The Equity Premium Puzzle and the Effect of Distributional Uncertainty

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Senior Thesis

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1. Introduction

Rajnish Mehra and Edward Prescott's 1985 paper, "The Equity Premium: A Puzzle" proposed a question that still has no definitive answer. Why does the standard consumption based asset pricing model fail to match the observed average equity premium? Mehra and Prescott studied the rates of return for U.S. equities and U.S. treasury bills from 1889 to 1978 and found that the excess equity return, or equity premium, averaged 5.71%[1]. However, for this same time period the standard model predicts an average equity premium of 1.8%. The discrepancy between the observed and predicted values of the equity premium constitutes what Mehra and Prescott dubbed "The Equity Premium Puzzle."

Since its introduction, numerous explanations have been put forth in order to explain the puzzle[2]. Explanations have included alternative preference structures, survivorship bias, borrowing constraints, tax rate changes, and loss aversion, to name a few. These hypotheses, except for changes in tax rates, all assume that the equity premium is a risk premium, in that it is driven by investor aversion to consumption growth variation. A higher risk aversion parameter suggests that investors are more averse to variation in consumption growth. This leads them to invest in less volatile, safer assets, like treasury bills, instead of equities. As investors demand more risk-free assets, the price of these assets rise, driving their return down. Additionally, companies must now offer higher rates of return through dividends in order to incentivize investors to enter the equity market. This process creates the equity premium that we observe.

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[1] This value is based on Shiller's data

[2] For a more detailed examination of these hypotheses, see Mehra, 2003, and Delong and Magin, 2009

Implicit in these proposed explanations is the assumption that investors have complete knowledge about the true consumption growth process. However, as Martin Weitzman notes, "what is learnable about the future stochastic consumption-growth process from any number of past empirical observations must fall far short of full structural knowledge"[3].

If investors do not have complete knowledge of the consumption growth process, as Weitzman states, then investor risk aversion accounts for only part of the observed equity premium. The rest is driven by investor uncertainty about the distribution of consumption growth. As long as investors have some degree of risk aversion, this lack of knowledge will lead them to invest more money in "risk-free assets," like treasury bills, than they otherwise would. This should drive the equity rate of return higher and the risk-free rate of return lower.

The analysis of the equity premium in this paper focuses on investor uncertainty about the consumption growth process, which has been recently examined by both Barro (2006) and Weitzman (2007). This paper seeks to build off of this existing research by considering several different methods for modeling investor uncertainty. One significant difference between this paper and prior research is the use of a distribution that accounts for the negative skewness present in the consumption growth data. Furthermore, the final section of this paper throws out the assumption of perfect correlation between the consumption growth and dividend growth processes and considers the fact that agents may have uncertainty about the dividend growth process.

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[3] Weitzman, Martin L. "Subjective Expectations and Asset-Return Puzzles." *The American Economic Review*, September 2007, 1102

In addition, this paper seeks to not only match the observed average equity premium, but the average risk-free and equity rates as well. Under Mehra and Prescott's specifications, the observed average equity premium can in fact be obtained if the risk-aversion parameter is set to a high enough number. Such a high risk aversion parameter however, implies an average risk-free rate many times higher than what has been observed. An arbitrarily high risk aversion parameter can "solve" the equity premium puzzle but creates a risk-free rate puzzle instead. This paper seeks to adjust Mehra and Prescott's model so that solving one puzzle does not create another. By matching the average observed risk-free rate, equity rate, and equity premium, the model has more power than if it were to match only one of these rates.

This paper does not intend to present a definitive solution to the Equity Premium Puzzle. It is unlikely that uncertainty about the consumption growth process is the only factor driving the equity premium, as is assumed in models used in this paper. Instead, a "true solution" to the puzzle likely relies on some combination of the proposed explanations. This paper simply seeks to add a further dimension to the existing proposed solutions.

2. The Standard Model

We start with a frictionless economy and a single, infinitely lived, representative agent, who is a stand-in representative for all investors[4]. This agent seeks to maximize

$$E_t\left[\sum_{k=0}^{\infty}\beta^k U\left(c_{t+k}\right)\right] \tag{1}$$

[4] For a more detailed derivation see Mehra and Prescott, 1985, and Mehra, 2003.

where β is the discount rate, E_t is the expected value at time t, and $U(c_t)$ denotes the period t utility derived from the consumption of c_t at time t. The representative agent's utility function is assumed to belong to the constant relative risk aversion class

$$U(c;\alpha) = \frac{c^{1-\alpha}}{1-\alpha} \tag{2}$$

The parameter α is the measure of the agent's risk aversion.

There is a single risky asset, or equity, in this economy that pays a dividend stream $\{y_t\}_{t=1}^{\infty}$. The agent is allocated one unit of this risky asset, so that $s_0=1$. The agent can purchase s_{t+1} units of this asset at time t which may later be sold to obtain a return of $\frac{p_{t+1}+y_{t+1}}{p_t}$ at time $t+1$. The agent can also purchase b_{t+1} units of the risk-free asset at time t which guarantees a gross return of $R_{f,t+1}$ at time $t+1$. The representative agent's problem is to choose $\{c_t, s_{t+1}, b_{t+1}\}$ so as to maximize (1) subject to the budget constraint

$$c_t + b_{t+1} + s_{t+1}p_t \leq R_{f,t}b_t + s_t\left[p_t + y_t\right] \tag{3}$$

for all t.

At time t the agent's total wealth is represented by the right-hand side of the equation. The agent can use this wealth to purchase goods, risk-free assets, or equities, all shown on the left-hand side of the equation. The agent will use his entire wealth to purchase one or several of these three objects in order to maximize utility. Therefore, the budget equation can be rewritten as follows

$$c_t + b_{t+1} + s_{t+1}p_t = R_{f,t}b_t + s_t[p_t + y_t] \tag{4}$$

Rearranging the budget constraint yields

$$c_t = R_{f,t}b_t + s_t\left[p_t + y_t\right] - b_{t+1} - s_{t+1}p_t \tag{5}$$

$$c_{t+1} = R_{f,t+1}b_{t+1} + s_{t+1}\left[p_{t+1} + y_{t+1}\right] - b_{t+2} - s_{t+2}p_{t+1} \tag{6}$$

Maximizing utility with respect to risk-free asset purchases and equity purchases yields the following respective pricing equations

$$1 = E_t\left[\beta \frac{U'(c_{t+1})}{U'(c_t)} R_{f,t+1}\right] \tag{7}$$

$$1 = E_t\left[\beta \frac{U'(c_{t+1})}{U'(c_t)} R_{e,t+1}\right] \tag{8}$$

where $R_{f,t+1}$ and $R_{e,t+1}$ are the risk-free asset and equity respectively. In order to find the equilibrium rate of return for the risk-free asset let $\frac{c_{t+1}}{c_t} = x_{t+1}$. Then using

$$1 = E_t\left[\beta U'(x_{t+1}) R_{f,t+1}\right] \tag{9}$$

and

$$U'(c) = c^{-\alpha} \tag{10}$$

we arrive at

$$1 = \beta R_{f,t+1} E_t\left[x_{t+1}^{-\alpha}\right] \tag{11}$$

This form can be reparameterized so that the expected value term becomes the formula for a moment generating function

$$R_{f,t+1} = \frac{1}{\beta} \frac{1}{E_t\left[e^{-\alpha \ln x_{t+1}}\right]} \tag{12}$$

Assuming $\ln(x_{t+1}) \sim N(\mu_x, \sigma_x{}^2)$, we then have

$$E(e^{tz}) = \exp(t\mu_z + \frac{t^2\sigma_z^2}{2}) \tag{13}$$

which combined with (12) yields

$$R_{f,t+1} = \frac{1}{\beta}\exp\left(\alpha\mu_x - \frac{1}{2}\alpha^2\sigma_x^2\right) \tag{14}$$

which is the equilibrium rate of return for the risk-free asset.

In order to derive the equilibrium rate of return for the equity we must combine (8) and

$$R_{e,t+1} = \frac{p_{t+1} + y_{t+1}}{p_{t+1}} \tag{15}$$

which yields the following

$$p_t = \beta E_t\left[(p_{t+1} + y_{t+1})x_{t+1}^{-\alpha}\right] \tag{16}$$

p_t is linearly increasing in y, so $p_t = wy_t$. Therefore

$$wy_t = \beta E_t\left[(wy_{t+1} + y_{t+1})x_{t+1}^{-\alpha}\right] \tag{17}$$

Let $\frac{y_{t+1}}{y_t} = z_{t+1}$, this simplifies the equation to

$$\frac{w+1}{w} = \frac{1}{\beta E_t\left[z_{t+1}x_{t+1}^{-\alpha}\right]} \tag{18}$$

Combining (15) and the fact that $p_t = wy_t$, we obtain

$$E_t(R_{e,t+1}) = \left(\frac{w+1}{w}\right)E_t(z_{t+1}) \tag{19}$$

Plugging in $\left(\frac{w+1}{w}\right)$ yields

$$E_t(R_{e,t+1}) = \frac{E_t(z_{t+1})}{\beta E_t\left[z_{t+1}x_{t+1}^{-\alpha}\right]} \tag{20}$$

which can be reparameterized in the following way so that the expected value terms take the form of a moment generating function

$$E_t\left(R_{e,t+1}\right)=\frac{E_t\left[\exp^{\ln z_{t+1}}\right]}{\beta E_t\left[\exp^{\ln z_{t+1}-\alpha\ln x_{t+1}}\right]} \tag{21}$$

Note that $a\ln z+b\ln x\sim N(a\mu_z+b\mu_x, a^2\sigma_z^2+b^2\sigma_x^2+2ab\sigma_z\sigma_x)$. Therefore

$$E_t(R_{e,t+1})=\frac{\exp(\mu_z+\frac{\sigma_z^2}{2})}{\beta\exp\left[\mu_z-\alpha\mu_x+\frac{1}{2}\left(\sigma_z^2+\alpha^2\sigma_x^2-2\alpha\sigma_z\sigma_x\right)\right]} \tag{22}$$

Imposing the equilibrium condition that consumption growth and dividend growth are perfectly correlated, i.e. $x_{t+1}=z_{t+1}$, results in

$$E_t(R_{e,t+1})=\frac{1}{\beta}\exp\left[\alpha\mu_x-\frac{1}{2}\alpha^2\sigma_x{}^2+\alpha\sigma_x{}^2\right] \tag{23}$$

which is the equilibrium rate of return for the equity.

2.1 The Equity Premium Puzzle

Taking the natural logarithm of (14) and (23) yields the following

$$\ln R_{f,t+1}=-\ln\beta+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2 \tag{24}$$

$$\ln E(R_{e,t+1})=-\ln\beta+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma^2 \tag{25}$$

Table 1 shows the U.S. Economy sample statistics from 1889-1978.

Table 1: U.S. Economy Sample Statistics, 1889-1978 (Shiller[5])

Statistic	Value
Risk-free rate, R_f	1.0160
Mean return on equity, $E(R_e)$	1.0731
Mean growth rate of log consumption, $ln[E(x)]$	0.02
Standard deviation of growth rate of log consumption, $ln[\sigma_x]$	0.040
Mean equity premium, $E(R_e)-R_f$	0.0571

For the risk aversion parameter α, and for the discount rate parameter β, Mehra and Prescott choose values of 10 and .99 respectively. As Mehra states in his 2003 paper, "I was very liberal in choosing the values for α and β. Most studies would indicate a value for α that is close to 2. If I were to pick a lower value for β, the risk-free rate would be even higher and the premium lower." The prior choice of these two variables, therefore, creates a reasonable upper bound for the predicted equity premium.

Plugging these values into (24) and (25) yields

$$R_{f,t+1} = 1.139$$

$$E(R_{e,t+1}) = 1.157$$

These results imply an average equity premium of 1.8%, significantly less than the 5.71% average equity premium observed in our sample.

2.2 Notes the Puzzle

Based on Mehra and Prescott's analysis, risk aversion alone cannot account for the so-called equity premium puzzle. Furthermore, as their choice of α shows, even unrealistically high value fails to explain the problem. The following analysis shows why.

[5] http://www.econ.yale.edu/~shiller/data.htm

A closer look at the equilibrium rate of return equations shows that they are only slightly different. Dividing the equilibrium equity rate of return, (23), by the equilibrium risk-free rate of return, (14), yields

$$\frac{R_{e,t+1}}{R_{f,t+1}} = \exp(\alpha\sigma^2) \tag{26}$$

Since the risk aversion parameter α is kept constant in Mehra and Prescott's formulation, the variance of consumption growth drives the equity premium. Yet the sample variance is too small to accurately match either the equity premium or the levels of the equity and risk-free asset returns. Investors must be, therefore, basing their beliefs about the variance of consumption growth on more than the corresponding sample statistic. These beliefs must lead investors to invest as if the variance of consumption growth is several times higher than what has been observed. One way to account for this is to remove the assumption that agents possess complete information about the consumption growth process. Instead, they have some degree of uncertainty about the true consumption growth process

Uncertainty about the consumption growth process will increase investors' perceived variance of this process and therefore the equity premium. However, if we are to also account for the levels of the observed average equity and risk-free returns we must put a restriction on the risk aversion parameter, α.

The risk-free rate of return (14) decreases as the variance of consumption growth increases, regardless of the value of the risk aversion parameter. The equity rate (23), by contrast, is expected to rise as the variance of consumption growth increases. A more volatile consumption growth process will lead investors to place their money in safer assets. In order to draw individuals into the equity market in this situation, companies must offer higher returns by offering higher dividends. This is why we expect the equity rate of return to increase as the variance of consumption growth rises. Yet it only does so as long as

$$\frac{1}{2}\alpha^2\sigma_x^2 < \alpha\sigma_x^2 \tag{27}$$

$$\alpha < 2$$

As long as the risk aversion parameter is set to a value less than two, the risk-free rate falls with increasing variance and the equity rate rises with increasing variance, which causes the equity premium to rise as well. This restriction of the value of the risk aversion parameter is not much of a restriction after all. Various empirical studies have suggested a risk aversion parameter ranging from zero to three[6]. This restriction, therefore, keeps the value of risk aversion consistent with empirical observations. For the remainder of this paper, the risk aversion parameter will be set equal to one.

3. Adding Distributional Uncertainty to the Mehra-Prescott Model

We begin by removing the assumption that agents have complete information about the consumption growth process. Instead we attempt to model agents as if they have some uncertainty about the true distribution of this process. The simplest way to account for this is to model agents in such a way that they believe the consumption growth process to be dictated by a convex combination of two distributions. The first distribution, *X*, is the normal distribution fitted with the sample moments. The second distribution, *Y*, is the normal distribution with the same mean as *X*, but a higher variance.

[6] For an extensive review of risk aversion literature see Mehra and Prescott (1985) and DeLong and Magin (2009)

$$X \sim N(\mu_x, \sigma_x^2) \tag{28}$$

$$Y \sim N(\mu_x, \sigma_y^2) \tag{29}$$

Where

$$\sigma_y^2 > \sigma_x^2 \tag{30}$$

Agents weight distribution *X* by *p*, and distribution *Y* by *1-p*. These two distributions are assumed to be uncorrelated.

The convex combination of these distributions results in a new distribution with the same mean but a new variance. The variance is dependent on *p*, which represents the weight agents assign to distribution *X*, and on the respective variances of distributions *X* and *Y*. The new distribution, *Z*, can be interpreted as the belief held by the representative agent about the true distribution of consumption growth, and is characterized as follows

$$Z = pX + (1-p)Y \tag{31}$$

$$Z \sim N(\mu_x, p^2\sigma_x^2 + (1-p)^2\sigma_y^2) \tag{32}$$

The representative agent now makes asset purchasing decisions based on his or her expectations about distribution Z. This yields the following asset return equations

$$R_{f,t+1} = \frac{1}{\beta}\exp\left(\alpha\mu_x - \frac{1}{2}\alpha^2\left(p^2\sigma_x^2 + (1-p)^2\sigma_y^2\right)\right) \tag{33}$$

and

$$R_{e,t+1} = \frac{1}{\beta}\exp\left(\alpha\mu_x - \frac{1}{2}\alpha^2\left(p^2\sigma_x^2 + (1-p)^2\sigma_y^2\right) + \alpha\left(p^2\sigma_x^2 + (1-p)^2\sigma_y^2\right)\right) \tag{34}$$

p=1 may be interpreted as the agent having perfect certainty about the consumption growth process. Under this condition these equations collapse into the equations found in Mehra and Prescott's paper. Using this modeling technique, and assuming a risk aversion parameter of one, the observed average equity premium can be matched. Table 2 shows these results compared to the Mehra-Prescott model assuming a risk aversion parameter of one.

Table 2: Equity Premium Results Comparison (α=1)

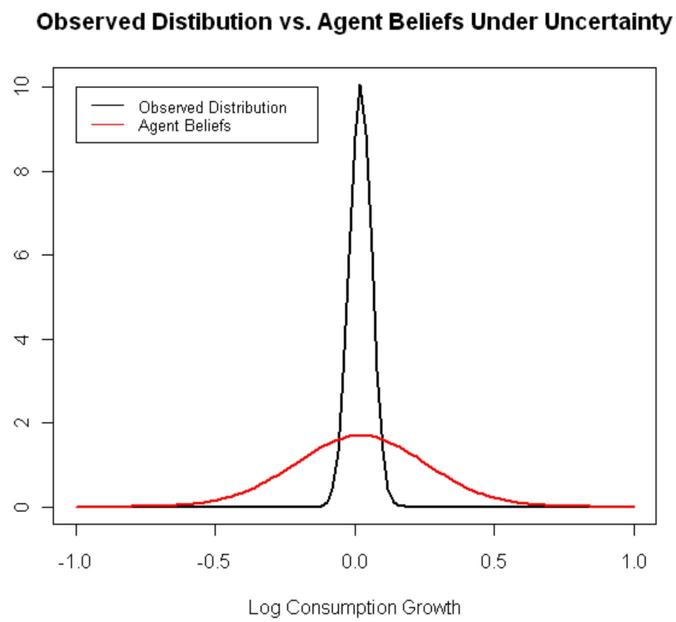
Agent Beliefs about the Log Consumption Growth Process	Standard Deviation	Return on Equity	Risk-Free Return	Equity Risk Premium
Convex Combination,Z	0.235	5.97%	0.26%	5.71%
Observed Distribution, X (Mehra-Prescott)	0.04	3.13%	2.97%	0.16%
Observed Data		7.31%	1.60%	5.71%

A standard deviation of approximately .235 is necessary to match the observed average equity premium under this model. This standard deviation can be obtained regardless of the weight the agent places on the observed distribution. As the agent puts more weight on the observed distribution of consumption growth, *X*, he or she must also believe distribution *Y* to be increasingly more volatile.

As this chart shows, accounting for investor uncertainty about the consumption growth process leads to a highly accurate estimate of the equity premium. Also of note is the fact that the Mehra-Prescott model generates almost no equity premium under a more reasonable assumption about the risk aversion parameter.

The standard deviation of the representative agent's beliefs under distributional uncertainty is nearly six times larger than the observed standard deviation of .04. In this scenario the agent believes the true distribution of consumption growth to be far more volatile than the data he or she has observed. A graphical comparison of the observed distribution, *X*, and the convex combination that represents the agent's beliefs, *Z*, is shown in Figure 1.

Figure 1: Observed Distribution of log-Consumption Growth vs. Agent Beliefs under Uncertainty, assuming Normally Distributed log-Consumption Growth



The convex combination that represents the agent's beliefs about the consumption growth process, shown in red, is far more volatile than the observed distribution of consumption growth, shown in black. The agent's believed consumption growth process presents a greater opportunity for large consumption growth than does the observed curve, but it also presents a greater risk of negative consumption growth. This riskiness leads agents to invest more heavily in risk-free assets, driving the price of these assets up, and their respective returns down. Correspondingly, the equity market must offer a significantly better rate of return, in the form of dividends, than risk-free assets in order to draw agents into this market. This process creates the equity premium that we observe.

This model, which accounts for the fact that agents do not have complete knowledge of the consumption growth process, matches the observed average equity premium. It also produces equity and risk-free returns that are closer to the observed data than the Mehra-Prescott model where agents have perfect certainty about the consumption growth process. These returns are not perfect however. The equity and risk-free asset returns generated by this model are 1.34% lower than the observed equity and risk-free returns respectively. Therefore an additional element must be added in order to accurately match the equity premium, equity rate, and risk-free rate.

4. The Skew Normal Distribution

The skew normal distribution is a generalization of the normal distribution to account for non-zero skewness (Azzalini, 1985). This distribution is parameterized in the following way

$$X \sim SN(\xi, \omega, \gamma) \tag{35}$$

where ξ is the location parameter, ω is the scale parameter, and γ is the shape parameter. When γ =0 the skew normal distribution collapses to a normal distribution with mean ξ and standard deviation ω.

The moment generating function for a skew normal distribution is

$$E[\exp(tx)] = 2\exp\left(\xi t + \frac{\omega^2 t^2}{2}\right)\Phi(\omega\delta t) \tag{36}$$

where

$$\delta = \frac{\gamma}{\sqrt{1+\gamma^2}} \tag{37}$$

and

$$\Phi = \text{Cumulative distribution function of the standard normal distribution}$$

In addition, the first three moments of the distribution are as follows[7]

$$E[X] = \xi + \omega\delta\sqrt{\frac{2}{\pi}} \tag{38}$$

$$Var[X] = \omega^2\left(1 - \frac{2\delta^2}{\pi}\right) \tag{39}$$

$$Skew[X] = \frac{4-\pi}{2}\frac{\left(\delta\sqrt{\frac{2}{\pi}}\right)^3}{\left(1 - \frac{2\delta^2}{\pi}\right)^{3/2}} \tag{40}$$

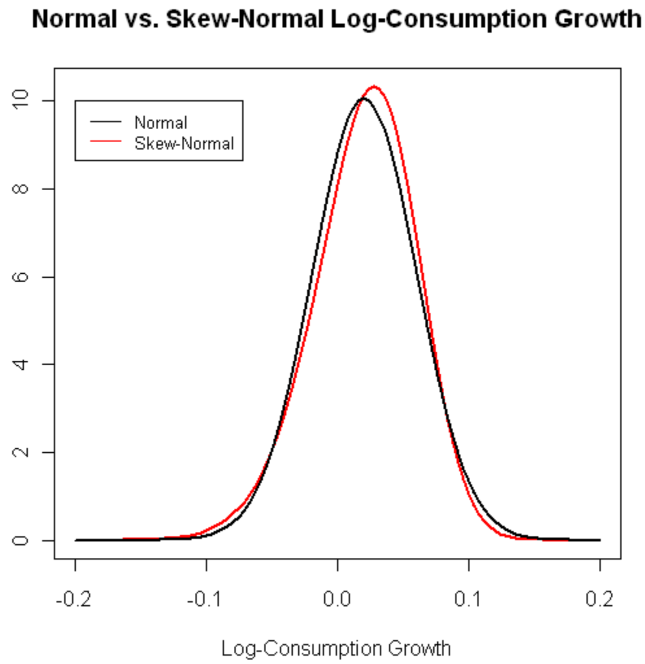
Recall that Mehra and Prescott assume log consumption growth to be normally distributed, which implies zero skewness. However, log consumption growth from 1889 to 1978 has a sample skewness of -.343. This observation suggests replacing the normal distribution assumption about log-consumption growth with a skew normal one, where

$$\ln\left(\frac{c_{t+1}}{c_t}\right) \sim SN(\xi_x, \omega_x, \gamma_x) \tag{41}$$

A comparison of the distribution of log-consumption growth under the two assumptions is shown in Figure 2.

Figure 2: A Comparison of Normally Distributed and Skew-Normally Distributed Log Consumption Growth

[7] http://azzalini.stat.unipd.it/SN/Intro/intro.html



The distribution of consumption growth under a skew-normal distribution is riskier from a large loss perspective than under a normal distribution. Therefore we would assume that investors would factor this riskiness into their investment decisions and this would lead to a higher equity premium than is obtained under the original Mehra-Prescott model. We must first however determine the rate of return equations for the equity and risk-free asset under the assumption of skew normally distributed log-consumption growth.

Plugging the moment generating function for the skew normal distribution, equation (29), into equations (12) and (21) yields

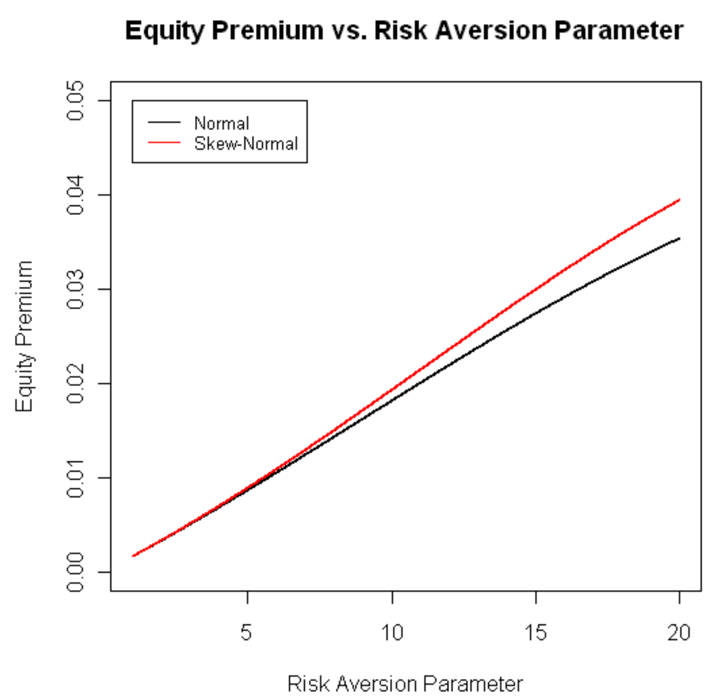
$$R_{f,t+1} = \frac{1}{2\beta}\frac{\exp\left(\alpha\xi_x - \frac{1}{2}\alpha^2\omega_x^2\right)}{\Phi(-\omega_x\delta_x\alpha)} \tag{42}$$

and

$$R_{e,t+1} = \frac{1}{\beta}\exp\left(\alpha\xi_x - \frac{1}{2}\alpha^2\omega_x^2 + \alpha\omega_x^2\right)\frac{\Phi(\omega_x\delta_x)}{\Phi(\omega_x\delta_x(1-\alpha))} \tag{43}$$

If γ_x=0, then δ_x=0, and these two equations collapse into the return equations found in the first part of this paper. Using these equations we can find the equity premium under the skew-normal assumption of log-consumption growth and compare this with the equity premium in Mehra and Prescott's paper. A plot of the equity premiums for both assumptions as a function of the risk-aversion parameter is shown in figure 3.

Figure 3: The Equity Premium as a Function of the Risk Aversion Parameter for Normally Distributed and Skew-Normally Distributed Log-Consumption Growth



Letting log-consumption growth be skew normally distributed has a small effect on the equity premium but only at unreasonably high risk-aversion levels. Since this assumption alone is not enough to generate the observed equity premium, we must also allow for uncertainty about the skew normally distributed log-consumption growth process.

5. Uncertainty about the Skew Normal Log-Consumption Growth Process

The methodology for adding uncertainty to this model is the same as under the assumption of normally distributed log-consumption growth. Agents believe that the true distribution of log-consumption growth is a convex combination of two distributions. These two distributions are *X*, where the skew-normal distribution is fitted with the observed sample statistics, and *Y*, another skew-normal distribution with a higher variance. The variance of a skew normal distribution (39) depends on the scale parameter, ω, and the shape parameter, δ. In order to increase the variance, the value of ω must increase and the absolute value of δ must decrease. Therefore we assume that *Y* shares the same location parameter as *X*, but has a larger scale parameter and a smaller absolute value of the shape parameter.

Let

$$X = \ln\left(\frac{c_{t+1}}{c_t}\right) \sim SN(\xi_x, \omega_x, \gamma_x) \tag{44}$$

$$Y = \ln\left(\frac{c'_{t+1}}{c'_t}\right) \sim SN(\xi_x, \omega_y, \gamma_y) \tag{45}$$

Where

$$\omega_x < \omega_y \tag{46}$$

and

$$|\gamma_x| > |\gamma_y| \tag{47}$$

Agents weight distribution *X* by *p,* and weight distribution *Y* by *1-p*[8].

$$Z = pX + (1-p)Y \tag{48}$$

$$Z \sim SN(\xi_x, \widetilde{\omega}, \widetilde{\gamma}) \tag{49}$$

Where

$$\widetilde{\omega} = (\widetilde{\Omega})^{1/2} \tag{50}$$

$$\widetilde{\gamma} = \frac{\widetilde{\omega}\widetilde{\Omega}^{-1}B'\gamma}{[1+\gamma'(\Omega_z - B\widetilde{\Omega}^{-1}B')\gamma]^{1/2}} = \frac{c_1}{[1+c_2-c_3]^{1/2}} \tag{51}$$

And

$$\widetilde{\Omega} = \begin{pmatrix} p & 1-p \end{pmatrix} \Omega \begin{pmatrix} p \\ 1-p \end{pmatrix} \tag{52}$$

$$\Omega = \omega \Omega_z \omega \tag{53}$$

$$\omega = diag(\omega_x, \omega_y) \tag{54}$$

$$\Omega_z = \begin{pmatrix} 1 & \rho_{xy} \\ \rho_{xy} & 1 \end{pmatrix} \tag{55}$$

$$B = \omega^{-1}\Omega \begin{pmatrix} p \\ 1-p \end{pmatrix} \tag{56}$$

Solving for the numerator of $\widetilde{\gamma}$ yields

$$\begin{aligned} \widetilde{\omega}\widetilde{\Omega}^{-1}B'\gamma = c_1 &= \left(p^2\omega_x^2 + 2p(1-p)\omega_x\omega_y\rho_{xy} + (1-p)^2\omega_y^2\right)^{-\frac{1}{2}} \\ &* \left(p\omega_x\gamma_x + (1-p)\omega_y\gamma_x\rho_{xy} + p\omega_x\gamma_y\rho_{xy} + (1-p)\omega_y\gamma_y\right) \end{aligned} \tag{57}$$

Solving for the denominator of $\widetilde{\gamma}$ yields

$$\gamma\Omega_z\gamma = c_2 = \gamma_x^2 + 2\gamma_x\gamma_y\rho_{xz} + \gamma_y^2 \tag{58}$$

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[8] A more detailed description of the following derivation can be found in Azzalini and Capitanio, 1999

$$\gamma' BB' \gamma \widetilde{\Omega}^{-1} = c_3 = \begin{pmatrix} \gamma_x^2 \left(p\omega_x + (1-p)\omega_y \rho_{xy} \right)^2 \\ + 2\gamma_x \gamma_y \left(p\omega_x + (1-p)\omega_y \rho_{xy} \right)\left(p\omega_x \rho_{xy} + (1-p)\omega_y \right) \\ + \gamma_y^2 \left(p\omega_x \rho_{xy} + (1-p)\omega_y \right)^2 \end{pmatrix} \\ * \left(p\omega_x^2 + 2p(1-p)\omega_x \omega_y \rho_{xy} + (1-p)^2 \omega_y^2 \right)^{-1} \tag{59}$$

$$\left[1 + \gamma' \Omega_z \gamma - \gamma' BB' \gamma \widetilde{\Omega}^{-1} \right]^{1/2} = \left[1 + c_2 - c_3 \right]^{1/2} \tag{60}$$

Solving for $\widetilde{\omega}$ yields

$$\widetilde{\omega} = \left(p^2 \omega_x^2 + 2p(1-p)\omega_x \omega_y \rho_{xy} + (1+p)^2 \omega_y^2 \right)^{1/2} \tag{61}$$

We assume that the correlation between the two distributions that the agent considers is zero. Therefore $\rho_{xy} = 0$. Using the parameters of the convex combination we can derive the equity and risk-free return equations under uncertainty about the skew normally distributed log-consumption growth process. The representative agent now bases his or her expectations on distribution *Z*, which represents his or her beliefs about the true distribution of consumption growth.

Taking the expectations of equations (12) and (21) with respect to distribution *Z* yields the following rate of return equations

$$R_{f,t+1} = \frac{1}{2\beta} \frac{\exp\left(\alpha \xi_x - \frac{1}{2} \alpha^2 \widetilde{\omega}^2 \right)}{\Phi\left(-\widetilde{\omega} \widetilde{\delta} \alpha \right)} \tag{62}$$

$$R_{e,t+1} = \frac{1}{\beta} \exp\left(\alpha \xi_x - \frac{1}{2} \alpha^2 \widetilde{\omega}^2 + \alpha \widetilde{\omega}^2 \right) \frac{\Phi\left(\widetilde{\omega} \widetilde{\delta} \right)}{\Phi\left(\widetilde{\omega} \widetilde{\delta} (1-\alpha) \right)} \tag{63}$$

Where

$$\widetilde{\delta} = \frac{\widetilde{\gamma}}{\sqrt{1 + \widetilde{\gamma}^2}} \tag{64}$$

p=1 can be interpreted as the representative agent having perfect certainty about the consumption growth process. Under this assumption the equilibrium rate of return equations collapse into the equations found in the previous section for skew normally distributed log-consumption growth. However, for p<1 and a risk aversion parameter of one, the observed average equity return, risk-free return, and equity premium can be matched. Table 3 compares these results to the uncertainty model under normally distributed log-consumption growth and to the Mehra-Prescott model, all assuming a risk aversion parameter of one.

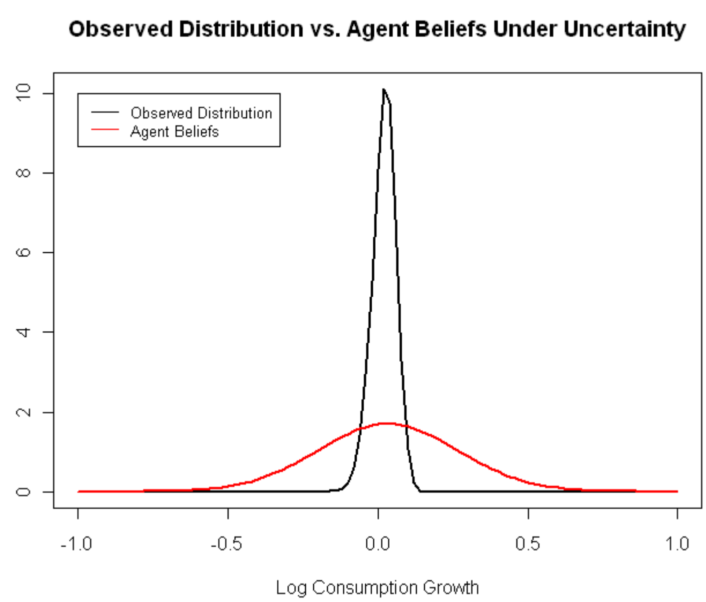
Table 3: Equity Premium Results for the Two Uncertainty Models and the Mehra-Prescott Model (α=1)

Agent Beliefs about the Log Consumption Growth Process	Scale Parameter	Shape Parameter	Return on Equity	Risk-Free Return	Equity Risk Premium
Convex Combination Z (Skew-Normal Assumption)	0.235	-.128	7.31%	1.60%	5.71%
Convex Combination Z (Normal Assumption)	0.235	0	5.97%	0.26%	5.71%
Observed Distribution X (Mehra-Prescott)	0.04	0	3.13%	2.97%	0.16%
Observed Data			7.31%	1.60%	5.71%

This model is a significant improvement over the model where the representative agent is uncertain about the log-consumption growth process but assumed it to be normally distributed. In that model the observed average equity premium could be matched, but the average equity and risk-free returns could not. In this case all three can be accurately matched.

The parameters necessary to match the observed values of the risk free rate, equity rate, and equity premium can be obtained regardless of the weight the agent places on the observed distribution. These two parameters imply a variance of .233, which is slightly less than the agent's beliefs about the true variance under the assumption of normally distributed log-consumption growth. A graphical comparison of the observed distribution and convex combination, which represents the agent's beliefs about the distribution of log-consumption growth, is shown in Figure 4.

Figure 4: Observed Distribution of log-Consumption Growth vs. Agent Beliefs under Uncertainty, Skew Normally Distributed log-Consumption Growth



As in the earlier example where log-consumption growth was assumed to be normally distributed, the convex combination that represents the agent's beliefs about the consumption growth process is more volatile than the observed distribution. These beliefs lead investors to invest in safer, risk-free assets, in order to reduce the volatility of future consumption growth.

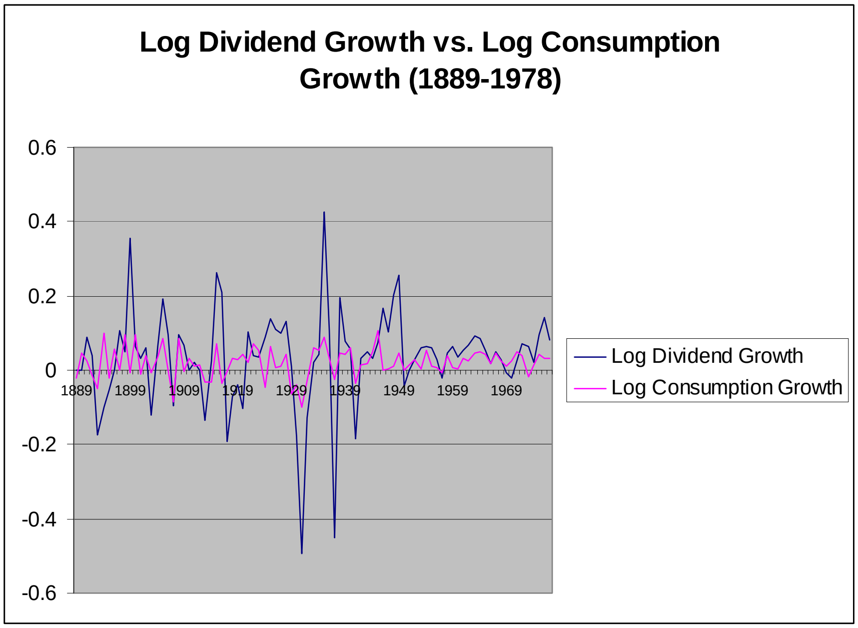
A slight degree of skewness in the agent's belief about the true distribution of consumption growth generates better estimates for the equity and risk-free rates of return than does the model where the agent assumes no skewness. This further justifies the earlier decision to transition from normally distributed log consumption growth to skew normally distributed log consumption growth.

6. The Skew Normal and Skew-T Mixed Model

By generalizing the log-consumption growth process to allow for non-zero skewness and by accounting for agent uncertainty about this process, the observed average equity rate, risk-free rate, and equity premium can be matched. However, this model assumes that log-consumption growth and log-dividend growth are perfectly correlated, when in fact log-dividend growth is far more volatile. Figure 5 shows log-consumption growth and log-dividend growth taken at year end from 1889 to 1978[9].

[9] This data is taken from Shiller

Figure 5: Log Dividend Growth vs. Log Consumption Growth from 1889 to 1978



This offers compelling evidence to treat log-dividend growth and log-consumption growth as if they are not perfectly correlated. From the representative agent's standpoint, the log-dividend growth process is a more volatile, and therefore more uncertain, process than log-consumption growth. Recall that $x_{t+1} = \frac{c_{t+1}}{c_t}$ is consumption growth and $z_{t+1} = \frac{y_{t+1}}{y_t}$ is dividend growth. Let[10]

$$W_{t+1} \sim SN(\xi_x = 0, \omega_x, \gamma_x) \tag{65}$$

And

$$V_{t+1} \sim \chi_v^2 / v \tag{66}$$

Where v represents the degrees of freedom associated with the χ^2 distribution.

Define

[10] The following formulation is taken from Azzalini and Capitanio, 2003.

$$\ln z_{t+1} = \xi_x + V_{t+1}^{-1/2} W_{t+1} \tag{67}$$

And

$$\ln x_{t+1} = \xi_x + W_{t+1} \tag{68}$$

Both z_{t+1} and x_{t+1} are log-Skew-t processes

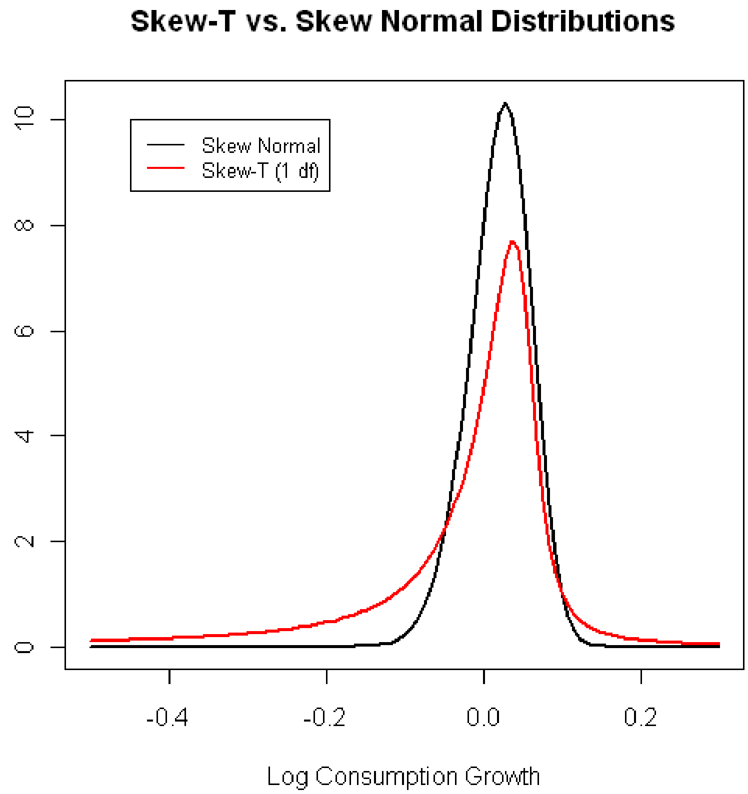
$$\ln z_{t+1} \sim ST(\xi_x, \omega_x, \gamma_x, v) \tag{69}$$

$$\ln x_{t+1} \sim ST(\xi_x, \omega_x, \gamma_x, \infty) \tag{70}$$

Note: $ST(\xi_x, \omega_x, \gamma_x, \infty) = SN(\xi_x, \omega_x, \gamma_x)$

We assume that the representative agent is certain that the log-consumption growth process follows a skew normal distribution but is less certain about the log-dividend growth process. The parameter *v* can be viewed as the posterior distribution for an agent with certainty about the log-dividend growth process up to the skewness and degrees of freedom parameters. For simplicity we assume that the agent holds dogmatic prior beliefs over these parameters. In order to model this we let *v*=1. As *v* increases the agent has greater certainty about the log-dividend growth process and his or her beliefs about this process approach a skew normal distribution. Figure 6 compares the skew-normal distribution to the skew-t distribution with one degree of freedom.

Figure 6: Skew-Normal vs. Skew-T Distribution with One Degree of Freedom



Note that the more uncertain the agent is about the dividend growth process the riskier it appears. Under this formulation, the log-dividend growth process has a much larger downside risk than the log-consumption growth process. This inherent riskiness should lead the agent to allocate his or her wealth to risk-free assets over equities. This would drive the risk-free rate down and the equity rate up, leading to an equity premium.

Under this formulation the risk-free rate of return equation is the same as in the case where log-consumption growth was assumed to be skew normally distributed and agents were completely certain about this process. The equity rate, however, will be different. This rate must calculated based on simulations since there is no joint moment generating function for a skew normal and skew-t distribution.

In simulating this moment generating function we operate under the assumption that it is well defined. Proving this mathematically is beyond the scope of this paper. However, we know that the skew-normal distribution has a well defined moment generating function. In addition, alternative parameterizations of the skew-t distribution, such as those examined by Kjersti and Haff (2006) and Kim (2007), show that the moment generating function is defined as long as the value of *t* in the moment generating function is larger than the skewness parameter. Based on this we assume that the parameterization used in this paper also has a well defined moment generating function and therefore that the joint moment generating function for the skew-normal and skew-t distributions is well defined.

With this in mind we present the following results which have been generated using a Monte Carlo analysis of 50,000 simulations.

$$R_{f,t+1} = 1.0298$$

$$R_{e,t+1} = 1.0935$$

$$R_{e,t+1} - R_{f,t+1} = .0637$$

While these numbers do not match the average observed rates of return or the average equity premium, they are more accurate than the Mehra-Prescott model where the agent has complete certainty about the log-consumption and log-dividend growth processes. In addition, this model eliminates the false assumption that consumption growth and dividend growth are perfectly correlated, which the previous models did not.

7. Conclusion

A simple formulation of uncertainty about the consumption growth process is all that is necessary to match the average observed average equity premium. However, while this formulation, which assumed normally distributed log consumption growth, matched the average equity premium, it failed to accurately match the average risk-free and equity rates of return. It predicted an average risk-free rate of .26% compared to the observed 1.60%, and predicted an average equity rate of 5.97% compared to the observed 7.31%.

The log consumption growth data from 1889-1978 was seen to experience negative skewness, which prompted a new assumption about the distribution of consumption growth. Under the Mehra and Prescott formulation, log-consumption growth was assumed to be normally distributed. However, in light of the skewness present in the data, the normal distribution was generalized to the skew-normal distribution. This assumption, along with agent uncertainty about the true consumption growth process, successfully matched the average observed equity premium, risk-free rate of return, and equity rate of return from 1889 to 1978.

Finally the assumption about the perfect correlation between consumption growth and dividend growth was relaxed. In this case the agent was modeled as having complete certainty about the consumption growth process but some uncertainty about the true dividend growth process. This case generated an equity premium of 6.37% with an equity return of 9.35% and a risk-free return of 2.98%. While these numbers do not match the observed average rates as well as prior cases did, this model does remove a false assumption that the prior cases did not address.

There appears to be reason for further research in analyzing the effect of distributional uncertainty upon the equity premium. This paper put forth simple models in order to examine the equity premium, but more sophisticated models may better describe investor beliefs about the consumption growth process. In addition, while the relaxation of the assumption of perfect correlation between log-consumption growth and log-dividend growth is a step towards a more realistic asset pricing model, this model still failed to match the equity premium as the simpler models did. The initial results from this model are promising however, and suggest the need for further analysis.

Appendix

Figure 4: A Comparison of the Models Used in this Paper (α=1)

Agent Beliefs about the Consumption Growth Process	Scale Parameter	Shape Parameter	Return on Equity	Risk-Free Return	Equity Risk Premium
Observed Distribution X (Mehra-Prescott)	0.04	0	3.13%	2.97%	0.16%
Convex Combination Z (Normal Assumption)	0.235	0	5.97%	0.26%	5.71%
Convex Combination Z (Skew-Normal Assumption)	0.235	-.128	7.31%	1.60%	5.71%
Skew Normal/Skew-T Mixed Model (Uncertainty about the Dividend Growth Process)	NA	NA	9.35%	2.98%	6.37%
Observed			7.31%	1.60%	5.71%

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