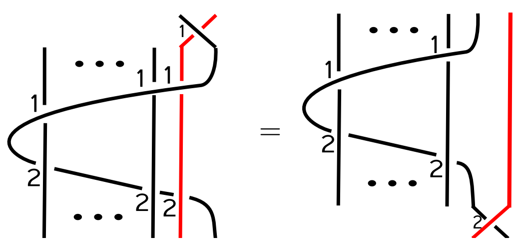
$$\Phi_{n+1}\left(\beta_n \frac{1}{u - \tau_n} \beta_n^{-1}\right) = \frac{Z_n}{u} \quad \text{(V.54)}$$

Recall that we identify an n-box as an $(n+1)$-box by adding a through string to the right.

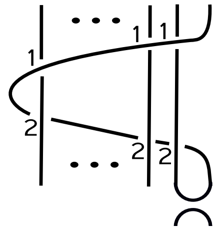
Proof. Equation (V.45) follows from



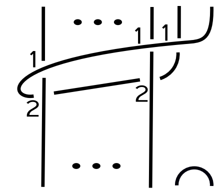
Equation (V.46) follows from



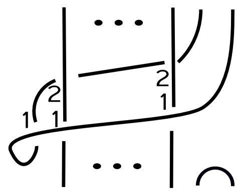
Equation (V.47) follows from



$= ir$

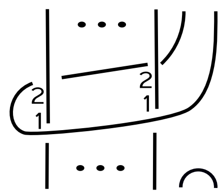


$= ir$

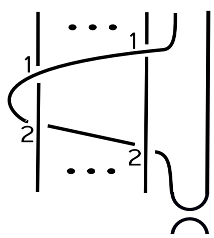


by Proposition V.5.3

$= -$



$= -$



by Proposition V.5.2

Similarly we have equation (V.48).

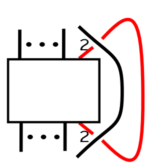
Equation (V.49) follows from Proposition V.5.3,

By Equation (V.47), (V.49), we have $h_n \tau_{n+1}^k = h_n(-\tau_n)^k$. So Equation (V.50) holds.

Similarly by Equation (V.48), (V.49), Equation (V.51) holds.

Equation (V.52), (V.53) follow from the definitions.

By Proposition V.5.2, we have



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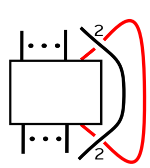
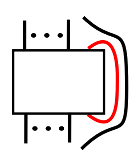


$=$



.

So by Equation (V.44),

$$\Phi_{n+1}(\beta_n \frac{1}{u-\tau_n} \beta_n^{-1}) = \Phi_n(\frac{1}{u-\tau_n}) = \frac{Z_n}{u}.$$

□

Lemma V.5.11. *For $n \geq 1$,*

$$Z_{n+1} - \frac{\delta}{2} = (Z_n - \frac{\delta}{2})\frac{(u-\tau_n)^2(u+q^{-2}\tau_n)(u+q^2\tau_n)}{(u+\tau_n)^2(u-q^{-2}\tau_n)(u-q^2\tau_n)}.$$

Proof. By Equation (V.45), we have

$$\beta_n^{-1}(u-\tau_{n+1}) = (u-\tau_n)\beta_n^{-1} + \tau_n(\beta_n^{-1} - \alpha_n).$$

So

$$\frac{1}{u-\tau_n}\beta_n^{-1} = \beta_n^{-1}\frac{1}{u-\tau_{n+1}} + \frac{\tau_n}{u-\tau_n}(\beta_n^{-1}-\alpha_n)\frac{1}{u-\tau_{n+1}}. \tag{V.55}$$

Therefore

$$\beta_n\frac{1}{u-\tau_n}\beta_n^{-1} = \frac{1}{u-\tau_{n+1}} + \beta_n\frac{\tau_n}{u-\tau_n}(\beta_n^{-1}-\alpha_n)\frac{1}{u-\tau_{n+1}}.$$

Applying Equation (V.52), (V.49), (V.50) to the right side, we have

$$\begin{aligned}\beta_n\frac{1}{u-\tau_n}\beta_n^{-1} &= \frac{1}{u-\tau_{n+1}} - (q-q^{-1})\beta_n\frac{\tau_n}{u-\tau_n}\frac{1}{u-\tau_{n+1}} + i(q-q^{-1})\beta_n\frac{\tau_n}{u-\tau_n}h_n\frac{1}{u-\tau_{n+1}}\\ &= \frac{1}{u-\tau_{n+1}} - (q-q^{-1})\beta_n\frac{1}{u-\tau_{n+1}}\frac{\tau_n}{u-\tau_n} + i(q-q^{-1})\beta_n\frac{\tau_n}{u-\tau_n}h_n\frac{1}{u+\tau_n}\end{aligned} \tag{V.56}$$

By Equation (V.55), (V.52), (V.50), we have

$$\begin{aligned}\beta_n\frac{1}{u-\tau_{n+1}} &= (\beta_n-\beta_n^{-1})\frac{1}{u-\tau_{n+1}} + \beta_n^{-1}\frac{1}{u-\tau_{n+1}}\\ &= (q-q^{-1})\frac{1}{u-\tau_{n+1}} + \frac{1}{u-\tau_n}\beta_n^{-1} - \frac{\tau_n}{u-\tau_n}(\beta_n^{-1}-\alpha_n)\frac{1}{u-\tau_{n+1}}\\ &= (q-q^{-1})\frac{1}{u-\tau_{n+1}} + \frac{1}{u-\tau_n}\beta_n^{-1}\\ &\quad + (q-q^{-1})\frac{\tau_n}{u-\tau_n}\frac{1}{u-\tau_{n+1}} - i(q-q^{-1})\frac{\tau_n}{u-\tau_n}h_n\frac{1}{u+\tau_n}\end{aligned} \tag{V.57}$$

By Equation (V.46), we have

$$(u-\tau_{n+1})\beta_n = \beta_n(u-\tau_n) - \tau_{n+1}(\beta_n - \alpha_n^{-1}).$$

So

$$\beta_n\frac{1}{u-\tau_n} = \frac{1}{u-\tau_{n+1}}\beta_n - \frac{\tau_{n+1}}{u-\tau_{n+1}}(\beta_n-\alpha_n^{-1})\frac{1}{u-\tau_n}.$$

Therefore

$$\beta_n\frac{\tau_n}{u-\tau_n} = \frac{\tau_{n+1}}{u-\tau_{n+1}}\beta_n - \frac{u\tau_{n+1}}{u-\tau_{n+1}}(\beta_n-\alpha_n^{-1})\frac{1}{u-\tau_n}.$$

Note that $\beta_n h_n = -q^{-1} h_n$, so

$$\beta_n \frac{\tau_n}{u-\tau_n} h_n = -q^{-1} \frac{\tau_{n+1}}{u-\tau_{n+1}} h_n - \frac{u\tau_{n+1}}{u-\tau_{n+1}} (\beta_n - \alpha_n^{-1}) \frac{1}{u-\tau_n} h_n.$$

By Equation (V.53), (V.49), (V.48), (V.51), (V.44), we have

$$\beta_n \frac{\tau_n}{u-\tau_n} h_n = q^{-1} \frac{\tau_n}{u+\tau_n} h_n + (q-q^{-1}) \frac{u\tau_n}{(u-\tau_n)(u+\tau_n)} h_n + i(q-q^{-1}) \frac{u\tau_n}{u+\tau_n} \frac{Z_n}{u} h_n \tag{V.58}$$

Applying Equation (V.57), (V.58) to the right side of (V.56), and applying Φ_{n+1} on both sides, we have

$$\begin{aligned}
&\Phi_{n+1}(\beta_n \frac{1}{u-\tau_n} \beta_n^{-1}) \\
=&\Phi_{n+1}(\frac{1}{u-\tau_{n+1}}) \\
&-(q-q^{-1})^2 \Phi_{n+1}(\frac{1}{u-\tau_{n+1}}) \frac{\tau_n}{u-\tau_n} - (q-q^{-1}) \frac{1}{u-\tau_n} \Phi_{n+1}(\beta_n^{-1}) \frac{\tau_n}{u-\tau_n} \\
&-(q-q^{-1})^2 \frac{\tau_n}{u-\tau_n} \Phi_{n+1}(\frac{1}{u-\tau_{n+1}}) \frac{\tau_n}{u-\tau_n} + i(q-q^{-1})^2 \frac{\tau_n}{u-\tau_n} \Phi_{n+1}(h_n) \frac{1}{u+\tau_n} \frac{\tau_n}{u-\tau_n} \\
&+i(q-q^{-1}) q^{-1} \frac{\tau_n}{u+\tau_n} \Phi_{n+1}(h_n) \frac{1}{u+\tau_n} + i(q-q^{-1})^2 \frac{u\tau_n}{(u-\tau_n)(u+\tau_n)} \Phi_{n+1}(h_n) \frac{1}{u+\tau_n} \\
&-(q-q^{-1})^2 \frac{u\tau_n}{u+\tau_n} \frac{Z_n}{u} \Phi_{n+1}(h_n) \frac{1}{u+\tau_n}
\end{aligned}$$

By Proposition V.5.3, τ_n, Z_n, Z_{n+1} commutes with each other. By Equation (V.54), (V.44), we have

$$\begin{aligned}
\frac{Z_n}{u} = &\frac{Z_{n+1}}{u} - (q-q^{-1})^2 \frac{Z_{n+1}}{u} \frac{\tau_n}{u-\tau_n} - iq^{-1}(q-q^{-1}) \frac{\tau_n}{(u-\tau_n)^2} \\
&-(q-q^{-1})^2 \frac{Z_{n+1}}{u} \frac{\tau_n^2}{(u-\tau_n)^2} + i(q-q^{-1})^2 \frac{\tau_n^2}{(u-\tau_n)^2(u+\tau_n)} \\
&+ i(q-q^{-1}) q^{-1} \frac{\tau_n}{(u+\tau_n)^2} + i(q-q^{-1})^2 \frac{u\tau_n}{(u-\tau_n)(u+\tau_n)^2} \\
&-(q-q^{-1})^2 \frac{Z_n}{u} \frac{u\tau_n}{(u+\tau_n)^2}
\end{aligned}$$

Recall that $\delta = \dfrac{i(q+q^{-1})}{q-q^{-1}}$. The above equation can be simplified as

$$\frac{Z_n - \frac{\delta}{2}}{u} \left(1 + (q-q^{-1})^2 \frac{u\tau_n}{(u+\tau_n)^2}\right) = \frac{Z_{n+1} - \frac{\delta}{2}}{u} \left(1 - (q-q^{-1})^2 \frac{u\tau_n}{(u-\tau_n)^2}\right).$$

Therefore

$$Z_{n+1} - \frac{\delta}{2} = (Z_n - \frac{\delta}{2}) \frac{(u-\tau_n)^2 (u+q^{-2}\tau_n)(u+q^2\tau_n)}{(u+\tau_n)^2 (u-q^{-2}\tau_n)(u-q^2\tau_n)}.$$

□

Notation V.5.12. *For a Young diagram μ, let us define*

$$\mu_+ = \{\lambda - \mu \mid \lambda > \mu\};$$
$$\mu_- = \{\mu - \lambda \mid \lambda < \mu\}.$$

Lemma V.5.13. *For a Young diagram μ, $|\mu| = n$, $n \geq 0$,*

$$Z(\mu,u) - \frac{\delta}{2} = \frac{\delta}{2} \prod_{c \in \mu_+} \frac{u+b_c}{u-b_c} \prod_{c \in \mu_-} \frac{u-b_c}{u+b_c}.$$

Proof. Note that

$$Z(\emptyset,u) = \sum_{i \geq 0} \delta u^{-i} = \frac{\delta u}{u-1},$$

so

$$Z(\emptyset,u) - \frac{\delta}{2} = \frac{\delta}{2} \frac{u+1}{u-1}.$$

The statement is true for $n = 0$.

For $|\mu| = n$, $n \geq 1$ and $\nu < \mu$, take $W = \tilde{y}_\mu \rho_{\mu > \nu}(\tilde{y}_\nu \otimes 1)$. Then by the definitions of Z_n and $Z_n(\cdot,u)$ and Proposition V.5.8, we have

$$WZ_n = Z_n(\nu,u)W, \quad WZ_{n+1} = Z(\mu,u)W, \quad W\tau_n = b_{\mu-\nu}W.$$

By Lemma V.5.11, we obtain the recursive formula

$$Z_{\mu,u} - \frac{\delta}{2} = (Z_{\nu,u} - \frac{\delta}{2}) \frac{(u-b_{\mu-\nu})^2(u+q^{-2}b_{\mu-\nu})(u+q^2 b_{\mu-\nu})}{(u+b_{\mu-\nu})^2(u-q^{-2}b_{\mu-\nu})(u-q^2 b_{\mu-\nu})}. \tag{V.59}$$

Therefore

$$Z(\mu,u) - \frac{\delta}{2} = \frac{\delta}{2} \prod_{c \in \mu_+} \frac{u+b_c}{u-b_c} \prod_{c \in \mu_-} \frac{u-b_c}{u+b_c}.$$

□

Theorem V.5.14 (trace formula)**.**

$$< \lambda >= \prod_{c \in \lambda} \frac{i(q^{h(c)} + q^{-h(c)})}{q^{h(c)} - q^{-h(c)}},$$

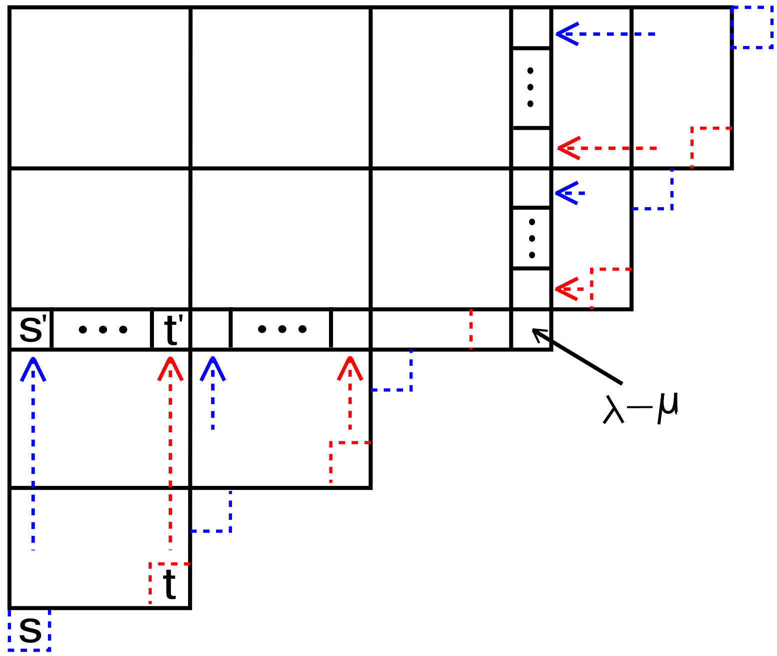
where $h(c)$ is the hook length of the cell c in λ.

Remark V.5.15. If we assume that $q = e^{i\theta}$, then $\delta = \cot\theta$ and

$$< \lambda >= \prod_{c \in \lambda} \cot(h(c)\theta).$$

Proof. For $|\mu| = n$, $n \geq 0$, $\lambda > \mu$, by Lemma V.5.9, V.5.13 and Proposition V.5.8, we have

$$\frac{<\lambda>}{<\mu>} = \delta \prod_{c\in\mu_+, c\neq\lambda-\mu} \frac{b_{\lambda-\mu}+b_c}{b_{\lambda-\mu}-b_c} \prod_{c\in\mu_-} \frac{b_{\lambda-\mu}-b_c}{b_{\lambda-\mu}+b_c}. \tag{V.60}$$



Without loss of generality, let λ be the above Young diagram. The cell $\lambda-\mu$ is marked in the diagram. Let C be the set of cells in μ located in the same row or column as $\lambda-\mu$. The cells in μ_+ except $\lambda-\mu$ are marked by dotted boxes outside μ, and s is the leftmost one. The cells in μ_- are marked by dotted boxes in side μ, and t is the left most one. The cells in C located in the same column as s and t are denoted by s' and t' respectively. Then

$$\frac{b_{\lambda-\mu}+b_s}{b_{\lambda-\mu}-b_s} = \frac{q^{h(s')}+q^{-h(s')}}{q^{h(s')}-q^{-h(s')}};$$
$$\frac{b_{\lambda-\mu}-b_t}{b_{\lambda-\mu}+b_t} = \frac{q^{h(t')-1}-q^{-(h(t')-1)}}{q^{h(t')-1}+q^{-(h(t')-1)}}.$$

So

$$\frac{b_{\lambda-\mu}+b_s}{b_{\lambda-\mu}-b_s}\frac{b_{\lambda-\mu}-b_t}{b_{\lambda-\mu}+b_t} = \prod_{k=h(s')}^{h(t')} \frac{i(q^k+q^{-k})}{q^k-q^{-k}} \times \left(\frac{i(q^{k-1}+q^{-(k-1)})}{q^{k-1}-q^{-(k-1)}}\right)^{-1}$$

Therefore the recursive formula (V.60) can be written as

$$\frac{<\lambda>}{<\mu>} = \delta \prod_{c\in C} \frac{i(q^{h(c)}+q^{-h(c)})}{q^{h(c)}-q^{-h(c)}} \times \left(\frac{i(q^{h(c)-1}+q^{-(h(c)-1)})}{q^{h(c)-1}-q^{-(h(c)-1)}}\right)^{-1}.$$

Note that $<\emptyset>=1$, $\delta=\frac{i(q+q^{-1})}{q-q^{-1}}$ and $h(\lambda-\mu)=1$, so

$$<\lambda>=\prod_{c\in\lambda}\frac{i(q^{h(c)}+q^{-h(c)})}{q^{h(c)}-q^{-h(c)}}.$$

□

V.6 Positivity

We have constructed the matrix units and computed the trace formula of $\mathscr{P}_\bullet$ over the field $\mathbb{C}(q)$. In this subsection, we consider q as a scalar and $\mathscr{P}_\bullet$ as a planar algebra over $\mathbb{C}$. We are going to find out all values of q, such that (a proper quotient of) $\mathscr{P}_\bullet$ is subfactor planar algebra. While working on the field $\mathbb{C}$, we need to be careful about Wenzl's formula (V.38), as it only works for an idempotent with a non-zero trace. When q is not a root of unit, from Theorem V.5.14, $<\lambda>$ is non-zero for any λ. Therefore we have the following:

Proposition V.6.1. *When q is not a root of unit, we have $\mathscr{P}_\bullet \cong YL_\bullet$ as a filtered algebra over the field $\mathbb{C}$. Moreover, $\mathscr{P}_\bullet$ is a semisimple monoidal linear category.*

Proof. Follows from Theorem V.4.5, V.5.14. □

When q is a root of unit, $\mathscr{P}_\bullet$ is no longer semisimple. We need to consider $(\mathscr{P}/\mathrm{Ker})_\bullet$, where Ker is the kernel of the partition function of $\mathscr{P}_\bullet$. If we expect $(\mathscr{P}/\mathrm{Ker})_\bullet$ to be a subfactor planar algebra, then it requires a convolution $*$ which reflects planar tangles vertically and a positive definite trace. In this case, each $(\mathscr{P}/\mathrm{Ker})_m$ is a C^* algebra.

Lemma V.6.2. *If $(\mathscr{P}/Ker)_\bullet$ is a subfactor planar algebra, then $q=e^{\frac{i\pi}{2N+2}}$, for $N\in\mathbb{N}^+$; and $R=R^*$ for the uncappable generator R.*

Proof. Recall that $R^2=id-e$, so $R^*=R$.

To obtain a subfactor planar algebra, δ has to be a positive number. Recall that $q=\frac{i+\delta}{\sqrt{1+\delta^2}}$. So $q=e^{i\theta}$, for some $0<\theta<\frac{\pi}{2}$. When $\frac{\pi}{2N+2}<\theta<\frac{\pi}{2N}$, $N\geq 1$, the minimal idempotents $\tilde{y}_{[i]}$, $1\leq i\leq N$, can be constructed inductively as in Theorem V.4.5, where $[i]$ is the Young diagram with 1 row and N columns. However, by Theorem V.5.14, $<[N]>=\cot(N\theta)<0$. So the trace is not positive semi-definite and we will not obtain a subfactor planar algebra. □

When $q=e^{\frac{i\pi}{2N+2}}$, $N\in\mathbb{N}^+$, let us define $*$ to be the conjugate-linear map on the universal planar algebra generated by R which fixes R and reflect planar tangles vertically. It is easy to check that $*$ fixes the relations of R. So it is well defined on $\mathscr{P}_\bullet$. Moreover, $*$ is a convolution.

We will show the trace of $\mathscr{P}_\bullet$ is positive semi-definite with respect to $*$. Then $(\mathscr{P}/\mathrm{Ker})_\bullet$ is a subfactor planar algebra. However, it becomes more tricky to construct the "matrix units" of $\mathscr{P}_\bullet$, since the basic construction and Wenzl's formula do not always work and s_m as the complement of the support of the basic construction ideal is not defined.

Recall that $\tilde{y}_\lambda$ is defined as $s_{|\lambda|}y_\lambda$ over $\mathbb{C}(q)$. If $\tilde{y}_\lambda$ is well defined over $\mathbb{C}$, then we have the trace formula V.5.14,

$$tr(y_\lambda) = \prod_{c\in\lambda} \cot(h(c)\theta).$$

Observe that the maximal hook length $h(c)$ is obtained on the $(1,1)$ cell, denoted by c_λ. Thus

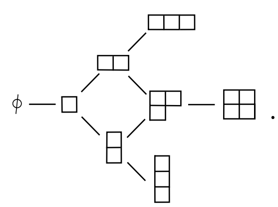
$$\begin{cases} tr(y_\lambda) > 0, \text{ when } h(c_\lambda) \leq N; \\ tr(y_\lambda) = 0, \text{ when } h(c_\lambda) = N+1. \end{cases}$$

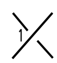
Notation V.6.3. *The $(1,1)$ cell of a Young diagram is denoted by c_λ. Take*

$$\begin{aligned} Y(N) &= \{\lambda \mid h(c_\lambda) \leq N\}; \\ B(N) &= \{\kappa \mid \kappa > \lambda,\ \lambda \in Y(N),\ \kappa \notin Y(N)\}. \end{aligned}$$

Let us define $YL(N)$ to be the sub lattice of Young's lattice YL consisting of $Y(N)$, and $YL(N)_\bullet$ to be the string algebra of $YL(N)$ starting from $\emptyset$.

For example, $YL(4)$ is given by



Let $H_\bullet$ be the Hecke algebra generated by  over $\mathbb{C}$. By the arguments in Section V.1, for any $\mu,\lambda \in Y(N)\cup b(N)$, such that $\mu < \lambda$, we can construct idempotents y_μ, y_λ and morphisms $\rho_{\mu<\lambda}$ from $y_\mu \otimes 1$ to y_λ, $\rho_{\lambda>\mu}$ from y_λ to $y_\mu \otimes 1$. Moreover $y_\mu^* = y_\mu$, $y_\lambda^* = y_\lambda$ and $\rho_{\mu<\lambda}^* = \rho_{\lambda>\mu}$. Then we have the branching formula V.38 for $\mu \in Y(N)$,

$$y_\mu \otimes 1 = \sum_{\lambda>\mu} \rho_{\mu<\lambda}\rho_{\lambda>\mu}.$$

Now let us construct $\tilde{y}_\lambda$, for $\lambda \in Y(N)\cup B(N)$, inductively without applying s_m as follows.

Set up $\tilde{y}_\emptyset = \emptyset$. Suppose $\mu \in Y(N)$ and $\tilde{y}_\lambda$ is constructed. For $\kappa \in Y(N)\cup B(N)$, $\kappa > \mu$, let us define $\tilde{y}_\kappa$ as

$$\tilde{y}_\kappa = \rho_{\kappa>\mu}\left(\tilde{y}_\mu \otimes 1 - \sum_{\lambda<\mu} \frac{<\lambda>}{<\mu>}(\tilde{y}_\mu\otimes 1)(\rho'_{\mu>\lambda}\otimes 1)(\tilde{y}_\lambda\otimes\cap)(\tilde{y}_\lambda\otimes\cup)(\rho'_{\lambda<\mu}\otimes 1)(\tilde{y}_\mu\otimes 1)\right)\rho_{\mu<\kappa}.$$

Recall that ρ and ρ' are renormalizations of $\dot{\rho}$ over $\mathbb{C}(q)$ and $\mathbb{C}$ respectively. So

$$\tilde{y}_\kappa = \rho_{\kappa>\mu}\left(\tilde{y}_\mu \otimes 1 - \sum_{\lambda<\mu} \frac{<\lambda>}{<\mu>}(\tilde{y}_\mu\otimes 1)(\rho_{\mu>\lambda}\otimes 1)(\tilde{y}_\lambda\otimes\cap)(\tilde{y}_\lambda\otimes\cup)(\rho_{\lambda<\mu}\otimes 1)(\tilde{y}_\mu\otimes 1)\right)\rho_{\mu<\kappa}$$

which is also defined over $\mathbb{C}(q)$. By Wenzl's formula V.38, we have $\tilde{y}_\kappa = s_m y_\kappa$ over $\mathbb{C}(q)$. Therefore the definition of $\tilde{y}_\kappa$ over $\mathbb{C}$ is independent of the choice of μ.

We have constructed $\tilde{y}_\lambda$, for $\lambda \in Y(N) \cup B(N)$. Thus Wenzl's formula V.38 holds for $\tilde{y}_\mu$, $\mu \in Y(N)$, over $\mathbb{C}$ as follows

$$\tilde{y}_\mu \otimes 1 = \sum_{\lambda<\mu} \frac{<\lambda>}{<\mu>} (\tilde{y}_\mu \otimes 1)(\rho'_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap)(\tilde{y}_\lambda \otimes \cup)(\rho'_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)$$
$$+ \sum_{\lambda>\mu} (\tilde{y}_\mu \otimes 1)\rho'_{\mu<\lambda}\tilde{y}_\lambda \rho'_{\lambda>\mu}(\tilde{y}_\mu \otimes 1).$$

Lemma V.6.4. *For a spherical planar algebra $\mathscr{P}_\bullet$, if y is a trace zero minimal idempotent in $\mathscr{P}_m$, then y is in the kernel of the partition function of $\mathscr{P}_\bullet$.*

Proof. By spherical isotopy, any closed diagram containing y is of the form $tr(px)$ for some x in $\mathscr{P}_m$. By assumption p is a trace zero minimal idempotent, so $tr(px) = 0$. Therefore y is in the kernel of the partition function of $\mathscr{P}_\bullet$. □

Note that $h(c_\kappa) = N+1$, for any $\kappa \in B(N)$. So $tr(y_\kappa) = 0$. By Lemma V.6.4, we have $y_\kappa \in \mathrm{Ker}$. Therefore in $(\mathscr{P}/\mathrm{Ker})_\bullet$, Wenzl's formula for $\tilde{y}_\mu$, $\mu \in Y(N)$, is given by

$$\tilde{y}_\mu \otimes 1 = \sum_{\lambda<\mu} \frac{<\lambda>}{<\mu>} (\tilde{y}_\mu \otimes 1)(\rho'_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap)(\tilde{y}_\lambda \otimes \cup)(\rho'_{\lambda<\mu} \otimes 1)(\tilde{y}_\mu \otimes 1)$$
$$+ \sum_{\lambda>\mu, \lambda \in Y(N)} (\tilde{y}_\mu \otimes 1)\rho'_{\mu<\lambda}\tilde{y}_\lambda \rho'_{\lambda>\mu}(\tilde{y}_\mu \otimes 1). \tag{V.61}$$

Now let us construct the matrix units of $(\mathscr{P}/\mathrm{Ker})_\bullet$ and show that it is a subfactor planar algebra.

Theorem V.6.5. *When $q = e^{\frac{i\pi}{2N+2}}$, $N \geq 1$, $(\mathscr{P}/Ker)_\bullet$ is a subfactor planar algebra, denoted by $\mathscr{E}_{N+2}$. Its principal graph is $YL(N)$.*

Remark V.6.6. Recall that there is a choice from the complex conjugate for the generator and relations. So for each $q = e^{\frac{i\pi}{2N+2}}$, we obtained a pair of complex conjugate subfactor planar algebras.

Proof. Let $\mathrm{Path}(m)$ be the set of all length m paths t in $YL(N)$ starting from $\emptyset$. For $t \in \mathrm{Path}(m)$ from $\emptyset$ to λ, take t' to be the first length $(m-1)$ path of t from $\emptyset$ to μ. Let us define $\tilde{P}_t^\pm$ inductively as follows,

$$P_\emptyset^\pm = \emptyset;$$
$$\tilde{P}_t^+ = (\tilde{P}_{t'}^+ \otimes 1)\rho'_{\mu<\lambda}\tilde{y}_\lambda, \qquad \text{when } \mu < \lambda;$$
$$\tilde{P}_t^+ = \sqrt{\frac{<\lambda>}{<\mu>}} (\tilde{P}_{t'}^+ \otimes 1)(\rho'_{\mu>\lambda} \otimes 1)(\tilde{y}_\lambda \otimes \cap), \qquad \text{when } \mu > \lambda;$$
$$\tilde{P}_t^- = \sqrt{\frac{<\lambda>}{<\mu>}} (\tilde{P}_{t'}^- \otimes \cup)(\rho'_{\mu<\lambda} \otimes 1)(\tilde{y}_\lambda \otimes 1), \qquad \text{when } \mu < \lambda;$$
$$\tilde{P}_t^- = \tilde{P}_{t'}^+ \rho'_{\mu>\lambda}(\tilde{y}_\lambda \otimes 1), \qquad \text{when } \mu > \lambda.$$

By definitions, we have $y_\lambda^* = y_\lambda$ and $(\tilde{P}_t^+)^* = \tilde{P}_t^-$. By Theorem V.4.5, the map $\omega_m : YL(N)_m \to \mathscr{P}_m$ as a linear extension of

$$\omega_m(t\tau^{-1}) = \tilde{P}_t^+ \tilde{P}_\tau^-$$

is an injective *-homomorphism. Recall that $tr(y_\lambda) > 0$, for any $\lambda \in Y(N)$, so ω_m is still injective passing to quotient $(\mathscr{P}/\mathrm{Ker})_m$.

Applying Wenzl's formula (V.61) to the identity 1_m of $(\mathscr{P}/\mathrm{Ker})_m$, we have

$$1_m = \sum_{t \in \mathrm{Path}(m)} \tilde{P}_t^+ \tilde{P}_t^-.$$

For an m-box x, if $t, \tau \in \mathrm{Path}(m)$ are paths from $\emptyset$ to different vertices, then $\tilde{P}_t^- x \tilde{P}_\tau^+ = 0$ by Theorem V.4.5. If $t, \tau \in \mathrm{Path}(m)$ are paths from $\emptyset$ to μ, then $tr(\tilde{P}_t^+ \tilde{P}_\tau^- \tilde{P}_\tau^+ \tilde{P}_t^-) =< \mu > \neq 0$. Take

$$x_{t,\tau} = \frac{tr(\tilde{P}_t^+ \tilde{P}_t^- x \tilde{P}_\tau^+ \tilde{P}_\tau^- \tilde{P}_\tau^+ \tilde{P}_t^-)}{tr(\tilde{P}_t^+ \tilde{P}_\tau^- \tilde{P}_\tau^+ \tilde{P}_t^-)}.$$

By Theorem V.4.5, we have

$$\tilde{P}_t^+ \tilde{P}_t^- x \tilde{P}_\tau^+ \tilde{P}_\tau^- = x_{t,\tau} \tilde{P}_t^+ \tilde{P}_\tau^-,$$

Let $\mathrm{Pair}(m)$ be the set of all pairs of paths (t, τ) in $\mathrm{Path}(m)$ from $\emptyset$ to the same vertex. Then

$$x = \sum_{(t,\tau) \in \mathrm{Pair}(m)} x_{t,\tau} \tilde{P}_t^+ \tilde{P}_\tau^-.$$

Therefore ω_m is onto $(\mathscr{P}/\mathrm{Ker})_m$.

Since $\omega_m : YL(N)_m \to (\mathscr{P}/\mathrm{Ker})_m$ is *-isomorphism and the trace is positive definite, we have that $(\mathscr{P}/\mathrm{Ker})_\bullet$ is subfactor planar algebra. Moreover, its principal graph is $YL(N)$. □

Corollary V.6.7. *For each m, we have $\mathscr{P}_m = YL(N)_m \oplus Ker_m$, where Ker_m is the two sided ideal of $\mathscr{P}_m$ generated by the trace zero minimal idempotents $\{\tilde{y}_\lambda\}_{\lambda \in B(N), |\lambda| \leq m}$.*

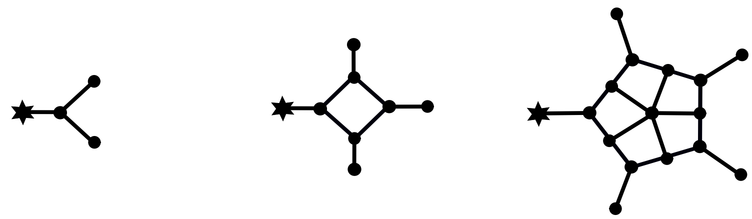
Proof. Note that $\mathrm{Ker}_m \subset \mathrm{Ker}$ and the decomposition

$$1_m = \sum_{t \in \mathrm{Path}(m)} \tilde{P}_t^+ \tilde{P}_t^-$$

also holds in $\mathscr{P}_m/\mathrm{Ker}_m$, so $\mathscr{P}_m = YL(N)_m \oplus \mathrm{Ker}_m$. □

Remark V.6.8. Our strategy of decomposing the non-semisimple algebra $\mathscr{P}_m$ into a direct sum of a semisimple algebra $(\mathscr{P}/\mathrm{Ker})_m$ and an ideal Ker_m also works in other cases, such as Temperley-Lieb, BMW, Bisch-Jones algebras etc. In general, the (planar) algebra $\mathscr{P}_\bullet$ given by generators and relations is semisimple over the field of rational functions in some parameters, but may not be semisimple over $\mathbb{C}$ when the parameters are scalars, in particular roots of unity. First we construct the matrix units for the algebra over rational functions and identify them as loops of a (directed) graph Γ starting from a distinguished vertex $\emptyset$. Then we find out the subgraph Y such that the statistical dimensions of vertices in Y are non-zero and the statistical dimensions of vertices in the boundary B of Y are zero. Then we have the decomposition of $\mathscr{P}_\bullet$ as a direct sum of the string algebra of Y and an ideal generated by trace zero idempotents corresponding to vertices in B.

When $N = 1$, the planar algebra has index 1. When $N = 2$, the planar algebra is the group subfactor planar algebra $\mathbb{Z}_3$. When $N = 3$, it is exactly the extra example in the classification of planar algebras generated by 2-box with at most 14 dimensional 3-boxes, but not in the two families Bisch-Jones and BMW planar algebras. We give the principal graphs for $N = 2, 3, 4$.

.

Proposition V.6.9. *When $q = e^{\frac{ik\pi}{2N+2}}$, $(k, 2N+2) = 1$, the quotient $(\mathscr{P}/Ker)_\bullet$ is a pivotal spherical fusion category. Moreover, the simple objects are given by $Y(N)$.*

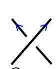
Proof. The argument is similar to the case for $q = e^{\frac{i\pi}{2N+2}}$. □

V.7 Dihedral symmetry

For $N \in \mathbb{N}^+$, $\theta = \frac{\pi}{2N+2}$, $q = e^{i\theta}$, we have constructed the unshaded subfactor planar algebra $\mathscr{E}_\bullet = (\mathscr{P}/\mathrm{Ker})_\bullet$. Its principal graph is $YL(N)$. While considering $\mathscr{E}_\bullet$ as a fusion category, its simple objects are given by $Y(N)$. The dimension of the object $\lambda \in Y(N)$ is given in Lemma V.5.14. Let G be the set of invertible objects, i.e. $G = \{\lambda \in Y(N) \mid <\lambda>= 1\}$. Then G forms a group under $\otimes$. Moreover, G is a subgroup of the automorphism group $\mathrm{Aut}(YL(N))$ of the graph $YL(N)$.

Proposition V.7.1. *Let $r_0 = \emptyset$ and r_k, $1 \leq k \leq N$, be the Young diagram with k rows and each row has $N+1-k$ cells. Then $G = \{r_k \mid 0 \leq k \leq N\}$.*

Proof. Note that $\emptyset$ is in G and it is a univalent vertex in $YL(N)$. So each vertex in G is univalent in $YL(N)$. Then for any vertex λ in G, $\lambda \neq \emptyset$, and any $\kappa > \lambda$, we have $\kappa \in B(N)$. Thus the Young diagram λ is a square with k rows and $N+1-k$ columns, for some $1 \leq k \leq N$, denoted by r_k. Conversely applying the trace formula in Lemma V.5.14, it is easy to check that $<r_k>= 1$ by the central symmetry of the Young diagram r_k and the fact $\cot(n\theta)\cot((N+1-k)\theta) = 1$. □

Since $\mathscr{E}_\bullet$ is a quotient of $\mathscr{P}_\bullet$, we keep the notations $\alpha =$ , α_i, $H_\bullet$, y_λ and $\tilde{y}_\lambda$ for $\mathscr{E}_\bullet$. Let s_m be the complement of the support of the basic construction ideal of $\mathscr{E}_m$, $m \geq 0$. Then $\overline{s_m} = s_m$ and $s_{|\lambda|}y_\lambda = \tilde{y}_\lambda$, for any $\lambda \in Y(N)$.

By Equation V.1 and $V.2$, it is easy to show that (by braided relations)

$$f^{(l)} = 1 \otimes f^{(l-1)} - \frac{[l-1]}{[l]}(1 \otimes f^{(l-1)})(q-\sigma)(1 \otimes f^{(l-1)}); \quad \text{(V.62)}$$

$$g^{(l)} = 1 \otimes g^{(l-1)} - \frac{[l-1]}{[l]}(1 \otimes g^{(l-1)})(q^{-1}+\sigma)(1 \otimes g^{(l-1)}). \quad \text{(V.63)}$$

Recall that $\overline{R} = -R$, so $\overline{s_2(q-\sigma)} = s_2(q^{-1}+\sigma)$. Therefore $\overline{s_l f^{(l)}} = s_l g^{(l)}$ by the recursive formulas (V.1) and (V.63). In particular, $\overline{\tilde{y}_{[N]}} = \tilde{y}_{[1^N]}$. Thus $r_N \otimes r_1 = r_0$ in G.

Proposition V.7.2. *For $N \geq 2$, we have $G = \mathbb{Z}_{N+1}$ and $r_k \otimes r_1 = r_{k+1}$, for $0 \leq k \leq N-1$.*

Proof. Let $d(v,w)$ be the distance of vertices v and w in the graph $YL(N)$. Then $r_k \otimes (\cdot)$ as an automorphism of $YL(N)$ preserves d, for $0 \leq k \leq N$.

Recall that $r_0 = \emptyset$, so $d(r_0, r_l) = |r_l| = (N+1-l)l$. Then

$$d(r_0, r_l) \begin{cases} = N & \text{for } l = 1, N; \\ > N & \text{for } 1 \leq l \leq N. \end{cases}$$

Therefore

$$d(r_k, r_k \otimes r_l) \begin{cases} = N & \text{for } l = 1, N; \\ > N & \text{for } 1 \leq l \leq N. \end{cases}$$

There is a length N path from r_k to r_{k+1} by removing the last column then adding one row. So

$$d(r_k, r_{k+1}) = N.$$

Since $N \geq 2$, we have $r_1 \neq r_N$. Thus $r_k \otimes r_1 \neq r_k \otimes r_N$. Therefore

$$\begin{cases} r_k \otimes r_1 = r_{k+1} \\ r_k \otimes r_N = r_{k-1} \end{cases} \quad \text{or} \quad \begin{cases} r_k \otimes r_1 = r_{k-1} \\ r_k \otimes r_N = r_{k+1} \end{cases}.$$

Note that $r_1 \otimes r_N = r_0$ and

$$r_k \otimes r_1 = r_{k+1} \Rightarrow r_{k+1} \otimes r_N = r_k,$$

so $r_k \otimes r_1 = r_{k+1}$, for $0 \leq k \leq N-1$. □

Observe that the map Ω switching R to $-R$ preserves the relations of R. Thus Ω extends to a $\mathbb{Z}_2$ automorphism of $\mathscr{P}_\bullet$ and $\mathscr{E}$. Moreover, $\Omega(s_m) = s_m$ and $\overline{s_2(q-\sigma)} = s_2(q^{-1}+\sigma)$. By the recursive formulas (V.1) and (V.2), we have $\Omega(s_l f^{(l)}) = \omega(s_l g^{(l)})$. By the construction of y_λ and the fact $\tilde{y}_\lambda = s_{|\lambda|} y_\lambda$, we have that the minimal projection $\Omega(\tilde{y}_\lambda)$ is equivalent to $\tilde{y}_{\Omega(\lambda)}$, where $\Omega(\lambda)$ is the reflection of the Young diagram λ by the diagonal. Thus Ω induces an $\mathbb{Z}_2$ automorphism on the principal graph $YL(N)$ by reflecting the Young diagrams by the diagonal. In particular, $\Omega(r_k) = r_{N+1-k}$. Then $\Omega(r_k \otimes \Omega(\lambda)) = r_{N+1-k} \otimes \lambda$. So G and $\{\Omega\}$ generates the Dihedral group $\mathrm{D}_{2(N+1)}$ in $\mathrm{Aut}(YL(N))$. The Dihedral Symmetries of $YL(N)$ was discovered by Suter in [Sut02]. In our case, it is realized as the invertible objects and automorphisms of $\mathscr{E}$. Furthermore, we have the following

Proposition V.7.3. *Suppose Γ is a sublattice of the Young lattice TL, such that for any $\lambda \in \Gamma$ and $\mu < \lambda$, we have $\mu \in \Gamma$. Then any automorphism of the graph Γ fixing $\emptyset$ is either the identity or the reflection by the diagonal. Consequently*

$$Aut(YL(N)) = D_{2(N+1)}.$$

Proof. Note that the distance from $\emptyset$ to λ is $|\lambda|$. If an automorphism Δ of the graph Γ fixes $\emptyset$, then $|\Delta(\lambda)| = |\lambda|$. Thus $\Delta([1]) = [1]$ and $\delta([2]) = [2]$ or $\delta([2]) = [1,1]$. For a vertex $\lambda \in \Gamma$, the vertices adjacent to λ with $|\lambda| - 1$ cells are given by $\lambda_< := \{\mu \mid \mu < \lambda\}$. Observe that if $\lambda_< = \lambda'_<$, for $|\lambda| \geq 3$, then $\lambda = \lambda'$. So Δ is either the identity or the reflection by the diagonal.

When $\Gamma = YL(N)$, the automorphism Δ fixes the set of univalent vertices $Y(N)$. Note that G acts transitively on $Y(N)$, so $\mathrm{Aut}(YL(N)) = \mathrm{D}_{2(N+1)}$.

□

Corollary V.7.4. *In particular, by the automorphism of YLN given in [Sut02], we have the fusion rule for $\mu \otimes [1^N]$. More precisely, $\mu \otimes [1^N]$ is obtained by removing the first row of μ and adding one column with $N-k$ cells on the left, where k is the number of cells in the first row of μ.*

From the Z_2 automorphism Ω of $\mathscr{E}_\bullet$, we obtain another subfactor planar algebra $\mathscr{E}_\bullet^\Omega$ as the fixed point algebra. This process is also known as orbifold construction or equivariantization. The fusion rules of equivariantizations of fusion categories are given in [BN13]. Thus we can derive the principal graph $YL(N)^\Omega$ of $\mathscr{E}_\bullet^\Omega$ from the principal graph $YL(N)$ of $\mathscr{E}_\bullet$ as follows.

For a vertex $\lambda \in YL(N)$,

(1) if $\Omega(\lambda) = \lambda$, then it splits into two vertices λ_0 and λ_1 in $YL(N)^\Omega$.

(2) If $\Omega(\lambda) \neq \lambda$, then λ and $\Omega(\lambda)$ combine as one vertex $(\lambda, \Omega(\lambda))$ in $YL(N)^\Omega$.

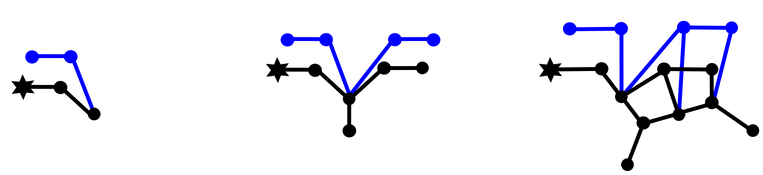
For an edge between μ and λ in $YL(N)$,

(3) if $\Omega(\mu) = \mu$ and $\Omega(\lambda) = \lambda$, then there is an edge between μ_k and λ_k, for $k = 0, 1$.

(4) If $\Omega(\mu) \neq \mu$ and $\Omega(\lambda) = \lambda$, then there is an edge between $(\mu, \Omega(\mu))$ and λ_k, for $k = 0, 1$.

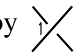
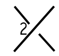
(5) If $\Omega(\mu) \neq \mu$ and $\Omega(\lambda) \neq \lambda$, then there is an edge between $(\mu, \Omega(\mu))$ and $(\lambda, \Omega(\lambda))$.

The Young diagrams invariant under Ω are the ones in the middle of the graph $YL(N)$. So $TL(N)^\Omega$ is the bottom half of $YL(N)$ with one more copy of the vertices in the middle and adjacent edges. We give the principal graph $YL(N)^\Omega$, for $N = 2, 3, 4$.



When $N=3$, it is a near group subfactor planar algebras. (Its even part is a near group fusion category.) It is proved in [LMP] that its invertible objects forms the group $\mathbb{Z}_4$. This near group subfactor planar algebra was first constructed by Izumi in [Izu93]. Therefore we obtain a sequence of complex conjugate pair of subfactor planar algebras which is an extension of the near group subfactor planar algebra for $\mathbb{Z}_4$.

V.8 Quantum subgroups

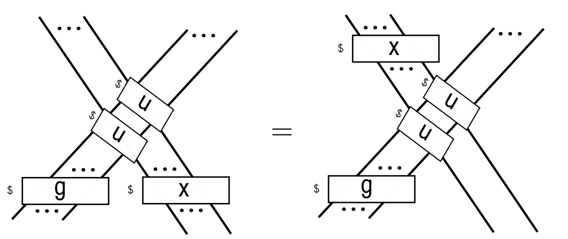
When $q = e^{\frac{i\pi}{2N+2}}$, $\mathscr{E}_\bullet = \mathscr{P}_\bullet/\mathrm{Ker}$ forms a fusion category. Its subcategory generated by  is the HOMFLY category for quantum $SU(N)_{N+2}$. Thus $\mathscr{E}_\bullet$ can be thought as (the representation category of) a subgroup of quantum $SU(N)_{N+2}$ in the sense of Onceanu [Ocn00]. The subcategory generated by  is the HOMFLY category for quantum $SU(N+2)_N$. Thus $\mathscr{E}_\bullet$ can also be thought as a subgroup of quantum $SU(N)$. These quantum subgroups are close related to conformal inclusions $SU(N)_{N+2} \subset SU(\frac{N(N+1)}{2})_1$ and $SU(N+2)_N \subset SU(\frac{(N+2)(N+1)}{2})_1$.

Remark V.8.1. For $n=3,4$, they are listed in Ocneanu's classification of subgroups of quantum $SU(n)$ [Ocn00]. While checking Ocneanu's list with Noah Snyder, we realized that that the zero-graded part of the subgroup E_9 of $SU(3)$ is a near group category with simple objects $1, g, g^2, X$, such that $X \otimes X = \oplus_{k=0}^{2} g^k \oplus 6X$. This example is particularly interesting, because 6 is a non-trivial multiple of the order of the group $\mathbb{Z}_3$.

The subalgebra $H_\bullet/\mathrm{Ker}$ in $\mathscr{E}_\bullet$ modulo the antisymmetrizer $g^{(N)}$ is the representation category of quantum $SU(N)$ at level $N+2$. Note that the trace of $g^{(N)} = y_{[1^N]}$ is one. It has a trace one subprojection $\tilde{y}_{[1^N]}$. Thus $\tilde{y}_{[1^N]} = g^{(N)}$. We are going to prove that $\mathscr{E}_\bullet$ modulo $g^{(N)}$ forms a $\mathbb{Z}_N$ graded pivotal spherical unitary fusion category which can be thought as the representation category of a subgroup of quantum $SU(N)$ at level $N+2$.

Remark V.8.2. The notion of modulo $g^{(N)}$ will be clear in the following arguments.

Definition V.8.3. For an unshaded subfactor planar algebra $\mathscr{S}_\bullet$, a trace one projection g in $\mathscr{E}_m$ is called a $\mathbb{Z}_m$ grading operator if there is a partial isometry u from $g \otimes 1$ to $1 \otimes g$, such that for any $x \in \mathscr{S}_k$. we have

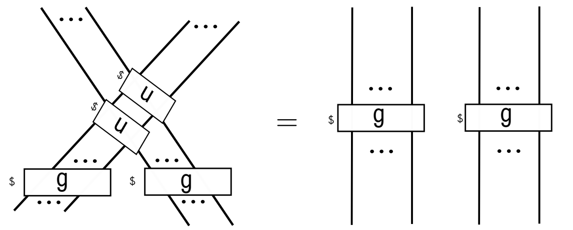


(V.64)

The Jones projection is a $\mathbb{Z}_2$ grading operator.

Note that g has trace one, so both $g \otimes 1$ and $1 \otimes g$ are minimal projections in $\mathscr{E}_{m+1}$. Thus the operator u is unique up to a phase if it exists. Moreover, Equation V.64 is independent of the choice of the phase. Observe that if h is a minimal projection equivalent to g in $\mathscr{E}_m$, then h is also a grading operator. Therefore the definition only depends on the equivalence class of g.

Since $g \otimes g$ is also a minimal projection, we can modify the isometry u by a phase, such that



(V.65)

Proposition V.8.4. *The antisymmetrizer $g^{(N)}$ is a $\mathbb{Z}_N$ grading operator for $\mathscr{E}_\bullet$.*

Proof. Take $U = (g^{(N)} \otimes 1)\alpha_N \alpha_{N-1} \cdots \alpha_1$. Then U is a partial isometry from $g^{(N)} \otimes 1$ to $1 \otimes g^{(N)}$ by type III Reidemester moves of α. By Proposition V.5.3, Equation V.64 holds for any x. □

Definition V.8.5. A $\mathbb{Z}_m$ grading operator g has periodicity k, if k is the smallest positive integer, such that $g^{\otimes k}$ is equivalent to $e^{\otimes \frac{mk}{2}}$.

Note that the equivalent classes of $\mathscr{E}_m$ are presented by minimal projections $y_\lambda \otimes e^{\otimes k}$, for all $\lambda \in Y(N)$, $k < \frac{N(N+1)}{2}$, $|\lambda| + 2k = m$. We are going to switch the grading operator e by $g^{[N]}$.

Let us take $g = g^{(N)}$. Recall that $g^{(N)} = y_{[1^N]}$, and $y_{[1^N]}$ is the generator of the group $\mathbb{Z}_{N+1}$ of invertible objects, so $e^{\otimes \frac{N(N+1)}{2}} \sim g^{\otimes N+1}$. Take

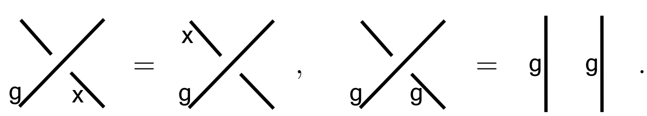
$$Y_g(N) = \{\tilde{y}_\lambda \otimes e^{\otimes k}, \lambda \in Y(N),\ k \geq 0 \mid \tilde{y}_\lambda e^{\otimes k} \nsim \tilde{y}_\mu e^{\otimes l} \otimes g \text{ in } \mathscr{E}_\bullet,\ \forall\ \mu \in Y(N), l \geq 0\}.$$

Recall that $\mu \otimes [1^N]$ is shown in Corollay V.7.4, and

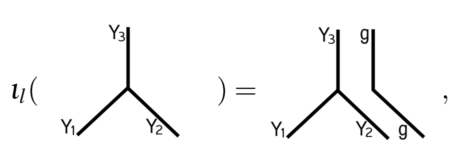
$$\lambda = \mu \otimes [1^N] \iff \tilde{y}_\lambda \sim \tilde{y}_\mu \otimes g^{[N]}.$$

Thus we can use the minimal projections $\tilde{y}_\lambda \otimes e^{\otimes k} \otimes g^{\otimes l}$, for all $\tilde{y}_\lambda \otimes e^{\otimes k} \in Y_g(N)$, $|\lambda| + 2k + Nl = m.$ to present the equivalent classes of $\mathscr{E}_m$. Let us consider $\mathscr{E}_\bullet$ as a $\mathbb{N} \cup \{0\}$ graded (rigid semisimple monoidal) tensor category with simple objects $y_\lambda \otimes e^{\otimes k} \otimes g^{\otimes l}$ graded by $|\lambda| + 2k + Nl = m$, for all $\lambda \in Y(N)$, $k < \frac{N(N+1)}{2}$, $l \geq 0$.

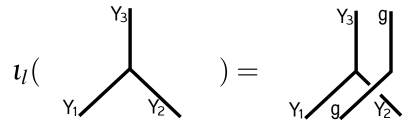
Now we fix the isometry u from $g \otimes 1$ to $1 \otimes g$, such that Equation (V.65) holds. We simplify Equation (V.64) and (V.65) by the following notations,



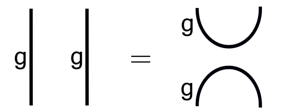
For objects $Y_k, 1 \leq k \leq 3$, let us define $\iota_l :\ \hom(Y_1 \otimes Y_2, Y_3) \to \hom((Y_1 \otimes g) \otimes Y_2, Y_3 \otimes g)$ as



and $\iota_r:\ \hom(Y_1\otimes Y_2,Y_3)\to\hom(Y_1\otimes(Y_2\otimes g),Y_3\otimes g)$ as



Then $\iota_l\iota_r=\iota_r\iota_l$. Recall that g is a trace one projection, thus



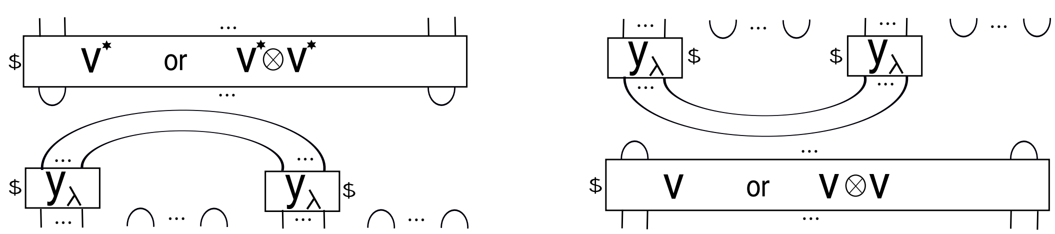
By this relation, it is easy to check that both ι_l and ι_r are invertible by capping off the g string.

We define a relation for objects and morphisms of the $\mathbb{N}\cup\{0\}$ graded tensor category $\mathscr{E}_\bullet$ as follows, for an object Y and a morphism $x\in\hom(Y_1\otimes Y_2,Y_3)$ as follows

$$
\begin{aligned}
&Y\sim Y\otimes g^l && \text{for any object } Y;\\
&\iota_l^{k_1}\iota_r^{k_2}(x)\sim\iota_r^{k_3}\iota_r^{k_4}(x), && \text{for any morphism } x \text{ and } k_j\geq 0,\ 1\leq 4.
\end{aligned}
$$

Since both ι_l and ι_r are invertible, it is easy to check that $\sim$ is an equivalence relation. Moreover, by the above braided relations of g, the $6j$-symbol is preserved under the equivalence relation. Therefore the quotient of $\mathscr{E}_\bullet$ by $\sim$ is a $\mathbb{Z}_N$ graded tensor category. Its simple objects are given by $Y(N)$ and the simple object $y_\lambda\otimes e^{\otimes k}$ is graded by $|\lambda|+2k=m$ modulo N. Therefore the quotient is a fusion category, called $\mathscr{E}_\bullet$ modulo g.

Since $[1^N]^{\otimes N+1}=\emptyset$, we have a non-zero morphism v from $g^{\otimes N+1}$ to $e^{\otimes\frac{N(N+1)}{2}}$. Recall that Ω is the reflection of Young diagrams by the diagonal. For a simple object $y_\lambda\otimes e^{\otimes k}$, it is easy to check that the dual object is given by $y_{\Omega(\lambda)}\otimes\otimes e^{\otimes l}$, such that $2|\lambda|+2k+2l=N(N+1)$ or $2N(N+1)$ with evaluation and coevaluation maps (up to a scalar) as follows

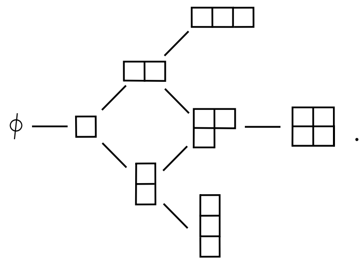


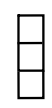
Thus $\mathscr{E}_\bullet$ modulo g is pivotal. Since $\mathscr{E}_\bullet$ is spherical, we have $\mathscr{E}_\bullet$ modulo g is spherical.

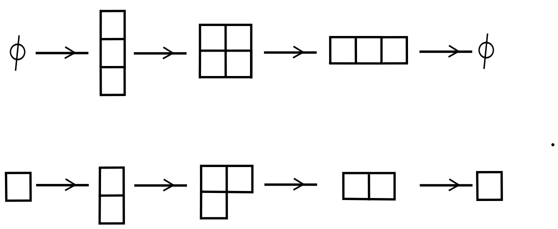
If we consider $\mathscr{E}_\bullet$ as a $\mathbb{N}\cup\{0\}$ graded tensor category, $\oplus_{k=0}^{\infty}g^{\otimes k}$ as a commutative algebra $\oplus_{k=0}^{\infty}g^{\otimes k}$ with a half braiding, then $\mathscr{E}_\bullet$ modulo g can be thought as the deequivariantization of the commutative algebra.

Note that $\tilde{y}_\mu\otimes g=\tilde{y}_{\mu\otimes[1^N]}\otimes e^{\otimes k}$, where $|\mu|+N=|\mu\otimes[1^N]|+2k$. Moreover $g^{\otimes N+1}=e^{\otimes\frac{N(N+1)}{2}}$. Let us fix one Young diagram λ_c in each equivalence class of $Y(N)$ under the action of $(\cdot)\otimes[1^N]$. Then it is more convenient to express the simple objects of $\mathscr{E}_\bullet$ modulo g as $\lambda_c \otimes e^{\otimes j}$ for all λ_c and $0 \le j < \frac{N(N+1)}{2}$.

For example, when $N = 3$, the principal graph of $\mathscr{E}_\bullet$ is



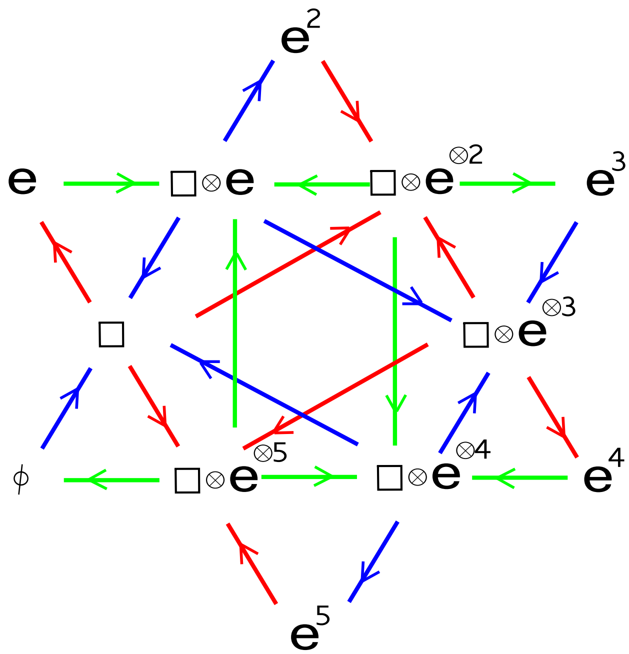
The grading operator is given by $[1^3] =$  . Its action on $Y(3)$ is



The fusion rule is given by

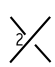
$$
\begin{aligned}
e^{\otimes 6} &= g^{\otimes 4} \sim \emptyset \\
[1] \otimes e &= e \otimes [1] \\
[1] \otimes [1] &= e \oplus [2] \oplus [1^2] \\
&\sim e \oplus ([2] \otimes g) \oplus ([1^2] \otimes g^{\otimes 3}) \\
&= e \oplus ([1] \otimes e^{\otimes 2}) \oplus ([1] \otimes e^{\otimes 5})
\end{aligned}
$$

Thus the $\mathbb{Z}_3$ graded branching rule of $\mathscr{E}_\bullet$ modulo g is



Note that $g^{\otimes k} \otimes e^{\otimes l}$ is also a grading operator with periodicity $\frac{N+1}{(N+1,k)}$. Thus we also obtain a $\mathbb{Z}_{kN+2l}$ graded fusion category as $\mathscr{E}_\bullet$ modulo $g^{\otimes k} \otimes e^{\otimes l}$. For example, when $N=3$, there are only two equivalent classes of $Y(3)$ corresponding to Young diagrams $\emptyset$ and $[1]$. When $k=1$, the simple objects of $\mathscr{E}_\bullet$ modulo $g \otimes e^{\otimes l}$ are given by $e^{\otimes j}$, $[1] \otimes e^{\otimes j}$, for $0 \leq j < 6+4l$. The grading of $e^{\otimes j}$ and $[1] \otimes e^{\otimes j}$ are $2j$ and $2j+1$ modulo $3+2l$ respectively. Moreover the fusion rule is given by

$$
\begin{aligned}
&e^{\otimes 6+4l} \sim \emptyset \\
&[1] \otimes e = e \otimes [1] \\
&[1] \otimes [1] = e \oplus ([1] \otimes e^{\otimes 2+l}) \oplus ([1] \otimes e^{\otimes 5+3l})
\end{aligned}
$$

Recall that $\mathscr{E}_\bullet$ has another braid $\beta =$  which is the generator of the Hecke algebra for quantum $SU(N+2)$ at level N. Thus we can construct the antisymmetrizer $h^{(l)}$, $1 \leq l \leq N+2$ from β_i as follows,

$$h^{(l)} = h^{(l-1)} - \frac{[l-1]}{[l]} h^{(l-1)} (q^{-1} + \beta_i) h^{(l-1)},$$

where $h^{(1)} = 1$. In particular, $h^{(N+2)}$ is a trace one projection. By Proposition V.5.3, $h^{(N+2)}$ is a grading operator for $\mathscr{E}_\bullet$. The $\mathbb{Z}_{N+2}$ graded pivotal spherical unitary fusion category $\mathscr{E}_\bullet$ modulo $h^{(N+2)}$ can be thought as the representation category of a subgroup of quantum $SU(N)$ at level $N+2$.

Let Φ be the trace preserving condition expectation from $\mathscr{E}_{N+2}$ to $\mathscr{E}_N$, i.e. adding two caps on the right of a $N+2$ box. Then it is also a trace preserving condition expectation on the Hecke algebra and $\Phi(h^{(N+2)}) = \frac{tr(h^{(N+2)})}{tr(h^{(N)})} h^{(N)}$.

Recall that s_m is the complement of the support of the basic construction ideal of $\mathscr{E}_m$, so $s_m\alpha_i = s_m\beta_i$. By the inductive construction of the antisymmetrizer, we have $s_m g^{(l)} = s_m h^{(l)}$, for $1 \leq l \leq N$. Recall that $s_m g^{(N)} = g^{(N)}$ which is the grading operator g, so

$$\begin{aligned}\Phi(h^{(N+2)}(g\otimes 1\otimes 1)) =&\Phi(h^{(N+2)}g)\\ =&\frac{tr(h^{(N+2)})}{tr(h^{(N)})}h^{(N)}g\\ =&\frac{tr(h^{(N+2)})}{tr(h^{(N)})}h^{(N)}s_m g\\ =&\frac{tr(h^{(N+2)})}{tr(h^{(N)})}g\\ \neq&0.\end{aligned}$$

Therefore the trace one projection $h^{(N+2)}$ is subequivalent to $g\otimes 1\otimes 1$. Note that

$$1\otimes 1 = e+\tilde{y}_{[11]}+\tilde{y}_{[1^2]}.$$

When $N\geq 3$, $g\otimes 1\otimes 1$ only has one trace one subprojection $g\otimes e$, thus we have the following:

Proposition V.8.6.

$$g\otimes e \sim h^{(N+2)},$$
$$(h^{(N+2)})^{\otimes N+1} \sim (g\otimes e)^{\otimes N+1} \sim e^{\frac{(N+2)(N+1)}{2}}.$$

When $N=3$, the simple objects of the fusion category $\mathscr{E}_\bullet$ modulo $h^{(N+2)}$ are given by $e^{\otimes j}$, $[1]\otimes e^{\otimes j}$, for $0\leq j<10$. Moreover, the fusion rule is given by

$$\begin{aligned}e^{\otimes 10} &= \emptyset\\ [1]\otimes e &= e\otimes [1]\\ [1]\otimes[1] &= e\oplus([1]\otimes e^{\otimes 3})\oplus([1]\otimes e^{\otimes 8})\end{aligned}$$

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APPENDIX

The q-parameterized Yang-Baxter relation planar algebra constructed in Chapter V has the following algebraic presentation. ($\alpha =$ , $h =$ )

$$\alpha_i - \alpha_i^{-1} = (q - q^{-1})\alpha_i$$
$$\alpha_i \alpha_j = \alpha_j \alpha_i,\ \forall\, |i-j| \geq 2$$
$$\alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1}$$
$$h_i^2 = \frac{i(q+q^{-1})}{q-q^{-1}} h_i$$
$$h_i h_j = h_j h_i,\ \forall\, |i-j| \geq 2$$
$$h_i h_{i\pm 1} h_i = h_{i\pm 1} h_i h_{i\pm 1}$$
$$\alpha_i h_i = h_i \alpha_i = q h_i$$
$$\alpha_i h_j = h_j \alpha_i\ \forall\, |i-j| \geq 2$$
$$\alpha_i \alpha_{i+1} h_i = h_{i+1} \alpha_i \alpha_{i+1} = i h_{i+1} h_i$$
$$h_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i h_{i+1} = -i h_i h_{i+1}$$
$$\alpha_i h_{i\pm 1} \alpha_{i\pm 1}^{-1} = \alpha_{i\pm 1}^{-1} h_i \alpha_{i\pm 1}$$
$$h_i h_{i\pm 1} \alpha_i = h_i \alpha_{i\pm 1}^{-1}$$
$$\alpha_i h_{i\pm 1} h_i = \alpha_{i\pm 1} h_i$$
$$h_i \alpha_{i\pm 1} h_i = i q^{-1} h_i$$

where $\alpha_i^{-1} = \alpha_i - q - q^{-1}$